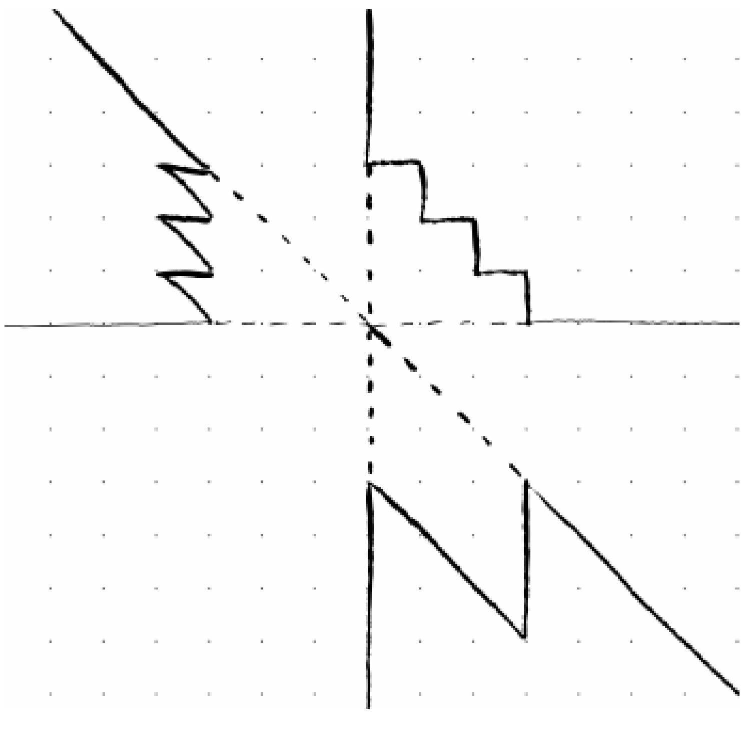
UNIVERSITAT DE BARCELONA

Vector bundles and sheaves on toric varieties

Tesi doctoral de Martí Salat Moltó



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Tesi de doctorat

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Programa de Doctorat en Matemàtiques i Informàtica de l'Escola de Doctorat de la Universitat de Barcelona, Facultat de Matemàtiques i Informàtica, Departament de Matemàtiques i Informàtica.

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Martí Salat Moltó.

Certifico que la present memòria ha estat desenvolupada per Martí Salat Moltó i dirigida per mi.

Dra. Rosa María Miró Roig
1 de setembre de 2022

Abstract

Framed within the areas of algebraic geometry and commutative algebra, this thesis contributes to the study of sheaves and vector bundles on toric varieties. From different perspectives, we take advantage of the theory on toric varieties to address two main problems: a better understanding of the structure of equivariant sheaves on a toric variety, and the Ein-Lazarsfeld-Mustopa conjecture concerning the stability of syzygy bundles on projective varieties.

After a preliminary Chapter 1, the core of this dissertation is developed along three main chapters. The plot line begins with the study of equivariant torsion-free sheaves, and evolves to the study of equivariant reflexive sheaves with an application towards the problem finding equivariant Ulrich bundles on a projective toric variety. Finally, we end this dissertation by addressing the stability of syzygy bundles on certain smooth complete toric varieties, and their moduli space, contributing to the Ein-Lazarsfeld-Mustopa conjecture.

More precisely, Chapter 1 contains the preliminary definitions and notions used in the main body of this work. We introduce the notion of a toric variety and its main features, highlighting the notion of a Cox ring and the algebraic-correspondence between modules and sheaves. Particularly, we focus our attention on equivariant sheaves on a toric variety. We recall the Klyachko construction describing torsion-free equivariant sheaves by means of a family of filtered vector spaces, and we illustrate it with many examples.

In Chapter 2, we focus our attention on the study of equivariant torsion-free sheaves, connected in a very natural way to the theory of monomial ideals. We introduce the notion of a Klyachko diagram, which generalizes the classical stair-case diagram of a monomial ideal. We pro-

vide many examples to illustrate the results throughout the two main sections of this chapter. After describing methods to compute the Klyachko diagram of a monomial ideal, we use it to describe the first local cohomology module, which measures the saturatedness of a monomial ideal. Finally, we apply the notion of a Klyachko diagram to the computation of the Hilbert function and the Hilbert polynomial of a monomial ideal. As a consequence, we characterize all monomial ideals having constant Hilbert polynomial, in terms of the shape of the Klyachko diagram.

Chapter 3 is devoted to the study of equivariant reflexive sheaves on a smooth complete toric variety. We describe a family of lattice polytopes encoding how the global sections of an equivariant reflexive sheaf change as we twist it by a line bundle. In particular, this gives a method to compute the Hilbert polynomial of an equivariant reflexive sheaf. We study in detail the case of smooth toric varieties with splitting fan. We are able to give bounds for the multigraded initial degree and for the multigraded regularity index of an equivariant reflexive sheaf on a smooth toric variety with splitting fan. From the latter result we give a method to compute explicitly the Hilbert polynomial of an equivariant reflexive sheaf on a smooth toric variety with splitting fan. Finally, we apply these tools to present a method aimed to find equivariant Ulrich bundles on a Hirzebruch surface, and we give an example of a rank 3 equivariant Ulrich bundle in the first Hirzebruch surface.

Chapter 4 treats the stability of syzygy bundles on a certain toric variety. We contribute to the Ein-Lazarsfeld-Mustopa conjecture, by proving the stability of the syzygy bundle of any polarization of a blow-up of a projective space along a linear subspace. Finally, we study the rigidness of the syzygy bundles in this setting, all of which correspond to smooth points in their associated moduli space.

Per els pares

Per a la Liena

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Notation

$\mathbb{C}$	an algebraically closed field of characteristic zero
$\mathbb{T}$	an algebraic torus $\mathbb{T} \cong (\mathbb{C}^*)^n$
$n(\rho)$	the first lattice point of a ray ρ
M	the character lattice $M = \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*)$
$M_{\mathbb{R}}$	$M \otimes \mathbb{R}$
N	the lattice $N = \mathrm{Hom}(M, \mathbb{Z})$ of one-parameter subgroups
$N_{\mathbb{R}}$	$N \otimes \mathbb{R}$
σ	a cone in $N_{\mathbb{R}}$
$\sigma^{\vee}$	the dual cone of σ
Σ	a fan in $N_{\mathbb{R}}$
$\sigma(k)$	the set of $k-$dimensional faces of σ
$\Sigma(k)$	the set of $k-$dimensional cones
$\tau \prec \sigma$	a face τ of a cone σ
D_{ρ}	the $\mathbb{T}-$invariant Weil divisor associated to a ray ρ
$\mathrm{Cl}(X)$	the class group of a toric variety X
R	the Cox ring of a toric variety

B	the irrelevant ideal of R
$\tilde{E}$	the quasi-coherent sheaf associated to a homogeneous module E on R
$\mathrm{H}^i_B(E)$	the ith local cohomology module of a module E on R with respect to B
$\mathrm{H}^i(X, \mathcal{E})$	the ith cohomology vector space of a quasi-coherent sheaf on X
$\chi(\mathcal{E})$	the Euler characteristic of a sheaf $\mathcal{E}$
$P_{\mathcal{E}}$	the Hilbert polynomial of a sheaf $\mathcal{E}$
$\{(\mathcal{C}^\sigma_I, \Delta^\sigma_I)\}_{\sigma\in\Sigma}$	Klyachko diagram of a monomial ideal I in R
$\Omega_{\mathcal{E}}$	The polytope in $M_{\mathbb{R}}$ bounding the non-zero global sections of an equivariant reflexive sheaf $\mathcal{E}$.
$\Omega_{\underline{\lambda}}$	A subpolytope in the tesselation of Ω
$\mu_H(\mathcal{E})$	slope of a vector bundle in a polarized variety (X, H)
$\bar{P}_{\mathcal{E}}(m)$	the reduced Hilbert polynomial of a sheaf $\mathcal{E}$.
$M_{X,P}$	moduli functor of vector bundles with fixed reduced Hilbert polynomial P
$M^s_{X,H,P}$	moduli functor of $H-$stable vector bundles on a polarized projective variety (X, H)

Introduction

This thesis is framed in the crossroads of algebraic geometry and commutative algebra, with an ultimate purpose: the study of sheaves on toric varieties. With tight connections to other areas in mathematics, toric varieties are ubiquitous in algebra and geometry since they comprise and generalize many fundamental varieties such as the projective space or the Hirzebruch surface. Along this dissertation we intertwine methods of algebraic geometry, commutative algebra and combinatorics to address the study of different kinds of sheaves on toric varieties: equivariant torsion-free sheaves, reflexive sheaves, locally free sheaves and syzygy bundles.

One of the most remarkable features regarding toric varieties is their intrinsic algebro-combinatorial structure. One may benefit from this rich combinatorics to describe essential invariants of sheaves on toric varieties. In this thesis, we take advantadge of this distinctive trait to shed light on the Hilbert polynomial and the regularity of reflexive equivariant sheaves, making connections to the search of Ulrich bundles. On the other hand, the algebraic framework provide tools to study the syzygy bundles on toric varieties. From this point of view, we address the Ein-Lazarsfeld-Mustopa conjecture, regarding the stability of syzygy bundles in polarized projective varieties.

Toric varieties can be constructed from many points of view, linking lattice polytopes, monomial parametrizations, algebraic quotients or semigroup rings. Focusing in the latter, we take X a smooth complete toric variety of dimension n, containing a torus $\mathbb{T}$ as a dense open subset. The toric structure of X is described by a fan Σ, which is a collection of cones σ in $\mathbb{R}^n$ parameterizing the orbits of the action of $\mathbb{T}$ on X. We set $M = \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*)$ the character group of $\mathbb{T}$ and $N = \mathrm{Hom}(M, \mathbb{Z})$ its dual lattice. We have that

$$\mathbb{T} \cong \mathrm{Spec}(\mathbb{C}[M]) \cong N \otimes \mathbb{C}^*.$$

Each cone $\sigma \subset N_{\mathbb{R}}$ in the fan Σ corresponds to an affine toric variety

$$U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma]) \supset \mathrm{Spec}(\mathbb{C}[M]) \cong \mathbb{T},$$

where $S_\sigma := \sigma^\vee \cap M$ and $\sigma^\vee$ is the dual cone of σ. The collection of cones in the fan Σ intersects in subcones, also contained in Σ. Thus, we can recover X from the fan by gluing all the affine toric varieties U_σ with $\sigma \in \Sigma$. On the other hand, the cones in Σ are in correspondence with the $\mathbb{T}-$invariant subvarieties of X. In particular a ray $\rho \in \Sigma$ (i.e. a cone of dimension 1) corresponds to a $\mathbb{T}-$invariant codimension one subvariety $D_\rho \subset X$ and they generate the group of $\mathbb{T}-$invariant Weil divisors:

$$\mathrm{Div}_{\mathbb{T}}(X) \cong \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z} D_\rho$$

where $\Sigma(1)$ is the set of rays in Σ.

Our purpose in this thesis is to delve into the study of sheaves on the toric variety X. For instance, we can use the description of the group of $\mathbb{T}-$invariant Weil divisors, to take a glance on how equivariant line bundles on X look like. If we quotient $\mathrm{Div}_{\mathbb{T}}(X)$ by linear equivalence, $\mathbb{T}-$invariant Weil divisors describe the whole class group of X in the following exact sequence:

$$0 \longrightarrow M \xrightarrow{\phi} \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z} D_\rho \xrightarrow{\pi} \mathrm{Cl}(X) \longrightarrow 0,$$

where $M = \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*)$ is the character lattice of the torus $\mathbb{T}$. In particular, for any $\mathbb{T}-$invariant Weil divisor $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$, the line bundle $\mathcal{O}(D)$ is equivariant. To study other types of sheaves on X we introduce the notion of the Cox ring, which is a polynomial ring

$$R = \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$$

graded by the class group of divisors $\mathrm{Cl}(X)$ such that

$$\deg(x_\rho) = [D_\rho] \in \mathrm{Cl}(X).$$

When $X = \mathbb{P}^n$ is the projective space, the Cox ring is the standard graded polynomial ring $R = \mathbb{C}[x_0, \ldots, x_n]$, and we have a correspondence between quasi-coherent sheaves on $\mathbb{P}^n$ and standard graded modules on R, which is bijective up to saturation with respect to the homogeneous maximal ideal $\mathfrak{m} := (x_0, \ldots, x_n)$. For X a smooth complete toric variety this correspondence holds and there is an algebraic-geometric dictionary between quasi-coherent sheaves on X and $\mathrm{Cl}(X)-$graded modules on R, modulo saturation with respect to the maximal homogeneous monomial ideal

$$B = \left(x^{\hat{\sigma}} := \prod_{\rho \notin \sigma(1)} x_\rho \,\middle|\, \sigma \in \Sigma \right).$$

More precisely, we have the following $\mathrm{Cl}(X)-$graded isomorphism

$$R \cong \bigoplus_{\alpha \in \mathrm{Cl}(X)} \mathrm{H}^0(X, \mathcal{O}(\alpha)),$$

and a $\mathrm{Cl}(X)-$graded module E corresponds to a quasi-coherent sheaf $\mathcal{E} = \tilde{E}$. The module E and the sheaf $\mathcal{E}$ are related by the following $\mathrm{Cl}(X)-$graded exact sequence and isomorphisms:

$$0 \longrightarrow \mathrm{H}^0_B(E) \longrightarrow E \longrightarrow \bigoplus_{\alpha \in \mathrm{Cl}(X)} \mathrm{H}^0(X, \mathcal{E}(\alpha)) \longrightarrow \mathrm{H}^1_B(E) \to 0,$$

and for any $i \geq 1$

$$\mathrm{H}^{i+1}_B(E) \cong \bigoplus_{\alpha \in \mathrm{Cl}(X)} \mathrm{H}^i(X, \mathcal{E}(\alpha)).$$

We can use this correspondence as a powerful tool to study sheaves on a toric variety. In this dissertation, we impose extra assumptions to a sheaf $\mathcal{E}$ on X, and exploit the algebraic-geometric properties from this correspondence to study the sheaf $\mathcal{E}$ or the $\mathrm{Cl}(X)-$graded modules associated to it. One example of these assumptions is to impose the sheaf $\mathcal{E}$ to be equivariant. As it turns out, among all the $\mathrm{Cl}(X)-$graded modules, those which yield an equivariant sheaf are those which are additionally $\mathbb{Z}^{|\Sigma(1)|}-$graded. That is, those $R-$modules which are compatible with the natural $\mathbb{Z}^{|\Sigma(1)|}-$grading (or fine-grading) of $R = \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$, called fine-graded modules. For instance, monomial ideals are torsion-free fine-graded modules of rank one, thus one can think on fine-graded modules as a generalization of monomial ideals in a higher rank.

From a graded commutative algebra perspective, fine-graded modules are compatible with any other grading in the polynomial ring. They have been the center of many works in the last decades, such are [11, 14, 15, 44, 53, 59, 66]. In parallel, from a geometric point of view, equivariant sheaves on a toric variety have also been a focus of interest in the last decades. Beginning from the works of Kaneyama [43] and Klyachko [45] where they respectively classified equivariant vector bundles on a toric variety using different approaches. Afterwards, the methods of Klyachko were generalized in [46] to tackle more generally torsion-free sheaves, and in [57] this construction was formalized by Perling to describe any kind of quasi-coherent sheaf on any toric variety. Since then, many works have studied equivariant vector bundles and sheaves from a geometric point of view such as [21, 58, 47] Given a quasi-coherent equivariant sheaf, for any $\mathbb{T}-$invariant open subset $U_\sigma \subset X$ corresponding to a cone $\sigma \in \Sigma$, we have the so-called isotypical decomposition

$$\Gamma(U_\sigma, \mathcal{E}) = \bigoplus_{m\in M} \Gamma(U_\sigma, \mathcal{E})_m =: \bigoplus_{m\in M} E_m^\sigma.$$

With this decomposition, a quasi-coherent equivariant sheaf is associated to a collection of vector spaces

$$\{E_m^\sigma \mid m \in M,\, \sigma \in \Sigma\}$$

which describe the $M-$graded sections of $\mathcal{E}$ on each $\mathbb{T}-$invariant open subset of X. This collection of vector spaces is mainly the definition of the $\Sigma-$family of $\mathcal{E}$, and it endows some structure from the restriction maps, which are $M-$graded

$$\Gamma(U_\sigma, \mathcal{E})_m \to \Gamma(U_\tau, \mathcal{E})_m,$$

where $\tau \prec \sigma$ is a face of σ such that $U_\tau \subset U_\sigma$. On the other hand, it is worthwhile noticing that the open subset $U_{\{0\}}$ corresponding to the cone

$\{0\}$ (which is a face of any other cone $\sigma \in \Sigma$) is isomorphic to the torus $\mathbb{T}$. In particular we have

$$\Gamma(U_{\{0\}}, \mathcal{E}) \cong \Gamma(\mathbb{T}, \mathcal{E}) \cong \mathbb{C}[M]^{\ell},$$

where $\ell = \mathrm{rk}(\mathcal{E})$ is the rank of $\mathcal{E}$. Hence, there is an $\ell-$dimensional vector space

$$\mathbf{E} \cong \mathbb{C}[M]^{\ell}_{m} \cong \mathbb{C}^{\ell}$$

and for any character $m \in M$ we have an isomorphism

$$\Gamma(U_{\{0\}}, \mathcal{E}) \cong \mathbf{E}.$$

As we impose further properties to the equivariant sheaf $\mathcal{E}$, the intern structure of a $\Sigma-$family becomes more workable. For instance, when $\mathcal{E}$ is torsion-free, we can see each vector space in the $\Sigma-$family of $\mathcal{E}$ as a subspace of $E^{\sigma}_{m} \subset \mathbf{E}$. This allows to intersect or sum several vector spaces in the $\Sigma-$family treating them as a subspaces of $\mathbf{E}$. As a consequence we have some formulas for the cohomology of $\mathcal{E}$:

$$\mathrm{H}^0(X, \mathcal{E}) \cong \bigcap_{\sigma \in \Sigma(n)} E^{\sigma}_{m}$$

$$\mathrm{H}^n(X, \mathcal{E}) \cong \mathbf{E} / \sum_{\rho \in \Sigma(1)} E^{\rho}_{m}$$

$$\chi(\mathcal{E})_m = \sum_{\sigma \in \Sigma} (-1)^{\mathrm{codim}(\sigma)} \dim E^{\sigma}_{m}.$$

We have used this feature to study monomial ideals, which correspond to rank one torsion-free equivariant sheaves. In particular, the $\Sigma-$family of the torsion-free sheaf $\tilde{I}$ corresponding to a monomial ideal I is a set

$$\{I^{\sigma}_{m} \mid m \in M, \, \sigma \in \Sigma\}.$$

Since $\tilde{I}$ has rank one, each of the vector spaces I^{σ}_{m} can be seen as a subspace of a vector space $\mathbf{I} \cong \mathbb{C}$. As a consequence it can either be $I^{\sigma}_{m} = 0$ or $I^{\sigma}_{m} \cong \mathbf{I}$. Thus, for each cone σ we have the set

$$\{m \in M \mid I^{\sigma}_{m} \neq 0\} \subset \sigma^{\vee},$$

which altogether characterize the $\Sigma-$family of $\tilde{I}$. These sets of characters have the appearance of a cone with a bitten apex, or a staircase-like diagram inside a cone. They are the core of the definition of the Klyachko diagram, introduced in Chapter 2. We use the Klyachko diagram of a monomial ideal as a generalization of the classical staircase diagram, but allowing us to work with non-standard graded polynomial rings such as the Cox ring of a toric variety.

The combinatorial picture for general torsion-free equivariant sheaves of higher rank is much more involved. However, we may restrict our attention to torsion-free sheaves $\mathcal{E}$ which are in addition reflexive, that is satisfying $\mathcal{E} \cong \mathcal{E}^{\vee\vee}$. In this case we have

$$\Gamma(U_\sigma, \mathcal{E}) \cong \bigcap_{\rho \in \sigma(1)} \Gamma(U_\rho, \mathcal{E}).$$

Hence, the $\Sigma-$family of $\mathcal{E}$ is characterized by the collection of vector spaces corresponding to the rays. And for each ray $\rho \in \Sigma(1)$ we have

$$\{E_m^\rho \mid m \in M\}.$$

Furthermore, each ray ρ induces a preorder in M such that for two characters $m \leq_\rho m'$ whenever $\langle m, \rho \rangle \leq \langle m', \rho \rangle$. Then, we have a linear map

$$\chi^{m'-m} : E_m^\rho \to E_{m'}^\rho$$

which is an isomorphism when $m \leq_\rho m'$ and $m' \leq_q m$, or equivalently when $\langle m, \rho \rangle = \langle m', \rho \rangle$.

In other words, when $\mathcal{E}$ is a reflexive equivariant sheaf, not only the $\Sigma-$family is given by the collections corresponding to the rays, but also for each ray ρ it describes an increasing filtration of vector spaces:

$$\hat{E}^\rho = \{E^\rho(j)\}_{j \in \mathbb{Z}},$$

such that for any character $m \in M$, $E_m^\rho = E^\rho(\langle m, \rho \rangle)$, and for $j \ll 0$ (respectively $j \gg 0$) we have $E^\rho(j) = 0$ (respectively $E^\rho(j) = \mathbf{E}$). For each ray ρ the filtration of vector spaces corresponding to $\mathcal{E}$ can be written as:

$$E^\rho(j) = \begin{cases} 0, & j < i_1^\rho \\ E_1^\rho, & i_1^\rho \le j < i_2^\rho \\ \vdots & \\ E_{\ell-1}^\rho, & i_{\ell-1}^\rho \le j < i_\ell^\rho \\ E_\ell^\rho = \mathbf{E}, & i_\ell^\rho \le j \end{cases}$$

for suitable $k-$dimensional subvector spaces $E_k^\rho \subset \mathbf{E}$, and integers

$$i_1^\rho \le i_2^\rho \le \cdots \le i_\ell^\rho.$$

This filtration describes the sections of $\mathcal{E}$ over the $\mathbb{T}-$invariant open set U_ρ since for any character $m \in M$ we have

$$\Gamma(U_\rho, \mathcal{E})_m = E_m^\rho \cong E^\rho(\langle m, n(\rho)\rangle).$$

For any cone σ and any character $m \in M$, we have that

$$E_m^\sigma = \Gamma(U_\sigma, \mathcal{E})_m \cong \bigcap_{\rho\in\sigma(1)} \Gamma(U_\rho, \mathcal{E})_m \cong \bigcap_{\rho\in\sigma(1)} E_m^\rho.$$

Then, we can recover the global sections of $\mathcal{E}$ since, for each character $m \in M$, we have:

$$\mathrm{H}^0(X, \mathcal{E})_m \cong \bigcap_{\sigma\in\Sigma(n)} E_m^\sigma \cong \bigcap_{\rho\in\Sigma(1)} E_m^\rho \cong \bigcap_{\rho\in\Sigma(1)} E^\rho(\langle m, n(\rho)\rangle).$$

This gives a polytope in $M_\mathbb{R} = M \otimes \mathbb{R} \cong \mathbb{R}^n$

$$\Omega_\mathcal{E} = \{x \in \mathbb{R}^n \mid \langle x, n(\rho)\rangle \ge i_1^\rho,\ \rho \in \Sigma(1)\}.$$

The interior lattice points correspond to characters $m \in M$ such that E_m^ρ is non-zero for each ray $\rho \in \Sigma(1)$. However, the global value of $\mathrm{H}^0(X, \mathcal{E})_m$, which depends on intersecting these vector subspaces may be zero. In other words, we may say that the polytope $\Omega_\mathcal{E}$ bounds the non-zero global sections of $\mathcal{E}$.

For instance, the filtration corresponding to a line bundle $\mathcal{L} = \mathcal{O}(D)$, for a Weil divisor $D = \sum_{\rho\in\Sigma(1)} a_\rho D_\rho$ is

$$L^\rho(j) = \begin{cases} 0, & j < -a_\rho \\ \mathbf{L}, & -a_\rho \le j, \end{cases}$$

where $\mathbf{L} \cong \mathrm{H}^0(\mathbb{T}, \mathcal{L})_m \cong \mathbb{C}$, for any character $m \in M$. This yields the polytope

$$\Omega_{\mathcal{L}} = \{x \in \mathbb{R}^n \mid \langle x, n(\rho)\rangle \geq -a_\rho\}.$$

We observe that this polytope coincides with the polytope P_D associated to a Weil divisor introduced in [21].

In Chapter 3 we use this construction to study how the global sections of an equivariant reflexive sheaf $\mathcal{E}$ on a complete smooth toric variety change, as we twist by different line bundles, providing information about the Hilbert polynomial of a $\mathcal{E}$. We first observe that, when twisting the equivariant reflexive sheaf $\mathcal{E}$ by the line bundle $\mathcal{L}$, the filtration corresponding to $\mathcal{E}(D) := \mathcal{E} \otimes \mathcal{L}$ is given by

$$E(D)^\rho(j) = \begin{cases} 0, & j < i_1^\rho - a_\rho \\ E_1^\rho, & i_1^\rho - a_\rho \leq j < i_2^\rho - a_\rho \\ \vdots & \\ E_{\ell-1}^\rho, & i_{\ell-1}^\rho - a_\rho \leq j < i_\ell^\rho - a_\rho \\ E_\ell^\rho = \mathbf{E}, & i_\ell^\rho - a_\rho \leq j. \end{cases}$$

At the level of polytopes we see that

$$\Omega_{\mathcal{E}(D)} = \Omega_{\mathcal{E}}(D) := \{x \in \mathbb{R}^n \mid \langle x, n(\rho)\rangle \geq i_1^\rho - a_\rho\},$$

which is obtained by dilating, not necessarily in a symmetric way, the original polytope $\Omega_{\mathcal{E}}$. From now on, let us set

$$\Omega := \Omega_{\mathcal{E}} \quad \text{and} \quad \Omega(D) := \Omega_{\mathcal{E}}(D).$$

As pointed out before, each lattice point in the polytope $\Omega(D)$ corresponds to a character $m \in M$ such that

$$\mathrm{H}^0(X, \mathcal{E}(D))_m = \bigcap_{\rho \in \Sigma(1)} E^\rho(\langle m, n(\rho)\rangle).$$

Each subvector space $E^\rho(\langle m, n(\rho)\rangle)$ is one among the following possible subvector spaces

$$E_1^\rho,\ E_2^\rho, \ldots,\ E_{\ell-1}^\rho,\ E_\ell^\rho = \mathbf{E}.$$

Namely, we have that $E^\rho(\langle m, n(\rho)\rangle) = E^\rho_k$ whenever

$$i^\rho_k - a_\rho \leq \langle m, n(\rho)\rangle < i^\rho_{k+1} - a_\rho,$$

and by abuse of notation we set $i^\rho_{\ell+1} := \infty$. Thus, let us order the set of rays $\Sigma(1) = \{\rho_1, \dots, \rho_r\}$ and we can tesselate the polytope $\Omega(D)$ into a collection of subpolytopes

$$\Omega_{\underline{\lambda}}(D) = \{x \in \mathbb{R}^n \mid i^{\rho_k}_{\lambda_k} - a_{\rho_k} \leq \langle x, n(\rho_k)\rangle < i^{\rho_k}_{\lambda_k+1} - a_{\rho_k}\},$$

where $\underline{\lambda} = (\lambda_1, \dots, \lambda_r)$ is a tuple of integers such that $1 \leq \lambda_k \leq r$. These subpolytopes allow us to control the sections of $\mathcal{E}(D)$ in a finer way. Indeed, now we have that any lattice point in $\Omega_{\underline{\lambda}}$ corresponds to a character $m \in M$ such that

$$\mathrm{H}^0(X, \mathcal{E}(D))_m \cong E^{\rho_1}_{\lambda_1} \cap \cdots \cap E^{\rho_r}_{\lambda_r}.$$

In other words, the subpolytopes $\Omega_{\underline{\lambda}}$ are in correspondence with the constant dimensional $M-$graded components of $\mathrm{H}^0(X, \mathcal{E}(D))$.

In general counting lattice points in a polytope, or even more, looking how the number of lattice points changes as we apply some dilations to a polytope, is a hard problem. With relations to the Ehrhart theory of polytopes, there have been many efforts to give new insights to these problems [9, 61, 23]. However, in Section 3.2, we are able to control the number of lattice points in this kind of polytopes, working on smooth toric varieties with a splitting fan. In fact, we are able to control how the number of lattice points changes as we dilate them accordingly to a twist by a line bundle, providing thus a way to compute the Hilbert polynomial $P_{\mathcal{E}}$ of $\mathcal{E}$. As an application of this approach, we describe a method to find Ulrich bundles on smooth toric varieties with splitting fans. Ulrich bundles on projective varieties have been a center of interest in the last decades, and have been studied from many different perspectives [1, 2, 17, 29, 64]. In particular, being $\mathcal{E}$ an Ulrich bundle on a projective variety imposes constraints on the Hilbert polynomial $P_{\mathcal{E}}$ of $\mathcal{E}$. Focusing in the Hirzebruch surface, when $\mathcal{E}$ is a reflexive equivariant sheaf, and thus an equivariant vector bundle, we can use the description of $P_{\mathcal{E}}$ in terms of the filtration, to fulfill the necessary conditions for being Ulrich. Using this method we are able to provide an example of a rank 3 equivariant Ulrich bundle on the Hirzebruch surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$

Finally, in the last part of this dissertation, we use the algebraic-geometric correspondence in a completely different way. Let (X, L) be a polarized projective variety, with L a very ample line bundle, and let M_L be the vector bundle defined as the syzygy bundle of the evaluation map

$$ev: \mathrm{H}^0(X, L) \otimes \mathcal{O} \longrightarrow L.$$

The stability of M_L with respect to the very ample line bundle L has been a center of interest in the last decades. In [25], Ein, Lazarsfeld and Mustopa posed the following conjecture regarding the asymptotical stability of syzygy bundles in polarized projective varieties.

Conjecture (Conjecture 4.2.1)**.** *Let A and P two line bundles on a smooth projective variety X. Assume that A is very ample and set $L_d := dA + P$ for any positive integer d. Then, the syzygy bundle M_{L_d} is $A-$stable for $d \gg 0$.*

This conjecture has been partially solved, for instance when

(1) $(X, L) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ (see [33] in characteristic zero and [10] for any characteristic),

(2) (X, L) where X is a smooth projective curve of genus $g \geq 1$ and $\deg(L) \geq 2g + 1$ (see [24, Proposition 1.5]),

(3) (X, L) where X is a simple abelian variety and L an ample globaly generated line bundle (see [13, Corollary 2.1]),

(4) when (X, L) is a sufficiently positive polarization of an algebraic surface X (see [25, Theorem A]) and

(5) (X, L) where X is an Enriques (resp. bielliptic) surface and L an ample globally generated line bundle (see [56, Theorem 3.5]).

In spite of these contributions, Conjecture 4.2.1 is far from being solved. In Chapter 4, we have tackled this conjecture when (X, L) is a polarized smooth toric variety. In that case the algebraic-geometric correspondence allows us to understand better the structure of the syzygy bundle M_L. We use this tool to study the case where $X = \mathrm{Bl}_Z(\mathbb{P}^n)$ is the blow up of the projective space $\mathbb{P}^n$ along a linear subspace $Z \subset \mathbb{P}^n$. In this setting, we contribute to the Ein-Lazarsfeld-Mustopa conjecture by proving that, for any very ample line bundle L on $\mathrm{Bl}_Z(\mathbb{P}^n)$, the syzygy bundle M_L is stable with respect to L.

On the other hand, the stability of the syzygy bundle M_L allows us to see it as a point $[M_L]$ in the corresponding moduli space. We can then use the cohomological properties of M_L to study how this moduli space looks like around the point $[M_L]$. In particular we find that it is always smooth. Even more, we are able to prove that M_L is almost always a rigid vector bundle, and therefore an isolated point in its moduli space. In the cases in which M_L is not rigid, we are able to compute explicitly the dimension of the irreducible component of the moduli space in which $[M_L]$ sits.

We will now outline the structure of this thesis.

Chapter 1 contains the preliminary definitions and notions used in the main body of this work. It is subdivided in three sections. In Section 1.1, we introduce the notion of a toric variety and its main features, highlighting the notion of a Cox ring and the algebraic-correspondence between modules and sheaves. In section 1.2, we recall the correspondence between sheaves on a toric variety and homogeneous modules on a polynomial ring graded by an abelian group. Finally, in Section 1.3, we focus our attention on equivariant sheaves on a toric variety. We recall the Klyachko construction describing torsion-free equivariant sheaves by means of a family of filtered vector spaces, and we illustrate it with many examples. Afterwards, we study how to compute the cohomology of equivariant torsion-free sheaves using the data provided by the Klyachko filtrations.

Chapter 2 introduces the notion of a Klyachko diagram, which generalizes the classical stair-case diagram of a monomial ideal. We provide many examples to illustrate the results throghout the two main sections of this chapter. Section 2.1 is the core of this chapter, and where Klyachko diagrams are introduced (see Definition 2.1.3). In this section, we show how to compute the Klyachko diagram of a given monomial ideal (see

Subsection 2.1.1); and how to retrieve a minimal set of generators from the data provided by a Klyachko diagram (see Subsection 2.1.2). Afterwards, we use the Klyachko diagram of a monomial ideal to describe its first local cohomology, which measures its saturatedness (see Subsection 2.1.3. Finally, in Section 2.2, we use the Klyachko diagram of a monomial ideal to describe its Hilbert function (see Proposition 2.2.1). In particular, we characterize all monomial ideals having constant Hilbert polynomial, in terms of the shape of the Klyachko diagram (see Corollary 2.2.4).

Chapter 3 is devoted to the study of equivariant reflexive sheaves on a toric variety. In Section 3.1, we describe how the global sections of an equivariant reflexive sheaf change as we twist it by a line bundle, using a family of lattice polytopes. In particular, this gives a method to compute the Hilbert polynomial of an equivariant reflexive sheaf. In Section 3.2, we focus on toric varieties with splitting fan. In this setting we deepen in the combinatorial study of the family of lattice polytopes introduced in the previous section. In particular, we are able to give bounds for the multigraded initial degree (Propositions 3.2.8 and 3.2.9) and for the multigraded regularity index (Theorem 3.2.18) of an equivariant reflexive sheaf. From the latter result we give a method to compute explicitly the Hilbert polynomial of an equivariant reflexive sheaf. Finally, in Section 3.3, we apply the tools described in the previous sections in this Chapter, to the theory of Ulrich bundles. In particular, we present a method aimed to find equivariant Ulrich bundles on a Hirzebruch surface, and we give an example of a rank 3 equivariant Ulrich bundle in the first Hirzebruch surface (Exmple 3.3.4).

Chapter 4 treats the stability of syzygy bundles on a toric variety, addressing the Ein-Lazarsfeld-Mustopa conjecture. In Section 4.1, we gather the basic notions about stability of vector bundles on a projective variety and moduli spaces. In Section 4.2 we recall the notion of a syzygy bundle, the Ein-Lazarsfeld-Mustopa conjecture and the cases for which this conjecture has been solved. Finally, in Section 4.3, we focus on solving the Ein-Lazarsfeld-Mustopa conjecture for the syzygy bundle of any polarization of a blow-up of a projective space along a linear subspace (Theorem 4.3.4). Afterwards, in Subsection 4.3.1, we study the rigidness of the syzygy bundles in this setting, all of which correspond to smooth points in their associated moduli space (Theorem 4.3.5).

Part of the results in this thesis are contained in the following papers and preprint:

1. R. M. Miró-Roig and M. Salat-Moltó, *Multigraded Castelnuovo–Mumford regularity via Klyachko filtrations.* Forum Mathematicum, **34:1** (2022) 41–60.

2. R. M. Miró-Roig and M. Salat-Moltó, *Klyachko diagrams of monomial ideals.* Algebras and Representation Theory. *To appear.*

3. R. M. Miró-Roig and M. Salat-Moltó, *Ein-Lazarsfeld-Mustopa conjecture for the blow-up of a projective space.* Preprint.

Chapter 1

Preliminaries

In this chapter, we gather the basic theory on toric varieties and sheaves, and the preliminary results needed in the forthcoming chapters. We do not claim any originality on this chapter, which is divided in three sections. In Section 1.1, we give a general overview on how toric varieties are constructed with an emphasis on the tools coming from the theory of semigroup rings. In Section 1.2, we recall the notion of the Cox ring of a toric variety, and the algebra-geometry correspondence between finitely generated modules and sheaves on toric varieties. Finally in Section 1.3, we focus on equivariant sheaves on toric varieties. We explain the construction of an equivariant torsion-free sheaf in terms of a collection of filtrations of vector spaces given by Klyachko in [45, 46].

1.1 Toric varieties

In this section, we gather the basic notation, definitions, properties and construction of toric varieties. An *algebraic* $n-$*torus* (or simply a *torus*) is an algebraic group $\mathbb{T} \cong (\mathbb{C}^*)^n$. The isomorphism of groups

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathrm{Hom}(\mathbb{C}^*, \mathbb{C}^*) \\ k & \longmapsto & (z \mapsto z^k) \end{array}$$

tells us that the character group of $\mathbb{T}$ is a lattice. Indeed,

$$\mathbb{X}(\mathbb{T}) = \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*) \cong \mathrm{Hom}((\mathbb{C}^*)^n, \mathbb{C}^*) \cong \mathbb{Z}^n.$$

It is customary to denote this *character lattice* by $M := \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*)$ and its dual lattice by $N = \mathrm{Hom}(M, \mathbb{Z})$, which we also call *lattice of*

1$-$*parameter subgroups.* We have the following isomorphisms

$$\mathbb{T} \cong \operatorname{Spec}(\mathbb{C}[M]) \cong \operatorname{Hom}(M, \mathbb{C}^*) \cong N \otimes \mathbb{C}^* \quad \text{and} \quad N \cong \operatorname{Hom}(\mathbb{C}^*, \mathbb{T}).$$

On the other hand, we can reverse this construction starting from a lattice N of rank n. We set $M = \operatorname{Hom}(N, \mathbb{Z})$ its dual lattice, and we define $\mathbb{T}_N := \operatorname{Spec}(\mathbb{C}[M])$, which is an $n-$torus. Therefore, when fixing a torus $\mathbb{T}_N$ we are fixing both the torus and the lattice N.

Now, let us take a closer look to the characters and 1$-$parameter subgroups of a $n-$torus $\mathbb{T}$. Fixed an isomorphism $\mathbb{T} \cong (\mathbb{C}^*)^n$, we write any element of $t \in \mathbb{T}$ as an $n-$uple $t = (t_1, \ldots, t_n)$, and the characters

$$\begin{array}{rccl} \chi_i : & \mathbb{T} & \longrightarrow & \mathbb{C}^* \\ & t & \longmapsto & \chi_i(t) := t_i \end{array}$$

form a $\mathbb{Z}-$basis of M. Then, we identify each character χ by an $n-$tuple of integers $m = (m_1, \ldots, m_n)$. As a group homomorphism, we write $\chi = \chi^m$ meaning the character

$$\begin{array}{rccl} \chi^m : & \mathbb{T} & \longrightarrow & \mathbb{C}^* \\ & t & \longmapsto & \chi^m(t) := t_1^{m_1} \cdots t_n^{m_n}. \end{array}$$

Similarly, we set $\{\lambda_1, \ldots, \lambda_n\}$ the basis of N dual to M, and in particular we have

$$\begin{array}{rccl} \lambda_i : & \mathbb{C}^* & \longrightarrow & \mathbb{T} \\ & z & \longmapsto & \lambda_i(z) := (1, \ldots, z, \ldots, 1) \in \mathbb{T}. \end{array}$$

We identify any 1$-$parameter subgroup λ with an $n-$uple of integers $u = (u_1, \ldots, u_n)$, and we write $\lambda = \lambda^u$ meaning

$$\begin{array}{rccl} \lambda^u : & \mathbb{C}^* & \longrightarrow & \mathbb{T} \\ & z & \longmapsto & \lambda^u(z) := (z^{u_1}, \ldots, z^{u_n}). \end{array}$$

Finally, the dual pairing between characters and 1$-$parameter subgroups is given in those basis by

$$\begin{array}{rccl} \langle \chi^m, \lambda^u \rangle = \chi^m(\lambda^u) : & \mathbb{C}^* & \longrightarrow & \mathbb{C}^* \\ & z & \longmapsto & z^{\langle m,u \rangle} \end{array}$$

which is identified with the usual scalar product $\langle m, u \rangle \in \mathbb{Z}$. Hereafter, these two types of notation will be used simultaneously depending on the needed point of view.

