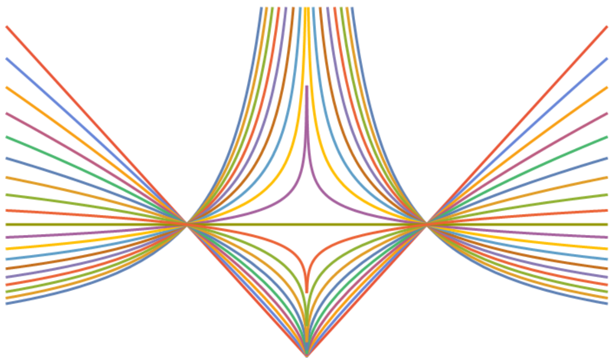
UNIVERSITAT DE BARCELONA
Doctorat de Matemàtiques i Informàtica

DOCTORAL DISSERTATION

Weighted restricted weak-type extrapolation on classical Lorentz spaces



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Resum

TESI DOCTORAL

Weighted restricted weak-type extrapolation on classical Lorentz spaces

per SERGI BAENA MIRET

Un resultat important en anàlisi harmònica és el teorema d'extrapolació de Rubio de Francia. En la seva versió original diu que si T és un operador sublineal que està acotat en $L^{p_0}(v)$, per a algun $p_0 \geqslant 1$ i per cada $v \in A_{p_0}$, llavors T està acotat en $L^p(v)$ per a qualsevol $p > 1$ i $v \in A_p$.

Ara bé, tot i que el teorema de Rubio de Francia ha demostrat ser molt útil en la pràctica, no permet obtenir estimacions en $p = 1$. És per això que en [61] es va desenvolupar una nova teoria d'extrapolació per tal de donar una solució a aquest problema, mostrant que les estimacions ponderades de tipus feble restringit (p, p) per a $p > 1$ i per a una classe una mica més gran que A_p (denotada per $\widehat{A}_p$) permeten arribar a $p = 1$.

De fet, en aquesta tesi comencem per veure que el recíproc del resultat anterior també és cert; és a dir, estudiem les propietats de les acotacions per als operadors T que són de tipus feble restringit $(1, 1)$ per a pesos en A_1 i demostrem que aquesta condició és una condició "norma", ja que és equivalent a estimacions ponderades restringides de tipus feble (p, p) per a pesos $\hat{A}_p$. Com a conseqüència obtenim, per exemple, acotacions d'operadors que es donen com a promig d'operadors del tipus anterior.

A més a més, presentem noves estimacions ponderades de tipus restringit en espais de Lorentz clàssics per a operadors que satisfan estimacions ponderades de tipus feble restringit (p, p), $p \geqslant 1$, i estenem després aquests resultats a l'extrapolació limitada i, a més, a l'extrapolació multi-variable. Com a resultat, obtenim noves acotacions ponderades d'espais de Lorentz clàssics per a operadors importants en l'anàlisi harmònica com ara aquests que satisfan una desigualtat de Fefferman-Stein, multiplicadors de Fourier de tipus Hörmander, operadors *rough*, operadors *sparse*, l'operador de Bochner-Riesz, entre d'altres. A més, a partir de les acotacions anteriors concluïm estimacions puntuals per a la reordenada decreixent d'aquests operadors.

Finalment, també estudiem estimacions de tipus fort sobre espais de Lorentz clàssics ponderats.

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Abstract

DOCTORAL DISSERTATION

Weighted restricted weak-type extrapolation on classical Lorentz spaces

by SERGI BAENA MIRET

An important result in Harmonic Analysis is the extrapolation theorem of Rubio de Francia. In its original version says that if T is a sublinear operator that is bounded in $L^{p_0}(v)$, for some $p_0 \geqslant 1$ and every $v \in A_{p_0}$, then T is bounded in $L^p(v)$ for any $p > 1$ and $v \in A_p$.

Although the Rubio de Francia theorem has proven to be very useful in practice, it does not allow to get estimates at $p = 1$. That is why in [61] it was developed a new extrapolation theory in order to give a solution to this issue, showing that weighted restricted weak-type (p, p) estimates for $p > 1$ and for an slightly bigger class than A_p (denoted by $\widehat{A}_p$) yield estimates at $p = 1$.

Indeed, in this thesis we start by seeing that the converse of the previous result is also true; that is, we study boundedness properties for operators T that are of restricted weak-type $(1, 1)$ for weights in A_1 and we prove that this condition is a "norm" condition since it is equivalent to weighted restricted weak-type (p, p) for $\hat{A}_p$ weights. As a consequence, we can obtain, for instance, boundedness for operators which are given as an average of operators of the above type.

As well, we present new weighted restricted estimates on classical Lorentz spaces for operators that satisfy weighted restricted weak-type (p, p) estimates, $p \geqslant 1$, extending then these results to the limited setting and, as well, to the multi-variable setting. As a consequence, we obtain new weighted estimates on classical Lorentz spaces for important operators in Harmonic Analysis such as operators that satisfy a Fefferman-Stein's inequality, Fourier multipliers of Hörmander type, rough operators, sparse operators, the Bochner-Riesz operator, among others. Further, from the previous estimates we prove pointwise estimates for the decreasing rearrangement of such operators.

Finally, we also study strong-type estimates on weighted classical Lorentz spaces.

Contents

Acknowledgements

“Life is very short and anxious for those who forget the past, neglect the present, and fear the future.” - Seneca

Memento mori (Latin for ‘remember that you [have to] die’) is a symbolic trope acting as a reminder of the inevitability of death. We all are going to die, we are all going to fall (which is not necessarily bad) so we should all be grateful for each breath, each experience and all the advice, help, love and affection that we have received from all the people that go along with us in each moment of our live.

It is said that we are the average of the five closest people around us, with whom we spend the most time. However, that influence does not stop anywhere near them. In fact, it is far more dispersed and includes people we have not even met yet.

Here, I will try to be fair and honest (and I apologize if I forget someone) with all the people who have influenced all the journey that I have gone through during the last years and that we can call now *doctoral dissertation*.

Raquel, my dear, what to tell you I have not told you yet. You are the person after me who has lived the most the process of my thesis (with all its ups and downs) and with whom I am most grateful. In all my moments of doubt, you have been like a lighthouse that has illuminated my path. You are compassionate, loving and attentive. I still remember the first result I proved, and it was by your side, and since then everything has been easier. You make me feel safe, happy and give me strength. Perhaps each of us can move faster on our own, but together we go further. We are and will be always अहिंसा (*ahimsa*) team. Thank you for having accompanied me up to that moment and for willing to continue this journey with me so we can live new and (if possible) better experiences together. I love you.

I would like to give special thanks to my advisor Dra. María Jesús Carro. I know that maybe at the beginning it was not so easy with me (I used to think too fast, and I needed to relax and reconsider what I was saying) and I admit that the first year of PhD was not the best one. However, you trusted me and motivated me to work harder and think twice. By your side I have improved a lot as a researcher and as a mathematician. You have supported and helped me in all the moments of despair, and when you have decided to give me a warning you always aimed to my overcoming. All in all, working with you have been easy

from the beginning, and I know that you have cared a lot about me. And now here we are, at the end of the journey, and I am sure I could have not done it without you.

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I would like to mention the first day I came to my office and I sit to my chair in front of my table. I was excited and scared equally, but among all the emotions I could feel, especially I was proud. I always thought myself as a mathematician, but that moment I knew it for real. There I made new special friends as Carlos Cruz, Eduard Roure, Gladston Duarte, Adrián Hinojosa and Camilo Torres, and I also had the opportunity to deepen with already well known friends as Andrés Rojas, Daniel Gil, Martí Salat, Liena Colarte, Dan Paraschiv, Víctor de la Torre, Sergi Arias and Juan Pello. Indeed, I also remember the day that some of us decided to share what we know (and also what we do not know) so we could discuss about mathematics, no matter about which area we could talk. That day we recovered SIMBa, an informal seminar of mathematics helded in Barcelona that allowed me to meet the following incredible people: Salvador Borrós, Clara Cufí, Enric Florit, Andriana Karuk, David Martínez, Pau Mir, Ainoa Murillo, Ignasi Sánchez, Damià Torres and Eduard Vilalta. Thanks for sharing your love for mathematics with me.

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as Fundación El Hogar Animal Sanctuary, Birds Friends, ADE, and more. There I have met incredible people (Elena Tova, Gemma Alegre, Yuyu, Luís, Shandra, Eric, Aldo, Sandra, Bernat...), I have helped many beautiful animals who have suffered and who have become invisible for years (Ahimsa, Zorte, Ruth, Justice, Greta, Calabaza, Zoe, Félix, Olivia, Elur, Kat...), and I have been filled with energy, love and smiles.

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Sergi Baena Miret
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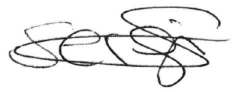

Chapter 1

Introduction

This short chapter is intended to be a brief description of our project. In Section 1.1, we include general notation and conventions. In Section 1.2, we introduce the starting point and motivation of this manuscript. To do so, we start with a short review on Rubio de Francia extrapolation, without going into the details, where we talk about the original Rubio de Francia theorem, the Hardy-Littlewood maximal operator, the Hilbert transform and a more recent result based on the Rubio de Francia extrapolation theorem. Further, to end this section we explain how we have organized this thesis. Finally, in Sections 1.3, 1.4, 1.5 and 1.6 we state the main extrapolation results on this manuscript (which correspond to Chapters 3, 4, 6 and 7 respectively) and its more interesting consequences.

1.1 Notation and conventions

Let (R, Σ, μ) be a σ-finite nonatomic measure space. In all the thesis we will use the following notation:

$\mathbb{N}$	the set of all natural numbers, not including 0
$\mathbb{Z}$	the set of all integers numbers
$\mathbb{R}$	the set of all real numbers
$\mathbb{R}^+$	the set of all positive real numbers
$\mathbb{R}^n$	the n-fold product of $\mathbb{R}$
χ_E	the characteristic function of a set E
dt, dr, ds dy, dz, dx	the Lebesgue measure
$\mathcal{M}$	the set of all μ-measurable functions
$\mathcal{M}^+$	the set of all nonnegative μ-measurable functions
$\log$	the logarithm with base e
$\inf$	the infimum
$\sup$	the supremum
$\operatorname{ess\,inf}$	the essential infimum
$\operatorname{ess\,sup}$	the essential supremum
$\mathcal{C}(E)$	the set of continuous functions on $E \subseteq \mathbb{R}^n$

$\mathcal{C}_c(E)$	the set of continuous functions on $E \subseteq \mathbb{R}^n$ with compact support
$\mathcal{C}^k(E)$	the set of k-times differentiable functions on $E \subseteq \mathbb{R}^n$
$\mathcal{C}^\infty(E)$	the set of infinitely differentiable (or smooth) functions on $E \subseteq \mathbb{R}^n$
$\mathcal{C}_c^\infty(E)$	the set of functions of $\mathcal{C}^\infty(E)$ on $E \subseteq \mathbb{R}^n$ with compact support
$\mathcal{S}(E)$	the Schwartz space of $\mathcal{C}^\infty(E)$ functions on $E \subseteq \mathbb{R}^n$ which decrease rapidly
$L^p(E)$	the Lebesgue space of $p \in (0,\infty)$ integrable functions on $E \subseteq \mathbb{R}^n$
$L^1_{\text{loc}}(E)$	the Lebesgue space of locally integrable functions on $E \subseteq \mathbb{R}^n$
p'	the conjugate exponent of $p > 1$; that is $\frac{1}{p} + \frac{1}{p'} = 1$,

Given a μ-measurable set E, we will use the notation

$$\mu(E) = \int_E d\mu.$$

Indeed, if μ is the Lebesgue measure, then we will simply write $|E|$.

For a given operator T and real or complex function spaces $\mathbb{X}$ and $\mathbb{Y}$ endowed with quasi-norms $\|\cdot\|_{\mathbb{X}}$ and $\|\cdot\|_{\mathbb{Y}}$ respectively, then $T : \mathbb{X} \to \mathbb{Y}$ will stand for that T is well defined on $\mathbb{Y}$ and

$$\|T\|_{\mathbb{X}\to\mathbb{Y}} = \sup_{f\in\mathbb{X}} \frac{\|Tf\|_{\mathbb{Y}}}{\|f\|_{\mathbb{X}}} < \infty.$$

When $\mathbb{X} = \mathbb{Y}$, we will write $\|T\|_{\mathbb{X}} := \|T\|_{\mathbb{X}\to\mathbb{X}}$. Further, we will say that $\mathbb{X} \subseteq \mathbb{Y}$ *continuously* if for every $f \in \mathbb{X}$, then $f \in \mathbb{Y}$ and $\|Id\|_{\mathbb{X}\to\mathbb{Y}} < \infty$, where $Id(f) := f$ (or, what is the same, $\|f\|_{\mathbb{Y}} \leqslant \|Id\|_{\mathbb{X}\to\mathbb{Y}} \|f\|_{\mathbb{X}}$). Besides, we will say that T is a *linear* operator (resp. *sublinear*) if $T(f+g) = Tf + Tg$ (resp. $T(f+g) \leqslant Tf + Tg$). More generally, given a multi-variable operator T, we will say that T is a *multilinear* operator (resp. *submultilinear*) if is linear (resp. sublinear) in each variable.

In general, we will work in $\mathbb{R}^n$, with $n \in \mathbb{N}$. Unless otherwise specified, by a function f we will mean a real or complex-valued function on $\mathbb{R}^n$. If we say that a function is measurable, but we don't specify any measure, then it will be with respect to the Lebesgue measure on $\mathbb{R}^n$. The same applies to a measurable set and also to the expression a.e. (that is, *almost everywhere*). By a *cube* in $\mathbb{R}^n$ with side length $\ell > 0$, we mean a cube open on the right

$$Q = [x_1, x_1 + \ell) \times \cdots \times [x_n, x_n + \ell)$$

with sides parallel to the axes and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. As well, we will say that $\widetilde{T}$ is the *adjoint operator* of the linear operator T if for every measurable functions f, g,

$$\int_{\mathbb{R}^n} Tf(x)g(x)\,dx = \int_{\mathbb{R}^n} f(x)\widetilde{T}g(x)\,dx.$$

We will call *weight* a nonnegative locally integrable function in $\mathbb{R}^n$. Further, if v is a weight, then by considering the measure $v(x)\,dx$, we will denote for every measurable set E,

$$v(E) = \int_E v(x)\,dx.$$

Finally, as usual we shall use the symbol $A \lesssim B$ to indicate that there exists a universal positive constant C, independent of all important parameters, such that $A \leqslant CB$. When $A \lesssim B$ and $B \lesssim A$, we will write $A \approx B$. The constant C is called the *implicit constant*. Usually, we will denote implicit constants by c, $\tilde{c}$, c', C, $\tilde{C}$, or C'. In many occasions, they will depend on some parameters $\gamma_1, \ldots, \gamma_m$, $m \in \mathbb{N}$, and if we want to point out that dependence, we shall do it by using subscripts as $A \leqslant C_{\gamma_1,\ldots,\gamma_m} B$.

1.2 Background and Motivation

In Harmonic Analysis, given an operator T, the study of whether

$$T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n), \qquad \text{for some } p \geqslant 1,$$

appears many times. There are many techniques to face these kind of problems and one that has gained increased attention in the area, as it has proven to be very useful and effective, is the well-known extrapolation theorem of Rubio de Francia (see [162, 163]). Roughly speaking, it says that if T is a sublinear operator which satisfies that for some $1 \leqslant p_0 < \infty$ and every weight $v \in A_{p_0}$ (see Definition 2.2.1)

$$T : L^{p_0}(v) \to L^{p_0}(v), \tag{1.1}$$

with constant depending on $\|v\|_{A_{p_0}}$, then for any $1 < p < \infty$ and $v \in A_p$,

$$T : L^p(v) \to L^p(v), \tag{1.2}$$

with constant depending on $\|v\|_{A_p}$, where

$$\|f\|_{L^p(v)} = \left(\int_{\mathbb{R}^n} |f(x)|^p v(x)\, dx \right)^{\frac{1}{p}}$$

is the *weighted Lebesgue space*. Note that, in particular, this is true if we let $v = 1$ in (1.2), so $L^p(\mathbb{R}^n)$ estimates follow from weighted L^{p_0} estimates.

The case $p = 1$ is usually called the *endpoint* exponent and from (1.1) one can not expect to obtain (1.2) for it. For example, this is the case of the *Hardy-Littlewood maximal operator* M (first introduced by G.H. Hardy and J.E. Littlewood [104] for $n = 1$ and then by N. Wiener [180] for $n > 1$) defined as

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy, \qquad f \in L^1_{\text{loc}}(\mathbb{R}^n),$$

where the supremum is taken over all cubes of Q in $\mathbb{R}^n$ containing $x \in \mathbb{R}^n$. Indeed, it is known that

$$M : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n), \qquad \forall p > 1,$$

but $M : L^1(\mathbb{R}^n) \nrightarrow L^1(\mathbb{R}^n)$ and the only function $f \in L^1(\mathbb{R}^n)$ for which $Mf \in L^1(\mathbb{R}^n)$ is $f = 0$ (see for instance [88]). However, if we introduce the *weak*-$L^1(\mathbb{R}^n)$ space, which is defined by those functions f such that

$$\|f\|_{L^{1,\infty}(\mathbb{R}^n)} = \sup_{t>0} t\, |\{x \in \mathbb{R}^n : |f(x)| > t\}| < \infty,$$

then from the Chebyshev's inequality is easy to see that $L^1(\mathbb{R}^n) \subseteq L^{1,\infty}(\mathbb{R}^n)$ continuously and it can be proved that now

$$M : L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n).$$

That is also the case of the *Hilbert Transform* H (first introduced by D. Hilbert [105, 106], although it was not until 1924 that G.H. Hardy called it Hilbert's operator due to its contributions to this operator [102, 103]) defined as

$$Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\, dy, \qquad f \in \mathcal{C}^\infty(\mathbb{R}),\, x \in \mathbb{R},$$

whenever this limit exists almost everywhere (see for instance [88]).

This naturally arises the question about what could we obtain if we weaken the hypothesis in (1.1) by

$$T : L^{p_0}(v) \to L^{p_0,\infty}(v),$$

with $L^{p_0,\infty}(v)$ being the weighted Lorentz space of measurable functions such that

$$\|f\|_{L^{p_0,\infty}(v)} = \sup_{t>0} t v\left(\{x \in \mathbb{R}^n : |f(x)| > t\}\right)^{\frac{1}{p_0}} < \infty,$$

but it turns out that even in this case, it is not possible to extrapolate until the endpoint $p = 1$ (just consider, for example, the operator $M^2 := M \circ M$).

That is why in [61] it was developed a new extrapolation theory in order to give a solution to this issue, where it was needed to introduce the weighted Lorentz spaces $L^{p,1}(v)$, $p \geqslant 1$, defined by those functions f such that

$$\|f\|_{L^{p,1}(v)} = p \int_0^\infty v\left(\{x \in \mathbb{R}^n : |f(x)| > t\}\right)^{\frac{1}{p}} dt < \infty$$

and which satisfy the chain of inclusions $L^{p,1}(v) \subseteq L^p(v) \subseteq L^{p,\infty}(v)$ continuously. Further, the authors also need to consider a bigger class of weights than A_p denoted by $\hat{A}_p$ (see Definition 2.2.7). Then, its main result reads as follows: if T is an operator (not necessarily sublinear) which satisfies that for some $1 \leqslant p_0 < \infty$ and every weight $v \in \hat{A}_{p_0}$

$$T : L^{p_0,1}(v) \to L^{p_0,\infty}(v), \tag{1.3}$$

with constant depending on $\|v\|_{\hat{A}_{p_0}}$, then for $v \in A_1$, T is of *weighted restricted weak-type* $(1,1)$; that is, for every measurable set $E \subseteq \mathbb{R}^n$,

$$\|T\chi_E\|_{L^{1,\infty}(v)} \leqslant C_v v(E), \tag{1.4}$$

with C_v depending on n, p_0 and $\|v\|_{A_1}$. Furthermore, if T belongs to the subclass of sublinear operators called (ε, δ)-*atomic approximable* operators (see Definition 2.3.10) in [61] it was shown that, in fact, (1.4) holds for every function in $L^1(v)$; that is,

$$T : L^1(v) \to L^{1,\infty}(v), \qquad \forall v \in A_1,$$

with constant depending on $\|v\|_{A_1}$.

Here, we have to point out that one of the main differences between the endpoint $p = 1$ and the other cases fall on the function space $L^{1,\infty}(v)$. Unlike $L^p(v)$ for $p \geqslant 1$, or even $L^{p,\infty}(v)$ for $p > 1$, the space $L^{1,\infty}(v)$ cannot be normed for any weight v to become a Banach function space (see Section 2.1.1 for the notion of (quasi)-Banach function spaces). Therefore, in particular, the previous result allows to obtain boundedness for general operators T (at least for characteristic functions) with arrival space the quasi-Banach function space $L^{1,\infty}(\mathbb{R}^n)$ from boundedness with arrival space the Banach function space $L^{p,\infty}(v)$, for $p > 1$ and $v \in \hat{A}_p$. Indeed, that result is the starting point and motivation of our project, which is organized as follows.

In Chapter 2 we introduce some notions and definitions, as well as some important (new and old) results that we will use later on throughout this manuscript based on classical Lorentz spaces (see Section 2.1), several classes of weights (see Section 2.2) and the Rubio de Francia extrapolation theory (see Section 2.3).

In Chapter 3 we show that if (1.4) holds then (1.3) holds, so that, indeed, both conditions are equivalent and we conclude that (1.4) is a "norm" condition although $L^{1,\infty}(v)$ can not be normed (see Section 3.3). This allows us to get interesting estimates on average operators, Fourier multipliers (as, for instance, in the context of restriction multipliers), integral operators and the Bochner-Riesz operator (see Section 3.5).

In Chapter 4 we see that the condition (1.3) also yields restricted weak-type boundedness on the setting of classical Lorentz spaces and we also generalize it to what is known as *limited-(range) extrapolation* (see Section 4.2). Moreover, we also show that having restricted weak-type boundedness on classical Lorentz spaces for a given sublinear operator T is equivalent to a pointwise estimate on its decreasing rearrangement by a Calderón admissible type operator (see Section 4.3).

In Chapter 5, we prove new boundedness of important operators in Harmonic Analysis over classical Lorentz spaces and we also obtain interesting pointwise estimates on its decreasing rearrangement. In particular, we study operators that satisfy a Fefferman-Stein's inequality (see Section 5.1), Fourier multipliers (see Section 5.2) such as Fourier multipliers of Hörmander type (see Section 5.2.1), Fourier multipliers that satisfy a Fefferman-Stein's type inequality (see Section 5.2.2) and radial Fourier multipliers with a derivative condition (see Section 5.2.3), rough singular integrals (see Section 5.3), intrinsic square functions (see Section 5.4), sparse operators (see Section 5.5), the Assani operator (see Section 5.6) and the Bochner-Riesz operator (see Section 5.7).

In Chapter 6 we prove extrapolation results on classical Lorentz spaces but, this time, for multi-variable operators based on weighted restricted weak-type estimates (see Section 6.2) and weighted mixed-type estimates (see Section 6.3). As a consequence, we obtain new estimates on bilinear Fourier multipliers and multilinear sparse operators (see Section 6.4).

Finally, in Chapter 7 we consider weighted classical Lorentz spaces by assuming (1.1).

This open a new line of research that could extend the results shown on this project (but that need new and different techniques that the ones proposed on the previous chapters) that, undoubtedly, we will be working in the near future.

We now continue to state the main extrapolation results on this thesis and its more interesting consequences. For technical reasons, in all our extrapolation theorems we require that the constants in each bound must behave in a nondecreasing way on the constants of the weights involved. We refer to Chapter 2 for all the notions and definitions.

As far as possible, we have tried to provide precise bibliographic information about all the known results. Besides, some of the results of this monograph are included in [6, 15, 16, 17].

1.3 Weak-type $(1,1)$ for weights in A_1

The main goal of this chapter is to prove that if (1.4) holds then (1.3) does also. The keystone of its proof consists on a Sawyer-type inequality (which can be found in Lemma 3.2.2) and the main result can be stated as follows.

Theorem 3.3.1. *Assume that for some pair of nonnegative functions (f,g),*

$$\|g\|_{L^{1,\infty}(u)} \leqslant \varphi(\|u\|_{A_1}) \|f\|_{L^1(u)}, \qquad \forall u \in A_1,$$

with φ being a nondecreasing function on $[1,\infty)$. Then, for every $1<p<\infty$,

$$\|g\|_{L^{p,\infty}(v)} \leqslant \Phi(\|v\|_{\widehat{A}_p}) \|f\|_{L^{p,1}(v)}, \qquad \forall v \in \widehat{A}_p,$$

where

$$\Phi(r) = C_1\varphi(C_2 r^p) r^{p-1} (1+\log r)^{\frac{2}{p'}}, \qquad r \geqslant 1,$$

with C_1 and C_2 being two positive constants independent of all parameters involved.

This has as a consequence, together with [61, Theorem 2.11], that up to some constants we have that (1.3) and (1.4) are equivalent (see Corollary 3.3.2). Therefore, we obtain that (1.4) is a "norm" condition, from which, in particular, we can get restricted weak-type $(1,1)$ estimates for average operators of operators satisfying (1.4).

Corollary 3.5.1. *Assume that $\{T_\theta\}_\theta$ is a family of operators indexed in a probability measure space such that the average operator*

$$T_A f(x) = \int T_\theta f(x)\,dP(\theta), \qquad x \in \mathbb{R}^n,$$

is well defined and that

$$T_\theta : L^1(u) \to L^{1,\infty}(u), \qquad \forall u \in A_1,$$

with constant less than or equal to $\varphi(\|u\|_{A_1})$, where φ is a positive nondecreasing function on $[1,\infty)$. Then, for every measurable set $E \subseteq \mathbb{R}^n$,

$$\|T_A \chi_E\|_{L^{1,\infty}(u)} \lesssim \varphi(C\|u\|_{A_1})(1+\log\|u\|_{A_1}) u(E), \qquad \forall u \in A_1.$$

Moreover, if T_A is a sublinear (ε,δ)-atomic approximable operator, then

$$T_A : L^1(u) \to L^{1,\infty}(u), \qquad \forall u \in A_1,$$

with constant less than or equal to $C_1\varphi(C_2\|u\|_{A_1})\|u\|_{A_1}(1+\log\|u\|_{A_1})$.

By virtue of Theorem 3.3.1, we also deduce boundedness of Fourier multipliers (see Section 3.5.2) and integral operators (see Section 3.5.3), and we completely answer an open question formulated in [46] about the weighted restricted weak-type (p,p) boundedness of the Bochner-Riesz B_λ at the critical index $\lambda = \frac{n-1}{2}$ (see Section 3.5.4).

Moreover, there are some operators for which (1.3) does not hold for every $p_0 \geqslant 1$ but, at least, it does for a limited range. Hence, our second main result on this chapter consists on a generalization of Theorem 3.3.1 which contains also those kind of operators and reads as follows.

Theorem 3.4.1. *Assume that for some pair of nonnegative functions (f,g) and for some $1 \leqslant p_0 < \infty$ and $0 < \alpha \leqslant 1$,*

$$\|g\|_{L^{p_0,\infty}(u^\alpha)} \leqslant \varphi\left(\|u\|_{A_1}^\alpha\right)\|f\|_{L^{p_0,1}(u^\alpha)}, \qquad \forall u \in A_1,$$

with φ being a nondecreasing function on $[1,\infty)$. Then, for any $p_0 \leqslant p < \frac{p_0}{1-\alpha}$,

$$\|g\|_{L^{p,\infty}(v)} \leqslant \Psi\left(\|v\|_{\hat{A}_{p;(\alpha(p),\beta(p))}}\right)\|f\|_{L^{p,1}(v)}, \qquad \forall v \in \hat{A}_{p;(\alpha(p),\beta(p))},$$

where $\alpha(p) = 1 - \frac{p(1-\alpha)}{p_0}$, $\beta(p) = \frac{p-p_0}{p_0(p-1)}$ and

$$\Psi(r) = C_1\left(\frac{1}{p_0 - p(1-\alpha)}\right)^{\frac{p-p_0}{p}} \varphi\left(C_2 r^{\frac{\alpha p}{p_0-p(1-\alpha)}}\right) r^{\frac{\alpha(p-p_0)}{p_0-p(1-\alpha)}}(1+\log r)^{\frac{2(p-p_0)}{p}}, \qquad r \geqslant 1,$$

with C_1 and C_2 being two positive constants independent of all parameters involved.

As a consequence of Theorem 3.4.1 we get new weighted estimates for the Bochner-Riesz operator B_λ below the critical index; that is, for $0 < \lambda < \frac{n-1}{2}$ (see Section 3.5.4).

However, under the hypothesis of Theorem 3.4.1 we observe that although p can be set to p_0, it must be strictly less than $\frac{p_0}{1-\alpha}$. To solve this issue, motivated by (3.4.2) we have changed the hypothesis of Theorem 3.4.1 so we have obtained the following result.

Theorem 3.4.3. *Assume that for some pair of nonnegative functions (f,g) and for some $1 \leqslant p_0 < \infty$ and $0 < \alpha < 1$,*

$$\|g\|_{L^{p_0,\infty}((Mh)^\alpha)} \leqslant C_{n,p_0,\alpha}\|f\|_{L^{p_0,1}((Mh)^\alpha)}, \qquad \forall h \in L^1_{loc}(\mathbb{R}^n).$$

Then, for $p = \frac{p_0}{1-\alpha}$,

$$\|g\|_{L^{p,\infty}\left(u^{1-\frac{p}{p_0}}\right)} \leqslant (1-\alpha)^{\frac{p_0-1}{p_0}} C_{n,p_0,\alpha}\Phi\left(\|u\|_{A_1}^{\frac{p}{p_0}-1}\right)^{\frac{1}{p_0}-\frac{1}{p}}\|f\|_{L^{p,1}\left(u^{1-\frac{p}{p_0}}\right)}, \qquad \forall u \in A_1,$$

for some nondecreasing function Φ on $[1,\infty)$.

1.4 Weighted restricted weak-type estimates on $\Lambda^p(w)$

We have aimed this chapter to obtain weighted restricted weak-type estimates on the setting of classical Lorentz spaces. Indeed, the first main result of this section is the following.

Theorem 4.2.2. *Let T be an operator satisfying that for some $1 \leqslant p_0 < \infty$,*

$$T : L^{p_0,1}(v) \to L^{p_0,\infty}(v), \qquad \forall v \in \hat{A}_{p_0},$$

with constant less than or equal to $\varphi(\|v\|_{\hat{A}_{p_0}})$, where φ is a positive nondecreasing function on $[1,\infty)$. Let $0 < p < \infty$.

(i) If $p_0 = 1$, then

$$T : \Lambda^{p,1}(w) \to \Lambda^{p,\infty}(w), \qquad \forall w \in B_p^R \cap B_\infty^*,$$

with constant less than or equal to $C_1 \|w\|_{B_p^R} \varphi\left(C_2 \|w\|_{B_\infty^*}\right)$.

(ii) If $p_0 > 1$ and T is sublinear, then, for every $0 < q < 1$,

$$T : \Lambda^{p,q}(w) \to \Lambda^{p,\infty}(w), \qquad \forall w \in B_p^R \cap B_\infty^*,$$

with constant less than or equal to $\dfrac{C_1}{1-q} \|w\|_{B_p^R} \max\left(1, \|w\|_{B_\infty^*}^{q-\frac{1}{p_0}}\right) \varphi\left(C_2 \|w\|_{B_\infty^*}^{\frac{1}{p_0}}\right)$.

For Theorem 4.2.2 we make a couple of observations.

- Comparing it with the restricted weak-type extrapolation on Lorentz spaces (see [56, Theorem 3.1] and [61, Corollary 2.15]), we observe that we obtain, in particular, that for a general operator T, if $p_0 = 1$ and $p \geqslant 1$,

  $$T : L^{p,1}(\mathbb{R}^n) \to L^{p,\infty}(\mathbb{R}^n)$$

  and if $p_0 > 1$ and T is sublinear, for every $0 < q < 1$ and $p > 1$,

  $$T : L^{1,q}(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n) \qquad \text{and} \qquad T : L^{p,1}(\mathbb{R}^n) \to L^{p,\infty}(\mathbb{R}^n).$$

- By means of the Hilbert transform (see (2.2.25)) the condition $B_p^R \cap B_\infty^*$ on the weight w of Theorem 4.2.2 (i) is sharp in the sense that it can not be found a greater class for w.

Now, it turns out that for a sublinear operator T, having for some $0 < q \leqslant 1$,

$$T : \Lambda^{1,q}(w) \to \Lambda^{1,\infty}(w), \qquad \forall w \in B_1^R \cap B_\infty^*,$$

for a suitable control of the norm constant is equivalent to an estimate on the decreasing rearrangement of T by a Calderón admissible type operator (see (4.3.3) for its definition). Indeed, we have the next result.

Theorem 4.3.10. *Given $0 < q \leqslant 1$. Let T be a sublinear operator and let φ be an admissible function (see Definition 4.3.1). Then,*

$$T : \Lambda^{1,q}(w) \to \Lambda^{1,\infty}(w), \qquad \forall w \in B_1^R \cap B_\infty^*,$$

with constant less than or equal to $C\|w\|_{B_1^R}\,\varphi(\|w\|_{B_\infty^*})$ if and only if for every locally integrable function f and for every $t > 0$,

$$(Tf)^*(t) \lesssim \left(\frac{1}{t^q}\int_0^t f^*(s)^q \frac{ds}{s^{1-q}}\right)^{\frac{1}{q}} + \int_t^\infty \frac{\varphi\left(1+\log\frac{s}{t}\right)}{1+\log\frac{s}{t}} f^*(s)\,\frac{ds}{s}. \tag{1.5}$$

Besides, while working on the proof Theorem 4.3.10 we realized about that, in fact, we have also the following result.

Theorem 4.3.15. *Given $0 < q \leqslant 1$. Let T be a sublinear operator and let φ be an admissible function. Then, $T : L^{1,q}(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)$ and for every $1 < p < \infty$,*

$$\|T\chi_E\|_{L^{p,\infty}(\mathbb{R}^n)} \leqslant C\varphi\,(p)\,|E|^{\frac{1}{p}}, \qquad \forall E \subseteq \mathbb{R}^n,$$

with C independent of p if and only if for every locally integrable function f and every $t > 0$, (1.5) holds.

Therefore, as a consequence of Theorems 4.2.2, 4.3.10 and 4.3.15, we obtain new estimates on the setting of classical Lorentz spaces and on the decreasing rearrangement for important operators in Harmonic Analysis such as

- operators that satisfy a Fefferman-Stein's inequality (see Section 5.1),
- radial Fourier multipliers with a derivative condition (see Section 5.2.3),
- rough singular integrals (see Section 5.3),
- intrinsic square functions (see Section 5.4),
- the Assani operator (see Section 5.6)
- and the Bochner-Riesz operator (see Section 5.7).

Furthermore, we have managed to extend the above extrapolation results on classical Lorentz spaces to the limited setting, and then our next main result reads as follows.

Theorem 4.2.5. *Let T be an operator satisfying that for some $1 \leqslant p_0 < \infty$ and $0 \leqslant \alpha, \beta \leqslant 1$ (not both identically zero),*

$$T : L^{p_0,1}(v) \to L^{p_0,\infty}(v), \qquad \forall v \in \hat{A}_{p_0;(\alpha,\beta)}, \tag{1.6}$$

with constant less than or equal to $\varphi(\|v\|_{\hat{A}_{p_0;(\alpha,\beta)}})$, with φ being a positive nondecreasing function on $[1,\infty)$. Let $0 < p < \infty$ and set p_- and p_+ as in (2.3.5).

(i) If $p_0 = 1$ or $0 \leqslant \beta < 1$,

$$T : \Lambda^{p,1}(w) \to \Lambda^{p,\infty}(w), \qquad \forall w \in B^R_{\frac{p}{p_-}} \cap B^*_{\frac{p_+}{p}}.$$

(ii) If $p_0 > 1$ and $\beta = 1$ then, for every $0 < q < 1$,

$$T : \Lambda^{p,q}(w) \to \Lambda^{p,\infty}(w), \qquad \forall w \in B^R_{\frac{p}{p_-}} \cap B^*_{\frac{p_+}{p}}.$$

For Theorem 4.2.5 we also make a couple of observations.

- If we take $w = 1$, then $w \in B^R_{\frac{p}{p_-}} \cap B^*_{\frac{p_+}{p}}$ whenever $p_- \leqslant p < p_+$, so as in Theorem 3.4.1, we are not able to extrapolate till $p = p_+$.
- We can not expect to get a bigger class of weights than $B^R_{\frac{p}{p_-}}$ since, for instance, the operator $M_{\frac{1}{p_-}} f := M(f^{p_-})^{\frac{1}{p_-}}$ satisfies (1.6) for $p_0 = p_-$, $\alpha = 1$ and $\beta = 0$ (see (2.2.8)) while $M_{\frac{1}{p_-}} : \Lambda^{p,1}(w) \to \Lambda^{p,\infty}(w)$ holds if and only if $w \in B^R_{\frac{p}{p_-}}$ (see (2.2.14)).

Further, if we assume a similar hypothesis as in Theorem 3.4.3, then we get the next result.

Theorem 4.2.8. *Let T be an operator satisfying that for some $1 \leqslant p_0 < \infty$ and $0 < \alpha < 1$,*

$$T : L^{p_0,1}((M\chi_F)^\alpha) \to L^{p_0,\infty}((M\chi_F)^\alpha), \qquad \forall F \subseteq \mathbb{R}^n,$$

with constant less than or equal to $C_{n,p_0,\alpha}$. Then, for every $0 < p < \infty$,

$$T : \Lambda^{p,1}(w) \to \Lambda^{p,\infty}(w), \qquad \forall w \in B^R_{\frac{p}{p_0}} \cap B^{*R}_{\frac{p_0}{(1-\alpha)p}},$$

with constant less than or equal to $\tilde{C}_{n,p_0,\alpha} \|w\|^{\frac{1}{p_0}}_{B^R_{\frac{p}{p_0}}} \log\left(1 + \|w\|_{B^{*R}_{\frac{p_0}{(1-\alpha)p}}}\right) \|w\|^{\frac{1-\alpha}{p_0}}_{B^{*R}_{\frac{p_0}{(1-\alpha)p}}}.$

In the limited setting, it also turns out that having for a sublinear operator T and for some $0 < q \leqslant 1$ and $1 \leqslant p_1 < p_2 \leqslant \infty$,

$$T : \Lambda^{1,q}(w) \to \Lambda^{1,\infty}(w), \qquad \forall w \in B^R_{\frac{1}{p_1}} \cap B^*_{p_2},$$

for a suitable control of the norm constant is equivalent to an estimate on the decreasing rearrangement of T by a Calderón admissible type operator. Indeed, we have the next result.

Theorem 4.3.17. *Given $0 < q \leqslant 1$ and $1 \leqslant p_1 < p_2 \leqslant \infty$. Let T be a sublinear operator and let φ be an admissible function (see Definition 4.3.1). If*

$$T : \Lambda^{1,q}(w) \to \Lambda^{1,\infty}(w), \qquad \forall w \in B^R_{\frac{1}{p_1}} \cap B^*_{p_2},$$

with constant less than or equal to $C\, \|w\|_{B^R_{\frac{1}{p_1}}}^{\frac{1}{p_1}} \varphi\left(\|w\|_{B^*_{p_2}}\right)$ *then, for every locally integrable function* f *and every* $t > 0$,

$$(Tf)^*(t) \lesssim \left(\frac{1}{t^{\frac{q}{p_1}}} \int_0^t f^*(s)^q \frac{ds}{s^{1-\frac{q}{p_1}}} \right)^{\frac{1}{q}} + \begin{cases} \dfrac{1}{t^{\frac{1}{p_2}}} \displaystyle\int_t^\infty \varphi\left(1 + \log \frac{s}{t}\right) f^*(s) \frac{ds}{s^{1-\frac{1}{p_2}}}, & p_2 < \infty, \\ \displaystyle\int_t^\infty \frac{\varphi\left(1+\log\frac{s}{t}\right)}{\left(1+\log\frac{s}{t}\right)} f^*(s) \frac{ds}{s}, & p_2 = \infty. \end{cases} \tag{1.7}$$

Conversely, suppose that (1.7) *holds. Then*

$$\|T\|_{\Lambda^{1,q}(w)\to\Lambda^{1,\infty}(w)} \lesssim \begin{cases} \|w\|_{B^R_{\frac{1}{p_1}}}^{\frac{1}{p_1}} \|w\|_{B^*_{p_2}} \varphi(\|w\|_{B^*_{p_2}}), & p_2 < \infty, \\ \|w\|_{B^R_{\frac{1}{p_1}}}^{\frac{1}{p_1}} \varphi(\|w\|_{B^*_\infty}), & p_2 = \infty. \end{cases}$$

Besides, while working on the proof Theorem 4.3.17 we realized about that, in fact, we also have the following result.

Theorem 4.3.22. *Given* $0 < q \leqslant 1$ *and* $1 \leqslant p_1 < p_2 \leqslant \infty$*. Let* T *be a sublinear operator and let* φ *be an admissible function. If* $T : L^{p_1,q}(\mathbb{R}^n) \to L^{p_1,\infty}(\mathbb{R}^n)$ *and for every* $p_1 < p < p_2$,

$$\|T\chi_E\|_{L^{p,\infty}(\mathbb{R}^n)} \leqslant C\varphi\left(\left[\frac{1}{p} - \frac{1}{p_2}\right]^{-1}\right) |E|^{\frac{1}{p}}, \qquad \forall E \subseteq \mathbb{R}^n,$$

with C *independent of* p *then* (1.7) *holds. Conversely, assume that we have* (1.7)*. Then, for every* $p_1 \leqslant p < p_2$,

$$\|T\|_{L^{p,q}(\mathbb{R}^n)\to L^{p,\infty}(\mathbb{R}^n)} \leqslant C\tilde{\varphi}\left(\left[\frac{1}{p} - \frac{1}{p_2}\right]^{-1}\right),$$

with C *independent of* p *and where, for* $x \geqslant 1$,

$$\tilde{\varphi}(x) = \begin{cases} x\varphi(x), & p_2 < \infty, \\ \varphi(x), & p_2 = \infty. \end{cases}$$

Therefore, as a consequence of Theorems 4.2.5, 4.3.17 and 4.3.22 we get new estimates on the setting of classical Lorentz spaces for important operators in Harmonic Analysis such as

- Fourier multipliers of Hörmander type (see Section 5.2.1),
- Fourier multipliers that satisfy a Fefferman-Stein's type inequality (see Section 5.2.2),
- rough singular integrals (see Section 5.3),
- sparse operators (see Section 5.5)
- and the Bochner-Riesz operator (see Section 5.7).

1.5 Multi-variable weighted estimates on $\Lambda^p(w)$

Continuing the work started in [161], where the author proved multi-variable extrapolation results for weighted Lorentz spaces, we now consider multi-variable extrapolation on classical Lorentz spaces.

Our first main result consists on multi-variable restricted weak-type estimates on classical Lorentz spaces.

Theorem 6.2.2. *Set $m \geqslant 1$ and let T be an operator satisfying that for some exponents $1 \leqslant p_1, \dots, p_m < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$,*

$$T : L^{p_1,1}(v_1) \times \cdots \times L^{p_m,1}(v_m) \to L^{p,\infty}(v_1^{p/p_1} \cdots v_m^{p/p_m}), \qquad \forall v_i \in A_{p_i}^R,\ i = 1, \dots, m,$$

with constant less than or equal to $\varphi\left(\|v_1\|_{A_{p_1}^R}, \dots, \|v_m\|_{A_{p_m}^R}\right)$, where $\varphi : [1,\infty)^m \to (0,\infty)$ is a nondecreasing function in each variable. Then, for all exponents $0 < q_1, \dots, q_m < \infty$ and $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$,

$$T : \Lambda^{q_1,\frac{1}{p_1}}(w_1) \times \cdots \times \Lambda^{q_m,\frac{1}{p_m}}(w_m) \to \Lambda^{q,\infty}(w), \qquad \forall w_i \in B_{q_i}^R \cap B_\infty^*,\ i = 1, \dots, m,$$

where w is a weight such that $W \lesssim W_1^{\frac{q}{q_1}} \cdots W_m^{\frac{q}{q_m}}$. Moreover, if T is a submultilinear operator and $\min\{p_1, \dots, p_m\} > m$, then, for every $0 < r < \frac{1}{m}$,

$$T : \Lambda^{q_1,r}(w_1) \times \cdots \times \Lambda^{q_m,r}(w_m) \to \Lambda^{q,\infty}(w), \qquad \forall w_i \in B_{q_i}^R \cap B_\infty^*,\ i = 1, \dots, m.$$

Let us make some observations for Theorem 6.2.2.

- An important key in the proof of Theorem 6.2.2 consists on obtaining restricted weak-type boundedness on classical Lorentz spaces of the m-fold product of Hardy-Littlewood maximal operator (see Theorem 6.1.1).

- If $m = 1$, we recover Theorem 4.2.2 although now in the hypothesis we have $v \in A_p^R$ instead of $v \in \hat{A}_p$. However, this is not a big deal since $\hat{A}_p \subseteq A_p^R$ (indeed, whether the equality of both classes hold or if the inclusion is strict is still an open question) and all examples that we study work for weights in the (a priori) bigger class A_p^R.

- If we let $w = w_1^{q/q_1} \dots w_m^{q/q_m}$, since $q_i > q$ for every $i = 1, \dots, m$, by virtue of the Hölder's inequality,

  $$W(t) \leqslant W_1^{q/q_1}(t) \dots W_m^{q/q_m}(t), \qquad \forall t > 0,$$

  so that Theorem 6.2.2 also holds for this w.

- If $p_1 = \cdots = p_m = 1$, then we obtain that

  $$T : \Lambda^{q_1,1}(w_1) \times \cdots \times \Lambda^{q_m,1}(w_m) \to \Lambda^{q,\infty}(w), \qquad \forall w_i \in B_{q_i}^R \cap B_\infty^*,\ i = 1, \dots, m,$$

  with constant less than or equal to

  $$C\varphi\left(\|w_1\|_{B_\infty^*}, \dots, \|w_m\|_{B_\infty^*}\right) \prod_{i=1}^m \|w_i\|_{B_{q_i}^R}$$

  (see Remark 6.2.3).

As a consequence of Theorem 6.2.2, we obtain new multi-variable restricted weak-type estimates on the setting of classical Lorentz spaces for multilinear sparse operators (see Section 6.4.2).

Our second main result consists on two-variable mixed-type estimates on classical Lorentz spaces.

Theorem 6.3.2. *Let T be an operator satisfying that for some exponents $1 < p_1, p_2 < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$,*

$$T : L^{p_1}(v_1) \times L^{p_2,1}(v_2) \to L^{p,\infty}(v_1^{p/p_1} v_2^{p/p_2}), \qquad \forall v_1 \in A_{p_1},\, v_2 \in A_{p_2}^R, \tag{1.8}$$

with constant less than or equal to $\varphi\left(\|v_1\|_{A_{p_1}}, \|v_2\|_{A_{p_2}^R}\right)$, where $\varphi : [1,\infty)^2 \to (0,\infty)$ is a nondecreasing function in each variable. Then, for every exponents $0 < q_1, q_2 < \infty$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$,

$$T : \Lambda^{q_1}(w_1) \times \Lambda^{q_2,\frac{1}{p_2}}(w_2) \to \Lambda^{q,\infty}(w), \qquad \forall w_1 \in B_{q_1} \cap B_\infty^*,\, w_2 \in B_{q_2}^R \cap B_\infty^*,$$

with w being such that $W \lesssim W_1^{\frac{q}{q_1}} W_2^{\frac{q}{q_2}}$. Moreover, if T is a submultilinear operator and $\min\{p_1, p_2\} > 2$, then, for every $0 < r < \frac{1}{2}$,

$$T : \Lambda^{q_1}(w_1) \times \Lambda^{q_2,r}(w_2) \to \Lambda^{q,\infty}(w), \qquad \forall w_1 \in B_{q_1} \cap B_\infty^*,\, w_2 \in B_{q_2}^R \cap B_\infty^*.$$

As before, for Theorem 6.3.2 we also make a couple of observations.

- In this case we need to impose that both exponents p_1 and p_2 to be greater than 1. In fact, we just need to ask that $p_2 > 1$ since, indeed, the case $p_1 = 1$ is handled by Theorem 6.2.2. That is due to the fact that one of the steps of the proof of Theorem 6.3.2 consists on translating the hypothesis (1.8) to the diagonal setting (i.e., to $p_1 = p_2$) and it is unknown at the present how to extrapolate from $p_2 = 1$ to a greater exponent and it is not possible to extrapolate from $p_1 > 1$ to 1.

- If $p_1 = p_2 = p_0 > 1$, then we obtain that

  $$T : \Lambda^{q_1}(w_1) \times \Lambda^{q_2,\frac{1}{p_0}}(w_2) \to \Lambda^{q,\infty}(w), \qquad \forall w_1 \in B_{q_1} \cap B_\infty^*,\, w_2 \in B_{q_2}^R \cap B_\infty^*,$$

  with constant less than or equal to

  $$C\, \|w_1\|_{B_{q_1}}^{\max\left(1,\frac{1}{q_1}\right)} \|w_2\|_{B_{q_2}^R}^{2-\frac{1}{p_0}} \varphi\left(\|w_1\|_{B_\infty^*} \|w_1\|_{B_{q_1}}^{(p_0-1)\min\left(1,\frac{1}{q_1}\right)}, \|w_2\|_{B_\infty^*}^{\frac{1}{p_0}}\right).$$

As a consequence of Theorem 6.3.2, we obtain new mixed-type estimates on the setting of classical Lorentz spaces for bilinear Fourier multipliers (see Section 6.4.1).

1.6 Further results: weighted strong-type estimates on $\Lambda_u^p(w)$

Finally, this last chapter is aimed to develop a new extrapolation result but now on the more general spaces $\Lambda_u^p(w)$, in order to serve as an example for a future research on this type of extrapolation. In that line, it was already known the following important result (see [83]).

Theorem 7.1.1. *Assume that for some pair of nonnegative functions (f, g) and for some $1 \leqslant p_0 < \infty$,*

$$\left(\int_{\mathbb{R}^n} g(x)^{p_0} v(x)\, dx\right)^{\frac{1}{p_0}} \leqslant \varphi(\|v\|_{A_{p_0}}) \left(\int_{\mathbb{R}^n} f(x)^{p_0} v(x)\, dx\right)^{\frac{1}{p_0}}, \qquad \forall v \in A_{p_0}, \tag{1.9}$$

where φ is a nondecreasing function on $[1, \infty)$. Let $\mathbb{X}$ be a r.i. Banach function space and let $u \in A_\infty$ such that

$$M : \mathbb{X}(u) \to \mathbb{X}(u) \qquad \text{and} \qquad M_u' : \mathbb{X}'(u) \to \mathbb{X}'(u).$$

Then,

$$\|g\|_{\mathbb{X}(u)} \leqslant C_1 \varphi\left(C_2 \|M_u'\|_{\mathbb{X}'(u)} \|M\|_{\mathbb{X}(u)}^{p_0 - 1}\right) \|f\|_{\mathbb{X}(u)}.$$

Indeed, taking $\mathbb{X} = \Lambda^p(w)$, for $p \geqslant 1$, it was already known when $\mathbb{X}$ is a Banach function space (see [57, 166]) and when $M : \Lambda_u^p(w) \to \Lambda_u^p(w)$ holds (see [58]). Hence, in order to make use of Theorem 7.1.1 we have applied every effort to prove whenever $M_u' : (\Lambda_u^p(w))' \to (\Lambda_u^p(w))'$ is true, so we have obtained the following result.

Theorem 7.2.3. *Given $u \in A_\infty$. For every $0 < p < \infty$,*

$$M_u' : (\Lambda_u^p(w))' \to (\Lambda_u^p(w))', \qquad \forall w \in B_p(u) \cap B_\infty^*.$$

Therefore, as a consequence of Theorem 7.2.3 we get the next extrapolation result.

Corollary 7.3.1. *Assume that for some pair of nonnegative functions (f, g) and for some $1 \leqslant p_0 < \infty$, (1.9) holds. Let $1 \leqslant p < \infty$ and $u \in A_\infty$. Then,*

$$\|g\|_{\Lambda_u^p(w)} \leqslant C_1 \varphi\left(C_2 \|M_u'\|_{\left(\Lambda_u^p(w)\right)'} \|M\|_{\Lambda_u^p(w)}^{p_0 - 1}\right) \|f\|_{\Lambda_u^p(w)}, \qquad \forall w \in B_p(u) \cap B_\infty^*.$$

For Corollary 7.3.1 we should make some observations.

- The case $0 < p < 1$, where $\Lambda^p(w)$ is not a Banach function space (see, for instance, [172]) can also be settled. Indeed, arguing as in the proof of Theorem 7.1.1 for $\mathbb{X} = \Lambda^p(w)$, and using that $\Lambda_u^p(w) = (\Lambda_u^1(w))^p$, in addition to that, by means of Theorem 2.3.1, we can consider p_0 as big as we want, it can be seen that Corollary 7.3.1 also holds for this range of p (although with a different constant) since Theorem 7.2.3 is also true for those exponents.

- Operators such as Fourier multipliers of Hörmander type (see Section 5.2.1), rough singular integrals (see Section 5.3), intrinsic square functions (see Section 5.4), sparse operators (see Section 5.5), the Bochner-Riesz operator (see Section 5.7), among others satisfy (1.9) so we can obtain estimates on the setting of weighted classical Lorentz spaces for all of them.

Chapter 2

Preliminars

We devote this chapter to introduce some notions and definitions, as well as some important results that we will use later on throughout this thesis. Among them, we will talk about classical Lorentz spaces (Section 2.1) and different classes of weights (Section 2.2). Finally, we will make a review of known results on extrapolation, from the oldest to the newest (Section 2.3).

We will provide references for all the results for which it is easy to refer to a source and we will provide proofs for those which are new or that we were unable to find its proof.

2.1 Classical Lorentz spaces

In this section, we present some basic concepts about classical Lorentz spaces. We start by recalling well known facts about rearrangement invariant Banach (and quasi-Banach) function spaces (Section 2.1.1) so that then we can ease the introduction of the definition and properties of the classical Lorentz spaces (Section 2.1.2) in addition to its associate space (Section 2.1.3). This section is not intended to be exhaustive.

2.1.1 R.i. (quasi)-Banach function spaces

We start by gathering some known basic facts about rearrangement invariant Banach and quasi-Banach function spaces (for a complete account we refer to [23]).

Let (R, Σ, μ) be a σ-finite nonatomic measure space. A *Banach function norm* ρ is a mapping $\rho : \mathcal{M}^+ \to [0, \infty]$ such that the following properties hold:

(i) $\rho(f) = 0 \Leftrightarrow f = 0$ μ-a.e.;

(ii) $\rho(af) = a\rho(f)$, for $a \geqslant 0$;

(iii) $\rho(f + g) \leqslant \rho(f) + \rho(g)$;

(iv) if $0 \leqslant f \leqslant g$ μ-a.e., then $\rho(f) \leqslant \rho(g)$;

(v) if $0 \leqslant f_n \nearrow f$ μ-a.e., then $\rho(f_n) \nearrow \rho(f)$;

(vi) if E is a measurable set such that $\mu(E) < \infty$, then $\rho(\chi_E) < \infty$ and $\int_E f\,d\mu \leqslant C_E \rho(f)$ for some constant $0 < C_E < \infty$, depending on E and ρ, but independent of f.