1.1.1 Cones and affine toric varieties.

We start with the affine case. Here we recall how the (normal) affine toric varieties are constructed from a convex geometry object called cone. Let $N \cong \mathbb{Z}^n$ be a lattice and $M = \mathrm{Hom}(N, \mathbb{Z}) \cong \mathbb{Z}^n$ its dual lattice and denote

$$N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \quad \text{and} \quad M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$$

their associated real vector spaces. By a *convex rational polyhedral cone* (or simply a *cone*) σ generated by a set $S \subset N$, we mean

$$\sigma = \mathrm{cone}(S) = \{\sum_{u \in S} \lambda_u u \ \mid \ \lambda_u \geq 0\}.$$

The dimension of σ is the minimal dimension of a subspace of $N_{\mathbb{R}}$ containing σ. We define the *dual cone* $\sigma^\vee \subset M_{\mathbb{R}}$ of σ to be the cone

$$\sigma^\vee = \{m \in M \ \mid \ \langle m, u\rangle \geq 0, \text{ for any } u \in \sigma\} \subset M_{\mathbb{R}}.$$

We have $(\sigma^\vee)^\vee = \sigma$. To any $0 \neq m \in M_{\mathbb{R}}$, we associate the hyperplane

$$H_m = \{u \in N_{\mathbb{R}} \ \mid \ \langle u, m\rangle = 0\} \subset N_{\mathbb{R}}$$

and the closed half-space

$$H_m^+ = \{u \in N_{\mathbb{R}} \ \mid \ \langle u, m\rangle \geq 0\} \subset N_{\mathbb{R}}.$$

Then, $\sigma \subset H_m^+$ if and only if $m \in \sigma^\vee \setminus \{0\}$, and $\sigma = H_{m_1}^+ \cap \cdots \cap H_{m_r}^+$, if and only if $\sigma^\vee = \mathrm{cone}(m_1, \ldots, m_r)$. By a *face* τ of σ, we mean a subset of the form $\tau = \sigma \cap H_m$ with $m \in \sigma^\vee$. Notice that σ is a face of itself taking $m = 0 \in \sigma^\vee$. We write $\tau \prec \sigma$ for the face-inclusion relation satisfying the usual simplicial properties. We denote by $\sigma(k)$ the set of all $k-$dimensional faces of σ. In particular, we call *facet* a face τ with $\dim(\tau) = \dim(\sigma) - 1$.

Proposition 1.1.1. *Let $\sigma \subset N_{\mathbb{R}}$ be a cone. The following are equivalent:*

(i) $\{0\}$ is a face of σ.

(ii) σ contains no positive-dimensional subspace of $N_{\mathbb{R}}$.

(iii) $\sigma \cap (-\sigma) = \{0\}$.

(iv) $\dim(\sigma^\vee) = n$.

Proof. See [20, Proposition 1.2.12]. □

Any cone satisfying the equivalent conditions in Proposition 1.1.1 is called a *strongly convex cone.* In this case, we have $\sigma = \mathrm{cone}(\sigma(1))$, we call *rays* the elements in $\sigma(1)$ and for any ray $\rho \in \sigma(1)$, we denote by $n(\rho) \in N$ its minimal generator in N: $\mathbb{Z}_{\geq 0} n(\rho) = \rho \cap N$. Given a cone $\sigma \subset N_\mathbb{R}$, we denote by $S_\sigma := \sigma^\vee \cap M \subset M$ its associated *semigroup.*

Example 1.1.2. Let $N = \mathbb{Z}^2$ be a lattice with standard basis $\{e_1, e_2\}$, and consider the cone $\sigma = \mathrm{cone}(-e_1, -e_1 + 3e_2)$. Set $\{u_1, u_2\}$ the basis of M dual of N. Then we have $\sigma^\vee = \mathrm{cone}(-3u_1 - u_2, u_2)$ and its associated semigroup is

$$S_\sigma = \mathbb{Z}_{\geq 0}\langle -3u_1 - u_2, u_2\rangle.$$

Proposition 1.1.3. *Let $\sigma \subset N_\mathbb{R}$ be a strongly convex cone. Then, S_σ is a finitely generated semigroup.*

Proof. See [20, Proposition 1.2.17]. □

From now on unless otherwise said, we will restrict our attention to strongly convex rational polyhedral cones, which we call simply *cones.* Next, we define an affine toric variety associated to a cone $\sigma \in N_\mathbb{R}$. Notice that, since S_σ is a finitely generated semigroup, the *semigroup algebra* $\mathbb{C}[S_\sigma] \subset \mathbb{C}[M]$ is finitely generated semigroup algebra, and we have the following:

Definition 1.1.4. Let $\sigma \subset N_\mathbb{R}$ be a cone with semigroup S_σ. Then, $U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma])$ is an affine toric variety with torus

$$\mathbb{T}_N = \mathrm{Spec}(\mathbb{C}[M]) \subset \mathrm{Spec}(\mathbb{C}[S_\sigma]).$$

Proposition 1.1.5. *Let $\sigma \subset N_\mathbb{R}$ be a cone and U_σ the corresponding affine toric variety. Then,*

(i) $\dim(U_\sigma) = \dim(N_\mathbb{R})$.

(ii) U_σ is normal.

(iii) U_σ *is smooth if and only if the set* $\{n(\rho) \mid \rho \in \sigma(1)\}$ *is a* $\mathbb{Z}-$*basis of a sublattice of* N*.*

Proof. See [20, Theorem 1.2.18, Theorem 1.3.5 and Theorem 1.3.12]. □

Example 1.1.6. Continuing with Example 1.1.2, let $N = \mathbb{Z}^2$ be a lattice with standard basis $\{e_1, e_2\}$ and the cone $\sigma = \text{cone}(-e_1, -e_1+3e_2)$. Then we have

$$\begin{aligned}\mathbb{C}[S_\sigma] &= \mathbb{C}[\chi^{-3u_1-u_2}, \chi^{u_2}] \\ &= \mathbb{C}[(\chi^{u_1})^{-3}(\chi^{u_2})^{-1}, \chi^{u_2}] \\ &\cong \mathbb{C}[t_1^{-3}t_2^{-1}, t_2] \subset \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}].\end{aligned}$$

1.1.2 General toric varieties

The more general notion of a (normal) toric variety is obtained by glueing affine toric varieties, which are encoded by special sets of cones called fans:

Definition 1.1.7. A collection Σ of cones in $N_\mathbb{R}$ is called a *fan* if it satisfies:

(i) For any cone $\sigma \in \Sigma$, if $\tau \prec \sigma$ is a face, then $\tau \in \Sigma$.

(ii) For any two cones $\sigma_1, \sigma_2 \in \Sigma$, the intersection $\tau = \sigma_1 \cap \sigma_2$ is a cone. Hence, $\tau \prec \sigma_1$, $\tau \prec \sigma_2$ and $\tau \in \Sigma$.

We denote by $\Sigma(k)$ the set of $k-$dimensional cones in Σ. We say that a cone $\sigma \in \Sigma$ is *maximal* if it is not contained in any other cone in Σ. Notice that the set $\Sigma_{\max}$ of all maximal cones determines the whole fan Σ. For any cone $\sigma \in \Sigma$ we obtain an affine toric variety U_σ (Proposition 1.1.4). The following lemmas illustrate how the simplicial definition of a fan translates to inclusions among these affine varieties.

Lemma 1.1.8. *Let* $\sigma \subset N_\mathbb{R}$ *be a cone and* $\tau = \sigma \cap H_m$ *a face for some* $m \in \sigma^\vee$*. Then,*

(i) $S_\tau = S_\sigma + \mathbb{Z}(-m)$*, and in particular* $\mathbb{C}[S_\tau] = \mathbb{C}[S_\sigma]_{\chi^m}$*.*

(ii) The inclusion $S_\sigma \subset S_\tau$ *induces a morphism* $U_\tau \hookrightarrow U_\sigma$*.*

Proof. See [20, Proposition 1.3.16]. □

Lemma 1.1.9. *Let $\sigma_1, \sigma_2 \subset N_{\mathbb{R}}$ be two cones. If $\tau = \sigma_1 \cap \sigma_2$ is a cone, then*

(i) there is $m \in \sigma_1^\vee \cap (-\sigma_2)^\vee \cap M$ such that

$$\tau = \sigma_1 \cap H_m = \sigma_2 \cap H_{-m}.$$

(ii) In particular, $\mathbb{C}[S_{\sigma_1}]_{\chi^m} = \mathbb{C}[S_\tau] = \mathbb{C}[S_{\sigma_2}]_{\chi^{-m}}$.

Proof. See [20, Lemma 1.2.13]. □

In general, we have the following definition:

Definition 1.1.10. The *toric variety* assosiated to a fan Σ in $N_{\mathbb{R}}$ is the normal variety obtained by glueing the collection of affine toric varieties $\{U_\sigma \mid \sigma \in \Sigma\}$ along the intersections of the cones, using the maps $U_{\sigma_1} \hookleftarrow U_{\sigma_1} \cap U_{\sigma_2} \hookrightarrow U_{\sigma_2}$.

The following example illustrates how the projective space is constructed as a toric variety:

Example 1.1.11 (Projective space)**.** Let $N = \mathbb{Z}^n$ be a lattice and denote by $\{e_1, \ldots, e_n\}$ its standard basis. Let Σ be the fan in $N_{\mathbb{R}}$ such that $\Sigma_{\max} = \{\sigma_0, \ldots, \sigma_n\}$, where:

$$\sigma_0 := \operatorname{cone}(e_1, \ldots, e_n)$$

and

$$\sigma_i := \operatorname{cone}(\{e_j \mid j \neq i\} \cup \{-e_1 - \cdots - e_n\}) \quad \text{for} \quad 1 \leq i \leq n.$$

Then, $\sigma_0^\vee = \operatorname{cone}(u_1, \ldots, u_n)$ and $\sigma_i^\vee = \operatorname{cone}(\{u_j - u_i \mid j \neq i\} \cup \{-u_i\})$. Thus, the coordinate rings of the affine toric varieties U_{σ_i} are the semigroup rings

$$\mathbb{C}[U_{\sigma_0}] \cong \mathbb{C}[t_1, \ldots, t_n]$$

and

$$\mathbb{C}[U_{\sigma_i}] \cong \mathbb{C}[\{t_j t_i^{-1} \mid j \neq i\} \cup \{t_i^{-1}\}] \quad 1 \leq i \leq n.$$

Setting $t_i = \frac{z_i}{z_0}$, we can write

$$\mathbb{C}[U_{\sigma_k}] = \mathbb{C}[\frac{z_i}{z_k} \mid i \neq k] \quad \text{for} \quad 0 \leq k \leq n.$$

In particular $X = \mathbb{P}^n$, the $n-$dimensional projective space covered by the usual affine open sets $\{z_k \neq 0\} \subset \mathbb{P}^n$.

Proposition 1.1.12. *Let X be a toric variety with fan Σ in $N_{\mathbb{R}}$. Then,*

(i) X is a normal, separated variety.

(ii) X is smooth if and only if for any cone $\sigma \in \Sigma$, the open set U_σ is smooth, or equivalently, the set

$$\{n(\rho) \mid \rho \in \sigma(1)\}$$

is a $\mathbb{Z}-$basis of a sublattice of N.

(ii) X is compact if and only if the fan Σ is complete, that is

$$\bigcup_{\sigma \in \Sigma} \sigma = N_{\mathbb{R}}.$$

Proof. See, [20, Theorem 3.1.5 and Theorem 3.1.19]. □

Example 1.1.13 (Hirzebruch surface)**.** Let $N = \mathbb{Z}^2$ be a lattice and denote $\{e, f\}$ its standard basis. For $a \in \mathbb{Z}_{\geq 0}$, we consider the fan Σ with $\Sigma_{\max} = \{\sigma_{ij} \mid 0 \leq i, j \leq 1\}$ where

$$\begin{aligned}
\sigma_{00} &= \operatorname{cone}(u_1, v_1)\\
\sigma_{01} &= \operatorname{cone}(u_1, v_0)\\
\sigma_{10} &= \operatorname{cone}(u_0, v_1)\\
\sigma_{11} &= \operatorname{cone}(u_0, v_0),
\end{aligned}$$

and $u_0 := -e + af$, $u_1 := e$, $v_0 := -f$ and $v_1 := f$.

Then, every cone σ_{ij} is smooth and $\cup_{i,j}\sigma_{ij} = N_{\mathbb{R}}$. Hence, X is a smooth compact toric surface. Moreover, X is isomorphic to the Hirzebruch surface $\mathcal{H}_a = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a))$.

When $a = 1$, the Hirzebruch surface $\mathcal{H}_a$ is isomorphic to the blow-up of $\mathbb{P}^2$ on a point fixed by the torus action. More generally we have the following example.

Example 1.1.14. Let X be a smooth complete toric variety with fan σ, and let $\sigma \in \Sigma_{\max}$ be a maximal cone representing a torus-invariant point x_σ. The blow-up of X at x_σ is the toric variety with fan Σ'

$$\Sigma'_{\max} = (\Sigma_{\max} \setminus \{\sigma\}) \cup \tilde{\sigma},$$

where $\tilde{\sigma}$ is a collection of maximal cones defined as follows:

Assume that $\sigma = \operatorname{cone}(v_1, \dots, v_t)$ for some vectors $v_1, \dots, v_t \in N_{\mathbb{R}}$. We define

$$v_0 := v_1 + \cdots + v_t,$$

and we have

$$\tilde{\sigma} := \{\operatorname{cone}(v_0, v_{i_1}, \dots, v_{i_{t-1}} \mid 1 \leq i_1 < \cdots < i_{t-1} \leq t\}.$$

With the notation of Examples 1.1.11, we can construct the the fan of the blow-up $\operatorname{Bl}_{x_{\sigma_1}, x_{\sigma_2}} \mathbb{P}^2$ of $\mathbb{P}^2$ at two torus-invariant points (for instance we take x_{σ_2} and x_{σ_1}). We first blow up x_{σ_2} obtaining a toric variety X' with fan Σ' such that

$$\Sigma'_{\max} = \{\sigma_0\} \cup \{\sigma_1\} \cup \{\sigma_{2,1}, \sigma_{2,2}\},$$

where $\sigma_{2,1} = \operatorname{cone}(-e_2, e_1)$ and $\sigma_{2,2} = \operatorname{cone}(-e_2, -e_1 - e_2)$. Notice that the resulting fan is isomorphic to the fan of Example 1.1.13, of the first Hirzebruch surface $\mathcal{H}_1$.

Finally, to obtain $\operatorname{Bl}_{x_{\sigma_1}, x_{\sigma_2}} \mathbb{P}^2$ we blow-up X' at the torus-invariant point x_{σ_1} and we obtain a toric variety $X'' = \operatorname{Bl}_{x_{\sigma_1}, x_{\sigma_2}} \mathbb{P}^2$, with fan Σ'' such that

$$\Sigma''_{\max} = \{\sigma_0\} \cup \{\sigma_{1,1}, \sigma_{1,2}\} \cup \{\sigma_{2,1}, \sigma_{2,2}\},$$

where $\sigma_{1,1} = \operatorname{cone}(-e_1, e_2)$ and $\sigma_{1,2} = \operatorname{cone}(-e_1, -e_1 - e_2)$.

1.1.3 Divisors on toric varieties

Let X be a toric variety with torus $\mathbb{T}_N$ and fan Σ. In this subsection, we see how the class group, $\operatorname{Cl}(X)$, of Weil divisors modulo linear equivalence is encoded by $\mathbb{T}_N-$invariant Weil divisors. First, we recall a correspondence between cones of the fan Σ and $\mathbb{T}_N-$orbits of the toric variety X. For a point $x \in X$, we define the $\mathbb{T}_N-$*orbit* of x by

$$O_x := \{t \cdot x \mid t \in \mathbb{T}_N\}$$

The combinatorial nature of the fan gives rise to the following correspondence.

Proposition 1.1.15. *Let X be the $n-$dimensional toric variety with fan Σ in $N_{\mathbb{R}}$. Then, there is a bijective correspondence:*

$$\begin{array}{ccl} \Sigma & \longleftrightarrow & \{\mathbb{T}_N - \textit{orbits in}\, X_\Sigma\} \\ \sigma & \longleftrightarrow & O(\sigma) \end{array}$$

such that for each cone $\sigma \in \Sigma$:

(i) $O(\sigma) \cong \mathrm{Hom}(\sigma^\perp \cap M, \mathbb{C}^*)$, *where*

$$\sigma^\perp = \{m \in M \mid \langle m, u\rangle = 0, \textit{ for all}\, u \in \sigma\}.$$

(ii) $\dim O(\sigma) = \mathrm{codim}(\sigma) = n - \dim(\sigma)$.

(iii) The open affine subset U_σ decomposes as a union of orbits:

$$U_\sigma = \bigcup_{\tau \prec \sigma} O(\tau).$$

(iv) $\tau \prec \sigma$ if and only if $O(\sigma) \subset \overline{O(\tau)}$ and

$$\overline{O(\tau)} = \bigcup_{\tau \prec \sigma} O(\sigma).$$

Moreover, for any cone $\tau \in \Sigma$ we set $V(\tau) := \overline{O(\tau)}$. Then, $V(\tau)$ is a normal toric variety contained in X.

Proof. See [20, Theorem 3.2.6] for (i)-(iv), and [20, Proposition 3.2.7] for the second part. □

Proposition 1.1.15 describes all $\mathbb{T}_N-$invariant subvarieties of a toric variety X with fan $\Sigma \subset N_{\mathbb{R}}$. In particular, for any ray $\rho \in \Sigma(1)$ the irreducible subvariety $V(\rho)$ has codimension one and, hence, it is a prime divisor which we denote by $D_\rho = V(\rho)$. Moreover, any $\mathbb{T}_N-$invariant Weil divisor is a $\mathbb{Z}-$linear combination of $\{D_\rho \mid \rho \in \Sigma(1)\}$.

Next, we describe how to encode the class group of Weil divisors modulo linear equivalence, from the set of all $\mathbb{T}_N-$invariant divisors. Notice that for any character $m \in M$ we have a rational function $\chi^m : X \to \mathbb{C}^*$. We denote by $\operatorname{div}(\chi^m)$ the associated principal divisor. The next two results show how to compute the class group $\operatorname{Cl}(X)$ of X.

Lemma 1.1.16. *For any character $m \in M$ we have*

$$\operatorname{div}(\chi^m) = \sum_{\rho\in\Sigma(1)} \langle m, n(\rho)\rangle D_\rho \in \bigoplus_{\rho\in\Sigma(1)} \mathbb{Z}D_\rho.$$

Proof. See [20, Proposition 4.1.2]. □

Proposition 1.1.17. *We have the exact sequence:*

$$M \xrightarrow{\phi} \bigoplus_{\rho\in\Sigma(1)} \mathbb{Z}D_\rho \xrightarrow{\pi} \operatorname{Cl}(X) \longrightarrow 0,$$

where for any character $m \in M$,

$$\phi(m) = \operatorname{div}(\chi^m) = \sum_{\rho\in\Sigma(1)} \langle m, n(\rho)\rangle D_\rho$$

and for any $\mathbb{T}_N$-invariant Weil divisor D, $\pi(D) = [D]$ is its class in $\operatorname{Cl}(X)$. *Moreover, ϕ is injective and the sequence*

$$0 \longrightarrow M \xrightarrow{\phi} \bigoplus_{\rho\in\Sigma(1)} \mathbb{Z}D_\rho \xrightarrow{\pi} \operatorname{Cl}(X) \longrightarrow 0, \tag{1.1}$$

is exact if and only if X has no torus factors or, equivalently, the set

$$\{n(\rho) \mid \rho \in \Sigma(1)\}$$

spans $N_{\mathbb{R}}$. In particular, $\operatorname{Cl}(X_\Sigma)$ *is a finitely generated abelian group.*

Proof. See [20, Theorem 4.1.3]. □

Example 1.1.18. Let us check that $\mathrm{Cl}(\mathbb{P}^n) \cong \mathbb{Z}$. Continuing with Example 1.1.11, we set $\rho_0 = \mathrm{cone}(-e_1 - \cdots - e_n)$ and $\rho_i = \mathrm{cone}(e_i)$ for $1 \le i \le n$. The exact sequence in Proposition 1.1.17 is

$$0 \longrightarrow M \xrightarrow{\phi} \bigoplus_{i=0}^{n} \mathbb{Z}D_{\rho_i} \xrightarrow{\pi} \mathrm{Cl}(\mathbb{P}^n) \longrightarrow 0$$

where $\phi\colon\ (d_1, \dots, d_n) \mapsto (-d_1 - \cdots - d_n, d_1, \dots, d_n)$. Therefore,

$$\mathrm{Im}\,\phi \cong \mathbb{Z}\langle D_{\rho_1} - D_{\rho_0}, \dots, D_{\rho_n} - D_{\rho_0}\rangle \subset \mathbb{Z}\langle D_{\rho_0}, D_{\rho_1}, \dots, D_{\rho_n}\rangle,$$

and we obtain

$$\mathrm{Cl}(\mathbb{P}^n) \cong \frac{\mathbb{Z}\langle D_{\rho_0}, \dots, D_{\rho_n}\rangle}{\mathbb{Z}\langle D_{\rho_1} - D_{\rho_0}, \dots, D_{\rho_n} - D_{\rho_0}\rangle} \cong \mathbb{Z}\langle [D_{\rho_0}]\rangle,$$

with $[D_{\rho_i}] = [D_{\rho_0}]$. In particular,

$$\mathrm{Cl}(\mathbb{P}^n) \cong \mathbb{Z}\langle [D_{\rho_0}]\rangle \cong \mathbb{Z}.$$

Example 1.1.19. Let us now compute the class group of the Hirzebruch surface $\mathcal{H}_a$ with $a \in \mathbb{Z}_{\ge 0}$. Continuing with Example 1.1.13, we set

$$\begin{aligned} \rho_0 &= \mathrm{cone}(-e + af) \\ \rho_1 &= \mathrm{cone}(e) \\ \eta_0 &= \mathrm{cone}(-f) \\ \eta_1 &= \mathrm{cone}(f). \end{aligned}$$

The exact sequence in Proposition 1.1.17 is

$$0 \longrightarrow M \xrightarrow{\phi} \begin{array}{ccc} \mathbb{Z}D_{\rho_0} & \oplus & \mathbb{Z}D_{\rho_1} \\ & \oplus & \\ \mathbb{Z}D_{\eta_0} & \oplus & \mathbb{Z}D_{\eta_1} \end{array} \xrightarrow{\pi} \mathrm{Cl}(\mathcal{H}_a) \longrightarrow 0,$$

where $\phi\colon\ (d_1, d_2) \mapsto (-d_1 + ad_2, d_1, -d_2, d_2)$. Hence,

$$\mathrm{Im}\,\phi = \mathbb{Z}\langle D_{\rho_1} - D_{\rho_0}, D_{\eta_1} - D_{\eta_0} + aD_{\rho_1}\rangle \subset \mathbb{Z}\langle D_{\rho_0}, D_{\rho_1}, D_{\eta_0}, D_{\eta_1}\rangle,$$

and we have

$$\mathrm{Cl}(\mathcal{H}_a) \cong \frac{\mathbb{Z}\langle D_{\rho_0}, D_{\rho_1}, D_{\eta_0}, D_{\eta_1}\rangle}{\mathbb{Z}\langle D_{\rho_1} - D_{\rho_0}, D_{\eta_1} - D_{\eta_0} + aD_{\rho_1}\rangle} \cong \mathbb{Z}\langle [D_{\rho_0}], [D_{\eta_0}]\rangle,$$

with $[D_{\rho_1}] = [D_{\rho_0}]$ and $[D_{\eta_1}] = [D_{\eta_0}] - a[D_{\rho_0}]$. Thus, the class group of $\mathcal{H}_a$ is

$$\mathrm{Cl}(\mathcal{H}_a) \cong \mathbb{Z}\langle [D_{\rho_0}], [D_{\eta_0}]\rangle \cong \mathbb{Z}^2.$$

1.2 Coherent sheaves and graded modules

A quasi-coherent sheaf on a projective space $\mathbb{P}^n$ corresponds to a homogeneous module over the polynomial ring $\mathbb{C}[x_0, \dots, x_n]$. In this section, we recall how this correspondence generalizes for toric varieties using the theory of non-standard graded algebras. First, in Subsection 1.2.1 we gather the basic notion and results on modules and algebras graded by general abelian groups. Afterwards in Subsection 1.2.2, we introduce the so-called Cox ring of a toric variety X, which is a $\mathrm{Cl}(X)-$graded polynomial ring, and its connection to quasi-coherent sheaves over X. We mainly follow [19], [28] and [20, Ch. 5].

1.2.1 Graded algebras and modules

Through this subsection we fix an abelian group G and a commutative ring R.

Definition 1.2.1. A $R-$algebra A is $G-$*graded* if it decomposes into $R-$submodules as $A = \bigoplus_{g\in G} A_g$ such that $A_g \cdot A_h \subseteq A_{g+h}$, for any $g, h \in G$. Similarly, an $A-$module E is $G-$graded if it decomposes as $E = \bigoplus_{g\in G} E_g$ such that $A_g E_h \subset E_{g+h}$, for any $g, h \in G$. The submodules E_g are called $G-$*homogeneous components* of E. We say that an element $f \in E$ is *homogeneous* if $f \in E_g$ for some $g \in G$, and we define the *degree* of f to be $\deg(f) = g$.

Given $\pi : G \to H$ a surjective morphism of abelian groups, any

$G-$graded $R-$algebra A is also endowed with an $H-$grading:

$$A = \bigoplus_{h \in H} A_h, \quad \text{with} \quad A_h := \bigoplus_{\substack{g \in G \\ \pi(g) = h}} A_g \quad \text{for any} \quad h \in H.$$

As an application, we have the following lemma:

Lemma 1.2.2. *Let* $0 \longrightarrow K \xrightarrow{\phi} G \xrightarrow{\pi} H \longrightarrow 0$ *be a short exact sequence of abelian groups, and* A *a* $G-$*graded* $R-$*algebra. Then, for any* $h \in H$*, the* $H-$*homogeneous graded component* A_h *of degree* h *of* A *is a* $K-$*graded* $R-$*module.*

Proof. We fix $h \in H$ and let $x, y \in G$ be such that $\pi(x) = \pi(y) = h$. By exactness, $x - y \in \operatorname{Im}(\phi)$ and thus

$$\pi^{-1}(h) = \{x + \phi(k) \mid k \in K\} = \{y + \phi(k) \mid k \in K\}.$$

On the other hand, we consider the $R-$module A_h as before and we have

$$A_h = \bigoplus_{g \in \pi^{-1}(h)} A_g = \bigoplus_{k \in K} A_{x+\phi(k)}.$$

Notice that this decomposition does not depend (up to permutation) on the chosen preimage $x \in G$, and it structures A_h as a $K-$graded $R-$module. $\square$

Example 1.2.3. (i) Let $A = R[x_1, \ldots, x_n]$ be the polynomial ring over R in n variables. Then, A endows a standard $\mathbb{Z}-$grading as well as a standard $\mathbb{Z}^n-$grading.

(ii) Let us consider $G = \mathbb{Z}/2$, and set

$$A_{\overline{0}} := R\langle x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha_1 + \cdots + \alpha_n \equiv 0 \mod 2\rangle$$
$$A_{\overline{1}} := R\langle x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha_1 + \cdots + \alpha_n \equiv 1 \mod 2\rangle.$$

Then, $A = A_{\overline{0}} \oplus A_{\overline{1}}$ is a $\mathbb{Z}/2-$graded $R-$algebra such that

$$\deg(x_i) = \overline{1}.$$

(iii) More in general, for any abelian group G and $w_1, \ldots, w_n \in G$, we set
$$A_g := R\langle x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha_1 w_1 + \cdots + \alpha_n w_n = g\rangle.$$
The $R-$modules A_g structure A as a $G-$graded $R-$algebra. Notice that this $G-$grading is determined by setting
$$\deg(x_i) = w_i.$$

Next, we describe when the localization of an algebra is graded.

Lemma 1.2.4. *Let A be a $G-$graded $R-$algebra and $f \in A$ a homogeneous element of degree d. Then, the localized $R-$algebra A_f at f is $G-$graded.*

Proof. For any $g \in G$, we define the $R-$submodule
$$(A_f)_g := \{\frac{a}{f^n} \mid \deg(a) - nd = g\} \subset A_f.$$
Then we have that
$$A_f = \bigoplus_{g \in G} (A_f)_g,$$
structuring A_f as a $G-$graded $R-$algebra. □

1.2.2 The Cox ring of a toric variety

In this subsection, we recall the correspondence between graded modules over the Cox ring R of a toric variety X and quasi-coherent sheaves on X.

Definition 1.2.5. The *Cox ring* of a toric variety X with fan Σ is the polynomial ring $R = \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$. R is $\mathrm{Cl}(X)-$graded by setting $\deg(x_\rho) := [D_\rho]$. We write $R = \mathbb{C}[x_1, \ldots, x_r]$ whenever we have fixed an ordered set of rays $\Sigma(1) = \{\rho_1, \ldots, \rho_r\}$. For any cone $\sigma \in \Sigma$, we set
$$x^{\hat{\sigma}} := \prod_{\rho_i \in \Sigma(1) \setminus \sigma(1)} x_i, \quad \text{and} \quad B := (x^{\hat{\sigma}} \mid \sigma \in \Sigma).$$

The ideal B is called the *irrelevant ideal* and we have
$$B = (x^{\hat{\sigma}} \mid \sigma \in \Sigma_{\max}).$$

Notice that for the cone $\{0\}$ we have that $R_{x^{\{\hat{0}\}}} = \mathbb{C}[x_\rho^{\pm} \mid \rho \in \Sigma(1)]$, which is the Laurent polynomial ring associated to R. Finally, for any $\mathbb{T}_N-$invariant Weil divisor $D = \sum_\rho a_\rho D_\rho$ we write

$$x^D := \prod_\rho x_\rho^{a_\rho} \in R_{x^{\{\hat{0}\}}}.$$

Example 1.2.6. (i) Continuing with Examples 1.1.11 and 1.1.18, the Cox ring of $\mathbb{P}^n$ is $\mathbb{C}[x_0, \dots, x_n]$, $\mathbb{Z}-$graded with $\deg(x_i) = 1$. Its irrelevant ideal is $B = (x_0, \dots, x_n)$.

(ii) Continuing with Examples 1.1.13 and 1.1.19, the Cox ring of the Hirzebruch surface $\mathcal{H}_a$ with $a \in \mathbb{Z}_{\geq 0}$ is $\mathbb{C}[x_0, x_1, y_0, y_1]$, $\mathbb{Z}^2-$graded with

$$\deg(x_0) = \deg(x_1) = (1,0), \ \deg(y_0) = (0,1) \text{ and } \deg(y_1) = (-a,1).$$

Its irrelevant ideal is $B = (x_0y_0, x_0y_1, x_1y_0, x_1y_1)$.

Recall that for any cone $\sigma \in \Sigma$, the coordinate ring of the affine open toric variety U_σ is $\mathbb{C}[U_\sigma] = \mathbb{C}[S_\sigma]$ and we have $\mathcal{O}_{U_\sigma} \cong \widetilde{\mathbb{C}[S_\sigma]}$. On the other hand, a character $m \in M$ belongs to S_σ if and only if $\langle m, n(\rho) \rangle \geq 0$ for all $\rho \in \sigma(1)$. In particular, we have an isomorphism

$$\mathbb{C}[S_\sigma] \cong (R_{x^{\hat{\sigma}}})_0$$

sending a character $\chi^m \in \mathbb{C}[S_\sigma]$ to the monomial $x^m := x^{\mathrm{div}(\chi^m)} \in (R_{x^{\hat{\sigma}}})_0$. Thus, we can recover the sheaf $\mathcal{O}_{U_\sigma}$ from the localization of R at $x^{\hat{\sigma}}$:

$$\widetilde{(R_{x^{\hat{\sigma}}})_0} \cong \mathcal{O}_{U_\sigma}.$$

Glueing all these sheaves using the structure of the fan Σ, we obtain the sheaf $\mathcal{O}_X$: if $\tau \prec \sigma$ is a face, there is a character $m \in S_\sigma$ such that $S_\tau = S_\sigma + \mathbb{Z}(-m)$. We have isomorphisms $\mathbb{C}[S_\sigma]_{\chi^m} \cong \mathbb{C}[S_\tau]$ and $(R_{x^{\hat{\sigma}}})_{x^m} \cong R_{x^{\hat{\tau}}}$, and the following commutative diagram

$$\begin{array}{ccc} (R_{x^{\hat{\sigma}}})_0 & \xrightarrow{\cong} & \mathbb{C}[S_\sigma] \\ \downarrow & & \downarrow \\ (R_{x^{\hat{\tau}}})_0 & \xrightarrow[\cong]{} & \mathbb{C}[S_\tau]. \end{array}$$

For the purpose of this thesis, we focus on a smooth compact toric variety X. We have the following correspondence generalizing the above construction.

Proposition 1.2.7. *Let X be a smooth compact toric variety, and R its Cox ring.*

(i) For any Weil divisor $D = \sum_\rho a_\rho D_\rho$, there is an isomorphism

$$R_{[D]} \cong \Gamma(X, \mathcal{O}_X(D)).$$

(ii) If E is a $\mathrm{Cl}(X)-$graded R-module, there is a quasi-coherent sheaf $\widetilde{E}$ on X such that

$$\Gamma(U_\sigma, \widetilde{E}) = (E_{x^{\hat{\sigma}}})_0,$$

for any $\sigma \in \Sigma$.

(iii) If $\mathcal{E}$ is a quasicoherent sheaf on X, there is a $\mathrm{Cl}(X)-$graded $R-$module such that $\widetilde{E} = \mathcal{E}$. $\widetilde{E}$ is coherent if and only if E is finitely generated.

(iv) $\widetilde{E} = 0$ if and only if $B^l E = 0$ for all $l \gg 0$.

(v) There is an exact sequence of $\mathrm{Cl}(X)-$graded modules

$$0 \longrightarrow H^0_B(E) \longrightarrow E \longrightarrow H^0_*(X, \widetilde{E}) \longrightarrow H^1_B(E) \longrightarrow 0.$$

(vi) There are $\mathrm{Cl}(X)-$ graded isomorphisms

$$H^{i+1}_B(E) \cong H^i_*(X, \widetilde{E}) = \bigoplus_{\alpha \in \mathrm{Cl}(X)} H^i(X, \widetilde{E}(\alpha)).$$

Proof. (i)–(iv) follow from [20, Proposition 5.3.3, Proposition 5.3.6, Proposition 5.3.7 and Proposition 5.3.10]. (v) and (vi) follows from [28, Proposition 2.3]. □

The module $\Gamma E = H^0_*(X, \widetilde{E})$ is called the $B-$*saturation* of E. We say that E is $B-$*saturated* if $E \cong \Gamma E$, or equivalently if $H^0_B(E) = H^1_B(E) = 0$. If $H^0_B(E) = (0 :_E B^\infty) = 0$, we say that E is $B-$*torsion free.*

1.3 Equivariant sheaves and fine-graded modules

In the previous section, we have seen that any quasicoherent sheaf $\mathcal{E}$ on a smooth compact toric variety X corresponds to a $\mathrm{Cl}(X)-$graded module E over the Cox ring R satisfying that $\mathcal{E} = \widetilde{E}$ (Proposition 1.2.7). In this section, we focus on equivariant sheaves corresponding to $\mathbb{Z}^{|\Sigma(1)|}-$graded modules. This restriction allows us to describe the module E with a filtered family of vector spaces. When $\mathcal{E}$ is reflexive, these filtrations are particularly well behaved. This construction was initially developed for locally free sheaves by Klyachko in [45], and further generalized for any quasi-coherent sheaf by Perling in [57]. In what follows, we mainly use the results as presented in [57].

We start with the definition of an equivariant sheaf on a toric variety X.

Definition 1.3.1. For any $t \in \mathbb{T}_N$, let $\mu_t : X \to X$ be the morphism given by the action of $\mathbb{T}_N$ on X. A quasi-coherent sheaf $\mathcal{E}$ on X is *equivariant* if there is a family of isomorphisms $\{\phi_t : \mu_t^*\mathcal{E} \cong \mathcal{E}\}_{t\in\mathbb{T}_N}$ such that

$$\phi_{t_1\cdot t_2} = \phi_{t_2} \circ \mu_{t_2}^*\phi_{t_1},$$

for any $t_1, t_2 \in \mathbb{T}_N$.

Next, we see which $\mathrm{Cl}(X)-$graded modules correspond to equivariant sheaves. We say that an $R-$module is *fine-graded* if it is $\mathbb{Z}^{|\Sigma(1)|}-$graded. Notice that fine-graded modules are also $\mathrm{Cl}(X)-$graded. In [6], Batyrev and Cox proved the following result:

Proposition 1.3.2. *Let E be a $\mathrm{Cl}(X)-$graded $R-$module. The quasi-coherent sheaf $\widetilde{E}$ is equivariant if and only if E is also fine-graded.*

Proof. See [6, Proposition 4.17]. □

In [45] and [46], Klyachko observed that any equivariant torsion-free sheaf is associated to a family of filtered vector spaces, the so-called Klyachko filtration. In what follows, we recall how this family can be constructed. Let $\mathcal{E}$ be an equivariant sheaf on X corresponding to a fine-graded module E. By Lemma 1.2.2 applied on the exact sequence (1.1), we have that any homogeneous piece of E is endowed with an $M-$grading:

$$E_\alpha = \bigoplus_{m\in M} E_{z+\phi(m)}, \quad \text{for any} \quad z \in \pi^{-1}(\alpha). \tag{1.2}$$

Now, for any $\sigma \in \Sigma$ we consider the monomial $x^{\hat{\sigma}}$, then the localized $R_{x^\sigma}-$module E_{x^σ} remains fine-graded by Lemma 1.2.4. As in (1.2), for any $\alpha \in \mathrm{Cl}(X)$, $(E_{x^{\hat{\sigma}}})_\alpha$ is $M-$graded. In particular, taking $\alpha = 0$ we have:

$$E^\sigma := (E_{x^{\hat{\sigma}}})_0 = \bigoplus_{m\in M} (E_{x^{\hat{\sigma}}})_{\phi(m)} =: \bigoplus_{m\in M} E^\sigma_m. \tag{1.3}$$

Since $(E_{x^{\hat{\sigma}}})_0$ is isomorphic to the $\mathbb{C}[S_\sigma]-$module $\Gamma(U_\sigma, \mathcal{E})$, geometrically we can see (1.3) as the isotypical decomposition of $\Gamma(U_\sigma, \mathcal{E})$ into $\mathbb{T}_N-$eigenspaces of sections:

$$\Gamma(U_\sigma, \mathcal{E}) = \bigoplus_{m\in M} \Gamma(U_\sigma, \mathcal{E})_m.$$

Recall that the semigroup S_σ induces a preorder on the character lattice M: for any $m, m' \in M$ we say that $m \leq_\sigma m'$ if and only if $m' - m \in S_\sigma$ or, equivalently, if $\langle m' - m, u\rangle \geq 0$ for all $u \in \sigma$. For any two characters $m \leq_\sigma m'$, the multiplication by $\chi^{m'-m} \in \mathbb{C}[S_\sigma]$ yields the map

$$\chi^\sigma_{m,m'} : E^\sigma_m \to E^\sigma_{m'}.$$

For any $m \leq_\sigma m' \leq_\sigma m''$, we have

$$\chi^\sigma_{m,m} = 1 \quad \text{and} \quad \chi^\sigma_{m,m''} = \chi^\sigma_{m',m''} \circ \chi^\sigma_{m,m'}.$$

In particular, $\chi^\sigma_{m,m'}$ is an isomorphism if $m \leq_\sigma m'$ and $m' \leq_\sigma m$ or, equivalently, if $m' - m \in \sigma^\perp$. We call $\hat{E}^\sigma := \{E^\sigma_m, \chi^\sigma_{m,m'}\}$ a $\sigma-$*family* (see [57, Definition 4.2]).

On the other hand, let $\tau \prec \sigma$ be two cones in Σ and $m \in M$ the character such that $S_\tau = S_\sigma + \mathbb{Z}(-m)$. There are isomorphisms

$$\mathbb{C}[S_\tau] \cong \mathbb{C}[S_\sigma]_{\chi^m} \quad \text{and} \quad E^\tau \cong E^\sigma_{x^m}$$

given by the localization at $\chi^m \in \mathbb{C}[S_\sigma]$ and $x^m \in (R_{x^{\hat{\sigma}}})_0$, respectively. Thus, we have a morphism $i^{\sigma\tau} : E^\sigma \to E^\tau$ corresponding geometrically to the restriction map of section modules $\Gamma(U_\sigma, \mathcal{E}) \to \Gamma(U_\tau, \mathcal{E})$. The morphism $i^{\sigma\tau}$ is $M-$graded since it is defined by the localization at a monomial. Thus, for any character $m' \in M$ we have a linear map

$$i^{\sigma\tau}_{m'} : E^\sigma_{m'} \to E^\tau_{m'}.$$

We call the set of $\sigma-$families $\{\hat{E}^\sigma\}_{\sigma\in\Sigma}$ a $\Sigma-$*family* (see [57, Definition 4.8]). In [57, Theorem 4.9] it is proved that $\Sigma-$families characterize equivariant quasi-coherent sheaves on X or, equivalently, $B-$saturated fine-graded $R-$modules.

1.3.1 Torsion-free equivariant sheaves

When $\mathcal{E}$ is an equivariant torsion-free sheaf, the restriction maps

$$\Gamma(U, \mathcal{E}) \to \Gamma(V, \mathcal{E}),$$

with $V \subset U \subset X$ are injective. Moreover, all the morphisms among the vector spaces in the above description of the $\Sigma-$family of $\mathcal{E}$ are injective. We have the following result:

Proposition 1.3.3. *Let $\mathcal{E}$ be an equivariant torsion-free sheaf of rank ℓ and $\{\hat{E}^\sigma\}$ its associated $\Sigma-$family. The following holds:*

(i) For any $m' \leq_\sigma m$, the linear map $\chi^\sigma_{m',m} : E^\sigma_{m'} \to E^\sigma_m$ is injective.

(ii) For any character $m \in M$, and any cones $\tau \prec \sigma$ in Σ, the linear map $i^{\sigma\tau}_m : E^\sigma_m \to E^\tau_m$ is injective.

(iii) There is a vector space $\mathbf{E} \cong \mathbb{C}^\ell$ such that $E^{\{0\}}_m \cong \mathbf{E}$ for any $m \in M$.

We have the following commutative diagram:

$$\begin{tikzcd}
 & E^{\{0\}}_{m'} \arrow[dl, "\varphi_{m'}"'] \arrow[d, "\chi^{\{0\}}_{m',m}"] & E^\sigma_{m'} \arrow[l] \arrow[d, "\chi^\sigma_{m',m}"] \\
\mathbf{E} \arrow[r, phantom, "\cong"] & {} & \\
 & E^{\{0\}}_m \arrow[ul, "\varphi_m"] & E^\sigma_m. \arrow[l]
\end{tikzcd}$$

Proof. See [57, Section 4.4]. $\square$

Remark 1.3.4. (i) By Proposition 1.3.3, the $\Sigma-$family $\{\hat{E}^\sigma\}_{\sigma\in\Sigma}$ of a torsion-free sheaf $\mathcal{E}$ of rank ℓ can be seen as a filtered collection of linear subspaces of a fixed ambient vector space $\mathbf{E}$. Geometrically, the vector space $\mathbf{E}$ can be identified with the $\ell-$dimensional vector space $\Gamma(\mathbb{T}_N, \mathcal{E})_m$ for any character $m \in M$.

(ii) The description of equivariant torsion-free sheaves given above is based on [57, Section 4]. We note that our order of filtrations is reverse of that of Klyachko [45, 46]. In these references, the filtration is taken as a collection of linear subspaces of $\mathcal{E}(x_0)$, the fiber of $\mathcal{E}$ at a point in the open orbit $U_{\{0\}} = \mathbb{T}_N \subset X$ (see [57, Remark 4.25]).

Let us illustrate how to compute the Klyachko filtrations following Proposition 1.3.3.

Example 1.3.5. Let $R = \mathbb{C}[x_0, x_1, x_2]$ be the Cox ring of $\mathbb{P}^2$ with fan Σ as in Example 1.1.11 and $I = (x_2^2, x_0x_2, x_0x_1)$ a monomial ideal. Let us compute the $\Sigma-$family associated to I viewed as a torsion-free fine-graded module and presented as follows:

$$R(0,0,-2) \oplus R(-1,0,-1) \oplus R(-1,-1,0) \xrightarrow{(x_2^2\ x_0x_2\ x_0x_1)} I \longrightarrow 0. \quad (1.4)$$

First, we want to determine the one-dimensional vector space $\mathbf{I} \cong \mathbb{C}$ such that $I^{\{0\}} \cong \mathbf{I}$. We start localizing at $x^{\widehat{\{0\}}} = x_0x_1x_2$ and we set

$$R^{\{0\}} := R_{x^{\widehat{\{0\}}}} = \mathbb{C}[x_0^{\pm}, x_1^{\pm}, x_2^{\pm}].$$

For any multidegree $(\alpha_0, \alpha_1, \alpha_2) \in \mathbb{Z}^3$, we denote by

$$R^{\{0\}}_{(\alpha_0,\alpha_1,\alpha_2)} = \mathbb{C}\langle x_0^{\alpha_0} x_1^{\alpha_1} x_2^{\alpha_2}\rangle$$

the vector space spanned by the (Laurent) monomial $x_0^{\alpha_0} x_1^{\alpha_1} x_2^{\alpha_2}$. On the other hand, any character $m = (d_1, d_2) \in M$ is embedded via the exact sequence (1.1) as

$$\phi(m) = (-d_1 - d_2, d_1, d_2) \in \mathbb{Z}^3.$$

To compute $I^{\{0\}}_m$, we take the degree$-\phi(m)$ component of (1.4). This yields the following exact sequence of vector spaces

$$R^{\{0\}}(0,0,-2)_m \oplus R^{\{0\}}(-1,0,-1)_m \oplus R^{\{0\}}(-1,-1,0)_m \xrightarrow{(x_2^2\ x_0x_2\ x_0x_1)} I^{\{0\}}_m \longrightarrow 0.$$

Thus, $I_m^{\{0\}} = \mathbb{C}\langle x_0^{-d_1-d_2} x_1^{d_1} x_2^{d_2}\rangle$ and there we have an isomorphism

$$\phi_m^{\{0\}} : I_m^{\{0\}} \cong \mathbf{I}.$$

Let us now fix the ray $\rho_0 \in \Sigma(1)$ and compute the vector subspaces

$$I_m^{\rho_0} \subset \mathbf{I},$$

for any character $m = (d_1, d_2) \in M$. As before, we set $R^{\rho_0} := R_{x^{\widehat{\rho_0}}}$ the localization at $x^{\widehat{\rho_0}} = x_1 x_2$. Now, we have that

$$R^{\rho_0} \cong \mathbb{C}[x_0, x_1^{\pm}, x_2^{\pm}],$$

and for any multidegree $(\alpha_0, \alpha_1, \alpha_2) \in \mathbb{Z}^3$,

$$R^{\rho_0}_{(\alpha_0,\alpha_1,\alpha_2)} = \begin{cases} \mathbb{C}\langle x_0^{\alpha_0} x_1^{\alpha_1} x_2^{\alpha_2}\rangle, & \text{if} \quad \alpha_0 \geq 0 \\ 0, & \text{if} \quad \alpha_0 \leq -1. \end{cases}$$

Taking the component of degree $\phi(m) = (-d_1 - d_2, d_1, d_2)$ of the exact sequence (1.4), we have

$$I_m^{\rho_0} = \begin{cases} \mathbb{C}\langle x_0^{-d_1-d_2} x_1^{d_1} x_2^{d_2}\rangle \cong \mathbf{I}, & \text{if} \quad -d_1 - d_2 \geq 0 \\ 0, & \text{if} \quad -d_1 - d_2 \leq -1. \end{cases}$$

Similarly for the rays ρ_1 and ρ_2, we obtain

$$I_m^{\rho_1} \cong \begin{cases} \mathbf{I}, & \text{if} \quad d_1 \geq 0 \\ 0, & \text{if} \quad d_1 \leq -1 \end{cases}$$

$$I_m^{\rho_2} \cong \begin{cases} \mathbf{I}, & \text{if} \quad d_2 \geq 0 \\ 0, & \text{if} \quad d_2 \leq -1. \end{cases}$$

It only remains to compute the components of the $\Sigma-$family associated to the two dimensional cones in Σ. Let us consider $\sigma_0 \in \Sigma(2)$ with rays $\sigma_0(1) = \{\rho_1, \rho_2\}$. We set $R^{\sigma_0} := R_{x^{\widehat{\sigma_0}}}$ the localization at $x^{\widehat{\sigma_0}} = x_0$. We have $R^{\sigma_0} \cong \mathbb{C}[x_0^{\pm}, x_1, x_2]$, and for any multidegree $(\alpha_0, \alpha_1, \alpha_2) \in \mathbb{Z}^3$,

$$R^{\sigma_0}_{(\alpha_0,\alpha_1,\alpha_2)} = \begin{cases} \mathbb{C}\langle x_0^{\alpha_0} x_1^{\alpha_1} x_2^{\alpha_2}\rangle, & \text{if} \quad \alpha_1 \geq 0,\ \alpha_2 \geq 0 \\ 0, & \text{if} \quad \alpha_1 \leq -1 \text{ or } \alpha_2 \leq -1. \end{cases}$$

As before, taking the component of degree $m=(d_1,d_2)$ of (1.4) we obtain

$$I_m^{\sigma_0} \cong \begin{cases} \mathbf{I}, & \text{if } \; d_1 = 0 \text{ and } d_2 \geq 1, \text{ or} \\ & \quad\; d_1 \geq 1 \text{ and } d_2 \geq 0 \\ 0, & \quad\; \text{otherwise.} \end{cases}$$

Similarly for the remaining cones σ_1 and σ_2, we have

$$I_m^{\sigma_1} \cong \begin{cases} \mathbf{I}, & \text{if } \; -d_1 - d_2 = 0 \text{ and } d_2 \geq 2, \text{ or} \\ & \quad\; -d_1 - d_2 \geq 1 \text{ and } d_2 \geq 0 \\ 0, & \quad\; \text{otherwise} \end{cases}$$

$$I_m^{\sigma_2} \cong \begin{cases} \mathbf{I}, & \text{if } \; -d_1 - d_2 \geq 0 \text{ and } d_1 \geq 0 \\ 0, & \text{otherwise;} \end{cases}$$

obtaining in this way a complete description of the $\Sigma-$family $\{I_m^\sigma\}$ of I.

1.3.2 Reflexive equivariant sheaves

We now assume that $\mathcal{E}$ is a reflexive equivariant sheaf of rank ℓ on X. In particular, $\mathcal{E}^{\vee\vee} \cong \mathcal{E}$ and for any subset $Z \subset X$ of codimension at least 2, we have $\mathrm{H}^0(X,\mathcal{E}) \cong \mathrm{H}^0(X \setminus Z, \mathcal{E})$. For any cone $\sigma \in \Sigma$, we set

$$Z := \bigcup_{\substack{\dim \tau \geq 2 \\ \tau \prec \sigma}} V(\tau) \subset U_\sigma,$$

which has codimension 2 by Proposition 1.1.15, and we have

$$U_\sigma \setminus Z = \bigcup_{\rho \in \sigma(1)} U_\rho.$$

Thus, by reflexivity of $\mathcal{E}$,

$$\Gamma(U_\sigma, \mathcal{E}) \cong \Gamma(\bigcup_{\rho \in \sigma(1)} U_\rho, \mathcal{E}) \cong \bigcap_{\rho \in \sigma(1)} \Gamma(U_\rho, \mathcal{E}), \text{ for any } \sigma \in \Sigma.$$

Translating this fact to the language of $\sigma-$families, for any character $m \in M$ we have

$$E_m^\sigma \cong \bigcap_{\rho \in \sigma(1)} E_m^\rho.$$

Hence, each $\sigma-$family of $\mathcal{E}$ is determined completely by the $\rho-$families with $\rho \in \sigma(1)$.

Let us fix a ray $\rho \in \Sigma(1)$. For any characters $m' \leq_\rho m$, the multiplication map $\chi^\rho_{m',m} : E^\rho_{m'} \to E^\rho_m$ is an isomorphism if $\langle m' - m, n(\rho)\rangle = 0$. Hence, for any $i \in \mathbb{Z}$ there is a subspace $E^\rho(i) \subset \mathbb{C}^\ell$ such that $E^\rho_m \cong E^\rho(i)$ if $\langle m, n(\rho)\rangle = i$. Consequently, any $\rho-$family $\hat{E}^\rho$ can be seen as an increasing filtration $\{E^\rho(i)\}_{i\in\mathbb{Z}}$ of vector subspaces of $\mathbf{E} \cong \mathbb{C}^\ell$ such that that $E^\rho(i) = 0$ for $i \ll 0$ and $E^\rho(i) = \mathbf{E}$ for $i \gg 0$. We introduce the following notation to express this collection of increasing filtrations:

Notation 1.3.6. For each ray ρ, let us denote the associated filtration as

$$\hat{E}^\rho = E^\rho(i^\rho_1, \dots, i^\rho_{\ell-1}, i^\rho_\ell; E^\rho_1, \dots, E^\rho_{\ell-1}, E^\rho_\ell).$$

We have,

$$E^\rho(i) = \begin{cases} 0, & i \leq i^\rho_1 - 1 \\ E^\rho_1, & i^\rho_1 \leq i \leq i^\rho_2 - 1 \\ \vdots & \\ E^\rho_{\ell-1}, & i^\rho_{\ell-1} \leq i \leq i^\rho_\ell - 1 \\ E^\rho_\ell, & i^\rho_\ell \leq i \end{cases}$$

where

$$0 \subseteq E^\rho_1 \subseteq \cdots \subseteq E^\rho_{\ell-1} \subseteq E^\rho_\ell = \mathbb{C}^\ell,$$

is an increasing filtration of vector spaces, and

$$i^\rho_1 \leq \cdots \leq i^\rho_{\ell-1} \leq i^\rho_\ell$$

is an increasing sequence of integers (the *steps of the filtration*) such that

$$i^\rho_j = i^\rho_{j+1} \quad \text{if and only if} \quad E^\rho_j = E^\rho_{j+1}.$$

If $\Sigma(1) = \{\rho_1, \dots, \rho_r\}$ is the (ordered) set of rays, we write

$$\begin{aligned} \hat{E}^k &:= \hat{E}^{\rho_k} \\ i^k_t &:= i^{\rho_k}_t \\ E^k_t &:= E^{\rho_k}_t, \end{aligned}$$

for any $1 \leq k \leq r$ and $1 \leq t \leq \ell$.

Example 1.3.7. Given a Weil divisor $D = \sum_\rho a_\rho D_\rho$, the line bundle $\mathcal{O}(D)$ corresponds to the following set of filtrations of a one-dimensional vector space $\mathbf{O} \cong \mathbb{C}$:

$$O^\rho(i) = \begin{cases} 0, & i < -a_\rho \\ \mathbf{O}, & i \geq -a_\rho. \end{cases}$$

This follows from Proposition 1.2.7, since $\mathcal{O}(D) \cong R(\widetilde{[\sum_\rho a_\rho D_\rho]})$. In the above notation, for any ray $\rho \in \Sigma(1)$, we write this filtration as

$$\hat{O}^\rho = O^\rho(-a_\rho; \mathbf{O}).$$

More generally, the following result shows the effect of twisting by $\mathcal{O}(D)$ for any Weil divisor $D = \sum_\rho a_\rho D_\rho$ on the set of filtrations.

Proposition 1.3.8. *Let $\mathcal{E}$ be an equivariant reflexive sheaf of rank ℓ with filtration given by*

$$E^\rho(i_1^\rho, \ldots, i_\ell^\rho; E_1^\rho, \ldots, E_\ell^\rho).$$

Let $D = \sum_\rho a_\rho D_\rho$ a $\mathbb{T}_N-$invariant Weil divisor. Then, the filtration of $\mathcal{E}([D]) = \mathcal{E} \otimes \mathcal{O}(D)$ is given by

$$E^\rho(i_1^\rho - a_\rho, \ldots, i_\ell^\rho - a_\rho; E_1^\rho, \ldots, E_\ell^\rho).$$

Proof. See [57, Section 4.7]. □

Example 1.3.9. If Σ is the fan of the Hirzebruch surface $\mathcal{H}_3$ and its Cox ring $R = \mathbb{C}[x_0, x_1, y_0, y_1]$ as in Example 1.2.6 (ii). Then, any character $m = (d_1, d_2)$ is embedded in $\mathbb{Z}^4$ as a multidegree

$$\phi(m) = (-d_1 + 3d_2, d_1, -d_2, d_2).$$

We consider the reflexive module E presented as:

$$R \xrightarrow{\phi} R(0,0,1,1) \oplus R(0,1,0,0) \oplus R(1,0,0,0) \longrightarrow E \longrightarrow 0 \qquad (1.5)$$

where ϕ is given by the following matrix:

$$\begin{pmatrix} 3y_0y_1 \\ x_1 \\ -x_0 \end{pmatrix}$$

and we want to determine its associated $\Sigma-$family. First of all, we want to determine $\mathbf{E} \cong E_m^{\{0\}}$ for any character $m \in M$. We localize the multi-graded sequence (1.5) at $x^{\widehat{\{0\}}} = x_0x_1y_0y_1$ and we set $R^{\{0\}} := R_{x^{\widehat{\{0\}}}}$. Any character $m = (d_1, d_2)$ yields an exact sequence of $\mathbb{C}-$linear spaces

$$R_m^{\{0\}} \xrightarrow{\phi} R^{\{0\}}(0,0,1,1)_m \oplus R^{\{0\}}(0,1,0,0)_m \oplus R^{\{0\}}(1,0,0,0)_m \longrightarrow E_m^{\{0\}} \longrightarrow 0.$$

Thus, we obtain $E_m^{\{0\}}$ as the cokernel of a $\mathbb{C}-$linear map:

$$E_m^{\{0\}} = \frac{\mathbb{C}\langle x_0^{3d_2-d_1}x_1^{d_1}y_0^{-d_2+1}y_1^{d_2+1}, x_0^{3d_2-d_1}x_1^{d_1+1}y_0^{-d_2}y_1^{d_2}, x_0^{3d_2-d_1+1}x_1^{d_1}y_0^{-d_2}y_1^{d_2}\rangle}{\langle 3x_0^{3d_2-d_1}x_1^{d_1}y_0^{-d_2+1}y_1^{d_2+1} + x_0^{3d_2-d_1}x_1^{d_1+1}y_0^{-d_2}y_1^{d_2} - x_0^{3d_2-d_1+1}x_1^{d_1}y_0^{-d_2}y_1^{d_2}\rangle}$$

and we have isomorphisms

$$\phi_m : E_m^{\{0\}} \longrightarrow \frac{\mathbb{C}\langle u_1, u_2, u_3\rangle}{\langle 3u_1 + u_2 - u_3\rangle} = \mathbb{C}\langle \overline{u_1}, \overline{u_2}\rangle =: \mathbf{E}$$

such that

$$\overline{u_3} = 3\overline{u_1} + \overline{u_2}.$$

Since E is a reflexive module, it only remains to compute the vector subspaces $E_m^\tau \subset \mathbf{E}$ for each ray $\tau \in \Sigma(1)$ and any character $m = (d_1, d_2)$. Let us consider the ray $\rho_0 \in \Sigma(1)$ and the localization at $x^{\hat{\rho_0}}$ of R. We set $R^{\rho_0} := R_{x^{\hat{\rho_0}}}$, and for any character $m = (d_1, d_2)$ we have

$$R_m^{\rho_0} = \begin{cases} \mathbb{C}\langle x_0^{3d_2-d_1}x_1^{d_1}y_0^{-d_2}y_1^{d_2}\rangle, & \text{if} \quad 3d_2 - d_1 \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, localizing the exact sequence (1.5) we obtain

$$E_m^{\rho_0} \cong \begin{cases} \mathbf{E}^0, & \text{if} \quad 3d_2 - d_1 \geq 0 \\ \langle \overline{u_3}\rangle = \langle 3\overline{u_1} + \overline{u_2}\rangle, & \text{if} \quad 3d_2 - d_1 = -1 \\ 0, & \text{if} \quad 3d_2 - d_1 \leq -2. \end{cases}$$

Similarly localizing with respect to the other rays, we obtain

$$E_m^{\rho_1} \cong \begin{cases} \mathbf{E}^0, & \text{if} \quad d_1 \geq 0 \\ \langle \overline{u_2} \rangle, & \text{if} \quad d_1 = -1 \\ 0, & \text{if} \quad d_1 \leq -2 \end{cases}$$

$$E_m^{\eta_0} \cong \begin{cases} \mathbf{E}^0, & \text{if} \quad -d_2 \geq 0 \\ \langle \overline{u_1} \rangle, & \text{if} \quad -d_2 = -1 \\ 0, & \text{if} \quad -d_2 \leq -2 \end{cases}$$

$$E_m^{\eta_0} \cong \begin{cases} \mathbf{E}^0, & \text{if} \quad d_2 \geq 0 \\ \langle \overline{u_1} \rangle, & \text{if} \quad d_2 = -1 \\ 0, & \text{if} \quad d_2 \leq -2. \end{cases}$$

In the notation introduced above, the filtrations of subspaces of $\mathbf{E}^0 \cong \mathbb{C}^2$ associated to E are

$$\begin{aligned} &E^{\rho_0}(-1, 0; \langle 3\overline{u_1} + \overline{u_2} \rangle, \mathbf{E}^0) \\ &E^{\rho_1}(-1, 0; \langle \overline{u_2} \rangle, \mathbf{E}^0) \\ &E^{\eta_0}(-1, 0; \langle \overline{u_1} \rangle, \mathbf{E}^0) \\ &E^{\eta_1}(-1, 0; \langle \overline{u_1} \rangle, \mathbf{E}^0). \end{aligned}$$

In [43, Theorem 3.5] (see also [45, Proposition 2.5]) it was proved that any equivariant locally free sheaf on an affine toric variety is decomposable as a direct sum of line bundles. Moreover, this result was generalized in [36] for any locally free sheaf on an affine toric variety. In particular, equivariant reflexive sheaves which are locally free are characterized in the following proposition:

Proposition 1.3.10. *Let $\mathcal{E}$ be an equivariant reflexive sheaf of rank ℓ with associated filtrations $E^\rho(i_1^\rho, \ldots, i_\ell^\rho; E_1^\rho, \ldots, E_\ell^\rho)$. Then $\mathcal{E}$ is locally free if and only if for any cone $\sigma \in \Sigma$, there is an $M/\sigma^\perp$-graded decomposition*

$$\mathbf{E}^0 = \bigoplus_{[m] \in M/(\sigma^\perp \cap M)} E^0_{[m]}$$

such that for each ray $\rho \in \sigma(1)$ we have

$$E^\rho(i) = \sum_{\substack{[m] \in M/(\sigma^\perp \cap M) \\ \langle m, n(\rho) \rangle \leq i}} E^0_{[m]}.$$

Proof. See [57, Proposition 4.24] and [45, Theorem 2.2.1]. □

We end this subsection with some examples to illustrate how to compute the Klyachko filtrations.

Example 1.3.11 (Tensor product)**.** Let $\mathcal{F}$ and $\mathcal{G}$ be locally free sheaves of rank ℓ_1 and ℓ_2, respectively and with filtrations

$$F^\rho(i_1^\rho, \dots, i_{\ell_1}^\rho; F_1^\rho, \dots, F_{\ell_1}^\rho)$$
$$G^\rho(j_1^\rho, \dots, j_{\ell_2}^\rho; G_1^\rho, \dots, G_{\ell_2}^\rho).$$

We write $F_{k+1}^\rho = F_k^\rho \oplus \mathbb{C}\langle f_{k+1}^\rho \rangle$ and $G_{k+1}^\rho = G_k^\rho \oplus \mathbb{C}\langle g_{k+1}^\rho \rangle$. Then $\mathcal{F} \otimes \mathcal{G}$ is an equivariant locally free sheaf of rank $\ell_1\ell_2$ and its filtration is given by:

$$(\mathcal{F} \otimes \mathcal{G})^\rho(j) = \bigoplus_{\substack{1 \le k_1 \le \ell_1 \\ 1 \le k_2 \le \ell_2 \\ i_{k_1}^\rho + j_{k_2}^\rho \le j}} \mathbb{C}\langle f_{k_1}^\rho \otimes g_{k_2}^\rho \rangle. \tag{1.6}$$

Indeed, for any ray $\rho \in \Sigma(1)$, the equivariant vector bundles $\mathcal{F}$ and $\mathcal{G}$ decompose as a direct sum of line bundles:

$$\mathcal{F}_{|U_\rho} \cong \mathcal{O}(-i_1^\rho D_\rho) \oplus \cdots \oplus \mathcal{O}(-i_{\ell_1}^\rho D_\rho)$$
$$\mathcal{G}_{|U_\rho} \cong \mathcal{O}(-j_1^\rho D_\rho) \oplus \cdots \oplus \mathcal{O}(-j_{\ell_2}^\rho D_\rho).$$

Therefore,

$$\begin{aligned}(\mathcal{F} \otimes \mathcal{G})_{|U_\rho} & \\ &\cong (\mathcal{O}(-i_1^\rho D_\rho) \oplus \cdots \oplus \mathcal{O}(-i_{\ell_1}^\rho D_\rho)) \otimes (\mathcal{O}(-j_1^\rho D_\rho) \oplus \cdots \oplus \mathcal{O}(-j_{\ell_2}^\rho D_\rho)) \\ &\cong \bigoplus_{\substack{1 \le k_1 \le \ell_1 \\ 1 \le k_2 \le \ell_2}} \mathcal{O}(-i_{k_1}^\rho D_\rho) \otimes \mathcal{O}(-j_{k_2}^\rho D_\rho).\end{aligned}$$

(1.6) follows since for any pair (k_1, k_2) and any character $m \in M$ we have

$$\begin{aligned}\Gamma(U_\rho, \mathcal{O}(-i_{k_1}^\rho D_\rho) \otimes \mathcal{O}(-j_{k_2}^\rho D_\rho))_m & \\ &\cong \begin{cases} 0, & \langle m, n(\rho) \rangle < i_{k_1}^\rho + j_{k_2}^\rho \\ \mathbb{C}\langle f_{i_{k_1}}^\rho \otimes g_{j_{k_2}}^\rho \rangle, & \langle m, n(\rho) \rangle \ge i_{k_1}^\rho + j_{k_2}^\rho. \end{cases}\end{aligned}$$

Example 1.3.12 (Exterior power)**.** Let $\mathcal{E}$ be an equivariant locally free sheaf on X of rank ℓ with filtration given by

$$E^\rho(i_1^\rho, \dots, i_\ell^\rho; E_1^\rho, \dots, E_\ell^\rho).$$

We write recursively $E_{k+1}^\rho = E_k^\rho \oplus \mathbb{C}\langle e_{k+1}^\rho \rangle$. Then, for any $1 \le q \le \ell$ the sheaf $\bigwedge^q \mathcal{E}$ is a locally free sheaf on X of rank $\binom{\ell}{q}$ and its filtration is given by

$$\left[\bigwedge^q \mathcal{E}\right]^\rho (j) = \bigoplus_{\substack{1 \le k_1 < \cdots < k_q \le \ell \\ i_{k_1} + \cdots + i_{k_q} \le j}} \mathbb{C}\langle e_{i_{k_1}}^\rho \otimes \cdots \otimes e_{i_{k_q}}^\rho \rangle. \tag{1.7}$$

Indeed, for each ray $\rho \in \Sigma(1)$, the restriction of $\mathcal{E}$ at the affine toric variety U_ρ decomposes as $\mathcal{E}_{|U_\rho} \cong \mathcal{O}_{U_\rho}(-i_1^\rho D_\rho) \oplus \cdots \oplus \mathcal{O}_{U_\rho}(-i_\ell^\rho D_\rho)$. Therefore,

$$\begin{aligned} (\textstyle\bigwedge^q \mathcal{E})_{|U_\rho} &\cong \textstyle\bigwedge^q \left(\mathcal{O}_{U_\rho}(-i_1^\rho D_\rho) \oplus \cdots \oplus \mathcal{O}_{U_\rho}(-i_\ell^\rho D_\rho)\right) \\ &\cong \textstyle\bigoplus_{1 \le k_1 < \cdots < k_q \le \ell} \mathcal{O}_{U_\rho}(-i_{k_1}^\rho D_\rho) \otimes \cdots \otimes \mathcal{O}_{U_\rho}(-i_{k_q}^\rho D_\rho). \end{aligned}$$

Finally, for each $q-$ple $(k_1, \dots k_q)$ and any character $m \in M$, we have

$$\begin{aligned} &\Gamma(U_\rho, \mathcal{O}(-i_{k_1}^\rho D_\rho) \otimes \cdots \otimes \mathcal{O}(-i_{k_q}^\rho D_\rho))_m \\ &\qquad\qquad \cong \begin{cases} 0, & \langle m, n(\rho) \rangle < i_{k_1}^\rho + \cdots + i_{k_q}^\rho \\ \mathbb{C}\langle e_{i_{k_1}}^\rho \otimes \cdots \otimes e_{i_{k_q}}^\rho \rangle, & \langle m, n(\rho) \rangle \ge i_{k_1}^\rho + \cdots + i_{k_q}^\rho, \end{cases} \end{aligned}$$

and (1.7) follows.

Example 1.3.13 (Symmetric power)**.** Let $\mathcal{E}$ be an equivariant locally free sheaf on X of rank ℓ with filtration given by

$$E^\rho(i_1^\rho, \dots, i_\ell^\rho; E_1^\rho, \dots, E_\ell^\rho).$$

We write recursively $E_{k+1}^\rho = E_k^\rho \oplus \mathbb{C}\langle e_{k+1}^\rho \rangle$. Then, for any integer $q \ge 1$ the sheaf $\mathrm{Sym}^q \mathcal{E}$ is a locally free sheaf on X of rank $\binom{\ell+q-1}{q}$ and its filtration is given by

$$(\mathrm{Sym}^q \mathcal{E})^\rho (j) = \bigoplus_{\substack{k_1, \dots, k_\ell \ge 0 \\ k_1 + \cdots + k_\ell = q \\ k_1 i_1 + \cdots + k_\ell i_\ell \le j}} \mathbb{C}\langle \overbrace{e_{i_1}^\rho \otimes \cdots \otimes e_{i_1}^\rho}^{k_1} \otimes \cdots \otimes \overbrace{e_{i_\ell}^\rho \otimes \cdots \otimes e_{i_\ell}^\rho}^{k_\ell} \rangle. \tag{1.8}$$

As before, for each ray $\rho \in \Sigma(1)$, the restriction of $\mathcal{E}$ at the affine toric variety U_ρ decomposes as $\mathcal{E}_{|U_\rho} \cong \mathcal{O}_{U_\rho}(-i_1^\rho D_\rho) \oplus \cdots \oplus \mathcal{O}_{U_\rho}(-i_\ell^\rho D_\rho)$. Therefore,

$$\begin{aligned}[\operatorname{Sym}^q \mathcal{E}]_{|U_\rho} &\cong \operatorname{Sym}^q \left(\mathcal{O}_{U_\rho}(-i_1^\rho D_\rho) \oplus \cdots \oplus \mathcal{O}_{U_\rho}(-i_\ell^\rho D_\rho)\right) \\ &\cong \bigoplus_{\substack{k_1,\cdots,k_l \geq 0 \\ k_1+\cdots+k_\ell = q}} \mathcal{O}_{U_\rho}(-i_1^\rho D_\rho)^{\otimes k_1} \otimes \cdots \otimes \mathcal{O}_{U_\rho}(-i_\ell^\rho D_\rho)^{\otimes k_\ell}.\end{aligned}$$

Finally, for each $\ell-$uple $(k_1, \dots k_\ell)$ and any character $m \in M$, we have

$$\Gamma(U_\rho, \mathcal{O}(-i_1^\rho D_\rho)^{\otimes k_1} \otimes \cdots \otimes \mathcal{O}(-i_\ell^\rho D_\rho)^{\otimes k_\ell})_m$$

$$\cong \begin{cases} 0, & \langle m, n(\rho)\rangle < k_1 i_1^\rho + \cdots + k_\ell i_\ell^\rho \\ \mathbb{C}\langle \overbrace{e_{i_1}^\rho \otimes \cdots \otimes e_{i_1}^\rho}^{k_1} \otimes \cdots \otimes \overbrace{e_{i_\ell}^\rho \otimes \cdots \otimes e_{i_\ell}^\rho}^{k_\ell} \rangle, & \langle m, n(\rho)\rangle \geq k_1 i_1^\rho + \cdots + k_\ell i_\ell^\rho, \end{cases}$$

which yields (1.8).

1.3.3 Cohomology of equivariant sheaves

In this last subsection, we recall how to compute the cohomology of an equivariant torsion-free sheaf using the associated $\Sigma-$family. In [53, Section 6] (see also [50, Section 3]) it is introduced the *canonical Čech complex*, which can be attached to the Cox ring R of a toric variety X with irrelevant ideal B:

$$\mathcal{C}^\bullet : 0 \longrightarrow R \xrightarrow{\delta^0} \bigoplus_{\sigma \in \Sigma(n)} R_{x^{\hat\sigma}} \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{n-1}} \bigoplus_{\sigma \in \Sigma(1)} R_{x^{\hat\sigma}} \xrightarrow{\delta^n} R_{x^{\{\hat 0\}}} \longrightarrow 0,$$

with the natural Čech differentials

$$\delta^i : \bigoplus_{\sigma \in \Sigma(i)} R_{x^{\hat\sigma}} \to \bigoplus_{\sigma \in \Sigma(i)} R_{x^{\hat\sigma}}, \quad \text{such that} \quad \delta^i(v_\sigma) = \sum_{k=1}^{i} (-1)^{i-1} \imath(v)_{\sigma^k}.$$

Here, we have fixed an order of the set of rays $\Sigma(1) = \{\rho_1, \dots, \rho_r\}$ and we can write an $i-$dimensional cone as

$$\sigma = \operatorname{cone}(n(\rho_{j_1}), \dots, n(\rho_{j_i})).$$

For any $1 \leq k \leq i$, $\sigma^k \prec \sigma$ is the face

$$\sigma^k = \operatorname{cone}(n(\rho_{j_1}), \ldots, \widehat{n(\rho_{j_k})}, \ldots, n(\rho_{j_i})),$$

and we have the inclusion $\imath : R_{x^{\hat{\sigma}}} \hookrightarrow R_{x^{\widehat{\sigma^k}}}$.

This complex generalizes the Čech complex used for the study of local cohomology over the standard graded polynomial ring, and for any homogeneous $R-$module E we have that

$$\mathrm{H}^i_B(E) \cong \mathrm{H}^i(\mathcal{C}^\bullet \otimes E).$$

In particular, we have the following result:

Proposition 1.3.14. *Let $\mathcal{E}$ be an equivariant quasi-coherent sheaf with $\Sigma-$family $\{E^\sigma_m\}$. Then, for any $m \in M$ we have a chain complex*

$$\mathcal{C}^\bullet(\mathcal{E})_m :\ 0 \longrightarrow \bigoplus_{\sigma \in \Sigma(n)} E^\sigma_m \xrightarrow{\partial^0} \cdots \xrightarrow{\partial^{n-2}} \bigoplus_{\sigma \in \Sigma(1)} E^\sigma_m \xrightarrow{\partial^{n-1}} E^{\{0\}}_m \xrightarrow{\partial^n} 0,$$

such that $\mathrm{H}^i(X, \mathcal{E})_m \cong \mathrm{H}^i(\mathcal{C}^\bullet(\mathcal{E})_m)$. *In particular, when $\mathcal{E}$ is torsion-free we have:*

(i) $H^0(X, \mathcal{E})_m \cong \bigcap_{\sigma \in \Sigma(n)} E^\sigma_m$.

(ii) $H^n(X, \mathcal{E})_m \cong \mathbf{E} / \sum_{\rho \in \Sigma(1)} E^\rho_m$.

(iii) $\chi(\mathcal{E})_m = \sum_{\sigma \in \Sigma} (-1)^{\operatorname{codim} \sigma} \dim_k E^\sigma_m$.

Proof. Let us consider the saturated module $E = \mathrm{H}^0_*(X, \mathcal{E})$, then we have

$$\mathcal{C}^\bullet \otimes E : 0 \longrightarrow E \xrightarrow{\delta^0} \bigoplus_{\sigma \in \Sigma(n)} E_{x^{\hat{\sigma}}} \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{n-1}} \bigoplus_{\sigma \in \Sigma(1)} E_{x^{\hat{\sigma}}} \xrightarrow{\delta^n} E_{x^{\{\hat{0}\}}} \longrightarrow 0.$$

Since we assume that E is saturated, then $\mathrm{H}^0_B(E) = \mathrm{H}^1_B(E) = 0$. Hence,

$$0 \longrightarrow E \xrightarrow{\delta^0} \bigoplus_{\sigma \in \Sigma(n)} E_{x^{\hat{\sigma}}} \xrightarrow{\delta^1} \bigoplus_{\sigma \in \Sigma(n-1)} E_{x^{\hat{\sigma}}}$$

is exact. Thus, we have $\mathrm{H}^0_*(X,\mathcal{E}) = E = \ker\delta^1$. On the other hand, by Proposition 1.2.7 we have that $\mathrm{H}^i(X,\mathcal{E}) \cong \mathrm{H}^{i+1}_B(E)_0$, which we may compute taking the degree 0 component of $\mathcal{C}^\bullet \otimes E$. Shifting $(\mathcal{C}^\bullet \otimes E)_0$ by -1, we have constructed a chain complex

$$\mathcal{C}^\bullet(\mathcal{E}) :\ 0 \longrightarrow \bigoplus_{\sigma\in\Sigma(n)} E^\sigma \xrightarrow{\partial^0=\delta^1} \bigoplus_{\sigma\in\Sigma(n-1)} E^\sigma \xrightarrow{\partial^1=\delta^2} \cdots$$
$$\cdots \xrightarrow{\partial^{n-2}=\delta^{n-1}} \bigoplus_{\sigma\in\Sigma(1)} E^\sigma \xrightarrow{\partial^{n-1}=\delta^n} E^{\{0\}} \xrightarrow{\partial^n} 0,$$

which is $M-$graded and satisfies that $\mathrm{H}^i(X,\mathcal{E})_m \cong \mathrm{H}^i(\mathcal{C}^\bullet(\mathcal{E})_m)$.

For the remaining part of the proof we notice that if $\mathcal{E}$ is torsion-free, all the vector spaces in the $\Sigma-$family $\{E^\sigma_m\}$ are actually sub-vector spaces of $\mathbf{E} \cong E^{\{0\}}_m \cong \mathbb{C}^\ell$. Then, the result follows using the definition of $\mathcal{C}^\bullet(\mathcal{E})$. □

Remark 1.3.15. In [45, Theorem 4.1.1, Remark 4.1.2 and Corollary 4.1.3] and [46, Metatheorem 1.3.3], Klyachko proved Proposition 1.3.14 for equivariant vector bundles and sketched how it worked for equivariant reflexive or, more generally, torsion-free sheaves.

Example 1.3.16. Let Σ be the fan of the projective plane $\mathbb{P}^2$ and its Cox ring $R = \mathbb{C}[x_0,x_1,x_2]$ as in Examples 1.1.11 and 1.2.6. We consider the monomial ideal $I = (x_2^2, x_0x_2, x_0x_1)$ of Example 1.3.5. For any character $m = (d_1,d_2)$, the chain complex of Proposition 1.3.14 is

$$\mathcal{C}^\bullet(I)_m : 0 \longrightarrow I^{\sigma_0}_m \oplus I^{\sigma_1}_m \oplus I^{\sigma_2}_m \xrightarrow{\partial^0} I^{\rho_0}_m \oplus I^{\rho_1}_m \oplus I^{\rho_2}_m \xrightarrow{\partial^1} I^{\{0\}}_m \longrightarrow 0, \quad (1.9)$$

with maps

$$\partial^0 = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \partial^1 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}.$$

However, depending on the character m some of the vector subspaces $I^{\sigma_i}_m$ or $I^{\rho_j}_m$ may be 0, and this corresponds to delete some rows or columns in the matrices of ∂^0 and ∂^1. For instance, for $m_1 = (0,0)$ we have

$$I^{\sigma_0}_{m_1} = I^{\sigma_1}_{m_1} = 0 \quad \text{and} \quad I^{\sigma_2}_{m_1} \cong \mathbf{I},$$

but

$$I_{m_1}^{\rho_j} \cong \mathbf{I} \quad \text{for} \quad 0 \leq j \leq 2.$$

Therefore, the chain complex (1.9) for $m_1 = (0,0)$ is equivalent to

$$\mathcal{C}^\bullet(I)_{m_1} : 0 \longrightarrow \mathbf{I} \xrightarrow{\partial^0} \mathbf{I} \oplus \mathbf{I} \oplus \mathbf{I} \xrightarrow{\partial^1} \mathbf{I} \longrightarrow 0,$$

with maps

$$\partial^0 = \begin{pmatrix} \cdot & \cdot & -1 \\ \cdot & \cdot & 1 \\ \cdot & \cdot & 0 \end{pmatrix} \quad \text{and} \quad \partial^1 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}.$$

Thus, we have

$$\begin{aligned} \mathrm{h}^0(X, \tilde{I})_{m_1} &= \dim \ker \partial^0_{m_1} = 0 \\ \mathrm{h}^1(X, \tilde{I})_{m_1} &= \dim \tfrac{\ker \partial^1_{m_1}}{\operatorname{Im} \partial^0_{m_1}} = 1 \\ \mathrm{h}^2(X, \tilde{I})_{m_1} &= \dim \tfrac{\mathbf{I}}{\operatorname{Im} \partial^1} = 0. \end{aligned}$$

Chapter 2

Monomial ideals

Lying in the crossroads of commutative algebra and combinatorics, monomial ideals play a prominent role in the study of ideals in a polynomial ring R. Indeed, many properties of an arbitrary ideal $I \subset R$ are reduced to considering its *initial ideal* $\mathrm{in}_>(I)$, which is a monomial ideal. For instance, it is a result due to Macaulay in [48], that the Hilbert function of an ideal $I \subset R$ coincides with the Hilbert function of its *initial ideal* $\mathrm{in}_>(I)$ (see for instance [26, Theorem 15.3]). Since the advent of combinatorial commutative algebra, the theory of monomial ideals has been linked with various topics in discrete mathematics, such as enumerative combinatorics, graph theory, simplicial geometry or lattice polytopes (see [30, 37, 5, 62, 39, 32, 22]).