The collection $\mathbb{X} = \mathbb{X}(\rho)$ defined by

$$\mathbb{X} = \{f \in \mathcal{M} : \rho(|f|) < \infty\}$$

is called a *Banach function space.* If for each $f \in \mathbb{X}$ we define $\|f\|_{\mathbb{X}} = \rho(|f|)$, then $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ becomes a Banach function space. Besides, by means of a function norm ρ, we can define its *associate norm* $\rho' : \mathcal{M}^+ \to [0,\infty]$ by

$$\rho'(f) = \sup\left\{\int_R fg\,d\mu : g \in \mathcal{M}^+, \rho(g) \leqslant 1\right\},$$

which is itself a function norm. This allows us to define the *associate space* of $\mathbb{X} = \mathbb{X}(\rho)$ to be the Banach function space $\mathbb{X}' = \mathbb{X}(\rho')$ (see [23, Ch. 1 - Theorem 2.2]). Further, $\mathbb{X} = \mathbb{X}''$ with equality of norms (see [23, Ch. 1 - Theorem 2.7]) and, by definition, it follows the following estimate known as *Hölder's Inequality*:

$$\int_R |fg|\,d\mu \leqslant \|f\|_{\mathbb{X}} \|g\|_{\mathbb{X}'}.$$

The *distribution function* μ_f of a measurable function f is

$$\mu_f(y) = \mu\{x \in R : |f(x)| > y\}, \qquad y \geqslant 0,$$

and, when $d\mu = dx$, we denote it by $\lambda_f(y)$. A function norm ρ is called *rearrangement invariant* (*r.i.* in short) if $\rho(f) = \rho(g)$ for every pair of functions f and g that satisfy $\mu_f(y) = \mu_g(y)$ for every $y > 0$. In this case, we say that $\mathbb{X} = \mathbb{X}(\rho)$ is a *r.i. Banach function space* and it follows that $\mathbb{X}'$ is also a r.i. Banach function space.

The *decreasing rearrangement* of f is the function f^* defined on $(0,\infty)$ by

$$f^*(t) = \inf\{y \geqslant 0 : \mu_f(y) \leqslant t\}, \qquad t \geqslant 0,$$

and satisfy $\lambda_{f^*}(y) = \mu_f(y)$ for every $y > 0$. This allows to obtain a representation of $\mathbb{X}$ on $(\mathbb{R}^+, dt)$ (see [23, Ch. 2 - Theorem 4.10]) as follows: there exists a r.i. Banach function space $\overline{\mathbb{X}}$ over $(\mathbb{R}^+, dt)$ such that $f \in \mathbb{X}$ if and only if $f^* \in \overline{\mathbb{X}}$ with

$$\|f\|_{\mathbb{X}} = \|f^*\|_{\overline{\mathbb{X}}} := \sup_{\|g\|_{\mathbb{X}'} \leqslant 1} \int_0^\infty f^*(t) g^*(t)\,dt.$$

Moreover, the associate space $\mathbb{X}'$ of $\mathbb{X}$ is represented in the same way by the associate space $\overline{\mathbb{X}}'$ of $\overline{\mathbb{X}}$ with $\|f\|_{\mathbb{X}'} = \|f^*\|_{\overline{\mathbb{X}}'}$.

Now we define the *Boyd indices* of a r.i. Banach function space. These indices were introduced by Boyd in a series of papers [29, 30, 31, 32, 33]. First, *the dilatation operator* is $E_t f(s) = f(st)$, for $s, t > 0$ and $f \in \mathcal{M}$. Then, if we set $h_{\mathbb{X}}(t) = \left\|E_{1/t}\right\|_{\overline{\mathbb{X}}}$, for $t > 0$, the Boyd indices are defined as follows (see, for instance, [23, Ch. 3 - Definition 5.12] and [107]): the *lower Boyd index* $\beta_{\mathbb{X}}$ is

$$\beta_{\mathbb{X}} = \sup_{0<t<1} \frac{\log h_{\mathbb{X}}(t)}{\log t} = \lim_{t\to 0^+} \frac{\log h_{\mathbb{X}}(t)}{\log t}$$

and the *upper Boyd index* $\alpha_{\mathbb{X}}$ is

$$\alpha_{\mathbb{X}} = \sup_{1<t<\infty} \frac{\log h_{\mathbb{X}}(t)}{\log t} = \lim_{t\to\infty} \frac{\log h_{\mathbb{X}}(t)}{\log t}$$

and they satisfy $0 \leqslant \beta_{\mathbb{X}} \leqslant \alpha_{\mathbb{X}} \leqslant 1$. Further, the relationship between the Boyd indices of $\mathbb{X}$ and $\mathbb{X}'$ is the following: $\alpha_{\mathbb{X}'} = 1 - \beta_{\mathbb{X}}$ and $\beta_{\mathbb{X}'} = 1 - \alpha_{\mathbb{X}}$.

In general, when restricted to a r.i. Banach function space $\mathbb{X}$ in $(\mathbb{R}^n, dx)$, it is possible to define a weighted version of $\mathbb{X}$. Take u being a *weight* (that is, u is a nonnegative locally integrable function) and consider the measure space $(\mathbb{R}^n, u(x)\,dx)$. The distribution function and the decreasing rearrangement with respect to u are given by

$$\lambda_f^u(y) = u\left(\{x \in \mathbb{R}^n : |f(x)| > y\}\right); \qquad f_u^*(t) = \inf\{y \geqslant 0 : \lambda_f^u(y) \leqslant t\}.$$

Then the weighted version of the space $\mathbb{X}$ can be defined as

$$\mathbb{X}(u) = \{f \in \mathcal{M} : \|f_u^*\|_{\overline{\mathbb{X}}} < \infty\},$$

with the norm associated to it $\|f\|_{\mathbb{X}(u)} = \|f_u^*\|_{\overline{\mathbb{X}}}$. Hence, by construction, $\mathbb{X}(u)$ is a Banach function space built over $\mathcal{M}\,(\mathbb{R}^n, u(x)\,dx)$ with associate space $\mathbb{X}(u)' = \mathbb{X}'(u)$, so that

$$\|f\|_{\mathbb{X}(u)'} = \|f_u^*\|_{\overline{\mathbb{X}}'} = \sup_{\|g\|_{\mathbb{X}(u)}\leqslant 1} \int_0^\infty f_u^*(t) g_u^*(t)\,dt = \sup_{\|g\|_{\mathbb{X}(u)}\leqslant 1} \int_{\mathbb{R}^n} |f(x)g(x)|u(x)\,dx.$$

Finally, we deal with the r.i. quasi-Banach function spaces. To do so, we define a *quasi-Banach function norm* similar as we did for the Banach function norm but with a weaker version of property (iii); that is,

(iii') $\rho(f+g) \leqslant C(\rho(f) + \rho(g))$, for some $C \geqslant 1$.

Then, the definition of *r.i. quasi-Banach function space* follows the same lines as the one for the Banach function spaces but with this new function norm. However, the constant in (iii') forces several changes in the properties of the space. That is why we consider r.i. quasi-Banach function spaces that are *p-convex* for some $0 < p \leqslant 1$ (see [116]) which are those such that

$$\mathbb{X}^{\frac{1}{p}} = \left\{ f \in \mathcal{M} \,:\, \|f\|_{\mathbb{X}^{\frac{1}{p}}} = \left\| |f|^{\frac{1}{p}} \right\|_{\mathbb{X}}^p < \infty \right\}$$

is a r.i. Banach function space. In particular, using that $\|f\|_{\mathbb{X}} = \||f|^p\|_{\mathbb{X}^{\frac{1}{p}}}^{\frac{1}{p}}$ and that $(\mathbb{X}^{\frac{1}{p}})'' = \mathbb{X}^{\frac{1}{p}}$, then

$$\|f\|_{\mathbb{X}} = \sup_{\|g\|_{\left(\mathbb{X}^{\frac{1}{p}}\right)'}\leqslant 1} \left(\int_{\mathbb{R}^n} |f(x)|^p |g(x)|\,dx \right)^{\frac{1}{p}} = \sup_{\|g\|_{\left(\mathbb{X}^{\frac{1}{p}}\right)'}\leqslant 1} \left(\int_0^\infty f^*(t)^p g^*(t)\,dt \right)^{\frac{1}{p}}, \tag{2.1.1}$$

and all the notions of r.i. Banach function spaces can be extended to r.i. quasi-Banach function spaces through (2.1.1). (We refer to [100, 146] for more details on this topic.)

2.1.2 The $\Lambda^p(w)$ spaces

The classical Lorentz spaces $\Lambda^p(w)$ (called like this to distinguish them from the Lorentz spaces $L^{p,q}(\mathbb{R}^n)$) were introduced and studied by Lorentz in [142, 143] for the measure space $((0,\ell), dx)$ and $\ell < \infty$. They are rearrangement invariant and generalize the $L^p(\mathbb{R}^n)$ Lebesgue spaces and $L^{p,q}(\mathbb{R}^n)$ (see [23, 99] for more details on Lebesgue and Lorentz spaces).

Given a *weight* w in $\mathbb{R}^+$ (that is, $w \in L^1_{\text{loc}}(\mathbb{R}^+)$ is nonnegative), denote $W(t) = \int_0^t w(r)\,dr$, $t > 0$. For $0 < p < \infty$, the *classical Lorentz spaces* $\Lambda^p(w)$ are defined as the set of measurable functions f such that

$$\|f\|_{\Lambda^p(w)} = \left(\int_0^\infty f^*(t)^p w(t)\,dt\right)^{\frac{1}{p}} = \left(\int_0^\infty p y^{p-1} W\left(\lambda_f(y)\right)\,dy\right)^{\frac{1}{p}} < \infty.$$

Example 2.1.1. (1) If $w = 1$, we recover the Lebesgue spaces $\Lambda^p(1) = L^p(\mathbb{R}^n)$.

(2) If $0 < p, q < \infty$ and $w = t^{\frac{q}{p}-1}$, we retrieve the Lorentz spaces $\Lambda^q(w) = L^{p,q}(\mathbb{R}^n)$.

Furthermore, we observe that $\|f\|_{\Lambda^p(w)} = \|f^*\|_{L^p(w)}$. This allows us to extend the previous definition to the space $\Lambda^{p,q}(w)$ by $\|f\|_{\Lambda^{p,q}(w)} = \|f^*\|_{L^{p,q}(w)}$ for every $0 < p < \infty$ and $0 < q \leqslant \infty$ (see [58]); that is $\Lambda^{p,q}(w)$ consists of all the measurable functions f that satisfy

$$\|f\|_{\Lambda^{p,q}(w)} = \left(\int_0^\infty f^*(t)^q W(t)^{\frac{q}{p}-1} w(t)\,dt\right)^{\frac{1}{q}} = \left(\int_0^\infty p t^{q-1} W\left(\lambda_f(t)\right)^{\frac{q}{p}}\,dt\right)^{\frac{1}{q}} < \infty,$$

for $0 < q < \infty$, and

$$\|f\|_{\Lambda^{p,\infty}(w)} = \sup_{t>0} W(t)^{\frac{1}{p}} f^*(t) = \sup_{y>0} y W\left(\lambda_f(y)\right)^{\frac{1}{p}} < \infty.$$

Observe that, direct from the definition, for $0 < p, q < \infty$,

$$\|f\|_{\Lambda^{p,q}(w)} = \|f\|_{\Lambda^q(\tilde{w})} \qquad \text{and} \qquad \|f\|_{\Lambda^{p,\infty}(w)} = \left(\frac{q}{p}\right)^{\frac{1}{p}} \|f\|_{\Lambda^{q,\infty}(\tilde{w})}, \tag{2.1.2}$$

where $\tilde{w}(t) = W(t)^{\frac{q}{p}-1} w(t)$, $t > 0$ (see [58, Remark 2.2.6]). Therefore, every Lorentz space as defined here reduces to $\Lambda^p(w)$ and its "weak version" $\Lambda^{p,\infty}(w)$. Besides, one elementary property of the Lorentz spaces is that $\Lambda^{p,q_0}(w) \subseteq \Lambda^{p,q_1}(w)$ continuously for $0 < q_0 \leqslant q_1 \leqslant \infty$ (see [41] for more information on embeddings between Lorentz spaces).

Moreover, although the spaces $\Lambda^p(w)$ are not necessarily Banach function spaces, at least when $w \in \Delta_2$ (that is, $W(2t) \leqslant CW(t)$, for every $t > 0$ and where $C > 0$ is independent of t) for every $0 < p < \infty$, $\Lambda^p(w)$ and $\Lambda^{p,\infty}(w)$ are quasi-Banach function spaces. Further, since $\Lambda^p(w) = (\Lambda^1(w))^p$ and $\Lambda^{p,\infty}(w) = (\Lambda^{1,\infty}(w))^p$, these spaces are also p-convex. For more details on these topics, we refer the reader to [58, Chapter 2] (see also [55]).

Besides, given a *weight* u in $\mathbb{R}^n$ (that is, $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ is nonnegative) it can also be defined a weighted version of the classical Lorentz spaces, denoted by $\Lambda^{p,q}_u(w) = \left(\Lambda^{p,q}(w)\right)(u)$, as the set of all measurable functions f such that $\|f\|_{\Lambda^{p,q}_u(w)} = \|f^*_u\|_{\Lambda^{p,q}(w)} < \infty$. Hence, if $w = 1$ we recover the weighted Lorentz spaces $\Lambda^{p,q}_u(1) = L^{p,q}(u)$ and if $u = 1$ we get again the classical Lorentz spaces $\Lambda^{p,q}_1(w) = \Lambda^{p,q}(w)$.

Finally, whenever $\Lambda^{p,\infty}_u(w)$ is a Banach function space, we have the following result for submultilinear operators that satisfy a weighted restricted weak-type estimate.

Proposition 2.1.2. *Let $m \geqslant 1$, exponents $0 < p, p_1, \dots, p_m < \infty$, weights $w, w_1, \dots, w_m \in \Delta_2$ such that $w_i \notin L^1(\mathbb{R}^+)$, $i = 1, \dots, m$, and weights $u, u_1, \dots, u_m$ in $\mathbb{R}^n$. Suppose that T is a submultilinear operator and $\Lambda_u^{p,\infty}(w)$ is a Banach function space under the norm $\|\cdot\|^*_{\Lambda_u^{p,\infty}(w)}$. If*

$$\|T(\chi_{E_1}, \dots, \chi_{E_m})\|^*_{\Lambda_u^{p,\infty}(w)} \leqslant C \prod_{i=1}^m W_i(u_i(E_i))^{\frac{1}{p_i}}, \qquad \forall E_1, \dots, E_m \subseteq \mathbb{R}^n,$$

then

$$T : \Lambda_{u_1}^{p_1,1}(w_1) \times \cdots \times \Lambda_{u_m}^{p_m,1}(w_m) \to \Lambda_u^{p,\infty}(w) \tag{2.1.3}$$

with constant less than or equal to $\frac{C}{p_1 \cdots p_m}$.

Proof. Assume first that $m = 1$. Since $w_1 \notin L^1(\mathbb{R}^+)$ but $w_1 \in \Delta_2$, the simple functions with support in a set of finite measure are dense in $\Lambda_{u_1}^{p_1,1}(w_1)$ (see [58, Theorem 2.3.12]). Then, since $\Lambda_u^{p,\infty}(w)$ is a Banach function space under the norm $\|\cdot\|^*_{\Lambda_u^{p,\infty}(w)}$ and T is sublinear, is enough to prove that (2.1.3) holds for positive simple function with support in a set of finite measure. Hence, without loss of generality, let F_j be sets that form an increasing sequence $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_l$ with $u_1(F_l) < \infty$ and let $a_j > 0$, $j = 1, \dots, l$, so that

$$f = \sum_{j=1}^l a_j \chi_{F_j} \quad \Longrightarrow \quad f^*_{u_1} = \sum_{j=1}^l a_j \chi_{(0, u_1(F_j)]}$$

and

$$\begin{aligned}
\|Tf\|^*_{\Lambda_u^{p,\infty}(w)} &\leqslant \sum_{j=1}^l a_j \left\|T\chi_{F_j}\right\|^*_{\Lambda_u^{p,\infty}(w)} \leqslant C \sum_{j=1}^l a_j W_1(u_1(F_j))^{\frac{1}{p_1}} \\
&= \frac{C}{p_1} \sum_{j=1}^l a_j \int_0^\infty \chi_{(0,u_1(F_j)]}(t) W_1(t)^{\frac{1}{p_1}-1} w_1(t)\, dt = \frac{C}{p_1} \|f\|_{\Lambda_{u_1}^{p_1,1}(w_1)}.
\end{aligned}$$

Now, assume that the result is true for each $1 \leqslant m \leqslant k$, for some $k \geqslant 1$, and let us see that it also holds for $m = k+1$. Fix sets $E_2, \dots, E_m \subseteq \mathbb{R}^n$ and let $T_1 = T(\,\cdot\,, \chi_{E_2}, \dots, \chi_{E_m})$, which satisfies

$$\|T_1 \chi_F\|^*_{\Lambda_u^{p,\infty}(w)} \leqslant C \left(\prod_{i=2}^m W_i(u_i(E_i))^{\frac{1}{p_i}} \right) W_1(u_1(F))^{\frac{1}{p_1}}, \qquad \forall F \subseteq \mathbb{R}^n.$$

Hence, by the induction hypothesis, for every locally integrable function f,

$$\|T_1 f\|^*_{\Lambda_u^{p,\infty}(w)} \leqslant \frac{C}{p_1} \left(\prod_{i=2}^m W_i(u_i(E_i))^{\frac{1}{p_i}} \right) \|f\|_{\Lambda_{u_1}^{p_1,1}(w_1)}. \tag{2.1.4}$$

Since the sets $E_2, \dots, E_m$ were arbitrary, if we now fix some locally integrable function f and set $T_2 = T(f, \cdot, \dots, \cdot)$, then (2.1.4) can be rewritten as

$$\|T_2(\chi_{E_2}, \dots, \chi_{E_m})\|^*_{\Lambda_u^{p,\infty}(w)} \leqslant \frac{C}{p_1} \|f\|_{\Lambda_{u_1}^{p_1,1}(w_1)} \prod_{i=2}^m W_i(u_i(E_i))^{\frac{1}{p_i}}, \qquad \forall E_2, \dots, E_m \subseteq \mathbb{R}^n,$$

so (2.1.3) follows by making use again of the induction hypothesis.

□

Remark 2.1.3. *When $w, w_1, \dots, w_m = 1$ and $p > 1$, it is well known that there exists a function norm $\|\cdot\|_{(p,\infty,v)}$ such that*

$$\|\cdot\|_{L^{p,\infty}(v)} \leqslant \|\cdot\|_{(p,\infty,v)} \leqslant \frac{p}{p-1}\|\cdot\|_{L^{p,\infty}(v)}$$

with which $L^{p,\infty}(v)$ is a Banach function space (see [23, Ch. 4 - Theorem 4.6]). Hence, by Proposition 2.1.2, if

$$\|T(\chi_{E_1}, \dots, \chi_{E_m})\|_{L^{p,\infty}(u)} \leqslant C\prod_{i=1}^m u_i(E_i)^{\frac{1}{p_i}}, \qquad \forall E_1, \dots, E_m \subseteq \mathbb{R}^n,$$

then

$$T : L^{p_1,1}(u_1) \times \cdots \times L^{p_m,1}(u_m) \to L^{p,\infty}(u)$$

with constant $\frac{Cp}{(p-1)p_1\cdots p_m}$.

2.1.3 The associate space $(\Lambda^p(w))'$

The associate space of the classical Lorentz spaces have been widely studied in [58, Chapter 2] (see also [166]). Indeed, it turns out that whenever $\Lambda^p(w)$ is a quasi-Banach function space (that is, $w \in \Delta_2$) then $(\Lambda^p(w))'$ is a Banach function space [58, Theorem 2.4.4].

For every $0 < p < \infty$, the associate space $(\Lambda^p(w))'$ is defined to be the set of measurable functions f such that

$$\|f\|_{(\Lambda^p(w))'} = \sup_{\|g\|_{\Lambda^p(w)}\leqslant 1}\int_{\mathbb{R}^n}|f(x)g(x)|\,dx = \sup_{\|g\|_{\Lambda^p(w)}\leqslant 1}\int_0^\infty f^*(t)g^*(t)\,dt. \tag{2.1.5}$$

From (2.1.5), the authors described in [58, Theorem 2.4.7] the associate space $(\Lambda^p(w))'$ in terms of the *maximal function*

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds, \qquad t > 0,$$

and identified when they are the trivial space (see [58, Theorem 2.4.9]). Indeed:

(i) If $0 < p \leqslant 1$, then

$$\|f\|_{(\Lambda^p(w))'} = \sup_{t>0}\frac{t}{W(t)^{\frac{1}{p}}}f^{**}(t), \tag{2.1.6}$$

and $(\Lambda^p(w))' \neq \{0\}$ if and only if

$$\sup_{0<t<1}\frac{t}{W(t)^{\frac{1}{p}}} < \infty. \tag{2.1.7}$$

(ii) If $p > 1$ and $w \notin L^1(\mathbb{R}^+)$,

$$\|f\|_{(\Lambda^p(w))'} = \left(\int_0^\infty f^{**}(t)^{p'}t^{p'}W(t)^{-p'}w(t)\,dt\right)^{\frac{1}{p'}}, \tag{2.1.8}$$

and $(\Lambda^p(w))' \neq \{0\}$ if and only if

$$\int_0^1\left(\frac{t}{W(t)}\right)^{p'-1}dt < \infty. \tag{2.1.9}$$

2.2 Several classes of weights

This section is aimed to describe the classes of weights that we will use in the following chapters and that, in fact, characterize the weighted strong-type and weak-type boundedness of the Hardy-Littlewood maximal operator and the Hilbert transform. We will distinguish between weights in $\mathbb{R}^n$ (Section 2.2.1) that will be related to weighted Lebesgue and Lorentz spaces, and weights in $\mathbb{R}^+$ (Sections 2.2.2, 2.2.3 and 2.2.4) that will be related to classical Lorentz spaces. Further, we will study some properties and technical lemmas involving these classes of weights which will be important on the next chapters.

2.2.1 A_p, A_p^R and $\hat{A}_p$

Given a weight v in $\mathbb{R}^n$, B. Muckenhoupt [148] showed that, for $n = 1$, the characterization of the weighted *strong-type boundedness* for $1 < p < \infty$ is given by

$$M : L^p(v) \to L^p(v) \qquad \Longleftrightarrow \qquad v \in A_p,$$

(see Definition 2.2.1 below) while it is known to be false for $p = 1$. Later, R. Coifman and C. Fefferman [73] extended it to higher dimensions, and, by a weighted norm inequality due to C. Fefferman and E.M. Stein [94], it was also seen that, for $1 \leqslant p < \infty$, the A_p class characterizes the weighted *weak-type boundedness*, that is

$$M : L^p(v) \to L^{p,\infty}(v) \qquad \Longleftrightarrow \qquad v \in A_p.$$

Now, for the *Hilbert Transform* H, in [111] R. Hunt, B. Muckenhoupt and R. Wheeden, characterized the weighted strong-type boundedness for $1 < p < \infty$ by

$$H : L^p(v) \to L^p(v) \qquad \Longleftrightarrow \qquad v \in A_p,$$

and the same condition also characterizes the weighted weak-type boundedness for $1 \leqslant p < \infty$. (For more details on both operators, we refer the reader to [88, 99].)

Definition 2.2.1. Given $1 < p < \infty$, we say that $v \in A_p$ if

$$\|v\|_{A_p} = \sup_Q \left(\frac{1}{|Q|} \int_Q v \right) \left(\frac{1}{|Q|} \int_Q v^{\frac{1}{1-p}} \right)^{p-1} < +\infty,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$. For $p = 1$, we say that $v \in A_1$ if

$$Mv(x) \leqslant Cv(x), \qquad \text{a.e. } x \in \mathbb{R}^n,$$

and the infimum of all such constants C is denoted by $\|v\|_{A_1}$.

In particular, we should mention the work of S. Buckley [35] (see also [127]), who proved that

$$\|M\|_{L^p(v)} \leqslant C_n p'^{\frac{1}{p}} p^{\frac{1}{p'}} \|v\|_{A_p}^{\frac{1}{p-1}} \text{ (for } p > 1) \quad \text{ and } \quad \|M\|_{L^p(v) \to L^{p,\infty}(v)} \leqslant C_n \|v\|_{A_p}^{\frac{1}{p}} \text{ (for } p \geqslant 1).$$

Example 2.2.2. For $-n < \gamma < n(p-1)$, and $\gamma = 0$ if $p = 1$, then $v(x) = |x|^\gamma$ is an A_p weight.

Direct from the definition, and by means of the Hölder's inequality, it holds that if $v \in A_p$, then $v \in A_q$, where $1 \leqslant p \leqslant q < \infty$. Hence, in view of the inclusions of the A_p weights, it is natural to denote

$$A_\infty = \bigcup_{1\leqslant p<\infty} A_p. \tag{2.2.1}$$

This class first appeared in [73] and [149], and can be characterized (see for instance [88, Corollary 7.6]) by those weights v for which there exists $\delta \in (0,1)$ such that

$$\sup_{E\subseteq Q} \left(\frac{|Q|}{|E|}\right)^\delta \frac{v(E)}{v(Q)} < \infty,$$

where the supremum is taken over all cubes Q and all measurable sets $E \subseteq Q$. (See [91, 95, 97, 110] for more details on this class of weights.)

Note that if $v \in A_\infty$ then v is non-integrable. Indeed, let $E = [0,1]^n$ and $Q_m = [-m,m]^n$, $m \geqslant 1$. Then there exists $\delta \in (0,1)$ such that

$$|Q_m|^\delta \leqslant C\frac{v(Q_m)}{v(E)},$$

and by taking the limit when m tends to infinity we get that $\|v\|_{L^1(\mathbb{R}^n)} = \infty$.

Further, similar as in (2.2.1), we can write $A_p = \bigcup_{1\leqslant q<p} A_q$ for every $1 < p < \infty$, since the weights belonging into the A_p class of weights satisfy the important property that there exists some $\varepsilon = \varepsilon(p,v) > 0$ (which decreases to 0 as $\|v\|_{A_p}$ grows to infinity) such that (see [158, Corollary 8.1])

$$v \in A_{p-\varepsilon} \qquad \text{with} \qquad \|v\|_{A_{p-\varepsilon}} \leqslant 2^{p-1}\|v\|_{A_p}$$

(see also [74, 88, 98]). Equivalently, there exists a decreasing function Φ_p on $[1,\infty)$ satisfying $1 < \Phi_p(r) < p$ for every $r \geqslant 1$, $\lim_{r\to\infty}\Phi_p(r) = 1$ and

$$v \in A_{\frac{p}{\Phi_p(\|v\|_{A_p})}} \qquad \text{with} \qquad \|v\|_{A_{\frac{p}{\Phi_p(\|v\|_{A_p})}}} \leqslant 2^{p-1}\|v\|_{A_p}. \tag{2.2.2}$$

Another property that we want to recall is called Jones' factorization: every A_p weight can be factored as the product of two A_1 weights. It was first conjectured by B. Muckenhoupt [150] at the Williamstown conference in 1979, and proved by P. Jones [117] at the same conference. Later, R. Coifman, P. Jones and J.L. Rubio de Francia gave a simpler proof of it in [72].

Proposition 2.2.3 (Jones' factorization)**.** *If $v \in A_p$ then there exist $v_0, v_1 \in A_1$ such that $v = v_0 v_1^{1-p}$. Moreover, given $1 \leqslant q < p < \infty$, if $v_0 \in A_q$ and $v_1 \in A_1$ then $v_0 v_1^{q-p} \in A_p$ and*

$$\left\|v_0 v_1^{q-p}\right\|_{A_p} \leqslant \|v_0\|_{A_q}\|v_1\|_{A_1}^{p-q}.$$

Finally, let us state some well known facts of the class A_1:

(i) For every $v_0, v_1 \in A_1$ and any $0 \leqslant \delta \leqslant 1$, by means of the Hölder's inequality, then $v_0^\delta v_1^{1-\delta} \in A_1$ with

$$\left\|v_0^\delta v_1^{1-\delta}\right\|_{A_1} \leqslant \|v_0\|_{A_1}^\delta \|v_1\|_{A_1}^{1-\delta}.$$

(ii) ([88, Theorem 7.7]) A weight u belongs to A_1 if and only if there exists $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and K such that $K, K^{-1} \in L^\infty(\mathbb{R}^n)$ satisfying that, for some $0 < \delta < 1$,

$$u(x) = K(x)(Mf(x))^\delta, \qquad \text{a.e. } x \in \mathbb{R}^n,$$

where $L^\infty(\mathbb{R}^n)$ consists of all measurable functions f such that

$$\|f\|_\infty := \|f\|_{L^\infty(\mathbb{R}^n)} = \operatorname{ess\,sup} f < \infty.$$

(iii) ([56, Lemma 2.12]) For every $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, every $v \in A_1$ and $0 \leqslant \delta < 1$, then $(Mf)^\delta v^{1-\delta} \in A_1$ with

$$\left\|(Mf)^\delta v^{1-\delta}\right\|_{A_1} \lesssim \frac{\|v\|_{A_1}}{1-\delta}. \tag{2.2.3}$$

(iv) ([158, Lemma 5.1]) If $1 \leqslant t \leqslant 1 + \frac{1}{2^{n+1}\|v\|_{A_1}}$, then

$$v^t \in A_1 \quad \text{with} \quad \left\|v^t\right\|_{A_1} \lesssim \|v\|_{A_1}. \tag{2.2.4}$$

Now we define, for $0 < \delta \leqslant 1$,

$$M_\delta f = M\left(|f|^{1/\delta}\right)^\delta, \qquad |f|^{1/\delta} \in L^1_{\mathrm{loc}}(\mathbb{R}^n). \tag{2.2.5}$$

Hence, easy computations show that for $\frac{1}{\delta} < p < \infty$,

$$M_\delta : L^p(v) \to L^p(v) \qquad \Longleftrightarrow \qquad v \in A_{\delta p} \subseteq A_p.$$

Besides, when $\frac{1}{\delta} \leqslant p < \infty$ the same happens for the weighted weak-type boundedness $M_\delta : L^p(v) \to L^{p,\infty}(v)$. This, together with Proposition 2.2.3, motivates the following subclass of the A_p weights (for more details see [59]).

Definition 2.2.4. Given $1 \leqslant p < \infty$ and $0 \leqslant \alpha, \beta \leqslant 1$, we say that $v \in A_{p;(\alpha,\beta)}$ if there exist $v_0, v_1 \in A_1$ such that $v = v_0^\alpha v_1^{\beta(1-p)}$. Moreover, for $v \in A_{p;(\alpha,\beta)}$ we define

$$\|v\|_{A_{p;(\alpha,\beta)}} = \inf\left\{\|v_0\|_{A_1}^\alpha \|v_1\|_{A_1}^{\beta(p-1)} : v = v_0^\alpha v_1^{\beta(1-p)}\right\}.$$

Observe that then, for $0 < \delta \leqslant 1$ and $\frac{1}{\delta} \leqslant p < \infty$, $A_{\delta p} = A_{p;\left(1, \frac{\delta p - 1}{p-1}\right)}$. Further, from Proposition 2.2.3, we have that $A_{p;(1,1)} = A_p$ and, indeed, $A_{p;(\alpha,\beta)} \subseteq A_p$ with $\|v\|_{A_p} \leqslant \|v\|_{A_{p;(\alpha,\beta)}}$. Besides, if $\alpha > 0$ and $q = 1 + \frac{\beta}{\alpha}(p-1)$, $v \in A_{p;(\alpha,\beta)}$ if and only if $v^{1/\alpha} \in A_q$ with $\|v\|_{A_{p;(\alpha,\beta)}} = \left\|v^{1/\alpha}\right\|_{A_q}^\alpha$.

In [70, 119], it was characterized the following weighted *restricted weak-type boundedness* for $1 \leqslant p < \infty$:

$$M : L^{p,1}(v) \to L^{p,\infty}(v) \qquad \Longleftrightarrow \qquad v \in A_p^R,$$

where A_p^R is the restricted A_p class of weights (see Definition 2.2.5 below). Further, for every $p \geqslant 1$, it turns out that $v \in A_p^R$ is equivalent to the weighted estimate

$$\|M\chi_E\|_{L^{p,\infty}(v)} \leqslant Cv(E)^{\frac{1}{p}}, \qquad \forall E \subseteq \mathbb{R}^n. \tag{2.2.6}$$

As well, for $1 \leqslant p < \infty$,

$$H : L^{p,1}(v) \to L^{p,\infty}(v) \qquad \Longleftrightarrow \qquad v \in A_p^R,$$

which can be seen, for instance, as a consequence of the pointwise domination of Calderón-Zygmund operators by the sparse operators (see [123]) since it is not so difficult to check that the sparse operators (see Section 5.5 for its definition) satisfy such estimate for every $p > 1$. In fact, is actually true for any operator with such control, not just for the Hilbert transform.

Definition 2.2.5 ([61])**.** Given $1 \leqslant p < \infty$, we say that $v \in A_p^R$ if

$$\|v\|_{A_p^R} = \sup_{E \subseteq Q} \frac{|E|}{|Q|} \left(\frac{v(Q)}{v(E)} \right)^{\frac{1}{p}} < \infty,$$

where the supremum is taken over all cubes Q and all measurable sets $E \subseteq Q$.

In particular, in [119] it was seen that

$$\|M\|_{L^{p,1}(v) \to L^{p,\infty}(v)} \approx \|v\|_{A_p^R}. \tag{2.2.7}$$

Example 2.2.6. For $-n < \gamma \leqslant n(p-1)$ then $v(x) = |x|^\gamma$ is an A_p^R weight.

When $p = 1$, this class coincides with $A_1 = A_1^R$. Further, for $p > 1$, the relation of the A_p^R with the A_p weights is the following [61]: for every $q > p$, $A_p \subsetneq A_p^R \subsetneq A_q$ with

$$\|v\|_{A_q} \lesssim \frac{1}{q-p} \|v\|_{A_p^R}^p \qquad \text{and} \qquad \|v\|_{A_p^R} \leqslant \|v\|_{A_p}^{\frac{1}{p}}.$$

Now, it is known that (see [88, Theorem 7.7]) for $0 \leqslant \delta < 1$, $(Mf)^\delta \in A_1$, while this is not true for $\delta = 1$. In particular, by means of Proposition 2.2.3, $v_0(Mf)^{\delta(1-p)} \in A_p$ for every $v_0 \in A_1$ but $v_0(Mf)^{1-p} \notin A_p$. However, it does belong to the A_p^R class of weights (see [61, Corollary 2.8]). This fact raises the question of whether every weight in A_p^R can be written in this way, which motivates the definition of the (a priori) subclass of weights for which this factorization holds.

Definition 2.2.7 ([56, 61])**.** Given $1 \leqslant p < \infty$, we say that $v \in \hat{A}_p$ if there exist $v_0 \in A_1$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that $v = v_0(Mf)^{1-p}$. Moreover, for $v \in \hat{A}_p$ we define

$$\|v\|_{\hat{A}_p} = \inf \left\{ \|v_0\|_{A_1}^{\frac{1}{p}} : v = v_0(Mf)^{1-p} \right\}.$$

Although the classes $A_p \subsetneq \hat{A}_p \subseteq A_p^R$ (with $\|v\|_{A_p^R} \lesssim \|v\|_{\hat{A}_p}$) need not be the same in general, it holds that

$$A_\infty = \bigcup_{1\leqslant p<\infty} A_p = \bigcup_{1\leqslant p<\infty} \hat{A}_p = \bigcup_{1\leqslant p<\infty} A_p^R.$$

Now, when willing to study the operator M_δ (see (2.2.5)) easy computations show that for $\frac{1}{\delta} \leqslant p < \infty$,

$$M_\delta : L^{p,1}(v) \to L^{p,\infty}(v) \qquad \Longleftrightarrow \qquad v \in A_{\delta p}^R \subseteq A_p^R. \tag{2.2.8}$$

This motivates the following subclass of the $\hat{A}_p$ class of weights:

Definition 2.2.8 ([51])**.** Given $1 \leqslant p < \infty$ and $0 \leqslant \alpha, \beta \leqslant 1$, we say that $v \in \hat{A}_{p;(\alpha,\beta)}$ if there exist $v_0 \in A_1$ and $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that $v = v_0^\alpha (Mf)^{\beta(1-p)}$. Moreover, for $v \in \hat{A}_{p;(\alpha,\beta)}$ we define

$$\|v\|_{\hat{A}_{p;(\alpha,\beta)}} = \inf\left\{\|v_0\|_{A_1}^{\frac{\alpha}{1+\beta(p-1)}} : v = v_0^\alpha (Mf)^{\beta(1-p)}\right\}.$$

Observe that then, for $0 < \delta \leqslant 1$ and $\frac{1}{\delta} \leqslant p < \infty$, $\hat{A}_{\delta p} = \hat{A}_{p;\left(1,\frac{\delta p-1}{p-1}\right)}$. Further, we have that $\hat{A}_{p;(1,1)} = \hat{A}_p$ and, indeed, $\hat{A}_{p;(\alpha,\beta)} \subseteq \hat{A}_p$ with $\|v\|_{\hat{A}_p} \leqslant \|v\|_{\hat{A}_{p;(\alpha,\beta)}}$.

2.2.2 B_p and B_p^R

Given a weight w in $\mathbb{R}^+$. M.A. Ariño and B. Muckenhoupt [10] characterized the weighted *strong-type boundedness* on classical Lorentz spaces $\Lambda^p(w)$ of the Hardy-Littlewood maximal operator for every $1 < p < \infty$ by

$$M : \Lambda^p(w) \to \Lambda^p(w) \qquad \Longleftrightarrow \qquad w \in B_p,$$

(see Definition 2.2.9 below). Moreover, in [153], C.J. Neugebauer saw that the same holds for the corresponding weighted weak-type boundedness; that is, for every $1 < p < \infty$,

$$M : \Lambda^p(w) \to \Lambda^{p,\infty}(w) \qquad \Longleftrightarrow \qquad w \in B_p,$$

which implies that the weighted strong-type and the weak-type boundedness are equivalent in that range of p. Later, in [54], M.J. Carro and J. Soria characterized the weighted strong-type boundedness in the case $0 < p \leqslant 1$ with the B_p class of weights as well. Besides, the authors observe that the B_p class of weights was sufficient for the weighted weak-type boundedness on this range of p, but not necessary.

Definition 2.2.9. Given $0 < p < \infty$, we say that $w \in B_p$ if

$$\|w\|_{B_p} = \sup_{t>0}\left(1 + \frac{1}{W(t)}\int_t^\infty \left(\frac{t}{r}\right)^p w(r)\,dr\right) < \infty.$$

Examples 2.2.10. (1) Let $\gamma > 0$, then $w(t) = t^{\gamma-1} \in B_p$ if and only if $\gamma < p$. Moreover, $\|w\|_{B_p} = \frac{p}{p-\gamma}$. In particular, $1 \in B_p$ for every $p > 1$.

(2) Let $a > 0$. Then, $w(t) = \chi_{(0,a)}(t) \in B_p$ if and only if $p > 1$. Moreover, in that case, $\|w\|_{B_p} = \frac{p}{p-1}$.

(3) Let $0 < \gamma < p$ and let Φ be a decreasing function. Then, $w(t) = \Phi(t)t^{\gamma-1} \in B_p$ since for every $t > 0$,

$$t^p \int_t^\infty \frac{w(r)}{r^p}\,dr = t^p \int_t^\infty \frac{\Phi(r)}{r^{p-\gamma+1}}\,dr \leqslant \frac{1}{p-\gamma}\Phi(t)t^\gamma \leqslant \frac{\gamma}{p-\gamma}\int_0^t w(r)\,dr$$

so that $\|w\|_{B_p} \leqslant \frac{p}{p-\gamma}$.

Proposition 2.2.11 ([10, 34, 54]). *Given $0 < p < \infty$. If $w \in B_p$ then*

$$\|M\|_{\Lambda^p(w)} \lesssim \|w\|_{B_p}^{\max\left(1,\frac{1}{p}\right)}.$$

Similar as for the A_p weights, the B_p weights also satisfies the $p-\varepsilon$ property (see, for instance, [58, Corollary 3.3.4]). In particular, following the estimates used in [153, Theorem 2.5], at least for $p \geqslant 1$ and for $w \in B_p$, taking $\varepsilon = \frac{p}{2\|w\|_{B_p}}$ it can be seen that

$$\|w\|_{B_{p-\varepsilon}} \lesssim \|w\|_{B_p}. \tag{2.2.9}$$

Further, the B_p weights also follows a chain of inclusion in the sense that $B_p \subsetneq B_q$, for every $0 < p < q < \infty$, so that it is natural to denote

$$B_\infty = \bigcup_{0<p<\infty} B_p.$$

Indeed, it is known that $w \in B_\infty$ if and only if $w \in \Delta_2$ (see [64]) which, in turn, is equivalent to $\Lambda^p(w)$ and $\Lambda^{p,\infty}(w)$ being quasi-Banach function spaces (see Section 2.1.2 for this notions). Further, although for $0 < p < 1$, $\Lambda^p(w)$ is never a Banach function space (see [172, Remark 3.2]) for $p \geqslant 1$, $\Lambda^p(w)$ is a Banach function space when $w \in B_p$, and the reciprocal is also true whenever $p > 1$ (see [166]), while for every $0 < p < \infty$, that $w \in B_p$ is equivalent to $\Lambda^{p,\infty}(w)$ being a Banach function space (see [172]).

Now, for $p = 1$ we observe that even though $\Lambda^1(1) = L^1(\mathbb{R}^n)$ is a Banach function space, the weight $w = 1$ does not belong to B_1. Hence, in order to characterize when $\Lambda^1(w)$ is a Banach function space is needed a bigger class than B_1. In [57], M.J. Carro, A. García del Almo and J. Soria proved that this condition is fulfilled by the restricted B_1 class of weights B_1^R.

Definition 2.2.12. Given $0 < p < \infty$, we say that $w \in B_p^R$ if

$$\|w\|_{B_p^R} = \sup_{0<r\leqslant t} \frac{rW(t)^{\frac{1}{p}}}{tW(r)^{\frac{1}{p}}} < \infty.$$

In particular, $w \in B_p^R$ is said to be a p *quasi-concave* function.

Contrary to the B_p weights, there is no $p-\varepsilon$ property in the B_p^R class. Indeed, if we take $w(t) = t^{p-1} \in B_p^R$, then $w \in B_q^R$ if and only if $q \geqslant p$.

Examples 2.2.13. (1) Let $\gamma > 0$, then $w(t) = t^{\gamma-1} \in B_p^R$ if and only if $\gamma \leqslant p$. Moreover, $\|w\|_{B_p^R} = 1$.

(2) Let $a>0$. Then, $w(t)=\chi_{(0,a)}(t)\in B_p^R$ if and only if $p\geqslant 1$. Moreover, in that case, $\|w\|_{B_p^R}=1$.

One of the important properties of these weights is that for $0<p\leqslant 1$, the B_p^R class of weights characterizes the weighted weak-type boundedness of the Hardy-Littlewood maximal operator (see [54, Theorem 3.3 (b)] and [57, Theorem 2.3]]) and, in that case,

$$\|M\|_{\Lambda^p(w)\to\Lambda^{p,\infty}(w)}\lesssim \|w\|_{B_p^R}. \tag{2.2.10}$$

Further, similar as for the A_p^R weights in (2.2.6), it turns out that $w\in B_p^R$ is equivalent to

$$\|M\chi_E\|_{\Lambda^{p,\infty}(w)}\leqslant CW(|E|)^{\frac{1}{p}},\qquad \forall E\subseteq\mathbb{R}^n, \tag{2.2.11}$$

for every $0<p<\infty$.

Now, we observe that

$$M:\Lambda^{p,1}(w)\to\Lambda^{p,\infty}(w)\qquad\Longleftrightarrow\qquad M:\Lambda^1(\tilde w)\to\Lambda^{1,\infty}(\tilde w),$$

with $\tilde w=W^{1/p-1}w$ (see (2.1.2)) and, clearly,

$$\|w\|_{B_p^R}=\|\tilde w\|_{B_1^R}. \tag{2.2.12}$$

Thus,

$$M:\Lambda^{p,1}(w)\to\Lambda^{p,\infty}(w)\quad\Longleftrightarrow\quad \tilde w\in B_1^R\quad\Longleftrightarrow\quad w\in B_p^R \tag{2.2.13}$$

and, from (2.2.10), $\|M\|_{\Lambda^{p,1}(w)\to\Lambda^{p,\infty}(w)}\lesssim\|w\|_{B_p^R}$. Moreover, from (2.2.11) and (2.2.13), for every $0<\delta\leqslant 1$,

$$M_\delta:\Lambda^{p,1}(w)\to\Lambda^{p,\infty}(w)\quad\Longleftrightarrow\quad w\in B_{\delta p}^R, \tag{2.2.14}$$

with $\|M_\delta\|_{\Lambda^{p,1}(w)\to\Lambda^{p,\infty}(w)}\lesssim\|w\|_{B_{\delta p}^R}^{\delta}$.

The relation of the B_p^R with the B_p weights is the following [153]: for every $0<p\leqslant q<\infty$, $B_p\subsetneq B_p^R\subsetneq B_q$ with

$$\|w\|_{B_q}\leqslant\frac{q}{q-p}\|w\|_{B_p^R}^p\qquad\text{and}\qquad \|w\|_{B_p^R}\lesssim\|w\|_{B_p}^{\frac{1}{p}}. \tag{2.2.15}$$

Further, we have already seen how the weights w and $\tilde w$ are related in the B_p^R class of weights (see (2.2.12)), and it is also interesting to know what happen in the class of weights B_p.

Lemma 2.2.14. *Let $0<p,q<\infty$. If $w\in B_{qp}$ then $\tilde w=W^{\frac{1}{p}-1}w\in B_q$ with*

$$\|\tilde w\|_{B_q}\lesssim\begin{cases}\|w\|_{B_{qp}}^{\frac{1}{p}}, & 0<p\leqslant 1,\\ \|w\|_{B_{qp}}^{\frac{1}{p}+1}, & p>1.\end{cases}$$

Proof. First assume that $0<p\leqslant 1$. Then, integrating by parts we obtain that

$$\int_t^\infty\left(\frac{t}{r}\right)^q\tilde w(r)\,dr\leqslant\|w\|_{B_{qp}}^{\frac{1}{p}}\tilde W(t)+qpt^q\int_t^\infty W(r)^{\frac{1}{p}}\frac{dr}{r^{1+q}},\quad t>0, \tag{2.2.16}$$

with $\tilde{W}$ being the primitive of $\tilde{w}$. Hence, using the Minkowski's inequality we observe that, for every $t > 0$,

$$t^q \int_t^\infty W(r)^{\frac{1}{p}} \frac{dr}{r^{1+q}} \lesssim \tilde{W}(t) + \left(\int_t^\infty \left(\frac{t}{r} \right)^{qp} w(r)\, dr \right)^{\frac{1}{p}} \lesssim \|w\|_{B_{qp}}^{\frac{1}{p}} \tilde{W}(t), \tag{2.2.17}$$

so that putting together (2.2.16) and (2.2.17), we deduce that $\|\tilde{w}\|_{B_q} \lesssim \|w\|_{B_{qp}}^{\frac{1}{p}}$.

Now take $1 < p < \infty$. If we set $\varepsilon = \frac{qp}{2\|w\|_{B_{qp}}}$, by (2.2.9) and (2.2.15) we get that $w \in B_{qp-\varepsilon}^R$ with $\|w\|_{B_{qp-\varepsilon}^R} \lesssim \|w\|_{B_{qp}}^{1/(qp-\varepsilon)}$. Therefore, we get that for every $t > 0$,

$$t^q \int_t^\infty W(r)^{\frac{1}{p}} \frac{dr}{r^{1+q}} = t^q \int_t^\infty \left[\frac{W(r)}{r^{qp-\varepsilon}} \right]^{\frac{1}{p}} \frac{dr}{r^{1+\frac{\varepsilon}{p}}} \lesssim \frac{1}{\varepsilon} \|w\|_{B_{qp}}^{\frac{1}{p}} \tilde{W}(t) \lesssim \|w\|_{B_{qp}}^{\frac{1}{p}+1} \tilde{W}(t), \tag{2.2.18}$$

and the desired result follows by putting together (2.2.16) and (2.2.18).

□

As a consequence, for $p > 1$, if $w \in B_p$ and $w \notin L^1(\mathbb{R}^+)$ then, from Lemma 2.2.14 we get that $\tilde{w} = W^{p'-2}w \in B_{p'}$ and, in that case, since $\tilde{W} \approx W^{p'-1}$, in account of [172, Theorem 2.5], (2.1.9) holds and, thus, $(\Lambda^p(w))' \neq \{0\}$. Further, observe that for $0 < p \leqslant 1$, if $w \in B_p^R$, then clearly (2.1.7) holds, so we deduce that $(\Lambda^p(w))' \neq \{0\}$, while for $p > 1$, it is not true in general (just take $w = t^{p-1} \in B_p^R \backslash B_p$ with which (2.1.9) does not hold). Moreover, for every $0 < p < \infty$ and $w \in B_p^R$, $(\Lambda^{p,1}(w))' \neq \{0\}$.

Finally, to end this section we show a technical lemma that will be useful later.

Lemma 2.2.15. *Given $0 < p < \infty$, $0 < q \leqslant 1$ and $w \in B_p^R$,*

$$\|\chi_E\|_{(\Lambda^{p,q}(w))'} \leqslant \left(\frac{q}{p} \right)^{\frac{1}{q}} \|w\|_{B_p^R} \frac{|E|}{W(|E|)^{\frac{1}{p}}}, \qquad \forall E \subseteq \mathbb{R}^n.$$

Further, if $p > 1$ and $w \in B_p$ is such that $w \notin L^1(\mathbb{R}^n)$,

$$\|\chi_E\|_{(\Lambda^p(w))'} \lesssim \|w\|_{B_p} \frac{|E|}{W(|E|)^{\frac{1}{p}}}, \qquad \forall E \subseteq \mathbb{R}^n.$$

Proof. First, let $\tilde{w} = W^{\frac{q}{p}-1}w$. Hence, by means of (2.1.6),

$$\|\chi_E\|_{(\Lambda^{p,q}(w))'} = \|\chi_E\|_{(\Lambda^q(\tilde{w}))'} = \left(\frac{q}{p} \right)^{\frac{1}{q}} \sup_{t>0} \frac{\min(t, |E|)}{W(t)^{\frac{1}{p}}} \leqslant \left(\frac{q}{p} \right)^{\frac{1}{q}} \|w\|_{B_p^R} \frac{|E|}{W(|E|)^{\frac{1}{p}}}.$$

Finally, taking $\varepsilon = \frac{p}{2\|w\|_{B_p}}$ we obtain that (see (2.1.8) and (2.2.9))

$$\begin{aligned}
\|\chi_E\|_{(\Lambda^p(w))'} &\leqslant \left(\int_0^{|E|} \left(\frac{t}{W(t)} \right)^{p'} w(t)\, dt \right)^{\frac{1}{p'}} + |E| \left(\int_{|E|}^\infty \frac{w(t)}{W(t)^{p'}}\, dt \right)^{\frac{1}{p'}} \\
&= \left(\int_0^{|E|} \left(\frac{t}{W(t)^{\frac{1}{p-\varepsilon}}} \right)^{p'} W(t)^{p'(\frac{1}{p-\varepsilon}-1)} w(t)\, dt \right)^{\frac{1}{p'}} + (p-1)^{\frac{1}{p'}} \frac{|E|}{W(|E|)^{\frac{1}{p}}} \\
&\lesssim (p-1)^{\frac{1}{p'}} \left(\|w\|_{B_p}^{\frac{1}{p}} \left[\frac{p}{\varepsilon} - 1 \right]^{\frac{1}{p'}} + 1 \right) \frac{|E|}{W(|E|)^{\frac{1}{p}}} \lesssim \|w\|_{B_p} \frac{|E|}{W(|E|)^{\frac{1}{p}}}.
\end{aligned}$$

□

2.2.3 B_∞^*, B_p^* and B_p^{*R}

Given a weight w in $\mathbb{R}^+$. The boundedness of the Hardy-Littlewood maximal operator from $(\Lambda^p(w))'$ to itself is fully characterized by means of the Boyd indices. Given a r.i. Banach function space $\mathbb{X}$ on $\mathbb{R}^n$, the Lorentz-Shimogaki theorem (see [144, 170] and [23, Ch.3 - Theorem 5.17]) asserts that

$$M : \mathbb{X} \to \mathbb{X} \qquad \Longleftrightarrow \qquad \alpha_{\mathbb{X}} < 1.$$

Therefore, since $\alpha_{\mathbb{X}'} = 1 - \beta_{\mathbb{X}}$,

$$M : \mathbb{X}' \to \mathbb{X}' \qquad \Longleftrightarrow \qquad \beta_{\mathbb{X}} > 0.$$

In 2007, A.K. Lerner and C. Pérez [134] generalized the Lorentz-Shimogaki theorem for every quasi-Banach function space, not necessarily rearrangement invariant. Further, in [4, Proposition 2.6] it was seen that $\beta_{\Lambda^p(w)} > 0$ is equivalent to $w \in B_\infty^*$ (see Definition 2.2.16 below). Therefore, putting all together yield that for every $0 < p < \infty$ and $w \in \Delta_2$ (so that $\Lambda^p(w)$ is a quasi-Banach function space)

$$M : (\Lambda^p(w))' \to (\Lambda^p(w))' \qquad \Longleftrightarrow \qquad w \in B_\infty^*. \tag{2.2.19}$$

Besides, it is known [166] that the weighted strong-type boundedness of the Hilbert transform for $0 < p < \infty$ is characterized by

$$H : \Lambda^p(w) \to \Lambda^p(w) \qquad \Longleftrightarrow \qquad w \in B_p \cap B_\infty^*. \tag{2.2.20}$$

Furthermore, in [3] it was proved that the corresponding characterization of the weighted weak-type boundedness, for $p > 1$, is given by the same condition, while for $0 < p \leqslant 1$ [166] it holds that

$$H : \Lambda^p(w) \to \Lambda^{p,\infty}(w) \qquad \Longleftrightarrow \qquad w \in B_p^R \cap B_\infty^*. \tag{2.2.21}$$

Definition 2.2.16 ([154])**.** We say that $w \in B_\infty^*$ if

$$\|w\|_{B_\infty^*} = \sup_{t>0} \frac{1}{W(t)} \int_0^t \frac{W(r)}{r}\, dr < \infty.$$

Examples 2.2.17. (1) If $\gamma > 0$ then $t^{\gamma-1} \in B_\infty^*$ with $\|w\|_{B_\infty^*} = \frac{1}{\gamma}$.

(2) If w is a weight such that there exists some $0 \leqslant \gamma < 1$ and some $0 < p < \infty$ satisfying that $W(t)^{\frac{1}{p}} t^{\gamma-1}$ is increasing, then $w \in B_\infty^*$.

(3) For every $a > 0$, $\chi_{(0,a)} \notin B_\infty^*$. In fact, if $w \in L^1(\mathbb{R}_+)$ then $w \notin B_\infty^*$.

Now, recall that [23, Ch. 3 - Theorem 3.8]

$$(Mf)^*(t) \approx f^{**}(t) = \frac{1}{t} \int_0^t f^*(r)\, dr =: Pf^*(t), \qquad \forall t > 0, \tag{2.2.22}$$

where P is known to be the *Hardy operator*. Then, the class of weights B_∞^* appears naturally when studying the boundedness of the *adjoint of the Hardy operator* defined as

$$Qf(t) = \int_t^\infty f(r)\,\frac{dr}{r}, \qquad t > 0,\ f \in \mathcal{M}_+, \tag{2.2.23}$$

between $L^p_{\rm dec}(w)$ and $L^p(w)$ (see [23] for more details on these operators).

Indeed, keeping track on constants, in [8, Theorem 4] (or [154, Theorem 3.3] for $p \geqslant 1$) it was proved that for $0 < p < \infty$,

$$Q : L^p_{\rm dec}(w) \to L^p(w) \qquad \Longleftrightarrow \qquad w \in B_\infty^* \tag{2.2.24}$$

and $\|Q\|_{L^p_{\rm dec}(w)\to L^p(w)} \lesssim \|w\|_{B_\infty^*}$, which allows us to obtain an specific control of the norm constant of (2.2.19).

Proposition 2.2.18. *Let $w \in B_\infty^* \cap \Delta_2$. Then, for every $0 < p < \infty$,*

$$\|M\|_{(\Lambda^p(w))'} \lesssim \|w\|_{B_\infty^*}.$$

Proof. Applying the definition of associate space, for every $f \in (\Lambda^p(w))'$,

$$\begin{aligned}\|Mf\|_{(\Lambda^p(w))'} &= \sup_{\|h\|_{\Lambda^p(w)}\leqslant 1} \int_{\mathbb{R}^n} Mf(x)h(x)\,dx = \sup_{h\downarrow} \frac{\int_0^\infty (Mf)^*(t)h(t)\,dt}{\left(\int_0^\infty h(t)^p w(t)\,dt\right)^{\frac1p}} \\ &\approx \sup_{h\downarrow} \frac{\int_0^\infty Pf^*(t)h(t)\,dt}{\left(\int_0^\infty h(t)^p w(t)\,dt\right)^{\frac1p}} = \sup_{h\downarrow} \frac{\int_0^\infty f^*(t)Qh(t)\,dt}{\left(\int_0^\infty h(t)^p w(t)\,dt\right)^{\frac1p}} \lesssim \|w\|_{B_\infty^*} \|f\|_{(\Lambda^p(w))'},\end{aligned}$$

where in the last estimate we have used the Hölder's inequality and the proper control of $\|Q\|_{L^p_{\rm dec}(w)\to L^p(w)}$ by $\|w\|_{B_\infty^*}$. ☐

In particular, for $\tilde w = W^{1/p-1}w$,

$$M : \left(\Lambda^{p,1}(w)\right)' \to \left(\Lambda^{p,1}(w)\right)' \quad \Longleftrightarrow \quad M : \left(\Lambda^{1}(\tilde w)\right)' \to \left(\Lambda^{1}(\tilde w)\right)' \quad \Longleftrightarrow \quad \tilde w \in B_\infty^*,$$

so we should study when $\tilde w \in B_\infty^*$.

Lemma 2.2.19. *Let $w \in B_\infty^*$. Then, for every $0 < p < \infty$ and $\tilde w = W^{\frac1p-1}w$,*

$$\|\tilde w\|_{B_\infty^*} = \sup_{t>0} \frac{1}{W(t)^{\frac1p}} \int_0^t \frac{W(r)^{\frac1p}}{r}\,dr \leqslant \max(1,p)\,\|w\|_{B_\infty^*}.$$

Proof. First, since W is an increasing function, for every $0 < p \leqslant 1$,

$$\int_0^t \frac{W(r)^{\frac1p}}{r}\,dr \leqslant W(t)^{\frac1p-1}\int_0^t \frac{W(r)}{r}\,dr \leqslant \|w\|_{B_\infty^*} W(t)^{\frac1p}, \qquad t > 0.$$

So let us consider the case $p > 1$. Hence, for every $t > 0$, using integration by parts and the definition of B_∞^*,

$$\begin{aligned}\int_0^t \frac{W(r)^{\frac1p}}{r}\,dr &\leqslant W(t)^{\frac1p-1}\int_0^t \frac{W(r)}{r}\,dr + \frac{p-1}{p}\int_0^t W(r)^{\frac1p-2}w(r)\int_0^r \frac{W(s)}{s}\,ds\,dr \\ &\leqslant p\,\|w\|_{B_\infty^*} W(t).\end{aligned}$$

☐

Hence, as a direct consequence of Proposition 2.2.18 and Lemma 2.2.19, we obtain the following result:

Proposition 2.2.20. *Let $w \in B_\infty^* \cap \Delta_2$. Then, for every $0 < p < \infty$,*

$$\|M\|_{(\Lambda^{p,1}(w))'} \lesssim \|w\|_{B_\infty^*}.$$

Furthermore, from Lemma 2.2.19 and (2.2.21) we also deduce that

$$H : \Lambda^{p,1}(w) \to \Lambda^{p,\infty}(w) \qquad \Longleftrightarrow \qquad w \in B_p^R \cap B_\infty^*. \tag{2.2.25}$$

Finally, observe that if $w \in B_\infty^*$, for every $0 < r \leqslant t$,

$$W(r) \log\left(\frac{t}{r}\right) \leqslant \int_0^t \log\left(\frac{t}{s}\right) w(s)ds = \int_0^t \frac{W(s)}{s} ds \leqslant \|w\|_{B_\infty^*} W(t).$$

So, let $\overline{W} : (0,\infty) \to (0,\infty)$ be the increasing submultiplicative function defined as

$$\overline{W}(\mu) = \sup\left\{\frac{W(t)}{W(s)} : 0 < t \leqslant \mu s\right\} = \sup_{x\in[0,\infty)} \frac{W(\mu x)}{W(x)}. \tag{2.2.26}$$

Therefore, we get that

$$\overline{W}(\mu) \leqslant \|w\|_{B_\infty^*}\left(\log\frac{1}{\mu}\right)^{-1}, \qquad \forall 0 < \mu < 1. \tag{2.2.27}$$

Lemma 2.2.21. *If $w \in B_\infty^*$ then*

$$\overline{W}(\mu) \lesssim \mu^{\frac{1}{e\|w\|_{B_\infty^*}}}, \qquad \forall 0 < \mu < 1.$$

Proof. Let $\mu_0 = e^{-e\|w\|_{B_\infty^*}}$. Since $\overline{W}$ is submultiplicative, by (2.2.27) and induction on $k \in \mathbb{N} \cup \{0\}$ we get that

$$\overline{W}(\mu_0^k) \leqslant \left(\overline{W}(\mu_0)\right)^k \leqslant \left(\frac{1}{e}\right)^k = \left(\mu_0^k\right)^{\frac{1}{e\|w\|_{B_\infty^*}}}.$$

Now take $\mu \in (0,1)$ and choose $k \in \mathbb{N} \cup \{0\}$ such that $\mu_0^{k+1} \leqslant \mu < \mu_0^k$. Then, since $\overline{W}$ is increasing,

$$\overline{W}(\mu) \leqslant \overline{W}(\mu_0^k) \leqslant \left(\mu_0^k\right)^{\frac{1}{e\|w\|_{B_\infty^*}}} \leqslant e\mu^{\frac{1}{e\|w\|_{B_\infty^*}}}.$$

□

At this point, for a given $0 < \delta < 1$, consider the operator M_δ (see (2.2.5)). Hence, on account of (2.2.22), the Minkowski's inequality and [58, Lemma 2.4],

$$(M_\delta f)^*(t) \approx P([f^*]^{1/\delta})(t)^\delta \leqslant \frac{1}{t^\delta}\int_0^t f^*(r)\frac{dr}{r^{1-\delta}} = P_{\frac{1}{\delta}} f^*(t), \qquad t > 0, \tag{2.2.28}$$

where $P_{\frac{1}{\delta}}$ is the *generalized Hardy operator*, while the converse estimate in (2.2.28) is false in general (see [21, Theorem 3]). The adjoint operator of $P_{\frac{1}{\delta}}$ is

$$Q_{\frac{1}{1-\delta}}f(t) = \frac{1}{t^{1-\delta}}\int_t^\infty f(r)\,\frac{dr}{r^\delta}, \qquad t>0,\ f\in\mathcal{M}_+, \tag{2.2.29}$$

and is called the *adjoint of the generalized Hardy operator* $P_{\frac{1}{\delta}}$ (see [23] for more details on theses operators).

Then, arguing as in Proposition 2.2.18, we obtain that

$$Q_{\frac{1}{1-\delta}} : L^p_{\text{dec}}(w) \to L^p(w) \qquad \Longrightarrow \qquad M_\delta : (\Lambda^p(w))' \to (\Lambda^p(w))', \tag{2.2.30}$$

with $\|M_\delta\|_{(\Lambda^p(w))'} \lesssim \left\|Q_{\frac{1}{1-\delta}}\right\|_{L^p_{\text{dec}}(w)\to L^p(w)}$, and the boundedness of the left-hand side of (2.2.30) is known to be characterized by $w\in B^*_{\frac{1}{(1-\delta)p}}$ (see [125, 154]).

Definition 2.2.22. Given $0<p<\infty$, we say that $w\in B^*_p$ if

$$\|w\|_{B^*_p} = \sup_{t>0}\frac{1}{W(t)}\int_0^t\left(\frac{t}{r}\right)^{\frac{1}{p}} w(r)\,dr < \infty.$$

Example 2.2.23. If $\gamma > \dfrac{1}{p}$, then $t^{\gamma-1}\in B^*_p$ with $\|w\|_{B^*_p} = \frac{\gamma}{\gamma-\frac{1}{p}}$.

Proposition 2.2.24. *Given* $0<\delta<1$.

(i) [125, Theorem 2.2] *If* $0<p<1$, $\left\|Q_{\frac{1}{1-\delta}}\right\|_{L^p_{dec}(w)\to L^p(w)} \leqslant \frac{1}{1-\delta}\|w\|^{\frac{1}{p}}_{B^*_{\frac{1}{(1-\delta)p}}}$.

(ii) [154, Theorem 3.1] *If* $1\leqslant p<\infty$, $\left\|Q_{\frac{1}{1-\delta}}\right\|_{L^p_{dec}(w)\to L^p(w)} \leqslant \frac{1}{1-\delta}\|w\|_{B^*_{\frac{1}{(1-\delta)p}}}$.

Hence, as a direct consequence of (2.2.30) and Proposition 2.2.24, we obtain the following result:

Proposition 2.2.25. *Given* $0<p<\infty$ *and* $0<\delta<1$. *If* $w\in B^*_{\frac{1}{(1-\delta)p}}$ *then*

$$\|M_\delta\|_{(\Lambda^p(w))'} \lesssim \frac{\delta}{1-\delta}\|w\|^{\max\left(1,\frac{1}{p}\right)}_{B^*_{\frac{1}{(1-\delta)p}}}.$$

Now, similar as for the B_p weights, the B^*_p weights satisfy the chain of inclusions $B^*_p \subsetneq B^*_q$, $q>p$, and also satisfy the $p-\varepsilon$ property (see, for instance, [125, Theorem 4.3] or [145, Lemma 4 (2)]). In fact, following the estimates used in [154, Theorem 3.2], at least for $p\geqslant 1$ and for $w\in B^*_p$, taking $\varepsilon = \frac{p}{4\|w\|_{B^*_p}}$ it can be seen that

$$\|w\|_{B^*_{p-\varepsilon}} \lesssim \|w\|_{B^*_p}. \tag{2.2.31}$$

Hence, in particular, the B_p^* class of weights can also be written as the union of the B_q^* for $0 < q < p$; that is

$$B_p^* = \bigcup_{0<q<p} B_q^*.$$

Further, it is also known that for $p > 1$,

$$Q_{\frac{1}{1-\delta}} : L^p_{\text{dec}}(w) \to L^{p,\infty}(w) \qquad \Longleftrightarrow \qquad w \in B^*_{\frac{1}{(1-\delta)p}},$$

while this is not the case for $0 < p \leqslant 1$ (see for instance [8, 50]).

Definition 2.2.26. Given $0 < p < \infty$, we say that $w \in B_p^{*R}$ if

$$\|w\|_{B_p^{*R}} = \sup_{0<r\leqslant t} \frac{tW(r)^p}{rW(t)^p} < \infty.$$

Contrary to the B_p^* class of weights, there is no $p - \varepsilon$ property in the B_p^{*R} class. Indeed, if we take $w(t) = t^{\frac{1}{p}-1} \in B_p^{*R}$, then $w \in B_q^{*R}$ if and only if $q \geqslant p$. Besides, for this class of weights we have that for every $0 < p \leqslant 1$,

$$\left\| Q_{\frac{1}{1-\delta}} \right\|_{L^p_{\text{dec}}(w) \to L^{p,\infty}(w)} \lesssim \|w\|^{1-\delta}_{B^{*R}_{\frac{1}{(1-\delta)p}}}$$

(see [8, Theorem 1]).

Example 2.2.27. If $\gamma \geqslant \dfrac{1}{p}$, then $t^{\gamma-1} \in B_p^{*R}$ with $\|w\|_{B_p^{R*}} = 1$.

Easy computations show that the B_p^* and B_p^{*R} classes of weights are related as follows: for every $0 < p < q < \infty$, then $B_p^{*R} \subsetneq B_q^* \subsetneq B_q^{*R}$ with

$$\|w\|^{\frac{1}{q}}_{B_q^{*R}} \leqslant \|w\|_{B_q^*} \leqslant \frac{q}{q-p} \|w\|^{\frac{1}{p}}_{B_p^{*R}}. \tag{2.2.32}$$

Further, these both classes increases to B_∞^* as the following result shows:

Proposition 2.2.28. *For every $0 < p < \infty$, $B_p^{*R} \subsetneq B_\infty^*$ with $\|w\|_{B_\infty^*} \lesssim \log\left(1 + \|w\|_{B_p^{*R}}\right)$. Moreover,*

$$B_\infty^* = \bigcup_{0<p<\infty} B_p^* = \bigcup_{0<p<\infty} B_p^{*R}.$$

Proof. First, given $0 < q < \infty$ and $\tilde{w} = W^{\frac{1}{q}-1}w$ (so that $\tilde{W} = qW^{\frac{1}{q}}$) due to Proposition 2.2.19,

$$\|w\|_{B_\infty^*} = \sup_{t>0} \frac{1}{\tilde{W}(t)^q} \int_0^t \frac{\tilde{W}(r)^q}{r}\, dr \leqslant \max\left(1, \frac{1}{q}\right) \left(\sup_{t>0} \frac{1}{W(t)^{\frac{1}{q}}} \int_0^t \frac{W(r)^{\frac{1}{q}}}{r}\, dr \right).$$

Hence, if $w \in B_p^{*R}$ we obtain

$$\|w\|_{B_\infty^*} \leqslant \max\left(1, \frac{1}{q}\right) \|w\|^{\frac{1}{qp}}_{B_p^{*R}} \left(\sup_{t>0} \frac{1}{t^{\frac{1}{qp}}} \int_0^t r^{\frac{1}{qp}-1}\, dr \right) = p \max(1, q) \|w\|^{\frac{1}{qp}}_{B_p^{*R}},$$

and taking the infimum on $q > 0$ yields the first part of the statement.

Further, it was seen in [2, Lemma 2.6] that $w \in B^*_\infty$ is equivalent to the existence of some $\mu_0 \in (0,1)$ such that $\overline{W}(\mu_0) < 1$ (see (2.2.26) for the definition of $\overline{W}$). Moreover, arguing similar as in [2, Lemma 2.7], it can be seen that $w \in B^*_p$ is equivalent to the existence of some $\mu_1 \in (0,1)$ such that $\overline{W}(\mu_1) < \mu_1^{1/p}$.

Hence, take $w \in B^*_\infty$. Then, there exists some $\mu_0 \in (0,1)$ and $\varepsilon \in (0, 1-\mu_0)$ so that $\overline{W}(\mu_0) < 1-\varepsilon$. Then, for $p = \frac{\log \mu_0}{\log(1-\varepsilon)} > 1$, it holds that $\overline{W}(\mu_0) < 1-\varepsilon = \mu_0^{1/p}$, which implies that $w \in B^*_p$.

□

Now, in particular, if $\tilde{w} = W^{1/p-1}w$, we have already seen in Proposition 2.2.25 that

$$\tilde{w} \in B^*_{\frac{1}{(1-\delta)p}} \qquad \Longrightarrow \qquad M_\delta : \left(\Lambda^{p,1}(w)\right)' \to \left(\Lambda^{p,1}(w)\right)',$$

so we should study when $\tilde{w} \in B^*_{\frac{1}{(1-\delta)p}}$.

Lemma 2.2.29. *Let $0 < p, q < \infty$. If $w \in B^*_{\frac{q}{p}}$ then $\tilde{w} = W^{\frac{1}{p}-1}w \in B^*_q$ with*

$$\|\tilde{w}\|_{B^*_q} \lesssim \begin{cases} \|w\|^{\frac{1}{p}}_{B^*_{\frac{q}{p}}}, & 0 < p \leqslant 1, \\ \|w\|^{\frac{1}{p}+1}_{B^*_{\frac{q}{p}}}, & p > 1. \end{cases}$$

Proof. First assume that $0 < p \leqslant 1$. Then, integrating by parts we obtain that

$$\int_0^t \left(\frac{t}{r}\right)^{\frac{1}{q}} \tilde{w}(r)\, dr \leqslant \tilde{W}(t) + \frac{pt^{\frac{1}{q}}}{q} \int_0^t W(r)^{\frac{1}{p}} \frac{dr}{r^{1+\frac{1}{q}}}, \qquad t > 0, \tag{2.2.33}$$

with $\tilde{W}$ being the primitive of $\tilde{w}$. Hence, using the Minkowski's inequality we observe that

$$t^{\frac{1}{q}} \int_0^t W(r)^{\frac{1}{p}} \frac{dr}{r^{1+\frac{1}{q}}} \leqslant q \left(\int_0^t \left(\frac{t}{r}\right)^{\frac{p}{q}} w(r)\, dr \right)^{\frac{1}{p}} \leqslant \frac{q}{p} \|w\|^{\frac{1}{p}}_{B^*_{\frac{q}{p}}} \tilde{W}(t), \quad t > 0, \tag{2.2.34}$$

so that putting together (2.2.33) and (2.2.34), we deduce that $\|\tilde{w}\|_{B^*_q} \lesssim \|w\|^{\frac{1}{p}}_{B^*_{\frac{q}{p}}}$.

Now take $1 < p < \infty$. If we set $\varepsilon = \frac{q}{4p\|w\|_{B^*_{\frac{q}{p}}}}$, by (2.2.31) and (2.2.32) we get that $w \in B^{*R}_{\frac{q}{p}-\varepsilon}$ with $\|w\|_{B^{*R}_{\frac{q}{p}-\varepsilon}} \lesssim \|w\|^{(q-p\varepsilon)/p}_{B^*_{\frac{q}{p}}}$. Therefore, for every $t > 0$,

$$t^{\frac{1}{q}} \int_0^t W(r)^{\frac{1}{p}} \frac{dr}{r^{1+\frac{1}{q}}} = t^{\frac{1}{q}} \int_0^t \left[\frac{W(r)}{r^{\frac{p}{q-p\varepsilon}}}\right]^{\frac{1}{p}} \frac{dr}{r^{1+\frac{1}{q}-\frac{1}{q-p\varepsilon}}} \lesssim \frac{1}{\frac{1}{q-p\varepsilon} - \frac{1}{q}} \|w\|^{\frac{1}{p}}_{B^*_{\frac{q}{p}}} W(t)^{\frac{1}{p}} \lesssim \|w\|^{\frac{1}{p}+1}_{B^*_{\frac{q}{p}}} \tilde{W}(t), \tag{2.2.35}$$

so, putting together (2.2.33) and (2.2.35) yields the desired result.

□

Hence, as a direct consequence of Proposition 2.2.25 and Lemma 2.2.29, we obtain the following result:

Proposition 2.2.30. *Given $0<\delta<1$ and $0<p<\infty$. If $w \in B^*_{\frac{1}{(1-\delta)p}}$, then*

$$\|M_\delta\|_{(\Lambda^{p,1}(w))'} \lesssim \frac{1}{1-\delta}\begin{cases} \|w\|^{\frac{1}{p}}_{B^*_{\frac{1}{(1-\delta)p}}}, & 0<p\leqslant 1,\\ (1-\delta)^{-\frac{1}{p}}\|w\|^{\frac{1}{p}+1}_{B^*_{\frac{1}{(1-\delta)p}}}, & p>1.\end{cases}$$

However, when restricted to characteristic functions, we can consider weights on the bigger class $B^{*R}_{\frac{1}{(1-\delta)p}}$:

Proposition 2.2.31. *Given $0<\delta<1$ and $0<p<\infty$. If $w \in B^{*R}_{\frac{1}{(1-\delta)p}}$, then*

$$\|M_\delta\chi_F\|_{(\Lambda^{p,1}(w))'} \lesssim \log\left(1+\|w\|_{B^{*R}_{\frac{1}{(1-\delta)p}}}\right)\|w\|^{1-\delta}_{B^{*R}_{\frac{1}{(1-\delta)p}}}\|\chi_F\|_{(\Lambda^{p,1}(w))'}, \qquad \forall F\subseteq \mathbb{R}^n.$$

Proof. First, from (2.1.6) and Propositions 2.2.19 and 2.2.28, we have that

$$\begin{aligned}\|M_\delta\chi_F\|_{(\Lambda^{p,1}(w))'} &\approx \sup_{t>0}\frac{1}{W(t)^{\frac{1}{p}}}\int_0^t (M_\delta\chi_F)^*(r)\,dr \lesssim \|w\|_{B^*_\infty}\left(\sup_{t>0}\frac{t}{W(t)^{\frac{1}{p}}}(M_\delta\chi_F)^*(t)\right)\\ &\lesssim \log\left(1+\|w\|_{B^{*R}_{\frac{1}{(1-\delta)p}}}\right)\left(\sup_{t>0}\frac{t}{W(t)^{\frac{1}{p}}}(M_\delta\chi_F)^*(t)\right).\end{aligned} \tag{2.2.36}$$

Now, due to (2.2.22),

$$\begin{aligned}\sup_{t>0}\frac{t}{W(t)^{\frac{1}{p}}}(M_\delta\chi_F)^*(t) &\approx \sup_{t>0}\frac{t}{W(t)^{\frac{1}{p}}}\left(\frac{\min(t,|F|)}{t}\right)^{\delta}\\ &= \max\left(\sup_{0<t<|F|}\frac{t}{W(t)^{\frac{1}{p}}}, |F|^{\delta}\sup_{t\geqslant|F|}\frac{t^{1-\delta}}{W(t)^{\frac{1}{p}}}\right)\\ &\leqslant \|w\|^{1-\delta}_{B^{*R}_{\frac{1}{(1-\delta)p}}}\left(\sup_{0<t\leqslant|F|}\frac{t}{W(t)^{\frac{1}{p}}}\right) = \|w\|^{1-\delta}_{B^{*R}_{\frac{1}{(1-\delta)p}}}\|\chi_F\|_{(\Lambda^{p,1}(w))'},\end{aligned} \tag{2.2.37}$$

so that the desired result follows by putting together (2.2.36) and (2.2.37). □

Finally, given $w \in B^*_p \subseteq B^{*R}_p$, we have that,

$$\overline{W}(\mu) \leqslant \|w\|_{B^*_p}\mu^{\frac{1}{p}}, \qquad \forall 0<\mu<1, \tag{2.2.38}$$

so that as a consequence, we have the following result.

Lemma 2.2.32. *Let $1 < p < \infty$. If $w \in B_p^*$, then*

$$\overline{W}(\mu) \lesssim \|w\|_{B_p^*} \mu^{\frac{1}{p}+\frac{1}{4p\|w\|_{B_p^*}}}, \qquad \forall 0 < \mu < 1.$$

Proof. First of all, we know that if $w \in B_p^*$, taking $\varepsilon = \frac{p}{4\|w\|_{B_p^*}}$ we have that $\|w\|_{B_{p-\varepsilon}^*} \lesssim \|w\|_{B_p^*}$ (see (2.2.31)). Hence, from (2.2.38) we obtain that for every $0 < \mu < 1$,

$$\overline{W}(\mu) \leqslant \|w\|_{B_{p-\varepsilon}^*} \mu^{\frac{1}{p-\varepsilon}} \lesssim \|w\|_{B_p^*} \mu^{\frac{1}{p}+\frac{1}{4p\|w\|_{B_p^*}}}.$$

□

2.2.4 $B_{\frac{1}{p_1}} \cap B_{p_2}^*$ and $B_{\frac{1}{p_1}}^R \cap B_{p_2}^*$

Recall that for the characterization of the boundedness of the Hilbert transform H over classical Lorentz spaces (see (2.2.20) and (2.2.21)) is needed that the weight w belong not just in one class of weights but in the intersection of two. Indeed, in Sections 4.2.2 and 4.3.4, w is going to belong in the intersection of two classes of weights, which will be either $B_{\frac{1}{p_1}} \cap B_{p_2}^*$ or the bigger class $B_{\frac{1}{p_1}}^R \cap B_{p_2}^*$, for some $0 < p_1 < \infty$ and $0 < p_2 \leqslant \infty$, so it is natural to ask when these intersection classes are non empty.

Proposition 2.2.33. *If $p_1 < p_2$ then $B_{\frac{1}{p_1}} \cap B_{p_2}^* \neq \varnothing$. Otherwise, $B_{\frac{1}{p_1}}^R \cap B_{p_2}^* = \varnothing$.*

Proof. First observe that $t^{\gamma-1} \in B_{\frac{1}{p_1}} \cap B_{p_2}^*$ whenever $\frac{1}{p_2} < \gamma < p_1$. Hence, if $p_1 < p_2$ then $B_{\frac{1}{p_1}} \cap B_{p_2}^* \neq \varnothing$, and the same must hold for the bigger class $B_{\frac{1}{p_1}}^R \cap B_{p_2}^*$.

On the other side, if $w \in B_{\frac{1}{p_1}}^R$ and $p_2 < \infty$ then

$$\int_0^t \left(\frac{t}{r}\right)^{\frac{1}{p_2}} w(r)\,dr \geqslant \left(\frac{W(t)^{p_1}}{\|w\|_{B_{\frac{1}{p_1}}^R}}\right)^{\frac{1}{p_2}} \int_0^t \frac{w(r)}{W(r)^{\frac{p_1}{p_2}}}\,dr, \qquad t > 0, \tag{2.2.39}$$

and the right-hand side of (2.2.39) blows up when $p_1 \geqslant p_2$, so that, in that case, $B_{\frac{1}{p_1}} \cap B_{p_2}^* = B_{\frac{1}{p_1}}^R \cap B_{p_2}^* = \varnothing$.

□

Remark 2.2.34. $B_{\frac{1}{p_1}} \cap B_{\infty}^* \neq \varnothing \neq B_{\frac{1}{p_1}}^R \cap B_{\infty}^*$.

In particular, from Proposition 2.2.33 we deduce that the power functions are examples of weights belonging in these intersection classes. Here, we give some more:

Examples 2.2.35. (1) If $w(t) = t^{\gamma-1}$ then

$$w \in B_{\frac{1}{p_1}} \iff \gamma < \frac{1}{p_1} \quad \text{and} \quad w \in B_{\frac{1}{p_1}}^R \iff \gamma \leqslant \frac{1}{p_1} \tag{2.2.40}$$

with $\|w\|_{B_{\frac{1}{p_1}}} = \frac{1}{1-\gamma p_1}$ and $\|w\|_{B^R_{\frac{1}{p_1}}} = 1$. Moreover,

$$w \in B^*_{p_2} \quad \Longleftrightarrow \quad \gamma > \frac{1}{p_2} \quad \text{and} \quad w \in B^*_\infty \quad \Longleftrightarrow \quad \gamma > 0 \tag{2.2.41}$$

with $\|w\|_{B^*_{p_2}} = \frac{\gamma}{\gamma - \frac{1}{p_2}}$ and $\|w\|_{B^*_\infty} = \frac{1}{\gamma}$.

(2) Set for every $m \in \mathbb{N}$ and $0 < \gamma \leqslant 1$,

$$w_{m,\gamma}(t) = \left(1 + \log_+ \frac{1}{t}\right)^m t^{\gamma-1}, \qquad t > 0.$$

By induction on m, it is easy to see that for every $p_2 > p_1$, then $w_{m,\frac{1}{p_1}} \in B^R_{\frac{1}{p_1}} \cap B^*_{p_2}$ with

$$\left\|w_{m,\frac{1}{p_1}}\right\|_{B^R_{\frac{1}{p_1}}} = 1 \quad \text{and} \quad \left\|w_{m,\frac{1}{p_1}}\right\|_{B^*_{p_2}} \lesssim \left[\left(\frac{p_1 p_2}{p_2 - p_1}\right)\left(\frac{m+1}{m}\right)\right]^m (m+1)!.$$

(3) Similarly, if we set for every $m \in \mathbb{N}$ and $0 < \gamma \leqslant 1$,

$$\tilde{w}_{m,\gamma}(t) = (1 + \log_+ t)^{-m}\, t^{\gamma-1}, \qquad t > 0.$$

Then, for every $p_2 > p_1$, $\tilde{w}_{m,\frac{1}{p_1}} \in B^R_{\frac{1}{p_1}} \cap B^*_{p_2}$ with

$$\left\|\tilde{w}_{m,\frac{1}{p_1}}\right\|_{B^R_{\frac{1}{p_1}}} \lesssim 1 \quad \text{and} \quad \left\|\tilde{w}_{m,\frac{1}{p_1}}\right\|_{B^*_{p_2}} \lesssim \left[\frac{p_1 p_2}{p_2 - p_1}\right]^m (m+1)^{m+1}.$$

In particular, for every $0 < \gamma \leqslant 1$ and $m \in \mathbb{N}$, we have the following continuous embeddings,

$$\Lambda^{1,q}\left(w_{m,\gamma}\right) \subseteq \Lambda^{1,q}(t^{\gamma-1}) = L^{\frac{1}{\gamma},1}(\mathbb{R}^n) \subseteq \Lambda^{1,q}\left(\tilde{w}_{m,\gamma}\right), \qquad \text{for every } 0 < q \leqslant \infty.$$

Further, if we denote by $f \approx\downarrow$ meaning that f is a quasi-decreasing function and by $f \approx\uparrow$ when $-f$ is quasi-decreasing, we obtain the following interpretation of the weights:

$$w \in B_{\frac{1}{p_1}} \cap B^*_{p_2} \qquad \Longleftrightarrow \qquad \exists \varepsilon > 0 : \frac{W(t)}{t^{\frac{1}{p_1}-\varepsilon}} \approx\downarrow \quad \text{and} \quad \frac{W(t)}{t^{\frac{1}{p_2}+\varepsilon}} \approx\uparrow,$$

and

$$w \in B^R_{\frac{1}{p_1}} \cap B^*_{p_2} \qquad \Longleftrightarrow \qquad \frac{W(t)}{t^{\frac{1}{p_1}}} \approx\downarrow \quad \text{and} \quad \exists \varepsilon > 0 : \frac{W(t)}{t^{\frac{1}{p_2}+\varepsilon}} \approx\uparrow,$$

where we are assuming that $\frac{1}{p_2} = 0$ when $p_2 = \infty$.

2.3 The Rubio de Francia extrapolation

In this section we gather some known Rubio de Francia extrapolation results related with weighted Lebesgue and Lorentz spaces. We will begin with the original Rubio de Francia theorem (Section 2.3.1) and continue with more recent versions of it based on r.i. spaces extrapolation (Section 2.3.2) and limited extrapolation (Section 2.3.3), all of them assuming weighted strong-type estimates. Finally, we will study some extensions stemmed from weighted restricted weak-type estimates (Section 2.3.4).

2.3.1 The original Rubio de Francia extrapolation

An important property of the A_p weights is the extrapolation theorem of Rubio de Francia. It was announced [162] in 1982 and given [163] in 1984 with a detailed proof, both by J.L. Rubio de Francia. In its original version, reads as follows: if T is a sublinear operator which satisfies the weighted strong-type boundedness

$$T : L^{p_0}(v) \to L^{p_0}(v), \qquad \forall v \in A_{p_0}, \tag{2.3.1}$$

for some $1 \leqslant p_0 < \infty$ and with constant depending on $\|v\|_{A_{p_0}}$, then for any $1 < p < \infty$,

$$T : L^{p}(v) \to L^{p}(v), \qquad \forall v \in A_{p}, \tag{2.3.2}$$

with constant depending on $\|v\|_{A_p}$. Note that, in particular, this is true if we let $p_0 = 2$ in (2.3.1) and $v = 1$ in (2.3.2), so for instance $L^p(\mathbb{R}^n)$ estimates follow from weighted L^2 estimates. This led A. Cordoba [96] to assert that "there are not $L^p(\mathbb{R}^n)$ spaces, only weighted L^2". Since then, many results concerning this topic have been studied (see, for example, [60, 83, 84, 89]).

In fact, it is known that the operator T plays no role.

Theorem 2.3.1 ([89]). *Assume that for some pair of nonnegative functions (f, g) and for some $1 \leqslant p_0 < \infty$,*

$$\left(\int_{\mathbb{R}^n} g(x)^{p_0} v(x)\, dx \right)^{\frac{1}{p_0}} \leqslant \varphi(\|v\|_{A_{p_0}}) \left(\int_{\mathbb{R}^n} f(x)^{p_0} v(x)\, dx \right)^{\frac{1}{p_0}}, \qquad \forall v \in A_{p_0}, \tag{2.3.3}$$

where φ is a nondecreasing function on $[1, \infty)$. Then, for all $1 < p < \infty$,

$$\left(\int_{\mathbb{R}^n} g(x)^{p} v(x)\, dx \right)^{\frac{1}{p}} \leqslant \Phi(\|v\|_{A_{p}}) \left(\int_{\mathbb{R}^n} f(x)^{p} v(x)\, dx \right)^{\frac{1}{p}}, \qquad \forall v \in A_{p},$$

where

$$\Phi(r) = C_1 \varphi\left(C_2 r^{\max\left(1, \frac{p_0 - 1}{p - 1}\right)} \right), \qquad r \geqslant 1. \tag{2.3.4}$$

Besides, for a general operator T it can also be deduced an extrapolation result in the weighted weak-type setting (see [83, Chapter 2.2]) which reads as follows:

Corollary 2.3.2. *Given an operator T, suppose that for some $1 \leqslant p_0 < \infty$,*

$$T : L^{p_0}(v) \to L^{p_0,\infty}(v), \qquad \forall v \in A_{p_0},$$

with constant less than or equal to $\varphi(\|v\|_{A_{p_0}})$, where φ is a positive nondecreasing function on $[1,\infty)$. Then, for every $1 < p < \infty$,

$$T : L^p(v) \to L^{p,\infty}(v), \qquad \forall v \in A_p,$$

with constant less than or equal to $\Phi(\|v\|_{A_p})$, where Φ is as in (2.3.4).

2.3.2 The Rubio de Francia extrapolation on r.i. spaces

In [82, Theorem 3.1], the authors were able to extend Theorem 2.3.1 for r.i. Banach function spaces (see also [83, Theorem 4.10]). Here we state a different version of it involving the maximal operator M instead of the Boyd indices (as it is done in [79, Theorem 10.1]) and where we keep track of the constants.

Theorem 2.3.3. *Assume that for some pair of nonnegative functions (f,g) and for some $1 \leqslant p_0 < \infty$, (2.3.3) holds. Further, let $\mathbb{X}$ be a r.i. Banach function space such that $M : \mathbb{X} \to \mathbb{X}$ and $M : \mathbb{X}' \to \mathbb{X}'$. Then,*

$$\|g\|_{\mathbb{X}} \leqslant C_1 \varphi(C_2 \|M\|_{\mathbb{X}'} \|M\|_{\mathbb{X}}^{p_0-1}) \|f\|_{\mathbb{X}}.$$

Indeed, the proof of Theorem 2.3.3 (as also the one for Theorem 2.3.1) relies on the construction of an A_1 weight based on an iteration algorithm introduced in [163] by J.L. Rubio de Francia, now known as *Rubio de Francia algorithm*: let $\mathbb{X}$ be a r.i. Banach function space so that $M : \mathbb{X} \to \mathbb{X}$ and given $h \in L^1_{\text{loc}}(\mathbb{R}^n)$, then

$$Rh(x) = \sum_{k=0}^{\infty} \frac{M^k h(x)}{(2\|M\|_{\mathbb{X}})^k}, \qquad x \in \mathbb{R}^n,$$

satisfy that $|h(x)| \leqslant Rh(x)$, $\|Rh\|_{A_1} \leqslant 2\|M\|_{\mathbb{X}}$ and $\|Rh\|_{\mathbb{X}} \leqslant 2\|h\|_{\mathbb{X}}$, where M^k denotes the k-th iteration of M and $M^0 h(x) = |h(x)|$.

2.3.3 The limited Rubio de Francia extrapolation

There are some operators T for which (2.3.1) does not hold for some $p_0 \geqslant 1$. As a consequence, they are not bounded even for one $p \geqslant 1$ on $L^p(v)$ for every $v \in A_p$, so the results studied before do not apply to this kind of operators. However, it can be seen that they are bounded in the smaller class $A_{p_0;(\alpha,\beta)}$ (see Definition 2.2.4) with $\alpha, \beta \in [0,1]$ (see, for instance, [14, 51, 59, 83, 89]) and, as a consequence, one can get weighted estimates with weights in a subclass of A_p where now the exponent p does not varies in the whole range $(1,\infty)$ but in an interval whose endpoints depend on the triple $(p_0;\alpha,\beta)$.

Given $1 \leqslant p_0 < \infty$ and $0 \leqslant \alpha, \beta \leqslant 1$, let us define

$$p_+ = \frac{p_0}{1-\alpha} \qquad \text{and} \qquad p_-' = \frac{p_0'}{1-\beta} \quad \left(\text{or } p_- = \frac{p_0}{1+\beta(p_0-1)}\right), \tag{2.3.5}$$

where $p_+ = \infty$ if $\alpha = 1$ and $p_- = 1$ if $\beta = 1$. Then, $1 \leqslant p_- \leqslant p_+ \leqslant \infty$ and we can associate to every $p_- \leqslant p \leqslant p_+$ the indices

$$\alpha(p) = \frac{p_+ - p}{p_+} \qquad \text{and} \qquad \beta(p) = \frac{p - p_-}{p_-(p-1)},$$

so that $0 \leqslant \alpha(p), \beta(p) \leqslant 1$, $p_+ = \frac{p}{1-\alpha(p)}$, $p_-' = \frac{p'}{1-\beta(p)}$ and $\alpha(p_0) = \alpha$, $\beta(p_0) = \beta$. Although it seems natural to start with some fixed triple $(p_0; \alpha, \beta)$, alternatively, we could take the endpoints p_- and p_+ as the original data.

Hence, following the proofs of [59, Theorem 2.7] and [89, Theorem 7.1], and keeping track on their respective constants, the weighted limited strong-type extrapolation can be stated in this way:

Theorem 2.3.4 ([14])**.** *Assume that for some pair of nonnegative functions (f, g), for some $1 \leqslant p_0 < \infty$ and $0 \leqslant \alpha, \beta \leqslant 1$ (not both identically zero) we have*

$$\|g\|_{L^{p_0}(v)} \leqslant \varphi(\|v\|_{A_{p_0;(\alpha,\beta)}}) \, \|f\|_{L^{p_0}(v)}\,, \qquad \forall v \in A_{p_0;(\alpha,\beta)},$$

where φ is a nondecreasing function on $[1, \infty)$. Then, for $p \in (p_-, p_+)$ holds that

$$\|g\|_{L^p(v)} \leqslant C_1 \varphi\left(C_2 \, \|v\|_{A_{p;(\alpha(p),\beta(p))}}^{\max\left(\frac{p_+ - p_0}{p_+ - p}, \frac{p_0 - p_-}{p - p_-}\right)} \right) \|f\|_{L^p(v)}\,, \qquad \forall v \in A_{p;(\alpha(p),\beta(p))}.$$

Remark 2.3.5. *A weight v belongs to the reverse Hölder class RH_q if for every measurable cube $Q \subseteq \mathbb{R}^n$,*

$$\begin{cases} \left(\frac{1}{|Q|} \int_Q v(x)^q \, dx \right)^{\frac{1}{q}} \lesssim \frac{v(Q)}{|Q|}, & 1 < q < \infty, \\ v(x) \lesssim \frac{v(Q)}{|Q|} \ a.e. \ x \in Q, & q = \infty, \end{cases}$$

(see, for instance, [14, 83]) while for $q = 1$ then $RH_1 = A_\infty$. It is known (see [115, (P6)]) that $v \in A_p \cap RH_q$ if and only if $v^q \in A_{q_p}$ with $q_p = q(p-1) + 1$ so that

$$A_{p_0;(\alpha,\beta)} = A_{\frac{p_0}{p_-}} \cap RH_{\left(\frac{p_+}{p_0}\right)'}, \qquad \alpha > 0,$$

and, for $\alpha = 0$, it can be seen that, as well, $A_{p_0;(0,\beta)} = A_{\frac{p_0}{p_-}} \cap RH_\infty$. Indeed, Theorem 2.3.4 was first proved in [14] where the authors considered this intersection class of weights instead of $A_{p_0;(\alpha,\beta)}$.

2.3.4 The restricted Rubio de Francia extrapolation

Although p_0 can be set to 1 in Corollary 2.3.2, it is not possible, in general, to extrapolate till the endpoint $p = 1$ (take just $T = M \circ M$, the composition of two Hardy-Littlewood maximal operators, or see, for instance, [157] where a counterexample is given in the case of commutators). It is fair to say, though, that the original purpose of this result was to deduce estimates on $L^p(\mathbb{R}^n)$ for $p > 1$ just from weighted L^2 estimates. However, in the recent papers [56, 61], a Rubio de Francia extrapolation theory for operators satisfying a weighted restricted weak-type boundedness for the class of weights $\widehat{A}_p$ (see Definition 2.2.7) has been developed. The main advantage of this new class of weights is that allows to obtain boundedness estimates at the endpoint $p = 1$.

In particular, in [61, Theorem 2.11] the authors shown the following:

Theorem 2.3.6. *Let $1 < p_0 < \infty$ and let T be an operator. Assume that*

$$T : L^{p_0,1}(v) \to L^{p_0,\infty}(v), \qquad \forall v \in \hat{A}_{p_0}, \tag{2.3.6}$$

with constant less than or equal to $\varphi(\|v\|_{\hat{A}_{p_0}})$, where φ is a positive nondecreasing function on $[1,\infty)$. Then, for every measurable set $E \subseteq \mathbb{R}^n$ there exists a constant $C > 0$ independent of E such that

$$\|T\chi_E\|_{L^{1,\infty}(u)} \leqslant C\|u\|_{A_1}^{1-\frac{1}{p_0}} \varphi\left(\|u\|_{A_1}^{\frac{1}{p_0}}\right) u(E), \qquad \forall u \in A_1. \tag{2.3.7}$$

Now, for simplicity, whenever an operator T satisfies that for every measurable set E and for some weight u,

$$\|T\chi_E\|_{L^{1,\infty}(u)} \leqslant C_u u(E),$$

we shall say that

$$T : L^1_{\mathcal{R}}(u) \to L^{1,\infty}(u), \tag{2.3.8}$$

with constant less than or equal to C_u.

Remark 2.3.7. *We should emphasize here that the operator T need not to be sublinear. However, if it is sublinear then it was proved in [171] that*

$$T : L^1_{\mathcal{R}}(u) \to L^{1,\infty}(u)$$

is equivalent to have

$$T : B^*(u) \to L^{1,\infty}(u)$$

where

$$B^*(u) = \left\{ f \in \mathcal{M} : \int_0^\infty \lambda_f^u(t) \left(1 + \log \frac{\|f\|_1}{\lambda_f^u(t)}\right) dt < \infty \right\},$$

which can be endowed with a quasi-norm.

Remark 2.3.8. *The complete result that T is of weighted weak-type $(1,1)$ for every weight $u \in A_1$ (i.e., that the estimate in (2.3.7) holds for every $f \in L^1(u)$) is, in general, false (see [61, Remark 2.12]), even if T is a sublinear operator.*

However, it was proved in [61, Theorem 3.5] that, for a quite smaller class of operators, (2.3.7) does hold for every $f \in L^1(u)$, paying the price of adding one more power of $\|u\|_{A_1}$ on the norm constant.

Definition 2.3.9. Given $\delta > 0$, a function $a \in L^1(\mathbb{R}^n)$ is called a δ-*atom* if it satisfies the following properties:

(i) $\int_{\mathbb{R}^n} a(x)\, dx = 0$, and

(ii) there exists a cube $Q \subseteq \mathbb{R}^n$ such that $|Q| \leqslant \delta$ and supp $a \subseteq Q$.

Definition 2.3.10. A sublinear operator T is called (ε, δ)-*atomic* if, for every $\varepsilon > 0$, there exists $\delta > 0$ satisfying that

$$\|Ta\|_{L^1(\mathbb{R}^n)+L^\infty(\mathbb{R}^n)} \leqslant \varepsilon \|a\|_1,$$

for every δ-atom a. Further, T is said to be (ε, δ)-*atomic approximable* if there exists a sequence $\{T_j\}_j$ of (ε, δ)-atomic operators such that, for every measurable set $E \subseteq \mathbb{R}^n$, then $|T_j \chi_E| \leqslant |T\chi_E|$ and, for every $f \in L^1(\mathbb{R}^n)$ such that $\|f\|_\infty \leqslant 1$,

$$|Tf(x)| \leqslant \liminf_j |T_j f(x)|,$$

for almost every $x \in \mathbb{R}^n$.

Examples 2.3.11. In [44], the author showed that for sublinear operators, the property of being (ε, δ)-atomic is not a strong one. For instance, if

$$Tf(x) = K * f(x) = \int_{\mathbb{R}^n} K(y-x) f(y)\, dy, \qquad x \in \mathbb{R}^n,$$

with $K \in L^p(\mathbb{R}^n)$ for some $1 \leqslant p < \infty$, then T is (ε, δ)-atomic. Further, if

$$T^* f(x) = \sup_{j \in \mathbb{N}} \left| \int_{\mathbb{R}^n} K_j(x, y) f(y)\, dy \right|, \qquad x \in \mathbb{R}^n,$$

with

$$\lim_{y \to x} \|K_j(\cdot, y) - K_j(\cdot, x)\|_{L^1(\mathbb{R}^n)+L^\infty(\mathbb{R}^n)} = 0,$$

then T^* is (ε, δ)-atomic approximable (in particular, standard maximal Calderón-Zygmund operators are of this type). In general,

$$T^* f(x) = \sup_j |T_j f(x)|, \qquad x \in \mathbb{R}^n,$$

where $\{T_j\}_j$ is a sequence of (ε, δ)-atomic, is also (ε, δ)-atomic approximable and the same holds for

$$Tf(x) = \left(\sum_j |T_j f(x)|^q \right)^{\frac{1}{q}}, \qquad x \in \mathbb{R}^n,$$

with $q \in [1, \infty)$ and

$$Tf(x) = \sum_j T_j f(x), \qquad x \in \mathbb{R}^n.$$

(We refer the reader to [44, 61] for more examples.)

Theorem 2.3.12 ([61])**.** *Let T be a sublinear (ε, δ)-atomic approximable operator and let $u \in A_1$. If*

$$T : L^1_{\mathcal{R}}(u) \to L^{1,\infty}(u),$$

with constant less than or equal to $C_u > 0$, then

$$T : L^1(u) \to L^{1,\infty}(u)$$

with constant less than or equal to $2^n C_u \|u\|_{A_1}$.

Therefore, as a consequence of Theorems 2.3.6 and 2.3.12 we have the following result.

Corollary 2.3.13. *Let $1 < p_0 < \infty$ and let T be a sublinear (ε, δ)-atomic approximable operator. Assume that (2.3.6) holds. Then,*

$$T : L^1(u) \to L^{1,\infty}(u), \qquad \forall u \in A_1,$$

with constant less than or equal to $C\|u\|_{A_1}^{2-\frac{1}{p_0}} \varphi\left(\|u\|_{A_1}^{\frac{1}{p_0}}\right)$.

Moreover, from Theorem 2.3.6 and [56, Theorem 3.1], it is known that if for a sublinear operator T there exists some $1 < p_0 < \infty$ so that T satisfies (2.3.6), then, for every $1 < p < \infty$,

$$T : L^{p,1}(v) \to L^{p,\infty}(v), \qquad \forall v \in \hat{A}_p, \tag{2.3.9}$$

but it remained as an open question what happens when $p_0 = 1$, until now (see Theorem 3.3.1). Indeed, there are many operators in Harmonic Analysis for which the weighted weak-type $(1,1)$ boundedness for every weight in A_1 has been proved [61, 112, 122, 135, 140, 178], and hence, as a consequence of the classical Rubio de Francia extrapolation theory (see Corollary 2.3.2) it was known that they are also bounded in $L^p(v)$ for every $v \in A_p$; but, in general, (2.3.9) has been unknown for many examples.

Further, according with the limited setting, for a general operator T it can also be deduced an extrapolation result in the weighted weak-type setting of Theorem 2.3.4 by following the same lines on the proof of [83, Chapter 2.2]; that is, if for some $p_0 \geqslant 1$

$$T : L^{p_0}(v) \to L^{p_0,\infty}(v), \qquad \forall v \in A_{p_0;(\alpha,\beta)},$$

with constant less than or equal to $\varphi(\|v\|_{A_{p_0;(\alpha,\beta)}})$, then for every $p_- < p < p_+$,

$$T : L^p(v) \to L^{p,\infty}(v), \qquad \forall v \in A_{p;(\alpha(p),\beta(p))}.$$

However, even when $p_- > 1$, in this case is neither possible, in general, to extrapolate till the endpoint $p = p_-$ (see, for instance, [118] where a counterexample was given in the case of the disc multiplier when restricted to radial functions) and the same happens for the endpoint $p = p_+$.

Nevertheless, in [51, Theorem 3.7] the authors were able to obtain an estimate in the endpoint p_- by assuming that the operators satisfy a weighted restricted weak-type boundedness for the class of weights $\hat{A}_{p;(\alpha,\beta)}$ (see Definition 2.2.8).

Theorem 2.3.14 ([51])**.** *Let $1 \leqslant p_0 < \infty$, $0 \leqslant \alpha, \beta \leqslant 1$ (not both identically zero) and let T be a sublinear operator. Assume that*

$$T : L^{p_0,1}(v) \to L^{p_0,\infty}(v), \qquad \forall v \in \hat{A}_{p_0;(\alpha,\beta)}, \tag{2.3.10}$$

with constant less than or equal to $\varphi(\|v\|_{\hat{A}_{p_0;(\alpha,\beta)}})$, where φ is a positive nondecreasing function on $[1, \infty)$. Then:

(i) *If* $p_- > 1$,

$$T : L^{p_-,1}\left(u^{\alpha(p_-)}\right) \to L^{p_-,\infty}\left(u^{\alpha(p_-)}\right), \qquad \forall u \in A_1, \tag{2.3.11}$$

with constant less than or equal to $\frac{1}{p_- -1}\Phi_{p_-}(\|u\|_{A_1}^{\alpha(p_-)})$, *and where* Φ_{p_-} *is a positive nondecreasing function on* $[1,\infty)$.

(ii) *If* $p_- = 1$,

$$T : L^{1,\frac{1}{p_0}}\left(u^{\alpha(p_-)}\right) \to L^{1,\infty}\left(u^{\alpha(p_-)}\right), \qquad \forall u \in A_1, \tag{2.3.12}$$

with constant less than or equal to $\Phi_1(\|u\|_{A_1}^{\alpha(p_-)})$. *In particular,*

$$T : L^1_{\mathcal{R}}\left(u^{\alpha(p_-)}\right) \to L^{1,\infty}\left(u^{\alpha(p_-)}\right), \qquad \forall u \in A_1.$$

Here, in Theorem 3.4.1, we will see that, in fact, assuming that (2.3.10) holds, we can extrapolate until any $p_- \leqslant p < p_+$; that is,

$$T : L^{p,1}(v) \to L^{p,\infty}(v), \qquad \forall v \in \hat{A}_{p;(\alpha(p),\beta(p))}. \tag{2.3.13}$$

To do so, we will make use of Theorem 2.3.14 since then, by assuming that either (2.3.11) or (2.3.12) are satisfied, we will extrapolate from weighted estimates of restricted weak-type (p_-, p_-) to (2.3.13). Indeed, there are many operators in Harmonic Analysis for which the weighted restricted weak-type (p_-, p_-) boundedness for every weight in $A_{p_-;(\alpha(p_-),0)}$ has been proved (see [87, 120, 122, 140]) but, in general, (2.3.13) has been unknown for many examples.

Chapter 3

Weak-type $(1,1)$ for weights in A_1

This chapter is aimed to the study of boundedness properties for operators T that are of weighted restricted weak-type (1, 1) in the sense that there exists $C > 0$ such that, for every measurable set $E \subseteq \mathbb{R}^n$,

$$\|T\chi_E\|_{L^{1,\infty}(u)} \leqslant C\varphi(\|u\|_{A_1})u(E), \qquad \forall u \in A_1.$$

Indeed, we will prove that this condition is a "norm" condition since it is equivalent to

$$T : L^{p,1}(v) \longrightarrow L^{p,\infty}(v), \qquad \forall v \in \hat{A}_p.$$

As a consequence, we can obtain estimates for operators which are given as an average of operators of the above type. To do so, we begin in Section 3.1 by motivating and introducing the problem we want to address. In fact, we study a particular case of average operators consisting on Fourier multipliers with the multiplier being a right-continuous bounded variation function. Further, in Section 3.2 we prove what is going to be the keystone of our main results and which consists on a Sawyer-type inequality. Then, we will prove our main results in Sections 3.3 and 3.4 respectively. Indeed, we will see that, for a general operator T,

$$\left\{\begin{array}{l} \|T\chi_E\|_{L^{1,\infty}(u)} \lesssim u(E), \\ \forall E \subseteq \mathbb{R}^n,\ u \in A_1, \end{array}\right. \quad \Longleftrightarrow \quad \left\{\begin{array}{l} T : L^{p,1}(v) \to L^{p,\infty}(v), \\ \forall 1 < p < \infty,\ v \in \hat{A}_p, \end{array}\right.$$

and, for some $p_0 \geqslant 1$ and $0 < \alpha < 1$,

$$\left\{\begin{array}{l} \|T\chi_E\|_{L^{p_0,\infty}(u^\alpha)} \lesssim (u^\alpha)(E)^{\frac{1}{p_0}}, \\ \forall E \subseteq \mathbb{R}^n,\ u \in A_1, \end{array}\right. \quad \Longleftrightarrow \quad \left\{\begin{array}{l} T : L^{p,1}(v) \to L^{p,\infty}(v), \\ \forall p_0 < p < \frac{p_0}{1-\alpha},\ v \in \hat{A}_{p;(\alpha(p),\beta(p))}, \end{array}\right.$$

respectively, and both equivalences for a suitable control of the constants. This will yield interesting weighted estimates that have been unknown up to know for many operators such as averaging operators, Fourier multipliers, integral operators and the Bochner-Riesz operator (see Section 3.5).

The results of this chapter are included in [17].

3.1 Average operators and Fourier multipliers

Let $\{T_\theta\}_\theta$ be a family of operators indexed in a probability measure space such that

$$T_\theta : L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n) \tag{3.1.1}$$

with norm less than or equal to a uniform constant C. What can we say about the boundedness of the average operator

$$T_A f(x) = \int T_\theta f(x) dP(\theta), \qquad x \in \mathbb{R}^n,$$

whenever is well defined? The following trivial example shows that, at first sight, nothing of interest can be concluded: for $0 < \theta < 1$, set

$$T_\theta f(x) = \frac{\int_0^1 f(y)dy}{|x-\theta|}, \qquad x \in (0,1),$$

so clearly T_θ satisfies (3.1.1) with $C = 2$, but

$$T_A f(x) = \int_0^1 T_\theta f(x) d\theta \equiv \infty, \qquad \forall x \in (0,1).$$

However, things change completely, and this is one of the main goals of this chapter, if we assume that

$$T_\theta : L^1(u) \to L^{1,\infty}(u), \qquad \forall u \in A_1.$$

Let us start with a very simple and motivating example. To do so, we need to introduce some classical definitions (see [88, Chapter 3] and [164, Chapter 8]).

Let

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi}\, dx, \qquad \xi \in \mathbb{R},$$

be the *Fourier transform* of a function $f \in L^2(\mathbb{R})$. Given a function $m \in L^\infty(\mathbb{R})$, we define a bounded operator T_m on $L^2(\mathbb{R})$ (called *Fourier multiplier*) by

$$\widehat{T_m f}(\xi) = m(\xi)\hat{f}(\xi), \qquad \xi \in \mathbb{R},\ f \in L^2(\mathbb{R}),$$

and m is called a *multiplier*. Examples of multipliers are the bounded variation functions.