In this chapter, we introduce the *Klyachko diagram* of a monomial ideal, which can be seen as a generalization of the classical *staircase diagram*, suited to study monomial ideals inside the Cox ring R of a smooth complete toric variety X. Since a monomial ideal I is fine-graded, it corresponds to an equivariant ideal sheaf $\tilde{I}$ on X. The Klyachko diagram of I encodes the information of the $\Sigma-$family of $\tilde{I}$, and it provides a new set of tools to examine algebraic properties of I such as its Hilbert function or its local cohomology.

Next we explain how this chapter is organized. First in Section 2.1, we define the Klyachko diagram of a monomial ideal I. We give procedures to compute the Klyachko diagram using the monomial generators of I as initial data (Subsection 2.1.1) and, conversely, to determine the generators of a $B-$saturated ideal I^{sat} from a given Klyachko diagram (Subsection 2.1.2). Finally, in Subsection 2.1.3, we compute the first local cohomology module $\mathrm{H}^1_B(I)$, which measures the saturatedness of a monomial ideal I, using the Klyachko diagram. After illustrating these procedures with plenty of examples, in Section 2.2 we apply this construction. We compute the Hilbert function and the Hilbert polynomial of a $B-$saturated monomial ideal I from its Klyachko diagram, and we characterize all monomial ideals having constant Hilbert polynomial.

The results of this chapter have been published in [55].

2.1 Klyachko diagrams of monomial ideals

Using the theory of Klyachko filtrations, in this section we define the *Klyachko diagram* of a monomial ideal I in the Cox ring R of a smooth complete toric variety. We show how that the Klyachko diagram is determined combinatorially by the monomials generating I (Proposition 2.1.4). Conversely, in Subsection 2.1.2, we give a method to compute a minimal set of generators of a $B-$saturated monomial ideal I from its Klyachko diagram.

From now on, we fix a smooth complete toric variety X with fan Σ. Let $\Sigma(1) = \{\rho_1, \dots, \rho_r\}$ be the set of rays (i.e. one-dimensional cones), with $r = |\Sigma(1)|$. We denote by $R = \mathbb{C}[x_1, \dots, x_r]$ its associated $\mathrm{Cl}(X)-$graded Cox ring and by B its irrelevant ideal.

2.1.1 From a monomial ideal to a Klyachko diagram

The equivariant sheaves associated to monomial ideals are torsion-free equivariant sheaves of rank 1. Following Proposition 1.3.3, the $\Sigma-$family $\{\hat{I}^\sigma\}_{\sigma\in\Sigma}$ of a monomial ideal I is a system of vector space filtrations of a $1-$dimensional vector space $\mathbf{I}$, which can be identified with $I_m^{\{0\}}$ for any character $m \in M$. For each cone $\sigma \in \Sigma$, we want to describe the $\sigma-$family $\hat{I}^\sigma = \{I_m^\sigma \,|\, m \in M\}$. Since $\mathbf{I}$ has dimension 1, each linear subspace $I_m^\sigma \subset \mathbf{I}$ can be either

$$I_m^\sigma \cong \mathbf{I} \quad \text{or} \quad I_m^\sigma = 0.$$

Therefore, the $\sigma-$family $\hat{I}^\sigma$ is characterized by the set of characters $\{m \in M \mid I_m^\sigma \neq 0\}$, which can be seen as the staircase diagram for the inclusion $I^\sigma \subset R^\sigma$ as $\mathbb{C}[S_\sigma]-$modules.

Let $I = (m_1, \ldots, m_t)$ be a monomial ideal. We write the monomials

$$m_i = x_1^{k_1^i} \cdots x_r^{k_r^i} =: x^{\underline{k}^i},$$

for $\underline{k}^i := (k_1^i, \ldots, k_r^i) \in \mathbb{Z}^r_{\geq 0}$ and $1 \leq i \leq t$. We present I as the image of a $\mathbb{Z}^r-$graded map as follows:

$$\bigoplus_{i=1}^{t} R(-\underline{k}^i) \xrightarrow{(m_1,\ldots,m_t)} I \longrightarrow 0. \tag{2.1}$$

Thus, for any character $m \in M$ we have that

$$I_m^{\{0\}} \cong \mathbb{C}\langle x_1^{\langle m, n(\rho_1)\rangle} \cdots x_r^{\langle m, n(\rho_r)\rangle}\rangle$$

and there is an isomorphism $\phi_m^{\{0\}} : I_m^{\{0\}} \cong \mathbf{I}$. Our first objective is to describe the sets of characters

$$\{m \in M \mid I_m^\sigma \neq 0\}$$

for any cone $\sigma \in \Sigma$. We start analyzing the case $t = 1$, that is I being a principal monomial ideal.

Lemma 2.1.1. *Let $I = (x^{\underline{k}}) \subset R$, with $\underline{k} \in \mathbb{Z}^r_{\geq 0}$, be an ideal generated by a single monomial. Then, for any cone $\sigma = \mathrm{cone}(\rho_{i_1}, \ldots, \rho_{i_c}) \in \Sigma$,*

$$I_m^\sigma \cong \begin{cases} \mathbf{I}, & \textit{if} \quad m \in \mathcal{C}_{\underline{k}}^\sigma \\ 0, & \quad\textit{otherwise,} \end{cases}$$

where $\mathcal{C}_{\underline{k}}^\sigma := \{m \in M \mid \langle m, \rho_{i_j}\rangle \geq k_{i_j}, \text{ for } 1 \leq j \leq c\}$.

Proof. The lemma follows from (2.1) taking $t = 1$ and using that

$$R_m^\sigma(-\underline{k}) = \begin{cases} \mathbb{C}\langle x_1^{\langle m,\rho_1\rangle - k_1} \cdots x_r^{\langle m,\rho_r\rangle - k_r}\rangle, \text{ if } \langle m, \rho_{i_j}\rangle - k_{i_j} \geq 0, \text{ for } 1 \leq j \leq c \\ 0, \qquad \text{otherwise.} \end{cases}$$

□

We set $\mathcal{C}_0^\sigma := \mathcal{C}_{(0,\dots,0)}^\sigma = \sigma^\vee \cap M$. Notice that $\mathcal{C}_{\underline{k}}^\sigma \subset \mathcal{C}_0^\sigma$, corresponding to $(x^{\underline{k}})^\sigma \subset R^\sigma$ for any $\underline{k} \in \mathbb{Z}^r$, is just a translation of $\sigma^\vee \cap M$. Applying repeatedly Lemma 2.1.1, we have:

Proposition 2.1.2. *Let $I = (m_1, \dots, m_t) \subset R$ be a monomial ideal with $m_i = x^{\underline{k}^i}$ for $\underline{k}^i \in \mathbb{Z}^r_{\geq 0}$ and $1 \leq i \leq t$. Then, for any cone $\sigma \in \Sigma$,*

$$I_m^\sigma \cong \begin{cases} \mathbf{I}, & \text{if } \ m \in \bigcup_{i=1}^t \mathcal{C}_{\underline{k}^i}^\sigma \\ 0, & \quad \text{otherwise.} \end{cases}$$

Proof. It follows from (2.1) that $I_m^\sigma = 0$ if and only if $R_m^\sigma(-\underline{k}^i) = 0$ for all $1 \leq i \leq t$. By Lemma 2.1.1, this occurs if and only if

$$m \in M \setminus \bigcup_{i=1}^t \mathcal{C}_{\underline{k}}^\sigma,$$

and the result follows. $\square$

Notice that by Proposition 2.1.2 we already have a description of the collection of the sets of characters $\{m \in M \mid I_m^\sigma \neq 0\}$ of a monomial ideal. However, the information on the inclusion $I^\sigma \subset R^\sigma$ is encoded in

$$\mathcal{C}_0^\sigma \setminus \bigcup_{i=1}^t \mathcal{C}_{\underline{k}^i}^\sigma = \bigcap_{i=1}^t (\mathcal{C}_0^\sigma \setminus \mathcal{C}_{\underline{k}^i}^\sigma) = \bigcap_{i=1}^t \bigcup_{j=1}^c \{m \in M \mid 0 \leq \langle m, \rho_{i_j} \rangle < k_{i_j}^i\},$$

which is the union of c^t sets. Indeed, for each $1 \leq j_1, \dots, j_t \leq c$,

$$P_{j_1,\dots,j_t} := \{m \in M \mid 0 \leq \langle m, \rho_{i_{j_1}} \rangle < k_{j_1}^1, \dots, 0 \leq \langle m, \rho_{i_{j_t}} \rangle < k_{j_t}^t\}.$$

The *Klyachko diagram* defined below is used in Proposition 2.1.4 to give a more compact alternative characterization of the collection of Klyachko filtrations of a monomial ideal. We attach to the monomial ideal I a collection of pairs $\{(\mathcal{C}_I^\sigma, \Delta_I^\sigma)\}_{\sigma \in \Sigma}$ constructed as follows.

For any ray $\rho_j \in \Sigma(1)$, we set

$$s_j = s_{\rho_j} := \min\{\deg_{\rho_j}(m_1), \dots, \deg_{\rho_j}(m_t)\}.$$

We write $\underline{s} := (s_1, \ldots, s_r)$, and for any cone $\sigma = \mathrm{cone}(\rho_{i_1}, \ldots, \rho_{i_c})$ of dimension c, we set

$$\mathcal{C}_I^\sigma := \mathcal{C}_{\underline{s}}^\sigma = \{m \in M \mid \langle m, \rho_{i_p} \rangle \geq s_{i_p},\ 1 \leq p \leq c\} = \bigcap_{j=1}^{c} \mathcal{C}_I^{\rho_{i_j}}.$$

Next, we construct Δ_I^σ. First, for any subset of monomials

$$\mathcal{S} = \{n_1, \ldots, n_s\} \subset \{m_1, \ldots, m_t\}$$

with $0 \leq s \leq t$, we define $\Delta_I^\sigma(\mathcal{S}) \subset \mathcal{C}_I^\sigma$ recursively on s:
If $s = 0$, then $\mathcal{S} = \emptyset$ and we set:

$$\Delta_I^\sigma(\emptyset) := \{m \in M \mid s_{i_1} \leq \langle m, \rho_{i_1} \rangle, \ldots, s_{i_c} \leq \langle m, \rho_{i_c} \rangle\}.$$

Otherwise, $s \geq 1$ and there is a permutation $\epsilon_{i_c} \in \mathfrak{S}_s$ such that

$$\deg_{\rho_{i_c}}(n_{\epsilon_{i_c}(1)}) \leq \deg_{\rho_{i_c}}(n_{\epsilon_{i_c}(2)}) \leq \cdots \leq \deg_{\rho_{i_c}}(n_{\epsilon_{i_c}(s)}).$$

- If $c = 1$ (and thus σ is a ray), then

$$\Delta_I^\sigma(\mathcal{S}) := \{m \in M \mid s_{i_1} \leq \langle m, \rho_{i_1} \rangle < \deg_{\rho_{i_1}}(n_{\epsilon_{i_1}(1)})\}.$$

- Otherwise, $\Delta_I^\sigma(\mathcal{S}) := \bigcup_{j=0}^{s} \Delta_I^\sigma(\mathcal{S})_j$ where:

$$\Delta_I^\sigma(\mathcal{S})_0 := \{m \in M \mid s_{i_c} \leq \langle m, \rho_{i_c} \rangle < \deg_{\rho_{i_c}}(n_{\epsilon_{i_c}(1)})\} \cap \Delta_I^{\sigma'}(\emptyset),$$

$$\Delta_I^\sigma(\mathcal{S})_j := \{m \in M \mid \deg_{\rho_{i_c}}(n_{\epsilon_{i_c}(j)}) \leq \langle m, \rho_{i_c} \rangle < \deg_{\rho_{i_c}}(n_{\epsilon_{i_c}(j+1)}))\} \cap \Delta_I^{\sigma'}(\{n_{\epsilon_{i_c}(1)}, \ldots, n_{\epsilon_{i_c}(j)}\}),\ 1 \leq j \leq s-1,$$

$$\Delta_I^\sigma(\mathcal{S})_s := \{m \in M \mid \deg_{\rho_{i_c}}(n_{\epsilon_{i_c}(s)}) \leq \langle m, \rho_{i_c} \rangle\} \cap \Delta_I^{\sigma'}(\{n_{\epsilon_{i_c}(1)}, \ldots, n_{\epsilon_{i_c}(s)}\}),$$

with $\sigma' = \mathrm{cone}(\rho_{i_1}, \ldots, \rho_{i_{c-1}})$.

Finally, we define $\Delta_I^\sigma := \Delta_I^\sigma(\{m_1, \ldots, m_t\})$.

Definition 2.1.3. We call the collection of pairs $\{(\mathcal{C}_I^\sigma, \Delta_I^\sigma)\}_{\sigma\in\Sigma}$ the *Klyachko diagram* of I.

Observe that each of the pairs $(\mathcal{C}_I^\sigma, \Delta_I^\sigma)$ depicts a staircase diagram of the inclusion $I^\sigma \subset R^\sigma$ (see also Example 2.1.5). Precisely, we have the following proposition:

Proposition 2.1.4. *Let $I = (m_1, \ldots, m_t) \subset R$ be a monomial ideal with $m_i = x^{\underline{k}^i}$ for $\underline{k}^i \in \mathbb{Z}^r_{\geq 0}$ and $1 \leq i \leq t$. Let $\{(\mathcal{C}_I^\sigma, \Delta_I^\sigma)\}_{\sigma\in\Sigma}$ be the Klyachko diagram of I. Then, for any cone $\sigma \in \Sigma$,*

$$I_m^\sigma \cong \begin{cases} \mathbf{I}, & m \in \mathcal{C}_I^\sigma \setminus \Delta_I^\sigma \\ 0, & \textit{otherwise.} \end{cases}$$

In particular, it holds

$$\bigcup_{i=1}^t \mathcal{C}_{\underline{k}^i}^\sigma = \mathcal{C}_I^\sigma \setminus \Delta_I^\sigma \quad \textit{and} \quad \mathcal{C}_0^\sigma \setminus \bigcup_{i=1}^t \mathcal{C}_{\underline{k}^i}^\sigma = \Delta_I^\sigma \cup (\mathcal{C}_0^\sigma \setminus \mathcal{C}_I^\sigma). \tag{2.2}$$

Before proving this proposition, let us see an example that illustrates Proposition 2.1.4 and shows how to compute the Klyachko diagram of a monomial ideal.

Example 2.1.5. Let $R = \mathbb{C}[x_0, x_1, x_2]$ be the Cox ring of $\mathbb{P}^2$ with fan Σ as in Example 1.2.6(i), and let $I = (x_2^2, x_0x_2, x_0x_1)$ be the monomial ideal of Example 1.3.5. First, we compute $s_0 = s_1 = s_2 = 0$ and we have

$$\begin{aligned} \mathcal{C}_I^{\rho_0} &= \{(d_1, d_2) \mid d_1 + d_2 \leq 0\} \\ \mathcal{C}_I^{\rho_1} &= \{(d_1, d_2) \mid d_1 \geq 0\} \\ \mathcal{C}_I^{\rho_2} &= \{(d_1, d_2) \mid d_2 \geq 0\}. \end{aligned}$$

We compute $\Delta_I^{\sigma_0}$. We order the monomials with respect to ρ_2:

$$\deg_{\rho_2}(x_0x_1) = 0 < \deg_{\rho_2}(x_0x_2) = 1 < \deg_{\rho_2}(x_2^2) = 2.$$

We obtain

$$\begin{aligned} &\Delta_I^{\sigma_0}(\{x_2^2, x_0x_2, x_0x_1\})_0 = \emptyset \\ &\Delta_I^{\sigma_0}(\{x_2^2, x_0x_2, x_0x_1\})_1 = \{(d_1, d_2) \mid d_2 = 0\} \cap \Delta^{\rho_1}(\{x_0x_1\}) \\ &\Delta_I^{\sigma_0}(\{x_2^2, x_0x_2, x_0x_1\})_2 = \{(d_1, d_2) \mid d_2 = 1\} \cap \Delta^{\rho_1}(\{x_0x_1, x_0x_2\}) \\ &\Delta_I^{\sigma_0}(\{x_2^2, x_0x_2, x_0x_1\})_3 = \emptyset. \end{aligned}$$

Since $\Delta_I^{\rho_1}(\{x_0x_1\}) = \{(d_1, d_2) \mid d_1 = 0\}$ and $\Delta_I^{\rho_1}(\{x_0x_1, x_0x_2\}) = \emptyset$, we get $\Delta_I^{\sigma_0}(\{x_2^2, x_0x_2, x_0x_1\})_1 = \{(0,0)\}$ and $\Delta_I^{\sigma_0}(\{x_2^2, x_0x_2, x_0x_1\})_2 = \emptyset$, respectively. Therefore,

$$\Delta_I^{\sigma_0} = \{(0,0)\}.$$

Similarly, for the remaining cones we obtain

$$\Delta_I^{\sigma_1} = \{(0,0), (-1,1)\} \quad \text{and} \quad \Delta_I^{\sigma_2} = \emptyset.$$

Figures 2.1 and 2.2 illustrate the Klyachko diagram of this example.

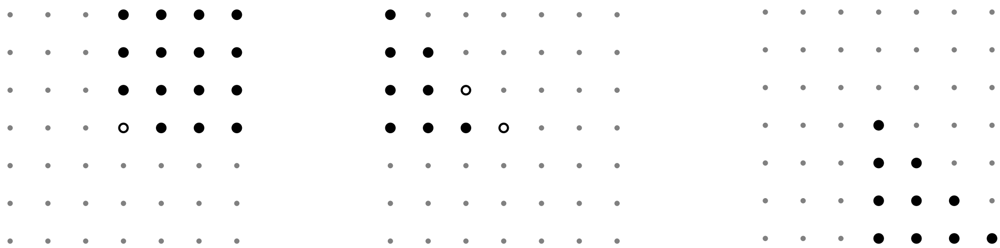

Figure 2.1: $\circ$ stand for points of $\Delta_I^{\sigma_0}$ (respectively $\Delta_I^{\sigma_1}$ and $\Delta_I^{\sigma_2}$) inside the points $\bullet$ of $\mathcal{C}_I^{\sigma_0}$ (respectively $\mathcal{C}_I^{\sigma_1}$ and $\mathcal{C}_I^{\sigma_2}$).

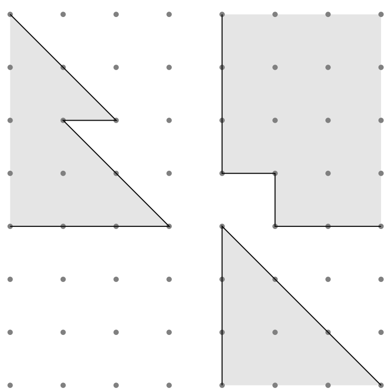

Figure 2.2: The Klyachko diagram of Figure 2.1 represented together in a single figure inside the M lattice. The shadowed region corresponds to each set $\mathcal{C}_I^{\sigma_i} \setminus \Delta_I^{\sigma_i}$ for $i = 0, 1, 2$.

By Proposition 2.1.4, we have

$$I_m^{\rho_0} \cong \begin{cases} \mathbf{I}, & m \in \{d_1 + d_2 \leq 0\} \\ 0, & \text{otherwise.} \end{cases}$$

$$I_m^{\rho_1} \cong \begin{cases} \mathbf{I}, & m \in \{d_1 \geq 0\} \\ 0, & \text{otherwise.} \end{cases}$$

$$I_m^{\rho_2} \cong \begin{cases} \mathbf{I}, & m \in \{d_2 \geq 0\} \\ 0, & \text{otherwise.} \end{cases}$$

and

$$I_m^{\sigma_0} \cong \begin{cases} \mathbf{I}, & m \in \{d_1 \geq 0, d_2 \geq 0\} \setminus \{(0,0)\} \\ 0, & \text{otherwise.} \end{cases}$$

$$I_m^{\sigma_1} \cong \begin{cases} \mathbf{I}, & m \in \{d_1 + d_2 \leq 0, d_2 \geq 0\} \setminus \{(0,0), (-1,1)\} \\ 0, & \text{otherwise.} \end{cases}$$

$$I_m^{\sigma_2} \cong \begin{cases} \mathbf{I}, & m \in \{d_1 \geq 0, d_1 + d_2 \leq 0\} \\ 0, & \text{otherwise} \end{cases}$$

which coincides with the $\Sigma-$family computed in Example 1.3.5.

Now we prove Proposition 2.1.4.

Proof of Proposition 2.1.4. First, we recall that for $1 \leq j \leq t$

$$R_m^{\rho_j} = \begin{cases} \mathbb{C}\langle x_1^{\langle m, n(\rho_1)\rangle} \cdots x_r^{\langle m, n(\rho_r)\rangle}\rangle, & \langle m, n(\rho_j)\rangle \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

If $\sigma = \rho_j$ is a ray, then we have

$$I_m^{\rho_j} \cong \begin{cases} \mathbf{I}, & \langle m, n(\rho_j)\rangle \geq s_j \\ 0, & \langle m, n(\rho_j)\rangle < s_j. \end{cases}$$

Otherwise, $\sigma = \text{cone}(\rho_{i_1}, \ldots, \rho_{i_c})$ for some $2 \leq c \leq r$. Assume that $m \in \Delta_I^\sigma(\{m_1, \ldots, m_t\})_j$ for some $1 \leq j \leq t$ and we want to see that $I_m^\sigma = 0$. Let $\epsilon_{i_c} \in \mathfrak{S}_t$ be a permutation such that

$$\deg_{\rho_{i_c}}(n_{\epsilon_{i_c}(1)}) \leq \deg_{\rho_{i_c}}(m_{\epsilon_{i_c}(2)}) \leq \cdots \leq \deg_{\rho_{i_c}}(m_{\epsilon_{i_c}(s)}),$$

and we have

$$\deg_{\rho_{i_c}}(m_{\epsilon_{i_c}(j)}) \leq \langle m, \rho_{i_c} \rangle < \deg_{\rho_{i_c}}(m_{\epsilon_{i_c}(j+1)})).$$

If $j < t$, we have

$$R^\sigma_m(-\underline{k}^{\epsilon_{i_c}(j+1)}) = \cdots = R^\sigma_m(-\underline{k}^{\epsilon_{i_c}(t)}) = 0.$$

So, it suffices to see that $R^\sigma_m(-\underline{k}^{\epsilon_{i_c}(p)}) = 0$ for $1 \leq p \leq j$. Recall that

$$\begin{aligned} &\Delta^\sigma_I(\{m_1, \ldots, m_t\})_j := \\ &\qquad \{m \in M \mid \deg_{\rho_{i_c}}(m_{\epsilon_{i_c}(j)}) \leq \langle m, \rho_{i_c} \rangle < \deg_{\rho_{i_c}}(m_{\epsilon_{i_c}(j+1)}))\} \cap \\ &\qquad\qquad \Delta^{\sigma'}_I(\{m_{\epsilon_{i_c}(1)}, \ldots, m_{\epsilon_{i_c}(j)}\}). \end{aligned}$$

In particular, we have that $m \in \Delta^{\sigma'}_I(\{m_{\epsilon_{i_c}(1)}, \ldots, m_{\epsilon_{i_c}(j)}\})$ where σ' is the cone $\sigma' = \text{cone}(\rho_{i_1}, \ldots, \rho_{i_{c-1}})$. We repeat the same argument for the cone σ' and the set of monomials $\{m_{\epsilon_{i_c}(1)}, \ldots, m_{\epsilon_{i_c}(j)}\}$. This procedure stops either when $\dim(\sigma') = 1$ (σ' is a ray) or when the set of monomials is empty. In the latter case, we already arrived at

$$R^\sigma_m(-\underline{k}^{\epsilon_{i_1}(1)}) = \cdots = R^\sigma_m(-\underline{k}^{\epsilon_{i_1}(t)}) = 0$$

and, hence, $I^\sigma_m = 0$. In the case $\dim(\sigma') = 1$, we may assume that the set of monomials is $\{n_{\epsilon_{i_1}(1)}, \ldots, n_{\epsilon_{i_1}(s)}\} \subset \{m_1, \ldots, m_t\}$ with $s \leq j$. Since we have

$$s_{i_1} \leq \langle m, \rho_{i_1} \rangle < \deg_{\rho_{i_1}}(n_{\epsilon_{i_1}(1)}),$$

then $R^\sigma_m(-\underline{k}^{\epsilon_{i_1}(1)}) = \cdots = R^\sigma_m(-\underline{k}^{\epsilon_{i_1}(s)}) = 0$, and we obtain $I^\sigma_m = 0$. By construction, if $m \in \mathcal{C}^\sigma_I \setminus \Delta^\sigma_I$ we get $R^\sigma_m(-\underline{k}^p) \neq 0$ for some $1 \leq p \leq t$, so $I^\sigma_m \cong \mathbf{I}$.

Finally, (2.2) follows from a comparison with the description of I^σ_m in Proposition 2.1.2. □

Combining Propositions 2.1.2 and 2.1.4, the next result shows how to obtain the Klyachko diagram of the sum of two monomial ideals.

Corollary 2.1.6. *Let* $\{(\mathcal{C}_I^\sigma, \Delta_I^\sigma)\}$ *and* $\{(\mathcal{C}_J^\sigma, \Delta_J^\sigma)\}$ *be the Klyachko diagrams of two monomial ideals* I *and* J*, respectively. Then, the Klyachko diagram of* $I+J$ *is given by*

$$\begin{cases} \mathcal{C}_{I+J}^\sigma = \{m \in M \mid \langle m, \rho\rangle \geq \min\{s_\rho^I, s_\rho^J\},\ \rho \in \sigma(1)\} \\ \Delta_{I+J}^\sigma = (\Delta_I^\sigma \cap \Delta_J^\sigma) \cup (\Delta_I \cap (\mathcal{C}_{I+J}^\sigma \setminus \mathcal{C}_J^\sigma)) \cup (\Delta_J \cap (\mathcal{C}_{I+J}^\sigma \setminus \mathcal{C}_I^\sigma)) \cup \\ \qquad (\mathcal{C}_{I+J}^\sigma \setminus (\mathcal{C}_I^\sigma \cup \mathcal{C}_J^\sigma)). \end{cases}$$

Proof. We write $I = (x^{\underline{k}^1}, \dots, x^{\underline{k}^t})$ and $J = (x^{\underline{l}^1}, \dots, x^{\underline{l}^s})$. Then we have that

$$s_j^I = \min\{k_j^1, \dots, k_j^t\} \quad \text{and} \quad s_j^J = \min\{l_j^1, \dots, l_j^s\}$$

Since $I+J = (x^{\underline{k}^1}, \dots, x^{\underline{k}^t}, x^{\underline{l}^1}, \dots, x^{\underline{l}^s})$, it follows that $s_j^{I+J} = \min\{s_j^I, s_j^J\}$. Thus,

$$\mathcal{C}_{I+J}^\sigma = \{m \in M \mid \langle m, \rho\rangle \geq \min\{s_\rho^I, s_\rho^J\},\ \rho \in \sigma(1)\}.$$

In particular, $\mathcal{C}_I^\sigma$ and $\mathcal{C}_J^\sigma$ are contained in $\mathcal{C}_{I+J}^\sigma$. By Propositions 2.1.2 and 2.1.4 we have $\mathcal{C}_{I+J}^\sigma \setminus \Delta_{I+J}^\sigma = (\mathcal{C}_I^\sigma \setminus \Delta_I^\sigma) \cup (\mathcal{C}_J^\sigma \setminus \Delta_J^\sigma)$. Taking complementaries with respect to $\mathcal{C}_{I+J}^\sigma$, it yields

$$\begin{aligned} \Delta_{I+J}^\sigma &= \mathcal{C}_{I+J}^\sigma \setminus ((\mathcal{C}_I^\sigma \setminus \Delta_I^\sigma) \cup (\mathcal{C}_J^\sigma \setminus \Delta_J^\sigma)) \\ &= (\mathcal{C}_{I+J}^\sigma \setminus (\mathcal{C}_I^\sigma \setminus \Delta_I^\sigma)) \cap (\mathcal{C}_{I+J}^\sigma \setminus (\mathcal{C}_J^\sigma \setminus \Delta_J^\sigma)) \\ &= (\Delta_I^\sigma \cup (\mathcal{C}_{I+J}^\sigma \setminus \mathcal{C}_I^\sigma)) \cap (\Delta_J^\sigma \cup (\mathcal{C}_{I+J}^\sigma \setminus \mathcal{C}_J^\sigma)), \end{aligned}$$

and the result follows. $\square$

The following example illustrates Corollary 2.1.6.

Example 2.1.7. Let $R = \mathbb{C}[x_0, x_1, x_2]$ be the Cox ring of $\mathbb{P}^2$ with fan Σ as in Example 1.2.6(i), and let $I = (x_1^2x_2^4, x_1^3x_2, x_1^5)$ and $J = (x_2^4, x_1x_2^3, x_1^4x_2^2)$ be two monomial ideals. Notice that

$$(s_0^I, s_1^I, s_2^I) = (0, 2, 0) \quad \text{and} \quad (s_0^J, s_1^J, s_2^J) = (0, 0, 2).$$

So, $(s_0^{I+J}, s_1^{I+J}, s_2^{I+J}) = (0, 0, 0)$. Hence, we have

$$\begin{aligned} \mathcal{C}_I^{\sigma_0} &= \{d_1 \geq 2, d_2 \geq 0\} \\ \mathcal{C}_J^{\sigma_0} &= \{d_1 \geq 0, d_2 \geq 2\} \\ \mathcal{C}_{I+J}^{\sigma_0} &= \{d_1 \geq 0, d_2 \geq 0\}, \end{aligned}$$

while

$$\begin{aligned}\mathcal{C}_I^{\sigma_1} &= \mathcal{C}_J^{\sigma_1} = \mathcal{C}_{I+J}^{\sigma_1} = \{d_1 + d_2 \leq 0, d_2 \geq 0\}\\ \mathcal{C}_I^{\sigma_2} &= \mathcal{C}_J^{\sigma_2} = \mathcal{C}_{I+J}^{\sigma_2} = \{d_1 + d_2 \leq 0, d_1 \geq 0\}.\end{aligned}$$

Computing the remaining Klyachko diagrams of I and J we obtain:

$$\begin{aligned}\Delta_I^{\sigma_0} &= \{(2,0),(2,1),(2,2),(2,3),(3,0),(4,0)\}\\ \Delta_I^{\sigma_1} &= \Delta_I^{\sigma_2} = \emptyset\\ \Delta_J^{\sigma_0} &= \{(0,2),(0,3),(1,2),(2,2),(3,2)\}\\ \Delta_J^{\sigma_1} &= \Delta_J^{\sigma_2} = \emptyset.\end{aligned}$$

Thus, applying Corollary 2.1.6 we get $\Delta_{I+J}^{\sigma_1} = \Delta_{I+J}^{\sigma_2} = \emptyset$ and

$$\begin{aligned}\Delta_{I+J}^{\sigma_0} = \{&(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2),\\ &(3,0),(4,0)\}.\end{aligned}$$

Figure 2.3 illustrates this example for the cone σ_0.

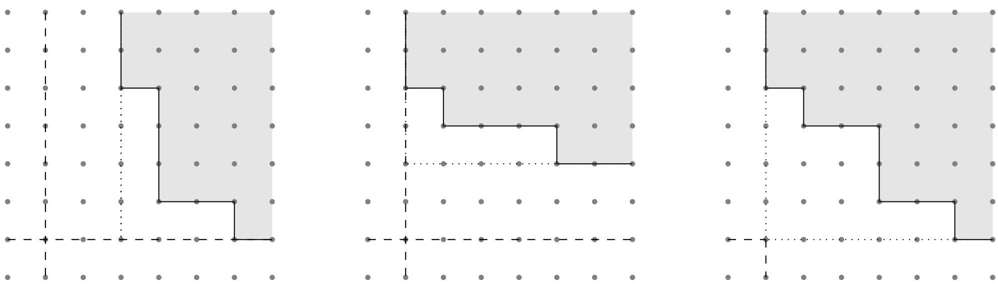

Figure 2.3: The part of the Klyachko diagram associated to the cone σ_0 of I, J and $I+J$, respectively. The dotted part corresponds to $\mathcal{C}_\bullet^{\sigma_0}$ and the shadowed part to $\mathcal{C}_\bullet^{\sigma_0} \setminus \Delta_\bullet^{\sigma_0}$.

We end this subsection with more examples on the compution of Klyachko diagrams.

Example 2.1.8. Let $R = \mathbb{C}[x_0, x_1, x_2, x_3]$ be the Cox ring of $\mathbb{P}^3$ with fan Σ as in Example 1.2.6(i), and let $I = (x_0x_1, x_1x_2x_3^2, x_2^2)$ be a monomial ideal. We have $s_0 = s_1 = s_2 = s_3 = 0$ and

$$\begin{aligned}
\mathcal{C}_I^{\rho_0} &= \{(d_1, d_2, d_3) \mid d_1 + d_2 + d_3 \leq 0\} \\
\mathcal{C}_I^{\rho_1} &= \{(d_1, d_2, d_3) \mid d_1 \geq 0\} \\
\mathcal{C}_I^{\rho_2} &= \{(d_1, d_2, d_3) \mid d_2 \geq 0\} \\
\mathcal{C}_I^{\rho_3} &= \{(d_1, d_2, d_3) \mid d_3 \geq 0\}.
\end{aligned}$$

We compute $\Delta_I^{\sigma_0}$. We order the monomials with respect to ρ_3:

$$\deg_{\rho_3}(x_0x_1) = \deg_{\rho_3}(x_2^2) = 0 < \deg_{\rho_3}(x_1x_2x_3^2) = 2$$

and we obtain

$$\begin{aligned}
&\Delta_I^{\sigma_0}(\mathcal{G})_0 = \emptyset \\
&\Delta_I^{\sigma_0}(\mathcal{G})_1 = \emptyset \\
&\Delta_I^{\sigma_0}(\mathcal{G})_2 = \{(d_1, d_2, d_3) \mid 0 \leq d_3 < 2\} \cap \Delta_I^{\sigma_0'}(\{x_0x_1, x_2^2\}) \\
&\Delta_I^{\sigma_0}(\mathcal{G})_3 = \{(d_1, d_2, d_3) \mid 2 \leq d_3\} \cap \Delta_I^{\sigma_0'}(\{x_0x_1, x_2^2, x_1x_2x_3^2\}),
\end{aligned}$$

where $\mathcal{G} = \{x_0x_1, x_1x_2x_3^2, x_2^2\}$ and $\sigma_0' = \operatorname{cone}(\rho_1, \rho_2)$. We proceed computing $\Delta_I^{\sigma_0'}(\{x_0x_1, x_2^2\})$ ordering the two monomials with respect to ρ_2:

$$\deg_{\rho_2}(x_0x_1) = 0 < \deg_{\rho_2}(x_2^2) = 2.$$

We get

$$\begin{aligned}
&\Delta_I^{\sigma_0'}(\{x_0x_1, x_2^2\})_0 = \emptyset \\
&\Delta_I^{\sigma_0'}(\{x_0x_1, x_2^2\})_1 = \{(d_1, d_2, d_3) \mid 0 \leq d_2 < 2\} \cap \Delta_I^{\rho_1}(\{x_0x_1\}) \\
&\Delta_I^{\sigma_0'}(\{x_0x_1, x_2^2\})_2 = \{(d_1, d_2, d_3) \mid 2 \leq d_2\} \cap \Delta_I^{\rho_1}(\{x_0x_1, x_2^2\}),
\end{aligned}$$

and $\Delta_I^{\rho_1}(\{x_0x_1\}) = \{(d_1, d_2, d_3) \mid d_1 = 0\}$, while $\Delta_I^{\rho_1}(\{x_0x_1, x_2^2\}) = \emptyset$. Hence, $\Delta_I^{\sigma_0'}(\{x_0x_1, x_2^2\}) = \{(d_1, d_2, d_3) \mid 0 \leq d_2 < 2,\ d_1 = 0\}$ and

$$\Delta_I^{\sigma_0}(\mathcal{G})_2 = \{(d_1, d_2, d_3) \mid 0 \leq d_3 < 2,\ 0 \leq d_2 < 2,\ d_1 = 0\}.$$

Similarly, we obtain

$$\begin{aligned}
\Delta_I^{\sigma_0'}(\{x_0x_1, x_2^2, &x_1x_2x_3^2\}) \\
&= \Delta_I^{\sigma_0'}(\{x_0x_1, x_2^2, x_1x_2x_3^2\})_1 \cup \Delta_I^{\sigma_0'}(\{x_0x_1, x_2^2, x_1x_2x_3^2\})_2 \\
&= \{(d_1, d_2, d_3) \mid 0 \leq d_2 \leq 1,\ d_1 = 0\}.
\end{aligned}$$

Hence, we have

$$\Delta_I^{\sigma_0}(\mathcal{G})_3 = \{(d_1, d_2, d_3) \mid 2 \le d_3,\ 0 \le d_2 \le 1,\ d_1 = 0\},\ \text{and}$$

$$\Delta_I^{\sigma_0} = \{(0, d_2, d_3) \mid 0 \le d_3 \le 1,\ 0 \le d_2 \le 1\} \cup \{(0, d_2, d_3) \mid 2 \le d_3,\ 0 \le d_2 \le 1\}.$$

Applying the same procedure for the remaining cones, we get

$$\begin{aligned} \Delta_I^{\sigma_1} &= \{(-d_2 - d_3, d_2, d_3) \mid 0 \le d_2, d_3 \le 1\} \cup \{(-d_3, 0, d_3) \mid 2 \le d_3\} \\ \Delta_I^{\sigma_2} &= \emptyset \\ \Delta_I^{\sigma_3} &= \{(0, d_2, d_3) \mid 0 \le d_2 \le 1,\ d_3 \le -d_2\} \cup \{(d_1, 0, -d_1) \mid 0 \le d_1\}. \end{aligned}$$

We notice that in this example $\Delta_I^{\sigma_1}$ and $\Delta_I^{\sigma_3}$ are unbounded.

Example 2.1.9. Let $R = \mathbb{C}[x_0, x_1, y_0, y_1]$ be the Cox ring of the Hirzebruch surface $\mathcal{H}_3$ with fan Σ as in Example 1.2.6(ii). R is endowed with a $\mathbb{Z}^2-$grading such that

$$\deg(x_0) = \deg(x_1) = (1, 0),\ \deg(y_0) = (0, 1)\ \text{and}\ \deg(y_1) = (-3, 1).$$

We consider the monomial ideal $I = (x_1, x_0^3 y_1)$, and we have

$$s_{\rho_0} = s_{\rho_1} = s_{\eta_0} = s_{\eta_1} = 0.$$

Therefore,

$$\begin{aligned} \mathcal{C}_I^{\rho_0} &= \{(d_1, d_2) \mid d_1 \le 3d_2\} \\ \mathcal{C}_I^{\rho_1} &= \{(d_1, d_2) \mid d_1 \ge 0\} \\ \mathcal{C}_I^{\eta_0} &= \{(d_1, d_2) \mid d_2 \le 0\} \\ \mathcal{C}_I^{\eta_1} &= \{(d_1, d_2) \mid d_2 \ge 0\}. \end{aligned}$$

We compute $\Delta^{\sigma_{00}}$. We order the monomials with respect to η_1:

$$\deg_{\eta_1}(x_1) = 0 < \deg_{\eta_1}(x_0^3 y_1) = 1.$$

We have

$$\begin{aligned} \Delta_I^{\sigma_{00}}(\mathcal{G})_0 &= \emptyset \\ \Delta_I^{\sigma_{00}}(\mathcal{G})_1 &= \{(d_1, d_2) \mid d_2 = 0\} \cap \Delta_I^{\rho_1}(\{x_1\}) \\ \Delta_I^{\sigma_{00}}(\mathcal{G})_2 &= \{(d_1, d_2) \mid d_2 = 1\} \cap \Delta_I^{\rho_1}(\{x_1, x_0^3 y_1\}), \end{aligned}$$

where $\mathcal{G} = \{x_1, x_0^3 y_1\}$. On the other hand, we have

$$\Delta_I^{\rho_1}(\{x_1\}) = \{(d_1, d_2) \mid d_1 \geq 0\}$$
$$\Delta_I^{\rho_1}(\{x_1, x_0^3 y_1\}) = \emptyset,$$

and we obtain that $\Delta_I^{\sigma_{00}} = \{(0,0)\}$. Applying the same procedure, we arrive at $\Delta_I^{\sigma_{01}} = \Delta_I^{\sigma_{10}} = \Delta_I^{\sigma_{11}} = \emptyset$ as shown in Figure 2.4.