Definition 3.1.1. Given a function $m : \mathbb{R} \to \mathbb{R}$, we say that m is of *bounded variation* if

$$V(m) = \sup \sum_{i=1}^N |m(x_i) - m(x_{i-1})| < \infty,$$

where the supremum is taken over all N and over all possible choices of $x_0, \dots, x_N$ such that $-\infty < x_0 < x_1 < \cdots < x_N < \infty$, and where $V(m)$ is called the *total variation* of m.

Now, let m be a bounded variation function on $\mathbb{R}$ that is right-continuous at every point $x \in \mathbb{R}$ and $\lim_{x\to-\infty} m(x) = 0$. If we denote by dm the Lebesgue-Stieltjes measure associated with m, we can write (see, for instance, [88, Corollary 3.8])

$$m(\xi) = \int_{-\infty}^{\xi} dm(t) = \int_{\mathbb{R}} \chi_{(-\infty,\xi)}(t)\, dm(t) = \int_{\mathbb{R}} \chi_{(t,\infty)}(\xi)\, dm(t), \qquad \forall \xi \in \mathbb{R},$$

and dm is a finite measure since

$$|dm| = \int_{\mathbb{R}} |dm(t)| = V(m) < \infty.$$

Hence, if we consider the Fourier multiplier operator

$$T_m f(x) = \int_{\mathbb{R}} m(\xi)\hat{f}(\xi)e^{2\pi i x\xi} d\xi, \qquad x \in \mathbb{R},$$

for every Schwartz function f (i.e., $f \in \mathcal{S}(\mathbb{R})$), a formal computation shows that

$$T_m f(x) = \int_{\mathbb{R}} H_t f(x) dm(t), \qquad \forall x \in \mathbb{R},$$

where

$$H_t f(x) := T_{\chi_{(t,\infty)}} f(x) = \int_t^{\infty} \hat{f}(\xi)e^{2\pi i x\xi} d\xi, \qquad x \in \mathbb{R}.$$

Now, H_t is essentially a Hilbert transform operator (since $Hf = T_m f$ with $m(\xi) = -i \operatorname{sgn} \xi$, $\xi \in \mathbb{R}$) because

$$\chi_{(t,\infty)}(\xi) = \frac{\operatorname{sgn}(\xi - t) + 1}{2}, \qquad \forall \xi \in \mathbb{R}.$$

Thus, since for every $p > 1$,

$$H_t : L^p(\mathbb{R}) \to L^p(\mathbb{R}),$$

we have, using the Minkowski's integral inequality and the density of the Schwartz functions on $L^p(\mathbb{R})$, that every right-continuous bounded variation function such that $\lim_{x\to-\infty} m(x) = 0$ is a Fourier multiplier on $L^p(\mathbb{R})$ for every $p > 1$. However, even though we also have

$$H_t : L^1(\mathbb{R}) \to L^{1,\infty}(\mathbb{R}),$$

we cannot deduce (at least not immediately) that the same boundedness holds for T_m due to the lack of the Minkowski's integral inequality for the space $L^{1,\infty}(\mathbb{R})$.

Indeed, the main theorem of this chapter (see Theorem 3.3.1) will show that since

$$H_t : L^1(u) \to L^{1,\infty}(u), \qquad \forall u \in A_1,$$

with constant less than or equal to $C \|u\|_{A_1} (1 + \log \|u\|_{A_1})$ (see [136]) with C independent of $t \in \mathbb{R}$, then

$$T_m : L^1_{\mathcal{R}}(u) \to L^{1,\infty}(u), \qquad \forall u \in A_1, \tag{3.1.2}$$

with constant less than or equal to $\tilde{C}|dm| \|u\|_{A_1} (1 + \log \|u\|_{A_1})^2$ (see (2.3.8) for the notation used in (3.1.2)). Certainly, (3.1.2) will be consequence of the fact that the converse of Theorem 2.3.6 is also true, that is

$$T : L^1_{\mathcal{R}}(u) \to L^{1,\infty}(u), \quad \forall u \in A_1 \qquad \Longleftrightarrow \qquad T : L^{p_0,1}(v) \to L^{p_0,\infty}(v), \quad \forall v \in \widehat{A}_{p_0}.$$

As a consequence, we will obtain the following results.

Corollary 3.1.2. *Let m be a bounded variation function on $\mathbb{R}$ that is right-continuous at every point $x \in \mathbb{R}$ and $\lim_{x\to-\infty} m(x) = 0$. Then,*

$$T_m : L^1_{\mathcal{R}}(u) \to L^{1,\infty}(u), \qquad \forall u \in A_1,$$

with constant less than or equal to $\tilde{C}|dm| \, \|u\|_{A_1} (1 + \log \|u\|_{A_1})^2$.

Corollary 3.1.3. *Let $c = (c_j)_j \in \ell^1(\mathbb{R})$ (that is, $\|c_j\|_{\ell^1(\mathbb{R})} := \sum_j |c_j| < \infty$) and let $\{T_j\}_j$ be such that*

$$T_j : L^1(u) \to L^{1,\infty}(u), \qquad \forall u \in A_1,$$

with constant less than or equal to $\varphi(\|u\|_{A_1})$, where φ is a positive nondecreasing function on $[1,\infty)$. Then,

$$\sum_j c_j T_j : L^1_{\mathcal{R}}(u) \to L^{1,\infty}(u), \qquad \forall u \in A_1,$$

with constant less than or equal to $C_1 \|c\|_{\ell^1(\mathbb{R})} \varphi(C_2 \|u\|_{A_1})(1 + \log \|u\|_{A_1})$.

3.2 A Sawyer-type inequality

"Sawyer-type inequalities" is a terminology coined in the paper [81], where the authors prove that

$$\left\| \frac{T(fv)}{v} \right\|_{L^{1,\infty}(uv)} \lesssim \|f\|_{L^1(uv)}, \qquad \forall u \in A_1 \text{ and } v \in A_1 \text{ or } uv \in A_\infty,$$

where T is either the Hardy-Littlewood maximal operator or a linear Calderón-Zygmund operator. This result extends some questions previously considered by B. Muckenhoupt and R. Wheeden in [152], and gives an affirmative answer for a conjecture formulated by E. Sawyer in [165], concerning the Hilbert transform. These kind of problems were advertised by B. Muckenhoupt in [151] and have been widely studied since then (see [56, 139, 141, 155, 156, 159, 160]).

In this section, we will study one of such estimates for weights belonging in the restricted class of weights $\hat{A}_p$. First, to do so, we need the following result.

Lemma 3.2.1. *Let $1 < p < \infty$ and $v \in \widehat{A}_p$. Take $\frac{1}{p'} < \theta \leqslant 1$ and set $u_0 = (Mh)^{\frac{(p-1)(1-\theta)}{\theta}}$. Then,*

$$M_{u_0} : L^{\frac{\theta p'}{\theta p'-1},1}(v) \to L^{\frac{\theta p'}{\theta p'-1},\infty}(v), \tag{3.2.1}$$

with constant less than or equal to

$$\frac{\theta^2 p' C_{n,p}}{1 - p(1-\theta)} \|v\|_{\hat{A}_p}^{\frac{2(\theta p'-1)}{\theta(p'-1)}},$$

and where

$$M_{u_0} f(x) = \sup_{Q \ni x} \frac{1}{u_0(Q)} \int_Q |f(y)| u_0(y)\, dy, \qquad x \in \mathbb{R}^n.$$

Proof. Observe that since $v \in \hat{A}_p$, then v is a doubling weight with constant $\Delta_v \leqslant C_1 \|v\|_{\hat{A}_p}^p$. Therefore, according to [56, Lemma 2.2 (i)], (3.2.1) is bounded with constant less than or equal to

$$C_1 \|v\|_{\hat{A}_p}^{\frac{\theta p'-1}{\theta(p'-1)}} \theta p' \left[\sup_{E \subseteq Q} \frac{u_0(E)}{u_0(Q)} \left(\frac{v(Q)}{v(E)} \right)^{\frac{\theta p'-1}{\theta p'}} \right],$$

where the supremum is taken over all cubes Q and all measurable sets $E \subseteq Q$. Now, given a cube Q and a measurable set $E \subseteq Q$,

$$\left(\frac{v(Q)}{v(E)} \right)^{\frac{\theta p'-1}{\theta p'}} = \left(\frac{|Q|}{|E|} \right)^{\frac{\theta p'-1}{\theta(p'-1)}} \left[\left(\frac{|E|}{|Q|} \right)^p \frac{v(Q)}{v(E)} \right]^{\frac{\theta p'-1}{\theta p'}} \leqslant C_2 \|v\|_{\hat{A}_p}^{\frac{\theta p'-1}{\theta(p'-1)}} \left(\frac{|Q|}{|E|} \right)^{\frac{\theta p'-1}{\theta(p'-1)}}$$

and, as well, due to [56, Lemma 2.5],

$$\sup_{E \subseteq Q} \frac{u_0(E)}{u_0(Q)} \left(\frac{|Q|}{|E|} \right)^{\frac{\theta p'-1}{\theta(p'-1)}} \leqslant \frac{\theta C_3}{1 - p(1-\theta)},$$

which yields the desired result. □

Now, we proceed to state which will be the cornerstone of the proof of our main results in this chapter. It was proved in [56, Lemma 2.6] for the case $\delta = 1$ and the extension to other δ's has been fundamental.

Lemma 3.2.2. *Let $1 < p < \infty$ and let $v = (Mh)^{1-p} u \in \widehat{A}_p$. Take θ and δ so that $\frac{1}{p'} < \theta < \delta \leqslant 1$ and set $v_\theta = (Mh)^{1-p} u^\theta$. Then, for every measurable set $E \subseteq \mathbb{R}^n$ we have*

$$\left\| \frac{M_\delta(\chi_E v_\theta)}{v_\theta} \right\|_{L^{p',\infty}(v)} \lesssim C_{n,p,\theta,\delta}(u) v(E)^{\frac{1}{p'}},$$

where

$$C_{n,p,\theta,\delta}(u) = \left(\frac{p^2}{(p-1)^2 (\delta - \theta)(\theta - \frac{1}{p'})^2} \right)^\theta \|u\|_{A_1}^{2\theta - \frac{2}{p'}}. \tag{3.2.2}$$

Proof. Observe that by virtue of the Kolmogorov's inequality [85] with $1 < q' = \frac{1}{\theta} < p'$, it is enough to prove that

$$\sup_{F \subseteq \mathbb{R}^n} \frac{1}{v(F)^{\frac{1}{q'} - \frac{1}{p'}}} \left(\int_F (Mh(x))^{(p-1)(q'-1)} \big(M_\delta(\chi_E (Mh)^{1-p} u^\theta)(x) \big)^{q'} dx \right)^{\frac{1}{q'}} \lesssim C_{n,p,\theta,\delta}(u) v(E)^{\frac{1}{p'}}.$$

Then, using the Fefferman-Stein's inequality [94], since $\delta q' > 1$, we obtain that

$$\begin{aligned} &\int_F (Mh(x))^{(p-1)(q'-1)} \big(M_\delta(\chi_E (Mh)^{1-p} u^\theta)(x) \big)^{q'} dx \\ \lesssim &\frac{\delta q'}{\delta q' - 1} \int_E (Mh(x))^{(1-p)q'} M(\chi_F (Mh)^{(p-1)(q'-1)})(x) u(x) dx. \end{aligned}$$

Now, since $u_0=(Mh)^{(p-1)(q'-1)}\in A_1$, by means of (2.2.3) we have that, for every $x\in E$ and every cube $Q\ni x$ in $\mathbb{R}^n$,

$$\begin{aligned}\frac{1}{|Q|}\int_Q \chi_F u_0(y)\,dy &\leqslant \frac{u_0(Q)}{|Q|}M_{u_0}(\chi_F)(x)\leqslant \|u_0\|_{A_1}u_0(x)M_{u_0}(\chi_F)(x)\\ &\lesssim \frac{1}{1-(p-1)(q'-1)}u_0(x)M_{u_0}(\chi_F)(x).\end{aligned}\tag{3.2.3}$$

Hence, taking the supremum over all cubes $Q\in\mathbb{R}^n$ such that $Q\ni x$ in (3.2.3), with $x\in E$, we deduce that

$$\begin{aligned}&\int_E (Mh(x))^{(1-p)q'}M(\chi_F(Mh)^{(p-1)(q'-1)})(x)u(x)dx\\ &\lesssim \frac{1}{1-(p-1)(q'-1)}\int_E M_{u_0}(\chi_F)(x)v(x)\,dx.\end{aligned}$$

Therefore, since $q'=\frac{1}{\theta}$, the inequality we want to prove will hold if we see that

$$\sup_{E\subseteq\mathbb{R}^n}\frac{1}{v(E)^{\frac{1}{p'}}}\left(\int_E M_{u_0}(\chi_F)(x)v(x)\,dx\right)^{\theta}\lesssim\left(\frac{(\delta-\theta)(1-p(1-\theta))}{\delta\theta}\right)^{\theta}C_{n,p,\theta,\delta}(u)v(F)^{\theta-\frac{1}{p'}}$$

or equivalently,

$$\sup_{E\subseteq\mathbb{R}^n}\frac{1}{v(E)^{1-\left(1-\frac{1}{\theta p'}\right)}}\int_E M_{u_0}(\chi_F)(x)v(x)\,dx\lesssim\left(\frac{(\delta-\theta)(1-p(1-\theta))}{\delta\theta}\right)C_{n,p,\theta,\delta}(u)^{\frac{1}{\theta}}v(F)^{1-\frac{1}{\theta p'}}.\tag{3.2.4}$$

Finally, using again the Kolmogorov's inequality in (3.2.4), it is enough to prove that

$$M_{u_0}:L^{\frac{\theta p'}{\theta p'-1},1}(v)\to L^{\frac{\theta p'}{\theta p'-1},\infty}(v)$$

with constant less than or equal to

$$\frac{c_{n,p}}{\theta p'}\left(\frac{(\delta-\theta)(1-p(1-\theta))}{\delta\theta}\right)C_{n,p,\theta,\delta}(u)^{\frac{1}{\theta}}.$$

According to Lemma 3.2.1, this will happen if

$$C_{n,p,\theta,\delta}(u)\gtrsim\left(\frac{p^2}{(p-1)^2(\delta-\theta)(1-p(1-\theta))^2}\right)^{\theta}\|u\|_{A_1}^{\frac{2(\theta p'-1)}{p'}},$$

from which the desired result follows by taking $C_{n,p,\theta,\delta}(u)$ as in (3.2.2).

□

3.3 Weighted restricted weak-type $(1,1)$

We are now ready to state and prove our first main result in this chapter.

Theorem 3.3.1. *Assume that for some pair of nonnegative functions (f,g),*

$$\|g\|_{L^{1,\infty}(u)} \leqslant \varphi(\|u\|_{A_1}) \|f\|_{L^1(u)}, \qquad \forall u \in A_1, \tag{3.3.1}$$

with φ being a nondecreasing function on $[1,\infty)$. Then, for every $1<p<\infty$,

$$\|g\|_{L^{p,\infty}(v)} \leqslant \Phi(\|v\|_{\widehat{A}_p}) \|f\|_{L^{p,1}(v)}, \qquad \forall v \in \widehat{A}_p,$$

where

$$\Phi(r) = C_1\varphi(C_2 r^p) r^{p-1} (1+\log r)^{\frac{2}{p'}}, \qquad r \geqslant 1, \tag{3.3.2}$$

with C_1 and C_2 being two positive constants independent of all parameters involved.

Proof. Let $h \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $u \in A_1$ so that $v = (Mh)^{1-p}u \in \widehat{A}_p$. Further, let us take

$$\frac{1}{p'} < \theta < 1, \qquad \delta = 1 - \frac{1-\theta}{t} \qquad \text{and} \qquad v_\theta = (Mh)^{1-p}u^\theta,$$

where $t = 1 + \frac{1}{2^{n+1}\|u\|_{A_1}}$ satisfies $u^t \in A_1$ with $\|u^t\|_{A_1} \lesssim \|u\|_{A_1}$ (see (2.2.4)). Then, $\theta < \delta < 1$ and, by (2.2.3), for every measurable set $F \subseteq \mathbb{R}^n$,

$$u_0 = M_\delta(\chi_F v_\theta)u^{1-\theta} = M(\chi_F v_\theta^{1/\delta})^\delta (u^t)^{1-\delta} \in A_1, \qquad \|u_0\|_{A_1} \leqslant \frac{C\|u\|_{A_1}}{1-\delta}.$$

Hence, taking $y>0$ and $F = \{x : g(x) > y\}$ so that $v(F) = \lambda_g^v(y)$, by hypothesis we obtain that

$$\begin{aligned} y\lambda_g^v(y) &= y\int_{\{x\,:\,g(x)>y\}} v(x)\,dx \leqslant y\int_F M_\delta(\chi_F v_\theta)(x)u(x)^{1-\theta}dx \\ &\leqslant \varphi\left(\frac{C\|u\|_{A_1}}{1-\delta}\right)\int_{\mathbb{R}^n} f(x)M_\delta(\chi_F v_\theta)(x)u(x)^{1-\theta}dx \\ &= \varphi\left(\frac{Ct\|u\|_{A_1}}{1-\theta}\right)\int_{\mathbb{R}^n} f(x)\frac{M_\delta(\chi_F v_\theta)(x)}{v_\theta(x)}v(x)\,dx \\ &\leqslant \varphi\left(\frac{Ct\|u\|_{A_1}}{1-\theta}\right)\left\|\frac{M_\delta(\chi_F v_\theta)}{v_\theta}\right\|_{L^{p',\infty}(v)} \|f\|_{L^{p,1}(v)}, \end{aligned} \tag{3.3.3}$$

where in the last estimate we have used the Hölder's inequality for Lorentz spaces with respect to the measure $v(x)\,dx$.

Now, by virtue of Lemma 3.2.2,

$$\left\|\frac{M_\delta(\chi_F v_\theta)}{v_\theta}\right\|_{L^{p',\infty}(v)} \lesssim C_{n,p,\theta,\delta}(u)v(F)^{\frac{1}{p'}} = C_{n,p,\theta,\delta}(u)\lambda_g^v(y)^{\frac{1}{p'}}.$$

Therefore, observe that if $\lambda_g^v(y) < \infty$ for every $y > 0$, then we can divide by $\lambda_g^v(y)^{\frac{1}{p'}}$ the left-hand side of (3.3.3), so taking the supremum over all $y > 0$, in particular, we obtain that

$$\|g\|_{L^{p,\infty}(v)} \lesssim C_{n,p,\theta,\delta}(u)\varphi\left(\frac{Ct\,\|u\|_{A_1}}{1-\theta}\right)\|f\|_{L^{p,1}(v)}\,.$$

Otherwise, for each $N \in \mathbb{N}$, let $g_N = g\chi_{B(0,N)}$. Then,

$$\lambda_{g_N}^v(y) \leqslant v(B(0,N)) < \infty, \qquad \forall y > 0,$$

and the pair of functions g_N and f satisfies also (3.3.1), so that arguing as above but now with g_N instead of g we obtain that, for every $N \in \mathbb{N}$,

$$\|g_N\|_{L^{p,\infty}(v)} \lesssim C_{n,p,\theta,\delta}(u)\varphi\left(\frac{Ct\,\|u\|_{A_1}}{1-\theta}\right)\|f\|_{L^{p,1}(v)}\,,$$

and so the same result hold for g by taking the supremum over all $N \in \mathbb{N}$.

Finally, concerning about the constant $C_{p,\theta,\delta}(u)$, we observe that

$$\begin{aligned} C_{n,p,\theta,\delta}(u) &= \left(\frac{p^2}{(p-1)^2(\delta-\theta)(\theta-\frac{1}{p'})^2}\right)^{\theta}\|u\|_{A_1}^{2\theta-\frac{2}{p'}} \\ &\approx \left(\frac{p^2}{(p-1)^2(1-\theta)(\theta-\frac{1}{p'})^2}\right)^{\theta}\|u\|_{A_1}^{3\theta-\frac{2}{p'}}\,. \end{aligned}$$

Therefore, letting

$$\theta = \frac{1}{p'}\left(1+\frac{1}{(p+1)R}\right), \qquad 1 \leqslant R < \infty,$$

then

$$C_{n,p,\theta,\delta}(u) \lesssim \left(\frac{p^5(p+1)^3R^2}{(p-1)^4}\right)^{\frac{1}{p'}\left(1+\frac{1}{(p+1)R}\right)}\|u\|_{A_1}^{\frac{1}{p'}}\|u\|_{A_1}^{\frac{3}{Rp'(p+1)}} \lesssim R^{\frac{2}{p'}}\|u\|_{A_1}^{\frac{1}{p'}}\|u\|_{A_1}^{\frac{3}{R}}\,.$$

Furthermore, with the same choice of θ,

$$\varphi\left(\frac{Ct\,\|u\|_{A_1}}{1-\theta}\right) \leqslant \varphi\left(\tilde{C}\,\|u\|_{A_1}\right).$$

Thus, the result follows by setting $R = 1 + \log\|u\|_{A_1}$ and then taking the infimum on $\|u\|_{A_1}$ over all the possible representations of $v \in \widehat{A}_p$. □

Therefore, by virtue of Theorems 2.3.6 and 3.3.1, the next result follows directly:

Corollary 3.3.2. *Given an operator T. If*

$$T : L^1_{\mathcal{R}}(u) \to L^{1,\infty}(u), \qquad \forall u \in A_1, \tag{3.3.4}$$

with constant less than or equal to $\varphi(\|u\|_{A_1})$, where φ is a positive nondecreasing function on $[1,\infty)$, then, for every $1 < p < \infty$ and every measurable set $E \subseteq \mathbb{R}^n$,

$$\|T\chi_E\|_{L^{p,\infty}(v)} \leqslant \Phi(\|v\|_{\hat{A}_p})v(E)^{\frac{1}{p}}, \qquad \forall v \in \widehat{A}_p, \tag{3.3.5}$$

where Φ is as in (3.3.2). Further, if (3.3.5) holds for every $1 < p < \infty$, then we have (3.3.4) but now with the norm constant less than or equal to $C_1\varphi(C_2\,\|u\|_{A_1})$.

Further, if T is a sublinear (ε, δ)-atomic approximable operator (see Definition 2.3.9) then, by means of Theorem 2.3.12, Corollary 3.3.2 can be improved in this wise:

Corollary 3.3.3. *Let T be a sublinear (ε, δ)-atomic approximable operator. If*

$$T : L^1(u) \to L^{1,\infty}(u), \qquad \forall u \in A_1, \tag{3.3.6}$$

with constant less than or equal to $\varphi(\|u\|_{A_1})$, where φ is a positive nondecreasing function on $[1, \infty)$, then, for every $1 < p < \infty$,

$$T : L^{p,1}(v) \to L^{p,\infty}(v), \qquad \forall v \in \hat{A}_p, \tag{3.3.7}$$

with constant less than or equal to $\Phi(\|v\|_{\hat{A}_p})$ and where Φ is as in (3.3.2). Further, if (3.3.7) holds for every $1 < p < \infty$, then we have (3.3.6) with constant less than or equal to $C_1 \|u\|_{A_1} \varphi(C_2\|u\|_{A_1})$.

3.4 Weighted restricted weak-type (p_-, p_-)

Now, we continue by stating and proving our second main result in this chapter.

Theorem 3.4.1. *Assume that for some pair of nonnegative functions (f, g) and for some $1 \leqslant p_0 < \infty$ and $0 < \alpha \leqslant 1$,*

$$\|g\|_{L^{p_0,\infty}(u^\alpha)} \leqslant \varphi\left(\|u\|_{A_1}^\alpha\right) \|f\|_{L^{p_0,1}(u^\alpha)}, \qquad \forall u \in A_1,$$

with φ being a nondecreasing function on $[1, \infty)$. Then, for any $p_0 \leqslant p < \frac{p_0}{1-\alpha}$,

$$\|g\|_{L^{p,\infty}(v)} \leqslant \Psi\left(\|v\|_{\hat{A}_{p;(\alpha(p),\beta(p))}}\right) \|f\|_{L^{p,1}(v)}, \qquad \forall v \in \hat{A}_{p;(\alpha(p),\beta(p))},$$

where $\alpha(p) = 1 - \frac{p(1-\alpha)}{p_0}$, $\beta(p) = \frac{p-p_0}{p_0(p-1)}$ and

$$\Psi(r) = C_1 \left(\frac{1}{p_0 - p(1-\alpha)}\right)^{\frac{p-p_0}{p}} \varphi\left(C_2 r^{\frac{\alpha p}{p_0 - p(1-\alpha)}}\right) r^{\frac{\alpha(p-p_0)}{p_0 - p(1-\alpha)}} (1 + \log r)^{\frac{2(p-p_0)}{p}}, \qquad r \geqslant 1,$$

with C_1 and C_2 being two positive constants independent of all parameters involved.

Proof. Let $h \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $u \in A_1$ so that $v = (Mh)^{\beta(p)(1-p)} u^{\alpha(p)} \in \hat{A}_{p;(\alpha(p),\beta(p))}$. Further, take $t = 1 + \frac{1}{2^{n+1}\|u\|_{A_1}}$ so that $u^t \in A_1$ with $\|u^t\|_{A_1} \lesssim \|u\|_{A_1}$ (see (2.2.4)) and, since $t > 1$,

$$\frac{p - p_0}{p} = \frac{\alpha - \alpha(p)}{1 - \alpha(p)} < \frac{t\alpha - \alpha(p)}{t - \alpha(p)} = \alpha - \frac{\alpha(p)(1 - \alpha)}{t - \alpha(p)} < \alpha.$$

Hence, we can take

$$v_\theta = (Mh)^{\beta(1-p)} u^{\alpha(p)\theta} \qquad \text{with} \qquad \frac{p - p_0}{p} < \theta < \frac{t\alpha - \alpha(p)}{t - \alpha(p)}.$$

Besides, since $\alpha(p) \leqslant \alpha$, letting

$$\delta = 1 - \frac{\alpha(p)(1-\theta)}{\alpha t} \in (0,1),$$

then $\theta < \alpha\delta < 1$ and, by (2.2.3), for every measurable set $F \subseteq \mathbb{R}^n$,

$$u_0 = \left(M_{\alpha\delta}(\chi_F v_\theta) u^{\alpha(p)(1-\theta)}\right)^{\frac{1}{\alpha}} = M\left(\chi_F v_\theta^{\frac{1}{\alpha\delta}}\right)^{\delta} (u^t)^{1-\delta} \in A_1, \quad \|u_0\|_{A_1} \leqslant \frac{C\,\|u\|_{A_1}}{1-\delta}.$$

Hence, taking $y > 0$ and $F = \{x : g(x) > y\}$ so that $v(F) = \lambda_g^v(y)$, by hypothesis we obtain that

$$\begin{aligned}
y^{p_0}\lambda_g^v(y) &= y^{p_0}\int_{\{x:g(x)>y\}} v(x)\,dx \leqslant y^{p_0}\int_F M_{\alpha\delta}(\chi_F v_\theta)(x)u(x)^{\alpha(p)(1-\theta)}dx = y^{p_0}\lambda_g^{u_0^\alpha}(y) \\
&\leqslant \varphi\left(\|u_0\|_{A_1}^\alpha\right)^{p_0}\left[p_0\int_0^\infty\left(\int_{\{f(x)>z\}} M_{\alpha\delta}(\chi_F v_\theta)(x)u(x)^{\alpha(p)(1-\theta)}\,dx\right)^{\frac{1}{p_0}} dz\right]^{p_0} \\
&\leqslant \varphi\left(\left[\frac{C\,\|u\|_{A_1}}{1-\delta}\right]^\alpha\right)^{p_0}\left\|\frac{M_{\alpha\delta}(\chi_F v_\theta)}{v_\theta}\right\|_{L^{(\frac{p}{p_0})',\infty}(v)}\left[p_0\int_0^\infty \left\|\chi_{\{f(x)>z\}}\right\|_{L^{\frac{p}{p_0},1}(v)}^{\frac{1}{p_0}} dz\right]^{p_0} \\
&\approx \varphi\left(\left[\frac{C\alpha t\,\|u\|_{A_1}}{\alpha(p)(1-\theta)}\right]^\alpha\right)^{p_0}\left\|\frac{M_{\alpha\delta}(\chi_F v_\theta)}{v_\theta}\right\|_{L^{(\frac{p}{p_0})',\infty}(v)}\|f\|_{L^{p,1}(v)}^{p_0},
\end{aligned}$$

where in the penultimate estimate we have used the Hölder's inequality for Lorentz spaces with respect to the measure $v(x)\,dx$.

Now, since $\beta(p)(1-p) = 1 - \frac{p}{p_0}$, then $v \in \hat{A}_{\frac{p}{p_0}}$ and, by virtue of Lemma 3.2.2,

$$\left\|\frac{M_{\alpha\delta}(\chi_F v_\theta)}{v_\theta}\right\|_{L^{(\frac{p}{p_0})',\infty}(v)} \lesssim C_{n,\frac{p}{p_0},\theta,\alpha\delta}(u)v(F)^{\frac{p-p_0}{p}} = C_{n,\frac{p}{p_0},\theta,\alpha\delta}(u)\lambda_g^v(y)^{\frac{p-p_0}{p}},$$

so arguing (if necessary) with $g_N = g\chi_{B(0,N)}$ as we did in the proof of Theorem 3.3.1 and taking the supremum over all $y > 0$, in particular, we obtain that

$$\|g\|_{L^{p,\infty}(v)} \lesssim C_{n,\frac{p}{p_0},\theta,\alpha\delta}(u)^{\frac{1}{p_0}}\varphi\left(\tilde{C}\,\|u\|_{A_1}^\alpha\right)\|f\|_{L^{p,1}(v)}.$$

Finally, concerning about the constant $C_{\frac{p}{p_0},\theta,\alpha\delta}(u)$, we observe that

$$\begin{aligned}
C_{n,\frac{p}{p_0},\theta,\alpha\delta}(u) &= \left(\frac{p^2}{(p-p_0)^2(\alpha\delta-\theta)(\theta-\frac{p-p_0}{p})^2}\right)^\theta \|u\|_{A_1}^{2\theta-\frac{2(p-p_0)}{p_0}} \\
&\lesssim \left(\frac{p^2}{(p-p_0)^2(\alpha-\theta)(\theta-\frac{p-p_0}{p})^2}\right)^\theta \|u\|_{A_1}^{3\theta-\frac{2(p-p_0)}{p_0}},
\end{aligned}$$

so the behaviour of the constant $C_{n,\frac{p}{p_0},\theta,\alpha\delta}(u)$ follows similar as in the proof of Theorem 3.3.1.

□

Therefore, by virtue of Theorems 2.3.14 and 3.4.1, the next result follows directly.

Corollary 3.4.2. *Let $1 \leqslant p_0 < \infty$, $0 \leqslant \alpha, \beta \leqslant 1$ (not both identically zero) and let T be a sublinear operator satisfying*

$$T : L^{p_0,1}(v) \to L^{p_0,\infty}(v), \qquad \forall v \in \hat{A}_{p_0;(\alpha,\beta)}, \tag{3.4.1}$$

with constant less than or equal to $\varphi(\|v\|_{\hat{A}_{p_0;(\alpha,\beta)}})$, where φ is a positive nondecreasing function on $[1, \infty)$. Then, for every $p_- \leqslant p < p_+$ (with p_- and p_+ as in (2.3.5)) but $p > 1$,

$$T : L^{p,1}(v) \to L^{p,\infty}(v), \qquad \forall v \in \hat{A}_{p;(\alpha(p),\beta(p))},$$

with constant less than or equal to $\frac{1}{p-1}\Phi_p\left(\|v\|_{\hat{A}_{p;(\alpha(p),\beta(p))}}\right)$, where Φ_p is a positive nondecreasing function on $[1, \infty)$. Further, if $p_- = 1$, then for every $0 < q < 1$,

$$T : L^{1,q}(u^{\alpha(1)}) \to L^{1,\infty}(u^{\alpha(1)}), \qquad \forall u \in A_1,$$

with constant less than or equal to $\frac{q}{1-q}\tilde{\Phi}_q\left(\|u\|_{A_1}^{\alpha(1)}\right)$, where $\tilde{\Phi}_q$ is a positive nondecreasing function on $[1, \infty)$.

Even though we have shown that, in particular, is possible to extrapolate from any p_0 till all $p \in [p_-, p_+)$, is important to point out that we have not been able to reach the exponent p_+ (at least when this should be possible, that is, $p_+ < \infty$). However, we also observe that p_0 could be set to p_+ (this is the case when $\alpha = 0$) and in that case (3.4.1) reduces to

$$T : L^{p_0,1}((Mh)^{\beta(1-p_0)}) \to L^{p_0,\infty}((Mh)^{\beta(1-p_0)}), \qquad \forall h \in L^1_{\mathrm{loc}}(\mathbb{R}^n), \tag{3.4.2}$$

where we should note that for these kind of weights, $\left\|(Mh)^{\beta(1-p_0)}\right\|_{\hat{A}_{p_0;(0,\beta)}} = 1$. Hence, motivated by (3.4.2) we have the following.

Theorem 3.4.3. *Assume that for some pair of nonnegative functions (f, g) and for some $1 \leqslant p_0 < \infty$ and $0 < \alpha < 1$,*

$$\|g\|_{L^{p_0,\infty}((Mh)^\alpha)} \leqslant C_{n,p_0,\alpha} \|f\|_{L^{p_0,1}((Mh)^\alpha)}, \qquad \forall h \in L^1_{loc}(\mathbb{R}^n).$$

Then, for $p = \frac{p_0}{1-\alpha}$,

$$\|g\|_{L^{p,\infty}\left(u^{1-\frac{p}{p_0}}\right)} \leqslant (1-\alpha)^{\frac{p_0-1}{p_0}} C_{n,p_0,\alpha} \Phi\left(\|u\|_{A_1}^{\frac{p}{p_0}-1}\right)^{\frac{1}{p_0}-\frac{1}{p}} \|f\|_{L^{p,1}\left(u^{1-\frac{p}{p_0}}\right)}, \qquad \forall u \in A_1,$$

for some nondecreasing function Φ on $[1, \infty)$.

Proof. Take $v = u^{1-\frac{p}{p_0}}$ for $u \in A_1$. Let $y > 0$ and define $F = \{x \in \mathbb{R}^n : g(x) > y\}$ so that $v(F) = \lambda_g^v(y)$. Then, by hypothesis,

$$\lambda_g^v(y) = \int_F v(x)\,dx \leqslant \int_F M_\alpha(\chi_F v)(x)\,dx \leqslant \frac{C^{p_0}_{n,p_0,\alpha}}{y^{p_0}} \|f\|^{p_0}_{L^{p_0,1}(M_\alpha(\chi_F v))},$$

where $M_\alpha(\chi_F v) = M\left(\chi_F u^{-\frac{p}{p_0}}\right)^\alpha$.

Now, due to the Hölder's inequality for Lorentz spaces with respect to the measure $v(x)\,dx$,

$$\|f\|_{L^{p_0,1}(M_\alpha(\chi_F v))} \leqslant \left\|\frac{M_\alpha(\chi_F v)}{v}\right\|^{\frac{1}{p_0}}_{L^{(\frac{p}{p_0})',\infty}(v)} \left[p_0 \int_0^\infty \left\|\chi_{\{f(x)>z\}}\right\|^{\frac{1}{p_0}}_{L^{\frac{p}{p_0},1}(v)} dz\right]$$

$$= (1-\alpha)^{\frac{p_0-1}{p_0}} \left\|\frac{M\left(\chi_F u^{-\frac{p}{p_0}}\right)}{u^{-\frac{p}{p_0}}}\right\|^{\frac{1}{p_0}-\frac{1}{p}}_{L^{1,\infty}(v)} \|f\|_{L^{p,1}(v)},$$

and, since $u^{-\frac{p}{p_0}} = v/u$, by means of [81, Theorem 1.3 and Remark 2.2] (see also [160]) there exists a nondecreasing function Φ on $[1,\infty)$ such that

$$\|f\|_{L^{p_0,1}(M_\alpha(\chi_F v))} \leqslant (1-\alpha)^{\frac{p_0-1}{p_0}} \Phi\left(\|u\|_{A_1}^{\frac{p}{p_0}-1}\right)^{\frac{1}{p_0}-\frac{1}{p}} \lambda_g^v(y)^{\frac{1}{p_0}-\frac{1}{p}} \|f\|_{L^{p,1}(v)}.$$

Therefore, putting all together and arguing (if necessary) with $g_N = g\chi_{B(0,N)}$ as we did in the proof of Theorem 3.3.1, we obtain that

$$y\lambda_g^v(y)^{\frac{1}{p}} \leqslant (1-\alpha)^{\frac{p_0-1}{p_0}} C_{n,p_0,\alpha} \Phi\left(\|u\|_{A_1}^{\frac{p}{p_0}-1}\right)^{\frac{1}{p_0}-\frac{1}{p}} \|f\|_{L^{p,1}(v)},$$

so that the result follows by taking the supremum on $y > 0$.

□

3.5 Applications

In this section, we present some applications for our extrapolation results previously introduced in this chapter. In particular, in Section 3.5.1 we will study the average operators that appeared on Section 3.1. Further, in Section 3.5.2 we will work with Fourier multipliers while in Section 3.5.3 we will focus on integral operators, where we will see that both, in fact, can be handled as particular cases of average operators. Finally, we will deal in Section 3.5.4 with the Bochner-Riesz operator. Indeed, similar estimates could be obtained for a large list of operators such as Fourier multipliers of Hörmander type (see Section 5.2.1), rough singular integrals (see Section 5.3), intrinsic square functions (see Section 5.4), among many others.

3.5.1 Average operators

Corollary 3.5.1. *Assume that $\{T_\theta\}_\theta$ is a family of operators indexed in a probability measure space such that the average operator*

$$T_A f(x) = \int T_\theta f(x) dP(\theta), \qquad x \in \mathbb{R}^n,$$

is well defined and that

$$T_\theta : L^1(u) \to L^{1,\infty}(u), \qquad \forall u \in A_1, \tag{3.5.1}$$

with constant less than or equal to $\varphi(\|u\|_{A_1})$*, where* φ *is a positive nondecreasing function on* $[1,\infty)$*. Then,*

$$T_A : L^1_{\mathcal{R}}(u) \to L^{1,\infty}(u), \qquad \forall u \in A_1, \tag{3.5.2}$$

with constant less than or equal to $C_1\varphi(C_2 \|u\|_{A_1})(1+\log \|u\|_{A_1})$*. Moreover, if* T_A *is a sublinear* (ε,δ)*-atomic approximable operator, then*

$$T_A : L^1(u) \to L^{1,\infty}(u), \qquad \forall u \in A_1, \tag{3.5.3}$$

with constant less than or equal to $\tilde{C}_1\varphi(C_2 \|u\|_{A_1}) \|u\|_{A_1} (1+\log \|u\|_{A_1})$*.*

Proof. Set $1 < p < \infty$. Using Theorem 3.3.1, we have that (3.5.1) implies

$$T_\theta : L^{p,1}(v) \to L^{p,\infty}(v), \qquad \forall v \in \widehat{A}_p,$$

with norm less than or equal to $\Phi(\|v\|_{\widehat{A}_p})$. Now, recall that $L^{p,\infty}(v)$ is a Banach function space since there exists a norm $\|\cdot\|_{(p,\infty,v)}$ so that

$$\|f\|_{L^{p,\infty}(v)} \leqslant \|f\|_{(p,\infty,v)} \leqslant \frac{p}{p-1} \|f\|_{L^{p,\infty}(v)} .$$

Hence, by the Minkowski's integral inequality (see [20, Theorem 4.4] and [167, Proposition 2.1]) T_A satisfies that, for every $1 < p < \infty$,

$$T_A : L^{p,1}(v) \to L^{p,\infty}(v), \qquad \forall v \in \widehat{A}_p,$$

with norm less than or equal to $\frac{p}{p-1}\Phi(\|v\|_{\widehat{A}_p})$. Therefore, using Theorem 2.3.6 the desired result (3.5.2) follows by taking the infimum in $p > 1$. Finally, (3.5.3) is just a consequence of Theorem 2.3.12. □

In particular, the next results stated in the introduction follow:

Proof of Corollary 3.1.2. This result is just a direct consequence of Corollary 3.5.1 since $\left\{\frac{1}{|dm|}H_t\right\}_t$ is a family of operators indexed in a probability measure. □

Proof of Corollary 3.1.3. This result is just a direct consequence of Corollary 3.5.1 since $\left\{\frac{c_j}{\|c\|_{\ell^1(\mathbb{R})}}T_j\right\}_j$ is a family of operators indexed in the counting probability measure. □

3.5.2 Fourier multipliers

Our first application for Fourier multipliers is in the context of *restriction multipliers* from $\mathbb{R}^{n+k}$ to $\mathbb{R}^n$, for some $k \geqslant 1$. First, we say that $m \in L^\infty(\mathbb{R}^n)$ is *normalized* if

$$\lim_j \widehat{\psi_j} * m(x) = m(x), \qquad \forall x \in \mathbb{R}^n,$$

where for each j, $\psi_j(x) = \psi(x/j)$, and $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $\hat{\psi} \geqslant 0$ and $\|\hat{\psi}\|_{L^1(\mathbb{R}^n)} = 1$.

It is easy to see that then for every Lebesgue point x of m, $\lim_j \widehat{\psi_j} * m(x) = m(x)$. In particular, every continuous and bounded function is normalized.

Before going into details, we first need the following density result about weighted Lorentz spaces.

Lemma 3.5.2. *Given $p \geqslant 1$ and a weight v in $\mathbb{R}^n$. Then, $\mathcal{C}_c^\infty(\mathbb{R}^n) \cap L^{p,1}(v)$ is dense in $L^{p,1}(v)$.*

Proof. The proof follows the same lines as [63, Lemma 2.2] with the necessary modifications. First, since finite linear combinations of characteristic functions of measurable sets $E \subseteq \mathbb{R}^n$ with finite measure are dense in $L^{p,1}(v)$ (see [99, Theorem 1.4.13]) it is enough to see that we can approximate χ_E by any function in $C_c^\infty(\mathbb{R}^n) \cap L^{p,1}(v)$ under the norm in $L^{p,1}(v)$.

In turn, we may assume that E is bounded (so that, $v(E) < \infty$) and contained in an open ball B sufficiently large (since $\chi_{E_k} \to \chi_E$ in $L^{p,1}(v)$ with $E_k = E \cap \mathbb{B}(0,k)$, where $\mathbb{B}(0,k)$ denotes the n-th dimensional open ball of center 0 and radius k).

Now, given $\varepsilon > 0$. Since $v\chi_B \in L^1(\mathbb{R}^n)$, there exists $\delta > 0$ such that if $A \subseteq \mathbb{R}^n$ is a measurable set satisfying $|A| < \delta$ then $v(A \cap B) < \varepsilon^p$. So let U be an open set and K a compact set, both in $\mathbb{R}^n$, such that $K \subseteq E \subseteq U \subseteq B$ and $|U \setminus K| < \delta$, and let $h \in C_c^\infty(\mathbb{R}^n)$ be a function with values in $[0,1]$ such that $|\chi_E(x) - h(x)| \leqslant \chi_{U\setminus K}(x)$ (see [76, Theorem 2.6.1]). Then

$$\|\chi_E - h\|_{L^{p,1}(v)} \leqslant \left\|\chi_{U\setminus K}\right\|_{L^{p,1}(v)} = v(U\setminus K)^{\frac{1}{p}} < \varepsilon,$$

as desired.

□

Proposition 3.5.3. *Let $k \geqslant 1$ and assume that a normalized function $m \in L^\infty(\mathbb{R}^{n+k})$ satisfies that*

$$T_m : L^1(u) \to L^{1,\infty}(u), \qquad \forall u \in A_1(\mathbb{R}^{n+k}),$$

with constant less than or equal to $\varphi(\|u\|_{A_1})$, where φ is a positive nondecreasing function on $[1,\infty)$. Let $\phi \in L^1(\mathbb{R}^k)$ and define

$$m_\phi(x) = \int_{\mathbb{R}^k} m(x,y)\phi(y)\, dy, \qquad x \in \mathbb{R}^n.$$

Then,

$$T_{m_\phi} : L^1_{\mathcal{R}}(v) \to L^{1,\infty}(v), \qquad \forall v \in A_1(\mathbb{R}^n),$$

with constant less than or equal to $C_1\varphi(C_2\|v\|_{A_1})\|v\|_{A_1}(1 + \log\|v\|_{A_1})$.

Proof. Take $v \in A_1(\mathbb{R}^n)$ and define $u = v \otimes \chi_{\mathbb{R}^k}$, so that

$$\begin{aligned} u :\ \mathbb{R}^n \times \mathbb{R}^k &\to \mathbb{R}, \\ (x,\, y) &\mapsto u(x,y) = v(x), \end{aligned}$$

satisfies $u \in A_1(\mathbb{R}^{n+k})$ with $\|u\|_{A_1} \leqslant \|v\|_{A_1}$. Then, $T_m : L^1(u) \to L^{1,\infty}(u)$ and, by [52, Theorem 4.4] (where here is used that m is normalized),

$$T_{m(\cdot,y)} : L^1(v) \to L^{1,\infty}(v), \qquad \forall y \in \mathbb{R}^k,$$

with

$$\sup_{y\in\mathbb{R}^k} \left\|T_{m(\cdot,y)}\right\|_{L^1(v)\to L^{1,\infty}(v)} \lesssim \|u\|_{A_1} \|T_m\|_{L^1(u)\to L^{1,\infty}(u)} \leqslant \|v\|_{A_1} \varphi(\|v\|_{A_1}).$$

Now, take $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. Then, for every $y \in \mathbb{R}^k$ we have that $m(\cdot,y)\hat{f} \in L^1(\mathbb{R}^n)$ and, as well, $m_\phi \hat{f} \in L^1(\mathbb{R}^n)$, so that, by the properties of the Fourier transform,

$$T_{m(\cdot,y)}f(x) = \left(m(\cdot,y)\hat{f}\right)^\vee(x) \quad \text{and} \quad T_{m_\phi}f(x) = (m_\phi\hat{f})^\vee(x), \qquad \forall x\in\mathbb{R}^n,$$

where for every $g \in L^1(\mathbb{R}^n)$,

$$\check{g}(x) = \int_{\mathbb{R}^n} g(\xi)e^{2\pi i\xi\cdot x}\,d\xi, \qquad x\in\mathbb{R}^n,$$

is the *inverse Fourier transform* of the function g. Hence, by Fubini's theorem,

$$\begin{aligned} T_{m_\phi}f(x) &= \int_{\mathbb{R}^n} m_\phi(\xi)\hat{f}(\xi)e^{2\pi ix\cdot\xi}\,d\xi = \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^k} m(\xi,y)\phi(y)\,dy\right)\hat{f}(\xi)e^{2\pi ix\cdot\xi}\,d\xi \\ &= \int_{\mathbb{R}^k}\left(\int_{\mathbb{R}^n} m(\xi,y)\hat{f}(\xi)e^{2\pi ix\cdot\xi}\,d\xi\right)\phi(y)\,dy = \int_{\mathbb{R}^k} T_{m(\cdot,y)}f(x)\phi(y)\,dy, \end{aligned}$$

and the result follows as in Corollary 3.5.1 together with the density of $L^{p,1}(v)$ by functions in $\mathcal{C}_c^\infty(\mathbb{R}^n)\cap L^{p,1}(v)$ (see Lemma 3.5.2).

□

For our next application, we observe that if $\|T_m\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \leqslant \|m\|_\infty$, for some $1<p<\infty$, then by iteration

$$\|T_{m^j}\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \leqslant \|m\|_\infty^j, \quad \forall j\in\mathbb{N},$$

where m^j stands for the j-th power of m. However, this may not be the case when dealing with boundedness between the spaces $L^1(\mathbb{R}^n)$ and $L^{1,\infty}(\mathbb{R}^n)$, since, in general, we can not iterate the operator. Our next application gives a necessary condition for this to happen.

Proposition 3.5.4. *Assume that $m\in L^\infty(\mathbb{R}^n)$ satisfies that for every $j\geqslant 1$,*

$$T_{m^j}: L^1(u)\to L^{1,\infty}(u), \qquad \forall u\in A_1,$$

with constant less than or equal to $C\|m\|_\infty^j \varphi(\|u\|_{A_1})$ uniformly in j, where φ is a positive nondecreasing function on $[1,\infty)$. Then if

$$F = \{x\in\mathbb{R}^n : m(x) = \|m\|_\infty\},$$

we have that

$$T_{\chi_F}: L^1_{\mathcal{R}}(u)\to L^{1,\infty}(u), \qquad \forall u\in A_1,$$

with constant less than or equal to $C_1\varphi(C_2\|u\|_{A_1})(1+\log\|u\|_{A_1})$.

Proof. Fix $r \in (0,1)$ and define

$$m_r(x) = (1-r)\sum_{j=0}^{\infty}\left(r\frac{m(x)}{\|m\|_\infty}\right)^j, \qquad x \in \mathbb{R}^n.$$

Hence, observe that since for almost every $x \in \mathbb{R}^n$, $|rm(x)| < \|m\|_\infty$, then

$$m_r(x) = \frac{1-r}{1-r\frac{m(x)}{\|m\|_\infty}} \xrightarrow[r\to 1]{} \chi_F(x), \qquad \text{a.e. } x \in \mathbb{R}^n.$$

Now, take $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. Then, due to the properties of the Fourier transform we have that $m_r\hat{f} \in L^1(\mathbb{R}^n)$. Further, by means of the dominated convergence theorem we get that for every $0 < r < 1$,

$$\begin{aligned} T_{m_r}f(x) = \left(m_r\hat{f}\right)^\vee(x) &= (1-r)\int_{\mathbb{R}^n}\sum_{j=0}^{\infty}\frac{r^j}{\|m\|_\infty^j}m(\xi)^j\hat{f}(\xi)e^{2\pi i x\cdot\xi}\,d\xi \\ &= (1-r)\sum_{j=0}^{\infty}\frac{r^j}{\|m\|_\infty^j}\int_{\mathbb{R}^n}m(\xi)^j\hat{f}(\xi)e^{2\pi i x\cdot\xi}\,d\xi = (1-r)\sum_{j=0}^{\infty}\frac{r^j}{\|m\|_\infty^j}T_{m^j}f(x), \end{aligned}$$

for every $x \in \mathbb{R}^n$, and as well,

$$T_{m_r}f(x) = \int_{\mathbb{R}^n} m_r(\xi)\hat{f}(\xi)e^{-2\pi i x\cdot\xi}\,d\xi \xrightarrow[r\to 1]{} \int_{\mathbb{R}^n}\chi_F(\xi)\hat{f}(\xi)e^{-2\pi i x\cdot\xi}\,d\xi = T_{\chi_F}f(x), \qquad \forall x \in \mathbb{R}^n.$$

Therefore, taking $c_j = r^j$ and $T_j = \frac{1}{\|m\|_\infty^j}T_{m^j}$, the result follows as in Corollary 3.1.3 together with the density of $L^{p,1}(v)$ by functions in $\mathcal{C}_c^\infty(\mathbb{R}^n) \cap L^{p,1}(v)$ (see Lemma 3.5.2) and the Fatou's Lemma.

□

3.5.3 Integral operators

Let us now consider the operator

$$Tf(x) = \int_{\mathbb{R}^m} K(x,y)f(y)dy, \qquad x \in \mathbb{R}^n,$$

where the integral kernel K satisfies some size condition of the form $|K(x,y)| \lesssim |x-y|^{-n}$.

Proposition 3.5.5. *Assume that, for every $s > 0$,*

$$T_s f(x) = \int_{|x-y|\geqslant s} K(x,y)f(y)dy, \qquad x \in \mathbb{R}^n,$$

satisfies that

$$T_s : L^1_{\mathcal{R}}(u) \to L^{1,\infty}(u), \qquad \forall u \in A_1,$$

with constant less than or equal to $\varphi(\|u\|_{A_1})$*, where* φ *is a positive nondecreasing function on* $[1,\infty)$*. Then, if* ϕ *is a right-continuous bounded variation function on* $(0,\infty)$ *with* $\lim_{x\to 0^+}\phi(x)=0$*, we have that*

$$T_\phi f(x)=\int_{\mathbb{R}^m} K(x,y)\phi(|x-y|)f(y)dy, \qquad x\in\mathbb{R}^n,$$

satisfies that

$$T_\phi : L^1_{\mathcal{R}}(u)\to L^{1,\infty}(u), \qquad \forall u\in A_1.$$

Proof. We observe that, by hypothesis,

$$\phi(|x-y|)=\int_0^{|x-y|}\phi'(s)ds, \qquad \phi'\in L^1(\mathbb{R}^n),$$

and hence, for every $x\in\mathbb{R}^n$ and every $\varepsilon>0$, by Fubini's theorem, we have that

$$T_\phi f(x)-\phi(\varepsilon)Tf(x)=\int_0^\infty\left(\int_{|x-y|\geqslant s\geqslant\varepsilon}K(x,y)f(y)dy\right)\phi'(s)\,ds=\int_\varepsilon^\infty T_s f(x)\phi'(s)\,ds,$$

is an average operator, and so the result follows by Corollary 3.5.1 and letting ε tend to zero. $\square$

3.5.4 The Bochner-Riesz operator

Let $n>1$ and $\lambda>0$. The *Bochner-Riesz operator* is defined as

$$\widehat{B_\lambda f}(\xi)=\left(1-|\xi|^2\right)_+^\lambda\hat{f}(\xi), \qquad \xi\in\mathbb{R}^n.$$

These operators were first introduced by S. Bochner in [25] and, since then, they have been widely studied (see [28, 40, 59, 66, 98, 120, 175, 178]).

The case $\lambda=0$ corresponds to the *disc multiplier* which is unbounded in $L^p(\mathbb{R}^n)$ if $n\geqslant 2$ and $p\neq 2$ (see [93]). Indeed, in that case, the disc multiplier $S_n:=B_0$ is defined by

$$\widehat{(S_n f)}(\xi)=\chi_{\mathbb{B}(0,1)}(\xi)\hat{f}(\xi), \qquad \xi\in\mathbb{R}^n,$$

where $\mathbb{B}(0,1)=\{x\in\mathbb{R}^n : |x|<1\}$ denote the n-th dimensional open ball of center 0 and radius 1.

When $\lambda>\frac{n-1}{2}$, it is well known that B_λ is controlled by the Hardy-Littlewood maximal operator M. As a consequence, all weighted inequalities for M are also satisfied by B_λ, so that in this case,

$$\|B_\lambda f\|_{L^{p,\infty}(v)}\lesssim\|v\|_{A_p^R}\|f\|_{L^{p,1}(v)}, \qquad \forall v\in A_p^R,$$

(see (2.2.7)). The value $\lambda=\frac{n-1}{2}$ is called the critical index. In this case, X.L. Shi and Q.Y. Sun [169] proved that $B_{\frac{n-1}{2}}$ is bounded in $L^p(v)$ for every $1<p<\infty$ and $v\in A_p$. The unweighted weak-type inequality for $p=1$ was first settled by M. Christ [68], who showed that $B_{\frac{n-1}{2}}$ is bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$, and the corresponding weighted weak-type inequality was obtained by A. Vargas in [178], where she proved that $B_{\frac{n-1}{2}}$ is bounded from $L^1(v)$ to $L^{1,\infty}(v)$ for every $v\in A_1$. Further, in [140] the authors gave the following quantitative result:

Proposition 3.5.6. *Let $n > 1$. Then,*

$$B_{\frac{n-1}{2}} : L^1(u) \to L^{1,\infty}(u), \qquad \forall u \in A_1,$$

with constant less than or equal to $\|u\|_{A_1}^2 \log_2(C\,\|u\|_{A_1} + 1)$.

Thereby, by virtue of Theorem 3.3.1, we completely answer the open question formulated in [46] about the weighted restricted weak-type (p,p) boundedness of $B_{\frac{n-1}{2}}$.

Corollary 3.5.7. *Let $n > 1$. Then, for every $1 < p < \infty$,*

$$B_{\frac{n-1}{2}} : L^{p,1}(v) \to L^{p,\infty}(v), \qquad \forall v \in \widehat{A}_p,$$

with constant less than or equal to $C\,\|v\|_{\hat{A}_p}^{3p-1} (1 + \log \|v\|_{\hat{A}_p})^{1+\frac{2}{p'}}$.

Below the critical index, that is $0 < \lambda < \frac{n-1}{2}$, B_λ is not bounded in $L^p(\mathbb{R}^n)$ for the whole range $1 < p < \infty$. For instance, in dimension $n = 2$, L. Carleson and P. Sjölin [40] proved that B_λ is bounded in $L^p(\mathbb{R}^2)$ if and only if $p > 1$ and

$$\lambda > \max\left(2\left|\frac{1}{p} - \frac{1}{2}\right| - \frac{1}{2}, 0\right),$$

or equivalently, $0 < \lambda < \frac{1}{2}$ and

$$\frac{4}{3+2\lambda} < p < \frac{4}{1-2\lambda}.$$

Moreover, A. Seeger [168] showed that the corresponding unweighted weak-type inequality at the endpoint

$$B_\lambda : L^{\frac{4}{3+2\lambda}}(\mathbb{R}^2) \to L^{\frac{4}{3+2\lambda},\infty}(\mathbb{R}^2) \tag{3.5.4}$$

also holds.

For higher dimensions it is already well known that B_λ is not bounded in $L^p(\mathbb{R}^n)$ for $p \leqslant \frac{2n}{n+1+2\lambda}$ or $p \geqslant \frac{2n}{n-1-2\lambda}$ (see for instance [88, Theorem 8.15] or [98, Proposition 5.2.3]). Furthermore, it was conjectured the following:

Conjecture 3.5.8 (Bochner-Riesz Conjecture)**.** *B_λ is bounded in $L^p(\mathbb{R}^n)$ if $p > 1$ and*

$$\lambda > \lambda(p) = \max\left(n\left|\frac{1}{p} - \frac{1}{2}\right| - \frac{1}{2}, 0\right),$$

or equivalently, for $0 < \lambda < \frac{n-1}{2}$ and

$$\frac{2n}{n+1+2\lambda} < p < \frac{2n}{n-1-2\lambda}. \tag{3.5.5}$$

However, the Bochner-Riesz conjecture only has been partially answered and the best results known up to now are currently due to Bourgain and Guth [28] (see also Lee [126]).

We summarize it here: let $0 < \lambda < \frac{n-1}{2}$ and p in the range of (3.5.5), if $q = \max(p, p')$ satisfies

$$q > \begin{cases} \dfrac{2(4n+3)}{4n-3}, & n \equiv 0 \,(\mathrm{mod}\ 3), \\ \dfrac{2n+1}{n-1}, & n \equiv 2 \,(\mathrm{mod}\ 3), \\ \dfrac{4(n+1)}{2n-1}, & n \equiv 2 \,(\mathrm{mod}\ 3), \end{cases}$$

then the Bochner-Riesz Conjecture holds.

Further, under the condition $\frac{n-1}{2(n+1)} < \lambda < \frac{n-1}{2}$, the corresponding unweighted weak-type inequality at the endpoint

$$B_\lambda : L^{\frac{2n}{n+1+2\lambda}}(\mathbb{R}^n) \to L^{\frac{2n}{n+1+2\lambda}(\mathbb{R}^n),\infty} \tag{3.5.6}$$

was settled by M. Christ [67] and extended to $\lambda = \frac{n-1}{2(n+1)}$ by T. Tao [175], while it remains unknown for the range $0 < \lambda < \frac{n-1}{2(n+1)}$. This is the often-called endpoint Bochner-Riesz conjecture (see [176]) and we observe that if it holds for some λ, then by duality (since B_λ is essentially self-adjoint) and interpolation, the Bochner-Riesz conjecture is also true for such λ and any p in the range of (3.5.5).

Moreover, in two recent papers [120, 124], new weighted estimates for B_λ have been proved using the fact that the Bochner-Riesz operator can be dominated by sparse type operators. As far as we know, these are the best weighted estimates known for every $n \geqslant 2$, together with the results for the $(2,2)$-strong type inequality in [59, 66].

Proposition 3.5.9 ([120])**.** *Let $n = 2$ and $0 < \lambda < \frac{1}{2}$. Then,*

$$B_\lambda : L^{\frac{4}{3+2\lambda}}\left(u^{\frac{2\lambda}{3+2\lambda}}\right) \to L^{\frac{4}{3+2\lambda},\infty}\left(u^{\frac{2\lambda}{3+2\lambda}}\right), \qquad \forall u \in A_1,$$

with constant less than or equal to $c(n,\lambda)\,\|u\|_{A_1}^{\frac{\lambda(7+4\lambda)}{6+4\lambda}}$.

Proposition 3.5.10 ([66])**.** *Let $n > 2$ and $\frac{n-1}{2(n+1)} < \lambda < \frac{n-1}{2}$. Then*

$$B_\lambda : L^2\left(u^{\frac{1+2\lambda}{n}}\right) \to L^2\left(u^{\frac{1+2\lambda}{n}}\right), \qquad \forall u \in A_1,$$

with constant less than or equal to $\varphi\left(\|u\|_{A_1}^{\frac{1+2\lambda}{n}}\right)$, where φ is a positive nondecreasing function on $[1,\infty)$.

Therefore, as a consequence of Theorem 3.4.1, we present some new weighted estimates for the Bochner-Riesz operator below the critical index.

Corollary 3.5.11. *Let $n = 2$ and $0 < \lambda < \frac{1}{2}$. For every $\frac{4}{3+2\lambda} \leqslant p < \frac{4}{3}$,*

$$B_\lambda : L^{p,1}(v) \to L^{p,\infty}(v), \qquad \forall v \in \hat{A}_{p;\left(\frac{4-3p}{4}, \frac{(3+2\lambda)p-4}{4(p-1)}\right)}.$$

Now, let $n > 2$ and $\frac{n-1}{2(n+1)} < \lambda < \frac{n-1}{2}$. For every $2 \leqslant p < \frac{2n}{n-1-2\lambda}$,

$$B_\lambda : L^{p,1}(v) \to L^{p,\infty}(v), \qquad \forall v \in \hat{A}_{p;\left(\frac{2n-p(n-1-2\lambda)}{2n}, \frac{p-2}{2(p-1)}\right)}.$$

Chapter 4

Weighted restricted weak-type estimates on $\Lambda^p(w)$

We dedicate this chapter to prove weighted restricted weak-type estimates on classical Lorentz spaces ($\Lambda^p(w)$ for $0 < p < \infty$). We start in Section 4.1 by introducing known results about boundedness of operators on classical Lorentz spaces and also by motivating the weighted restricted weak-type estimates that we will study. Indeed, in Section 4.2 we will make use of the theory of Rubio de Francia extrapolation (see Section 2.3) to study boundedness of operators from $\Lambda^{p,1}(w)$ to $\Lambda^{p,\infty}(w)$, and we will also adapt the ideas that we will use there to the limited setting. Finally, in Section 4.3, we will see that, in fact, weighted restricted weak-type estimates on classical Lorentz spaces for sublinear operators T (even in the limited setting) are equivalent to pointwise estimates on the decreasing rearrangement of T by a Calderón admissible type operator (see Sections 4.3.1 and 4.3.2). Besides, we will figure it out that weighted restricted weak-type estimates on classical Lorentz spaces, actually follow from unweighted restricted weak-type estimates on Lorentz spaces for a suitable control of the norm operator constant.

The results of this chapter are included in [6, 15, 16].

4.1 An introduction about boundedness on $\Lambda^p(w)$

There are many results in the literature about boundedness of operators on $\Lambda^p(w)$ or even on r.i. spaces. For instance, let us just mention the classical paper [166] including the case of the Hilbert transform and Riesz transforms, and a recent one [92] which contains a rather complete list of papers on this topic, among which we should include [2, 30, 71].

Now, the operators that appear on the references mentioned above have one important property in common: they all satisfy that for some (and hence for all) $1 < p_0 < \infty$,

$$T : L^{p_0}(v) \to L^{p_0}(v), \qquad \forall v \in A_{p_0}, \tag{4.1.1}$$

with constant less than or equal to $\varphi(\|v\|_{A_{p_0}})$, where φ is a positive nondecreasing function on $[1, \infty)$, so that they all fulfill the hypothesis of Theorem 2.3.3 for $g = |Th|$ and $f = |h| \in L^{p_0}(v)$. Actually, that theorem is very useful to prove the boundedness of operators for which condition (4.1.1) has been widely studied, while this is not the case in other contexts such as, for example, of classical Lorentz spaces or more generally r.i. spaces. Let us explain, as an example, the case of the sparse operators $A_{\mathcal{S}}$ introduced by A.K. Lerner in [132] (see Section 5.5) which have become very useful since they dominate many operators such as Calderón-Zygmund operators. Since $A_{\mathcal{S}}$ are known to satisfy (4.1.1), Theorem 2.3.3 implies that, for example, if we consider classical Lorentz spaces whenever are Banach function spaces (that is, for $w \in B_p$ if $1 < p < \infty$ or $w \in B_1^R$ if $p = 1$) and if

$$M : \Lambda^p(w) \to \Lambda^p(w) \qquad \text{and} \qquad M : (\Lambda^p(w))' \to (\Lambda^p(w))', \tag{4.1.2}$$

then we have that

$$A_{\mathcal{S}} : \Lambda^p(w) \to \Lambda^p(w).$$

On many occasions the difficulty to apply Theorem 2.3.3 to a concrete space $\mathbb{X}$ is precisely to characterize when (4.1.2) holds for $\mathbb{X}$. In the case of $\mathbb{X} = \Lambda^p(w)$ it has been already settled that the boundedness of M over $\Lambda^p(w)$ is characterized by B_p (see Section 2.2.2) while the boundedness of M over $(\Lambda^p(w))'$ is done by B_∞^* (see Section 2.2.3) so we can concluded that, for $p \geqslant 1$,

$$w \in B_p \cap B_\infty^* \qquad \Longrightarrow \qquad A_{\mathcal{S}} : \Lambda^p(w) \to \Lambda^p(w).$$

In fact, the same result is true for every operator T satisfying (4.1.1) and this condition is sharp (in the sense that it can not be found a greater class for w) by means of the Hilbert transform (see (2.2.20)). However, up to now we have just considered $p \geqslant 1$ since Theorem 2.3.3 concerns the case of Banach function spaces. But, what can we say if $0 < p < 1$ (where $\Lambda^p(w)$ is "at most" a quasi-Banach function space)? Moreover, in the case $p = 1$, it is known that

$$A_{\mathcal{S}} : L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n),$$

and this case is not covered by Theorem 2.3.3, since it is known that (4.1.1) does not imply, in general, the unweighted weak-type boundedness of T from $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$ (see Remark 2.3.8). Nevertheless, in Theorem 2.3.6 is stated that if we assume a slightly stronger condition on T; that is, for $1 < p_0 < \infty$,

$$T : L^{p_0,1}(v) \to L^{p_0,\infty}(v), \qquad \forall v \in \hat{A}_{p_0}, \tag{4.1.3}$$

with constant less than or equal to $\varphi(\|v\|_{\hat{A}_{p_0}})$, where φ is a positive nondecreasing function on $[1, \infty)$, which is satisfied by the sparse operators (among many others) then we can arrive, at least for characteristic functions, to the endpoint $p = 1$.

The purpose of this chapter is to study, assuming that T satisfies (4.1.3), even for $p_0 = 1$, what conditions do we need on w in order to deduce weighted restricted weak-type boundedness on classical Lorentz spaces; that is,

$$T : \Lambda^{p,q}(w) \to \Lambda^{p,\infty}(w), \qquad \text{for some } 0 < q \leqslant 1, \tag{4.1.4}$$

where the exponent $q < 1$ appears for that operators for which the unweighted weak-type boundedness from $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$ holds just for characteristic functions. To do so, we will use an extrapolation argument based on an estimate on the distribution function of the operator acting to any measurable function of $\Lambda^{p,q}(w)$ (see Section 4.2.1).

Besides, we will see that, in particular, under the suitable conditions of w and if T is sublinear, (4.1.4) is going to be equivalent to an estimate on the decreasing rearrangement of T (see Section 4.3.3) which reads as follow: for every locally integrable function f,

$$(Tf)^*(t) \lesssim \left(\frac{1}{t^q}\int_0^t f^*(s)^q \frac{ds}{s^{1-q}}\right)^{\frac{1}{q}} + \int_t^\infty \left(1+\log\frac{s}{t}\right)^{-1} \varphi\left(1+\log\frac{s}{t}\right) f^*(s)\,\frac{ds}{s}, \qquad \forall t>0,$$

for some admissible function φ (see Definition 4.3.1) which is going to be related with the φ that controls the constant of (4.1.3) (i.e., with the behaviour of the weight norm). Although to assume that condition on φ could seem (a priori) to be restrictive, a lot of operators satisfy (4.1.3) with φ being a power function or even the product between a power function and the composition of the positive part of logarithms, and all these cases are admissible functions. Furthermore, we will show that actually, in order to have (4.1.4) we just have to impose that for every $p_0 \geqslant 1$,

$$T: L^{1,q}(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n) \qquad \text{and} \qquad \|T\chi_E\|_{L^{p_0,\infty}(\mathbb{R}^n)} \leqslant \varphi(p_0)u(E)^{\frac{1}{p_0}}, \qquad \forall E\subseteq \mathbb{R}^n.$$

These kind of pointwise estimates for the decreasing rearrangement are very interesting since obviously it have, as a consequence, boundedness properties of such operators on rearrangement invariant spaces (for more details, we refer to [23]). In particular, it will allow us to obtain a different approach (from using extrapolation) in order to get weighted restricted weak-type estimates on classical Lorentz spaces.

Now, if we consider the Bochner-Riesz operator B_λ (see Section 3.5.4) below the critical index (that is, $0<\lambda<\frac{n-1}{2}$) then it is known that B_λ is not bounded in $L^p(\mathbb{R}^n)$ for the whole range $1<p<\infty$ (see [88, 98]) and the same happens when studying the boundedness between $L^{p,1}(\mathbb{R}^n)$ and $L^{p,\infty}(\mathbb{R}^n)$, so there is not even one $p_0>1$ such that (4.1.3) could hold. However, it is true for a subclass of the $\hat{A}_{p_0}$ weights (see Definition 2.2.8) that is, there exists some $p_0>1$ and some $\alpha,\beta\in[0,1]$ (not both identically zero) such that

$$T: L^{p_0,1}(v) \to L^{p_0,\infty}(v), \qquad \forall v \in \hat{A}_{p_0;(\alpha,\beta)}. \tag{4.1.5}$$

This type of weighted inequalities are also satisfied by other operators such as the Hörmander Fourier multipliers $m\in M(s,l)$ with $l<n$ (see Section 5.2.1) among many others. We will adapt the ideas used on the results above mentioned for operators satisfying (4.1.3) (which as we have already said are included on Sections 4.2.1 and 4.3.3) to take into account also the operators that even though do not satisfy (4.1.3) they do (4.1.5), so that we will obtain weighted restricted weak-type estimates on classical Lorentz spaces for these cases as well (see Sections 4.2.2 and 4.3.4).

4.2 Weighted restricted weak-type estimates by extrapolation on $\Lambda^p(w)$

There are some results involving extrapolation aimed to obtain weighted estimates on classical Lorentz spaces [60, 84].

In this section, we use the extrapolation theory to obtain new results about weighted restricted weak-type estimates on $\Lambda^p(w)$ (see Section 4.2.1) and we adapt it to deduce also weighted estimates on the limited setting (see Section 4.2.2).

4.2.1 Weighted restricted weak-type extrapolation

Our first goal is to study the operators for which there exists some $p_0 \geqslant 1$ such that

$$T : L^{p_0,1}(v) \to L^{p_0,\infty}(v), \qquad \forall v \in \hat{A}_{p_0},$$

and see for which conditions on p and w the corresponding weighted weak-type estimate on classical Lorentz spaces holds. First, let us see a technical result.

Lemma 4.2.1. *Given the exponents* $0 \leqslant \beta_0 \leqslant 1$, $1 \leqslant p_0 < \infty$, $0 < q_0 \leqslant p_0$ *and* $0 < p < \infty$. *Define*

$$p_1 = p(\beta_0 + (1-\beta_0)p_0) \qquad \text{and} \qquad q_1 = \frac{q_0}{p_0}(\beta_0 + (1-\beta_0)p_0),$$

and take $f \in \Lambda^{p_1,q_1}(w)$. *If* $v = (Mf)^{\beta_0(1-p_0)}h$, *for some* $h \in (\Lambda^{p,1}(w))'$, *then* $f \in L^{p_0,q_0}(v)$ *with*

$$\|f\|_{L^{p_0,q_0}(v)}^{p_0} \leqslant p\left(\frac{p_0}{p_1}\right)^{\frac{p_0}{q_0}} \|f\|_{\Lambda^{p_1,q_1}(w)}^{\frac{p_1}{p}} \|h\|_{(\Lambda^{p,1}(w))'}.$$

Proof. First, we observe that, for every $t > 0$, if $x \in \{|f(x)| > t\}$ then

$$Mf(x)^{\beta_0(1-p_0)} \leqslant |f(x)|^{\beta_0(1-p_0)} \leqslant t^{\beta_0(1-p_0)}.$$

Hence,

$$\begin{aligned}
\|f\|_{L^{p_0,q_0}(v)}^{p_0} &= \left(p_0 \int_0^\infty t^{q_0-1}\left(\int_{\{|f(x)|>t\}} v(x)\,dx\right)^{\frac{q_0}{p_0}} dt\right)^{\frac{p_0}{q_0}} \\
&\leqslant \left(p_0 \int_0^\infty t^{q_1-1}\left(\int_{\{|f(x)|>t\}} h(x)\,dx\right)^{\frac{q_0}{p_0}} dt\right)^{\frac{p_0}{q_0}} \\
&\leqslant \|h\|_{(\Lambda^{p,1}(w))'}\left(p_0 \int_0^\infty t^{q_1-1}\left\|\chi_{\{|f(x)|>t\}}\right\|_{\Lambda^{p,1}(w)}^{\frac{q_0}{p_0}} dt\right)^{\frac{p_0}{q_0}},
\end{aligned}$$

where in the last estimate we have used the Hölder's inequality for classical Lorentz spaces. Finally, we see that

$$\int_0^\infty t^{q_1-1}\left\|\chi_{\{|f(x)|>t\}}\right\|_{\Lambda^{p,1}(w)}^{\frac{q_0}{p_0}} dt = p^{\frac{q_0}{p_0}} \int_0^\infty t^{q_1-1} W(\lambda_f(t))^{\frac{q_1}{p_1}}\,dt = \frac{p^{\frac{q_0}{p_0}}}{p_1}\|f\|_{\Lambda^{p_1,q_1}(w)}^{q_1},$$

from which the desired result follows. □

Now, we state and prove our first main result in this section.

Theorem 4.2.2. *Let T be an operator satisfying that for some* $1 \leqslant p_0 < \infty$,

$$T : L^{p_0,1}(v) \to L^{p_0,\infty}(v), \qquad \forall v \in \hat{A}_{p_0}, \tag{4.2.1}$$

with constant less than or equal to $\varphi(\|v\|_{\hat{A}_{p_0}})$, *where* φ *is a positive nondecreasing function on* $[1,\infty)$. *Let* $0 < p < \infty$.

(i) If $p_0 = 1$, then

$$T : \Lambda^{p,1}(w) \to \Lambda^{p,\infty}(w), \qquad \forall w \in B_p^R \cap B_\infty^*,$$

with constant less than or equal to $C_1 \|w\|_{B_p^R} \varphi\left(C_2 \|w\|_{B_\infty^*}\right)$.

(ii) If $p_0 > 1$ and T is sublinear, then, for every $0 < q < 1$,

$$T : \Lambda^{p,q}(w) \to \Lambda^{p,\infty}(w), \qquad \forall w \in B_p^R \cap B_\infty^*,$$

with constant less than or equal to $\dfrac{C_1}{1-q} \|w\|_{B_p^R} \max\left(1, \|w\|_{B_\infty^*}^{q-\frac{1}{p_0}}\right) \varphi\left(C_2 \|w\|_{B_\infty^*}^{\frac{1}{p_0}}\right)$.

Proof. First, by means of Proposition 2.2.20 we can define, for every measurable set $F \subseteq \mathbb{R}^n$,

$$R\chi_F(x) = \sum_{k=0}^{\infty} \frac{M^k \chi_F(x)}{\left(2\|M\|_{(\Lambda^{p,1}(w))'}\right)^k}, \qquad x \in \mathbb{R}^n.$$

Then,

(1) $\chi_F(x) \leqslant R\chi_F(x)$,

(2) $\|R\chi_F\|_{A_1} \leqslant 2\|M\|_{(\Lambda^{p,1}(w))'} \lesssim \|w\|_{B_\infty^*}$,

(3) $\|R\chi_F\|_{(\Lambda^{p,1}(w))'} \leqslant 2\|\chi_F\|_{(\Lambda^{p,1}(w))'} \leqslant \dfrac{2}{p}\|w\|_{B_p^R} \dfrac{|F|}{W(|F|)^{\frac{1}{p}}}$,

where on the right-hand side of (3) we have used Lemma 2.2.15.

Let $y > 0$ and set $F = \{x \in \mathbb{R}^n : |Tf(x)| > y\}$, so that $|F| = \lambda_{Tf}(y)$. Then,

$$v = (Mf)^{1-p_0} R\chi_F \in \hat{A}_{p_0}, \tag{4.2.2}$$

and, by hypothesis,

$$\begin{aligned}
\lambda_{Tf}(y) &= \int_{\{|Tf(x)|>y\}} dx \leqslant \lambda_{Mf}(\gamma y) + \int_{\{|Tf(x)|>y,\, Mf(x)\leqslant \gamma y\}} R\chi_F(x)\,dx \\
&\leqslant \lambda_{Mf}(\gamma y) + \gamma^{p_0-1}\frac{y^{p_0}}{y}\int_F v(x)\,dx \leqslant \lambda_{Mf}(\gamma y) + \frac{\gamma^{p_0-1}\varphi\left(\|v\|_{\hat{A}_{p_0}}\right)^{p_0}}{y}\|f\|_{L^{p_0,1}(v)}^{p_0},
\end{aligned}$$

so that by means of Lemma 4.2.1 (with $q_0 = \beta_0 = 1$ and $h = R\chi_F$) together with property (3) of R,

$$\lambda_{Tf}(y) \lesssim \lambda_{Mf}(\gamma y) + \frac{\gamma^{p_0-1}\varphi\left(\|v\|_{\hat{A}_{p_0}}\right)^{p_0}}{y}\|w\|_{B_p^R}\frac{\lambda_{Tf}(y)}{W(\lambda_{Tf}(y))^{\frac{1}{p}}}\|f\|_{\Lambda^{p,\frac{1}{p_0}}(w)}. \tag{4.2.3}$$

Further, since $w \in B_p^R$ and $\Lambda^{p,\frac{1}{p_0}}(w) \subseteq \Lambda^{p,1}(w)$ continuously,

$$\sup_{y>0} yW(\lambda_{Mf}(\gamma y))^{\frac{1}{p}} = \frac{1}{\gamma} \|Mf\|_{\Lambda^{p,\infty}(w)} \lesssim \|w\|_{B_p^R} \frac{1}{\gamma} \|f\|_{\Lambda^{p,\frac{1}{p_0}}(w)}.$$

Therefore, observe that if $\lambda_{Tf}(y) < \infty$ for every $y > 0$, then it is possible to divide by $\lambda_{Tf}(y)$ in (4.2.3) so that taking the supremum over all $y > 0$, in particular, we conclude that

$$\|Tf\|_{\Lambda^{p,\infty}(w)} \lesssim \|w\|_{B_p^R} \max\left(\frac{1}{\gamma}, \gamma^{p_0-1}\varphi\left(\|v\|_{\hat{A}_{p_0}}\right)^{p_0}\right) \|f\|_{\Lambda^{p,\frac{1}{p_0}}(w)},$$

and taking the infimum in $\gamma > 0$ yields that

$$\|Tf\|_{\Lambda^{p,\infty}(w)} \lesssim \|w\|_{B_p^R} \varphi\left(\|v\|_{\hat{A}_{p_0}}\right) \|f\|_{\Lambda^{p,\frac{1}{p_0}}(w)} \leqslant \|w\|_{B_p^R} \varphi\left(C \|w\|_{B_\infty^*}^{\frac{1}{p_0}}\right) \|f\|_{\Lambda^{p,\frac{1}{p_0}}(w)}, \quad (4.2.4)$$

where in the last estimate we have used the definition of v and property (2) of R. Otherwise, for each $N \in \mathbb{N}$, let $T_N f = |Tf|\chi_{B(0,N)}$. Then,

$$\lambda_{T_N f}(y) \leqslant |B(0,N)| < \infty, \qquad \forall y > 0,$$

and T_N satisfies also (4.2.1), so that arguing as above but now with T_N instead of T we obtain that, for every $N \in \mathbb{N}$,

$$\|T_N f\|_{\Lambda^{p,\infty}(w)} \lesssim \|w\|_{B_p^R} \varphi\left(C \|w\|_{B_\infty^*}^{\frac{1}{p_0}}\right) \|f\|_{\Lambda^{p,\frac{1}{p_0}}(w)},$$

and so the same result hold for T by taking the supremum over all $N \in \mathbb{N}$.

Finally, if $p_0 = 1$, (i) follows directly from (4.2.4). For (ii), we observe that if $0 < q \leqslant \frac{1}{p_0}$, then $\Lambda^{p,q}(w) \subseteq \Lambda^{p,\frac{1}{p_0}}(w)$ continuously and the result also follows. Otherwise, if we take $\frac{1}{p_0} < q < 1$, we have from [61, Corollary 2.15] that

$$T : L^{\frac{1}{q},1}(v) \to L^{\frac{1}{q},\infty}(v), \qquad \forall v \in \hat{A}_{\frac{1}{q}}, \quad (4.2.5)$$

with constant less than or equal to

$$\frac{C_1}{1-q} \|v\|_{\hat{A}_{\frac{1}{q}}}^{1-\frac{1}{qp_0}} \varphi\left(C_2 \|v\|_{\hat{A}_{\frac{1}{q}}}^{\frac{1}{qp_0}}\right).$$

Therefore, arguing as above but with (4.2.5) instead of (4.2.1), we obtain the desired result with constant

$$\frac{C_1}{1-q} \|w\|_{B_p^R} \|w\|_{B_\infty^*}^{q-\frac{1}{p_0}} \varphi\left(C_2 \|w\|_{B_\infty^*}^{\frac{1}{p_0}}\right).$$

□

Remark 4.2.3. *By means of the Hilbert transform (see (2.2.25)) the condition $B_p^R \cap B_\infty^*$ on the weight w of Theorem 4.2.2 (i) is sharp in the sense that it can not be found a greater class for w.*

As a consequence of Theorem 4.2.2, we have the next result.

Corollary 4.2.4. *Let T be a sublinear operator satisfying (4.2.1). Given $0 < p < \infty$ and $0 < r \leqslant \infty$, then*

$$T : \Lambda^{p,r}(w) \to \Lambda^{p,r}(w), \qquad \forall w \in B_p \cap B_\infty^*.$$

Proof. Observe that if $w \in B_p \cap B_\infty^*$, there exists some $\varepsilon > 0$ such that $w \in B_{p-\varepsilon}^R \cap B_\infty^*$ (see (2.2.9)) and, of course, $w \in B_{p+\varepsilon}^R \cap B_\infty^*$. Hence, by virtue of Theorem 4.2.2, for some $0 < q \leqslant 1$,

$$T : \Lambda^{p-\varepsilon,q}(w) \to \Lambda^{p-\varepsilon,\infty}(w) \qquad \text{and} \qquad T : \Lambda^{p+\varepsilon,q}(w) \to \Lambda^{p+\varepsilon,\infty}(w),$$

and the result follows by interpolation on classical Lorentz spaces [58, Theorem 2.6.5]. □

4.2.2 Weighted limited restricted weak-type extrapolation

Our next goal is to show that a similar result as in Section 4.2.1 holds true when dealing with weighted limited restricted weak-type estimates.

Theorem 4.2.5. *Let T be an operator satisfying that for some $1 \leqslant p_0 < \infty$ and $0 \leqslant \alpha, \beta \leqslant 1$ (not both identically zero),*

$$T : L^{p_0,1}(v) \to L^{p_0,\infty}(v), \qquad \forall v \in \hat{A}_{p_0;(\alpha,\beta)}, \tag{4.2.6}$$

with constant less than or equal to $\varphi(\|v\|_{\hat{A}_{p_0;(\alpha,\beta)}})$, with φ being a positive nondecreasing function on $[1,\infty)$. Let $0 < p < \infty$ and set p_- and p_+ as in (2.3.5).

(i) If $p_0 = 1$ or $0 \leqslant \beta < 1$,

$$T : \Lambda^{p,1}(w) \to \Lambda^{p,\infty}(w), \qquad \forall w \in B_{\frac{p}{p_-}}^R \cap B_{\frac{p_+}{p}}^*. \tag{4.2.7}$$

(ii) If $p_0 > 1$ and $\beta = 1$ then, for every $0 < q < 1$,

$$T : \Lambda^{p,q}(w) \to \Lambda^{p,\infty}(w), \qquad \forall w \in B_{\frac{p}{p_-}}^R \cap B_{\frac{p_+}{p}}^*.$$

Proof. Taking $\tilde{w} = W^{1/p-1}w$, from Lemmas 2.2.19 and 2.2.29, and the definition of $B_{\frac{p}{p_-}}^R$, we have that

$$w \in B_{\frac{p}{p_-}}^R \cap B_{\frac{p_+}{p}}^* \qquad \Longleftrightarrow \qquad \tilde{w} \in B_{\frac{1}{p_-}}^R \cap B_{p_+}^*,$$

so, together with (2.1.2), we can assume that $p = 1$. Further, we claim that (4.2.6) can be changed to this new assumption: for every $0 < q < 1$ (if $p_0 > 1$ and $\beta = 1$) or $q = 1$ (otherwise) then

$$T : L^{p_-,q}(u^{\alpha(p_-)}) \to L^{p_-,\infty}(u^{\alpha(p_-)}), \qquad \forall u \in A_1, \tag{4.2.8}$$

with constant less than or equal to $\tilde{\varphi}(\|u\|_{A_1}^{\alpha(p_-)})$, where $\tilde{\varphi}$ is a positive nondecreasing function on $[1,\infty)$. Indeed, if $p_0 = 1$ or $\beta = 0$ (so that $p_0 = p_-$) then $q = 1$ and $\tilde{\varphi} = \varphi$. Otherwise, if

$p_0 > 1$ and $0 < \beta < 1$ (so, as well, $p_- > 1$) we just have to make use of Theorem 2.3.14 so $q = 1$ and $\tilde{\varphi} = \frac{1}{p_- - 1}\Phi_{p_-}$. However, if $p_0 > 1$ and $\beta = 1$ (so $p_- = 1$) for every $0 < q < 1$, by virtue of Corollary 3.4.2 we get that (4.2.8) holds with $\tilde{\varphi} = \frac{q}{1-q}\tilde{\Phi}_q$.

Therefore, assume (4.2.8), take $p = 1$ and let $w \in B^R_{\frac{1}{p_-}} \cap B^*_{p_+}$. Since $p_+ = \frac{p_-}{1-\alpha(p_-)}$, by means of Propositions 2.2.20 and 2.2.30 (where we are assuming that for $\alpha(p_-) = 1$ then $p_+ = \infty$) we can define, for every measurable set $F \subseteq \mathbb{R}^n$,

$$R\chi_F(x) = \sum_{k=0}^{\infty} \frac{M^k \chi_F(x)}{\left(2\left\|M_{\alpha(p_-)}\right\|_{\left(\Lambda^{\frac{1}{p_-},1}(w)\right)'}\right)^{k/\alpha(p_-)}}, \qquad x \in \mathbb{R}^n,$$

where $M_{\alpha(p_-)} = M\left(|\cdot|^{1/\alpha(p_-)}\right)^{\alpha(p_-)}$. Then,

(1) $\chi_F(x) \leqslant R\chi_F(x)^{\alpha(p_-)}$,

(2) $\|R\chi_F\|_{A_1}^{\alpha(p_-)} \leqslant 2\left\|M_{\alpha(p_-)}\right\|_{\left(\Lambda^{\frac{1}{p_-},1}(w)\right)'} \lesssim \begin{cases} \|w\|_{B^*_{p_+}}^{p_-}, & 0 < \alpha(p_-) < 1, \\ \|w\|_{B^*_\infty}, & \alpha(p_-) = 1, \end{cases}$

(3) $\left\|(R\chi_F)^{\alpha(p_-)}\right\|_{\left(\Lambda^{\frac{1}{p_-},1}(w)\right)'} \leqslant 2\left\|\chi_F\right\|_{\left(\Lambda^{\frac{1}{p_-},1}(w)\right)'} \leqslant 2p_- \|w\|_{B^R_{\frac{1}{p_-}}} \dfrac{|F|}{W(|F|)^{p_-}}$,

where on the right-hand side of (3) we have used Lemma 2.2.15.

Let $y > 0$ and set $F = \{x \in \mathbb{R}^n : |Tf(x)| > y\}$, so that $|F| = \lambda_{Tf}(y)$. Then, by hypothesis,

$$y^{p_-}\lambda_{Tf}(y) \leqslant y^{p_-}\int_{\{|Tf(x)|>y\}} R\chi_F(x)^{\alpha(p_-)}\,dx \leqslant \tilde{\varphi}\left(\|R\chi_F\|_{A_1}^{\alpha(p_-)}\right)^{p_-} \|f\|^{p_-}_{L^{p_-,q}\left((R\chi_F)^{\alpha(p_-)}\right)}. \tag{4.2.9}$$

Hence, by means of Lemma 4.2.1 (with $p_0 = p_-$, $q_0 = q$, $\beta_0 = 0$, $p = \frac{1}{p_-}$ and $h = (R\chi_F)^{\alpha(p_-)}$) and the property (3) of R,

$$y^{p_-}\lambda_{Tf}(y) \lesssim \|w\|_{B^R_{\frac{1}{p_-}}} \tilde{\varphi}\left(\|R\chi_F\|_{A_1}^{\alpha(p_-)}\right)^{p_-} \frac{\lambda_{Tf}(y)}{W(\lambda_{Tf}(y))^{p_-}} \|f\|^{p_-}_{\Lambda^{1,q}(w)},$$

so arguing (if necessary) with $T_N f = |Tf|\chi_{B(0,N)}$ as we did in the proof of Theorem 4.2.2 and taking the supremum over all $y > 0$, in particular, we obtain that that,

$$\|Tf\|_{\Lambda^{1,\infty}(w)} \lesssim \|w\|^{\frac{1}{p_-}}_{B^R_{\frac{1}{p_-}}} \tilde{\varphi}\left(\|R\chi_F\|_{A_1}^{\alpha(p_-)}\right) \|f\|_{\Lambda^{1,q}(w)},$$

and the desired result follows by the property (2) of R and the fact that $\tilde{\varphi}$ is nondecreasing. □

Remark 4.2.6. *If $p_0 = 1$ or $\beta = 0$ in (4.2.6) (that is, $p_0 = p_-$) then we get that (4.2.7) holds with constant less than or equal to* $C_1 \|w\|_{B^R_{\frac{p}{p_0}}}^{1/p_0} \overline{\varphi}\left(C_2 \|w\|_{B^*_{\frac{p_0}{(1-\alpha)p}}}\right)$, *where*

$$\overline{\varphi}(x) = \begin{cases} \varphi\left(x^{\frac{p_0}{p}}\right), & 0<\alpha<1 \text{ and } 0<p\leqslant 1, \\ \varphi\left(x^{\frac{p_0(p+1)}{p}}\right), & 0<\alpha<1 \text{ and } p>1, \\ \varphi(x), & \alpha = 1, \end{cases} \tag{4.2.10}$$

for every $x \geqslant 1$ (here we are letting $\frac{p_0}{(1-\alpha)p} = \infty$ for $\alpha = 1$).

Remark 4.2.7. *(i) We can not expect to get a bigger class of weights than $B^R_{\frac{p}{p_-}}$ since, for instance, the operator $M_{\frac{1}{p_-}}$ satisfies (4.2.6) for $p_0 = p_-$, $\alpha = 1$ and $\beta = 0$ (see (2.2.8)) while $M_{\frac{1}{p_-}} : \Lambda^{p,1}(w) \to \Lambda^{p,\infty}(w)$ holds if and only if $w \in B^R_{\frac{p}{p_-}}$ (see (2.2.14)).*

(ii) Since $p_- < p_+$, we obtain that by Proposition 2.2.33, the class $B^R_{\frac{p}{p_-}} \cap B^*_{\frac{p_+}{p}}$ is nonempty. Further, if we take $w = 1$, then $w \in B^R_{\frac{p}{p_-}} \cap B^*_{\frac{p_+}{p}}$ whenever $p_- \leqslant p < p_+$, so as in Theorem 3.4.1, we are not able to extrapolate till $p = p_+$.

Nevertheless, if we assume a similar hypothesis as in Theorem 3.4.3, then we get the next result:

Theorem 4.2.8. *Let T be an operator satisfying that for some $1 \leqslant p_0 < \infty$ and $0 < \alpha < 1$,*

$$T : L^{p_0,1}((M\chi_F)^\alpha) \to L^{p_0,\infty}((M\chi_F)^\alpha), \qquad \forall F \subseteq \mathbb{R}^n,$$

with constant less than or equal to $C_{n,p_0,\alpha}$. Then, for every $0 < p < \infty$,

$$T : \Lambda^{p,1}(w) \to \Lambda^{p,\infty}(w), \qquad \forall w \in B^R_{\frac{p}{p_0}} \cap B^{*R}_{\frac{p_0}{(1-\alpha)p}},$$

with constant less than or equal to $\tilde{C}_{n,p_0,\alpha} \|w\|_{B^R_{\frac{p}{p_0}}}^{\frac{1}{p_0}} \log\left(1 + \|w\|_{B^{*R}_{\frac{p_0}{(1-\alpha)p}}}\right) \|w\|_{B^{*R}_{\frac{p_0}{(1-\alpha)p}}}^{\frac{1-\alpha}{p_0}}$.

Proof. The proof is analogous to the one for Theorem 4.2.5 (since in this case p_0 and p_- coincide) but with the following modification: in (4.2.9), instead of using the weight $(R\chi_F)^{\alpha(p_-)}$ we consider $(M\chi_F)^\alpha$ which, by means of Proposition 2.2.31, it also satisfies property (3) of R (although with a different constant). □

Remark 4.2.9. *If we take $w = 1$, then $w \in B^R_{\frac{p}{p_-}} \cap B^{*R}_{\frac{p_+}{p}}$ whenever $p_- \leqslant p \leqslant p_+$, and in this case we are able to extrapolate till the endpoint p_+ so as we did in Theorem 3.4.3.*

Finally, as a consequence of Theorem 4.2.5, we have the next result.

Corollary 4.2.10. *Let T be a sublinear operator satisfying (4.2.6). Given $0 < p < \infty$ and $0 < r \leqslant \infty$, then*

$$T : \Lambda^{p,r}(w) \to \Lambda^{p,r}(w), \qquad \forall w \in B_{\frac{p}{p_-}} \cap B^*_{\frac{p_+}{p}}.$$

Proof. Observe that if $w \in B_{\frac{p}{p_-}} \cap B^*_{\frac{p_+}{p}}$, there exists some $\varepsilon > 0$ such that $w \in B^R_{\frac{p-\varepsilon}{p_-}} \cap B^*_{\frac{p_+}{p-\varepsilon}}$ (see (2.2.9)) and $w \in B^R_{\frac{p+\varepsilon}{p_-}} \cap B^*_{\frac{p_+}{p+\varepsilon}}$ (see (2.2.31)). Hence, by virtue of Theorem 4.2.5, for some $0 < q \leqslant 1$,

$$T : \Lambda^{p-\varepsilon,q}(w) \to \Lambda^{p-\varepsilon,\infty}(w) \qquad \text{and} \qquad T : \Lambda^{p+\varepsilon,q}(w) \to \Lambda^{p+\varepsilon,\infty}(w),$$

and the result follows by interpolation on classical Lorentz spaces [58, Theorem 2.6.5]. □

Remark 4.2.11. *Recall that $A_{p_0;(\alpha,\beta)}$ can be written as $A_{\frac{p_0}{p_-}} \cap RH_{\left(\frac{p_+}{p_0}\right)'}$ (see Remark 2.3.5) so we have found interesting that this intersection class of weights keep some symmetry with the class of weights $B_{\frac{p}{p_-}} \cap B^*_{\frac{p_+}{p}}$.*

4.3 Weighted restricted weak-type estimates by pointwise estimates on $\Lambda^p(w)$

For a given sublinear operator T, there are many results involving pointwise estimates on the decreasing rearrangement of Tf from which it can be obtained boundedness results on r.i. spaces (see [7, 18, 23, 36, 43, 48, 49, 92]).

Indeed, in Section 4.3.3 we will see that the fact that a sublinear operator T satisfies weighted restricted weak-type estimates on classical Lorentz spaces as in Theorem 4.2.2, where in this case we will need to assume that φ is an admissible function (see Section 4.3.1), is equivalent to a pointwise estimate on $(Tf)^*$ by a Calderón admissible type operator (see Section 4.3.2). Further, in Section 4.3.4 we will show that a similar result holds in the limited setting.

In particular, we will obtain an extrapolation result between weighted restricted weak-type estimates on classical Lorentz spaces. For instance, results of this type but for weighted strong-type estimates have been studied in [47].

4.3.1 Admissible functions

There are many interesting operators in Harmonic Analysis satisfying that for some $p_0 \geqslant 1$,

$$T : L^{p_0,1}(v) \to L^{p_0,\infty}(v), \qquad \forall v \in A^R_{p_0},$$

with constant less than or equal to $C \|v\|^k_{A^R_{p_0}}$, $k \in \mathbb{N}$. However, on some occasions the behaviour of the constant has been improved from, let us say, $\|v\|^{1+\varepsilon}_{A^R_{p_0}}$ for some $\varepsilon > 0$, to an expression of the form $\varphi(\|v\|_{A^R_{p_0}})$ where φ is not a power function. This is, for example, the case when T is a Calderón-Zygmund operator, where for $p_0 = 1$ the best function φ known up to now is $\varphi(t) = t(1 + \log^+ t)$ (see [136]).

In order to cover this important class of operators we will introduce the concept of admissible function.

Definition 4.3.1. A function $\varphi : [1,\infty] \to [1,\infty]$ is called *admissible* if satisfies that $\varphi(1) = 1$ and that there exist some $\gamma_0, \gamma_1 > 0$ such that

$$\frac{\gamma_0}{x} \leqslant \frac{\varphi'(x)}{\varphi(x)} \leqslant \frac{\gamma_1}{x}, \qquad \forall x \geqslant 1. \tag{4.3.1}$$

Observe that (4.3.1) implies that for every $x \geqslant 1$,

(i) $\varphi(x)$ is increasing,

(ii) $\max\left\{\dfrac{\varphi'(x)}{\gamma_1}, 1\right\} \leqslant \varphi(x)$,

(iii) $x^{\gamma_0} \leqslant \varphi(x) \leqslant x^{\gamma_1}$.

Besides, since for every $x, y \geqslant 1$,

$$\log\varphi(xy) = \int_1^x (\log\varphi)'(s)\,ds + \int_x^{xy} (\log\varphi)'(s)ds \leqslant \log\varphi(x) + \gamma_1 \log y,$$

it also holds that

$$\varphi(xy) \leqslant y^{\gamma_1}\varphi(x), \qquad \forall x, y \geqslant 1. \tag{4.3.2}$$

Examples 4.3.2. (1) If φ_0 and φ_1 are admissible functions, then $\varphi = \varphi_0\varphi_1$ is an admissible function. Further, $\varphi = \varphi_0 \circ \varphi_1$ (i.e., φ is the composition of φ_0 and φ_1) is also an admissible function.

(2) Given $\gamma_0 > 0$ and $\gamma_1 \geqslant 0$, the function

$$\varphi(x) = x^{\gamma_0}(1+\log x)^{\gamma_1}, \qquad x \geqslant 1,$$

is admissible. In particular, given $k \in \mathbb{N}$, if we define for every $x \geqslant 1$,

$$\log_{(k)} x = \begin{cases} 1 + \log x, & \text{if } k = 1, \\ 1 + \log\big(\log_{(k-1)} x\big), & \text{if } k > 1, \end{cases}$$

then, for $\gamma_0 > 0$ and $\gamma_1, \dots, \gamma_k \geqslant 0$, the function

$$\varphi(x) = x^{\gamma_0}\prod_{i=1}^{k}\big(\log_{(i)} x\big)^{\gamma_i}, \qquad x \geqslant 1,$$

is also admissible.

The next lemmas are simple computations for admissible functions which shall be fundamental later on.

Lemma 4.3.3. *Given $0 < p \leqslant \infty$. If $r \geqslant 1$ then*

$$\begin{cases} \displaystyle\int_1^r \varphi\big(1+\log s\big)\frac{ds}{s^{1-\frac{1}{p}}} \approx \varphi\,(1+\log r)\, r^{\frac{1}{p}} - 1, & p < \infty, \\ \displaystyle\int_1^r \big(1+\log s\big)^{-1}\varphi\big(1+\log s\big)\frac{ds}{s} \approx \varphi\,(1+\log r) - 1, & p = \infty. \end{cases}$$

Proof. Assume first that $p = \infty$. Hence, taking $x = 1 + \log s$ and using that $\varphi'(x) \approx \dfrac{\varphi(x)}{x}$, we get that

$$\int_1^{1+\log r} \frac{\varphi(x)}{x}\, dx \approx [\varphi(x)]_1^{1+\log r} = \varphi\big(1+\log r\big) - 1.$$

Now suppose that $p < \infty$. Then, taking again $x = 1 + \log s$ and integrating by parts, we deduce that

$$\int_1^{1+\log r} \varphi(x) e^{\frac{x-1}{p}}\, dx = p\left(\varphi(1+\log r) r^{\frac{1}{p}} - 1\right) - p \int_1^{1+\log r} \varphi'(x) e^{\frac{x-1}{p}}\, dx,$$

so that

$$\int_1^{1+\log r} \left[\varphi(x) + p\varphi'(x)\right] e^{\frac{x-1}{p}}\, dx = p\left(\varphi(1+\log r) r^{\frac{1}{p}} - 1\right).$$

Finally, the result plainly follows from the fact that $\varphi(x) + p\varphi'(x) \approx \varphi(x)$ for every $x \geqslant 1$. □

Lemma 4.3.4. *Given $1 < p \leqslant \infty$. There exists some $\lambda > 1$ depending only on φ (and on p when that is finite) such that for every $t > 0$ and $r \geqslant \lambda t$,*

$$\begin{cases} \displaystyle\int_r^\infty \varphi\left(1+\log\frac{s}{t}\right) \frac{ds}{s^{2-\frac{1}{p}}} \approx \varphi\left(1+\log\frac{r}{t}\right)\frac{1}{r^{1-\frac{1}{p}}}, & p < \infty, \\ \displaystyle\int_r^\infty \left(1+\log\frac{s}{t}\right)^{-1} \varphi\left(1+\log\frac{s}{t}\right)\frac{ds}{s^2} \approx \frac{\varphi\left(1+\log\frac{r}{t}\right)}{\left(1+\log\frac{r}{t}\right)}\frac{1}{r}, & p = \infty. \end{cases}$$

Proof. Consider the function

$$g_p(s) = \begin{cases} -\varphi\left(1+\log\dfrac{s}{t}\right)\dfrac{1}{s^{1-\frac{1}{p}}}, & p < \infty, \\ -\left(1+\log\dfrac{s}{t}\right)^{-1}\varphi\left(1+\log\dfrac{s}{t}\right)\dfrac{1}{s}, & p = \infty. \end{cases}$$

Then, straightforward computations show that there exists some $\lambda > 1$ depending only on φ (and on p when that is finite) such that for every $s \geqslant \lambda t$,

$$g_p'(s) \approx \begin{cases} \varphi\left(1+\log\dfrac{s}{t}\right)\dfrac{1}{s^{2-\frac{1}{p}}}, & p < \infty, \\ \left(1+\log\dfrac{s}{t}\right)^{-1}\varphi\left(1+\log\dfrac{s}{t}\right)\dfrac{1}{s^2}, & p = \infty. \end{cases}$$

Thus, since $\lim_{s\to\infty} g_p(s) = 0$, the result follows for every $r \geqslant \lambda t$. □

Lemma 4.3.5. *Given $x \in \mathbb{R}$ and $0 < \mu \leqslant 1$. Then*

$$\inf_{y\in(0,\mu]} \varphi(y^{-1}) e^{yx} \leqslant \begin{cases} \mu^{-\gamma_1} e^{\mu x}, & \textit{if } x \leqslant 0, \\ \mu^{-\gamma_1} e^{\mu} \varphi\left(1+x\right), & \textit{if } x > 0. \end{cases}$$

Proof. If $x \leqslant 0$, the infimum is attained at $y = \mu$. On the other hand, if $x > 0$, we take $y = \mu/(1+x)$. Finally, the desired result follows by (4.3.2). $\square$

Lemma 4.3.6. *For every $y \geqslant 1$,*

$$\sup_{x\in[1,\infty)} \varphi(x)e^{-\frac{x}{y}} \leqslant \max\left\{1, \left(\frac{\gamma_1}{e}\right)^{\gamma_1}\right\} \varphi(y).$$

Proof. First, if $x \leqslant y$ then $\varphi(x)e^{-x/y} \leqslant \varphi(y)$. Otherwise, by means of (4.3.2),

$$\varphi(x)e^{-\frac{x}{y}} \leqslant \left(\frac{x}{y}\right)^{\gamma_1} e^{-\frac{x}{y}}\varphi(y) \leqslant \max\left\{1, \left(\frac{\gamma_1}{e}\right)^{\gamma_1}\right\} \varphi(y).$$

$\square$

4.3.2 Calderón admissible type operators

Given $1 \leqslant p_1, p_2 \leqslant \infty$, $1 \leqslant q_1 \leqslant p_1$, $q_1 < \infty$, and let φ be an admissible function. We define for every nonnegative measurable function f and every $t > 0$,

$$P_{p_1,q_1} f(t) = \left(\frac{1}{t^{\frac{1}{p_1}}}\int_0^t f(s)^{\frac{1}{q_1}} \frac{ds}{s^{1-\frac{1}{p_1}}}\right)^{q_1},$$

$$Q_{p_2,\varphi} f(t) = \begin{cases} \dfrac{1}{t^{\frac{1}{p_2}}}\displaystyle\int_t^\infty \varphi\left(1+\log\frac{s}{t}\right) f(s) \frac{ds}{s^{1-\frac{1}{p_2}}}, & p_2 < \infty, \\ \displaystyle\int_t^\infty \left(1+\log\frac{s}{t}\right)^{-1} \varphi\left(1+\log\frac{s}{t}\right) f(s)\frac{ds}{s}, & p_2 = \infty, \end{cases}$$

with $\frac{1}{p_1} = 0$ if $p_1 = \infty$. Then, the *Calderón admissible type operators* are defined as

$$S_{p_1,q_1,p_2,\varphi} f(t) = P_{p_1,q_1} f(t) + Q_{p_2,\varphi} f(t), \qquad t > 0. \tag{4.3.3}$$

In particular, if $p_1 = q_1 = 1$, $p_2 = \infty$, and $\varphi(x) = x$, we recover the Calderón operator [23]

$$S_{1,1,\infty,x} f(t) := Sf(t) = Pf(t) + Qf(t), \qquad t > 0,$$

where P and Q are respectively the Hardy operator and its adjoint (see (2.2.22) and (2.2.23)). Besides, if $p_1 = \frac{1}{\delta}$, $q_1 = 1$ and $p_2 = \frac{1}{1-\delta}$, $0 < \delta < 1$, we recover the generalized Hardy operator $P_{\frac{1}{\delta}}$ and its adjoint $Q_{\frac{1}{1-\delta}}$ seen in (2.2.28) and (2.2.29).

We observe that, in general,

$$S_{p_1,q_1,p_2,\varphi} f(t) = \left(\int_0^1 f(st)^{\frac{1}{q_1}} \frac{ds}{s^{1-\frac{1}{p_1}}}\right)^{q_1} + \begin{cases} \displaystyle\int_1^\infty \varphi(1+\log s) f(st) \frac{ds}{s^{1-\frac{1}{p_2}}}, & p_2 < \infty, \\ \displaystyle\int_1^\infty \frac{\varphi(1+\log s)}{1+\log s} f(st)\frac{ds}{s}, & p_2 = \infty. \end{cases} \tag{4.3.4}$$

Lemma 4.3.7. *Let $1 \leqslant p_1, p_2 \leqslant \infty$ and $0 < q \leqslant 1$. For every positive simple function f with bounded support,*

$$S_{p_1,1,p_2,\varphi}(f^*)^{**}(t) = S_{p_1,1,p_2,\varphi}(f^{**})(t) \leqslant 2^{\frac{1}{q}-1} S_{\frac{p_1}{q},\frac{1}{q},p_2,\varphi}(f^{**})(t), \qquad \forall t > 0.$$

Proof. By (4.3.4), clearly $S_{p_1,1,p_2,\varphi}(f^*)$ is a decreasing function. Then, it holds that

$$S_{p_1,1,p_2,\varphi}(f^*)^{**}(t) = P\left(S_{p_1,1,p_2,\varphi}(f^*)\right)(t), \qquad \forall t > 0,$$

so that by Fubini's theorem we deduce that

$$P\left(S_{p_1,1,p_2,\varphi}(f^*)\right)(t) = S_{p_1,1,p_2,\varphi}(f^{**})(t), \qquad \forall t > 0.$$

Now consider $a_1, \dots, a_m > 0$ and the sets with finite measure $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_m \subseteq \mathbb{R}^n$, so that

$$f = \sum_{j=1}^m a_j \chi_{F_j} \qquad \text{and (then)} \qquad f^* = \sum_{j=1}^m a_j (\chi_{F_j})^* = \sum_{j=1}^m a_j \chi_{[0,|F_j|)}. \tag{4.3.5}$$

Hence, since for every simple function χ_F ($F \subseteq \mathbb{R}^n$) with finite measure we have

$$S_{p_1,1,p_2,\varphi}((\chi_F)^{**})(t) \leqslant 2^{\frac{1}{q}-1} S_{\frac{p_1}{q},\frac{1}{q},p_2,\varphi}((\chi_F)^{**})(t), \qquad \forall t > 0,$$

and since by means of the reverse Minkowski's inequality ($0 < q \leqslant 1$) we also have that for every nonnegative measurable functions f_1, f_2,

$$S_{\frac{p_1}{q},\frac{1}{q},p_2,\varphi}(f_1 + f_2)(t) \geqslant S_{\frac{p_1}{q},\frac{1}{q},p_2,\varphi} f_1(t) + S_{\frac{p_1}{q},\frac{1}{q},p_2,\varphi} f_2(t), \qquad \forall t > 0,$$

we obtain that for every $t > 0$,

$$\begin{aligned} S_{p_1,1,p_2,\varphi}(f^{**})(t) &= \sum_{j=1}^m a_j S_{p_1,1,p_2,\varphi}((\chi_{F_j})^{**})(t) \leqslant 2^{\frac{1}{q}-1} \sum_{j=1}^m a_j S_{\frac{p_1}{q},\frac{1}{q},p_2,\varphi}((\chi_{F_j})^{**})(t) \\ &\leqslant 2^{\frac{1}{q}-1} S_{\frac{p_1}{q},\frac{1}{q},p_2,\varphi}\left(\sum_{j=1}^m a_j (\chi_{F_j})^{**}\right)(t) = 2^{\frac{1}{q}-1} S_{\frac{p_1}{q},\frac{1}{q},p_2,\varphi}(f^{**})(t). \end{aligned}$$

□

4.3.3 Weighted restricted weak-type estimates and decreasing rearrangement estimates

Let us first see an easy proposition which helps to motivate what follows:

Proposition 4.3.8. *Let T be an operator satisfying that*

$$T : \Lambda^{1,q}(w) \to \Lambda^{1,\infty}(w), \qquad \forall w \in B_1^R, \tag{4.3.6}$$

with constant less than or equal to $\varphi(\|w\|_{B_1^R})$. Then, for every locally integrable function f and for every $t > 0$,

$$(Tf)^*(t) \leqslant \varphi(1) P_{\frac{1}{q},\frac{1}{q}}(f^*)(t) = \varphi(1) \left(\frac{1}{t^q} \int_0^t f^*(s)^q \, \frac{ds}{s^{1-q}}\right)^{\frac{1}{q}}. \tag{4.3.7}$$

Proof. The hypothesis implies that, for every $w \in B_1^R$,

$$(Tf)^*(t)W(t) \leqslant \varphi(\|w\|_{B_1^R}) \left(\int_0^\infty s^{q-1} W(\lambda_f(s))^q \, ds\right)^{\frac{1}{q}}, \qquad \forall t > 0.$$

Hence, since

$$W(\lambda_f(s)) = W(\lambda_{f^*}(s)) = \int_0^{\lambda_{f^*}(s)} w(r)\, dr = \int_{\{f^*(r)>s\}} w(r)\, dr = \lambda_{f^*}^w(s), \qquad \forall s > 0,$$

in particular, taking $w = \chi_{[0,t]} \in B_1^R$ (so that $\|w\|_{B_1^R} = 1$) we get $\lambda_{f^*}^w = \lambda_{f^*\chi_{[0,t]}}$ and

$$(Tf)^*(t) \leqslant \frac{\varphi(1)}{t} \left(\int_0^\infty s^{q-1} \lambda_{f^*\chi_{[0,t]}}(s)^q \, ds\right)^{\frac{1}{q}} = \varphi(1) \left(\frac{1}{t^q} \int_0^t f^*(s)^q \, \frac{ds}{s^{1-q}}\right)^{\frac{1}{q}}, \qquad \forall t > 0,$$

where in the last equality we have used Fubini's theorem. $\square$

Remark 4.3.9. *(i) If an operator T satisfies (4.3.7), we have that (4.3.6) holds with constant less than or equal to $C\varphi(1)\,\|w\|_{B_1^R}$ and hence we can conclude that, under the hypothesis of the previous theorem,*

$$\|T\|_{\Lambda^{1,q}(w)\to\Lambda^{1,\infty}(w)} \leqslant \tilde{C} \min\left(\|w\|_{B_1^R}, \varphi\left(\|w\|_{B_1^R}\right)\right),$$

for some positive constant $\tilde{C}$ independent of w.