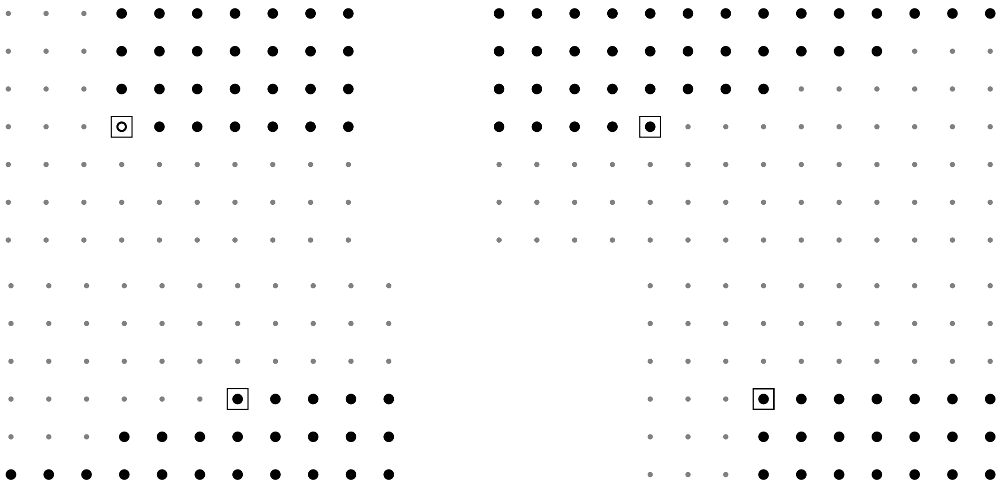

Figure 2.4: Klyachko diagram of Example 2.1.5 (iii). It displays $\Delta_I^{\sigma_{ij}}$ ($\circ$) inside $\mathcal{C}_I^{\sigma_{ij}}$ ($\bullet$) for $(i,j) = (0,0)$, $(1,0)$, $(0,1)$, $(1,1)$, clockwise. In each picture $\square$ places the origin $(0,0) \in M \cong \mathbb{Z}^2$.

2.1.2 From a Klyachko diagram to a monomial ideal

In this subsection, we find the minimal set of monomials generating the saturated ideal I associated to a Klyachko diagram $\{(\mathcal{C}_I^\sigma, \Delta_I^\sigma)\}_{\sigma\in\Sigma}$. We may assume that $\Delta_I^\sigma \neq \emptyset$ for some cone $\sigma \in \Sigma$. Otherwise, by Proposition 2.1.4 and Lemma 2.1.1 the $\Sigma-$family of I would be the $\Sigma-$family of a principal monomial ideal $I = (x_1^{s_1} \cdots x_r^{s_r})$, where $s_i \in \mathbb{Z}$ such that $\mathcal{C}_I^{\rho_j} = \{m \in M \mid \langle m, \rho_j \rangle \geq s_j\}$.

For any $D = (a_\rho)_{\rho\in\Sigma(1)} \in \mathbb{Z}^r$ representing a Weil divisor $\sum_{\rho\in\Sigma(1)} a_\rho D_\rho$, we denote by $[D] \in \mathrm{Cl}(X)$ its class in $\mathrm{Cl}(X)$. The following remark shows how shifting by D affects the Klyachko diagram.

Remark 2.1.10. Since X is smooth, for any $D = (a_\rho)_{\rho\in\Sigma(1)} \in \mathbb{Z}^r$ and any cone $\sigma \in \Sigma$ there is a character $\tau_\sigma \in M$ such that $\langle \tau_\sigma, \rho\rangle = a_\rho$ for all $\rho \in \sigma(1)$. Then, for any $m \in M$ and $\rho \in \sigma(1)$, $\langle m, \rho\rangle + a_\rho = \langle m + \tau_\sigma, \rho\rangle$, so

$$R^\sigma_m(D) = R^\sigma_{m+\tau_\sigma}.$$

It follows from (2.1) that

$$I^\sigma_m(D) = I^\sigma_{m+\tau_\sigma}.$$

Therefore, the Klyachko diagram of the shifted monomial ideal $I(D)$ is given by

$$\begin{cases} \mathcal{C}^\sigma_{I(D)} = \mathcal{C}^\sigma_I(D) := \mathcal{C}^\sigma_I + \tau_\sigma \\ \Delta^\sigma_{I(D)} = \Delta^\sigma_I(D) := \Delta^\sigma_I + \tau_\sigma, \end{cases}$$

which is obtained applying translations to the original Klyachko diagram.

Since I is $\mathrm{Cl}(X)-$graded and finitely generated, the monomials minimally generating I belong to a finite number of homogeneous pieces. We first start by providing a monomial basis of each homogeneous piece $I_{[D]} \subset R_{[D]}$, for $D \in \mathbb{Z}^r$.

Lemma 2.1.11. *Let $\{(\mathcal{C}^\sigma_I, \Delta^\sigma_I)\}_{\sigma\in\Sigma}$ be a Klyachko diagram of a $B-$saturated monomial ideal I, and $D = (a_\rho)_{\rho\in\Sigma(1)} \in \mathbb{Z}^r$. Then,*

$$I_{[D]} = \mathbb{C}\left\langle x^{m+D} | m \in \bigcap_{\sigma\in\Sigma_{\max}} (\mathcal{C}^\sigma_I(D) \setminus \Delta^\sigma_I(D)) \right\rangle.$$

Proof. For any cone $\sigma \in \Sigma$ we have that

$$I^\sigma_{[D]} = I^\sigma_0(D) = \bigoplus_{m\in M} I(D)_m$$

and

$$R^\sigma_{[D]} = \mathbb{C}\langle x^{m+D} \mid m \in \mathcal{C}^\sigma_0(D)\rangle.$$

On the other hand, by Proposition 1.3.14 for any character $m \in M$ we have

$$\begin{aligned} I(D)_m &\cong H^0(X, \tilde{I}(D))_m \\ &\cong \bigcap_{\sigma\in\Sigma_{\max}} I^\sigma_m(D). \end{aligned}$$

Applying Proposition 2.1.4, for any cone $\sigma \in \Sigma$ we have that $I_m^\sigma(D) \neq 0$ if and only if $m \in \mathcal{C}_I^\sigma(D) \setminus \Delta_I^\sigma(D)$. Hence,

$$I_{[D]} = \bigoplus_{m \in M} \bigcap_{\sigma \in \Sigma_{\max}} I_m^\sigma(D) \quad = \quad \bigoplus_{m \in \bigcap_{\sigma \in \Sigma_{\max}} (\mathcal{C}_I^\sigma(D) \setminus \Delta_I^\sigma(D))} \mathbb{C}\langle x^{m+D} \rangle$$

and the lemma follows. $\square$

As a result of Lemma 2.1.11, we know a basis of each homogeneous piece of I. Our next task is to characterize which monomials in a homogeneous piece $I_{[E]}$ are divisible by a single monomial x^{m+D}. The following Lemma answers this question.

Lemma 2.1.12. *Let $\{(\mathcal{C}_I^\sigma, \Delta_I^\sigma)\}_{\sigma \in \Sigma}$ be a Klyachko diagram of a $B-$saturated monomial ideal I, and $D = (a_\rho)_{\rho \in \Sigma(1)} \in \mathbb{Z}^r$. Consider a character*

$$m \in \bigcap_{\sigma \in \Sigma_{\max}} (\mathcal{C}_I^\sigma(D) \setminus \Delta_I^\sigma(D))$$

and let $E = (b_\rho)_{\rho \in \Sigma(1)} \in \mathbb{Z}^r$ be such that $b_\rho \geq a_\rho$ for all $\rho \in \Sigma(1)$. Then, the set of monomials in $I_{[E]}$ which are divisible by x^{m+D} is

$$T_E(x^{m+D}) := \left\{ x^{m'+E} \middle| \langle m', \rho \rangle \geq \langle m, \rho \rangle + a_\rho - b_\rho, \rho \in \Sigma(1) \right\}.$$

Proof. For any $m' \in M$, the monomial $x^{m'+E}$ is divisible by x^{m+D} if and only if $\langle m', \rho \rangle + b_\rho \geq \langle m, \rho \rangle + a_\rho$ for any $\rho \in \Sigma(1)$, and the lemma follows. $\square$

Now, let $\mathcal{G} = \{x^{m_1 + \underline{k}^1}, \ldots, x^{m_t + \underline{k}^t}\}$ be a finite set of monomials of possibly different degrees. For any $E = (b_\rho)_{\rho \in \Sigma(1)} \in \mathbb{Z}^r$ such that $b_\rho \geq k_\rho^i$ for $\rho \in \Sigma(1)$ and $1 \leq i \leq t$, we define

$$T_E \mathcal{G} := T_E(x^{m_1 + \underline{k}^1}) \cup \cdots \cup T_E(x^{m_t + \underline{k}^t}),$$

which describes the span of the monomials of $\mathcal{G}$ inside $I_{[E]}$.

Finally, we can describe a finite set of generators of a $B-$saturated monomial ideal I corresponding to a given Klyachko diagram. Since X is smooth, we can assume that

$$\mathrm{Cl}(X) \cong \mathbb{Z}\langle [D_{\rho_{i_1}}], \ldots, [D_{\rho_{i_\ell}}]\rangle \cong \mathbb{Z}^\ell.$$

Up to permutation of variables, we may also assume that

$$i_1 = 1, \ldots, i_\ell = \ell.$$

For any $a = (a_1, \ldots, a_\ell) \in \mathbb{Z}^\ell$ we set $\overline{a} := (a_1, \ldots, a_\ell, 0, \ldots, 0) \in \mathbb{Z}^r$. For any $u, v \in \mathbb{Z}^\ell$ we say that $v \preceq u$ if $u_i \geq v_i$ for $1 \leq i \leq \ell$, which defines a partial order. We set $\mathcal{G}_0 = \emptyset$ and for any $u \in \mathbb{Z}^\ell$, we define

$$\mathcal{G}_u := \left\{ x^{m+\overline{u}} \mid m \in \bigcap_{\sigma\in\Sigma_{\max}} \mathcal{C}_I^\sigma(\overline{u}) \setminus \Delta_I^\sigma(\overline{u}) \right\} \setminus \bigcup_{v \preceq u} T_{\overline{u}}\mathcal{G}_v,$$

assuming we have determined $\mathcal{G}_v$ for any $v \preceq u$. Since I is finitely generated, there are only finitely many degrees $u \in \mathbb{Z}^\ell$ such that $\mathcal{G}_u \neq \emptyset$.

If R is the Cox ring of $\mathbb{P}^{r-1}$, and so R has the standard $\mathbb{Z}-$grading, then by construction this method gives directly a minimal set of monomial generators for I. The following example illustrates it:

Example 2.1.13. (i) Let $R = \mathbb{C}[x_0, x_1, x_2]$ be the Cox ring of $\mathbb{P}^2$ (See Example 1.2.6(i)). Consider the following Klyachko diagram $\{(\mathcal{C}_I^i, \Delta_I^i)\}_{0\leq i\leq 2}$, where $\mathcal{C}_I^i$ and Δ_I^i stand for $\mathcal{C}_I^{\sigma_i}$ and $\Delta_I^{\sigma_i}$:

$$\begin{aligned}
\mathcal{C}_I^0 &= \{(d_1, d_2) \mid d_1 \geq 0,\ d_2 \geq 0\}\\
\mathcal{C}_I^1 &= \{(d_1, d_2) \mid d_1 + d_2 \leq 0,\ d_2 \geq 0\}\\
\mathcal{C}_I^2 &= \{(d_1, d_2) \mid d_1 + d_2 \leq 0,\ d_1 \geq 0\}\\
\Delta_I^0 &= \{(0,0), (1,0)\}\\
\Delta_I^1 &= \Delta_I^2 = \{(0,0)\}.
\end{aligned}$$

Applying the above procedure, we obtain that

$$\begin{aligned}
\mathcal{G}_0 &= \mathcal{G}_1 = \emptyset\\
\mathcal{G}_2 &= \{x_0x_2, x_1x_2\}\\
\mathcal{G}_3 &= \{x_0x_1^2\}\\
\mathcal{G}_j &= \emptyset \quad \text{for} \quad j \geq 4.
\end{aligned}$$

Hence, the saturated monomial ideal corresponding to this Klyachko diagram is $I = (x_0x_2, x_1x_2, x_0x_1^2)$.

(ii) Let $R = \mathbb{C}[x_0, x_1, x_2, x_3]$ be the Cox ring of $\mathbb{P}^3$ (See Example 1.2.6(i)). Consider the following Klyachko diagram $\{(\mathcal{C}_I^i, \Delta_I^i)\}_{0\leq i\leq 3}$, where $\mathcal{C}_I^i$ and Δ_I^i stand for $\mathcal{C}_I^{\sigma_i}$ and $\Delta_I^{\sigma_i}$, respectively:

$$\begin{aligned}
\mathcal{C}_I^0 &= \{(d_1, d_2, d_3) \mid d_1 \geq 0,\ d_2 \geq 0, d_3 \geq 0\}\\
\mathcal{C}_I^1 &= \{(d_1, d_2, d_3) \mid d_1 + d_2 + d_3 \leq 0,\ d_2 \geq 0, d_3 \geq 0\}\\
\mathcal{C}_I^2 &= \{(d_1, d_2, d_3) \mid d_1 + d_2 + d_3 \leq 0,\ d_1 \geq 0, d_3 \geq 0\}\\
\mathcal{C}_I^3 &= \{(d_1, d_2, d_3) \mid d_1 + d_2 + d_3 \leq 0,\ d_1 \geq 0, d_2 \geq 0\}\\
\Delta_I^0 &= \{(0, 0, d_3) \mid d_3 \geq 0\} \cup \{(0, 1, 0)\}\\
\Delta_I^1 &= \Delta_I^2 = \Delta_I^3 = \emptyset.
\end{aligned}$$

Applying the above procedure, we obtain that

$$\begin{aligned}
&\mathcal{G}_0 = \emptyset\\
&\mathcal{G}_1 = \{x_1\}\\
&\mathcal{G}_2 = \{x_0^2x_1, x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3\} \setminus T_{(2,0,0,0)}\{x_1\} = \{x_2^2, x_2x_3\}\\
&\mathcal{G}_j = \emptyset \quad \text{for} \quad j \geq 3.
\end{aligned}$$

Hence, the saturated monomial ideal corresponding to this Klyachko diagram is $I = (x_1, x_2^2, x_2x_3)$.

In more general gradings, we cannot assure that this method gives a *minimal* set of generators of I, but just a finite set of monomials generating I. However, we can extract from it a minimal set of monomials generating I by using suitable monomial divisions. The following example illustrates this situation:

Example 2.1.14. Let $R = \mathbb{C}[x_0, x_1, y_0, y_1]$ be the Cox ring of the Hirzebruch surface $\mathcal{H}_3$ (see Example 1.2.6(ii)). In particular, R is $\mathbb{Z}^2-$graded with

$$\deg(x_0) = \deg(x_1) = (1, 0),\ \deg(y_0) = (0, 1) \text{ and } \deg(y_1) = (-3, 1).$$

Let $\{(\mathcal{C}_I^{\sigma_{ij}}, \Delta_I^{\sigma_{ij}})\}_{0\leq i,j\leq 1}$ be the Klyachko diagram of Example 2.1.9. Applying the above procedure we obtain

$$\begin{aligned}
\mathcal{G}_{(1,0)} &= \{x_1\}\\
\mathcal{G}_{(0,1)} &= \{x_0^3y_1, x_0^2x_1y_1, x_0x_1^2y_1, x_1^3y_1\}\\
\mathcal{G}_u \ &= \emptyset \quad \text{for} \quad u \in \mathbb{Z}^2 \setminus \{(1, 0), (0, 1)\}.
\end{aligned}$$

Hence $\{x_1, x_0^3y_1, x_0^2x_1y_1, x_0x_1^2y_1, x_1^3y_1\}$ is a set of generators for a saturated monomial ideal I corresponding to this Klyachko diagram. However, the first monomial divides the three last monomials. Therefore, I is *minimally* generated by $\{x_1, x_0^3y_1\}$.

2.1.3 Non-saturated monomial ideals

The previous subsections have shown that the theory of Klyachko diagrams is well suited to describe saturated monomial ideals, but we cannot retrieve directly information of non-saturated monomial ideals. In this subsection, we describe the quotient $I^{\text{sat}}/I \cong H_B^1(I)$ using the Klyachko diagram $\{(\mathcal{C}_I^\sigma, \Delta_I^\sigma)\}_{\sigma\in\Sigma}$ and the generators of I.

Proposition 2.1.15. *Let $I = (x^{m_1+\underline{k}^t}, \dots, x^{m_t+\underline{k}^t})$ be a monomial ideal with Klyachko diagram $\{(\mathcal{C}_I^\sigma, \Delta_I^\sigma)\}_{\sigma\in\Sigma}$, such that for $1 \le i \le t$, $m_i \in M$ and $\underline{k}^i = (k_\rho^i)_{\rho\in\Sigma(1)} \in \mathbb{Z}^r$ satisfy $\langle m_i, \rho\rangle + k_\rho^t \ge 0$ for all $\rho \in \Sigma(1)$. Then, for any $D = (a_\rho)_{\rho\in\Sigma(1)} \in \mathbb{Z}^r$,*

$$H_B^1(I)_{[D]} \cong \mathbb{C}\left\langle \left\{ x^{m+D} \,\middle|\, m \in \bigcap_{\sigma\in\Sigma_{\max}} (\mathcal{C}_I^\sigma(D) \setminus \Delta_I^\sigma(D)) \right\} \setminus \bigcup_{i=1}^{t} T_D(x^{m_i+\underline{k}^i}) \right\rangle.$$

Proof. From Lemma 2.1.12, $I_{[D]} = \bigcup_{i=1}^t T_D(x^{m_i+\underline{k}^i}) \subset I_{[D]}^{sat}$. By Proposition 2.1.4, the Klyachko diagram characterizes the saturation of I, and by Lemma 2.1.11 we have that

$$I_{[D]}^{sat} = \mathbb{C}\langle x^{m+D} | m \in \bigcap_{\sigma\in\Sigma_{\max}} (\mathcal{C}_I^\sigma(D) \setminus \Delta_I^\sigma(D))\rangle,$$

and the result follows. □

Example 2.1.16. Let $R = \mathbb{C}[x_0, x_1, x_2]$ be the Cox ring of $\mathbb{P}^2$, with irrelevant ideal $B = (x_0, x_1, x_2)$ and let $I = (x_0^3x_1, x_0x_1x_2^2, x_2^3, x_1^3)$ be a monomial ideal. Computing its Klyachko diagram we obtain

$$\begin{aligned} &s_0 = s_1 = s_2 = 0 \\ &\Delta^0 = \{(0,0), (0,1), (0,2)\} \\ &\Delta^1 = \Delta^2 = \emptyset. \end{aligned}$$

From Proposition 2.1.15 we obtain:

$$\begin{aligned}
&H^1_B(I)_0 = 0\\
&H^1_B(I)_3 = \mathbb{C}\langle x_0^2x_1, x_0x_1^2, x_0x_1x_2, x_1^2x_2, x_1x_2^2\rangle\\
&H^1_B(I)_1 = \mathbb{C}\langle x_1\rangle\\
&H^1_B(I)_4 = \mathbb{C}\langle x_0^2x_1^2, x_0^2x_1x_2, x_0x_1^2x_2, x_1^2x_2^2\rangle\\
&H^1_B(I)_2 = \mathbb{C}\langle x_0x_1, x_1^2, x_1x_2\rangle\\
&H^1_B(I)_5 = \mathbb{C}\langle x_0^2x_1^2x_2\rangle\\
&H^1_B(I)_j = 0 \quad \text{for} \quad j \geq 6.
\end{aligned}$$

2.2 Hilbert function of monomial ideals

In this section, we show how to compute the Hilbert function and the Hilbert polynomial of a $B-$saturated monomial ideal from its Klyachko diagram. As a consequence, we develop a formula for the Hilbert polynomial in terms of the Klyachko diagram.

For any ray $\rho \in \Sigma(1)$, recall that we set

$$\mathcal{C}_0^\rho := \{m \in M | \langle m, \rho\rangle \geq 0\} \quad \text{and} \quad \mathcal{C}_0^\sigma = \bigcap_{\rho\in\sigma(1)} \mathcal{C}^\rho, \text{ for } \sigma \in \Sigma.$$

By Lemma 2.1.1, the Klyachko diagram of R, seen as the monomial ideal $R = (1)$ is $\{(\mathcal{C}_0^\sigma, \emptyset)\}_{\sigma\in\Sigma}$. For any multidegree $D \in \mathbb{Z}^r$, we define

$$\mathcal{C}_0(D) := \bigcap_{\sigma\in\Sigma_{\max}} \mathcal{C}_0^\sigma(D),$$

such that $\mathcal{C}_0(D)$ gives a monomial basis of $R_{[D]}$ (see Proposition 2.1.11). The following result tells us how to compute the value of the Hilbert function of I from this description.

Proposition 2.2.1. *Let I be a $B-$saturated monomial ideal with Klyachko diagram $\{(\mathcal{C}_I^\sigma, \Delta_I^\sigma)\}_{\sigma\in\Sigma}$. Then, the Hilbert function of I is given by*

$$h_{R/I}(\alpha) = \left| \bigcup_{\sigma\in\Sigma_{\max}} ((\Delta_I^\sigma(\overline{\alpha}) \cap \mathcal{C}_0(\overline{\alpha})) \cup (\mathcal{C}_0(\overline{\alpha}) \setminus \mathcal{C}_I^\sigma(\overline{\alpha}))) \right|$$

for any $\alpha \in \mathrm{Cl}(X)$.

Proof. By Lemma 2.1.11, there is a bijection between a monomial basis of I_α (respectively of R_α) and $\bigcap_{\sigma\in\Sigma_{\max}}(\mathcal{C}_I^\sigma(\overline{\alpha}) \setminus \Delta_I^\sigma(\overline{\alpha}))$ (respectively $\mathcal{C}_0(\overline{\alpha})$). Thus,

$$\begin{aligned} h_{R/I}(\alpha) &= \left| \mathcal{C}_0(\overline{\alpha}) \setminus \left(\bigcap_{\sigma\in\Sigma_{\max}} (\mathcal{C}_I^\sigma(\overline{\alpha}) \setminus \Delta_I^\sigma(\overline{\alpha})) \right) \right| \\ &= \left| \bigcup_{\sigma\in\Sigma_{\max}} \mathcal{C}_0(\overline{\alpha}) \setminus (\mathcal{C}_I^\sigma(\overline{\alpha}) \setminus \Delta_I^\sigma(\overline{\alpha})) \right| \\ &= \left| \bigcup_{\sigma\in\Sigma_{\max}} ((\Delta_I^\sigma(\overline{\alpha}) \cap \mathcal{C}_0(\overline{\alpha})) \cup (\mathcal{C}_0(\overline{\alpha}) \setminus \mathcal{C}_I^\sigma(\overline{\alpha}))) \right|. \end{aligned}$$

□

Remark 2.2.2. The above formula simplifies when I is an ideal generated by monomials with no common factor. Let $I \subset R$ be a monomial ideal and $x^{\underline{k}} \in R$ a monomial. We recall that the Hilbert function of the monomial ideal $J = x^{\underline{k}} I$ is

$$h_{R/J}(\alpha) = h_R(\alpha) - h_R(\alpha - [\underline{k}]) + h_{R/I}(\alpha - [\underline{k}]). \tag{2.3}$$

Thus, we can assume that the Klyachko diagram of I has $s_\rho = 0$ for any $\rho \in \Sigma(1)$ and, its Hilbert function is

$$h_{R/I}(\alpha) = \left| \bigcup_{\sigma\in\Sigma_{\max}} (\Delta^\sigma(\overline{\alpha}) \cap \mathcal{C}_0(\overline{\alpha})) \right|. \tag{2.4}$$

Otherwise, $I = \left(\prod_{\rho\in\Sigma(1)} x_\rho^{s_\rho}\right) I_0$ where I_0 is a monomial ideal with $s_\rho = 0$ for any $\rho \in \Sigma(1)$, and we can compute the Hilbert function of I using (2.3).

Example 2.2.3. Let $R = \mathbb{C}[x_0, x_1, x_2]$ be the Cox ring of $\mathbb{P}^2$, and consider the monomial ideal $I = (x_2^2, x_0x_2, x_0x_1)$ as in Example 2.1.5. For any $a \in \mathbb{Z}$, we set $\overline{a} = (a, 0, 0)$ and we have

$$\begin{aligned} \Delta_I^{\sigma_0}(\overline{a}) &= \{(0,0)\} \\ \Delta_I^{\sigma_1}(\overline{a}) &= \{(a,0), (a-1,1)\} \\ \Delta_I^{\sigma_2}(\overline{a}) &= \emptyset. \end{aligned}$$

Since $s_0 = s_1 = s_2 = 0$, by (2.4) we have the following Hilbert function:

$$h_{R/I}(t) = \begin{cases} 0, & t \leq -1 \\ 1, & t = 0 \\ 3, & t \geq 1. \end{cases}$$

In particular, the Hilbert polynomial of R/I is a constant $P_{R/I} \equiv 3$.

In the following result, we characterize the Klyachko diagram of a monomial ideal I with constant Hilbert polynomial. In particular, notice that I is necessarily generated by monomials without common factors.

Corollary 2.2.4. *Let I be a monomial ideal with Klyachko diagram $\{(\mathcal{C}_I^\sigma, \Delta_I^\sigma)\}_{\sigma\in\Sigma}$. Then, the Hilbert polynomial $P_{R/I}$ of I is constant if and only if $s_\rho = 0$ for any $\rho \in \Sigma(1)$ and Δ_I^σ is finite for any $\sigma \in \Sigma_{\max}$. Moreover,*

$$P_{R/I}(\alpha) = \sum_{\sigma\in\Sigma_{\max}} |\Delta_I^\sigma|.$$

Proof. The left implication follows directly from Proposition 2.2.1 and Remark 2.2.2. Conversely, if $s_\rho > 0$ for some $\rho \in \Sigma(1)$, then there is $\sigma \in \Sigma_{\max}$ such that $\rho \in \sigma(1)$ and $\mathcal{C}_0(\overline{\alpha}) \setminus \mathcal{C}_I^\sigma(\overline{\alpha})$ increases with α, and $P_{R/I}$ would not be constant. Now, assume that there is some $\sigma \in \Sigma_{\max}$ such that Δ_I^σ is not finite. By construction, Δ_I^σ contains a set Δ' of the form

$$\{m\in M \mid \langle m, \rho_{i_1}\rangle = k_1, \ldots, \langle m, \rho_{i_l}\rangle = k_l, \langle m, \rho_{i_{l+1}}\rangle \geq k_{l+1}, \ldots, \langle m, \rho_{i_c}\rangle \geq k_c\}.$$

Since $\Delta'(\overline{\alpha})$ is not bounded in $\mathcal{C}_0^\sigma(\overline{\alpha})$, the number of points in $\Delta'(\overline{\alpha})\cap\mathcal{C}_0(\overline{\alpha})$ increases with α for $\alpha \gg 0$. Therefore, it follows from (2.4) that the Hilbert function $h_{R/I}(\alpha)$ of I increases with α for $\alpha \gg 0$. □

Remark 2.2.5. Notice that by Corollary 2.2.4 we have characterized all monomial ideals $I \subset R$ with $\dim R/I = 1$ in terms of the Klyachko diagram.

We finish illustrating Corollary 2.2.4 with the following example.

Example 2.2.6. Let $R = \mathbb{C}[x_0, x_1, x_2, x_3]$ be the Cox ring of $\mathbb{P}^3$ and $I{=}(x_0x_1, x_2^2, x_1x_2x_3^2)$ as in Example 2.1.8. For any $a \in \mathbb{Z}$ we set $\overline{a}$ and we have,

$$\begin{aligned}
\Delta_I^{\sigma_0}(\overline{a}) =& \{(0,d_2,d_3) \in M \mid 0 \leq d_2, d_3 \leq 1\} \cup \\
& \{(0,d_2,d_3) \in M \mid d_3 \geq 2, 0 \leq d_2 \leq 1\} \\
\Delta_I^{\sigma_1}(\overline{a}) =& \{(a-d_2-d_3,d_2,d_3) \in M \mid 0 \leq d_2, d_3 \leq 1\} \cup \\
& \{(a-d_3,0,d_3) \in M \mid d_3 \geq 2\} \\
\Delta_I^{\sigma_2}(\overline{a}) =& \emptyset \\
\Delta_I^{\sigma_3}(\overline{a}) =& \{(0,d_2,d_3) \in M \mid 0 \leq d_2 \leq 1, d_3 \leq a-d_2\} \cup \\
& \{(d_1,0,a-d_1) \in M \mid d_1 \geq 1\}.
\end{aligned}$$

Counting the number of different points in $\bigcup_{i=0}^{3} \Delta_I^{\sigma_i}(\overline{a})$, we obtain that

$$\begin{aligned}
h_{R/I}(0) &= 1 \\
h_{R/I}(1) &= 4 \\
h_{R/I}(2) &= 8 \\
h_{R/I}(a) &= 3(a+1) \quad \text{for} \quad a \geq 3.
\end{aligned}$$

Thus, the Hilbert polynomial of I is $P_{R/I}(a) = 3(a+1)$.

Chapter 3

Equivariant reflexive sheaves

In this chapter, we study equivariant reflexive sheaves on a smooth toric variety X. Given an equivariant reflexive sheaf $\mathcal{E}$ on X, we address how the global sections of $\mathcal{E} \otimes \mathcal{L}$ change as we twist by different line bundles $\mathcal{L}$ on X. This problem can be encoded by counting lattice points of a collection of lattice polytopes, even though controlling the number of these lattice points can be a very difficult problem. However, for some smooth toric varieties with some extra structure on their fan, this problem can be overcome. Precisely, for toric varieties with splitting fan, the lattice polytopes can be sliced into smaller-dimensional polytopes. This in turn, sheds new light on the value of $\mathrm{h}^0(X, \mathcal{E} \otimes \mathcal{L})$, and ultimately on the Hilbert polynomial of $\mathcal{E}$, and its multigraded regularity.

Finally, we turn our attention to the problem of finding equivariant Ulrich bundles on a smooth toric variety. Ulrich bundles have been a center of interest in the last decades, and have been studied from many different perspectives [2, 17, 29, 64, 1]. Here we address this problem from the point of view of the Klyachko filtrations associated to equivariant reflexive sheaves. In [1] the Chern classes and the shape of a minimal locally free resolution of Ulrich bundles on a Hirzebruch surface were determined. However, it is not strightforward to see when these invariants yield an equivariant Ulrich bundle. In the last part of this chapter, we build an example of a rank 3 equivariant Ulrich bundle on the Hirzebruch surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ (see Example 3.3.4).

Next we explain how this chapter is organized. In Section 3.1, we present the collection of polytopes used to encode the global sections of an equivariant reflexive sheaf on a smooth toric variety (see Proposition 3.1.2). In Section 3.2, we restrict our attention to a smooth toric variety with splitting fan X. After fixing some notation, this section is divided into two subsections. In Subsection 3.2.1 we address the problem of bounding the initial degree of a reflexive equivariant sheaf (see Propositions 3.2.8 and 3.2.9). Afterwards, in Subsection 3.2.2 we study in more detail the Hilbert polynomial of an equivariant reflexive sheaf $\mathcal{E}$ on X. We find a sharp upper bound for the regularity index of $\mathcal{E}$ (see Theorem 3.2.18). Finally, in Section 3.3, we focus on finding equivariant Ulrich bundles on smooth toric surfaces. In particular, we apply the results of Sections 3.1 and 3.2 to finally find an equivariant Ulrich bundle on a Hirzebruch surface (see Example 3.3.4).

3.1 Equivariant reflexive sheaves and polytopes

In this section, we introduce a family of lattice polytopes describing the global sections of an equivariant reflexive sheaf. This description is then used as a tool to compute the Euler characteristic of an equivariant reflexive sheaf, and moreover its Hilbert polynomial. Let X be a complete smooth toric variety of dimension n and set $\Sigma_{\max} = \{\sigma_1, \ldots, \sigma_s\}$. We choose a basis $\{e_1, \ldots, e_n\}$ of $N = \mathbb{Z}^n$ such that $\sigma_1 = \mathrm{cone}(e_1, \ldots, e_n)$. We write $\Sigma(1) = \{\rho_1, \ldots, \rho_n, \rho_{n+1}, \ldots, \rho_r\}$ the set of rays such that

$$\begin{aligned} \rho_k &= \mathrm{cone}(e_k) && \text{for } 1 \le k \le n \quad \text{and} \\ \rho_{n+k} &= \mathrm{cone}(a_1^k e_1 + \cdots + a_n^k e_n) && \text{for } 1 \le k \le r-n. \end{aligned}$$

The class group of X is

$$\mathrm{Cl}(X) = \mathrm{coker}(\phi) \cong \mathbb{Z}^{r-n}$$

where $\phi : M \to \mathbb{Z}^r$ is given by the following matrix

$$\begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ a_1^1 & \cdots & a_n^1 \\ \vdots & & \vdots \\ a_1^{r-n} & \cdots & a_n^{r-n} \end{pmatrix}. \tag{3.1}$$

Thus, we have

$$\mathrm{Cl}(X) = \mathbb{Z}\langle [D_{\rho_{n+1}}], \ldots, [D_{\rho_r}]\rangle$$

and

$$[D_{\rho_k}] = a_k^1[D_{\rho_{n+1}}] + \cdots + a_k^{r-n}[D_{\rho_r}] \text{ for } 1 \leq k \leq n.$$

Let $\mathcal{E}$ be an equivariant reflexive sheaf of rank ℓ and $\{\hat{E}^k\}_{k=1}^r$ the associated $\Sigma-$family, where

$$\hat{E}^k = E^k(i_1^k, \ldots, i_\ell^k; E_1^k, \ldots, E_{\ell-1}^k, \mathbb{C}^\ell).$$

Next, we describe a family of lattice polytopes encoding the global sections of $\mathcal{E}$ twisted by any line bundle $\mathcal{L}_{\underline{t}} := \mathcal{O}(t_1D_{\rho_{n+1}} + \cdots + t_{r-n}D_{\rho_r})$. We set $\mathcal{E}(\underline{t}) := \mathcal{E} \otimes \mathcal{L}_{\underline{t}}$, which is a reflexive equivariant sheaf with filtrations

$$\begin{aligned} \hat{E}^k(\underline{t}) &= E^k(i_1^k, \ldots, i_\ell^k; E_1^k, \ldots, E_\ell^k) \qquad \text{for } 1 \leq k \leq n \quad \text{and} \\ \hat{E}^{n+k}(\underline{t}) &= E^{n+k}(i_1^{n+k} - t_k, \ldots, i_\ell^{n+k} - t_k; E_1^{n+k}, \ldots, E_\ell^{n+k}) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for } 1 \leq k \leq r-n. \end{aligned} \tag{3.2}$$

We introduce the following $n-$dimensional lattice polytope in $M_\mathbb{R}$:

$$\Omega(\underline{t}) := \left\{ x \in M_\mathbb{R} \,\middle|\, \begin{array}{l} i_1^k \leq \langle x, n(\rho_k)\rangle \text{ for } 1 \leq k \leq n, \\ i_1^{n+k} - t_k \leq \langle x, n(\rho_{n+k})\rangle \text{ for } 1 \leq k \leq r-n \end{array} \right\}.$$

The polytope $\Omega(\underline{t})$ can be also defined by the following linear system of inequalities:

$$\begin{pmatrix} i_1^1 \\ \vdots \\ i_1^n \\ i_1^{n+1} - t_1 \\ \vdots \\ i_1^r - t_{r-n} \end{pmatrix} \leq \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ a_1^1 & \cdots & a_n^1 \\ \vdots & & \vdots \\ a_1^{r-n} & \cdots & a_n^{r-n} \end{pmatrix} \cdot x \tag{3.3}$$

The polytope $\Omega(\underline{t})$ bounds the region of characters $m \in M$ for which $\mathcal{E}(\underline{t})$ has $m-$graded global sections. Precisely we have the following result:

Proposition 3.1.1. *Let $m \in M$ be a character. If $\mathrm{H}^0(X, \mathcal{E}(\underline{t}))_m \neq 0$, then $m \in \Omega(\underline{t}) \cap M$.*

Proof. It follows from Proposition 1.3.14, that

$$\begin{aligned}\mathrm{H}^0(X,\mathcal{E}(\underline{t}))_m \cong E^1(\langle m, n(\rho_1)\rangle) \cap \cdots \cap E^n(\langle m, n(\rho_1)\rangle)\cap \\ E^{n+1}(\langle m, n(\rho_1)\rangle + t_1) \cap \cdots \cap E^r(\langle m, n(\rho_1)\rangle + t_{r-n}).\end{aligned} \tag{3.4}$$

Thus, being $\mathrm{H}^0(X,\mathcal{E}(\underline{t}))_m \neq 0$ means that each of the vector spaces intersecting in (3.4) is non-zero. This in turn implies that $m \in M$ satisfies the inequalities (3.3). □

Our next task is to tessellate $\Omega(\underline{t})$ into a collection of lattice polytopes, each of them describing a subregion in which the $M-$graded pieces of $\mathrm{H}^0(X,\mathcal{E}(\underline{t}))$ have constant dimension. For any $\underline{\lambda} \in \{1,\dots,\ell\}^r$, and $\underline{t} = (t_1,\dots,t_{r-n}) \in \mathbb{Z}^{r-n}$ we consider the $n-$dimensional polytope $\Omega_{\underline{\lambda}}(\underline{t})$, defined by linear system of inequalities

$$\begin{pmatrix} i^1_{\lambda_1} \\ \vdots \\ i^n_{\lambda_n} \\ i^{n+1}_{\lambda_{n+1}} - t_1 \\ \vdots \\ i^r_{\lambda_r} - t_{r-n} \end{pmatrix} \leq \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ a^1_1 & \cdots & a^1_n \\ \vdots & & \vdots \\ a^{r-n}_1 & \cdots & a^{r-n}_n \end{pmatrix} \cdot x < \begin{pmatrix} i^1_{\lambda_1+1} \\ \vdots \\ i^n_{\lambda_n+1} \\ i^{n+1}_{\lambda_{n+1}+1} - t_1 \\ \vdots \\ i^r_{\lambda_r+1} - t_{r-n} \end{pmatrix} \tag{3.5}$$

where we set $i^k_{\ell+1} := \infty$. We denote by

$$\Psi_{\underline{\lambda}}(\underline{t}) := \Omega_{\underline{\lambda}}(\underline{t}) \cap M$$

the integer solutions of (3.5). We have the following result.

Proposition 3.1.2. *Let $\underline{t} \in \mathbb{Z}^{r-n}$.*

(i) For any $\underline{\lambda} \in \{1,\dots,\ell\}^r$. A character $m \in \Psi_{\underline{\lambda}}(\underline{t})$ if and only if

$$\mathrm{H}^0(X,\mathcal{E}(\underline{t}))_m \cong \bigcap_{k=1}^{r} E^k_{\lambda_k}.$$

(ii) If $m \in M$ is a character such that $\mathrm{H}^0(X,\mathcal{E}(\underline{t}))_m \neq 0$, then there is $\underline{\lambda} \in \{1,\dots,\ell\}^r$ such that $m \in \Psi_{\underline{\lambda}}(\underline{t})$.

(iii) We have,

$$\mathrm{h}^0(X, \mathcal{E}(\underline{t})) = \sum_{\underline{\lambda} \in \{1,\dots,\ell\}^r} |\Psi_{\underline{\lambda}}(\underline{t})| D(\underline{\lambda})$$

where for $\underline{\lambda} = (\lambda_1, \dots, \lambda_r)$, $D(\underline{\lambda}) = \dim \bigcap_{k=1}^r E^k_{\lambda_k}$.