(ii) We also observe that the operator T plays no role and hence the same can be formulated for couples of functions (f, g) in the following sense:

$$\|g\|_{\Lambda^{1,\infty}(w)} \leqslant \varphi\left(\|w\|_{B_1^R}\right) \|f\|_{\Lambda^{1,q}(w)}, \qquad \forall w \in B_1^R,$$

implies that

$$g^*(t) \leqslant \varphi(1) \left(\frac{1}{t^q} \int_0^t f^*(s)^q \, \frac{ds}{s^{1-q}}\right)^{\frac{1}{q}}, \qquad \forall t > 0.$$

Taking into account Theorem 4.2.2, our next goal is to include the hypothesis $w \in B_\infty^*$ in its statement.

Theorem 4.3.10. *Given $0 < q \leqslant 1$. Let T be a sublinear operator and let φ be an admissible function (see Definition 4.3.1). Then,*

$$T : \Lambda^{1,q}(w) \to \Lambda^{1,\infty}(w), \qquad \forall w B_1^R \cap B_\infty^*, \tag{4.3.8}$$

with constant less than or equal to $C\|w\|_{B_1^R}\,\varphi(\|w\|_{B_\infty^*})$ if and only if for every locally integrable function f and for every $t > 0$,

$$(Tf)^*(t) \lesssim S_{\frac{1}{q},\frac{1}{q},\infty,\varphi}(f^*)(t) = \left(\frac{1}{t^q} \int_0^t f^*(s)^q \, \frac{ds}{s^{1-q}}\right)^{\frac{1}{q}} + \int_t^\infty \frac{\varphi\left(1+\log\frac{s}{t}\right)}{1+\log\frac{s}{t}} f^*(s)\, \frac{ds}{s}. \tag{4.3.9}$$

Proof. We will first prove that if $f = \chi_F$ with $F \subseteq \mathbb{R}^n$ a measurable set of finite measure, then

$$(T\chi_F)^*(t) \lesssim S_{1,1,\infty,\varphi}((\chi_F)^*)(t), \qquad \forall t > 0. \tag{4.3.10}$$

Indeed, using our hypothesis with $w(t) = t^{\gamma-1}$, $0 < \gamma \leqslant 1$, due to (2.2.40) and (2.2.41), we get that

$$(T\chi_F)^*(t) \lesssim \varphi(\gamma^{-1}) \left(\frac{|F|}{t}\right)^{\gamma}, \qquad \forall t > 0. \tag{4.3.11}$$

Taking the infimum in $\gamma \in (0,1]$, and using Lemma 4.3.5 we obtain that

$$(T\chi_F)^*(t) \lesssim \left(\frac{|F|}{t}\right) \chi_{(|F|,\infty)}(t) + \varphi\left(1 + \log \frac{|F|}{t}\right) \chi_{(0,|F|)}(t), \qquad \forall t > 0,$$

and by Lemma 4.3.3, for every $t > 0$,

$$\begin{aligned}(T\chi_F)^*(t) &\lesssim \frac{1}{t}\int_0^t (\chi_F)^*(s)\,ds + \int_t^\infty \left(1 + \log\frac{s}{t}\right)^{-1} \varphi\left(1 + \log \frac{s}{t}\right) (\chi_F)^*(s)\,\frac{ds}{s} \\ &= S_{1,1,\infty,\varphi}((\chi_F)^*)(t),\end{aligned}$$

so that (4.3.10) holds.

Now, the proof of (4.3.9) for an arbitrary locally integrable function f follows the same lines as the proof of [23, Ch. 3 - Theorem 4.7]. For the sake of completeness, we include the computations for f being a positive simple function with bounded support adapted to our case. Consider f as in (4.3.5). Using what we have already proved for characteristic functions, together with the sublinearity of T, we get that for every $t > 0$,

$$\begin{aligned}(Tf)^{**}(t) &\leqslant \sum_{j=1}^m a_j \left(T(\chi_{F_j})\right)^{**}(t) \lesssim \sum_{j=1}^m a_j \left(S_{1,1,\infty,\varphi}((\chi_F)^*)\right)^{**}(t) \\ &= \left(S_{1,1,\infty,\varphi}\left(\sum_{j=1}^m a_j \chi_{[0,|F_j|)}\right)\right)^{**}(t) = S_{1,1,\infty,\varphi}(f^*)^{**}(t).\end{aligned} \tag{4.3.12}$$

Further, since $S_{1,1,\infty,\varphi}(f^*)^{**} \lesssim S_{\frac{1}{q},\frac{1}{q},\infty,\varphi}(f^{**})$ (see Lemma 4.3.7) we finally obtain that

$$(Tf)^{**}(t) \lesssim S_{\frac{1}{q},\frac{1}{q},\infty,\varphi}(f^{**})(t), \qquad \forall t > 0. \tag{4.3.13}$$

Fix $t > 0$ and consider the set $E = \{x \in \mathbb{R}^n : f(x) > f^*(t)\}$. Then, define for $x \in \mathbb{R}^n$,

$$g(x) = (f(x) - f^*(t))\chi_E(x) \qquad \text{and} \qquad h(x) = f^*(t)\chi_E(x) + f(x)\chi_{E^c}(x), \tag{4.3.14}$$

so we have that

$$g^*(r) = (f^*(r) - f^*(t))^+ \qquad \text{and} \qquad h^*(r) = \min\{f^*(r), f^*(t)\}, \qquad \forall r > 0.$$

Since $w = 1$ belong to $B_1^R \cap B_\infty^*$, the corresponding unweighted weak-type inequality leads to

$$(Tg)^*\left(\frac{t}{2}\right) \lesssim \frac{1}{t}\left(\int_0^\infty g^*(s)^q \frac{ds}{s^{1-q}}\right)^{\frac{1}{q}} \leqslant \left(\frac{1}{t^q}\int_0^t f^*(s)^q \frac{ds}{s^{1-q}}\right)^{\frac{1}{q}} = P_{\frac{1}{q},\frac{1}{q}}(f^*)(t). \tag{4.3.15}$$

On the other hand, using (4.3.13) we get

$$(Th)^{**}(t) \lesssim S_{\frac{1}{q},\frac{1}{q},\infty,\varphi}(h^{**})(t) = P_{\frac{1}{q},\frac{1}{q}}(h^{**})(t) + Q_{\infty,\varphi}(h^{**})(t) \lesssim P_{\frac{1}{q},\frac{1}{q}}(f^*)(t) + Q_{\infty,\varphi}(h^{**})(t),$$

where the last estimate holds because $h^*(r) = f^*(t)$ for every $r \in [0,t]$ and $f^*(t) \lesssim P_{\frac{1}{q},\frac{1}{q}}(f^*)(t)$. Besides, for simplicity on the notation, we consider the auxiliary function $\widetilde{\varphi}(x) = \frac{\varphi(x)}{x}$, for $x \geqslant 1$. By Fubini's theorem,

$$\begin{aligned} Q_{\infty,\varphi}(h^{**})(t) = \int_t^\infty \widetilde{\varphi}\left(1+\log\frac{s}{t}\right) h^{**}(s)\,\frac{ds}{s} = f^*(t)\int_0^t \left(\int_t^\infty \widetilde{\varphi}\left(1+\log\frac{s}{t}\right)\frac{ds}{s^2}\right)dr \\ + \int_t^\infty \left(\int_r^\infty \widetilde{\varphi}\left(1+\log\frac{s}{t}\right)\frac{ds}{s^2}\right) f^*(r)\,dr = f^*(t)I_1 + I_2. \end{aligned} \tag{4.3.16}$$

On the one hand, the first integral does not depend on t. Indeed, by the change of variable $y = s/t$,

$$I_1 = \int_0^t \left(\int_t^\infty \widetilde{\varphi}\left(1+\log\frac{s}{t}\right)\frac{ds}{s^2}\right)dr = \left(\int_1^\infty \widetilde{\varphi}\left(1+\log y\right)\frac{dy}{y^2}\right) = C(\varphi) < \infty.$$

On the other hand, given any $\lambda > 1$ we observe that

$$\begin{aligned} I_2 = \int_t^\infty \left(\int_r^\infty \widetilde{\varphi}\left(1+\log\frac{s}{t}\right)\frac{ds}{s^2}\right) f^*(r)\,dr = \int_t^{\lambda t} \left(\int_r^\infty \widetilde{\varphi}\left(1+\log\frac{s}{t}\right)\frac{ds}{s^2}\right) f^*(r)\,dr \\ + \int_{\lambda t}^\infty \left(\int_r^\infty \widetilde{\varphi}\left(1+\log\frac{s}{t}\right)\frac{ds}{s^2}\right) f^*(r)\,dr. \end{aligned}$$

The first part can be handled as we did for I_1. Indeed, using that f^* is decreasing we get

$$\int_t^{\lambda t} \left(\int_r^\infty \widetilde{\varphi}\left(1+\log\frac{s}{t}\right)\frac{ds}{s^2}\right) f^*(r)\,dr \leqslant \frac{f^*(t)}{t}\int_t^{\lambda t}\left(\int_{\frac{r}{t}}^\infty \widetilde{\varphi}\left(1+\log y\right)\frac{dy}{y^2}\right)dr \leqslant C(\varphi)\lambda f^*(t).$$

For the second part, by Lemma 4.3.4 we know that there exists some $\lambda = \lambda(\varphi) > 1$ such that

$$\int_r^\infty \widetilde{\varphi}\left(1+\log\frac{s}{t}\right)\frac{ds}{s^2} \approx \frac{1}{r}\,\widetilde{\varphi}\left(1+\log\frac{r}{t}\right).$$

Therefore, with such λ,

$$\int_{\lambda t}^\infty \left(\int_r^\infty \widetilde{\varphi}\left(1+\log\frac{s}{t}\right)\frac{ds}{s^2}\right) f^*(r)\,dr \lesssim \int_{\lambda t}^\infty \widetilde{\varphi}\left(1+\log\frac{r}{t}\right) f^*(r)\,\frac{dr}{r}.$$

In conclusion, putting I_1 and I_2 together we obtain that

$$Q_{\infty,\varphi}(h^{**})(t) \lesssim f^*(t) + \int_t^\infty \widetilde{\varphi}\left(1+\log\frac{r}{t}\right) f^*(r)\,\frac{dr}{r} = f^*(t) + Q_{\infty,\varphi}(f^*)(t).$$

Thus, bringing all together, using again that $f^*(t) \lesssim P_{\frac{1}{q},\frac{1}{q}}(f^*)(t)$, and since $t > 0$ is arbitrary we obtain that

$$(Tf)^*(t) \leqslant (Tg)^*\left(\frac{t}{2}\right) + (Th)^{**}\left(\frac{t}{2}\right) \lesssim S_{\frac{1}{q},\frac{1}{q},\infty,\varphi}(f^*)(t), \qquad \forall t > 0. \tag{4.3.17}$$

Conversely, suppose that

$$(Tf)^*(t) \lesssim S_{\frac{1}{q},\frac{1}{q},\infty,\varphi}(f^*)(t) = P_{\frac{1}{q},\frac{1}{q}}(f^*)(t) + Q_{\infty,\varphi}(f^*)(t), \qquad \forall t>0,$$

and take $w \in B_1^R \cap B_\infty^*$. Then, since $\Lambda^{1,q}(w) \subseteq \Lambda^1(w)$ continuously, if we set $\tilde{w} = qW^{q-1}w$ we have that

$$\|T\|_{\Lambda^{1,q}(w)\to\Lambda^{1,\infty}(w)} \leqslant \left\|P_{\frac{1}{q},1}\right\|^{\frac{1}{q}}_{L^1_{\text{dec}}(\tilde{w})\to L^{1,\infty}(\tilde{w})} + \|Q_{\infty,\varphi}\|_{L^1_{\text{dec}}(w)\to L^{1,\infty}(w)}. \tag{4.3.18}$$

The operators $P_{\frac{1}{q},1}$ and $Q_{\infty,\varphi}$ have the form

$$P_{\frac{1}{q},1}f(t) = \int_0^\infty k_1(t,s)f(s)ds \qquad \text{and} \qquad Q_{\infty,\varphi}f(t) = \int_0^\infty k_2(t,s)f(s)ds, \qquad \forall t>0,$$

where the kernels are

$$k_1(t,s) = \frac{1}{t^q}\chi_{[0,t)}(s)\frac{1}{s^{1-q}} \qquad \text{and} \qquad k_2(t,s) = \frac{1}{s}\left(1+\log\frac{s}{t}\right)^{-1}\varphi\left(1+\log\frac{s}{t}\right)\chi_{[t,\infty)}(s),$$

for every $t,s>0$. So, using [54, Theorem 3.3], the norm in (4.3.18) can be estimated by $A_{k_1}^{\frac{1}{q}} + A_{k_2}$, where

$$A_{k_i} = \sup_{t>0}\left(\sup_{r>0}\left(\int_0^r k_i(t,s)ds\right)W_i(r)^{-1}\right)W_i(t), \qquad i=1,2,$$

with $W_1 = \tilde{W}$ and $W_2 = W$.

Now note that if $0<r<t$ then we have

$$\int_0^r k_1(t,s)\,ds = \frac{1}{q}\left(\frac{r}{t}\right)^q \qquad \text{and} \qquad \int_0^r k_2(t,s)\,ds = 0,$$

while if $r>t$, by Lemma 4.3.3, we obtain

$$\int_0^r k_1(t,s)\,ds = \frac{1}{q} \qquad \text{and} \qquad \int_0^r k_2(t,s)\,ds \lesssim \varphi\left(1+\log\frac{r}{t}\right).$$

As a consequence we have that

$$A_{k_1} \approx \left[\sup_{t>0}\left(\sup_{0<r<t}\left(\frac{r}{t}\right)W(r)^{-1}\right)W(t)\right]^q \leqslant \|w\|^q_{B_1^R},$$

and

$$A_{k_2} \lesssim \sup_{t>0}\left(\sup_{r>t}\varphi\left(1+\log\frac{r}{t}\right)\frac{W(t)}{W(r)}\right).$$

Further, if $\mu = t/r < 1$, then by Lemma 2.2.21,

$$\frac{W(t)}{W(r)} = \frac{W(\mu r)}{W(r)} \leqslant e\mu^{1/(e\|w\|_{B_\infty^*})},$$

so that,

$$A_{k_2} \lesssim \sup_{\mu<1} \left(\varphi \left(1 + \log \frac{1}{\mu} \right) \mu^{1/(e\|w\|_{B_\infty^*})} \right) \lesssim \sup_{x>1} \left(\varphi(x) e^{-x/(e\|w\|_{B_\infty^*})} \right).$$

Finally, by Lemma 4.3.6 and the estimate (4.3.2), we obtain that

$$A_{k_2} \lesssim \varphi(e\,\|w\|_{B_\infty^*}) \lesssim \varphi(\|w\|_{B_\infty^*}),$$

and, since $\|w\|_{B_1^R}, \varphi(\|w\|_{B_\infty^*}) \geqslant 1$, we arrive to the desired result

$$\|T\|_{\Lambda^{1,q}(w)\to\Lambda^{1,\infty}(w)} \lesssim \|w\|_{B_1^R}\, \varphi(\|w\|_{B_\infty^*}).$$

□

Remark 4.3.11. *Observe that the sublinearity of T is just used in (4.3.12), so if we have (4.3.9) for T being a general operator, we will also obtain the boundedness in (4.3.8).*

As a first consequence of Theorem 4.3.10, we have the following result.

Corollary 4.3.12. *Given $0 < q \leqslant 1$. Let T be a sublinear operator and let φ be an admissible function. Then, for some $0 < p < \infty$,*

$$T : \Lambda^{p,q}(w) \to \Lambda^{p,\infty}(w), \qquad \forall w \in B_p^R \cap B_\infty^*, \tag{4.3.19}$$

with constant less than or equal to $C\,\|w\|_{B_p^R}\, \varphi(\|w\|_{B_\infty^*})$ if and only if for every locally integrable function f and for every $t > 0$, (4.3.9) holds.

Proof. Recall that $\Lambda^{p,q}(w) = \Lambda^{1,q}(\tilde{w})$ and $\Lambda^{p,\infty}(w) = \Lambda^{1,\infty}((1/p)\tilde{w})$, for $\tilde{w} = W^{1/p-1}w$ (see (2.1.2)). Thus, since $\|w\|_{B_p^R} = \|\tilde{w}\|_{B_1^R}$ and $\|w\|_{B_\infty^*} \approx \|\tilde{w}\|_{B_\infty^*}$ (see Lemma 2.2.19) the result follows by means of Theorem 4.3.10. □

Remark 4.3.13. *Observe that by virtue of Corollary 4.3.12, if we have for some $0 < p < \infty$ that (4.3.19) holds then, indeed, it does also for every $p > 0$.*

Besides, from Theorem 4.2.2 and Corollary 4.3.12, we deduce the next result.

Corollary 4.3.14. *Let T be a sublinear operator satisfying that for some $1 \leqslant p_0 < \infty$,*

$$T : L^{p_0,1}(v) \to L^{p_0,\infty}(v), \qquad \forall v \in \hat{A}_{p_0},$$

with constant less than or equal to $\varphi(\|v\|_{\hat{A}_{p_0}})$, where φ is an admissible function.

(i) If $p_0 = 1$, for every locally integrable function f,

$$(Tf)^*(t) \lesssim S_{1,1,\infty,\varphi}(f^*)(t), \qquad \forall t > 0.$$

(ii) If $p_0 > 1$, for every $0 < q < 1$ and every locally integrable function f,

$$(Tf)^*(t) \lesssim \frac{1}{1-q} S_{\frac{1}{q},\frac{1}{q},\infty,\varphi_q}(f^*)(t), \qquad \forall t > 0,$$

with $\varphi_q(x) = \max\left(1, x^{q-\frac{1}{p_0}}\right) \varphi\left(x^{\frac{1}{p_0}}\right)$, $x \geqslant 1$.

Finally, to end this section, we want to point out that in order to prove that (4.3.8) implies (4.3.9) we just have used power weights (see (4.3.11) and (4.3.15)). Hence, actually we only need to assume that $T : \Lambda^{1,q}(w_1) \to \Lambda^{1,\infty}(w_1)$ and

$$\|T\chi_E\|_{\Lambda^{1,\infty}(w_\gamma)} \lesssim \|w_\gamma\|_{B_1^R}\, \varphi\left(\|w_\gamma\|_{B_\infty^*}\right) W_\gamma(|E|), \qquad \forall E \subseteq \mathbb{R}^n,$$

where $w_\gamma(t) = t^{\gamma-1}$, for $0 < \gamma \leqslant 1$. Further, we observe that for these weights,

$$\Lambda^{1,q}(w_1) = L^{1,q}(\mathbb{R}^n), \qquad \|\cdot\|_{\Lambda^{1,\infty}(w_\gamma)} = \frac{1}{\gamma}\|\cdot\|_{L^{\frac{1}{\gamma},\infty}(\mathbb{R}^n)}, \qquad W_\gamma(t) = \frac{t^\gamma}{\gamma},$$

and $C\,\|w_\gamma\|_{B_1^R}\,\varphi(\|w_\gamma\|_{B_\infty^*}) = C\varphi(\gamma^{-1})$. Thus, Theorem 4.3.10 can be written in the following equivalent way.

Theorem 4.3.15. *Given $0 < q \leqslant 1$. Let T be a sublinear operator and let φ be an admissible function. Then, $T : L^{1,q}(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)$ and for every $1 < p < \infty$,*

$$\|T\chi_E\|_{L^{p,\infty}(\mathbb{R}^n)} \leqslant C\varphi(p)\,|E|^{\frac{1}{p}}, \qquad \forall E \subseteq \mathbb{R}^n,$$

with C independent of p if and only if for every locally integrable function f and every $t > 0$, (4.3.9) holds.

Furthermore, as a consequence of Theorem 4.3.15, we have the following result.

Corollary 4.3.16. *Given $0 < q \leqslant 1$ and a sublinear operator T. If $T : L^{1,q}(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)$ and there exists $p_0 > 1$ such that for some admissible function φ and for every measurable set $E \subseteq \mathbb{R}^n$,*

$$\|T\chi_E\|_{L^{p_0,\infty}(v)} \leqslant \varphi\left(\|v\|_{A_{p_0}}\right) v(E)^{\frac{1}{p_0}}, \qquad \forall v \in A_{p_0}, \tag{4.3.20}$$

then, for every locally integrable function f and for every $t > 0$, (4.3.9) holds.

Proof. Setting $w = 1$, actually the proof of Theorem 4.2.2 can be adapted to consider (4.3.20) instead of (4.2.1) as hypothesis. In fact, taking in (4.2.2) the weight $v = (R'f)^{1-p_0} R\chi_F \in A_{p_0}$, where

$$R'f(x) = \sum_{k=0}^{\infty} \frac{M^k f(x)}{(2\,\|M\|_{L^{p,\infty}(\mathbb{R}^n)})^k}, \qquad x \in \mathbb{R}^n,$$

and keeping track of the constants involved, it can be seen that, for every $1 < p < \infty$,

$$\|T\chi_E\|_{L^{p,\infty}(\mathbb{R}^n)} \leqslant C\,\|M\|_{L^{p,\infty}(\mathbb{R}^n)}^{\frac{p_0-1}{p_0}}\,\varphi\left(\|M\|_{L^{p,\infty}(\mathbb{R}^n)}^{p_0-1}\,\|M\|_{L^{p',\infty}(\mathbb{R}^n)}\right)|E|^{\frac{1}{p}}, \qquad \forall E \subseteq \mathbb{R}^n,$$

with C independent of p.

Now, easy computations show that $\|M\|_{L^{p,\infty}(\mathbb{R}^n)\to L^{p,\infty}(\mathbb{R}^n)} \leqslant C_n p'$, so that, for every $p \geqslant 2$,

$$\|T\chi_E\|_{L^{p,\infty}(\mathbb{R}^n)} \lesssim \varphi(p)\,|E|^{\frac{1}{p}}, \qquad \forall E \subseteq \mathbb{R}^n. \tag{4.3.21}$$

Besides, from interpolation between the unweighted restricted weak-type $(1,1)$ and $(2,2)$ estimates of T for characteristic functions (see, for instance, [88, 174]) we obtain that (4.3.21) also holds for $1 < p < 2$, and the desired result follows directly from Theorem 4.3.15. □

4.3.4 Weighted limited restricted weak-type estimates and decreasing rearrangement estimates

The next step is to consider the nonempty subclass $B^R_{\frac{1}{p_1}} \cap B^*_{p_2} \subseteq B^R_1 \cap B^*_\infty$ (for $p_1 < p_2$) so that we obtain the following generalization of Theorem 4.3.10.

Theorem 4.3.17. *Given $0 < q \leqslant 1$ and $1 \leqslant p_1 < p_2 \leqslant \infty$. Let T be a sublinear operator and let φ be an admissible function (see Definition 4.3.1). If*

$$T : \Lambda^{1,q}(w) \to \Lambda^{1,\infty}(w), \qquad \forall w \in B^R_{\frac{1}{p_1}} \cap B^*_{p_2}, \tag{4.3.22}$$

with constant less than or equal to $C \|w\|^{\frac{1}{p_1}}_{B^R_{\frac{1}{p_1}}} \varphi\left(\|w\|_{B^*_{p_2}}\right)$ then, for every locally integrable function f and every $t > 0$,

$$\begin{aligned}(Tf)^*(t) &\lesssim S_{\frac{p_1}{q},\frac{1}{q},p_2,\varphi}(f^*)(t)\\ &= \left(\frac{1}{t^{\frac{q}{p_1}}}\int_0^t f^*(s)^q \frac{ds}{s^{1-\frac{q}{p_1}}}\right)^{\frac{1}{q}} + \begin{cases} \dfrac{1}{t^{\frac{1}{p_2}}}\displaystyle\int_t^\infty \varphi\left(1+\log\frac{s}{t}\right) f^*(s)\frac{ds}{s^{1-\frac{1}{p_2}}}, & p_2<\infty,\\ \displaystyle\int_t^\infty \frac{\varphi\left(1+\log\frac{s}{t}\right)}{\left(1+\log\frac{s}{t}\right)} f^*(s)\frac{ds}{s}, & p_2=\infty.\end{cases}\end{aligned} \tag{4.3.23}$$

Conversely, suppose that $(Tf)^*(t) \lesssim S_{\frac{p_1}{q},\frac{1}{q},p_2,\varphi}(f^*)(t)$, for every $t > 0$. Then

$$\|T\|_{\Lambda^{1,q}(w)\to\Lambda^{1,\infty}(w)} \lesssim \begin{cases} \|w\|^{\frac{1}{p_1}}_{B^R_{\frac{1}{p_1}}} \|w\|_{B^*_{p_2}} \varphi(\|w\|_{B^*_{p_2}}), & p_2<\infty,\\ \|w\|^{\frac{1}{p_1}}_{B^R_{\frac{1}{p_1}}} \varphi(\|w\|_{B^*_\infty}), & p_2=\infty.\end{cases}$$

Proof. First, assume that (4.3.22) holds. Note that by (2.2.40) and (2.2.41), $w(t) = t^{\gamma-1}$ belongs to $B^R_{\frac{1}{p_1}} \cap B^*_{p_2}$ for every $\gamma \in \left(\frac{1}{p_2}, \frac{1}{p_1}\right]$. Hence, using our hypothesis, we obtain that for every measurable set E,

$$(T\chi_E)^*(t) \lesssim \varphi\left(\left[\gamma - \frac{1}{p_2}\right]^{-1}\right)\left(\frac{|E|}{t}\right)^\gamma = \left[\varphi\left(\tilde{\gamma}^{-1}\right)\left(\frac{|E|}{t}\right)^{\tilde{\gamma}}\right]\left(\frac{|E|}{t}\right)^{\frac{1}{p_2}}, \qquad \forall t > 0, \tag{4.3.24}$$

with $\tilde{\gamma} = \gamma - \frac{1}{p_2}$. Hence, taking the infimum in $\tilde{\gamma} \in \left(0, \frac{1}{p_1} - \frac{1}{p_2}\right]$ and using Lemma 4.3.5 we get that

$$(T\chi_E)^*(t) \lesssim \left(\frac{|E|}{t}\right)^{\frac{1}{p_1}} \chi_{(|E|,\infty)}(t) + \varphi\left(1+\log\frac{|E|}{t}\right)\left(\frac{|E|}{t}\right)^{\frac{1}{p_2}} \chi_{(0,|E|)}(t), \qquad \forall t > 0,$$

so that by Lemma 4.3.3,

$$(T\chi_E)^*(t) \lesssim S_{p_1,1,p_2,\varphi}(\chi_E)^*(t), \qquad \forall t > 0.$$

The desired result for positive simple functions with bounded support follows the same lines as the proof of the necessity of Theorem 4.3.10 with few modifications. First of all, we consider a positive simple function like the one in (4.3.5). Hence, as in (4.3.12), using what we have already proved for characteristic functions together with the sublinearity of T and Lemma 4.3.7, we obtain that

$$(Tf)^{**}(t) \lesssim S_{\frac{p_1}{q},\frac{1}{q},p_2,\varphi}(f^{**})(t), \qquad \forall t > 0. \tag{4.3.25}$$

So fix $t > 0$ and take functions g and h from f defined as in (4.3.14). Since the weight $w(r) = r^{\frac{1}{p_1}-1}$ is in $B^R_{\frac{1}{p_1}} \cap B^*_{p_2}$, the corresponding weighted weak-type inequality leads to

$$\begin{aligned}(Tg)^*\left(\frac{t}{2}\right) &\lesssim \frac{1}{t^{\frac{1}{p_1}}}\left(\int_0^t (f^*(s) - f^*(t))^q \frac{ds}{s^{1-\frac{q}{p_1}}}\right)^{\frac{1}{q}} \\ &\leqslant \left(\frac{1}{t^{\frac{q}{p_1}}}\int_0^t f^*(s)^q \frac{ds}{s^{1-\frac{q}{p_1}}}\right)^{\frac{1}{q}} = P_{\frac{p_1}{q},\frac{1}{q}}(f^*)(t).\end{aligned} \tag{4.3.26}$$

On the other hand, using (4.3.25) for h instead of f we get

$$\begin{aligned}(Th)^{**}(t) &\lesssim S_{\frac{q}{p_1},\frac{1}{q},p_2,\varphi}(h^{**})(t) = P_{\frac{p_1}{q},\frac{1}{q}}(h^{**})(t) + Q_{p_2,\varphi}(h^{**})(t) \\ &\lesssim P_{\frac{p_1}{q},\frac{1}{q}}(f^*)(t) + Q_{p_2,\varphi}(h^{**})(t),\end{aligned}$$

where the last estimate holds because $h^*(r) = f^*(t)$ for every $r \in [0,t]$ and due to the estimate $f^*(t) \lesssim P_{\frac{p_1}{q},\frac{1}{q}}(f^*)(t)$. Besides, arguing as we did to bound (4.3.16), but now with the auxiliary function

$$\widetilde{\varphi}(x) = \begin{cases} \varphi(x)e^{\frac{x-1}{p_2}}, & 1 < p_2 < \infty, \\ \dfrac{\varphi(x)}{x}, & p_2 = \infty, \end{cases}$$

we deduce that

$$Q_{p_2,\varphi}(h^{**})(t) \lesssim f^*(t) + Q_{p_2,\varphi}(f^*)(t) \lesssim P_{\frac{p_1}{q},\frac{1}{q}}(f^*)(t) + Q_{p_2,\varphi}(f^*)(t),$$

so putting all together, and since $t > 0$ is arbitrary, the result follows as in (4.3.17).

Conversely, assume that $(Tf)^*(t) \lesssim S_{\frac{q}{p_1},\frac{1}{q},p_2,\varphi}(f^*)(t)$, for every $t > 0$, and take $w \in B^R_{\frac{1}{p_1}} \cap B^*_{p_2}$. Hence, arguing as in the proof of the sufficiency of Theorem 4.3.10, we have that

$$\|T\|_{\Lambda^{1,q}(w)\to\Lambda^{1,\infty}(w)} \leqslant \left\|P_{\frac{p_1}{q},1}\right\|^{\frac{1}{q}}_{L^1_{\text{dec}}(\tilde{w})\to L^{1,\infty}(\tilde{w})} + \|Q_{p_2,\varphi}\|_{L^1_{\text{dec}}(w)\to L^{1,\infty}(w)} \lesssim A^{\frac{1}{q}}_{k_1} + A_{k_2},$$

where $\tilde{w} = qW^{q-1}w$ and

$$A_{k_i} = \sup_{t>0}\left(\sup_{r>0}\left(\int_0^r k_i(t,s)ds\right) W_i(r)^{-1}\right) W_i(t), \qquad i = 1, 2,$$

with $W_1 = \tilde{W}$, $W_2 = W$ and

$$k_1(t,s) = \frac{1}{t^{\frac{q}{p_1}}}\chi_{[0,t)}(s)\frac{1}{s^{1-\frac{q}{p_1}}}, \quad k_2(t,s) = \begin{cases} \dfrac{\varphi\left(1+\log\frac{s}{t}\right)}{s}\left(\dfrac{s}{t}\right)^{\frac{1}{p_2}}\chi_{[t,\infty)}(s), & p_2<\infty, \\ \dfrac{\varphi\left(1+\log\frac{s}{t}\right)}{s(1+\log\frac{s}{t})}\chi_{[t,\infty)}(s), & p_2=\infty, \end{cases}$$

for every $t, s > 0$.

First suppose that $p_2 < \infty$. Hence, if $0 < r < t$ then we have

$$\int_0^r k_1(t,s)\,ds = \frac{p_1}{q}\left(\frac{r}{t}\right)^{\frac{q}{p_1}} \qquad \text{and} \qquad \int_0^r k_2(t,s)\,ds = 0,$$

while if $r > t$, by Lemma 4.3.3, we obtain

$$\int_0^r k_1(t,s)\,ds = \frac{p_1}{q} \qquad \text{and} \qquad \int_0^r k_2(t,s)\,ds \lesssim \varphi\left(1+\log\frac{r}{t}\right)\left(\frac{r}{t}\right)^{\frac{1}{p_2}}.$$

As a consequence we have that

$$A_{k_1} \approx \left[\sup_{t>0}\left(\sup_{0<r<t}\left(\frac{r}{t}\right)^{\frac{1}{p_1}} W(r)^{-1}\right) W(t)\right]^q \leqslant \|w\|_{B^R_{\frac{1}{p_1}}}^{\frac{q}{p_1}},$$

and

$$A_{k_2} \lesssim \sup_{t>0}\left(\sup_{r>t}\varphi\left(1+\log\frac{r}{t}\right)\left(\frac{r}{t}\right)^{\frac{1}{p_2}}\frac{W(t)}{W(r)}\right).$$

Hence, if $\mu = t/r < 1$, then by Lemma 2.2.32,

$$\left(\frac{r}{t}\right)^{\frac{1}{p_2}}\frac{W(t)}{W(r)} = \mu^{-\frac{1}{p_2}}\frac{W(\mu r)}{W(r)} \lesssim \|w\|_{B^*_{p_2}}\mu^{\frac{1}{4p_2\|w\|_{B^*_{p_2}}}}.$$

Therefore

$$A_{k_2} \lesssim \|w\|_{B^*_{p_2}}\sup_{\mu<1}\left(\varphi\left(1+\log\frac{1}{\mu}\right)\mu^{\frac{1}{4p_2\|w\|_{B^*_{p_2}}}}\right) \lesssim \|w\|_{B^*_{p_2}}\sup_{x>1}\left(\varphi(x)e^{-\frac{x}{4p_2\|w\|_{B^*_{p_2}}}}\right).$$

Finally, by Lemma 4.3.6 and the inequality (4.3.2) we obtain that

$$A_{k_2} \lesssim \|w\|_{B^*_{p_2}}\varphi\big(4p_2\|w\|_{B^*_{p_2}}\big) \lesssim \|w\|_{B^*_{p_2}}\varphi\big(\|w\|_{B^*_{p_2}}\big),$$

and, since $\|w\|_{B^R_{\frac{1}{p_1}}}, \|w\|_{B^*_{p_2}}\varphi\big(\|w\|_{B^*_{p_2}}\big) \geqslant 1$, we arrive to the desired result

$$\|T\|_{\Lambda^{1,q}(w)\to\Lambda^{1,\infty}(w)} \lesssim \|w\|_{B^R_{\frac{1}{p_1}}}^{\frac{1}{p_1}}\|w\|_{B^*_{p_2}}\varphi\big(\|w\|_{B^*_{p_2}}\big).$$

If $p_2 = \infty$, the proof is a combination of the proof for the case $p_2 < \infty$ and the proof of the sufficiency in Theorem 4.3.10. □

Remark 4.3.18. *Observe that the sublinearity of T is just used in (4.3.25), so if we have (4.3.23) for T being a general operator, we will also obtain the boundedness in (4.3.22).*

As a first consequence, similar as for Corollary 4.3.12, but now making use of Lemma 2.2.29, we have the following result.

Corollary 4.3.19. *Given $0 < q \leqslant 1$ and $1 \leqslant p_1 < p_2 \leqslant \infty$. Let T be a sublinear operator and let φ be an admissible function. If for every $0 < p < \infty$,*

$$T : \Lambda^{p,q}(w) \to \Lambda^{p,\infty}(w), \qquad \forall w \in B^R_{\frac{p}{p_1}} \cap B^*_{\frac{p_2}{p}}, \tag{4.3.27}$$

with constant less than or equal to $C \|w\|^{\frac{1}{p_1}}_{B^R_{\frac{p}{p_1}}} \varphi(\|w\|_{B^*_{\frac{p_2}{p}}})$, then, for every locally integrable function f and every $t > 0$,

(i) if $p_2 < \infty$,

$$(Tf)^*(t) \lesssim S_{\frac{p_1}{q},\frac{1}{q},p_2,\varphi_p}(f^*)(t), \qquad \varphi_p(x) = \begin{cases} \varphi(x^{p+1}), & 0 < p \leqslant 1, \\ \varphi(x^p), & 1 < p < \infty, \end{cases}$$

(ii) while if $p_2 = \infty$, $(Tf)^*(t) \lesssim S_{\frac{p_1}{q},\frac{1}{q},\infty,\varphi}(f^*)(t)$.

Conversely, suppose that $(Tf)^*(t) \lesssim S_{\frac{p_1}{q},\frac{1}{q},p_2,\varphi}(f^*)(t)$, for every $t > 0$. Then,

(i) if $p_2 < \infty$,

$$\|T\|_{\Lambda^{p,q}(w)\to\Lambda^{p,\infty}(w)} \lesssim \|w\|^{\frac{1}{p_1}}_{B^R_{\frac{p}{p_1}}} \tilde{\varphi}_p(\|w\|_{B^*_{\frac{p_2}{p}}}), \quad \tilde{\varphi}_p(x) = \begin{cases} x^{\frac{1}{p}}\varphi\left(x^{\frac{1}{p}}\right), & 0 < p \leqslant 1, \\ x^{\frac{1}{p}+1}\varphi\left(x^{\frac{1}{p}+1}\right), & 1 < p < \infty, \end{cases}$$

(ii) while if $p_2 = \infty$, $\|T\|_{\Lambda^{p,q}(w)\to\Lambda^{p,\infty}(w)} \lesssim \|w\|^{\frac{1}{p_1}}_{B^R_{\frac{p}{p_1}}} \varphi(\|w\|_{B^*_\infty})$.

Remark 4.3.20. *Observe that by virtue of Corollary 4.3.19, if we have for some $0 < p < \infty$ that (4.3.27) holds then, indeed, it does also for every $p > 0$.*

Besides, from Theorems 4.2.5 and 4.3.17, Remark 4.2.6 and observing that, given an admissible function φ, the function $\overline{\varphi}$ as in (4.2.10) is also an admissible function, we deduce the next result.

Corollary 4.3.21. *Let T be a sublinear operator such that, for some $1 \leqslant p_0 < \infty$ and $0 < \alpha \leqslant 1$,*

$$T : L^{p_0,1}(u^\alpha) \to L^{p_0,\infty}(u^\alpha), \qquad \forall u \in A_1,$$

with constant less than or equal to $\varphi(\|u\|^\alpha_{A_1})$, where φ is an admissible function. Then, for every locally integrable function f and $t > 0$,

$$\begin{aligned}(Tf)^*(t) &\lesssim S_{p_0,1,\frac{p_0}{1-\alpha},\tilde{\varphi}}(f^*)(t) \\ &= \frac{1}{t^{\frac{1}{p_0}}}\int_0^t f^*(s)\,\frac{ds}{s^{1-\frac{1}{p_0}}} + \frac{1}{t^{\frac{1-\alpha}{p_0}}}\int_t^\infty \tilde{\varphi}\left(1+\log\frac{s}{t}\right) f^*(s)\,\frac{ds}{s^{1-\frac{1-\alpha}{p_0}}},\end{aligned}$$

with

$$\tilde{\varphi}(x) = \begin{cases} \varphi(x^{p_0}), & 0 < \alpha < 1, \\ x^{-1}\varphi(x), & \alpha = 1. \end{cases}$$

Finally, to end this section, we want to point out that in order to prove that (4.3.22) implies (4.3.23) we just have used power weights (see (4.3.24) and (4.3.26)). Hence, similar as we did in Theorem 4.3.15, we have that Theorem 4.3.17 can be written in the following equivalent way.

Theorem 4.3.22. *Given $0 < q \leqslant 1$ and $1 \leqslant p_1 < p_2 \leqslant \infty$. Let T be a sublinear operator and let φ be an admissible function. If $T : L^{p_1,q}(\mathbb{R}^n) \to L^{p_1,\infty}(\mathbb{R}^n)$ and for every $p_1 < p < p_2$,*

$$\|T\chi_E\|_{L^{p,\infty}(\mathbb{R}^n)} \leqslant C\varphi\left(\left[\frac{1}{p} - \frac{1}{p_2}\right]^{-1}\right)|E|^{\frac{1}{p}}, \qquad \forall E \subseteq \mathbb{R}^n,$$

with C independent of p then, for every locally integrable function f and for every $t > 0$, (4.3.23) holds. Conversely, assume that we have (4.3.23). Then, for every $p_1 \leqslant p < p_2$,

$$\|T\|_{L^{p,q}(\mathbb{R}^n)\to L^{p,\infty}(\mathbb{R}^n)} \leqslant C\tilde{\varphi}\left(\left[\frac{1}{p} - \frac{1}{p_2}\right]^{-1}\right),$$

with C independent of p and where, for $x \geqslant 1$,

$$\tilde{\varphi}(x) = \begin{cases} x\varphi(x), & p_2 < \infty, \\ \varphi(x), & p_2 = \infty. \end{cases}$$

Remark 4.3.23. *If $p > 1$ and $0 < q < 1$, then that $T : L^{p,q}(\mathbb{R}^n) \to L^{p,\infty}(\mathbb{R}^n)$ is equivalent to $T : L^{p,1}(\mathbb{R}^n) \to L^{p,\infty}(\mathbb{R}^n)$ (see Remark 2.1.3). Therefore, if $p_1 > 1$ and we have (4.3.22) for some $0 < q < 1$, taking power weights $w_\gamma(t) = t^{\gamma-1}$, for $\frac{1}{p_2} < \gamma \leqslant \frac{1}{p_1}$, due to Theorem 4.3.22 we also have it for $q = 1$ (although for a different constant). This is interesting by itself since $\Lambda^{1,\infty}(w)$ may not be a Banach function space for every $w \in B^{*R}_{\frac{1}{p_1}} \cap B^*_{p_+}$ and we can not argue as in Proposition 2.1.2.*

It is known that if for $1 \leqslant p_1 < p_2 < \infty$ we have both $T : L^{p_1,1}(\mathbb{R}^n) \to L^{p_1,\infty}(\mathbb{R}^n)$ and $T : L^{p_2,1}(\mathbb{R}^n) \to L^{p_2,\infty}(\mathbb{R}^n)$ then (see [23, Ch. 4 - Theorem 4.11]) we get that for every locally integrable function f and every $t > 0$,

$$(Tf)^*(t) \lesssim S_{p_1,1,p_2,1}(f^*)(t) = \frac{1}{t^{\frac{1}{p_1}}}\int_0^t f^*(s)\,\frac{ds}{s^{1-\frac{1}{p_1}}} + \frac{1}{t^{\frac{1}{p_2}}}\int_t^\infty f^*(s)\,\frac{ds}{s^{1-\frac{1}{p_2}}}.$$

Now observe that we have obtained weaker estimates in Theorem 4.3.22 since we have allowed p to be p_1 but not p_2 (just as close as we want). Thus, it would be natural to consider the case where we allow p to be p_2 (for $p_2 < \infty$) and not p_1. Indeed, this was exactly the motivation and what it was studied in [7].

This kind of results belong to the Yano's and Zygmund extrapolation theory (see [9, 11, 42, 43, 182, 183]). However, we want to emphasize here that, contrary to what happens in the proof of Yano's and Zygmund's results, where the function f is decomposed on an infinite sum of functions f_n, we have used a simple proof that follows the ideas of [23, Ch. 3 - Theorem 4.7], where the function f is decomposed as the sum of just two functions.

Chapter 5

Boundedness of operators on $\Lambda^p(w)$

We focus this chapter to apply the results obtained in Chapter 4 to some important operators in Harmonic Analysis, so that we will obtain interesting boundedness on the setting of the classical Lorentz spaces together with unknown estimates on the decreasing rearrangement of such operators. In particular, we will study operators that satisfy a Fefferman-Stein's inequality (see Section 5.1), Fourier multipliers (see Section 5.2) such as Fourier multipliers of Hörmander type (see Section 5.2.1), Fourier multipliers that satisfy a Fefferman-Stein's type inequality (see Section 5.2.2) and radial Fourier multipliers with a derivative condition (see Section 5.2.3), rough singular integrals (see Section 5.3), intrinsic square functions (see Section 5.4), sparse operators (see Section 5.5), the Assani operator (see Section 5.6) and the Bochner-Riesz operator (see Section 5.7).

Some of the results of this chapter are included in [6, 15, 16].

5.1 Fefferman-Stein's inequality

An operator T is said to satisfy a *Fefferman-Stein's inequality* [94] if

$$\int_{\{|Tf(x)|>y\}} u(x)dx \lesssim \int |f(x)|Mu(x)dx, \qquad \text{for every weight } u. \tag{5.1.1}$$

Clearly, for an operator satisfying (5.1.1) we have that

$$T : L^1(u) \to L^{1,\infty}(u), \qquad \forall u \in A_1,$$

with constant less than or equal to $C\,\|u\|_{A_1}$ (that is, with a linear norm constant).

This is the case (among many others operators) of the area function [65] defined by

$$Sf(x) = \left(\int_{|x-y|\leqslant t} |\nabla_{y,t}(f * P_t)(y)|^2\right)^{\frac{1}{2}}, \qquad x \in \mathbb{R}^n,$$

where

$$\nabla_{y,t} = \left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \cdots, \frac{\partial}{\partial y_n}, \frac{\partial}{\partial t}\right) \qquad \text{and} \qquad P_t(y) = \frac{c_n t}{(t^2 + |y|^2)^{\frac{n+1}{2}}}, \qquad (y,t) \in \mathbb{R}^{n+1}_+.$$

As a consequence of Theorem 4.2.2 and Corollary 4.3.14, we get the following result.

Corollary 5.1.1. *Let T be an operator satisfying a Fefferman-Stein's inequality. For every $0 < p < \infty$,*

$$T : \Lambda^{p,1}(w) \to \Lambda^{p,\infty}(w), \qquad \forall w \in B_p^R \cap B_\infty^*,$$

with constant less than or equal to $C \|w\|_{B_p^R} \|w\|_{B_\infty^*}$. Further, if T is sublinear, for every locally integrable function f and for every $t > 0$,

$$(Tf)^*(t) \lesssim \frac{1}{t}\int_0^t f^*(s)\,ds + \int_t^\infty f^*(s)\,\frac{ds}{s}.$$

5.2 Fourier multipliers

Recall that given $m \in L^\infty(\mathbb{R}^n)$, then T_m is a *Fourier multiplier* if, for every Schwartz function f (that is, $f \in \mathcal{S}(\mathbb{R}^n)$),

$$\widehat{T_m f}(\xi) = m(\xi)\hat{f}(\xi), \qquad \xi \in \mathbb{R}^n,$$

and m is called a multiplier (see Section 3.1).

5.2.1 Fourier multipliers of Hörmander type

Given $n \geqslant 1$, let us use the standard notation $|\alpha| = \alpha_1 + \cdots + \alpha_n$ for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$ and if $x \in \mathbb{R}^n$,

$$\left(\frac{\partial}{\partial x}\right)^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}.$$

Given $k \in \mathbb{N}$ such that $k > \frac{n}{2}$, $1 < s \leqslant 2$ and $m : \mathbb{R}^n \to \mathbb{R}$ be a bounded function in $\mathcal{C}^k(\mathbb{R}^n \backslash \{0\})$, we say that m satisfies the *Hörmander condition* with respect to s and k, and denote it by $m \in HC(s,k)$, if

$$\sup_{R>0} R^{|\alpha| - \frac{n}{s}} \left(\int_{R<|x|<2R} \left| \left(\frac{\partial}{\partial x}\right)^\alpha m(x)\right|^s dx \right)^{\frac{1}{s}} < \infty, \qquad |\alpha| \leqslant k.$$

Then, the *Fourier multipliers operators of Hörmander type* are those defined by

$$\widehat{T_m f}(\xi) = m(\xi)\widehat{f}(\xi), \qquad \xi \in \mathbb{R}^n,$$

where $m \in HC(s,k)$. The classical Hörmander theorem (see for example [99, Theorem 6.2.7]) says that when $s = 2$, T_m is bounded in $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ and satisfies the unweighted weak-type $(1,1)$ inequality whenever $m \in HC(2,k)$. The generalization of the condition to $1 < s < 2$ was introduced in [37], where the authors see that for a given $m \in HC(s,k)$ need $k > \frac{n}{s}$. In [108, 121, 177], the authors introduce power weights to the problem, and later in [122, Theorem 1], it was proved for A_p weights.

Proposition 5.2.1 ([122]). *Let $1 < s \leqslant 2$, $\frac{n}{s} < k \leqslant n$, $k \in \mathbb{N}$, and $m \in HC(s,k)$. Then, there exists a positive nondecreasing function φ on $[1,\infty)$ such that*

(i)

$$T_m : L^1\left(v^{\frac{k}{n}}\right) \to L^{1,\infty}\left(v^{\frac{k}{n}}\right), \qquad \forall v \in A_1,$$

with constant less than or equal to $\varphi\left(\|v\|_{A_1}^{k/n}\right)$,

(ii) and if $k < n$,

$$T_m : L^{\frac{n}{k}}(v) \to L^{\frac{n}{k}}(v), \qquad \forall v \in A_1,$$

with constant less than or equal to $\varphi\left(\|v\|_{A_1}\right)$.

We should mention that although the boundedness results of Proposition 5.2.1 are known, the behaviour of the function φ (that is nondecreasing) has not been written explicitly anywhere, and since we want to make use of Theorem 4.2.5, we need such behaviour. That is why we will dedicate what follows to keep track of the constants of the weights involved on the previous results to find the desired behaviour of φ.

With that aim in mind, following the ideas of [109, 122], we will work with a truncation K_N of the kernel K, where $K = \check{m}$. To do so, we take $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$ to be a nonnegative function supported in $\{\frac{1}{2} < |x| < 2\}$ and we select, for each $j \in \mathbb{Z}$, $\varphi_j(\,\cdot\,) = \varphi(2^{-j}\,\cdot\,)$, that satisfy

$$\sum_{j\in\mathbb{Z}} \varphi_j(x) = \sum_{j\in\mathbb{Z}} \varphi(2^{-j}x) = 1, \qquad \forall x \neq 0.$$

Now, for each $j \in \mathbb{Z}$, set

$$m_j(x) = m(x)\varphi_j(x), \qquad x \in \mathbb{R}^n,$$

so that m_j is supported in $\{2^{j-1} < |x| < 2^{j+1}\}$ and

$$m(x) = \sum_{j\in\mathbb{Z}} m_j(x), \qquad \forall x \neq 0.$$

Besides, define for all $j \in \mathbb{Z}$, $k_j = \check{m}_j$ and let, for any $N \in \mathbb{N}$ and every $x \in \mathbb{R}^n$,

$$m_N(x) = \sum_{j=-N}^{N} m_j(x) \qquad \text{and} \qquad K_N(x) = (m^N)^\vee(x) \;=\; \sum_{j=-N}^{N} k_j(x).$$

Hence, it follows that $\left\|m^N\right\|_\infty \leqslant \|m\|_\infty$ and that

$$m^N(x) \to m(x), \qquad \forall x \neq 0.$$

At this point, we present some technical lemmas that will help us to pursue in our goal, where we have been keeping track of their respectively constants involved. First, as a consequence of [122, Lemma 1] it can be derived the following estimate for weights in A_1.

Lemma 5.2.2. *Let* $1 < s \leqslant 2$, $\frac{n}{s} < k \leqslant n$, $k \in \mathbb{N}$, *and* $m \in HC(s,k)$. *Let* Q *be a cube and take* y_Q *and* D_Q *to be its center an diameter respectively. Then, for every* $y \in Q$ *and* $N \in \mathbb{N}$,

$$\int_{|x-y_Q|>2D_Q} |K_N(x-y) - K_N(x-y_Q)| v(x)\,dx \leqslant \frac{C_{n,s,k}\,\|v\|_{A_1}}{k\Phi(\|v\|_{A_1}) - n}\left(\operatorname*{ess\,inf}_{x\in Q} v(x)\right), \quad \forall v \in A_1,$$

uniformly on N, *where* Φ *is a decreasing function on* $[1,\infty)$ *defined by*

$$\Phi(x) = \min\left(\frac{n+sk}{2k}, \frac{n}{k}\left[1 + \frac{1}{2^{n+1}x}\right], \frac{2n}{2k-1}\right) > \frac{n}{k}, \qquad x \geqslant 1.$$

Further, we will also need two estimates involving the *Fefferman-Stein maximal operator* $M^{\#}$ (also known as *sharp maximal operator*) defined for every $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ as

$$M^{\#}f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q \left| f(y) - \left(\frac{1}{|Q|} \int_Q f(z)\, dz \right) \right| dy, \qquad x \in \mathbb{R}^n, \tag{5.2.1}$$

where the supremum is taken over all cubes Q with sides parallel to the axes containing the point x (see [88, 158]).

The first estimate is a consequence of [129, Theorem 1.1].

Lemma 5.2.3. *For every $f \in L^1_{loc}(\mathbb{R}^n)$,*

$$\|Mf\|_{L^2(v)} \leqslant C_n \|v\|_{A_2} \left\|M^{\#}f\right\|_{L^2(v)}, \qquad \forall v \in A_2.$$

On the other side, following the proof of [122, Estimate (3.1)] we also have the next result.

Lemma 5.2.4. *Let $1 < s \leqslant 2$, $\frac{n}{s} < k \leqslant n$, $k \in \mathbb{N}$, and $m \in HC(s,k)$. Take $\frac{n}{k} < r < \min\left(\frac{n}{k-1}, s\right)$. Then, for every $f \in L^1_{loc}(\mathbb{R}^n)$ and every $N \in \mathbb{N}$,*

$$M^{\#}(K_N * f)(x) \leqslant \frac{C_{n,s,k}}{(s-r)(rk-n)^3(n-r(k-1))} M_{\frac{1}{r}} f(x), \qquad \forall x \in \mathbb{R}^n,$$

uniformly on N and where $M_r f = M(|f|^r)^{\frac{1}{r}}$.

At this point, we are now ready to settle our main goal in this section:

Proof of Proposition 5.2.1. First, to prove (i) (and, as we will see below, also (ii)) we just need to see that

$$T_m : L^2(v) \to L^2(v), \qquad \forall v \in A_{\frac{2k}{n}}, \tag{5.2.2}$$

with constant less than or equal to $\Psi\left(\|v\|_{A_{\frac{2k}{n}}}\right)$, with Ψ being a nonincreasing function on $[1, \infty)$, since then the proof will follow by means of standard techniques based on the Calderón-Zygmund decomposition of a function $f \in L^1(\mathbb{R}^n) \cap L^1(v^{k/n})$, together with the fact that $v^{k/n} \in A_1$ is doubling with constant controlled by $C\|v\|_{A_1}^{k/n}$ and, as well, Lemma 5.2.2 (see [122, pp. 354-356]).

So let us see (5.2.2). Take $v \in A_{\frac{2k}{n}}$ and define

$$r = \min\left(\frac{3}{2}, \frac{n+sk}{2k}, \frac{2n}{2k-1}, \frac{n\Phi_{\frac{2k}{n}}\left(\|v\|_{A_{\frac{2k}{n}}}\right)}{k}\right),$$

with $\Phi_{\frac{2k}{n}}$ being a decreasing function on $[1, \infty)$ as in (2.2.2). Hence,

$$\frac{n}{k} < r < \min\left(\frac{n}{k-1}, s\right) \qquad \text{and} \qquad v \in A_{\frac{2}{r}} \text{ with } \|v\|_{A_{\frac{2}{r}}} \lesssim \|v\|_{A_{\frac{2k}{n}}}.$$

Thus, due to Lemmas 5.2.3 and 5.2.4,

$$\|K_N * f\|_{L^2(v)} \lesssim \|v\|_{A_{\frac{2k}{n}}} \left\|M^{\#}(K_N * f)\right\|_{L^2(v)} \lesssim \frac{1}{(rk-n)^3} \|v\|_{A_{\frac{2k}{n}}} \left\|M_{\frac{1}{r}} f\right\|_{L^2(v)}$$
$$\lesssim \frac{1}{(rk-n)^3} \|v\|_{A_{\frac{2}{r}}}^{\frac{3-r}{2-r}} \|f\|_{L^2(v)} \lesssim \frac{1}{(rk-n)^3} \|v\|_{A_{\frac{2k}{n}}}^4 \|f\|_{L^2(v)},$$

uniformly on N. Therefore, arguing as in [122, Remark 2], we obtain (5.2.2) by taking

$$\Psi(x) \approx \frac{x^4}{(rk-n)^3}, \qquad \forall x \geqslant 1.$$

Now, in order to prove (ii) we will make use of an interpolation argument. First, we shall note that by means of extrapolation (see for instance [89, Theorem 7.1]) for every $\frac{n}{k} < p \leqslant 2$,

$$T_m : L^p(v) \to L^p(v), \qquad \forall v \in A_{\frac{pk}{n}}, \tag{5.2.3}$$

with constant less than or equal to

$$\Psi_p\left(\|v\|_{A_{\frac{pk}{n}}}\right) = C_{n,k}\Psi\left(\frac{C'_{n,k}}{(pk-n)^2}\|v\|_{A_{\frac{pk}{n}}}\right).$$

Hence, assume that $k < n$ and fix $1 < p_0 < \frac{n}{k}$. Let $\overline{m}$ be the complex conjugate of m, so that $\widetilde{T_m} = T_{\overline{m}}$ is the adjoint operator of T_m and $\overline{m}$ satisfies the same estimates as m. Then, since $n < 2k$, we have that $p_0' > \frac{n}{k}$ and by duality together with (5.2.3),

$$\|T_m f\|_{L^{p_0}(\mathbb{R}^n)} = \sup_{\|g\|_{L^{p_0'}(\mathbb{R}^n)} \leqslant 1} \int_{\mathbb{R}^n} f(x) T_{\overline{m}} g(x)\, dx \lesssim \|f\|_{L^{p_0}(\mathbb{R}^n)}.$$

Further, take $v \in A_1$ and set

$$t = \min\left(1 + \frac{1}{2^{n+1}\|v\|_{A_1}}, \frac{n(2-p_0)}{2(n-kp_0)}\right) > 1,$$

so that $v^t \in A_1$ with $\|v^t\|_{A_1} \lesssim \|v\|_{A_1}$ (see (2.2.4)). Take $\theta = \frac{1}{t}$, $v_0 = 1$, $v_1 = v^t$ and

$$\frac{n}{k} < p_1 := \frac{np_0}{n - t(n-kp_0)} \leqslant 2.$$

Hence, taking into account (5.2.3),

$$\|T_m\|_{L^{p_1}(v_1)} \leqslant \Psi_{p_1}\left(\|v\|_{A_{\frac{p_1 k}{n}}}\right) \lesssim \frac{1}{(p_1 k - n)^8 (rk-n)^3} \|v\|_{A_1}^4$$
$$\lesssim \frac{1}{(t-1)^8 (r_1 k - n)^3} \|v\|_{A_1}^4 =: \Phi(\|v\|_{A_1}),$$

with

$$r_1 = \min\left(\frac{3}{2}, \frac{n+sk}{2k}, \frac{2n}{2k-1}, \frac{n\Phi_{\frac{2k}{n}}\left(\|v\|_{A_1}\right)}{k}\right),$$

so that Φ is a nondecreasing function on $[1,\infty)$. Therefore, by interpolation with change of measure (see, for instance, [23, Theorem 3.6])

$$T_m : L^{\frac{n}{k}}(v) \to L^{\frac{n}{k}}(v)$$

with constant less than or equal to

$$C\Phi(\|v\|_{A_1})^{\frac{1}{t}} \lesssim \max\left(1, \Phi(\|v\|_{A_1})\right).$$

$\square$

Therefore, as a consequence of Theorem 4.2.5, together with Proposition 5.2.1, we get the following result.

Corollary 5.2.5. *Let $1 < s \leqslant 2$, $\frac{n}{s} < k \leqslant n$, $k \in \mathbb{N}$, and $m \in HC(s,k)$. For every $0 < p < \infty$,*

$$T_m : \Lambda^{p,1}(w) \to \Lambda^{p,\infty}(w), \qquad \forall w \in \left(B^R_{\frac{pk}{n}} \cap B^*_\infty\right) \cup \left(B^R_p \cap B^*_{\frac{n}{(n-k)p}}\right).$$

5.2.2 Fourier multipliers with a Fefferman-Stein's type inequality

It has been of great interest to identify, when possible, for which maximal operators $\mathcal{M}$ the operator T_m satisfies a *Fefferman-Stein's type inequality* in $L^2(\mathbb{R}^n)$ of the form

$$\int_{\mathbb{R}^n} |T_m f(x)|^2 u(x)\, dx \leqslant \int_{\mathbb{R}^n} |f(x)|^2 \mathcal{M}u(x)\, dx, \qquad \text{for every weight } u,$$

(see for instance [24, 66, 77, 78, 88, 173]). In particular, we present the following interesting case:

Proposition 5.2.6 ([24]). *If $m : \mathbb{R} \to \mathbb{R}$ is a bounded function which is uniformly of bounded variation on dyadic intervals; that is*

$$\sup_{R>0} \int_{R\leqslant|\xi|\leqslant 2R} |m'(\xi)|\, d\xi < \infty, \tag{5.2.4}$$

then, for every locally integrable function f,

$$\int_{\mathbb{R}} |T_m f(x)|^2 u(x)\, dx \leqslant C \int_{\mathbb{R}} |f(x)|^2 M^7 u(x)\, dx, \qquad \textit{for every weight } u,$$

where $M^7 = M \underbrace{\circ \cdots \circ}_{7} M$ is the 7-fold composition of M with itself.

Hence, if $m : \mathbb{R} \to \mathbb{R}$ is a bounded function satisfying (5.2.4), then

$$T_m : L^{2,1}(u) \to L^{2,\infty}(u), \qquad \forall u \in A_1,$$

with constant less than or equal to $C\|u\|^7_{A_1}$ and, as a consequence of Theorem 4.2.5, Remark 4.2.6 and Corollary 4.3.21, we obtain the following result.

Corollary 5.2.7. *Let $m : \mathbb{R} \to \mathbb{R}$ be a bounded function satisfying (5.2.4). For every $0 < p < \infty$,*

$$T_m : \Lambda^{p,1}(w) \to \Lambda^{p,\infty}(w), \qquad \forall w \in B^R_{\frac{p}{2}} \cap B^*_\infty,$$

with constant less than or equal to $C \|w\|^{1/2}_{B^R_{\frac{p}{2}}} \|w\|^7_{B^*_\infty}$. Further, for every locally integrable function f and every $t > 0$,

$$(T_m f)^*(t) \lesssim \frac{1}{t^{\frac{1}{2}}} \int_0^t f^*(s) \frac{ds}{t^{\frac{1}{2}}} + \int_t^\infty \left(1 + \log \frac{s}{t}\right)^6 f^*(s) \frac{ds}{s}.$$

In this context of Fourier multipliers, let us now consider, for each $\gamma_0, \gamma_1 \in \mathbb{R}$, the class $\mathcal{C}(\gamma_0, \gamma_1)$ of functions $m : \mathbb{R} \to \mathbb{R}$ for which

$$\operatorname{supp}(m) \subseteq \{\xi : |\xi|^{\gamma_0} \geqslant 1\}, \qquad \sup_{\xi \in \mathbb{R}} |\xi|^{\gamma_1} |m(\xi)| < \infty,$$

and

$$\sup_{R^{\gamma_0} \geqslant 1} \; \sup_{I \subseteq [R,2R],\, \ell(I) = R^{-\gamma_0+1}} R^{\gamma_1} \int_{\pm I} |m'(\xi)| \, d\xi < \infty,$$

where I denotes any interval of $\mathbb{R}$.

Proposition 5.2.8 ([24]). *Let $\gamma_0, \gamma_1 \in \mathbb{R}$ such that $\gamma_0 \geqslant 2\gamma_1$ and $\gamma_0 \cdot \gamma_1 > 0$ or $\gamma_1 = \gamma_2 = 0$. If $m \in \mathcal{C}(\gamma_0, \gamma_1)$ then, for every locally integrable function f,*

$$\int_{\mathbb{R}} |T_m f(x)|^2 u(x) \, dx \leqslant C \int_{\mathbb{R}} |f(x)|^2 M^6 \left(\left[M^5 \left(u^{\frac{\gamma_0}{2\gamma_1}} \right) \right]^{\frac{2\gamma_1}{\gamma_0}} \right)(x) \, dx, \qquad \text{for every weight } u,$$

where we are assuming that $\frac{\gamma_0}{2\gamma_1} = 1$ for $\gamma_0 = \gamma_1 = 0$.

Therefore, under the hypothesis of the previous result,

$$T_m : L^{2,1}\left(u^{\frac{2\gamma_1}{\gamma_0}}\right) \to L^{2,\infty}\left(u^{\frac{2\gamma_1}{\gamma_0}}\right), \qquad \forall u \in A_1,$$

with constant less than or equal to

$$\begin{cases} C_1 \left(\frac{\gamma_0}{\gamma_0 - 2\gamma_1}\right)^6 \|u\|^{\frac{10\gamma_1}{\gamma_0}}_{A_1}, & \gamma_0 > 2\gamma_1,\ \gamma_0 \cdot \gamma_1 > 0, \\ C_2 \|u\|^{11}_{A_1}, & \gamma_0 = 2\gamma_1, \end{cases}$$

so that as a consequence of Theorem 4.2.5, Remark 4.2.6 and Corollary 4.3.21, we have the following results.

Corollary 5.2.9. *Let $\gamma_0, \gamma_1 \in \mathbb{R}$ such that $\gamma_0 > 2\gamma_1$ and $\gamma_0 \cdot \gamma_1 > 0$, and let $m \in \mathcal{C}(\gamma_0, \gamma_1)$. For every $0 < p < \infty$,*

$$T_m : \Lambda^{p,1}(w) \to \Lambda^{p,\infty}(w), \qquad \forall w \in B^R_{\frac{p}{2}} \cap B^*_{\frac{2\gamma_0}{(\gamma_0 - 2\gamma_1)p}},$$

with constant less than or equal to

$$C\left(\frac{\gamma_0}{\gamma_0-2\gamma_1}\right)^6 \|w\|_{B^R_{\frac{p}{2}}}^{\frac12}\begin{cases}\|w\|_{B^*_{\frac{2\gamma_0}{(\gamma_0-2\gamma_1)p}}}^{\frac{10}{p}}, & 0<p\leqslant 1,\\ \|w\|_{B^*_{\frac{2\gamma_0}{(\gamma_0-2\gamma_1)p}}}^{\frac{10(p+1)}{p}}, & p>1.\end{cases}$$

Further, for every locally integrable function f and every $t>0$,

$$(T_m f)^*(t) \lesssim \frac{1}{t^{\frac12}}\int_0^t f^*(s)\,\frac{ds}{t^{\frac12}} + \frac{1}{t^{\frac{\gamma_0-2\gamma_1}{2\gamma_0}}}\int_t^\infty \left(1+\log\frac{s}{t}\right)^{10} f^*(s)\,\frac{ds}{s^{\frac{\gamma_0+2\gamma_1}{2\gamma_0}}}.$$

Corollary 5.2.10. *Let $\gamma\in\mathbb{R}$ and $m\in\mathcal{C}(2\gamma,\gamma)$. For every $0<p<\infty$,*

$$T_m : \Lambda^{p,1}(w)\to\Lambda^{p,\infty}(w), \qquad \forall w\in B^R_{\frac{p}{2}}\cap B^*_\infty,$$

with constant less than or equal to $C\,\|w\|_{B^R_{\frac{p}{2}}}^{1/2}\|w\|_{B^*_\infty}^{11}$. Further, for every locally integrable function f and every $t>0$,

$$(T_m f)^*(t) \lesssim \frac{1}{t^{\frac12}}\int_0^t f^*(s)\,\frac{ds}{t^{\frac12}} + \int_t^\infty \left(1+\log\frac{s}{t}\right)^{10} f^*(s)\,\frac{ds}{s}.$$

Finally, given $a\in\mathbb{R}$, the *Sobolev space* $L^2_a(\mathbb{R}^n)$ is defined as the set of measurable functions f such that $(1+|\cdot|^2)^{a/2}\hat f\in L^2(\mathbb{R}^n)$ and, in that case, its norm is defined by

$$\|f\|_{L^2_a(\mathbb{R}^n)} = \left\|(1+|\cdot|^2)^{\frac{a}{2}}\hat f\right\|_{L^2(\mathbb{R}^n)}.$$

Proposition 5.2.11 ([88]). *Given $a>\frac{n}{2}$ and $m\in L^2_a(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n}|Tf(x)|^2u(x)\,dx \leqslant C\int_{\mathbb{R}^n}|f(x)|^2 Mu(x)\,dx, \qquad \textit{for every weight } u,$$

where the constant C is independent of u.

Hence, if $a>\frac{n}{2}$ and $m\in L^2_a(\mathbb{R}^n)$,

$$T : L^{2,1}(u)\to L^{2,\infty}(u), \qquad \forall u\in A_1,$$

with constant less than or equal to $C\,\|u\|_{A_1}$ and, as a consequence of Theorem 4.2.5, Remark 4.2.6 and Corollary 4.3.21, we obtain the following result.

Corollary 5.2.12. *Given $a>\frac{n}{2}$ and $m\in L^2_a(\mathbb{R}^n)$. For every $0<p<\infty$,*

$$T : \Lambda^{p,1}(w)\to\Lambda^{p,\infty}(w), \qquad \forall w\in B^R_{\frac{p}{2}}\cap B^*_\infty,$$

with constant less than or equal to $C\,\|w\|_{B^R_{\frac{p}{2}}}^{1/2}\|w\|_{B^*_\infty}$. Further, for every locally integrable function f and every $t>0$,

$$(Tf)^*(t) \lesssim \frac{1}{t^{\frac12}}\int_0^t f^*(s)\,\frac{ds}{t^{\frac12}} + \int_t^\infty f^*(s)\,\frac{ds}{s}.$$

5.2.3 Radial Fourier multipliers with a derivative condition

Given a multiplier m, we say that T_m is a *radial Fourier multiplier* if m is a radial function; that is $m(\xi) = m_0(|\xi|)$, $\forall \xi \in \mathbb{R}^n$, for some function $m_0 : [0, \infty) \to \mathbb{R}$.

In this section, we will study a particular case where the fractional derivative of some order of the multiplier m satisfies some constrains.

One can define fractional derivatives in multiple ways. However, the definition that we will need is in the sense of Weyl: given $0 \leqslant \delta < 1$ and $r > 0$, we define the *truncated fractional integral* of order $1 - \delta$ for $f \in L^1_{\text{loc}}(\mathbb{R})$ by

$$I_r^{1-\delta} f(t) = \begin{cases} \dfrac{1}{\Gamma(1-\delta)} \displaystyle\int_{-r}^{r} (s-t)_+^{-\delta} f(s)\, ds, & t < r, \\ 0, & t \geqslant r, \end{cases}$$

for every $t \in \mathbb{R}$, with Γ being the Gamma function

$$\Gamma(y) = \int_0^\infty x^{y-1} e^{-x}\, dx, \qquad y > 0.$$

Moreover, if $\alpha = [\alpha] + \delta > 0$, with $[\alpha]$ being its integer part and δ its fractional part, we define the fractional derivative of f of order α by

$$D^\alpha f(t) = -\left(\frac{d}{dt}\right)^{[\alpha]} \lim_{r\to\infty} \frac{d}{dt} I_r^{1-\delta} f(t), \qquad t \in \mathbb{R},$$

whenever the limit and the derivatives exist.

Proposition 5.2.13 ([46]). *Fix $n \geqslant 2$ and $\alpha = \frac{n+1}{2}$. Let $m \in L^\infty(0,\infty) \cap \mathcal{C}(0,\infty)$ which vanishes at infinity and satisfies that*

$$D^{\alpha-j} m \in AC_{loc}, \qquad \forall j = 1, \dots, [\alpha],$$

with AC_{loc} being the space of functions that are absolutely continuous on every compact subset of $(0,\infty)$. Then, if $D^\alpha m$ exists and $\Phi(t) = t^{\alpha-1} D^\alpha m(t) \in L^1(0,\infty)$, the operator T_m defined by

$$\widehat{T_m f}(\xi) = m(|\xi|^2) \widehat{f}(\xi), \qquad \xi \in \mathbb{R}^n, \tag{5.2.5}$$

satisfies that

$$T_m : L^1(v) \to L^{1,\infty}(v), \qquad \forall u \in A_1,$$

with constant less than or equal to $C \|\Phi\|_{L^1(0,\infty)} \|u\|_{A_1}^5$.

Therefore, as a consequence of Theorem 4.2.2 and Corollary 4.3.14, we get the following result.

Corollary 5.2.14. *Fix $n \geqslant 2$ and $\alpha = \frac{n+1}{2}$. Let $m \in L^\infty(0,\infty) \cap \mathcal{C}(0,\infty)$ satisfying the hypotheses of Proposition 5.2.13 and let T_m be defined as (5.2.5). For every $0 < p < \infty$,*

$$T : \Lambda^{p,1}(w) \to \Lambda^{p,\infty}(w), \qquad \forall w \in B_p^R \cap B_\infty^*,$$

with constant less than or equal to $C \|\Phi\|_{L^1(0,\infty)} \|w\|_{B_p^R} \|w\|_{B_\infty^*}^5$. Further, for every locally integrable function f and for every $t > 0$,

$$(Tf)^*(t) \lesssim \frac{1}{t} \int_0^t f^*(s)\, ds + \int_t^\infty \left(1 + \log\frac{s}{t}\right)^4 f^*(s) \frac{ds}{s}.$$

5.3 Rough singular integrals

Let $n > 1$ and denote $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ the n-th dimensional sphere with center 0 and radius 1. Given $\Omega \in L^\infty(\mathbb{S}^{n-1})$ a positive homogeneous function of degree zero such that

$$\int_{\mathbb{S}^{n-1}} \Omega(x)\, dx = 0, \tag{5.3.1}$$

the *rough singular integral* is defined by

$$T_\Omega f(x) = \text{p.v.} \int_{\mathbb{R}^n} \Omega\left(\frac{y}{|y|}\right) f(x-y) \frac{dy}{|y|^n} = \lim_{\varepsilon \to 0^+} \int_{|y|>\varepsilon} \Omega\left(\frac{y}{|y|}\right) f(x-y) \frac{dy}{|y|^n},$$

whenever this limit exists. This operator was first introduced by Calderón and Zygmund who proved that (see [38, 39]) T_Ω is bounded in $L^p(\mathbb{R}^n)$ if the even part of Ω belongs to $L \log_+ L(\mathbb{R}^n)$ and its odd part belongs to $L^1(\mathbb{R}^n)$. Since then, this operator has been widely studied (see [68, 69, 178]). In [90], J. Duoandikoetxea and J.L. Rubio de Francia proved, for $p > 1$, its weighted strong-type (p,p) boundedness for every weight in the Muckenhoupt class A_p, later improved in [87], [114] and [179]. Further, in [140, Theorem 1.6] the authors obtained the following result.

Proposition 5.3.1. *Let $n > 1$ and $\Omega \in L^\infty(\mathbb{S}^{n-1})$ satisfying (5.3.1). Then,*

$$T_\Omega : L^2(v) \to L^{2,\infty}(v), \qquad \forall v \in A_2,$$

with constant less than or equal to $\|v\|_{A_1}^2$.

Therefore, together with its corresponding unweighted weak-type $(1,1)$ estimate; that is

$$T_\Omega : L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n),$$

(see, for instance, [122, 140, 178]) as a consequence of Corollaries 4.3.12 and 4.3.16, we get the following result.

Corollary 5.3.2. *Let $n > 1$ and $\Omega \in L^\infty(\mathbb{S}^{n-1})$ satisfying (5.3.1). For every $0 < p < \infty$,*

$$T_\Omega : \Lambda^{p,1}(w) \to \Lambda^{p,\infty}(w), \qquad \forall w \in B_p^R \cap B_\infty^*,$$

with constant less than or equal to $C\, \|w\|_{B_p^R} \|w\|_{B_\infty^*}^2$. Further, for every locally integrable function f and for every $t > 0$,

$$(T_\Omega f)^*(t) \lesssim \frac{1}{t} \int_0^t f^*(s)\, ds + \int_t^\infty \left(1 + \log \frac{s}{t}\right) f^*(s) \frac{ds}{s}.$$

In [87, Theorem 5], the authors considered $\Omega \in L^q(\mathbb{S}^{n-1})$, for $1 < q < \infty$, and proved the following result.

Proposition 5.3.3 ([87])**.** *Let $n > 1$, $q > 1$ and $\Omega \in L^q(\mathbb{S}^{n-1})$ satisfying (5.3.1). Then, there exists a positive nondecreasing function φ on $[1,\infty)$ such that*

$$T_\Omega : L^{q'}(v) \to L^{q'}(v), \qquad \forall v \in A_1,$$

with constant less than or equal to $\varphi\left(\|v\|_{A_1}\right)$.

We should mention that although the boundedness result of Proposition 5.3.3 is known, the behaviour of the function φ (that is nondecreasing) has not been written explicitly anywhere, and since we want to make use of Theorem 4.2.5, we need such behaviour. Hence, as we did for the Fourier multipliers of Hörmander type (see Section 5.2.1) we will dedicate what follows to find the desired behaviour on φ.

To do so, we refer to [75, Corollary A.1], where it was considered the case that Ω belongs to the Orlicz-Lorentz space $L^{q,1}\log L(\mathbb{S}^{n-1})$, $q>1$, with norm

$$\|\Omega\|_{L^{q,1}\log L(\mathbb{S}^{n-1})} = \int_0^1 \Omega^*(t) t^{\frac{1}{q}-1} \log\frac{1}{t}\, dt < \infty.$$

Besides, in [140, Corollary 1.15] it was proved the following result.

Proposition 5.3.4. *Let $n>1$, $q>1$ and $\Omega \in L^{q,1}\log L(\mathbb{S}^{n-1})$ satisfying (5.3.1). Then, for every $p>q'$,*

$$\|T_\Omega\|_{L^p(v)} \leqslant c_n q \|\Omega\|_{L^{q,1}\log L(\mathbb{S}^{n-1})}\, p \left(\frac{p}{q'}\right)' (c_n p')^{\frac{q'(p-1)}{p-q'}} \|v\|_{A_1}, \qquad \forall v \in A_1.$$

Now observe that, indeed, we can avoid the Lorentz norm in the statement of Proposition 5.3.4 by appealing to the continuous embeddings $L^q(\mathbb{S}^{n-1}) \subseteq L^{q-\varepsilon,1}\log L(\mathbb{S}^{n-1})$ for all $q>1$ and $0<\varepsilon<q-1$, since by means of the Hölder's inequality,

$$\|\Omega\|_{L^{q-\varepsilon,1}\log L(\mathbb{S}^{n-1})} \leqslant \|\Omega\|_{L^q(\mathbb{S}^{n-1})} \left(\int_0^1 t^{q'\left(\frac{1}{q-\varepsilon}-1\right)} \log\frac{1}{t}\, dt\right)^{\frac{1}{q'}} \leqslant \frac{c_q}{\varepsilon^2} \|\Omega\|_{L^q(\mathbb{S}^{n-1})}. \tag{5.3.2}$$

Therefore, we are now ready to settle our first main goal in this section:

Proof of Proposition 5.3.3. First, we will see that for $p>q'$,

$$T_\Omega : L^p(v) \to L^p(v), \qquad \forall v \in A_1, \tag{5.3.3}$$

with constant less than or equal to $\varphi_{p,q}(\|v\|_{A_1})$, where $\varphi_{p,q}$ is a positive nondecreasing function on $[1,\infty)$.

Let $0<\varepsilon<q-p'$ so that $p>(q-\varepsilon)'$. Then, by Proposition 5.3.4 and (5.3.2), we have that for every $v\in A_1$,

$$\begin{aligned}
\|T_\Omega\|_{L^p(v)} &\lesssim (q-\varepsilon)\|\Omega\|_{L^{q-\varepsilon,1}\log L(\mathbb{S}^{n-1})}\, p\left(\frac{p}{(q-\varepsilon)'}\right)' (c_n p')^{\frac{(q-\varepsilon)'(p-1)}{p-(q-\varepsilon)'}} \|v\|_{A_1} \\
&\lesssim \frac{\|\Omega\|_{L^q(\mathbb{S}^{n-1})}}{\varepsilon^2}\left(\frac{p^2}{(q-\varepsilon)(p-1)-p}\right)(c_n q)^{\frac{q}{q-\varepsilon-p'}} \|v\|_{A_1}.
\end{aligned}$$

In particular, for $\varepsilon = \frac{q-p'}{2}$,

$$\begin{aligned}
\|T_\Omega\|_{L^p(v)} &\lesssim \|\Omega\|_{L^q(\mathbb{S}^{n-1})} \frac{p}{(q-p')^3} (c_n q)^{\frac{2q}{q-p'}} \|v\|_{A_1} \\
&\lesssim \|\Omega\|_{L^q(\mathbb{S}^{n-1})} \frac{p(p-1)^3}{(p-q')^3} (c_n q)^{\frac{2q'(p-1)}{p-q'}} \|v\|_{A_1} := \varphi_{p,q}(\|v\|_{A_1}),
\end{aligned}$$

so that we obtain (5.3.3).

Now, in order to prove Proposition 5.3.3 we will make use of an interpolation argument. Hence, fix $1 < p_0 = \min\left(\frac{q+1}{2}, \frac{q'+1}{2}\right)$ (so that $p_0 < \min(q, q')$). Therefore, from (5.3.3) and by duality (since T_Ω is, essentially, self-adjoint) we obtain that

$$\|T_\Omega\|_{L^{p_0}(v)} \leqslant C_{n,q,p_0}.$$

Further, take $v \in A_1$ and set

$$t = \min\left(1 + \frac{1}{2^{n+1}\|v\|_{A_1}}, \frac{2q'-p_0}{2(q'-p_0)}\right) > 1$$

so that $v^t \in A_1$ with $\|v^t\|_{A_1} \lesssim \|v\|_{A_1}$ (see (2.2.4)). Moreover, let $\theta = \frac{1}{t}$, $v_0 = 1$, $v_1 = v^t$ and

$$q' < p_1 := \frac{q'p_0}{q' - t(q'-p_0)} \leqslant 2q'.$$

Hence, from (5.3.3) we get

$$\begin{aligned}
\|T_\Omega\|_{L^{p_1}(v_1)} \lesssim \varphi_{p_1,q}\left(\|v_1\|_{A_1}\right) &= \|\Omega\|_{L^q(\mathbb{S}^{n-1})} \frac{p_1(p_1-1)^3}{(p_1-q')^3} (c_n q)^{\frac{2q'(p_1-1)}{p_1-q'}} \|v\|_{A_1} \\
&\lesssim \|\Omega\|_{L^q(\mathbb{S}^{n-1})} \frac{(c_n q)^{\frac{2q'(2q'-1)}{(t-1)(q'-p_0)}}}{(t-1)^3} \|v\|_{A_1} =: \Phi_q(\|v\|_{A_1}),
\end{aligned}$$

where Φ_q is a nondecreasing function on $[1,\infty)$. Therefore, by interpolation with change of measure (see, for instance, [23, Theorem 3.6]),

$$\|T_\Omega\|_{L^{q'}(v)} \lesssim \Phi_q(\|v\|_{A_1})^{\frac{1}{t}} \leqslant \max\left(1, \Phi_q(\|v\|_{A_1})\right).$$

□

Therefore, as a consequence of Theorem 4.2.5 and Proposition 5.3.3, we get the following result.

Corollary 5.3.5. *Let $n > 1$, $q > 1$ and $\Omega \in L^q(\mathbb{S}^{n-1})$ satisfying (5.3.1). For every $0 < p < \infty$,*

$$T_\Omega : \Lambda^{p,1}(w) \to \Lambda^{p,\infty}(w), \qquad \forall w \in B^R_{\frac{p}{q'}} \cap B^*_\infty.$$

Furthermore, if we assume that Ω also satisfies the L^q-Dini condition; that is

$$\int_0^1 \omega_q(y)\frac{dy}{y} < +\infty \qquad \text{with} \qquad \omega_q(y) = \sup_{|\rho|<y}\left(\int_{\mathbb{S}^{n-1}} |\Omega(\rho x) - \Omega(x)|^q \, d\sigma\right), \tag{5.3.4}$$

where the supremum is taken over all rotations ρ of $\mathbb{S}^{n-1}$ and where $|\rho| = \sup_{x\in\mathbb{S}^{n-1}} |\rho x - x|$, in [122, Theorem 4] it was shown the following result.

Proposition 5.3.6 ([122]). *Let $n > 1$, $q > 1$ and $\Omega \in L^q(\mathbb{S}^{n-1})$ satisfying (5.3.1) and the L^q-Dini condition (5.3.4). Then, there exists a positive nondecreasing function φ on $[1,\infty)$ such that*

$$T_\Omega : L^1\left(v^{\frac{1}{q'}}\right) \to L^{1,\infty}\left(v^{\frac{1}{q'}}\right), \qquad \forall v \in A_1,$$

with constant less than or equal to $\varphi\left(\|v\|_{A_1}^{\frac{1}{q'}}\right)$.

Remark 5.3.7. *As in Proposition 5.3.3, the behaviour of φ in Proposition 5.3.6 is unknown. However, to see that φ is a nondecreasing function on $[1,\infty)$ follows the same lines as the one for Proposition 5.2.1 (i) except for few modifications due to the different kernel, and it is based in [122, Lemma 5] instead of [122, Lemma 1] and it makes use of (5.3.3) (for $p = 2$ and weights $v^{\frac{1}{q'}} \in A_1$) instead of (5.2.2).*

Therefore, as a consequence of Theorem 4.2.5 and Proposition 5.3.6, we get the following result.

Corollary 5.3.8. *Let $n > 1$, $q > 1$ and $\Omega \in L^q(\mathbb{S}^{n-1})$ satisfying (5.3.1) and the L^q-Dini condition (5.3.4). For $0 < p < \infty$,*

$$T_\Omega : \Lambda^{p,1}(w) \to \Lambda^{p,\infty}(w), \qquad \forall w \in \left(B^R_{\frac{p}{q'}} \cap B^*_\infty\right) \cup \left(B^R_p \cap B^*_{\frac{q}{p}}\right).$$

5.4 Intrinsic square functions

For $0 < \alpha \leqslant 1$, let $\mathcal{C}_\alpha$ be the family of functions ϕ supported in $\mathbb{B}(0,1)$ (the n-th dimensional open ball of center 0 and radius 1) such that

$$\int_{\mathbb{B}(0,1)} \phi(x)\,dx = 0 \qquad \text{and} \qquad |\phi(x) - \phi(x')| < |x - x'|^\alpha, \quad \forall x, x' \in \mathbb{R}^n.$$

Then, given $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, set

$$A_\alpha f(y,t) = \sup_{\phi \in \mathcal{C}^\alpha} |(\phi_t * f)(y)|, \qquad (y,t) \in \mathbb{R}^{n+1}_+,$$

where we are using ϕ_t to denote the usual $L^1(\mathbb{R}^n)$ dilatation of ϕ; that is $\phi_t(x) = t^{-n}\phi\left(\frac{x}{t}\right)$.

The *intrinsic square function* (of order α) introduced by M. Wilson in [181] is defined by

$$G_\alpha f(x) = \left(\int_{\Gamma_\alpha(x)} |A_\alpha(f)(y,t)|^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}, \qquad x \in \mathbb{R}^n,$$

with $\Gamma_\alpha(x) = \{(y,t) : |x-y| < \alpha t\}$. In [130] it was proved that

$$\|G_\alpha f\|_{L^3(v)} \lesssim \|v\|_{A_3}^{\frac{1}{2}} \|f\|_{L^3(v)}, \qquad \forall v \in A_3,$$

and using the extrapolation of Rubio de Francia (see Theorem 2.3.1) it was obtained that, for every $1 < p < \infty$,

$$\|G_\alpha f\|_{L^p(v)} \lesssim \|v\|_{A_p}^{\max\left(\frac{1}{2},\frac{1}{p-1}\right)} \|f\|_{L^p(v)}, \qquad \forall v \in A_p,$$

and the exponent $\max\left(\frac{1}{2}, \frac{1}{p-1}\right)$ is the best possible (see [128]). Moreover, in [61, 181] it was shown that G_α also satisfies the weighted weak-type $(1,1)$ inequality

$$\|G_\alpha f\|_{L^{1,\infty}(v)} \lesssim \|v\|_{A_1}^{\frac{5}{2}} \|f\|_{L^1(v)}, \qquad \forall v \in A_1.$$

Therefore, as a consequence of Theorem 4.2.2 and Corollary 4.3.14, we get the following result.

Corollary 5.4.1. *Let $0 < \alpha \leqslant 1$. For every $0 < p < \infty$,*

$$G_\alpha : \Lambda^{p,1}(w) \to \Lambda^{p,\infty}(w), \qquad \forall w \in B_p^R \cap B_\infty^*,$$

with constant less than or equal to $C \|w\|_{B_p^R} \|w\|_{B_\infty^*}^{5/2}$. Further, for every locally integrable function f and for every $t > 0$,

$$(G_\alpha f)^*(t) \lesssim \frac{1}{t}\int_0^t f^*(s)\,ds + \int_t^\infty \left(1 + \log\frac{s}{t}\right)^{\frac{3}{2}} f^*(s)\,\frac{ds}{s}.$$

Remark 5.4.2. *In [181] was proved that G_α dominates pointwise (modulo constant) operators such as the Lusin area integral, the Littlewood-Paley g-function and the continuous square function (see also [61]). Therefore, analogous results as in Corollary 5.4.1 can be derived for those operators as well.*

5.5 Sparse operators

These operators have become very popular due to their role in the often-called A_2 conjecture consisting in proving that if T is a Calderón-Zygmund operator then

$$\|Tf\|_{L^2(v)} \lesssim \|v\|_{A_2} \|f\|_{L^2(v)}, \qquad \forall v \in A_2.$$

This result was first obtained by T.P. Hytönen [113] and then simplified by A.K. Lerner [131, 132], who proved that the norm of a Calderón-Zygmund operator in a Banach function space $\mathbb{X}$ is dominated by the supremum of the norm in $\mathbb{X}$ of all the possible sparse operators, and then proved that every sparse operator is bounded in $L^2(v)$ for every weight $v \in A_2$ with sharp constant.

Let us give the precise definition. First, a general *dyadic grid* $\mathcal{D}$ is a collection of cubes in $\mathbb{R}^n$ satisfying the following properties:

(i) For any cube $Q \in \mathcal{D}$, its side length is 2^k for some $k \in \mathbb{Z}$.

(ii) Every two cubes in $\mathcal{D}$ are either disjoint or one is wholly contained in the other.

(iii) If $\mathcal{D}_k \subseteq \mathcal{D}$ is the subfamily of cubes formed by the cubes of exactly side length 2^k, $k \in \mathbb{Z}$, then $\mathcal{D}_k$ form a partition of $\mathbb{R}^n$.

Hence, let $0 < \eta < 1$ and let $\mathcal{D}$ be a family of dyadic cubes. A collection of cubes $\mathcal{S} \subseteq \mathcal{D}$ is called *η-sparse* if one can choose pairwise disjoint measurable sets $E_Q \subseteq Q$ with $|E_Q| \geqslant \eta|Q|$, where $Q \in \mathcal{S}$ (see [88, 133] for more details). Hence, given a η-sparse family of cubes $\mathcal{S} \subseteq \mathcal{D}$ and let $1 \leqslant r < \infty$, the *r-sparse operator* $\mathcal{A}_{r,\mathcal{S}}$ corresponding to the family $\mathcal{S}$ is defined by

$$\mathcal{A}_{r,\mathcal{S}} f(x) = \sum_{Q \in \mathcal{S}} \left(\frac{1}{|Q|} \int_Q |f(y)|^r \, dy \right)^{\frac{1}{r}} \chi_Q(x), \qquad x \in \mathbb{R}^n.$$

When $r = 1$, we recover the classical *sparse operator* denoted by $\mathcal{A}_{\mathcal{S}} = \mathcal{A}_{1,\mathcal{S}}$ (see [132]).

Proposition 5.5.1 ([112]). *The sparse operator $\mathcal{A}_{\mathcal{S}}$ satisfies*

$$\|\mathcal{A}_{\mathcal{S}} f\|_{L^{1,\infty}(v)} \lesssim \|v\|_{A_1} \log\left(1 + \|v\|_{A_1}\right) \|f\|_{L^1(v)}, \qquad \forall v \in A_1. \tag{5.5.1}$$

In [136] the same bound was proved for a Calderón-Zygmund operator and in [137] the authors showed that this last result was sharp. Hence, using the domination property of the Calderón-Zygmund operators by sparse operators, it can be concluded that (5.5.1) is also sharp. We thank A.K. Lerner for this information.

Further, for $p > 1$, the r-sparse operator also satisfies weighted restricted weak-type (p, p) estimates. Indeed, the proof follows by duality and using the same ideas as in [61, Theorem 4.1] with the necessary modifications (see also [45, Corollary 3.2]).

Proposition 5.5.2. *Given $r \geqslant 1$, for every $p \geqslant r$, $p > 1$,*

$$\|\mathcal{A}_{r,\mathcal{S}} f\|_{L^{p,\infty}(v)} \leqslant \frac{C_{n,p,r}}{p-1} \|v\|_{A^{\mathcal{R}}_{\frac{p}{r}}}^{\frac{r+p}{r}} \|f\|_{L^{p,1}(v)}, \qquad \forall v \in A^{\mathcal{R}}_{\frac{p}{r}}. \tag{5.5.2}$$

Therefore, as a consequence of Corollaries 4.3.14 and 4.3.21, and by means of Propositions 5.5.1 and 5.5.2, we can get estimates on the decreasing rearrangement of $\mathcal{A}_{r,\mathcal{S}}$. However, due to the weight constants involved in (5.5.1) and (5.5.2) are not linear, that estimates are far from being sharp.

Indeed, one can easily compute the norm from $L^p(\mathbb{R}^n)$ to $L^{p,\infty}(\mathbb{R}^n)$ directly, using the standard duality technique: given $r > 1$, for every $p \geqslant r$ and a measurable function $f \in L^p(\mathbb{R}^n)$,

$$\|\mathcal{A}_{r,\mathcal{S}} f\|_{L^{p,\infty}(\mathbb{R}^n)} \leqslant \frac{p}{p-1} \sup_{\|g\|_{L^{p',1}(\mathbb{R}^n)} \leqslant 1} \int \mathcal{A}_{r,\mathcal{S}} f(x)|g(x)|dx,$$

and, by taking such a function g, since $\|M\|_{L^{p',1}(\mathbb{R}^n)} \leqslant c_n p$, then

$$\begin{aligned}
\int \mathcal{A}_{r,\mathcal{S}} f(x)|g(x)|dx &= \sum_{Q \in \mathcal{S}} \left(\frac{1}{|Q|} \int_Q |f(y)|^r \, dy \right)^{\frac{1}{r}} \int_Q |g(x)|dx \\
&\leqslant \frac{1}{\eta} \sum_{Q \in \mathcal{S}} |E_Q| \left(\frac{1}{|Q|} \int_Q |f(y)|^r \, dy \right)^{\frac{1}{r}} \frac{1}{|Q|} \int_Q |g(x)|dx \\
&\leqslant \frac{1}{\eta} \int_{\mathbb{R}^n} M(f^r)(x)^{\frac{1}{r}} Mg(x)dx \leqslant \frac{1}{\eta} \|M(|f|^r)\|_{L^{\frac{p}{r},\infty}(\mathbb{R}^n)}^{\frac{1}{r}} \|Mg\|_{L^{p',1}(\mathbb{R}^n)} \\
&\leqslant \frac{C_n}{\eta} \|f\|_{L^{\frac{p}{r}}(\mathbb{R}^n)}^{\frac{1}{r}} p \|g\|_{L^{p',1}(\mathbb{R}^n)} \leqslant \frac{C_n}{\eta} p \|f\|_{L^p(\mathbb{R}^n)},
\end{aligned}$$

so we obtain that, in particular,

$$\|\mathcal{A}_{r,\mathcal{S}}\chi_E\|_{L^{p,\infty}(\mathbb{R}^n)} \leqslant \frac{C_n r}{\eta(r-1)} p|E|^{\frac{1}{p}}, \qquad \forall E \subseteq \mathbb{R}^n. \tag{5.5.3}$$

Further, if $r = 1$, we can estimate (5.5.3) for every $p \geqslant 2$ by

$$\|\mathcal{A}_{r,\mathcal{S}}\chi_E\|_{L^{p,\infty}(\mathbb{R}^n)} \leqslant \frac{2C_n}{\eta} p|E|^{\frac{1}{p}}, \qquad \forall E \subseteq \mathbb{R}^n,$$

so that due to interpolation between the unweighted restricted $(1,1)$ and $(2,2)$ estimates of $A_{1,\mathcal{S}}$ for characteristic functions (see, for instance, [88, 174]) we have that for $1 < p < 2$, the corresponding constant norm of the unweighted restricted (p,p) estimate for characteristic functions is also linear on p.

Therefore, as a consequence of Theorem 4.3.22, we get the following result.

Corollary 5.5.3. *Given $r \geqslant 1$. For every locally integrable function f and every $t > 0$,*

$$\left(\mathcal{A}_{r,\mathcal{S}}f\right)^*(t) \lesssim \frac{1}{t^{\frac{1}{r}}} \int_0^t f^*(s)\, \frac{ds}{s^{1-\frac{1}{r}}} + \int_t^\infty f^*(s)\, \frac{ds}{s}. \tag{5.5.4}$$

Now, it is known (see [22] and [23, Ch. 3 - Theorem 4.7]) that, for every locally integrable function f,

$$(Hf)^*(t) \lesssim \frac{1}{t}\int_0^t f^*(s)\, ds + \int_t^\infty f^*(s)\, \frac{ds}{s} \lesssim (Hg)^*(t), \qquad \forall t > 0,$$

for some g being an equimeasurable function with f and where H is the Hilbert transform. Hence, by the pointwise domination of the Hilbert transform by Sparse operators, we conclude that the estimate in (5.5.4), at least for $r = 1$, is sharp for the sparse operator $A_{\mathcal{S}}$.

Finally, by means of Corollaries 5.5.3, 4.3.12 and 4.3.19, we obtain the following result.

Corollary 5.5.4. *Given $r \geqslant 1$. For every $0 < p < \infty$,*

$$T : \Lambda^{p,1}(w) \to \Lambda^{p,\infty}(w), \qquad \forall w \in B^R_{\frac{p}{r}} \cap B^*_\infty,$$

with constant less than or equal to $C\,\|w\|^{\frac{1}{r}}_{B^R_{\frac{p}{r}}} \|w\|_{B^*_\infty}$.

5.6 The Assani operator

There is a very interesting operator, named *Assani operator* after being introduced in [13] by one of their authors, which is related with the Return time theorem of Bourgain [27] (see [12, 13] for a very interesting review on this topic) and so, as well, with the ergodic theory: branch of mathematics that studies statistical properties of deterministic dynamical systems that has its origins on the work of Boltzmann [26] in statistical mechanics problems (for more details see [147]).

The Assani operator is defined as

$$Af(x) = \left\| \frac{f(\cdot)\chi_{(0,x)}(\cdot)}{x-\cdot} \right\|_{L^{1,\infty}(0,1)}, \qquad x \in \mathbb{R},$$

where, for every locally integrable function f,

$$\|f\|_{L^{1,\infty}(0,1)} = \sup_{y>0} y\,|\{x \in (0,1) \,:\, |f(x)| > y\}|\,.$$

This operator satisfies two important properties:

(i) For every $0 < q < 1$,

$$A : L^{1,q}(\mathbb{R}) \to L^{1,\infty}(\mathbb{R}). \tag{5.6.1}$$

This boundedness was obtained in [62] and, as a consequence, it was proved that the space $L^{1,q}(\mathbb{R})$ satisfies the Return Time Property for the Tail, while this is not the case for $L^1(\mathbb{R})$ (see [13]) since A does not satisfy the unweighted weak-type $(1,1)$ inequality; that is

$$A : L^1(\mathbb{R}) \nrightarrow L^{1,\infty}(\mathbb{R}).$$

(ii) For every measurable set $E \subseteq \mathbb{R}$, $A\chi_E \leqslant M\chi_E$, and all the restricted weighted inequalities satisfied by M also holds for A. In particular, let $p > 1$ and $v \in A_p^R$. Further, take $f \in L^{p,1}(v)$ and define

$$E_i = \left\{x \in \mathbb{R}^n : 2^{i-1} < |f(x)| \leqslant 2^i\right\}, \qquad \forall i \in \mathbb{Z},$$

and $f_i = f\chi_{E_i}$. Then, since A is sublinear on disjointly supported functions and monotone, we have that

$$Af(x) \leqslant \sum_{i\in\mathbb{Z}} Af_i(x) \leqslant \sum_{i\in\mathbb{Z}} 2^i A\chi_{E_i}(x), \qquad \forall x \in \mathbb{R},$$

so that

$$\begin{aligned}
\|Af\|_{L^{p,\infty}(v)} &\leqslant \frac{p}{p-1}\sum_{i\in\mathbb{Z}} 2^i \|A\chi_{E_i}\|_{L^{p,\infty}(v)} \leqslant \frac{p}{p-1}\sum_{i\in\mathbb{Z}} 2^i \|M\chi_{E_i}\|_{L^{p,\infty}(v)} \\
&\lesssim \frac{1}{p-1}\|v\|_{A_p^R}\sum_{i\in\mathbb{Z}} 2^i v(E_i)^{\frac{1}{p}} \lesssim \frac{1}{p-1}\|v\|_{A_p^R}\sum_{i\in\mathbb{Z}}\int_{2^{i-2}}^{2^{i-1}} v\left(\{|f|>t\}\right)^{\frac{1}{p}}\,dt \\
&= \frac{1}{p-1}\|v\|_{A_p^R}\int_{\mathbb{R}} v\left(\{|f|>t\}\right)^{\frac{1}{p}}\,dt = \frac{1}{p-1}\|v\|_{A_p^R}\|f\|_{L^{p,1}(v)}\,.
\end{aligned}$$

Therefore, as a consequence of Theorem 4.2.2 we get the following result.

Corollary 5.6.1. *Let $0 < q < 1$. Then, for every $0 < p < \infty$,*

$$A : \Lambda^{p,q}(w) \to \Lambda^{p,\infty}(w), \qquad \forall w \in B_p^R \cap B_\infty^*,$$

with constant less than or equal to $\frac{C_{p,q,n}}{(1-q)^2}\|w\|_{B_p^R}\|w\|_{B_\infty^*}^q$.

Hence, observe that in the case $w = 1$, we recover the boundedness for every $0 < q < 1$ of (5.6.1), so again we obtain that $L^{1,q}(\mathbb{R})$ satisfies the Return Time Property for the Tail. On the other hand, it is an interesting open question in the area whether the space

$$L \log\log\log L = \Lambda^1\left(1 + \log\left(1 + \log\left(1 + \log\frac{1}{\cdot}\right)\right)\right), \tag{5.6.2}$$

satisfies this property. Hence, it will be very interesting to study for which weights w the Assani operator is bounded from $\Lambda^1(w)$ to $\Lambda^{1,\infty}(w)$. Indeed, since for $w \in B_1$ then $\Lambda^{1,\infty}(w)$ is a Banach function space, from Proposition 2.1.2 we deduce that

$$A : \Lambda^1(w) \to \Lambda^{1,\infty}(w), \qquad \forall w \in B_1 \cap B_\infty^*.$$

However, (5.6.2) is neither covered for this class of weights, since

$$w(t) = 1 + \log\left(1 + \log\left(1 + \log\frac{1}{t}\right)\right) \in (B_1^R \backslash B_1) \cap B_\infty^*.$$

5.7 The Bochner-Riesz operator

Let $n > 1$ and $\lambda > 0$. Recall that the Bochner-Riesz operator B_λ (see Section 3.5.4) above the critical index (that is, for $\lambda > \frac{n-1}{2}$) is controlled by the Hardy-Littlewood maximal operator M. Hence, for every locally integrable function f and for every $t > 0$,

$$(B_\lambda f)^*(t) \lesssim \frac{1}{t}\int_0^t f^*(s)\,ds$$

(see (2.2.22)) and, for any $0 < p < \infty$,

$$B_\lambda : \Lambda^{p,1}(w) \to \Lambda^{p,\infty}(w), \qquad \forall w \in B_p^R,$$

with constant less than or equal to $C\,\|w\|_{B_p^R}$ (see (2.2.13)).

Now, at the critical index, in [140, Theorem 1.6] the authors obtained the following quantitative result.

Proposition 5.7.1. *Let $n > 1$. Then,*

$$B_{\frac{n-1}{2}} : L^2(v) \to L^{2,\infty}(v), \qquad \forall v \in A_2,$$

with constant less than or equal to $C\,\|v\|_{A_2}^2$.

Therefore, as a consequence of Corollaries 4.3.12 and 4.3.16, we get the following result.

Corollary 5.7.2. *Let $n > 1$. For every $0 < p < \infty$,*

$$B_{\frac{n-1}{2}} : \Lambda^{p,1}(w) \to \Lambda^{p,\infty}(w), \qquad \forall w \in B_p^R \cap B_\infty^*,$$

with constant less than or equal to $C\,\|w\|_{B_p^R}\,\|w\|_{B_\infty^*}^2$. Further, for every locally integrable function f and for every $t > 0$,

$$\left(B_{\frac{n-1}{2}} f\right)^*(t) \lesssim \frac{1}{t}\int_0^t f^*(s)\,ds + \int_t^\infty \left(1 + \log\frac{s}{t}\right) f^*(s)\,\frac{ds}{s}.$$

Finally, below the critical index, we already know that B_λ satisfies the unweighted weak-type estimates at the endpoint (3.5.4) and (3.5.6). Hence, by duality we obtain the unweighted restricted weak-type estimate for $n = 2$,

$$B_\lambda : L^{\frac{4}{1-2\lambda},1}(\mathbb{R}^2) \to L^{\frac{4}{1-2\lambda},\infty}(\mathbb{R}^2), \qquad \forall\, 0 < \lambda < \frac{1}{2},$$

and, for $n > 2$, we get

$$B_\lambda : L^{\frac{2n}{n-1-2\lambda},1}(\mathbb{R}^n) \to L^{\frac{2n}{n-1-2\lambda},\infty}(\mathbb{R}^n), \qquad \forall\, \frac{n-1}{2(n+1)} \leqslant \lambda < \frac{n-1}{2}.$$

Then, as a consequence of [23, Ch. 4 - Theorem 4.11], for every locally integrable function f and every $t > 0$,

$$(B_\lambda f)^*(t) \lesssim \frac{1}{t^{\frac{3+2\lambda}{4}}} \int_0^t f^*(s)\, \frac{ds}{s^{\frac{1-2\lambda}{4}}} + \frac{1}{t^{\frac{1-2\lambda}{4}}} \int_t^\infty f^*(s)\, \frac{ds}{s^{\frac{3+2\lambda}{4}}}, \qquad \forall\, 0 < \lambda < \frac{1}{2},$$

and, for $n > 2$,

$$(B_\lambda f)^*(t) \lesssim \frac{1}{t^{\frac{n+1+2\lambda}{2n}}} \int_0^t f^*(s)\, \frac{ds}{s^{\frac{n-1-2\lambda}{2n}}} + \frac{1}{t^{\frac{n-1-2\lambda}{2n}}} \int_t^\infty f^*(s)\, \frac{ds}{s^{\frac{n+1+2\lambda}{2n}}}, \quad \forall\, \frac{n-1}{2(n+1)} \leqslant \lambda < \frac{n-1}{2}.$$

Therefore, from Corollary 4.3.19 we deduce the following result.

Corollary 5.7.3. *Given* $0 < p < \infty$.