Moreover, if $\underline{t} \gg 0$, *then*

$$\chi(\mathcal{E}(\underline{t})) = \sum_{\underline{\lambda} \in \{1,\dots,\ell\}^r} |\Psi_{\underline{\lambda}}(\underline{t})| D(\underline{\lambda}). \tag{3.6}$$

Proof. By Proposition 1.3.14, for any character $m \in M$ we have

$$\begin{aligned} \mathrm{H}^0(X, \mathcal{E}(\underline{t}))_m \cong E^1(\langle m, n(\rho_1)\rangle) \cap \dots \cap E^n(\langle m, n(\rho_1)\rangle) \cap \\ E^{n+1}(\langle m, n(\rho_1)\rangle + t_1) \cap \dots \cap E^r(\langle m, n(\rho_1)\rangle + t_{r-n}). \end{aligned} \tag{3.7}$$

On the other hand, m is a solution of (3.5) for some $\underline{\lambda} \in \{1, \dots, \ell\}^r$ if and only if

$$\begin{aligned} &i^k_{m_k} \le \langle m, n(\rho_k)\rangle < i^k_{m_k+1} && \text{for } 1 \le k \le n \\ &i^{n+k}_{m_k} - t_k \le \langle m, n(\rho_{n+k})\rangle < i^{n+k}_{m_{n+k}+1} - t_k && \text{for } 1 \le k \le r-n. \end{aligned}$$

Therefore we have

$$\mathrm{H}^0(X, \mathcal{E}(\underline{t}))_m \cong \bigcap_{k=1}^r E^k_{\lambda_k}$$

and (i) follows.

By construction, the polytope $\Omega(\underline{t})$ is covered by the subpolytopes $\{\Omega_{\underline{\lambda}}(\underline{t})\}_{\underline{\lambda} \in \{1,\dots,\ell\}^r}$ therefore, (ii) is a consequence of Proposition 3.1.1. On the other hand, (iii) follows from (i), since we have that

$$\mathrm{h}^0(X, \mathcal{E}(\underline{t})) = \sum_{m \in \Omega(\underline{t})} \dim(\mathrm{H}^0(X, \mathcal{E}(\underline{t}))_m).$$

Finally, for $\underline{t} \gg 0$, we have:

$$\mathrm{h}^0(X, \mathcal{E}(\underline{t})) = \chi(\mathcal{E}(\underline{t})).$$

From this, (3.6) follows. □

Remark 3.1.3. After reversing some signs in the construction (see Remark 1.3.4), the family of polytopes $\{\Omega(\underline{t})_{\underline{\lambda}}\}_{\underline{\lambda} \in \{1,\dots,\ell\}^r}$ contains the *Parliament of polytopes*, introduced in [21] for studying positivity properties of equivariant vector bundles on a toric variety.

3.2 Equivariant reflexive sheaves on toric varieties with splitting fans

In this section, we study in more detail reflexive sheaves on smooth toric varieties with $1-$splitting fans. Particular instances of these kind of toric varieties are products of projective spaces $\mathbb{P}^r \times \mathbb{P}^s$, the Hirzebruch surfaces and projective bundles $\mathbb{P}(\bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^s}(a_i))$.

Definition 3.2.1. Let Σ be the fan of a $d-$dimensional smooth complete toric variety.

i) A set of rays $\{\rho_1, \ldots, \rho_k\}$ is called a *primitive collection* if it holds

$$\mathrm{cone}(n(\rho_1), \ldots, n(\rho_k)) \notin \Sigma,$$

but for any $1 \leq j \leq k$, we have

$$\mathrm{cone}(n(\rho_1), \ldots, \widehat{n(\rho_j)}, \ldots, n(\rho_k)) \in \Sigma.$$

ii) If $\{\rho_1, \ldots, \rho_k\}$ is a primitive collection, then there is a unique cone $\sigma = \mathrm{cone}(v_1, \ldots, v_t) \in \Sigma$ and integers $a_1, \ldots, a_t > 0$ such that

$$n(\rho_1) + \cdots + n(\rho_k) = a_1 v_1 + \cdots + a_t v_t,$$

which is called the associated *primitive relation*.

iii) We say that Σ is a *splitting fan* if any two primitive collections have no common elements.

The following proposition gives a complete geometric description of the toric varieties associated to an splitting fan:

Proposition 3.2.2. *Let Σ be the fan of a $d-$dimensional smooth complete toric variety. The fan Σ is a splitting fan if and only if there is a sequence of toric varieties $X = X_k, \ldots, X_0$ such that $X_0 = \mathbb{P}^n$ for some n and for $1 \leq i \leq k$, X_i is a projectivization of a decomposable vector bundle over X_{i-1}. Moreover, we say then that the fan is $k-$splitting.*

Proof. See [4, Theorem 4.3 and Corollary 4.4]. $\square$

In particular, a toric variety has a 1−splitting fan if and only if X is a $(r+s)$−dimensional toric variety

$$V_s(a_1,\ldots,a_r)=\mathbb{P}(\mathcal{O}_{\mathbb{P}^s}\oplus\mathcal{O}_{\mathbb{P}^s}(a_1)\oplus\cdots\oplus\mathcal{O}_{\mathbb{P}^s}(a_r)).$$

Let us introduce the fan $\Sigma\subset\mathbb{R}^{r+s}$ of $V_s(a_1,\ldots,a_r)$. We take the standard basis $\{e_1,\ldots,e_s,f_1,\ldots,f_r\}$ for $\mathbb{Z}^{s+r}$, and we define

$$\begin{aligned}
\rho_0 &:= \operatorname{cone}(-e_1-\cdots-e_s+a_1f_1+\cdots+a_rf_r)\\
\rho_i &:= \operatorname{cone}(e_i)\quad \text{for } 1\le i\le s\\
\eta_0 &:= \operatorname{cone}(-f_0-\cdots-f_r)\\
\eta_j &:= \operatorname{cone}(f_j)\quad \text{for } 1\le j\le r.
\end{aligned}$$

For each $1\le i\le s$ and $1\le j\le r$, we define the $r+s$−dimensional cones

$$\sigma_{ij}:=\operatorname{cone}(n(\rho_0),\ldots,\widehat{n(\rho_i)},\ldots,n(\rho_s),n(\eta_0),\ldots,\widehat{n(\eta_j)},\ldots,n(\eta_r)).$$

Then, the fan Σ is characterized by

$$\Sigma_{max}=\{\sigma_{ij}\,|\,1\le i\le s,\ 1\le j\le r\},$$

and the set of rays is

$$\Sigma(1)=\{\rho_0,\ldots,\rho_s,\eta_0,\ldots,\eta_r\}.$$

By Proposition 1.1.17, the class group of $X=V_s(a_1,\ldots,a_r)$ is given by $\mathrm{Cl}(X)\cong\operatorname{coker}\phi$ where

$$\phi:\mathbb{Z}^{s+r}\to\mathbb{Z}^{s+r+2}$$

is given by the following matrix:

$$\begin{pmatrix}
-1 & \cdots & -1 & a_1 & \cdots & a_r\\
1 & \cdots & 0 & 0 & \cdots & 0\\
\vdots & \ddots & \vdots & \vdots & & \vdots\\
0 & \cdots & 1 & 0 & \cdots & 0\\
0 & \cdots & 0 & -1 & \cdots & -1\\
0 & \cdots & 0 & 1 & \cdots & 0\\
\vdots & & \vdots & \vdots & \ddots & \vdots\\
0 & \cdots & 0 & 0 & \cdots & 1
\end{pmatrix}$$

In particular, $\mathrm{Cl}(X) \cong \mathbb{Z}\langle [D_{\rho_0}], [D_{\eta_0}]\rangle$ and we have

$$[D_{\rho_i}] = [D_{\rho_0}] \quad \text{for} \quad 1 \leq i \leq s, \tag{3.8}$$

and

$$[D_{\eta_j}] = -a_j[D_{\rho_0}] + [D_{\eta_0}] \quad \text{for} \quad 1 \leq j \leq r.$$

Remark 3.2.3. There is a conflicting notation in the literature on projective bundles. We define the projective bundle of a vector bundle $\mathcal{E}$ by

$$\mathbb{P}(\mathcal{E}) := \mathrm{Proj}(\mathrm{Sym}\,\mathcal{E}),$$

as found in [20]. In this setting, $[D_{\rho_0}]$ is the class of the projective fiber $\pi^*\mathcal{O}_{\mathbb{P}^s}(1)$ and $[D_{\eta_0}]$ is the class of the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. It is worthwile noticing, to avoid any confusion, that in some texts, like [27], some authors define a projective bundle as "$\mathbb{P}(\mathcal{E}) = \mathrm{Proj}(\mathrm{Sym}\,\mathcal{E}^\vee)$" that would be, in our notation, $\mathbb{P}(\mathcal{E}^\vee)$.

A line bundle $\mathcal{O}_X(a,b) := \mathcal{O}_X(a[D_{\rho_0}] + b[D_{\eta_0}])$ is ample (respectively, nef) if and only if $a, b > 0$ (respectively, $a, b \geq 0$). The anticanonical divisor is given by

$$\begin{aligned} -K_X &= D_{\rho_0} + \cdots + D_{\rho_s} + D_{\eta_0} + \cdots + D_{\eta_r} \\ &= (s + 1 - a_1 - \cdots - a_r)D_{\rho_0} + (r+1)D_{\eta_0}. \end{aligned}$$

In particular, X is Fano (i.e. $-K_X$ is ample) if and only if

$$a_1 + \cdots + a_r < s + 1.$$

On the other hand, since

$$[D_{\rho_0}] \cdots [D_{\rho_s}] = [D_{\eta_0}] \cdots [D_{\eta_r}] = 0,$$

we have that $[D_{\eta_0}]([D_{\eta_0}] - a_1[D_{\rho_0}]) \cdots ([D_{\eta_0}] - a_r[D_{\rho_0}]) = 0$ and $[D_{\rho_0}]^k = 0$ for $k \geq s+1$. From this we deduce that

$$[D_{\rho_0}]^{s-j}[D_{\eta_0}]^{r+j} = \begin{cases} 0 & j < 0 \\ s_0 = 1 & j = 0 \\ s_j = \sigma_1 s_{j-1} - \sigma_2 s_{j-1} + \cdots + (-1)^{j+1}\sigma_j s_0 & 1 \leq j \leq \min\{r,s\} \\ s_j = \sigma_1 s_{r-1} - \sigma_2 s_{j-1} + \cdots + (-1)^{r+1}\sigma_r s_0 & \min\{r,s\} < j \leq s \end{cases} \tag{3.9}$$

where $\sigma_k = \sigma_k(a_1, \ldots, a_r) = \sum_{1 \leq i_1 < \cdots < i_k \leq r} a_{i_1} \cdots a_{i_k}$ are the elementary symmetric polynomials.

Finally, we introduce the following notation to work with equivariant reflexive sheaves on X, which takes advantage on the splitting fan structure.

Notation 3.2.4. Let $\mathcal{E}$ be an equivariant reflexive sheaf of rank ℓ on X. We write its associated filtration as follows:

$$\begin{aligned} \hat{E}^{\rho_k} &= F^k(i_1^k, \ldots, i_\ell^k; F_1^k, \ldots, F_\ell^k), \quad \text{for } 0 \leq k \leq r \\ \hat{E}^{\eta_k} &= G^k(j_1^k, \ldots, j_\ell^k; G_1^k, \ldots, G_\ell^k), \quad \text{for } 0 \leq k \leq s. \end{aligned}$$

For any line bundle $\mathcal{O}(p,q)$, we set $\mathcal{E}(p,q) := \mathcal{E} \otimes \mathcal{O}(p,q)$. By Proposition 1.3.8, $\mathcal{E}(p,q)$ has the same filtrations as $\mathcal{E}$ except from the rays ρ_0 and η_0. For these rays we have:

$$\begin{aligned} \hat{E}^{\rho_0}(p,q) &= F^0(i_1^0 - p, \ldots, i_\ell^0 - p; F_1^0, \ldots, F_\ell^0) \\ \hat{E}^{\eta_0}(p,q) &= G^0(j_1^0 - q, \ldots, j_\ell^0 - q; G_1^0, \ldots, G_\ell^0). \end{aligned}$$

3.2.1 Lower and upper bounds for the multigraded initial degree

In this section, following Proposition 3.3 we ask for any equivariant reflexive sheaf on a smooth complete toric variety X with a $1-$splitting fan, for which integers $(p,q) \in \mathbb{Z}^2$ we have

$$\mathrm{H}^0(X, \mathcal{E}(p,q)) \neq 0.$$

Precisely, we address finding the *multigraded initial degree* of $\mathcal{E}$, which is a generalization of the notion of the *initial degree* of a reflexive sheaf $\mathcal{G}$ on $\mathbb{P}^n$, defined as:

$$\operatorname{indeg}(\mathcal{G}) = \min\{t \in \mathbb{Z} \mid \mathrm{H}^0(\mathbb{P}^n, \mathcal{G}(t)) \neq 0\}.$$

Recall that when X is a smooth toric variety, by Proposition 1.1.17, we have $\mathrm{Cl}(X) \cong \mathbb{Z}^c$, but often $c > 1$. Following [50], we introduce the following notion of *multigraded initial degree*:

$$\operatorname{indeg}(\mathcal{G}) = \{\alpha \in \mathrm{Cl}(X) \mid \mathrm{H}^0(X, \mathcal{G}(\alpha)) \neq 0\}.$$

Given a set $\mathcal{A} \subset \mathbb{Z}^c$, we say that $B \subset \mathbb{Z}^c$ is a *lower bound* (respectively *upper bound*) of $\mathcal{A}$ if $\mathcal{A} \subset B$ (respectively $B \subset \mathcal{A}$). Our main goal, in this subsection, is to use the splitting fan structure of $X = V_s(a_1, \dots, a_r)$ to bound the multigraded initial degree of an equivariant reflexive sheaf on X.

We fix an equivariant reflexive sheaf $\mathcal{E}$ with filtrations as in Notation 3.2.4. By Proposition 3.1.1 the multigraded initial degree of $\mathcal{E}$ is related the study of the lattice points in the polytope $\Omega(p,q)$, which in this case is defined by the following inequalities:

$$\begin{pmatrix} -1 & \cdots & -1 & a_1 & \cdots & a_r \\ 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -1 & \cdots & -1 \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix} \cdot x \geq \begin{pmatrix} i_1^0 - p \\ i_1^1 \\ \vdots \\ i_1^k \\ \vdots \\ i_1^s \\ j_1^0 - q \\ j_1^1 \\ \vdots \\ j_1^r \end{pmatrix}. \tag{3.10}$$

Moreover, let us recall that for any $(p,q) \in \mathbb{Z}^2$ and any character $m \in M$, we have (see (3.7)):

$$\begin{aligned}[\mathrm{H}^0(X, \mathcal{E}(p,q)]_m \cong \hat{E}^{\rho_0}(\langle m, \rho_0\rangle + p) \cap \hat{E}^{\rho_1}(\langle m, \rho_1\rangle) \cap \cdots \cap \hat{E}^{\rho_s}(\langle m, \rho_s\rangle) \cap \\ \hat{E}^{\eta_0}(\langle m, \eta_0\rangle + q) \cap \hat{E}^{\eta_1}(\langle m, \eta_1\rangle) \cap \cdots \cap \hat{E}^{\eta_r}(\langle m, \eta_r\rangle).\end{aligned} \tag{3.11}$$

To find the bounds of $\operatorname{indeg}(\mathcal{E})$, the following lemmas will be useful.

Lemma 3.2.5. *Let $0 \leq a_1 \leq \cdots \leq a_r$, A and $B \geq 0$ be integers. The system*

$$\left.\begin{aligned} a_1x_1 + \cdots + a_rx_r \geq A \\ x_1 + \cdots + x_r \leq B \end{aligned}\right\} \tag{3.12}$$

has a solution in $\mathbb{Z}^r_{\geq 0}$ if and only if $A \leq a_rB$.

Proof. First, we assume that $(x_1, \ldots, x_r) \in \mathbb{Z}^r_{\geq 0}$ is a solution of (3.12). In particular,

$$x_1 + \cdots + x_r \leq B \quad \text{and} \quad A \leq a_1 x_1 + \cdots + a_r x_r.$$

Now, since $0 \leq a_1 \leq \cdots \leq a_r$, we get

$$A \leq a_1 x_1 + \cdots + a_r x_r \leq a_r x_1 + \cdots + a_r x_r \leq a_r B,$$

yielding $A \leq a_r B$. Conversely, if $A \leq a_r B$ then $(0, \ldots, 0, B) \in \mathbb{Z}^r_{\geq 0}$ is a solution of (3.12). $\square$

Lemma 3.2.6. *Let $0 \leq a_1 \leq \cdots \leq a_r$, A and B be integers. The system*

$$\begin{pmatrix} 1 & \cdots & 1 & -a_1 & \cdots & -a_r \\ 0 & \cdots & 0 & 1 & \cdots & 1 \end{pmatrix} \cdot \underline{e} \leq \begin{pmatrix} -A \\ B \end{pmatrix} \tag{3.13}$$

has a solution $\underline{e} \in \mathbb{Z}^{s+r}_{\geq 0}$ if and only if $B \geq 0$ and the system

$$\left.\begin{aligned} a_1 x_1 + \cdots + a_r x_r \geq & A \\ x_1 + \cdots + x_r \leq & B \end{aligned}\right\} \tag{3.14}$$

has a solution in $\mathbb{Z}^r_{\geq 0}$.

Proof. First we assume that $(e_1, \ldots, e_{s+r}) \in \mathbb{Z}^{s+r}_{\geq 0}$ is a solution of (3.13). Then

$$0 \leq e_{s+1} + \cdots + e_{s+r} \leq B,$$

hence $B \geq 0$. On the other hand,

$$a_1 e_{s+1} + \cdots + a_r e_{s+r} \geq e_1 + \cdots + e_s + A \geq A.$$

Thus, $(e_{s+1}, \ldots, e_{s+r}) \in \mathbb{Z}^r_{\geq 0}$ is a solution of (3.14). Conversely, we assume that $B \geq 0$ and $(x_1, \ldots, x_r)$ is a solution of (3.14), then

$$(0, \ldots, 0, x_1, \ldots, x_r)$$

is a solution of (3.13). $\square$

Lemma 3.2.7. *Let $0 \le a_1 \le \cdots \le a_r$, $\lambda_0, \ldots, \lambda_s, \mu_0, \ldots, \mu_r$ be integers. The system*

$$\begin{pmatrix} -1 & \cdots & -1 & a_1 & \cdots & a_r \\ 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -1 & \cdots & -1 \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix} \cdot m \ge \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_s \\ \mu_0 \\ \mu_1 \\ \vdots \\ \mu_r \end{pmatrix} \tag{3.15}$$

has a solution $m \in \mathbb{Z}^{s+r}$ if and only if the system

$$\begin{pmatrix} 1 & \cdots & 1 & -a_1 & \cdots & -a_r \\ 0 & \cdots & 0 & 1 & \cdots & 1 \end{pmatrix} \cdot \underline{e} \le \begin{pmatrix} -\lambda_0 - \cdots - \lambda_s + a_1\mu_1 + \cdots + a_r\mu_r \\ -\mu_0 - \cdots - \mu_r \end{pmatrix} \tag{3.16}$$

has solutions in $\mathbb{Z}^{s+r}_{\ge 0}$.

Proof. It follows using the change of variables

$$e_1 = d_1 - \lambda_1, \ldots,\ e_s = d_s - \lambda_s,\ e_{s+1} = d_{s+1} - \mu_1, \ldots,\ e_{s+r} = d_{s+r} - \mu_r,$$

where $m = (d_1, \ldots, d_{s+r})$. □

Proposition 3.2.8. *Let $\mathcal{E}$ be an equivariant reflexive sheaf on X. Then, the set*

$$L_E := \left\{ (p,q) \middle| \begin{array}{l} q \ge j_1^0 + \cdots + j_1^r \\ p + a_r q \ge i_1^0 + \cdots + i_1^s + a_r j_1^0 + \\ \qquad (a_r - a_1) j_1^1 + \cdots + (a_r - a_{r-1}) j_1^{r-1} \end{array} \right\}$$

is a lower bound of $\operatorname{indeg}(\mathcal{E})$.

Proof. We see that $\operatorname{indeg}(\mathcal{E}) \subset L_E$. Let $(p,q) \in \mathbb{Z}^2$ be a degree such that $\mathrm{H}^0(X, \mathcal{E}(p,q)) \ne 0$. By Proposition 3.1.1, the polytope $\Omega(p,q)$ defined in (3.10) cointains a lattice point $m \in \mathbb{Z}^{s+r}$. In other words, m is a solution of (3.15) setting

$$\lambda_0 = i_1^0 - p,\ \lambda_1 = i_1^1,\ \ldots,\ \lambda_s = i_1^s,\ \mu_0 = j_1^0 - q,\ \mu_1 = j_1^1,\ \ldots,\ \mu_r = j_1^r.$$

Then, by Lemma 3.2.7 the system

$$\left.\begin{array}{r} e_1 + \cdots + e_s - a_1 e_{s+1} + \cdots + a_r e_{s+r} \leq i_1^0 + \cdots + i_1^s - p - a_1 j_1^1 - \cdots - a_r j_1^r \\ e_{s+1} + \cdots + e_{s+r} \leq q - j_1^0 - \cdots - j_1^r \end{array}\right\}$$

has a solution in $\mathbb{Z}^{s+r}_{\geq 0}$. Finally, by Lemmas 3.2.6 and 3.2.5, this implies that

$$q - j_1^0 - \cdots - j_1^r \geq 0$$

and

$$i_1^0 + \cdots + i_1^s - p - a_1 j_1^1 - \cdots - a_r j_1^r \leq a_r(q - j_1^0 - \cdots - j_1^r).$$

The result then follows by rearranging this expression. □

To describe now an upper bound of $\operatorname{indeg}(\mathcal{E})$ we need to define the following subsets of $\mathbb{Z}^2$:

$$I(k) := \left\{ (p,q) \middle| \begin{array}{l} q \ \geq \ j_\ell^0 + \cdots + j_\ell^r \\ p + a_r q \geq i_\ell^0 + \cdots + i_1^k + \cdots + i_\ell^s + a_r j_\ell^0 + \\ \qquad\qquad (a_r - a_1) j_\ell^1 + \cdots + (a_r - a_{r-1}) j_\ell^{r-1} \end{array} \right\}$$

for $0 \leq k \leq s$.

$$J(k) := \left\{ (p,q) \middle| \begin{array}{l} q \ \geq \ j_\ell^0 + \cdots + j_1^k + \cdots + j_\ell^r \\ p + a_r q \geq i_\ell^0 + \cdots + i_\ell^s + a_r j_\ell^0 + (a_r - a_1) j_\ell^1 + \\ \qquad\qquad \cdots + (a_r - a_k) j_1^k \cdots + (a_r - a_{r-1}) j_\ell^{r-1} \end{array} \right\}$$

for $0 \leq k \leq r$.

Proposition 3.2.9. *Let $\mathcal{E}$ be an equivariant reflexive sheaf on X. If either $(p,q) \in I(k)$ for any $0 \leq k \leq s$, or $(p,q) \in J(k)$ for any $0 \leq k \leq r$, then $\mathrm{H}^0(X, \mathcal{E}(p,q)) \neq 0$. In particular,*

$$U_E := \left(\bigcup_{k=0}^{s} I(k) \right) \cup \left(\bigcup_{k=0}^{r} J(k) \right)$$

is an upper bound of $\operatorname{indeg}(\mathcal{E})$.

Proof. If $(p,q) \in I(k)$ for some integer $0 \le k \le s$, we have that

$$\begin{cases} q - j_\ell^0 - \cdots - j_\ell^r \ge 0 \\ i_\ell^0 + \cdots + i_1^k + \cdots + i_\ell^s - p - a_1 j_\ell^1 - \cdots - a_r j_\ell^r \le a_r(q - j_\ell^0 - \cdots - j_\ell^r). \end{cases}$$

Hence, by Lemmas 3.2.5, 3.2.6 and 3.2.7, the system of inequalities

$$\begin{pmatrix} -1 & \cdots & -1 & a_1 & \cdots & a_r \\ 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -1 & \cdots & -1 \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix} \cdot m \ge \begin{pmatrix} i_\ell^0 - p \\ i_\ell^1 \\ \vdots \\ i_1^k \\ \vdots \\ i_\ell^s \\ j_\ell^0 - q \\ j_l^1 \\ \vdots \\ j_\ell^r \end{pmatrix}$$

has a solution $m \in \mathbb{Z}^{s+r}$. This implies that that

$$\dim \hat{E}^{\rho_k}(p,q)(\langle m, n(\rho_k)\rangle) \ge 1,$$

and that the remaining vector spaces intersecting in (3.11) are $\mathbb{C}^\ell$. In particular

$$\mathrm{H}^0(X, \mathcal{E}(p,q))_m \cong \hat{E}^{\rho_k}(p,q)(\langle m, n(\rho_k)\rangle) \ne 0.$$

Symmetrically, if $(p,q) \in J(k)$ for some integer $0 \le k \le r$, we obtain that

$$\mathrm{H}^0(X, \mathcal{E}(p,q))_m \cong \hat{E}^{\eta_k}(p,q)(\langle m, n(\eta_k)\rangle),$$

and $\dim \hat{E}^{\eta_k}(p,q)(\langle m, n(\eta_k)\rangle) \ge 1$. □

3.2.2 Bounding the multigraded regularity

In this subsection, we investigate in more detail the Hilbert polynomial of a reflexive equivariant sheaf $\mathcal{E}$ on a smooth complete toric variety with 1−splitting fan $X = V_s(a_1, \ldots, a_r)$. Using the results in Section 3.1, we find a bound of the *multigraded regularity index* of $\mathcal{E}$, which is at the same time a bound for its *multigraded Castelnuovo-Mumford regularity.*

Let us first recall some basic facts on the multigraded regularity of sheaves on a smooth complete toric variety X, as introduced in [51]. Let Σ be the fan of X, such that $\Sigma(1) = \{\rho_1, \ldots, \rho_r\}$, and let us denote R the Cox ring of X. We set

$$\mathcal{A} = \{[D_{\rho_1}], \ldots, [D_{\rho_r}]\} \subset \mathrm{Cl}(X),$$

the set of degrees of the variables of R. For any cone σ we define

$$\mathcal{A}_{\hat{\sigma}} = \{[D_{\rho_i}] \mid \rho_i \notin \sigma\} \subset \mathcal{A}$$

and the subsemigroup

$$\mathcal{K} = \bigcap_{\sigma \in \Sigma} \mathbb{N}\mathcal{A}_{\hat{\sigma}} \subset \mathbb{N}\mathcal{A},$$

which is saturated because X is smooth. If $\Sigma_{\max} = \{\sigma_1, \ldots, \sigma_s\}$ is the set of maximal cones in Σ, then we have

$$\mathcal{K} = \mathbb{N}\mathcal{A}_{\hat{\sigma_1}} \cap \cdots \cap \mathbb{N}\mathcal{A}_{\hat{\sigma_s}}.$$

We have the following algebraic result:

Proposition 3.2.10. *Let X be a smooth complete toric variety and R its Cox ring. Let E be a finitely generated $\mathrm{Cl}(X)-$graded $R-$module*

(i) If $\alpha \in \mathcal{K}$, then $H^i_B(R)_\alpha = 0$ for all $i \geq 0$.

(ii) Assume that E has a $\mathrm{Cl}(X)-$graded minimal free resolution

$$0 \to \bigoplus_i R(-\alpha_{i,t}) \to \bigoplus_i R(-\alpha_{i,t-1}) \to \cdots$$
$$\cdots \to \bigoplus_i R(-\alpha_{i,1}) \to \bigoplus_i R(-\alpha_{i,0}) \to E \to 0.$$

If $\beta \in \bigcap_{i,j}(\alpha_{i,j} + \mathcal{K})$, then $H^i_B(E)_\beta = 0$ for $i \geq 0$.

(iii) There is a polynomial P_E in k variables such that

$$h_E(\alpha) - P_E(\alpha) = \sum_{i=0}^{d} (-1)^i \dim H^i_B(E)_\alpha.$$

Proof. (i) and (ii) follow from [50, Corollary 3.6]. For (iii), see [51, Proposition 2.8 and Proposition 2.14]. □

The polynomial P_E is then the *multigraded Hilbert polynomial* of E. In particular, by Proposition 3.2.10 (ii) and the arithmetic Nullstellensatz it follows that $P_E = P_{\Gamma E}$, and by Proposition 1.2.7 we have that

$$P_E(\alpha) = P_{\tilde{E}}(\alpha) = \chi(\tilde{E}(\alpha)),$$

for any $\alpha \in \mathrm{Cl}(X)$. We define the *multigraded regularity index* of E to be

$$\mathrm{r.\,i.}(E) = \{\alpha \in \mathrm{Cl}(X) \mid h_E(\alpha) = P_E(\alpha)\}.$$

In particular, by Proposition 3.2.10 (ii) and (iii) the set

$$\bigcap_{i,j} (\alpha_{i,j} + \mathcal{K})$$

is a lower bound of $\mathrm{r.\,i.}(E)$. If $\mathrm{Cl}(X) \cong \mathbb{Z}$, we abuse notation and write

$$\mathrm{r.\,i.}(E) := \min\{i \in \mathbb{Z} \mid h_E(i) = P_E(i)\},$$

and the definition coincides with the notion of regularity index found in [63] or [40] for the standard $\mathbb{Z}-$graded polynomial ring. Analogously, if $\mathcal{E}$ is a quasi-coherent sheaf on X we define its multigraded regularity index as

$$\mathrm{r.\,i.}(\mathcal{E}) = \{\alpha \in \mathrm{Cl}(X) \mid \mathrm{h}^0(X, \mathcal{E}(\alpha) = P_{\mathcal{E}}(\alpha)\}.$$

Following [50], the notion of Castelnuovo-Mumford multigraded regularity is closely related to the multigraded regularity index. We denote by $\mathcal{C} = \{c_1, \dots, c_s\} \subset \mathcal{K}$ the unique minimal generating subset such that $\mathcal{K} = \mathbb{N}\mathcal{C}$, which is called a *Hilbert basis* (see [60, IV.16.4]). For any integer $i \geq 0$ we set

$$\mathbb{N}\mathcal{C}[i] = \bigcup (-\lambda_1 c_1 - \cdots - \lambda_s c_s + \mathcal{K}),$$

where the union runs over all $\lambda_1, \dots, \lambda_s \in \mathbb{N}$ such that $\lambda_1 + \cdots + \lambda_s = i$.

Definition 3.2.11. Let X be a smooth complete toric variety, and let $\mathcal{E}$ be a quasi-coherent sheaf on X. For $\alpha \in \mathrm{Cl}(X)$, the quasi-coherent sheaf $\mathcal{E}$ is $\alpha-$*regular* if

$$H^i(X, \mathcal{E}(\beta)) = 0 \text{ for any } i \geq 1 \text{ and all } \beta \in \alpha + \mathbb{N}\mathcal{C}[i-1].$$

The set $\mathrm{reg}(\mathcal{E}) = \{\alpha \in \mathrm{Cl}(X) \mid \mathcal{E} \text{ is } \alpha - \text{regular}\}$ is called the *multigraded Castelnuovo-Mumford regularity* of $\mathcal{E}$.

On the other hand, we define

$$\mathrm{reg}(\mathcal{E})^\circ = \bigcup_{\alpha \in \mathrm{reg}(E)} ((\alpha + \mathcal{K}) \setminus \{\alpha\}) \subset \mathrm{reg}(E).$$

The following result relates the multigraded Castelnuovo-Mumford regularity with the multigraded regularity index:

Proposition 3.2.12. *Let X be a smooth complete toric variety. If a quasi-coherent sheaf $\mathcal{E}$ is $\alpha-$regular, then*

$$(\alpha + \mathcal{K}) \setminus \{\alpha\} \subset \mathrm{r.i.}(\mathcal{E}).$$

In particular $\mathrm{reg}(\mathcal{E})^\circ \subset \mathrm{r.i.}(\mathcal{E})$, *and thus* $\mathrm{r.i.}(\mathcal{E})$ *is a lower bound of* $\mathrm{reg}(\mathcal{E})^\circ$.

Proof. See [51, Corollary 2.15]. □

In the last part of this subsection, we deal with smooth complete toric varieties with $1-$splitting fan. We use the splitting fan structure of $X = V_s(a_1, \dots, a_r)$ to find a bound for the multigraded regularity index $\mathrm{r.i.}(\mathcal{E})$ by means of Proposition 3.1.2 (iii). We introduce the following notation, following the splitting structure of the fan. Let $\underline{m} \in \{1, \dots, \ell\}^s$ and $\underline{n} \in \{1, \dots, \ell\}^r$ and we write $\underline{m} := (m_0, \dots, m_s)$ and $\underline{n} := (n_0, \dots, n_r)$. The polytope $\Omega_{(\underline{m}, \underline{n})}(p, q)$ defined in (3.5) is given by the following system of linear inequalities:

$$\begin{pmatrix} i^0_{m_0+1}-p \\ i^1_{m_1+1} \\ \vdots \\ i^s_{m_s+1} \\ j^0_{n_0+1}-q \\ j^1_{n_1+1} \\ \vdots \\ j^r_{n_r+1} \end{pmatrix} > \begin{pmatrix} -1 & \cdots & -1 & a_1 & \cdots & a_r \\ 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -1 & \cdots & -1 \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix} \cdot x \geq \begin{pmatrix} i^0_{m_0}-p \\ i^1_{m_1} \\ \vdots \\ i^s_{m_s} \\ j^0_{n_0}-q \\ j^1_{n_1} \\ \vdots \\ j^r_{n_r} \end{pmatrix} \tag{3.17}$$

as before, we set $i^k_{\ell+1} = j^k_{\ell+1} := \infty$ and

$$\Psi_{(\underline{m},\underline{n})}(p,q) := \Omega_{(\underline{m},\underline{n})}(p,q) \cap \mathbb{Z}^{s+r}.$$

By Proposition 3.1.2 we have to study the behaviour of $|\Psi_{(\underline{m},\underline{n})}(p,q)|$ with respect to (p,q). We start by splitting the system (3.17) into two smaller systems. Let us first define $\Psi_{\underline{n}}(q) \subset \mathbb{Z}^r$ as the set of integer solutions of the system:

$$\begin{pmatrix} j^0_{n_0+1} \\ j^1_{n_1+1} \\ \vdots \\ j^r_{n_r+1} \end{pmatrix} > \begin{pmatrix} -1 & \cdots & -1 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \cdot \underline{c} \geq \begin{pmatrix} j^0_{n_0}-q \\ j^1_{n_1} \\ \vdots \\ j^r_{n_r} \end{pmatrix}. \tag{3.18}$$

For any $\underline{c} := (c_1, \ldots, c_r) \in \Psi_{\underline{n}}(q)$ we set $A(\underline{c}) := a_1c_1 + \cdots + a_rc_r$ and we define $\Psi_{\underline{m}}(p;\underline{c}) \subset \mathbb{Z}^s$ to be the set of integer solutions of the system:

$$\begin{pmatrix} i^0_{m_0+1}-p-A(\underline{c}) \\ i^0_{m_1+1} \\ \vdots \\ i^0_{m_s+1} \end{pmatrix} > \begin{pmatrix} -1 & \cdots & -1 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \cdot m \geq \begin{pmatrix} i^0_{m_0}-p-A(\underline{c}) \\ i^0_{m_1} \\ \vdots \\ i^0_{m_s} \end{pmatrix}. \tag{3.19}$$

Thus,

$$\Psi_{\underline{m}}(p;\underline{c}) \times \{\underline{c}\} \subset \Psi_{(\underline{m},\underline{n})}(p,q),$$

and we can *slice* the set $\Psi_{(\underline{m},\underline{n})}(p,q)$ as follows:

$$\Psi_{(\underline{m},\underline{n})}(p,q) = \bigcup_{\underline{c}\in\Psi_{\underline{n}}(q)} \Psi_{\underline{m}}(p;\underline{c}) \times \{\underline{c}\}. \tag{3.20}$$

Along the following lemmas we study both systems (3.19) and (3.18), and when their cardinality of solutions behaves as a polynomial. We start with system (3.18).

Lemma 3.2.13. *If* $q \geq j^0_\ell+\cdots+j^r_\ell-1$ *and* $n_0,\ldots,n_r < \ell$*, then* $\Psi_{\underline{n}}(q)=\emptyset$.

Proof. Let $\underline{c}\in\mathbb{Z}^r$ be a solution of

$$\left.\begin{array}{c} j^1_{n_1+1} > c_1 \geq j^1_{n_1} \\ \vdots \\ j^r_{n_r+1} > c_r \geq j^r_{n_r} \end{array}\right\}.$$

In particular we have,

$$j^1_{n_1+1}+\cdots+j^r_{n_r+1} > c_1+\cdots+c_r \geq j^1_{n_1}+\cdots+j^r_{n_r}.$$

On the other hand, the assumption on q provides that

$$q-j^0_\ell \geq j^1_\ell+\cdots+j^r_\ell-1.$$

If $n_0,\ldots,n_r<\ell$, then $j^0_{n_0+1},\ldots,j^r_{n_r+1}<\infty$ and we have

$$q-j^0_{n_0+1} \geq q-j^0_\ell \geq j^1_\ell+\cdots+j^r_\ell-1 \geq j^1_{n_1+1}+\cdots+j^r_{n_r+1}-1.$$

Therefore we obtain,

$$q-j^0_{n_0+1} \geq j^1_{n_1+1}+\cdots+j^r_{n_r+1}-1 > c_1+\cdots+c_r-1.$$

Hence, $q-j^0_{n_0+1} \geq c_1+\cdots+c_r$ and it does not hold that

$$q-j^0_{n_0} \geq c_1+\cdots+c_r > q-j^0_{n_0+1}.$$

$\square$

Lemma 3.2.14. *Let us fix $\underline{n} \in \{1, \dots, \ell\}^r$ and $p, q \in \mathbb{Z}$ such that*

$$p \geq i^0_\ell + \cdots + i^s_\ell - a_1 j^1_1 - \cdots - a_r j^r_1 - 1.$$

If $m_0, \dots, m_s < \ell$, then $\Psi_{\underline{m}}(p; \underline{c}) = \emptyset$ for any solution $\underline{c} \in \Psi_{\underline{n}}(q)$.

Proof. Let $(d_1, \dots, d_s) \in \mathbb{Z}^s$ be a solution of

$$\left.\begin{array}{c} i^1_{m_1+1} > d_1 \geq i^1_{m_1} \\ \vdots \\ i^s_{m_s+1} > d_s \geq i^s_{m_s} \end{array}\right\}.$$

In particular we have,

$$i^1_{m_1+1} + \cdots + i^s_{m_s+1} > d_1 + \cdots + d_s \geq i^1_{m_1} + \cdots + i^s_{m_s}.$$

On the other hand, the assumption on p implies that

$$p - i^0_\ell \geq i^1_\ell + \cdots + i^s_\ell - (a_1 j^1_1 + \cdots + a_r j^r_1) - 1.$$

Let us fix any solution $(c_1, \dots, c_r) \in \Psi_{\underline{n}}(q)$. If $m_0, \dots, m_s < \ell$, then $i^0_{m_0+1}, \dots, i^s_{m_s+1} < \infty$ and we have

$$\begin{aligned} p - i^0_{m_0+1} \geq p - i_\ell &\geq i^1_\ell + \cdots + i^s_\ell - (a_1 j^1_1 + \cdots + a_r j^r_1) - 1 \\ &\geq i^1_{m_1+1} + \cdots + i^s_{m_s+1} - (a_1 j^1_{n_1} + \cdots + a_r j^r_{n_r}) - 1 \\ &\geq i^1_{m_1+1} + \cdots + i^s_{m_s+1} - (a_1 c_1 + \cdots + a_r c_r) - 1 \end{aligned}$$

Therefore,

$$\begin{aligned} p - i^0_{m_0+1} &\geq i^1_{m_1+1} + \cdots + i^s_{m_s+1} - (a_1 c_1 + \cdots + a_r c_r) - 1 \\ &> d_1 + \ldots + d_s - (a_1 c_1 + \cdots + a_r c_r) - 1. \end{aligned}$$

Hence we obtain,

$$p - i^0_{m_0+1} \geq d_1 + \ldots + d_s - (a_1 c_1 + \cdots + a_r c_r)$$

implying that $(d_1, \dots, d_s)$ is not a solution of

$$p - i^0_{m_0} \geq d_1 + \cdots + d_s - (a_1 c_1 + \cdots + a_r c_r) > p - i^0_{m_0+1}.$$

$\square$

Lemma 3.2.15. *Let us fix* $\underline{m} \in \{1, \dots, \ell\}^s$, $\underline{n} \in \{1, \dots, \ell\}^r$ *and* $\underline{c} \in \Psi_{\underline{n}}(q)$ *a solution. If*

$$p \geq i_\ell^0 + \cdots + i_\ell^s - a_1 j_1^1 - \cdots - a_r j_1^r - 1,$$

then $|\Psi_{\underline{m}}(p; \underline{c})|$ *is a polynomial in* $\mathbb{C}[p, c_1, \dots, c_r]$.

Proof. By Lemma 3.2.14, we assume that $m_k = \ell$ for at least one integer $0 \leq k \leq s$. Let us first assume that

$$m_0 = m_{k_1} = \ldots = m_{k_t} = \ell,$$

for $0 \leq t \leq s$. Up to permutation of indices we can assume that

$$k_1 = 1, \dots, k_t = t.$$

Then $|\Psi_{\underline{m}}(p; \underline{c})|$ is the number of integer solutions of the system (3.19), which is equivalent to

$$\left.\begin{array}{rcl} p - i_\ell^0 \geq & d_1 + \cdots + d_s - A(\underline{c}) & > -\infty \\ \infty > & d_1 & \geq i_\ell^1 \\ & \vdots & \\ \infty > & d_t & \geq i_\ell^t \\ i_{m_{t+1}+1}^{t+1} > & d_{t+1} & \geq i_{m_{t+1}}^{t+1} \\ & \vdots & \\ i_{m_s+1}^s > & d_s & \geq i_{m_s}^s \end{array}\right\}.$$

After the change of variables $e_k = d_k - i_\ell^k$ for $1 \leq k \leq t$, we have that

$$|\Psi_{\underline{m}}(p; \underline{c})| = \sum_{d_{t+1} = i_{m_{t+1}}^{t+1}}^{i_{m_{t+1}+1}^{t+1} - 1} \cdots \sum_{d_s = i_{m_s}^s}^{i_{m_s+1}^s - 1} \binom{t + p - i_\ell^0 - i_\ell^1 - \cdots - i_\ell^t + A(\underline{c}) - d_{t+1} - \cdots - d_s}{t}$$

which is a polynomial. Indeed, fix any integer $(s-t)-$uple $(d_{t+1}, \dots, d_s)$ such that

$$i_{m_{t+1}+1}^{t+1} > d_{t+1} \geq i_{m_{t+1}}^{t+1}, \dots, i_{m_s+1}^s > d_s \geq i_{m_s}^s.$$

From the bound on p in the hypothesis, we have

$$p - i^0_\ell - i^1_\ell - \cdots - i^t_\ell + A(\underline{c}) - d_{t+1} - \cdots - d_s \geq 0,$$

which implies that

$$\binom{t + p - i^0_\ell - i^1_\ell - \cdots - i^t_\ell + A(\underline{c}) - d_{t+1} - \cdots - d_s}{t}$$

is a polynomial in $\mathbb{C}[p, c_1, \ldots, c_r]$, and the claim follows.

To finish the proof of the Lemma, we need to consider the case

$$1 \leq m_0 < \ell.$$

Up to permutation of indices we assume that $m_1 = \ldots = m_t = \ell$, but in this case $1 \leq t \leq s$. After the change of variables $e_1 := d_1 + \cdots + d_s$, the system (3.19) is equivalent to

$$\left.\begin{array}{rcl}
p - i^0_{m_0} \geq & e_1 - A(\underline{c}) & > p - i^0_{m_0+1} \\
e_1 - i^1_\ell - d_{t+1} - \cdots - d_s \geq & d_2 + \cdots + d_t & > -\infty \\
\infty > & d_2 & \geq i^2_\ell \\
 & \vdots & \\
\infty > & d_t & \geq i^t_\ell \\
i^{t+1}_{m_{t+1}+1} > & d_{t+1} & \geq i^{t+1}_{m_{t+1}} \\
 & \vdots & \\
i^s_{m_s+1} > & d_s & \geq i^s_{m_s}
\end{array}\right\}.$$

Thus,

$$\begin{aligned}
|\Psi_{\underline{m}}(p; \underline{c})| &= \sum_{e_1 = p - i^0_{m_0} + A(\underline{c}) + 1}^{p - i_{m^0_0+1} + A(\underline{c})} \sum_{d_{t+1} = i^{t+1}_{m_{t+1}}}^{i^{t+1}_{m_{t+1}+1} - 1} \cdots \\
&\quad \cdots \sum_{d_s = i^s_{m_s}}^{i^s_{m_s+1} - 1} \binom{t - 1 + e_1 - i^1_\ell - \cdots - i^t_\ell - d_{t+1} - \cdots - d_s}{t - 1}.
\end{aligned}$$

Since

$$\begin{aligned} e_1 - i^1_\ell - \cdots - i^t_\ell - d_{t+1} - \cdots - d_s &\\ &> p - i^0_{m_0} - i^1_\ell - \cdots - i^t_\ell - d_{t+1} - \cdots - d_s + A(\underline{c}) \\ &\geq p - i^0_\ell - \cdots - i^s_\ell + a_1 j^1_1 + \cdots + a_r j^r_1 \geq 0, \end{aligned}$$

then

$$P(e_1) = \sum_{d_{t+1}=i^{t+1}_{m_{t+1}}}^{i^{t+1}_{m_{t+1}+1}-1} \cdots \sum_{d_s=i^s_{m_s}}^{i^s_{m_s+1}-1} \binom{t-1+e_1-i^1_\ell-\cdots-i^t_\ell-d_{t+1}-\cdots-d_s}{t-1}.$$

is a polynomial in $\mathbb{C}[e_1]$. Therefore,

$$|\Psi_p(\underline{m},\underline{c})| = \sum_{e_1=p-i^0_{m_0}+A(\underline{c})+1}^{p-i^0_{m_0+1}+A(\underline{c})} P(e_1) = \sum_{k=0}^{i^0_{m_0}-i^0_{m_0+1}} P(k+p-i^0_{m_0}+A(\underline{c})),$$

which is a finite sum of polynomials in $\mathbb{C}[p, c_1, \ldots, c_r]$. $\square$

Definition 3.2.16. The nth Bernoulli polynomial is defined recursively as

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k,$$

where $B_0 := 1$. In particular Bernoulli polynomials satisfy the well known Faulhaber identity:

$$\sum_{k=0}^{q} k^t = \frac{B_{t+1}(q+1) - B_{t+1}(1)}{t+1}.$$

Lemma 3.2.17. *If $P(q, e_1, \ldots, e_k) \in \mathbb{C}[q, e_1, \ldots, e_k]$ is a polynomial, then*

$$\sum_{\substack{e_1+\ldots+e_k \leq q \\ e_i \geq 0}} P(q, e_1, \ldots, e_k)$$

is a polynomial in $\mathbb{C}[q]$.

Proof. We proceed by induction over k. If $k = 1$, we want to see that the function

$$f(q) := \sum_{e_1=0}^{q} P(q, e_1)$$

is a polynomial. Notice that we can write

$$P(q, e_1) = P_0(q) + P_1(q)e_1 + \cdots + P_d(q)e_1^d$$

with $P_i \in \mathbb{C}[q]$, where $d = \deg_{e_1} P$. Hence,

$$f(q) = P_0(q) \sum_{e_1=0}^{q} 1 + P_1(q) \sum_{e_1=0}^{q} e_1 + \cdots + P_d(q) \sum_{e_1=0}^{q} e_1^d,$$

which is a $\mathbb{C}[q]-$linear combination of the Bernoulli polynomials

$$B_1(q), \ldots, B_{d+1}(q).$$

Now we assume, by induction, that the result is true for some $k \geq 1$. We want to see that the function

$$f(q) := \sum_{\substack{e_1+\ldots+e_{k+1} \leq q \\ e_i \geq 0}} P(q, e_1, \ldots, e_{k+1})$$

is a polynomial. We can write $f(q)$ as follows:

$$\begin{aligned}
f(q) &:= \sum_{\substack{e_1+\cdots+e_{k+1} \leq q \\ e_i \geq 0}} P(q, e_1, \ldots, e_{k+1}) \\
&= \sum_{\substack{e_1+\cdots+e_k \leq q \\ e_i \geq 0}} \sum_{e_{k+1}=0}^{q-(e_1+\cdots+e_k)} P(q, e_1, \ldots, e_k, e_{k+1}) \\
&= \sum_{\substack{e_1+\cdots+e_k \leq q \\ e_i \geq 0}} g(q, e_1, \ldots, e_k).
\end{aligned}$$

By induction it is enough to see that $g(p, q, e_1, \ldots, e_k)$ is a polynomial. We write

$$\begin{aligned}
&P(q, e_1, \ldots, e_k, e_{k+1}) = \\
&\quad P_0(q, e_1, \ldots, e_k) + P_1(q, e_1, \ldots, e_k)e_{k+1} + \cdots + P_d(q, e_1, \ldots, e_k)e_{k+1}^d
\end{aligned}$$

with $P_i \in \mathbb{C}[q, e_1, \ldots, e_k]$ and $d = \deg_{e_{k+1}} P$. Therefore,

$$g(q, e_1, \ldots, e_k) = P_0(q, e_1, \ldots, e_k) \sum_{e_{k+1}=0}^{q-(e_1+\cdots+e_k)} 1 + \\ P_1(q, e_1, \ldots, e_k) \sum_{e_{k+1}=0}^{q-(e_1+\cdots+e_{kG})} e_{k+1} + \cdots + P_d(q, e_1, \ldots, e_k) \sum_{e_{k+1}=0}^{q-(e_1+\cdots+e_k)} e_{k+1}^d.$$

Again, this is a $\mathbb{C}[q, e_1, \ldots, e_k]-$linear combination of the Bernoulli polynomials

$$B_1(q - (e_1 + \cdots + e_k) + 1), \ldots, B_{d+1}(q - (e_1 + \cdots + e_k) + 1).$$

Hence, we have $g \in \mathbb{C}[q, e_1, \ldots, e_k]$. $\square$

We are now ready to prove the main result of this subsection.

Theorem 3.2.18. *Let $\mathcal{E}$ be a reflexive sheaf of rank ℓ with associated filtrations as in Notation 3.2.4. If $p \geq i_\ell^0 + \cdots + i_\ell^s - a_1 j_1^1 - \cdots - a_r j_1^r - 1$ and $q \geq j_\ell^0 + \cdots + j_\ell^r - 1$, then $h^0(X, \mathcal{E}(p, q))$ is a polynomial. In particular, the set*

$$\left\{ (p, q) \mid \begin{array}{l} p \geq i_\ell^0 + \cdots + i_\ell^s - a_1 j_1^1 - \cdots - a_r j_1^r - 1 \\ q \geq j_\ell^0 + \cdots + j_\ell^r - 1 \end{array} \right\}$$

is an upper bound of r. i.$(\mathcal{E})$.

Proof. By Proposition 3.1.2 and (3.20), for any integers p, q we have

$$\begin{aligned} \mathrm{h}^0(X, \mathcal{E}(p, q)) &= \sum_{\substack{1 \leq m_0, \ldots, m_s \leq \ell \\ 1 \leq n_0, \ldots, n_r \leq \ell}} |\Psi_{(\underline{m}, \underline{n})}(p, q)| D(\underline{m}, \underline{n}) \\ &= \sum_{1 \leq n_0, \cdots, n_r \leq \ell} \sum_{\mathbf{c} \in \Psi_{\underline{n}}(q)} \sum_{1 \leq m_0, \ldots, m_s \leq \ell} |\Psi_{\underline{m}}(p; \underline{c})| D(\underline{m}, \underline{n}) \end{aligned}$$

where $D(\underline{m}, \underline{n}) := \dim(F_{m_0}^0 \cap \cdots \cap F_{m_s}^s \cap G_{n_0}^0 \cap \cdots \cap G_{n_r}^r)$. Rearranging the sum it yields:

$$\mathrm{h}^0(X, \mathcal{E}(p, q)) = \sum_{1 \leq m_0, \ldots, m_s \leq \ell} \sum_{1 \leq n_0, \ldots, n_r \leq \ell} \left[\sum_{\underline{c} \in \Psi_{\underline{n}}(q)} |\Psi_{\underline{m}}(p; \underline{c})| \right] D(\underline{m}, \underline{n}).$$

Hence, the proof reduces to see that if $p \geq i^0_\ell + \cdots + i^s_\ell - a_1 j^1_1 - \cdots - a_r j^r_1 - 1$ and $q \geq j^0_\ell + \cdots + j^r_\ell - 1$, the sum

$$f(\underline{m}, \underline{n}, p, q) := \sum_{\underline{c} \in \Psi_{\underline{n}}(q)} |\Psi_{\underline{m}}(p; \underline{c})|$$

is a polynomial in $\mathbb{C}[p, q]$ for any $\underline{m}, \underline{n}$. Since $q \geq j^0_\ell + \cdots + j^r_\ell - 1$, Lemma 3.2.14 applies. In particular, we can assume that there is an integer $0 \leq k \leq r$ such that $n_k = \ell$, otherwise $f(\underline{m}, \underline{n}, p, q) = 0$. Let us first suppose that $n_0 = \ell$ and, reordering if necessary, we have

$$n_1 = \cdots = n_t = \ell,$$

for $0 \leq t \leq r$. Then the system (3.18) is equivalent to

$$\left.\begin{array}{ccccc} q - j^0_\ell & \geq & c_1 + \cdots + c_r & > & -\infty \\ \infty & > & c_1 & \geq & j^1_\ell \\ \vdots & & & & \\ \infty & > & c_t & \geq & j^t_\ell \\ j^{t+1}_{n_{t+1}+1} & > & c_{t+1} & \geq & j^{t+1}_{n_{t+1}} \\ \vdots & & & & \\ j^r_{n_r+1} & > & c_r & \geq & j^r_{n_r} \end{array}\right\}.$$

Applying the change of variables $e_i = c_i - j^i_\ell$ for $1 \leq i \leq t$ to this system we obtain

$$f(\underline{m}, \underline{n}, p, q) = \sum_{c_r = j^r_{n_r}}^{j^r_{n_r+1}-1} \cdots \sum_{c_r = j^{t+1}_{n_{t+1}}}^{j^{t+1}_{n_{t+1}+1}-1} \sum_{\substack{e_1 + \cdots + e_t = 0 \\ e_i \geq 0}}^{q - j^0_\ell - \cdots - j^t_\ell - c_{t+1} - \cdots - c_r} |\Psi_p(\underline{m}; e_1 + j^1_\ell, \ldots, e_t + j^t_\ell, c_{t+1}, \ldots, c_r)|.$$

By Lemma 3.2.15, since $p \geq i^0_\ell + \cdots + i^s_\ell - a_1 j^1_1 - \cdots - a_r j^r_1 - 1$, we have that

$$|\Psi_p(\underline{m}; e_1 + j^1_\ell, \ldots, e_t + j^t_\ell, c_{t+1}, \ldots, c_r)|$$

is a polynomial in $\mathbb{C}[p, e_1, \ldots, e_t, c_{t+1}, \ldots, c_r]$. Therefore, applying Lemma 3.2.17,

$$P(p, q, c_{t+1}, \ldots, c_r) := \sum_{\substack{e_1 + \cdots + e_t \leq q - j^0_\ell - \cdots - j^t_\ell - c_{t+1} - \cdots - c_r \\ e_i \geq 0}} |\Psi_p(\underline{m}; e_1 + j^1_\ell, \ldots, e_t + j^t_\ell, c_{t+1}, \ldots, c_r)|$$

is a polynomial in $\mathbb{C}[p,q,c_{t+1},\dots,c_r]$, which implies that $f(\underline{m},\underline{n},p,q)\in\mathbb{C}[p,q]$. To finish the proof we need to consider the case $1\le n_0<\ell$. Without loss of generality we can assume that

$$n_1=\cdots=n_t=\ell,$$

but now with $1\le t\le r$. Thus, after the change of variables

$$e_1=c_1+\cdots+c_r,$$

the system (3.18) is equivalent to

$$\left.\begin{array}{rcl}
q-j^0_{n_0}\geq & e_1 & > \ q-j^0_{n_0+1}\\
e_1-c_{t+1}-\cdots-c_r-j^1_\ell\geq & c_2+\cdots+c_t+ & > \ -\infty\\
\infty> & c_2 & \geq \ j^2_\ell\\
\vdots & & \\
\infty> & c_t & \geq \ j^t_\ell\\
j^{t+1}_{n_{t+1}+1}> & c_{t+1} & \geq \ j^{t+1}_{n_{t+1}}\\
\vdots & & \\
j^r_{n_r+1}> & c_r & \geq \ j^r_{n_r}
\end{array}\right\}.$$

Defining $e_i:=c_i-j^i_\ell$ for $2\le i\le t$ we obtain

$$f(\underline{m},\underline{n},p,q)=\sum_{c_r=j^r_{n_r}}^{j^r_{n_r+1}-1}\cdots\sum_{c_r=j^{t+1}_{n_{t+1}}}^{j^{t+1}_{n_{t+1}+1}-1}\ \sum_{e_1=q-j^0_{n_0+1}+1}^{q-j^0_{n_0}}$$

$$\sum_{\substack{e_2+\cdots+e_t=0\\ e_i\geq 0}}^{e_1-j^1_\ell-\cdots-j^t_\ell-c_{t+1}-\cdots-c_r}|\Psi_p(\underline{m};e_1-e_2-j^2_\ell-\cdots-e_t-j^t_\ell,e_2+j^2_\ell,\dots,e_t+j^t_\ell,c_{t+1},\dots,c_r)|.$$

Analogously as in the previous case, this last equality proves that

$$f(\underline{m},\underline{n},p,q)\in\mathbb{C}[p,q],$$

since $p\geq i^0_\ell+\cdots+i^s_\ell-a_1j^1_1-\cdots-a_rj^r_1-1$. □

The following example shows the sharpness of this bound.

Example 3.2.19. Let $\mathcal{H}_3$ be the Hirzebruch surface as in Example 1.1.13. Let us fix $\{e_1, e_2, e_3\}$ a basis of $\mathbb{C}^3$ and we consider the following subspaces:

$$\begin{aligned} F_1^0 &= \langle 3e_1 + 3e_2 + e_3 \rangle & F_1^1 &= \langle 9e_1 + 4e_2 + 8e_3 \rangle \\ F_2^0 &= F_1^0 + \langle 4e_1 + 2e_3 \rangle & F_2^1 &= F_1^1 + \langle 2e_1 + 8e_2 + 8e_2 \rangle \\ G_1^0 &= \langle 6e_2 + 3e_3 \rangle & G_1^1 &= \langle 4e_1 + 4e_3 \rangle \\ G_2^0 &= G_1^0 + \langle 7e_1 + e_2 + 3e_3 \rangle & G_2^1 &= G_1^1 + \langle 9e_1 + 8e_2 \rangle. \end{aligned}$$

Let $\mathcal{E}$ be a rank 3 reflexive sheaf with set of filtrations

$$\begin{aligned} \hat{E}^{\rho_0} &= F^0(-3, -1, 0; F_1^0, F_2^0, \mathbb{C}^3) & \hat{E}^{\rho_1} &= F^1(-9, -3, 0; F_1^1, F_2^1, \mathbb{C}^3) \\ \hat{E}^{\eta_0} &= G^0(-4, -1, 0; G_1^0, G_2^0, \mathbb{C}^3) & \hat{E}^{\eta_1} &= G^1(-2, -1, 0; G_1^1, G_2^1, \mathbb{C}^3). \end{aligned}$$

Using the package ToricVectorBundles of Macaulay2 [34], we obtain the following values for $H^1(\mathcal{H}_3, \mathcal{E}(p, q))$ for $-4 \leq q \leq 4$ and $2 \leq q \leq 10$:

$$\begin{pmatrix} 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 11 & 10 & 9 & 8 & 8 & 8 & 8 & 8 & 8 \\ 24 & 24 & 24 & 24 & 24 & 24 & 24 & 24 & 24 \\ 31 & 33 & 35 & 37 & 39 & 41 & 43 & 45 & 47 \end{pmatrix}.$$

The region highlighted in the figure corresponds to the bound in Theorem 3.2.18 for this case: $p \geq 5$ and $q \geq -1$.

Notice that the filtration corresponding to $\mathcal{E}$ is *general*, that is a filtration such that any intersection of the form $\bigcap_{k=0}^{r} E^k_{n_k}$ has minimal dimension. Using Macaulay2, we have checked in many cases that the bound of Theorem 3.2.18 is sharp for reflexive sheaves with general filtrations.

3.3 Application: finding equivariant Ulrich bundles

In this section, we apply the tools about equivariant reflexive sheaves developed in the previous sections to address the problem of finding equivariant Ulrich bundles. First of all, let us start with some definitions and properties about Ulrich bundles, we follow mainly [17]. Let X be variety and H a very ample line bundle on X. For any $t \in \mathbb{Z}$ we set $\mathcal{E}(t) := \mathcal{E} \otimes tH$.

Definition 3.3.1. Let $\mathcal{E}$ be a vector bundle on X.

(i) $\mathcal{E}$ is called *initialized* if

$$\mathrm{H}^0(X, \mathcal{E}) \neq 0 \quad \text{and} \quad \mathrm{H}^0(X, \mathcal{E}(t)) = 0 \quad \text{for all} \quad t < 0.$$

ii) $\mathcal{E}$ is called *arithmetically Cohen-Macaulay* (aCM for short), if

$$\mathrm{H}^i(X, \mathcal{E}(t)) = 0 \quad \text{for all} \quad t \in \mathbb{Z} \quad \text{and} \quad 1 \leq i \leq \dim(X) - 1.$$

(iii) $\mathcal{E}$ is called *Ulrich* if $\mathcal{E}$ is an initialized, aCM bundle satisfying

$$\mathrm{h}^0(X, \mathcal{E}) = \deg(X)\,\mathrm{rk}(\mathcal{E}).$$

Ulrich bundles first appeared in a purely algebraic context in 1984 in the study of maximal Cohen-Macaulay modules with maximal number of generators (see [64]). From the algebraic-geometric standpoint, in [7] Ulrich line bundles were connected to the problem of finding a linear determinantal representation of a hypersurface in $\mathbb{P}^n$. Moreover, in [29] the existence of an Ulrich bundle on a polarized projective variety (X, H) was used to construct the Cayley-Chow form of X. In this latter reference, the authors asked whether any polarized projective variety (X, H) carries an Ulrich bundle. In the last decades, the search for Ulrich bundles has been the center of an intense research, as one can see in, among many others, [2, 3, 8, 12, 31].

Ulrich bundles can be characterized cohomologically as follows:

Proposition 3.3.2. *Let (X, H) be a polarized $n-$dimensional smooth variety and let $\mathcal{E}$ be an initialized vector bundle on X. The following conditions are equivalent:*

(i) $\mathcal{E}$ is Ulrich.

(ii) $\mathrm{H}^i(X, \mathcal{E}(-t)) = 0$ *for $i \geq 0$ and $1 \leq t \leq n$.*

(iii) $\mathrm{H}^i(X, \mathcal{E}(-i)) = 0$ *for* $i > 0$ *and* $\mathrm{H}^i(X, \mathcal{E}(-i-1)) = 0$ *for* $i < n$.

Moreover, if $\mathcal{E}$ *is an Ulrich bundle, then*

$$\chi(\mathcal{E}(t)) = \deg(X)\,\mathrm{rk}(\mathcal{E})\binom{t+d}{d}. \tag{3.21}$$

Proof. See [17, Theorem 3.2.9 and Corollary 3.2.10]. □

In Section 3.1, we have introduced a method to study the Hilbert polynomial of an equivariant reflexive sheaf $\mathcal{E}$ on a smooth toric variety (see Proposition 3.1.2). Furthermore, in Subsection 3.2.2, this method has been applied when the toric variety X has a splitting fan. Being the Hirzebruch surface $\mathcal{H}_a$ a toric surface with splitting fan, we may combining (3.21) with the expression of the Euler characteristic obtained in Theorem 3.2.18 to find candidates of equivariant vector bundles on $\mathcal{H}_a$. In general Ulrich bundles on a Hirzebruch surface have been studied from other perspectives in [1]. However, it is not strightforward to see when the Ulrich bundles presented in [1] are equivariant.

Let us recall the notation from Example 1.1.13. We take $\{e, f\}$ the standard basis of $N = \mathbb{Z}^2$. For any integer $a \geq 0$, the fan $\Sigma \subset \mathbb{R}^2$ of $\mathcal{H}_a$ is defined as $\Sigma_{\max} = \{\sigma_{ij} \mid 0 \leq i, j \leq 1\}$ where

$$\begin{aligned}
\sigma_{00} &= \mathrm{cone}(u_1, v_1)\\
\sigma_{01} &= \mathrm{cone}(u_1, v_0)\\
\sigma_{10} &= \mathrm{cone}(u_0, v_1)\\
\sigma_{11} &= \mathrm{cone}(u_0, v_0)
\end{aligned}$$

and $u_0 := -e + af$, $u_1 := e$, $v_0 := -f$ and $v_1 := f$. We denote the four rays in $\Sigma(1)$ by:

$$\begin{aligned}
\rho_0 &= \mathrm{cone}(u_0)\\
\rho_1 &= \mathrm{cone}(u_1)\\
\eta_0 &= \mathrm{cone}(v_0)\\
\eta_1 &= \mathrm{cone}(v_1).
\end{aligned}$$

The class group of $\mathcal{H}_a$ is $\mathrm{Cl}(\mathcal{H}_a) = \mathbb{Z}\langle [D_{\rho_0}], [D_{\eta_0}]\rangle$ and we have

$$[D_{\rho_1}] = [D_{\rho_0}] \quad \text{and} \quad [D_{\eta_1}] = -a[D_{\rho_0}] + [D_{\eta_0}]. \tag{3.22}$$

A line bundle $H = \mathcal{O}(\alpha,\beta) := \mathcal{O}(\alpha[D_{\rho_0}] + \beta[D_{\eta_0}])$ is very ample if and only if

$$\alpha, \beta > 0.$$

On the other hand, we have

$$[D_{\rho_0}][D_{\rho_1}] = 0 \quad \text{and} \quad [D_{\eta_0}][D_{\eta_1}] = 0,$$

which combined with (3.22) it yields

$$[D_{\rho_0}]^2 = 0, \quad [D_{\rho_0}][D_{\eta_0}] = 1 \quad \text{and} \quad [D_{\eta_0}]^2 = a. \tag{3.23}$$

In particular, the degree of $(\mathcal{H}_a, H)$ is

$$\deg(\mathcal{H}_a) = (\alpha[D_{\rho_0}] + \beta[D_{\eta_0}])^2 = 2\alpha\beta + a\beta^2.$$

Now let $\mathcal{E}$ be a rank ℓ equivariant vector bundle on $\mathcal{H}_a$. As seen in Section 3.2, $\mathcal{H}_a = V_1(a_1)$. Thus, we use the same notation as in Notation 3.2.4 and we write the associated filtrations of $\mathcal{E}$ as follows:

$$\begin{array}{l} \hat{E}^{\rho_0} = F^0(i_1^0, \dots, i_\ell^0; F_1^0, \dots, F_\ell^0) \\ \hat{E}^{\rho_1} = F^1(i_1^1, \dots, i_\ell^1; F_1^1, \dots, F_\ell^1) \\ \hat{E}^{\eta_0} = G^0(j_1^0, \dots, j_\ell^0; G_1^0, \dots, G_\ell^0) \\ \hat{E}^{\eta_1} = G^1(j_1^1, \dots, j_\ell^1; G_1^1, \dots, G_\ell^1). \end{array} \tag{3.24}$$

Our goal is to see under which conditions, the filtrations (3.24) correspond to an equivariant vector bundle $\mathcal{E}$ with the Euler characteristic of an Ulrich bundle. In particular, (3.21) yields:

$$\chi(\mathcal{E}(t\alpha, t\beta)) = \ell(2\alpha\beta + a\beta^2)\binom{t+2}{2}. \tag{3.25}$$

Let us first use Proposition 3.1.2 to write $\chi(\mathcal{E}(p,q))$ for any p and q in terms of the filtration. For each $(\underline{\lambda}, \underline{\mu}) = (\lambda_0, \lambda_1, \mu_0, \mu_1) \in \{1, \dots, \ell\}^4$ we have a lattice polytope $\Omega_{(\underline{\lambda},\underline{\mu})}(p,q) \subset M_{\mathbb{R}} \cong \mathbb{R}^2$ given by the following inequalities (see also (3.17)):

$$\begin{pmatrix} i_{\lambda_0}^0 - p \\ i_{\lambda_1}^1 \\ j_{\mu_0}^0 - q \\ j_{\mu_1}^1 \end{pmatrix} \leq \begin{pmatrix} -1 & a \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{pmatrix} \cdot x < \begin{pmatrix} i_{\lambda_0+1}^0 - p \\ i_{\lambda_1+1}^1 \\ j_{\mu_0+1}^0 - q \\ j_{\mu_1+1}^1 \end{pmatrix}.$$

By Proposition 3.1.2, we can determine the value of $\mathrm{h}^0(X, \mathcal{E}(p,q))$ if we are able to count the number of lattice points in $\Psi_{(\underline{\lambda},\underline{\mu})}(p,q) = \Omega_{(\underline{\lambda},\underline{\mu})}(p,q) \cap M$. From Theorem 3.2.18 we know an upper bound for the regularity index of $\mathcal{E}$ and we can find an expression of $\chi(\mathcal{E}(p,q))$ in terms of the filtration of $\mathcal{E}$.

Proposition 3.3.3. *Let $\mathcal{E}$ be a vector bundle on $\mathcal{H}_a$ with filtrations as in (3.24). Then, for $p \geq i^0_\ell + i^1_\ell - aj^1_1 - 1$ and $q \geq j^0_\ell + j^1_\ell - 1$, the dimension $\mathrm{h}^0(X, \mathcal{E}(p,q))$ is a polynomial. In particular, the Hilbert polynomial of $\mathcal{E}$ is*

$$\begin{aligned}\chi(\mathcal{E}(p,q)) = {} & \chi_0(\mathcal{E}(p,q)) + \\ & \frac{a}{2}\Bigg(\ell\left[(q - j^0_\ell - j^1_\ell)(q - j^0_\ell - j^1_\ell + 1) + 2j^1_\ell(q - j^0_\ell - j^1_\ell + 1)\right] + \\ & \qquad \sum_{k=1}^{\ell-1} k(j^1_k + j^1_{k+1} - 1)(j^1_{k+1} - j^1_k) + \\ & \qquad \sum_{k=1}^{\ell-1} k(2(q+1) - j^0_{k+1} - j^0_k - 1)(j^0_{k+1} - j^0_k)\Bigg), \end{aligned} \tag{3.26}$$

where $\chi_0(\mathcal{E}(p,q))$ is the polynomial

$$\begin{aligned}\chi_0(\mathcal{E}(p,q)) = {} & \ell(p - i^0_\ell - i^1_\ell + 1)(q - j^0_\ell - j^1_\ell + 1) + \\ & \sum_{k=1}^{\ell-1} k\left[(p - i^0_\ell - i^1_\ell + 1)(j^0_{k+1} + j^1_{k+1} - j^0_k - j^1_k)\right] + \\ & \sum_{k=1}^{\ell-1} k\left[(q - j^0_\ell - j^1_\ell + 1)(i^0_{k+1} + i^1_{k+1} - i^0_k - i^1_k)\right] + \\ & \sum_{1 \leq k,m \leq \ell} (i^0_{k+1} - i^0_k)(j^0_{m+1} - j^0_m)\dim(F^0_k \cap G^0_m) + \\ & \sum_{1 \leq k,m \leq \ell} (i^1_{k+1} - i^1_k)(j^0_{m+1} - j^0_m)\dim(F^1_k \cap G^0_m) + \\ & \sum_{1 \leq k,m \leq \ell} (i^0_{k+1} - i^0_k)(j^1_{m+1} - j^1_m)\dim(F^0_k \cap G^1_m) + \\ & \sum_{1 \leq k,m \leq \ell} (i^1_{k+1} - i^1_k)(j^1_{m+1} - j^1_m)\dim(F^1_k \cap G^1_m)\end{aligned}$$

Proof. From Theorem 3.2.18 taking $r = s = 1$ and $a_1 = a$, it follows that for $p \geq i^0_\ell + i^1_\ell - aj^1_1 - 1$ and $q \geq j^0_\ell + j^1_\ell - 1$, the dimension $\mathrm{h}^0(X, \mathcal{E}(p, q))$ is a polynomial. On the other hand, (3.26) follows from the proofs of Theorem 3.2.18 and Lemma 3.2.15. □

We may use Proposition 3.3.3 together with (3.25), taking $p = t\alpha$ and $q = t\beta$ to search for equivariant vector bundles of a fixed rank ℓ. The system of equations resulting from this strategy can be very involved, even for small values of ℓ, α or β. Nevertheless, using this method and the computer algebra system Mathematica we have been able to find a rank 3 equivariant Ulrich vector bundle in $\mathcal{H}_1$ with respect to the ample.

Example 3.3.4. Let $\mathcal{E}$ be the rank 3 equivariant vector bundle in $\mathcal{H}_1$ associated to the following filtrations:

$$\begin{aligned}
&F^0(-2, -1, 0; \langle e_1 \rangle, \langle e_1, e_3 \rangle, \mathbb{C}^3)\\
&F^1(-2, -1, 0; \langle e_2 \rangle, \langle e_2, e_3 \rangle, \mathbb{C}^3)\\
&G^0(-1, 0, 0; \langle e_1 + e_2 - e_3 \rangle, \mathbb{C}^3, \mathbb{C}^3)\\
&G^1(0, 0, 0; \mathbb{C}^3, \mathbb{C}^3, \mathbb{C}^3),
\end{aligned}$$

where $\{e_1, e_2, e_3\}$ is a basis of $\mathbb{C}^3$. Let $H = \mathcal{O}(1, 2)$ be a very ample line bundle. Using the package ToricVectorBundles of Macaulay2 [34] we can check that $\mathrm{h}^0(X, \mathcal{E}) = 3 \cdot 8$

$$\begin{aligned}
\mathrm{H}^0(X, \mathcal{E}(-1, -2)) = \mathrm{H}^1(X, \mathcal{E}(-1, -2)) = 0\\
\mathrm{H}^1(X, \mathcal{E}(-2, -4)) = \mathrm{H}^2(X, \mathcal{E}(-2, -4)) = 0.
\end{aligned}$$

Hence, by Proposition 3.3.2(ii), $\mathcal{E}$ is an Ulrich bundle with respect to H.

Chapter 4

Stability of syzygy bundles

This chapter is devoted to the study of syzygy bundles on a polarized smooth projective toric variety (X, L), where L is a very ample line bundle on X. In particular, we study the stability of syzygy bundles M_L attached to the complete linear system $|L|$ and we address the Ein-Lazarsfeld-Mustopa conjecture [25] (see Conjecture 4.2.1 and Question 4.2.2).

Focusing on $X = \mathrm{Bl}_Z(\mathbb{P}^n)$, the blow-up of a projective space $\mathbb{P}^n$ along a linear subspace $Z \subset \mathbb{P}^n$, we prove that for any very ample line bundle L on $\mathrm{Bl}_Z(\mathbb{P}^n)$ its corresponding syzygy bundle M_L is $L-$stable. One remarkable consequence of M_L being $L-$stable is that it can be seen as a point $[M_L]$ in its corresponding moduli space $\mathcal{M} = \mathcal{M}_X(N-1; c_1, \ldots, c_{\min\{N-1,n\}})$ where

$$N = \mathrm{h}^0(X, L) - 1$$

is the rank of M_L and

$$c_i = c_i(M_L)$$

its Chern classes, for $1 \leq i \leq \min\{N-1, n\}$ (see Subsection 4.1). In our case, we find that for $n > 2$ the syzygy bundle M_L is infinitessimally rigid and hence $[M_L]$ is an isolated point in $\mathcal{M}$. Furthermore, for $n = 2$ M_L is always unobstructed and we can compute the dimension of the tangent space of $\mathcal{M}$ at $[M_L]$.

This chapter is organized as follows. In Section 4.1, we recall the basic definitions and results on stability of vector bundles and moduli spaces we need in the sequel. In Section 4.2, we recall the notion of a syzygy bundle and the known results regarding the stability of syzygy bundles, as well as recent contributions to the Ein-Lazarsfeld-Mustopa conjecture. Afterwards, in Section 4.3 we focus on the blow up $X = \mathrm{Bl}_Z(\mathbb{P}^n)$ of the projective space $\mathbb{P}^n$ along a linear subspace Z. First, an algebraic result on the syzygies of a monomial ideal yields information on the minimal locally free resolution of the syzygy bundle M_L (Corollary 4.3.3). Together with Coandă's Lemma (Lemma 4.1.4), it allows us to prove that the syzygy bundle M_L is $L-$stable for any ample line bundle L on $\mathrm{Bl}_Z(\mathbb{P}^n)$ (Theorem 4.3.4). Finally, in Subsection 4.3.1 we address the rigidness of the syzygy bundles M_L and the local properties of the moduli space $\mathcal{M}$ around the point $[M_L]$ (Theorem 4.3.5).

4.1 Preliminaries on stability of vector bundles and moduli spaces

Let (X, H) be a polarized smooth projective variety. For any torsion-free sheaf $\mathcal{E}$ we define the *reduced Hilbert polynomial* of $\mathcal{E}$ to be $\bar{P}_{\mathcal{E}}(t) \in \mathbb{Q}[t]$ such that for any integer $m \in \mathbb{Z}$ we have:

$$\bar{P}_{\mathcal{E}}(m) := \frac{\chi(\mathcal{E} \otimes \mathcal{O}_X(mH))}{\mathrm{rk}(\mathcal{E})}.$$

Now, for a fixed polynomial $P \in \mathbb{Q}[t]$, we define the *moduli functor of vector bundles* having P as its reduced Hilbert polynomial:

$$M_{X,P} : \underline{\mathrm{Sch}}/\mathbb{C} \to \underline{\mathrm{Set}}$$

For any scheme T over $\mathbb{C}$, we set

$$M_{X,P} := \left\{ T-\text{ flat families } F \to X \times T \,\middle|\, \begin{array}{l} F_t \text{ is a vector bundle with} \\ \bar{P}_{F_t} = P \quad \text{for any} \quad t \in T \end{array} \right\} \Big/ \sim$$

where two $T-$flat families $F \to X \times T$ and $F' \to X \times T$ are $\sim -$equivalent if and only if there is a line bundle $\mathcal{L} \in \mathrm{Pic}(T)$ such that

$$F' \cong F \otimes \pi_2^* \mathcal{L},$$

with $\pi_2 : X \times T \to T$ the natural projection.

On the other hand, for any morphism $f : T \to T'$ we set $f_X := \mathrm{Id}_X \times f$ and we define

$$\begin{array}{rccc} M_{X,P}(f) : & M_{X,P}(T) & \to & M_{X,P}(T') \\ & [F] & \mapsto & [f_X^* F]. \end{array}$$

We say that a scheme $\mathcal{M}_{X,P}$ is a *fine moduli space* of vector bundles with fixed reduced Hilbert polynomial P if $\mathcal{M}_{X,P}$ represents the functor $M_{X,P}$. In general, this functor is not representable and we have to impose further restrictions to the vector bundles in the definition of $M_{X,P}$. The notion of a slope stable vector bundle will give a solution to this issue.

Definition 4.1.1. Let (X, H) be a polarized smooth irreducible projective variety of dimension n. For a torsion-free sheaf $\mathcal{E}$ on X we set

$$\mu_H(\mathcal{E}) := \frac{c_1(\mathcal{E})H^{n-1}}{\operatorname{rk}(\mathcal{E})}$$

called the *slope* of $\mathcal{E}$ with respect to H. We say that $\mathcal{E}$ is $H-$*semistable* (resp. $H-$*stable*) if for any non-zero subsheaf $\mathcal{F} \subset \mathcal{E}$ with $\operatorname{rk}(\mathcal{F}) < \operatorname{rk}(\mathcal{E})$ we have

$$\mu_H(\mathcal{F}) \leq \mu_H(\mathcal{E}) \quad (\text{resp. } \mu_H(\mathcal{F}) < \mu_H(\mathcal{E})).$$

Now, for a fixed polynomial $P \in \mathbb{Q}[t]$ we define the *moduli* functor of $H-$stable vector bundles on (X, H):

$$M^s_{X,H,P} : \underline{\mathrm{Sch}}/\mathbb{C} \to \underline{\mathrm{Set}}.$$

For any scheme T over $\mathbb{C}$ we set

$$M^s_{X,H,P} := \\ \left\{ T-\text{flat families} \quad F \to X \times T \;\middle|\; \begin{array}{l} F_t \text{ is an } H-\text{stable} \\ \text{vector bundle with} \\ \bar{P}_{F_t} = P \quad \text{for any} \quad t \in T \end{array} \right\} \Big/ \sim$$

where $\sim$ is the same equivalence relation as before, and for any morphism $f : T \to T'$ we define $M^s_{X,H,P}(f)$ as $M_{X,P}(f)$. Now, we have the following result.

Proposition 4.1.2. *Let (X, H) be a polarized smooth projective variety of dimension n and let $P \in \mathbb{Q}[t]$ be a polynomial. Then, there is a separated and locally of finite type over $\mathbb{C}$ scheme $\mathcal{M}^s_{X,H,P}$ such that:*

i) There is a natural transformation

$$\Psi : M^s_{X,H,P} \to \operatorname{Hom}(-, \mathcal{M}^s_{X,H,P}),$$

which is bijective for any reduced point $\{x_0\}$.

ii) For every scheme $\mathcal{N}$ and every natural transformation

$$\Phi : M^s_{X,H,P} \to \mathrm{Hom}(-, \mathcal{N}),$$

there is a unique morphism $f : \mathcal{M}^s_{X,H,P} \to \mathcal{N}$ such that the following diagram

$$\begin{tikzcd} M^s_{X,H,P} \arrow[rr, "\Psi"] \arrow[dr, "\Phi"'] & & \mathrm{Hom}(-, \mathcal{M}^s_{X,H,P}) \arrow[dl, "f_*"] \\ & \mathrm{Hom}(-, \mathcal{N}) & \end{tikzcd}$$

commutes.

iii) The scheme $M^s_{X,H,P}$ decomposes into a disjoint union of schemes

$$\mathcal{M}_{X,H}(r, c_1, \ldots, c_{\min\{r,n\}}),$$

such that $\mathcal{M}_{X,H}(r, c_1, \ldots, c_{\min\{r,n\}})$ is the moduli space of $H-$stable rank r vector bundles with a fixed Chern classes $(c_1, \ldots, c_{\min\{r,n\}})$ up to numerical equivalence.

Proof. See [52, Theorem 5.6]. □

In general few results are known about the local and global structure of $\mathcal{M}_{X,H}(r, c_1, \ldots, c_{\min\{r,n\}})$. For instance, we have the following result:

Proposition 4.1.3. *Let (X, H) be a polarized smooth projective variety of dimension n, and $\mathcal{E}$ an $H-$stable vector bundle of rank r. Set $(c_1, \ldots, c_{\min\{r,n\}})$ the Chern classes of $\mathcal{E}$. In particular $\mathcal{E}$ represents a point $[\mathcal{E}]$ in the moduli space $\mathcal{M} = \mathcal{M}_{X,H}(r; c_1, \ldots, c_{\min\{r,n\}})$.*

i) The Zariski tangent space of $\mathcal{M}$ at $[\mathcal{E}]$ is canonically isomorphic to

$$T_{[E]}\mathcal{M} \cong \mathrm{Ext}^1(\mathcal{E}, \mathcal{E}) \cong \mathrm{H}^1(X, \mathcal{E} \otimes \mathcal{E}^\vee).$$

ii) Set $\mathcal{M}_{[E]}$ to be the irreducible component in $\mathcal{M}$ containing $[E]$, then we have

$$\mathrm{ext}^1(\mathcal{E}, \mathcal{E}) \geq \dim \mathcal{M}_{[E]} \geq \mathrm{ext}^1(\mathcal{E}, \mathcal{E}) - \mathrm{ext}^2(\mathcal{E}, \mathcal{E}),$$

where $\mathrm{ext}^i(\mathcal{E}, \mathcal{E}) = \dim \mathrm{Ext}^i(\mathcal{E}, \mathcal{E})$.

Proof. See [41] □

In particular, if $\mathrm{H}^2(X, \mathcal{E} \otimes \mathcal{E}^\vee) = 0$ we have that $[E]$ is a smooth point of $\mathcal{M}$ and we say that $\mathcal{E}$ is *unobstructed.* In addition, if $[E]$ is an isolated point, and thus $\dim T_{[E]}\mathcal{M} = 0$ we say that $\mathcal{E}$ is *infinitessimally rigid.*

We end this section by stating a cohomological characterization of the stability which will play an important role in the forthcoming sections.

Lemma 4.1.4. *Let (X, L) be a polarized smooth variety of dimension n. Let E be a vector bundle on X. Suppose that for any integer q and any line bundle G on X such that*

$$0 < q < \mathrm{rk}(E) \quad \text{and} \quad (G \cdot L^{n-1}) \geq q\mu_L(E)$$

one has $\mathrm{H}^0(X, \bigwedge^q E \otimes G^\vee) = 0$. Then, E is $L-$stable.

Proof. See [16, Lemma 2.1]. □

4.2 Syzygy bundles on a projective variety

In this section, we recall the basic definitions, problems and known results regarding the stability of syzygy bundles on a polarized projective variety (X, L). The *syzygy bundle* M_L is defined as the kernel of the evaluation map

$$ev : \mathrm{H}^0(X, L) \otimes \mathcal{O}_X \longrightarrow L.$$

Thus, M_L is a vector bundle of rank $\mathrm{h}^0(X, L) - 1$ fitting in the following short exact sequence

$$0 \longrightarrow M_L \longrightarrow \mathrm{H}^0(X, L) \otimes \mathcal{O}_X \longrightarrow L \longrightarrow 0. \tag{4.1}$$

In particular we have:

- $c_1(M_L) = -c_1(L)$,
- $\mathrm{rk}(M_L) = \mathrm{h}^0(X, L) - 1$,
- $\mu(M_L) = \frac{-L^n}{\mathrm{h}^0(X,L)-1}$.

Arising in a variety of geometric and algebraic problems, the syzygy bundles M_L have been extensively studied from different points of view. In particular, many efforts have been invested on knowing whether M_L is a stable vector bundle with respect to some polarization. As far as we know, the stability of M_L has been proved in the following cases:

(1) $(X, L) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ (see [33] in characteristic zero and [10] for any characteristic),

(2) (X, L) where X is a smooth projective curve of genus $g \geq 1$ and $\deg(L) \geq 2g + 1$ (see [24, Proposition 1.5]),

(3) (X, L) where X is a simple abelian variety and L an ample globaly generated line bundle (see [13, Corollary 2.1]),

(4) when (X, L) is a sufficiently positive polarization of an algebraic surface X (see [25, Theorem A]) and

(5) (X, L) where X is an Enriques (resp. bielliptic) surface and L an ample globally generated line bundle (see [56, Theorem 3.5]).

In [25, Corollary 2.6], L. Ein, R. Lazarsfeld and Y. Mustopa posed the following conjecture:

Conjecture 4.2.1. *Let A and P two line bundles on a smooth projective variety X. Assume that A is very ample and set $L_d := dA + P$ for any positive integer d. Then, the syzygy bundle M_{L_d} is $A-$stable for $d \gg 0$.*

Related to this conjecture, in [38, Question 7.8], M. Hering, M. Mustaţă and S. Payne consider the following question:

Question 4.2.2. Let L be an ample line bundle on a projective toric variety X. Is the syzygy bundle M_L (semi)stable, with respect to some choice of a polarization?

4.3 Syzygy bundles on blow-ups of projective spaces

The goal of this section is to answer positively Question 4.2.2 when X is the blow-up $\mathrm{Bl}_Z(\mathbb{P}^n)$ of a projective space $\mathbb{P}^n$ along a linear subspace

$Z \subset \mathbb{P}^n$ (see Theorem 4.3.4). Let us start by recalling how the blow-up $\mathrm{Bl}_Z(\mathbb{P}^n)$ can be seen as a toric variety. We start with the following classical result.

Proposition 4.3.1. *Let $Z \subset \mathbb{P}^n$ be a linear subspace of dimension $r-1$. Then, the blow-up of $\mathbb{P}^n$ along Z is isomorphic to the projective bundle $\mathbb{P}(\mathcal{O}^r_{\mathbb{P}^s} \oplus \mathcal{O}_{\mathbb{P}^s}(1))$, where $s = n - r$.*

In particular, $\mathrm{Bl}_Z(\mathbb{P}^n)$ is a smooth toric variety with splitting fan (see Definition 3.2.1).

Proof. See [27, Proposition 9.11]. □

Following Section 3.2, we fix the following notation to write the fan Σ of $X = \mathrm{Bl}_Z(\mathbb{P}^n) \cong \mathbb{P}(\mathcal{O}^r_{\mathbb{P}^s} \oplus \mathcal{O}_{\mathbb{P}^s}(1))$. We take the lattice $N = \mathbb{Z}^s \times \mathbb{Z}^r$ and set $\{e_1, \ldots, e_s\}$ and $\{f_1, \ldots, f_r\}$ be the standard basis of $\mathbb{Z}^s$ and $\mathbb{Z}^r$, respectively (see [20, Proposition 7.3.7]). We set

$$\begin{aligned}
&\rho_0 := \mathrm{cone}(-e_1 - \cdots - e_s + f_r)\\
&\rho_i := \mathrm{cone}(e_i) \quad 1 \le i \le s\\
&\eta_0 := \mathrm{cone}(-f_1 - \cdots - f_r)\\
&\eta_j := \mathrm{cone}(f_j) \quad 1 \le j \le r,
\end{aligned}$$

and for $1 \le i \le s$ and $1 \le j \le r$ we define the $r+s-$dimensional cones

$$\sigma_{ij} := \mathrm{cone}(\rho_0, \ldots, \widehat{\rho_i}, \ldots, \rho_s, \eta_0, \ldots, \widehat{\eta_j}, \ldots, \eta_r).$$

Then $\Sigma(1) = \{\rho_0, \ldots, \rho_s, \eta_0, \ldots, \eta_r\}$ and the fan Σ is given by the following set of maximal cones:

$$\Sigma_{max} = \{\sigma_{ij} \mid 1 \le i \le s,\ 1 \le j \le r\}.$$

From (3.8) we know the class group of X:

$$\mathrm{Cl}(X) \cong \mathbb{Z}\langle [D_{\rho_0}], [D_{\eta_0}] \rangle,$$

such that

$$\begin{aligned}
&[D_{\rho_i}] = [D_{\rho_0}], && \text{for } 1 \le i \le s,\\
&[D_{\eta_j}] = [D_{\eta_0}], && \text{for } 1 \le j \le r-1\\
&[D_{\eta_r}] = -[D_{\rho_0}] + [D_{\eta_0}].
\end{aligned}$$

In particular, the Cox ring of X is the polynomial ring

$$R = \mathbb{C}[x_0, \ldots, x_s, y_0, \ldots, y_r]$$

with a $\mathbb{Z}^2-$grading given by

$$\begin{array}{ll} \deg(x_i) = (1,0), & \text{for } 0 \le i \le s \\ \deg(y_j) = (0,1), & \text{for } 0 \le j \le r-1 \\ \deg(y_r) = (-1,1). & \end{array}$$

On the other hand, by (3.9) we have the following intersection numbers

$$[D_{\rho_0}]^{s-j}[D_{\eta_0}]^{r+j} = \begin{cases} 0 & j < 0 \\ 1 & 0 \le j \le s. \end{cases}$$

Finally, let us recall from Section 3.2, that a line bundle

$$L = \mathcal{O}(a,b) := \mathcal{O}(a[D_{\rho_0}] + b[D_{\eta_0}])$$

is ample (respectively effective) if and only if $a, b > 0$ $(a, b \ge 0)$. We fix now two integers $a, b > 0$ and an arbitrary ample line bundle $L = \mathcal{O}_X(a,b)$ on X. Our goal is to prove that the syzygy bundle M_L fitting into the exact sequence

$$0 \longrightarrow M_L \longrightarrow \mathrm{H}^0(X, \mathcal{O}_X(L)) \otimes \mathcal{O}_X \longrightarrow \mathcal{O}_X(L) \longrightarrow 0$$

is $L-$stable. We start with an algebraic result which plays an important role in the structure of M_L.

Proposition 4.3.2. *Let $S = \mathbb{C}[x_0, \ldots, x_s, y_0, \ldots, y_r]$ be the $\mathbb{Z}^2-$graded polynomial ring with*

$$\begin{array}{ll} \deg(x_i) = (1,0), & \text{for } 0 \le i \le s \\ \deg(y_j) = (0,1), & \text{for } 0 \le j \le r-1 \\ \deg(y_r) = (-1,1). & \end{array}$$

For any integers $a, b > 0$ we consider the syzygy module K_L of the monomial ideal

$$I_{a,b} = (x_0^{a_0} \cdots x_s^{a_s} y_0^{b_0} \cdots y_r^{b_r} \mid a_0 + \cdots + a_s = a + b_r,\ b_0 + \cdots + b_r = b).$$

Then, K_L is minimally generated by elements of degree $(a+1, b)$ and $(a, b+1)$.

Proof. Since $I_{a,b}$ is a monomial ideal generated by forms of degree (a, b), then K_L is generated by syzygies of degree $(a + p, b + q)$ of the form

$$fw_1 - gw_2 = 0,$$

with f, g monomials of degree (p, q) (with $q \geq 0$ and $p \geq -q$), and $w_1, w_2 \in I_{a,b}$ monomials of degree (a, b). First of all notice that if $p = -q$, then f and g would be monomials of degree $(-q, q)$, so

$$f = g = y_r^q.$$

In particular we have that

$$fw_1 - gw_2 = y_r^q(w_1 - w_2),$$

which cannot be a syzygy, since w_1 and w_2 are different monomials. Therefore we can assume from now on that $q \geq 0$ and $p \geq -q + 1$. Let us write

$$\begin{aligned} f &= x_0^{l_0} \cdots x_s^{l_s} y_0^{m_0} \cdots y_r^{m_r} \\ g &= x_0^{\lambda_0} \cdots x_s^{\lambda_s} y_0^{\mu_0} \cdots y_r^{\mu_r} \\ w_1 &= x_0^{a_0} \cdots x_s^{a_s} y_0^{b_0} \cdots y_r^{b_r} \\ w_2 &= x_0^{\alpha_0} \cdots x_s^{\alpha_s} y_0^{\beta_0} \cdots y_r^{\beta_r}. \end{aligned}$$

Let us consider any syzygy

$$fw_1 - gw_2 = 0$$

with $\deg(f) = \deg(g) = (-q + 1 + z, q)$ with either $z \geq 1$ or $q \geq 1$. We will see that

$$fw_1 - gw_2 = w(f'w_1' - g'w_2')$$

for some monomial w of degree either $(1, 0)$ or $(0, 1)$, and monomials f', g' of degree either $(-q + 1 + (z - 1), q)$ (assuming $z \geq 1$) or degree $(-q + 1 + z, q - 1)$ (assuming $q \geq 1$), and $w_1', w_2' \in I_{a,b}$. We distinguish two main cases:

<u>Case 1:</u> there is $0 \leq i \leq s$ such that $l_i \geq 1$.
<u>Case 2:</u> $l_0 = \cdots = l_s = 0$.

We start analyzing Case 1 and we distinguish two subcases (A) and (B) as follows:

(A) There is an index $0 \leq i_0 \leq s$ such that both $l_i \geq 1$ and $\lambda_{i_0} \geq 1$. Then, we have

$$fw_1 - gw_2 = x_{i_0}\left(\frac{f}{x_{i_0}}w_1 - \frac{g}{x_{i_0}}w_2\right) = 0,$$

and $f' = \frac{f}{x_{i_0}}$, $g' = \frac{g}{x_{i_0}}$ are monomials of degree $(-q+1+(z-1), q)$.
(B) Otherwise, we may assume without loss of generality (permuting $\{x_0, \dots, x_s\}$, if necessary), that $l_0 \geq 1$, $\lambda_0 = 0$ and for any $1 \leq i \leq s$, $l_i\lambda_i = 0$. In particular, we have $\alpha_0 = a_0 + l_0 \geq 1$. In this case, we have two options:

(B.1) there is an index j with $1 \leq j \leq s$ such that $\lambda_j \geq 1$, or else
(B.2) $\lambda_0 = \cdots = \lambda_s = 0$.

If (B.1) holds, then we may write

$$fw_1 - gw_2 = x_0\left(\frac{f}{x_0}w_1 - \frac{g}{x_j}\frac{w_2x_j}{x_0}\right).$$

If $w_1 \neq \frac{w_2x_j}{x_0}$, then we may set

$$f' = \frac{f}{x_0}, \quad w_1' = w_1, \quad g' = \frac{g}{x_j} \quad \text{and} \quad w_2' = \frac{w_2x_j}{x_0}.$$

Otherwise, we would have $f = \frac{gx_0}{x_j}$, so

$$fw_1 - gw_2 = \frac{g}{x_j}(x_0w_1 - x_jw_2)$$

would be a multiple of a syzygy of degree $(a+1, b)$.

On the other hand, if (B.2) holds it implies that $q \geq 1$. Indeed, if $q = 0$ then

$$\mu_0 = \cdots = \mu_r = 0$$

and we would have $0 = 1 + z$, which cannot occur since in this case $z \geq 1$. We distinguish four more subcases:

(B.2.1) There is $0 \leq j \leq r-1$ such that $m_j \geq 1$ and $\mu_j \geq 1$
(B.2.2) There is $0 \leq j_0 \leq r-1$ such that $m_{j_0} \geq 1$ and $m_j\mu_j = 0$ for all $0 \leq j \leq r-1$.

(B.2.3) There is $0 \leq j_0 \leq r-1$ such that $\mu_{j_0} \geq 1$ and $m_j \mu_j = 0$ for all $0 \leq j \leq r-1$.

(B.2.4) $m_0 = \cdots m_{r-1} = \mu_0 = \cdots = \mu_{r-1} = 0$.

If case (B.2.1) holds, then we may write

$$fw_1 - gw_2 = y_0 \left(\frac{f}{y_0} w_1 - \frac{g}{y_0} w_2 \right),$$

and we have $\deg(\frac{f}{y_0}) = \deg(\frac{g}{y_0}) = (-q+1+z, q-1)$.

On the other hand, in both cases (B.2.2) and (B.2.3) we may assume without loss of generality that $j_0 = 0$, permuting $\{y_0, \ldots, y_{r-1}\}$ if necessary. If (B.2.2) holds, then $m_0 \geq 1$ and $\mu_0 = 0$. In particular, $\beta_0 = b_0 + m_0 \geq 1$ and we distinguish two situations:

- There is $1 \leq k \leq r-1$ such that $\mu_k \geq 1$. We have

  $$fw_1 - gw_2 = y_0 \left(\frac{f}{y_0} w_1 - \frac{g}{y_k} \frac{w_2 y_k}{y_0} \right).$$

  If $w_1 \neq \frac{w_2 y_k}{y_0}$, then we take

  $$f' = \frac{f}{y_0}, \quad w_1' = w_1, \quad g' = \frac{g}{y_k} \quad \text{and} \quad w_2' = \frac{w_2 y_k}{y_0}.$$

  If not, we have $f = \frac{g y_0}{y_k}$, so

  $$fw_1 - gw_2 = \frac{g}{y_k}(y_0 w_1 - y_k w_2)$$

  is a multiple of a syzygy of degree $(a, b+1)$.

- Otherwise we may suppose

  $$\mu_0 = \cdots = \mu_{r-1} = 0,$$

  but then $\deg(g) = (-q, q)$ which is a contradiction.

In the case (B.2.3), we have $m_0 = 0$ and $\mu_0 \geq 1$, which in particular gives $b_0 = \beta_0 + \mu_0 \geq 1$. We have two subcases:

- There is $1 \leq k \leq r-1$ such that $m_k \geq 1$, then we have
$$fw_1 - gw_2 = y_0 \left(\frac{f}{y_k} \frac{w_1 y_k}{y_0} - \frac{g}{y_0} w_2 \right)$$
and it follows as before.

- Otherwise, we have
$$m_0 = \cdots = m_{r-1} = 0,$$
in particular we have $m_r = \mu_0 + \cdots + \mu_r \geq 1$. Since we are assuming that $l_0 \geq 1$, there is $0 \leq i \leq s$ such that $a_i \geq 1$. Hence, we have
$$fw_1 - gw_2 = y_0 \left(\frac{f}{x_0 y_r} \frac{w_1 y_r x_0}{y_0} - \frac{g}{y_0} w_2 \right)$$
and this subcase follows as before, since $\deg(y_r x_0) = (0, 1)$.

Finally, if (B.2.3) holds, then we would have $\deg(g) = (-\mu_r, \mu_r)$, which is a contradiction.

To finish the proof, we have to analyze Case 2. Now, we assume that
$$l_0 = \cdots = l_s = \lambda_0 = \cdots = \lambda_s = 0,$$
and we distinguish two cases (A) and (B) as follows.
(A) There is $0 \leq j \leq r-1$ such that $m_j \geq 1$ and $\mu_j \geq 1$. Then, we have
$$fw_1 - gw_2 = y_j \left(\frac{f}{y_j} w_1 - \frac{g}{y_j} w_2 \right).$$
(B) Otherwise, we assume that for any $0 \leq j \leq r-1$ we have
$$m_j \lambda_j = 0.$$
Then, there are indices $0 \leq j, \nu \leq r-1$ such that $m_j \geq 1$ and $\mu_\nu \geq 1$. Indeed, if either $m_0 = \cdots = m_{r-1} = 0$ or $\mu_0 = \cdots = \mu_{r-1} = 0$ we would have $\deg(f) = (-m_r, m_r)$ or $\deg(g) = (-\mu_r, \mu_r)$, which is a contradiction. Thus, without loss of generality we may suppose $j = 0$ and $\nu = 1$. Hence,
$$fw_1 - gw_2 = y_0 \left(\frac{f}{y_0} w_1 - \frac{g}{y_1} \frac{w_2 y_1}{y_0} \right).$$

As before, if $w_1 \neq \frac{w_2 y_1}{y_0}$ the result follows directly. Otherwise, we have

$$w_1 y_0 = w_2 y_1$$

and then $fw_1 - gw_2$ is a multiple of a syzygy of degree $(a, b+1)$. Now the proof is complete. $\square$

This result has a geometric consequence on the structure of the syzygy bundle M_L:

Corollary 4.3.3. *Let $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^s}^r \oplus \mathcal{O}_{\mathbb{P}^s}(1))$ be a blow-up of $\mathbb{P}^{s+r}$ at a linear subspace of dimension $r-1$. Let $a, b > 0$ be two integers and $L = \mathcal{O}_X(a,b)$ an ample line bundle on X. Then, the minimal locally free resolution of the syzygy bundle M_L associated to L begins as*

$$\mathcal{O}(-1,0)^{\lambda} \oplus \mathcal{O}(0,-1)^{\mu} \to M_L \to 0,$$

for some integers $\lambda, \mu > 0$.

In particular, for any $q \leq 1$ we have the beginning of the minimal locally free resolution of $\bigwedge^q M_L$:

$$\bigoplus_{q_1+q_2=q} \mathcal{O}(-q_1,-q_2)^{\beta_{q_1,q_2}} \to \bigwedge^q M_L \to 0,$$

for some integers β_{q_1,q_2}.

Proof. The Cox ring of X is the $\mathbb{Z}^2-$graded polynomial ring

$$S = \mathbb{C}[x_0, \ldots, x_s, y_0, \ldots, y_r]$$

with a $\mathbb{Z}^2$ grading given by

$$\begin{array}{ll} \deg(x_i) = (1,0), & \text{for } 0 \leq i \leq s \\ \deg(y_j) = (0,1), & \text{for } 0 \leq j \leq r-1 \\ \deg(y_r) = (-1,1). & \end{array}$$

On the other hand, $M_L(-L) \cong \widetilde{K_L}$ is the sheaffification of the syzygy module of the monomial ideal

$$I_{a,b} = (x_0^{a_0} \cdots x_s^{a_s} y_0^{b_0} \cdots y_r^{b_r} \mid a_0 + \cdots + a_s = a + b_r,\ b_0 + \cdots + b_r = b).$$

By Proposition 4.3.2, we have that the minimal free resolution of K_L begins as:

$$S(-a-1,-b)^{\lambda} \oplus S(-a,-b-1)^{\mu} \to K_L \to 0.$$

Hence, the result follows by sheaffifying and then twisting this presentation by $\mathcal{O}_X(a,b)$. □

Finally, we are able to establish the main result of this section.

Theorem 4.3.4. *Let $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^s}^r \oplus \mathcal{O}_{\mathbb{P}^s}(1))$ be the blow-up of $\mathbb{P}^{r+s}$ along a linear subspace $Z \subset \mathbb{P}^{r+s}$ of dimension $r-1$. Consider any ample line bundle $L = \mathcal{O}_X(a,b)$ on X with $a,b>0$. Then, the syzygy bundle M_L is $L-$stable.*

Proof. Let us denote $N = N(s,r,a,b) = \mathrm{h}^0(X, \mathcal{O}_X(L))$. By Lemma 4.1.4, it is enough to see that for any $0<q<N-1$ and any line bundle $G = \mathcal{O}_X(x,y)$ satisfying that

$$G \cdot L^{r+s-1} \leq q\frac{L^{r+s}}{N-1},$$

we have

$$\mathrm{H}^0(X, \bigwedge^q M_L(x,y)) = 0.$$

Notice that G needs to be effective, thus we may assume that $x+y \geq 0$ and $y \geq 0$. Moreover, by Corollary 4.3.3, if $x+y<q$ then we have already $\mathrm{H}^0(X, \bigwedge^q M_L(x,y)) = 0$. Hence, we may also assume that G satisfies $x+y \geq q$, and next we see that in this case we have the inequality:

$$G \cdot L^{r+s-1} > q\frac{L^{r+s}}{N-1} \tag{4.2}$$

finishing the proof.

We use the description of the intersection products on X given before to express both sides of (4.2):

$$
\begin{aligned}
&G \cdot L^{r+s-1} = \\
&\quad (x[D_{\rho_0}] + y[D_{\eta_0}]) \sum_{i=0}^{s} \binom{r+s-1}{i} a^i b^{r+s-1-i} [D_{\rho_0}]^i [D_{\eta_0}]^{r+s-1-i} \\
&\quad = (x+y) \sum_{i=0}^{s-1} \binom{r+s-1}{i} a^i b^{r+s-1-i} + \binom{r+s-1}{s} a^s b^{r-1} y \\
&\quad = b^{r-1} \left((x+y) \sum_{i=0}^{s-1} \binom{r+s-1}{i} a^i b^{s-i} + \binom{r+s-1}{s} a^s y \right).
\end{aligned}
$$

Similarly, we have

$$
L^{r+s-1} = b^r \left((a+b) \sum_{i=0}^{s-1} \binom{r+s-1}{i} a^i b^{s-1-i} + \binom{r+s-1}{s} a^s \right). \tag{4.3}
$$

Thus (4.2) is equivalent to

$$
\begin{aligned}
&(x+y) \sum_{i=0}^{s-1} \binom{r+s-1}{i} a^i b^{s-i} + \binom{r+s-1}{s} a^s y > \\
&\qquad q \frac{b \left((a+b) \sum_{i=0}^{s-1} \binom{r+s-1}{i} a^i b^{s-1-i} + \binom{r+s-1}{s} a^s \right)}{N-1}.
\end{aligned} \tag{4.4}
$$

Since $x + y \geq q$ and $y \geq 0$ we can bound the left hand side of (4.4) as

$$
\begin{aligned}
&(x+y) \sum_{i=0}^{s-1} \binom{r+s-1}{i} a^i b^{s-i} + \binom{r+s-1}{s} a^s y \geq \\
&\qquad qb \sum_{i=0}^{s-1} \binom{r+s-1}{i} a^i b^{s-1-i}.
\end{aligned} \tag{4.5}
$$

Thus, reducing the proof of (4.4) into seeing

$$
(N-1-a-b) \sum_{i=0}^{s-1} \binom{r+s-1}{i} a^i b^{s-i} > \binom{r+s-1}{s} a^s. \tag{4.6}
$$

Let now $S = \mathbb{C}[x_0, \dots, x_s, y_0, \dots, y_r]$ be the Cox ring of X. Then, the vector space $\mathrm{H}^0(X, \mathcal{O}_X(a,b))$ is isomorphic to the degree$-(a,b)$ homogeneous piece of S. In particular,

$$\begin{aligned}
N &= \sum_{i=0}^{b} \binom{r-1+b-i}{r-1}\binom{s+a+i}{s} \\
&= \binom{r-1+b}{r-1}\binom{s+a}{s} + \sum_{i=1}^{b-1} \binom{r-1+b-i}{r-1}\binom{s+a+i}{s} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \binom{s+a+b}{s} \\
&\geq \binom{r-1+b}{r-1}\binom{s+a}{s} + 1 + a + b \\
&\quad + \frac{(a+b)(a+b+1)\sum_{i=2}^{s}(2+a+b)\cdots(\widehat{i+a+b})\cdots(s+a+b)}{s!} \\
&\geq \binom{r-1+b}{r-1}\binom{s+a}{s} + 1 + a + b.
\end{aligned}$$

Applying this inequality and the fact that $b \geq 1$ we can finally show (4.6), ending the proof:

$$\begin{aligned}
(N-1-a-b)\sum_{i=0}^{s-1}\binom{r+s-1}{i}a^i b^{s-i} &\geq r\binom{s+a}{s}\sum_{i=0}^{s-1}\binom{r+s-1}{i}a^i \\
&\geq r\binom{s+a}{s}\binom{r+s-1}{s-1}a^{s-1} \\
&= r\frac{(s+a)(s-1+a)\cdots(1+a)}{s(s-1)\cdots 1}\frac{s}{r}\binom{r+s-1}{s}a^{s-1} \\
&> (s+a)\binom{r+s-1}{s}a^{s-1} \\
&> \binom{r+s-1}{s}a^s.
\end{aligned}$$

□

4.3.1 Rigidness of syzygy bundles

In Theorem 4.3.4 we have seen that the syzygy bundle M_L corresponding to an ample line bundle L on a blow-up $X = \mathrm{Bl}_Z(\mathbb{P}^n)$ of $\mathbb{P}^n$ along a linear subspace $Z \subset \mathbb{P}^n$ of codimension $r-1$ is L−stable. Thus, we may consider the moduli space

$$\mathcal{M} = \mathcal{M}_X(N-1; c_1, \ldots, c_{\min\{N-1,n\}})$$

of stable vector bundles E on X of rank $N-1 = \mathrm{h}^0(X, \mathcal{O}(L)) - 1$ and Chern classes $c_i(E) = c_i := c_i(M_L)$ for $1 \leq i \leq \min\{N-1, n\}$. In this subsection, we use the stability of M_L to study the moduli space $\mathcal{M}$ locally around $[M_L]$, and we see that the syzygy bundles M_L are infinitesimally rigid unless $n = 2$ and $L = \mathcal{O}_X(a,b)$ with $a \geq 1$ and $b \geq 2$. In this particular case we prove that M_L is unobstructed, so $[M_L]$ is a smooth point in $\mathcal{M}$, and we compute the dimension of the Zariski tangent space $T_{[M_L]}\mathcal{M}$ of the moduli space $\mathcal{M}$ at $[M_L]$.

Let us recall (see Proposition 4.1.3) that the Zariski tangent space of $\mathcal{M}$ at $[M_L]$ is canonically given by

$$T_{[M_L]}\mathcal{M} \cong \mathrm{Ext}^1(M_L, M_L) \cong \mathrm{H}^1(X, M_L \otimes M_L^{\vee}).$$

In particular, we say that M_L is infinitesimally rigid if $[M_L]$ is an isolated point or, equivalently, $\dim T_{[M_L]}\mathcal{M} = 0$. We have the following result.

Theorem 4.3.5. *Let $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^s}^r \oplus \mathcal{O}_{\mathbb{P}^s}(1))$ be the blow-up of $\mathbb{P}^{r+s}$ along a linear subspace $Z \subset \mathbb{P}^{r+s}$ of dimension $r-1$. Fix any ample line bundle $L = \mathcal{O}_X(a,b)$ on X with $a, b > 0$. Then, the syzygy bundle M_L is infinitesimally rigid unless $r+s=2$ and $b \geq 2$.*

If $r+s=2$ and $b \geq 2$, then M_L is unobstructed and we have

$$\dim_{\mathbb{C}} T_{[M_L]}\mathcal{M} = \left[\sum_{i=0}^{b-2}(a+i)^{b-2-i}\right]^{\mathrm{h}^0(X,\mathcal{O}_X(L))}.$$

Proof. Let us start studying $\mathrm{H}^1(X, M_L \otimes M_L^{\vee})$. We consider the exact sequence

$$0 \to M_L \to \mathcal{O}_X^N \to \mathcal{O}_X(L) \to 0. \tag{4.7}$$

Taking the long exact sequence of cohomology we obtain that

$$\mathrm{H}^i(X, M_L) = 0,$$

for $i \geq 0$. On the other hand, twisting the exact sequence (4.7) by $\mathcal{O}_X(-L)$ we have the following description of the cohomology of $M_L(-L)$:

$$\begin{aligned}
&\mathrm{H}^0(X, M_L(-L)) = 0\\
&\mathrm{H}^1(X, M_L(-L)) \cong \mathrm{H}^0(X, \mathcal{O}_X) \oplus \mathrm{H}^1(X, \mathcal{O}_X(-L))^N\\
&\mathrm{H}^i(X, M_L(-L)) \cong \mathrm{H}^i(X, \mathcal{O}_X(-L))^N, \text{ for } i \geq 2.
\end{aligned}$$

On the other hand, dualizing the exact sequence (4.7) and tensoring it by M_L, we obtain:

$$0 \to M_L(-L) \to M_L^N \to M_L \otimes M_L^\vee \to 0.$$

Taking again the long exact sequence of cohomology and using the above vanishings, we obtain that

$$\mathrm{H}^i(X, M_L \otimes M_L^\vee) \cong \mathrm{H}^{i+1}(X, M_L(-L)). \tag{4.8}$$

In particular $\mathrm{H}^1(X, M_L \otimes M_L^\vee) \cong \mathrm{H}^2(X, \mathcal{O}_X(-L))^N$, and by Kodaira's vanishing theorem we have that if $\dim(X) > 2$, then $\mathrm{H}^2(X, \mathcal{O}_X(-L)) = 0$ and hence M_L is infinitesimally rigid.

It only remains to study the case $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$. Using the projection formula we obtain that

$$\begin{aligned}
\mathrm{H}^i(X, \mathcal{O}_X(a,b)) &= \mathrm{H}^i(\mathbb{P}(\mathcal{E}), \pi^*\mathcal{O}_{\mathbb{P}^1}(a) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(b)) \cong\\
&\begin{cases} \mathrm{H}^i(\mathbb{P}^1, \operatorname{Sym}^b \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^1}(a)), & b \geq 0\\ 0, & b = -1\\ \mathrm{H}^{2-i}(\mathbb{P}^1, \operatorname{Sym}^{-b-2} \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^1}(-1-a))^\vee, & b \leq -2. \end{cases}
\end{aligned} \tag{4.9}$$

Hence for any ample line bundle of the form $L = \mathcal{O}_X(a, 1)$ in X, we already have that M_L is infinitesimally rigid. Now, we consider an ample line bundle $L = \mathcal{O}_X(a, b)$ with $a \geq 1$ and $b \geq 2$. Since $\mathcal{E} = \mathcal{O}_{\mathbb{P}^s}^r \oplus \mathcal{O}_{\mathbb{P}^s}(1)$, we have that

$$\operatorname{Sym}^\ell \mathcal{E} \cong \bigoplus_{i=0}^{\ell} \mathcal{O}_{\mathbb{P}^1}(i)^{\ell - i},$$

so (4.9) yields

$$\begin{aligned}
\mathrm{H}^2(X, \mathcal{O}_X(-L)) &\cong \mathrm{H}^0(\mathbb{P}^1, \operatorname{Sym}^{b-2} \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^1}(-1+a))^\vee \cong\\
&\bigoplus_{i=0}^{b-2} \left[\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(a+i-1))^\vee\right]^{b-2-i}.
\end{aligned} \tag{4.10}$$

Thus, we have that $\mathrm{H}^2(X, \mathcal{O}_X(-L)) = 0$ if and only if $a + b < 3$ which cannot happen. Hence, to finish the proof, we ought to see that if $L = \mathcal{O}_X(a, b)$ is an ample line bundle on X with $b \geq 2$, then M_L is unobstructed. Since the obstruction space at $[M_L]$ is a subspace of $\mathrm{Ext}^2(M_L, M_L) \cong \mathrm{H}^2(X, M_L \otimes M_L)$, it is enough to use (4.8) and see that $\mathrm{H}^3(X, \mathcal{O}_X(-L)) = 0$. Again, by (4.9) we have

$$\mathrm{H}^3(X, \mathcal{O}_X(-L)) \cong \bigoplus_{i=0}^{b-2} \left[\mathrm{H}^1(\mathbb{P}^1, \mathcal{O}(a+i-1))^\vee\right]^{b-2-i}.$$

Since for all $0 \leq i \leq b-2$, we have that $a + i - 1 \geq a - 1 \geq 0$, we obtain that $\mathrm{H}^1(\mathbb{P}^1, \mathcal{O}(a+i-1)) = 0$. Therefore $\mathrm{H}^3(X, \mathcal{O}_X(-L)) = 0$ and M_L is unobstructed. In this case we may use (4.10) to compute the dimension of the Zariski tangent space $T_{[M_L]}\mathcal{M}$ of the moduli space $\mathcal{M}$ at $[M_L]$:

$$\begin{aligned} \dim_{\mathbb{C}} T_{[M_L]}\mathcal{M} &= \mathrm{h}^1(X, M_L \otimes M_L^\vee) \\ &= \mathrm{h}^2(X, \mathcal{O}_X(-L))^N \\ &= \left[\sum_{i=0}^{b-2} (a+i)^{b-2-i}\right]^N. \end{aligned}$$

□

Resum en català

En l'àmbit de la geometria algebraica i l'àlgebra commutativa, aquesta tesi contribueix a l'estudi dels feixos i fibrats vectorials en varietats tòriques. Al llarg de la tesi s'utilitza, des de diferents perspectives, la teoria de varietats tòriques en relació amb dos objectius principals: aprofundir en el coneixement de l'estructura dels feixos equivariants en varietats tòriques i contribuir a la conjectura d'Ein-Lazarsfeld-Mustopa sobre l'estabilitat dels fibrats de sizígies en varietats projectives.

El contingut principal d'aquesta tesi es desenvolupa al llarg dels tres capítols posteriors al Capítol 1 de preliminars. A mesura que s'avança en el text s'imposen estructures més concretes sobre els feixos que s'investiguen. Començant per feixos lliures de torsió, i passant per l'estudi de feixos reflexius, la dissertació acaba amb una contribució a la conjectura d'Ein-Lazarsfeld-Mustopa, estudiant l'estabilitat dels fibrats de sizígies en certes varietats tòriques projectives. A continuació es descriu el contingut de cada capítol amb més detall:

En el Capítol 1, s'exposen les definicions i resultats preliminars sobre varietats tòriques que s'utilitzaran al llarg de la tesi. Amb especial èmfasi es recull la noció d'anell de Cox d'una varietat tòrica, així com la correspondència algebro-geomètrica entre feixos i mòduls que se'n deriva. Finalment, dirigim la nostra atenció als feixos equivariants sobre una varietat tòrica i la construcció de Klyachko, que els descriu mitjançant una família de filtracions d'un espai vectorial.

En el Capítol 3, ens centrem en l'estudi dels feixos equivariants lliures de torsió, que tenen una estreta relació amb la teoria d'ideals monomials. Es comença introduïnt la noció de diagrama de Klyachko d'un ideal monomial, que en generalitza el clàssic diagrama en escaleta. Després d'introduir mètodes per a calcular el diagrama de Klyachko, l'utilitzem per descriure el primer mòdul de cohomologia local d'un ideal monomial. Finalment, apliquem aquesta noció al càlcul de la funció de Hilbert i polinomi de Hilbert d'un ideal monomial. Com a coneqüència, caracteritzem tots els ideals monomials amb polinomi de Hilbert constant, en termes del seu diagrama de Klyachko.

El capítol 4 se centra en l'estudi dels feixos reflexius equivariants en una varietat tòrica. Per tal de descriure com varia, en torçar per un fibrat de línia, la dimensió de les seccions globals d'un feix reflexiu equivariant, introduïm una família de politops reticul·lars. En particular, d'aquesta família de politops, se'n deriva un mètode per a calcular el polinomi de Hilbert d'un feix reflexiu equivariant. Quan la varietat tòrica subjacent té un *splitting fan*, aquest estudi es pot fer més precís. En aquest cas, aportem cotes superior i inferior pel grau (multigraduat) inicial d'un feix reflexiu equivariant, així com una cota superior del seu índex (multigraduat) de regularitat. Finalment, apliquem aquests resultats per a introduir un mètode per tal de trobar fibrats d'Ulrich equivariants sobre una superfície de Hirzebruch. A partir d'aquest mètode, donem un exemple d'un fibrat Ulrich equivariant de rang 3 sobre la primera superfície de Hirzebruch.

El Capítol 4 tracta l'estabilitat dels fibrats de sizígies sobre una varietat tòrica projectiva. Demostrem que, per a qualsevol polarització d'un *blow-up* d'un espai projectiu al llarg d'un subespai lineal, el fibrat de sizígies resultant és estable, contribuïnt així a la conjectura d'Ein-Lazarsfeld-Mustopa. Finalment, estudiem la rigidesa d'aquests fibrats de sizígies i demostrem que sempre es corresponen amb punts llisos en el seu espai de mòduli.

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