(i) If $n = 2$ *and* $0 < \lambda < \frac{1}{2}$,

$$B_\lambda : \Lambda^{p,1}(w) \to \Lambda^{p,\infty}(w), \qquad \forall w \in B^R_{\frac{p(3+2\lambda)}{4}} \cap B^*_{\frac{4}{p(1-2\lambda)}},$$

with constant less than or equal to

$$C_1 \|w\|_{B^R_{\frac{p(3+2\lambda)}{4}}}^{\frac{3+2\lambda}{4}} \begin{cases} \|w\|_{B^*_{\frac{4}{p(1-2\lambda)}}}^{\frac{1}{p}}, & 0 < p \leqslant 1, \\ \|w\|_{B^*_{\frac{4}{p(1-2\lambda)}}}^{\frac{1}{p}+1}, & 1 < p < \infty. \end{cases}$$

(ii) If $n > 2$ *and* $\frac{n-1}{2(n+1)} \leqslant \lambda < \frac{n-1}{2}$,

$$B_\lambda : \Lambda^{p,1}(w) \to \Lambda^{p,\infty}(w), \qquad \forall w \in B^R_{\frac{p(n+1+2\lambda)}{2n}} \cap B^*_{\frac{2n}{p(n-1-2\lambda)}},$$

with constant less than or equal to

$$C_2 \|w\|_{B^R_{\frac{p(n+1+2\lambda)}{2n}}}^{\frac{n+1+2\lambda}{2n}} \begin{cases} \|w\|_{B^*_{\frac{2n}{p(n-1-2\lambda)}}}^{\frac{1}{p}}, & 0 < p \leqslant 1, \\ \|w\|_{B^*_{\frac{2n}{p(n-1-2\lambda)}}}^{\frac{1}{p}+1}, & 1 < p < \infty. \end{cases}$$

At this point, define the class

$$\mathcal{W}(B_\lambda) = \left\{ 0 \leqslant w \in L^1_{\mathrm{loc}}(\mathbb{R}^+) : \|w\|_{\mathcal{W}(B_\lambda)} = \sup_{f \in \Lambda^1(w)} \frac{\sup_{t>0} (B_\lambda f)^*(t) W(t)}{\int_0^\infty f^*(t) w(t)\, dt} < \infty \right\}.$$

Then, from (2.1.2), one can immediately see that

$$w_{n,\lambda}(t) = t^{\frac{n+1+2\lambda}{2n}-1} \in \mathcal{W}(B_\lambda) \quad \Longrightarrow \quad \text{the endpoint Bochner-Riesz conjecture holds for } B_\lambda,$$

and so does the Bochner-Riesz conjecture as well. On the other side (see (2.2.40) and (2.2.41)), given $q > \frac{2n}{n+1+2\lambda}$,

$$w_{n,\lambda} \in B^R_{\frac{n+1+2\lambda}{2n}} \cap B^*_q, \qquad \forall n > 1 \ \text{ and } \ 0 < \lambda < \frac{n-1}{2},$$

so an interesting open question is to study for which $n > 1$ and $0 < \lambda < \frac{n-1}{2}$,

$$B^R_{\frac{n+1+2\lambda}{2n}} \cap B^*_q \subseteq \mathcal{W}(B_\lambda), \qquad \text{for some } q > \frac{2n}{n+1+2\lambda}.$$

Chapter 6

Multi-variable weighted estimates on $\Lambda^p(w)$

In this chapter, we pursue on proving weighted restricted weak-type estimates over classical Lorentz spaces $\Lambda^p(w)$ but now in the multi-variable setting. With this aim, in Section 6.1 we will first introduce and study the m-fold product of Hardy-Littlewood maximal operators, which is deeply related to the multi-variable type extrapolation. Further, in Sections 6.2 and 6.3 we will present our main results on multi-variable extrapolation, based on weighted restricted weak-type and mixed-type estimates respectively. Finally, in Section 6.4 we will show some applications of the results on multi-variable extrapolation applied to bilinear Fourier multipliers (see Section 6.4.1) and multilinear sparse operators (see Section 6.4.2).

6.1 The m-fold product of Hardy-Littlewood maximal operators

Due to its close relation with multi-variable extrapolation, our first goal is to study the boundedness over classical Lorentz spaces of the m-fold product of Hardy-Littlewood maximal operators defined as

$$M^{\otimes}(f_1, \dots, f_m)(x) = \prod_{i=1}^{m} Mf_i(x), \qquad x \in \mathbb{R}^n,\ f_1, \dots, f_m \in L^1_{\text{loc}}(\mathbb{R}^n).$$

Indeed, the corresponding multi-variable weighted restricted weak-type boundedness of $M^{\otimes}$ on Lorentz spaces (see [161, Theorem 2.4.1] and [53, Theorem 3.3]) is characterized as follows: given $1 \leqslant p_1, \dots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and $v_1, \dots, v_m \in A_\infty$,

$$M^{\otimes} : L^{p_1,1}(v_1) \times \cdots \times L^{p_m,1}(v_m) \to L^{p,\infty}(v_1^{p/p_1} \cdots v_m^{p/p_m}) \quad \Longleftrightarrow \quad v_i \in A^R_{p_i},\ i = 1, \dots, m,$$

and the same holds with multi-variable weighted restricted weak-type boundedness on classical Lorentz spaces:

Theorem 6.1.1. *Set $m \geqslant 1$. Let $0 < q_1, \dots, q_m < \infty$ and $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$. Then,*

$$M^{\otimes} : \Lambda^{q_1,1}(w_1) \times \cdots \times \Lambda^{q_m,1}(w_m) \to \Lambda^{p,\infty}(w), \qquad \forall w_i \in B^R_{q_i},\ i = 1, \dots, m, \tag{6.1.1}$$

with

$$\left\|M^{\otimes}\right\| := \left\|M^{\otimes}\right\|_{\Lambda^{q_1,1}(w_1)\times\cdots\times\Lambda^{q_m,1}(w_m)\to\Lambda^{p,\infty}(w)} \lesssim m^m \prod_{i=1}^m \|w_i\|^2_{B^R_{q_i}}, \tag{6.1.2}$$

where w is such that $W \lesssim W_1^{q/q_1} \cdots W_m^{q/q_m}$. Moreover, if (6.1.1) holds, with now w being such that $W \gtrsim W_1^{q/q_1} \cdots W_m^{q/q_m}$, then $w_i \in B^R_{q_i}$ for every $i = 1, \dots, m$, and

$$\max\left(\|w_1\|_{B^R_{q_1}}, \dots, \|w_m\|_{B^R_{q_m}}\right) \lesssim \left\|M^{\otimes}\right\| \prod_{i=1}^m q_i.$$

Proof. First assume that, for $i = 1, \dots, m$, $w_i \in B^R_{q_i}$. Then, since

$$W(t)^{\frac{1}{q}}(M^{\otimes}(f_1, \dots, f_m))^*(t) \lesssim \prod_{i=1}^m W_i(t)^{\frac{1}{q_i}}(Mf_i)^*\left(\frac{t}{m}\right), \qquad \forall t > 0,$$

and, by virtue of (2.2.13),

$$\sup_{t>0} W_i(t)^{\frac{1}{q_i}}(Mf_i)^*\left(\frac{t}{m}\right) \leqslant m\, \|w_i\|_{B^R_{q_i}} \|Mf_i\|_{\Lambda^{q_i,\infty}(w_i)} \lesssim m\, \|w_i\|^2_{B^R_{q_i}} \|f_i\|_{\Lambda^{q_i,1}(w_i)},$$

for every $i = 1, \dots, m$, we deduce that

$$\left\|M^{\otimes}(f_1, \dots, f_m)\right\|_{\Lambda^{q,\infty}(w)} = \sup_{t>0} W(t)^{\frac{1}{q}}(M^{\otimes}(f_1, \dots, f_m))^*(t) \lesssim m^m \prod_{i=1}^m \|w_i\|^2_{B^R_{q_i}} \|f_i\|_{\Lambda^{q_i,1}(w_i)},$$

so that (6.1.2) holds.

On the other side, if (6.1.1) is bounded with constant $\|M^{\otimes}\|$, then for every cube Q in $\mathbb{R}^n$ and for every measurable function g with $\|g\|_{\Lambda^{q_1,1}(w_1)} = 1$,

$$\left\|M^{\otimes}(g, \chi_Q, \dots, \chi_Q)\right\|_{\Lambda^{q,\infty}(w)} \leqslant \left\|M^{\otimes}\right\| \prod_{i=2}^m q_i W_i(|Q|)^{\frac{1}{q_i}}.$$

Indeed, since $M\chi_Q \geqslant \chi_Q$, $i = 2, \dots, m$, and $Mg \geqslant \left(\frac{1}{|Q|}\int_Q g\right)\chi_Q$, we have that

$$\begin{aligned}\left(\prod_{i=1}^m W_i(|Q|)^{\frac{1}{q_i}}\right)\left(\frac{1}{|Q|}\int_Q g\right) &\lesssim W(|Q|)^{\frac{1}{q}}\left(\frac{1}{|Q|}\int_Q g\right) = \sup_{t>0} W(t)^{\frac{1}{q}}\left[\left(\frac{1}{|Q|}\int_Q g\right)\chi_Q\right]^*(t) \\ &\leqslant \left\|M^{\otimes}(g, \chi_Q, \dots, \chi_Q)\right\|_{\Lambda^{q,\infty}(w)} \leqslant \left\|M^{\otimes}\right\| \prod_{i=2}^m q_i W_i(|Q|)^{\frac{1}{q_i}}.\end{aligned}$$

Therefore, taking the supremum over all such measurable functions g, we obtain that

$$\|\chi_Q\|_{(\Lambda^{q_1,1}(w_1))'} \frac{W_1(|Q|)^{\frac{1}{q_1}}}{|Q|} \lesssim \left\|M^{\otimes}\right\| \prod_{i=2}^m q_i.$$

However, from the definition of associate space (see (2.1.6))

$$\|\chi_Q\|_{\left(\Lambda^{q_1,1}(w_1)\right)'} = \|\chi_Q\|_{\left(\Lambda^1\left(W_1^{\frac{1}{q_1}-1}w_1\right)\right)'} = \frac{1}{q_1}\sup_{t>0}\frac{\min(t,|Q|)}{W_1(t)^{\frac{1}{q_1}}} = \frac{1}{q_1}\sup_{0<t<|Q|}\frac{t}{W_1(t)^{\frac{1}{q_1}}},$$

and Q is any cube in $\mathbb{R}^n$, so we deduce that necessarily

$$w_1 \in B_{q_1}^R \qquad \text{with} \qquad \|w_1\|_{B_{q_1}^R} \lesssim \|M^\otimes\| \prod_{i=1}^m q_i.$$

Analogously, for every $i = 2,\dots,m$, $w_i \in B_{q_i}^R$ with $\|w_i\|_{B_{q_i}^R} \lesssim \|M^\otimes\| \prod_{i=1}^m q_i$.

□

Remark 6.1.2. *If we let $w = w_1^{q/q_1}\dots w_m^{q/q_m}$, since $q_i > q$ for every $i = 1,\dots,m$, by virtue of the Hölder's inequality,*

$$W(t) \leqslant W_1^{q/q_1}(t)\dots W_m^{q/q_m}(t), \qquad \forall t > 0,$$

so (6.1.1) holds for such w whenever $w_i \in B_{q_i}^R$, $i = 1,\dots,m$.

Remark 6.1.3. *Even if $w_i \in B_{q_i}^R$, $i = 1,\dots,m$, this does not necessarily imply that $w \in B_q^R$. For instance, take $w_i(t) = t^{q_i-1} \in B_{q_i}^R$ and observe that*

$$W(t)^{\frac{1}{q}} \lesssim W_1(t)^{\frac{1}{q_1}}\cdots W_m(t)^{\frac{1}{q_m}} = \left(\prod_{i=1}^m \frac{1}{q_i^{\frac{1}{q_i}}}\right)t^m \qquad \Longrightarrow \qquad t^{1-m} \lesssim \frac{t}{W(t)^{\frac{1}{q}}}$$

so that, for $m \geqslant 2$, $w \notin B_q^R$. Nevertheless, if $w \in B_q^R$, taking $w_1,\dots,w_m$ such that $W^{1/q} \approx W_1^{1/q_1}\cdots W_m^{1/q_m}$ we obtain, for every $0 < r \leqslant t$,

$$\left(\frac{W_i(t)}{W_i(r)}\right)^{\frac{1}{q_i}} \leqslant \prod_{j=1}^m \frac{W_j(t)^{\frac{1}{q_j}}}{W_j(r)^{\frac{1}{q_j}}} \approx \frac{W(t)^{\frac{1}{q}}}{W(r)^{\frac{1}{q}}} \leqslant \|w\|_{B_q^R}\frac{t}{r},$$

so that $w_i \in B_{q_i}^R$ with $\|w_i\|_{B_{q_i}^R} \lesssim \|w\|_{B_q^R}$, $i = 1,\dots,m$.

Now observe that in order to prove (6.1.1) we have essentially used that $w_i \in B_{q_i}^R$, $i = 1,\dots,m$, and the weighted restricted weak-type boundedness of M over classical Lorentz spaces (see (2.2.13)), since the other properties would hold also for any product type operator

$$T^\otimes(f_1,\dots,f_m)(x) = \prod_{i=1}^m T_i f_i(x), \qquad x \in \mathbb{R}^n.$$

Therefore, arguing identically as for the proof of (6.1.1), we also have the following result for the more general operator $T^\otimes$.

Lemma 6.1.4. *Set $m \geqslant 1$. Let $0 < q_1, \dots, q_m < \infty$, $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$ and, for each $i = 1, \dots, m$, take $w_i \in B^R_{q_i}$. If for every $i = 1, \dots, m$,*

$$T_i : \Lambda^{q_i,1}(w_i) \to \Lambda^{q_i,\infty}(w_i),$$

with constant controlled by C_{T_i,w_i,q_i}, then

$$\left\| T^{\otimes}(f_1, \dots, f_m) \right\|_{\Lambda^{q,\infty}(w)} \lesssim m^m \prod_{i=1}^m \|w_i\|_{B^R_{q_i}} C_{T_i,w_i,q_i} \|f_i\|_{\Lambda^{q_i,1}(w_i)},$$

where w is such that $W \lesssim W_1^{q/q_1} \cdots W_m^{q/q_m}$. Moreover, if there exists some $1 \leqslant \ell \leqslant m$ such that for every $i = 1, \dots, \ell$,

$$T_i : \Lambda^{q_i}(w_i) \to \Lambda^{q_i,\infty}(w_i)$$

with constant controlled by $\tilde{C}_{T_i,w_i,q_i}$, then

$$\left\| T^{\otimes}(f_1, \dots, f_m) \right\|_{\Lambda^{q,\infty}(w)} \lesssim m^m \prod_{i=1}^m \|w_i\|_{B^R_{q_i}} \prod_{i=1}^{\ell} \tilde{C}_{T_i,w_i,q_i} \|f_i\|_{\Lambda^{q_i}(w_i)} \prod_{i=\ell+1}^{m} C_{T_i,w_i,q_i} \|f_i\|_{\Lambda^{q_i,1}(w_i)}.$$

6.2 Multi-variable weighted restricted weak-type extrapolation

Recall that in Section 4.2.1 we have considered one-variable operators for which there exists some $p_0 \geqslant 1$ such that

$$T : L^{p_0,1}(v) \to L^{p_0,\infty}(v), \qquad \forall v \in A^R_{p_0}.$$

Now, we will consider the multi-variable setting; that is, there exists some exponents $1 \leqslant p_1, \dots, p_m < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ such that

$$T : L^{p_1,1}(v_1) \times \cdots \times L^{p_m,1}(v_m) \to L^{p,\infty}(v_1^{p/p_1} \cdots v_m^{p/p_m}), \qquad \forall v_i \in A^R_{p_i},\ i = 1, \dots, m. \quad (6.2.1)$$

However, to proof our main result we will have to translate our hypothesis to the "diagonal case", i.e., when all the exponents $p_1, \dots, p_m$ are equal. With this aim, we will make use of the following particular version of [161, Theorem 4.2.6].

Theorem 6.2.1. *Set $m \geqslant 1$ and let T be an operator satisfying (6.2.1) with constant less than or equal to $\varphi\left(\|v_1\|_{A^R_{p_1}}, \dots, \|v_m\|_{A^R_{p_m}} \right)$, where $\varphi : [1, \infty)^m \to (0, \infty)$ is a nondecreasing function in each variable. Then, for every $1 \leqslant s \leqslant \min\{p_1, \dots, p_m\}$,*

$$T : L^{s,\frac{s}{p_1}}(v_1) \times \cdots \times L^{s,\frac{s}{p_m}}(v_m) \to L^{\frac{s}{m},\infty}(v_1^{1/m} \cdots v_m^{1/m}), \qquad \forall v_i \in \hat{A}_s,\ i = 1, \dots, m, \quad (6.2.2)$$

with constant less than or equal to $\Psi(\|v_1\|_{\hat{A}_s}, \dots, \|v_m\|_{\hat{A}_s})$, where $\Psi : [1, \infty)^m \to (0, \infty)$ is a nondecreasing function in each variable.

Now we are in conditions to settle our first main result in this chapter.

Theorem 6.2.2. *Set $m \geqslant 1$ and let T be an operator satisfying (6.2.1) with constant less than or equal to $\varphi\left(\|v_1\|_{A_{p_1}^R}, \dots, \|v_m\|_{A_{p_m}^R}\right)$, where $\varphi : [1,\infty)^m \to (0,\infty)$ is a nondecreasing function in each variable. Then, for all exponents $0 < q_1, \dots, q_m < \infty$ and $\frac{1}{q} = \frac{1}{q_1} + \dots + \frac{1}{q_m}$,*

$$T : \Lambda^{q_1, \frac{1}{p_1}}(w_1) \times \dots \times \Lambda^{q_m, \frac{1}{p_m}}(w_m) \to \Lambda^{q,\infty}(w), \qquad \forall w_i \in B_{q_i}^R \cap B_\infty^*,\ i = 1, \dots, m,$$

where w is a weight such that $W \lesssim W_1^{\frac{q}{q_1}} \cdots W_m^{\frac{q}{q_m}}$. Moreover, if T is a submultilinear operator and $\min\{p_1, \dots, p_m\} > m$, then, for every $0 < r < \frac{1}{m}$,

$$T : \Lambda^{q_1,r}(w_1) \times \dots \times \Lambda^{q_m,r}(w_m) \to \Lambda^{q,\infty}(w), \qquad \forall w_i \in B_{q_i}^R \cap B_\infty^*,\ i = 1, \dots, m.$$

Proof. First take any $1 \leqslant s \leqslant \min\{p_1, \dots, p_m\}$. Hence, by virtue of Theorem 6.2.1, we obtain that (6.2.2) holds.

Now, by means of Proposition 2.2.20, we can define, for each $i = 1, \dots, m$ and for every measurable set $F \subseteq \mathbb{R}^n$,

$$R_i\chi_F(x) = \sum_{k=0}^{\infty} \frac{M^k \chi_F(x)}{\left(2\,\|M\|_{\left(\Lambda^{q_i,1}(w_i)\right)'}\right)^k}, \qquad x \in \mathbb{R}^n. \tag{6.2.3}$$

Then, for every $i = 1, \dots, m$,

(1) $\chi_F(x) \leqslant R_i\chi_F(x)$,

(2) $\|R_i\chi_F\|_{A_1} \leqslant 2\,\|M\|_{\left(\Lambda^{q_i,1}(w_i)\right)'} \lesssim \|w_i\|_{B_\infty^*}$,

(3) $\|R_i\chi_F\|_{\left(\Lambda^{q_i,1}(w_i)\right)'} \leqslant 2\,\|\chi_F\|_{\left(\Lambda^{q_i,1}(w_i)\right)'} \leqslant \dfrac{2}{q_i}\,\|w_i\|_{B_{q_i}^R} \dfrac{|F|}{W_i(|F|)^{\frac{1}{q_i}}}$,

where on the right-hand side of (3) we have used Lemma 2.2.15.

Let $y > 0$ and, for every locally integrable functions $f_1, \dots, f_m$, set $F = \{x \in \mathbb{R}^n : |T(f_1, \dots, f_m)(x)| > y\}$, so that $|F| = \lambda_{T(f_1,\dots,f_m)}(y)$. Then, taking

$$M^{\otimes}(f_1, \dots, f_m)^{\frac{1-s}{m}} \prod_{i=1}^m (R_i\chi_F)^{\frac{1}{m}} = \prod_{i=1}^m \left((Mf_i)^{1-s} R_i\chi_F\right)^{\frac{1}{m}} = \prod_{i=1}^m v_i^{\frac{1}{m}}, \quad v_i \in \hat{A}_s,\ i = 1, \dots, m,$$

we get that

$$\begin{aligned}
\lambda_{T(f_1,\dots,f_m)}(y) &\leqslant \lambda_{M^{\otimes}(f_1,\dots,f_m)}(\gamma y) + \int_{\{|T(f_1,\dots,f_m)(x)|>y,\, M^{\otimes}(f_1,\dots,f_m)(x)\leqslant \gamma y\}} \prod_{i=1}^m R_i\chi_F(x)^{\frac{1}{m}}\,dx \\
&\leqslant \lambda_{M^{\otimes}(f_1,\dots,f_m)}(\gamma y) + \gamma^{\frac{s-1}{m}} \frac{y^{\frac{s}{m}}}{y^{\frac{1}{m}}} \int_F v_1(x)^{\frac{1}{m}} \cdots v_m(x)^{\frac{1}{m}}\,dx \\
&\leqslant \lambda_{M^{\otimes}(f_1,\dots,f_m)}(\gamma y) + \frac{\gamma^{\frac{s-1}{m}}\,\Psi\left(\|v_1\|_{\hat{A}_s}, \dots, \|v_m\|_{\hat{A}_s}\right)^{\frac{s}{m}}}{y^{\frac{1}{m}}} \prod_{i=1}^m \|f_i\|_{L^{s,\frac{s}{p_i}}(v_i)}^{\frac{s}{m}}.
\end{aligned}$$

Hence, by means of Lemma 4.2.1 (with $p_0 = s$, $q_0 = \frac{s}{p_i}$, $\beta_0 = 1$, $p = q_i$ and $h = R_i\chi_F$), the property (3) of R_i and the definition of the weight w,

$$\lambda_{T(f_1,\dots,f_m)}(y) \lesssim \lambda_{M^{\otimes}(f_1,\dots,f_m)}(\gamma y) + \left(\frac{\lambda_{T(f_1,\dots,f_m)}(y)}{W(\lambda_{T(f_1,\dots,f_m)}(y))^{\frac{1}{mq}}}\right)\left[\frac{\gamma^{s-1}\Psi\left(\|v_1\|_{\hat{A}_s},\dots,\|v_m\|_{\hat{A}_s}\right)^s}{y}\prod_{i=1}^m \|w_i\|_{B^R_{q_i}}\|f_i\|_{\Lambda^{q_i,\frac{1}{p_i}}(w_i)}\right]^{\frac{1}{m}}.$$

Now, we observe that since for every $i = 1,\dots,m$, $w_i \in B^R_{q_i}$ and $\Lambda^{q_i,\frac{1}{p_i}}(w_i) \subseteq \Lambda^{q_i,1}(w_i)$ continuously, from Theorem 6.1.1 it follows that

$$yW(\lambda_{M^{\otimes}(f_1,\dots,f_m)}(\gamma y))^{\frac{1}{q}} \leqslant \frac{1}{\gamma}\left\|M^{\otimes}(f_1,\dots,f_m)\right\|_{\Lambda^{q,\infty}(w)} \lesssim m^m\frac{1}{\gamma}\prod_{i=1}^m \|w_i\|^2_{B^R_{q_i}}\|f_i\|_{\Lambda^{q_i,1}(w_i)}.$$

Thus, arguing (if necessary) with $T_N(f_1,\dots,f_m) = |T(f_1,\dots,f_m)|\chi_{B(0,N)}$ similar as we did in the proof of Theorem 4.2.2, in particular, we obtain that

$$\begin{aligned} &yW(\lambda_{T(f_1,\dots,f_m)}(y))^{\frac{1}{q}} \\ &\lesssim \max\left(\frac{m^m}{\gamma}\prod_{i=1}^m\|w_i\|_{B^R_{q_i}}, \gamma^{s-1}\Psi\left(\|v_1\|_{\hat{A}_s},\dots,\|v_m\|_{\hat{A}_s}\right)^s\right)\prod_{i=1}^m \|w_i\|_{B^R_{q_i}}\|f_i\|_{\Lambda^{q_i,\frac{1}{p_i}}(w_i)}, \end{aligned}$$

so that taking the infimum in $\gamma > 0$ and the supremum in $y > 0$ we get that

$$\begin{aligned} \|T(f_1,\dots,f_m)\|_{\Lambda^{q,\infty}(w)} &\lesssim m^{\frac{m(s-1)}{s}}\Psi\left(\|v_1\|_{\hat{A}_s},\dots,\|v_m\|_{\hat{A}_s}\right)\prod_{i=1}^m \|w_i\|^{2-\frac{1}{s}}_{B^R_{q_i}}\|f_i\|_{\Lambda^{q_i,\frac{1}{p_i}}(w_i)} \\ &\leqslant m^{\frac{m(s-1)}{s}}\Psi\left(\|w_1\|^{\frac{1}{s}}_{B^*_\infty},\dots,\|w_m\|^{\frac{1}{s}}_{B^*_\infty}\right)\prod_{i=1}^m \|w_i\|^{2-\frac{1}{s}}_{B^R_{q_i}}\|f_i\|_{\Lambda^{q_i,\frac{1}{p_i}}(w_i)}, \end{aligned}$$

where in the last estimate we have used the definition of each v_i and property (2) of each R_i.

Now assume that T is a submultilinear operator and $\min\{p_1,\dots,p_m\} > m$. Besides, this time take any $m < s \leqslant \min\{p_1,\dots,p_m\}$. Then, (6.2.2) can be rewritten as (see Remark 2.1.3)

$$T : L^{s,1}(v_1)\times\cdots\times L^{s,1}(v_m) \to L^{\frac{s}{m},\infty}(v_1^{1/m}\cdots v_m^{1/m}), \qquad \forall v_i \in \hat{A}_s,\ i = 1,\dots,m,$$

with constant $\frac{1}{(s-m)s^{m-1}}\Psi\left(\|v_1\|_{\hat{A}_s},\dots,\|v_m\|_{\hat{A}_s}\right)$. Therefore, arguing as before (but now using Lemma 4.2.1 with $q_0 = 1$) we obtain that, for every $m < s \leqslant \min\{p_1,\dots,p_m\}$,

$$T : \Lambda^{q_1,\frac{1}{s}}(w_1)\times\cdots\times\Lambda^{q_m,\frac{1}{s}}(w_m) \to \Lambda^{q,\infty}(w), \qquad \forall w_i \in B^R_{q_i}\cap B^*_\infty,\ i = 1,\dots,m,$$

with constant less than or equal to

$$\frac{m^{\frac{m(s-1)}{s}}}{(s-m)s^{m-1}}\Psi\left(\|w_1\|^{\frac{1}{s}}_{B^*_\infty},\dots,\|w_m\|^{\frac{1}{s}}_{B^*_\infty}\right)\prod_{i=1}^m\|w_i\|^{2-\frac{1}{s}}_{B^R_{q_i}}.$$

Finally, the desired result follows by taking any $0 < r \leqslant \frac{1}{s} < \frac{1}{m}$ and using the continuous inclusion $\Lambda^{q_i,r}(w_i) \subseteq \Lambda^{q_i,\frac{1}{s}}(w_i)$, $i = 1,\dots,m$.

□

Remark 6.2.3. *Observe that if $p_1 = \cdots = p_m = p_0$, then there is not need in Theorem 6.2.2 of using Theorem 6.2.1 so that, in particular, we obtain that*

$$T : \Lambda^{q_1,\frac{1}{p_0}}(w_1) \times \cdots \times \Lambda^{q_m,\frac{1}{p_0}}(w_m) \to \Lambda^{q,\infty}(w), \qquad \forall w_i \in B_{q_i}^R \cap B_\infty^*,\ i = 1,\ldots,m,$$

with constant less than or equal to

$$C\varphi\left(\|w_1\|_{B_\infty^*}^{\frac{1}{p_0}}, \ldots, \|w_m\|_{B_\infty^*}^{\frac{1}{p_0}}\right)\prod_{i=1}^m \|w_i\|_{B_{q_i}^R}^{2-\frac{1}{p_0}}.$$

Remark 6.2.4. *If we let $w = w_1^{q/q_1} \ldots w_m^{q/q_m}$, due to Remark 6.1.2 we have that Theorem 6.2.2 also holds for this w.*

As a consequence of Theorem 6.2.2, we have the next result.

Corollary 6.2.5. *Under the hypothesis of Theorem 6.2.2 and if T is a submultilinear operator, then, for all exponents $0 < q_1, \ldots, q_m < \infty$ and $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$,*

$$T : \Lambda^{q_1,1}(w_1) \times \cdots \times \Lambda^{q_m,1}(w_m) \to \Lambda^{q,\infty}(w), \qquad \forall w_i \in B_{q_i}^R \cap B_\infty^*,\ i = 1,\ldots,m,$$

where $w \in B_q$ satisfies $W \lesssim W_1^{\frac{q}{q_1}} \cdots W_m^{\frac{q}{q_m}}$.

Proof. Since $w \in B_q$, then $\Lambda^{q,\infty}(w)$ is a Banach function space under the norm $\|\cdot\|_{\Lambda^{q,\infty}(w)}$, and the result follows by means of Proposition 2.1.2 and Theorem 6.2.2. $\square$

6.3 Two-variable weighted mixed-type extrapolation

In this section, we work in the two-variable setting and we relax the hypothesis of Theorem 6.2.2 by introducing some weighted Lebesgue space (with weight in A_{p_1}) in addition to a weighted Lorentz space (with weight in $A_{p_2}^R$) in (6.2.1). As before, we will need to go through the diagonal setting. With this aim, we will make use of the following particular version of [161, Corollary 3.3.29].

Theorem 6.3.1. *Let T be an operator satisfying that for some exponents $1 < p_1, p_2 < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$,*

$$T : L^{p_1}(v_1) \times L^{p_2,1}(v_2) \to L^{p,\infty}(v_1^{p/p_1} v_2^{p/p_2}), \qquad \forall v_1 \in A_{p_1},\ v_2 \in A_{p_2}^R, \tag{6.3.1}$$

with constant less than or equal to $\varphi\left(\|v_1\|_{A_{p_1}}, \|v_2\|_{A_{p_2}^R}\right)$, where $\varphi : [1,\infty)^2 \to (0,\infty)$ is a nondecreasing function in each variable. Then, for every exponent $1 < s \leqslant \min\{p_1, p_2\}$,

$$T : L^s(v_1) \times L^{s,\frac{s}{p_2}}(v_2) \to L^{\frac{s}{2},\infty}(v_1^{1/2} v_2^{1/2}), \qquad \forall v_1 \in A_s,\ v_2 \in \hat{A}_s, \tag{6.3.2}$$

with constant less than or equal to $\Psi(\|v_1\|_{A_s}, \|v_2\|_{\hat{A}_s})$, where $\Psi : [1,\infty)^2 \to (0,\infty)$ is a nondecreasing function in each variable.

Now we are in conditions to settle our second main result in this chapter.

Theorem 6.3.2. *Let T be an operator satisfying (6.3.1). Then, for every exponents $0 < q_1, q_2 < \infty$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$,*

$$T : \Lambda^{q_1}(w_1) \times \Lambda^{q_2,\frac{1}{p_2}}(w_2) \to \Lambda^{q,\infty}(w), \qquad \forall w_1 \in B_{q_1} \cap B_\infty^*,\ w_2 \in B_{q_2}^R \cap B_\infty^*,$$

with w being such that $W \lesssim W_1^{\frac{q}{q_1}} W_2^{\frac{q}{q_2}}$. Moreover, if T is a submultilinear operator and $\min\{p_1, p_2\} > 2$, then, for every $0 < r < \frac{1}{2}$,

$$T : \Lambda^{q_1}(w_1) \times \Lambda^{q_2,r}(w_2) \to \Lambda^{q,\infty}(w), \qquad \forall w_1 \in B_{q_1} \cap B_\infty^*,\ w_2 \in B_{q_2}^R \cap B_\infty^*.$$

Proof. First take any $1 < s \leqslant \min\{p_1, p_2\}$. Hence, by virtue of Theorem 6.3.1, we obtain that (6.3.2) holds.

Now, by means of Proposition 2.2.20, we can define the Rubio de Francia operator R as in (6.2.3) for q_2 and w_2. On the other side, from Propositions 2.2.11 and 2.2.18 we define, for every measurable set $F \subseteq \mathbb{R}^n$ and every locally integrable function f,

$$R'\chi_F(x) = \sum_{k=0}^{\infty} \frac{M^k \chi_F(x)}{\left(2\, \|M\|_{(\Lambda^{q_1}(w_1))'}\right)^k} \qquad \text{and} \qquad Sf(x) = \sum_{k=0}^{\infty} \frac{M^k f(x)}{\left(2\, \|M\|_{\Lambda^{q_1}(w_1)}\right)^k},$$

for every $x \in \mathbb{R}^n$. Then,

(1)' $\chi_F(x) \leqslant R'\chi_F(x)$,

(2)' $\|R'\chi_F\|_{A_1} \lesssim \|w_1\|_{B_\infty^*}$,

(3)' $\|R'\chi_F\|_{(\Lambda^{q_1}(w_1))'} \lesssim \|w_1\|_{B_{q_1}}^{\max\left(1,\frac{1}{q_1}\right)} \dfrac{|F|}{W_1(|F|)^{\frac{1}{q_1}}}$,

(1)" $|f(x)| \leqslant Sf(x)$,

(2)" $\|Sf\|_{A_1} \lesssim \|w_1\|_{B_{q_1}}^{\max\left(1,\frac{1}{q_1}\right)}$,

(3)" $\|Sf\|_{\Lambda^{q_1,\infty}(w_1)} \lesssim \|f\|_{\Lambda^{q_1}(w_1)}$,

where on the right-hand side of (3)' we have used Lemma 2.2.15.

Let $y > 0$ and, for every locally integrable functions f_1, f_2, set

$$F = \{x \in \mathbb{R}^n : |T(f_1, f_2)(x)| > y\},$$

so that $|F| = \lambda_{T(f_1,f_2)}(y)$. Besides, denote

$$T^{\otimes}(f_1, f_2)(x) = Sf_1(x) Mf_2(x), \qquad x \in \mathbb{R}^n.$$

Then, taking

$$T^{\otimes}(f_1, f_2)^{\frac{1-s}{2}} (R'\chi_F)^{\frac{1}{2}} (R\chi_F)^{\frac{1}{2}} = \left[(Sf_1)^{1-s} R'\chi_F\right]^{\frac{1}{2}} \left[(Mf_2)^{1-s} R\chi_F\right]^{\frac{1}{2}} = v_1^{\frac{1}{2}} v_2^{\frac{1}{2}},$$

with $v_1 \in A_s$ and $v_2 \in \hat{A}_s$, we get that

$$\begin{aligned}\lambda_{T(f_1,f_2)}(y) &\leqslant \lambda_{T^{\otimes}(f_1,f_2)}(\gamma y) + \int_{\{|T(f_1,f_2)(x)|>y,\, T^{\otimes}(f_1,f_2)(x)\leqslant \gamma y\}} R'\chi_F(x)^{\frac{1}{2}} R\chi_F(x)^{\frac{1}{2}}\, dx \\ &\leqslant \lambda_{T^{\otimes}(f_1,f_2)}(\gamma y) + \gamma^{\frac{s-1}{2}} \frac{y^{\frac{s}{2}}}{y^{\frac{1}{2}}} \int_F v_1(x)^{\frac{1}{2}} v_2(x)^{\frac{1}{2}}\, dx \\ &\leqslant \lambda_{T^{\otimes}(f_1,f_2)}(\gamma y) + \frac{\gamma^{\frac{s-1}{2}} \Psi(\|v_1\|_{A_s}, \|v_2\|_{\hat{A}_s})^{\frac{s}{2}}}{y^{\frac{1}{2}}} \|f_1\|_{L^s(v_1)}^{\frac{s}{2}} \|f_2\|_{L^{s,\frac{s}{p_2}}(v_2)}^{\frac{s}{2}}.\end{aligned}$$

At this point, we observe that

$$\|v_1\|_{A_s} \lesssim \|w_1\|_{B_\infty^*} \|w_1\|_{B_{q_1}}^{(s-1)\min\left(1,\frac{1}{q_1}\right)}$$

and, since $(Sf_1)^{1-s} \leqslant f_1^{1-s}$,

$$\|f_1\|_{L^s(v_1)}^s \leqslant \int_{\mathbb{R}^n} f_1(x) R'\chi_F(x)\, dx \lesssim \|w_1\|_{B_{q_1}}^{\max\left(1,\frac{1}{q_1}\right)} \frac{\lambda_{T(f_1,f_2)}(y)}{W_1(\lambda_{T(f_1,f_2)}(y))^{\frac{1}{q_1}}} \|f_1\|_{\Lambda^{q_1}(w_1)},$$

while $\|v_2\|_{\hat{A}_s}^s \lesssim \|w_2\|_{B_\infty^*}$ and, due to Lemma 4.2.1 (with $p_0 = s$, $q_0 = \frac{s}{p_2}$, $\beta_0 = 1$, $p = q_2$ and $h = R\chi_F$),

$$\|f_2\|_{L^{s,\frac{s}{p_2}}(v_2)}^s \lesssim \|w_2\|_{B_{q_2}^R} \frac{\lambda_{T(f_1,f_2)}(y)}{W_2(\lambda_{T(f_1,f_2)}(y))^{\frac{1}{q_2}}} \|f_2\|_{\Lambda^{q_2,\frac{1}{p_2}}(w_2)}.$$

Further, from Lemma 6.1.4 we deduce that

$$\left\|T^{\otimes}(f_1,f_2)\right\|_{\Lambda^{q,\infty}(w)} \lesssim \|w_1\|_{B_{q_1}}^{\max\left(1,\frac{1}{q_1}\right)} \|w_2\|_{B_{q_2}^R}^2 \|f_1\|_{\Lambda^{q_1}(w_1)} \|f_2\|_{\Lambda^{q_2,1}(w_2)}.$$

Finally, the desired result follows the same lines as the proof of Theorem 6.2.2.

□

Remark 6.3.3. *Observe that if $p_1 = p_2 = p_0 > 1$, then there is not need in Theorem 6.3.2 of using Theorem 6.3.1 so that, in particular, we obtain that*

$$T : \Lambda^{q_1}(w_1) \times \Lambda^{q_2,\frac{1}{p_0}}(w_2) \to \Lambda^{q,\infty}(w), \qquad \forall w_1 \in B_{q_1} \cap B_\infty^*,\ w_2 \in B_{q_2}^R \cap B_\infty^*,$$

with constant less than or equal to

$$C\|w_1\|_{B_{q_1}}^{\max\left(1,\frac{1}{q_1}\right)} \|w_2\|_{B_{q_2}^R}^{2-\frac{1}{p_0}} \varphi\left(\|w_1\|_{B_\infty^*} \|w_1\|_{B_{q_1}}^{(p_0-1)\min\left(1,\frac{1}{q_1}\right)}, \|w_2\|_{B_\infty^*}^{\frac{1}{p_0}}\right).$$

Remark 6.3.4. *If we just had considered weighted Lebesgue spaces in (6.3.1) with weights in A_{p_1} and A_{p_2}, arguing identically as in the proof of Theorem 6.3.2, but instead of having called Theorem 6.3.1 we had used some two-variable weighted strong-type extrapolation (see, for instance, [101]) it could be deduced that then*

$$T : \Lambda^{q_1}(w_1) \times \Lambda^{q_2}(w_2) \to \Lambda^{q,\infty}(w), \qquad \forall w_1 \in B_{q_1} \cap B_\infty^*,\ w_2 \in B_{q_2} \cap B_\infty^*.$$

To end this section, similar as we did to prove Corollary 6.2.5, as a consequence of Theorem 6.3.2 we have the next result.

Corollary 6.3.5. *Under the hypothesis of Theorem 6.3.2 and if T is a submultilinear operator, then, for all exponents $0 < q_1, q_2 < \infty$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$,*

$$T : \Lambda^{q_1}(w_1) \times \Lambda^{q_2,1}(w_2) \to \Lambda^{q,\infty}(w), \qquad \forall w_1 \in B_{q_1} \cap B_\infty^*,\ w_2 \in B_{q_2} \cap B_\infty^*,$$

where $w \in B_q$ satisfies $W \lesssim W_1^{\frac{q}{q_1}} W_2^{\frac{q}{q_2}}$.

6.4 Applications

In this section, we present some applications for our multi-variable extrapolation results previously introduced. Indeed, in Section 6.4.1 we will study bilinear Fourier multipliers and in Section 6.4.2 multilinear sparse operators.

6.4.1 Bilinear Fourier multipliers

Recall that for a bounded variation function $m : \mathbb{R} \to \mathbb{R}$ (see Definition 3.1.1) that is right-continuous at every point $x \in \mathbb{R}$ and satisfies that $\lim_{x\to-\infty} m(x) = 0$, we have that for a given Schwartz function f (that is, $f \in \mathcal{S}(\mathbb{R}^n)$),

$$T_m f(x) = \int_{\mathbb{R}} H_t f(x)\, dm(t), \qquad \forall x \in \mathbb{R},$$

where dm denotes the Lebesgue-Stieltjes measure (which satisfies that $|dm| < \infty$) and, for every $t \in \mathbb{R}$,

$$H_t f(x) := T_{\chi_{(t,\infty)}} f(x) = \int_t^\infty \hat{f}(\xi) e^{2\pi i x \xi}\, d\xi, \qquad x \in \mathbb{R}.$$

In particular, since $Hf = T_m f$ with $m(\xi) = -i \operatorname{sgn} \xi$ (where H is the Hilbert transform) and

$$\chi_{(t,\infty)}(\xi) = \frac{\operatorname{sgn}(\xi - t) + 1}{2}, \qquad \forall \xi \in \mathbb{R},$$

we have that, for every $t \in \mathbb{R}$,

$$H_t f(x) = \int_{\mathbb{R}} \chi_{(t,\infty)} \hat{f}(\xi) e^{2\pi i x \xi}\, d\xi = \frac{1}{2}\left[f(x) + i e^{2\pi i x t} H(e^{-2\pi i t \cdot} f)(x)\right] \qquad \forall x \in \mathbb{R}.$$

Therefore, given $p > 1$ and $v \in A_p^R$, by means of Proposition 5.5.2 and the pointwise domination of the Hilbert transform by sparse operators, for every $t \in \mathbb{R}$,

$$\|H_t f\|_{L^{p,\infty}(v)} \leqslant \frac{p}{2(p-1)} \left(\|f\|_{L^{p,\infty}(v)} + \left\|H(e^{-2\pi i t \cdot} f)\right\|_{L^{p,\infty}(v)}\right) \leqslant C_{n,p} \|v\|_{A_p^R}^{p+1} \|f\|_{L^{p,1}(v)}, \tag{6.4.1}$$

with $C_{n,p} > 0$ independent of $t \in \mathbb{R}$.

Inspired by this result, let us take a measure μ on $\mathbb{R}^2$ such that $|\mu|(\mathbb{R}^2) < \infty$. Further, define the function

$$m_\mu(\xi_1, \xi_2) = \int_{\mathbb{R}^2} \chi_{(t_1,\infty)}(\xi_1)\chi_{(t_2,\infty)}(\xi_2)\, d\mu(t_1, t_2), \qquad (\xi_1, \xi_2) \in \mathbb{R}^2.$$

It is clear that $\|m_\mu\|_{L^\infty(\mathbb{R}^2)} \leqslant |\mu|(\mathbb{R}^2)$, so m_μ can be considered as a multiplier in $\mathbb{R}^2$ and we can define the *bilinear Fourier multiplier* operator

$$T_{m_\mu}(f_1, f_2)(x) = \int_{\mathbb{R}^2} m_\mu(\xi_1, \xi_2)\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)e^{2\pi i x\cdot(\xi_1+\xi_2)}\, d(\xi_1, \xi_2), \qquad x \in \mathbb{R}^n,$$

initially defined for Schwartz functions f_1, f_2 (that is, $f_1, f_2 \in \mathcal{S}(\mathbb{R}^n)$). Hence, applying Fubini's theorem, we have that, indeed, T_{m_μ} is a two-variable averaging operator since

$$\begin{aligned}
&T_{m_\mu}(f_1, f_2)(x)\\
&= \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} \chi_{(t_1,\infty)}(\xi_1)\hat{f}_1(\xi_1)e^{2\pi i x\cdot\xi_1}\, d\xi_1\right)\left(\int_{\mathbb{R}} \chi_{(t_2,\infty)}(\xi_2)\hat{f}_2(\xi_2)e^{2\pi i x\cdot\xi_2}\, d\xi_2\right) d\mu(t_1, t_2)\\
&= \int_{\mathbb{R}^2} H_{t_1}f_1(x)H_{t_2}f_2(x)\ d\mu(t_1, t_2),
\end{aligned}$$

for every $x \in \mathbb{R}^n$.

As a consequence of (6.4.1) together with Theorem 6.3.2, we obtain the following result.

Corollary 6.4.1. *Given exponents $0 < q_1, q_2 < \infty$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. For every $0 < r < 1$,*

$$T_{m_\mu} : \Lambda^{q_1}(w_1) \times \Lambda^{q_2,r}(w_2) \to \Lambda^{q,\infty}(w), \qquad \forall w_1 \in B_{q_1} \cap B_\infty^*,\ w_2 \in B_{q_2}^R \cap B_\infty^*,$$

with w being a weight such that $W \lesssim W_1^{\frac{q}{q_1}} W_2^{\frac{q}{q_2}}$. Further, if $w \in B_q$, then we can take $r = 1$.

Proof. Let $p_2 = \frac{1}{r} > 1$ and $p_1 > 1$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} < 1$, and take $v_1 \in A_{p_1}$ and $v_2 \in A_{p_2}^R$. Hence, by virtue of the Minkowski's integral inequality (see [20, Theorem 4.4] and [167, Proposition 2.1]) we have that, for every measurable functions f_1, f_2,

$$\left\|T_{m_\mu}(f_1, f_2)\right\|_{L^{p,\infty}(v_1^{p/p_1} v_2^{p/p_2})} \leqslant \frac{p}{p-1}\int_{\mathbb{R}^2} \|H_{t_1}f_1 H_{t_2}f_2\|_{L^{p,\infty}(v_1^{p/p_1} v_2^{p/p_2})}\, d|\mu|(t_1, t_2).$$

Therefore, due to [161, Proposition 3.4.1], we obtain that

$$T_{m_\mu} : L^{p_1,1}(v_1) \times L^{p_2,1}(v_2) \to L^{p,\infty}(v_1^{p/p_1} v_2^{p/p_2})$$

with constant less than or equal to $\frac{p}{p-1}|\mu|(\mathbb{R}^2)\Phi(\|v_1\|_{A_{p_1}}, \|v_2\|_{A_{p_2}^R})$, where $\Phi : [1,\infty)^2 \to (0,\infty)$ is a nondecreasing function in each variable. Finally, the result follows from Theorem 6.3.2 and Corollary 6.3.5.

$\square$

6.4.2 Multilinear sparse operators

Set $m \geqslant 2$. Given a η-sparse family of cubes $\mathcal{S} \subseteq \mathcal{D}$ (see Section 5.5), the m-linear sparse operator corresponding to the family $\mathcal{S}$ is defined by

$$\mathcal{A}_{\mathcal{S}}^m(f_1, \dots, f_m)(x) = \sum_{Q \in \mathcal{S}} \left(\prod_{i=1}^m \frac{1}{|Q|} \int_Q f_i(y)\, dy \right) \chi_Q(x), \qquad x \in \mathbb{R}^n.$$

In [160, Theorem 10, (5.4)], the authors proved the following result (where the constant that appears comes from by applying twice the Hölder's inequality on the constant of the same result).

Proposition 6.4.2. *Let $m \geqslant 2$. Then,*

$$\mathcal{A}_{\mathcal{S}}^m : L^1(v_1) \times \cdots \times L^1(v_m) \to L^{\frac{1}{m},\infty}(v_1^{1/m} \cdots v_m^{1/m}), \qquad \forall v_1, \dots, v_m \in A_1,$$

with constant less than or equal to $C \left(\sum_{i=1}^m \|v_i\|_{A_1}\right)^{2m} \prod_{i=1}^m \|v_i\|_{A_1}$.

As a consequence of Theorem 6.2.2 and Remark 6.2.3, we obtain the following result.

Corollary 6.4.3. *Let $m \geqslant 2$, $0 < q_1, \dots, q_m < \infty$ and $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$. Then,*

$$\mathcal{A}_{\mathcal{S}}^m : \Lambda^{q_1,1}(w_1) \times \cdots \times \Lambda^{q_m,1}(w_m) \to \Lambda^{q,\infty}(w), \qquad \forall w_i \in B_{q_i}^R \cap B_\infty^*,\ i = 1, \dots, m,$$

with constant less than or equal to

$$C \left(\sum_{i=1}^m \|w_i\|_{B_\infty^*} \right)^{2m} \prod_{i=1}^m \|w_i\|_{B_{q_i}^R} \|w_i\|_{B_\infty^*},$$

where w is a weight such that $W \lesssim W_1^{\frac{q}{q_1}} \cdots W_m^{\frac{q}{q_m}}$.

Remark 6.4.4. *Since any m-linear ω-Calderón-Zygmund operator with ω satisfying the Dini condition can be dominated by such sparse operators (see [99, Exercise 1.4.17] and [138, Theorem 1.2 and Proposition 3.1]) the same result can be derived for them.*

Chapter 7

Further results: weighted strong-type estimates on $\Lambda_u^p(w)$

The purpose of this chapter is the study of Rubio de Francia extrapolation results in the setting of weighted classical Lorentz spaces $\Lambda_u^p(w)$. We start in Section 7.1 by introducing known results about extrapolation on weighted r.i. Banach function spaces and also by motivating the weighted strong-type estimates that we want to study for $\Lambda_u^p(w)$, from which the class of weights $B_p(u)$ will come out. Indeed, we will see that the only condition to check, which has been unknown up to now, is the boundedness of the dual Hardy-Littlewood maximal function induced for some weight u over $(\Lambda_u^p(w))'$, which will be settled in Section 7.2. Thus, as a consequence, in Section 7.3 we will obtain boundedness of operators over weighted classical Lorentz spaces even when they are quasi-Banach function spaces.

7.1 An introduction about boundedness on $\Lambda_u^p(w)$ and the $B_p(u)$ weights

In this section, we will consider operators T that satisfies, for some $p_0 \geqslant 1$,

$$T : L^{p_0}(v) \to L^{p_0}(v), \qquad \text{for } v \text{ belonging in some class of weights.} \tag{7.1.1}$$

Indeed, these results are obtained using extrapolation theory by means of various versions of the Rubio de Francia theorem (see Theorem 2.3.1). For instance, in [80, 84] the authors study operators satisfying (7.1.1) for all weights $v \in A_\infty$, getting interesting estimates over weighted r.i. Banach function spaces $\mathbb{X}(u)$ for every weight $u \in A_\infty$.

Further, in [83, Theorem 4.10] the authors considered weights $v \in A_{p_0}$ in (7.1.1), proving a more general version of Theorem 2.3.3 involving weighted r.i. Banach function spaces $\mathbb{X}(u)$. Here, for the sake of simplicity, we state a different version of it consisting on introducing, instead of the Boyd indices, the maximal operator M and its dual induced by a weight $u \in A_\infty$ defined as

$$M'_u f(x) = \frac{M(fu)(x)}{u(x)}, \qquad x \in \mathbb{R}^n,\ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n), \tag{7.1.2}$$

and where we keep track of the constants.

Theorem 7.1.1. *Assume that for some pair of nonnegative functions (f, g) and for some $1 \leqslant p_0 < \infty$,*

$$\left(\int_{\mathbb{R}^n} g(x)^{p_0} v(x)\, dx\right)^{\frac{1}{p_0}} \leqslant \varphi(\|v\|_{A_{p_0}}) \left(\int_{\mathbb{R}^n} f(x)^{p_0} v(x)\, dx\right)^{\frac{1}{p_0}}, \qquad \forall v \in A_{p_0}, \tag{7.1.3}$$

where φ is a nondecreasing function on $[1, \infty)$. Let $\mathbb{X}$ be a r.i. Banach function space and let $u \in A_\infty$ such that

$$M : \mathbb{X}(u) \to \mathbb{X}(u) \qquad \textit{and} \qquad M_u' : \mathbb{X}'(u) \to \mathbb{X}'(u). \tag{7.1.4}$$

Then,

$$\|g\|_{\mathbb{X}(u)} \leqslant C_1 \varphi\left(C_2 \|M_u'\|_{\mathbb{X}'(u)} \|M\|_{\mathbb{X}(u)}^{p_0-1}\right) \|f\|_{\mathbb{X}(u)}.$$

In fact, the proof of Theorem 7.1.1 relies on the construction of the following two A_1 weights: given $h_1, h_2 \in L^1_{\text{loc}}(\mathbb{R}^n)$, then

$$Rh_1(x) = \sum_{k=0}^{\infty} \frac{M^k h_1(x)}{\left(2\|M\|_{\mathbb{X}(u)}\right)^k} \qquad \text{and} \qquad Sh_2(x) = \sum_{k=0}^{\infty} \frac{(M_u')^k h_2(x)}{\left(2\|M_u'\|_{\mathbb{X}'(u)}\right)^k}, \qquad x \in \mathbb{R}^n,$$

satisfy that

(1) $|h_1(x)| \leqslant Rh_1(x)$,

(2) $\|Rh_1\|_{A_1} \leqslant 2\|M\|_{\mathbb{X}(u)}$,

(3) $\|Rh_1\|_{\mathbb{X}(u)} \leqslant 2\|h_1\|_{\mathbb{X}(u)}$,

(1)' $|h_2(x)|u(x) \leqslant S(h_2 u)(x)$,

(2)' $\|S(h_2 u)\|_{A_1} \leqslant 2\|M_u'\|_{\mathbb{X}'(u)}$,

(3)' $\|S(h_2 u)/u\|_{\mathbb{X}'(u)} \leqslant 2\|h_2\|_{\mathbb{X}'(u)}$.

All in all, in order to get estimates over $\mathbb{X}(u)$ we first must study whether $\mathbb{X}$ is a r.i. Banach function space and (7.1.4) holds.

Now, take $u \in A_\infty$ and let $\mathbb{X} = \Lambda^p(w)$ (so that $\mathbb{X}(u) = \Lambda_u^p(w)$) for w being a weight in $\mathbb{R}^+$. In [58, Theorem 3.3.5] it was characterized the weighted strong-type boundedness on $\Lambda_u^p(w)$ of the Hardy-Littlewood maximal operator for every $0 < p < \infty$ by

$$M : \Lambda_u^p(w) \to \Lambda_u^p(w) \qquad \Longleftrightarrow \qquad w \in B_p(u), \tag{7.1.5}$$

(see Definition 7.1.2 below) while, for $p > 1$, in [5, Theorem 1.2] the authors showed that the same holds for the corresponding weighted weak-type boundedness; that is,

$$M : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w) \qquad \Longleftrightarrow \qquad w \in B_p(u).$$

Further, in [3, Theorem 1.1] and [4, Theorem 5.5] it was shown that for $p > 1$,

$$H : \Lambda_u^p(w) \to \Lambda_u^p(w) \quad \Longleftrightarrow \quad H : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w) \quad \Longleftrightarrow \quad w \in B_p(u) \cap B_\infty^*. \tag{7.1.6}$$

Definition 7.1.2. Given $0 < p < \infty$, we say that $w \in B_p(u)$ if there exists some $\varepsilon > 0$ such that

$$\sup_{E_j \subseteq Q_j, \forall 1 \leqslant j \leqslant J} \left(\inf_{1 \leqslant j \leqslant J} \frac{|E_j|}{|Q_j|} \right) \frac{W\left(u\left(\bigcup_{j=1}^J Q_j\right)\right)^{\frac{1}{p-\varepsilon}}}{W\left(u\left(\bigcup_{j=1}^J E_j\right)\right)^{\frac{1}{p-\varepsilon}}} < \infty, \tag{7.1.7}$$

where the supremum is taken over every finite family of cubes $\{Q_j\}_{j=1}^J$.

Remark 7.1.3. *(i) If $u = 1$, due to (2.2.9) we have that (7.1.7) is equivalent to $w \in B_p$.*

(ii) If $w = 1$ and $1 < p < \infty$, then (7.1.7) is equivalent to

$$\frac{u\left(\bigcup_{j=1}^J Q_j\right)}{u\left(\bigcup_{j=1}^J E_j\right)} \lesssim \max_{1 \leqslant j \leqslant J} \left(\frac{|Q_j|}{|E_j|} \right)^q,$$

which agrees with $u \in A_p$ (see, for instance, [70, 119]).

We observe that, direct from the definition, this class of weights increase with the exponent in the sense that for every $0 < q \leqslant p$, $B_q(u) \subseteq B_p(u)$, and, if $w \in B_p(u)$ then there exists some $\varepsilon > 0$ such that $w \in B_{p-\varepsilon}(u)$. Further, for every $0 < p < \infty$, $B_p(u) \subseteq B_p$, while it turns out to be an equality if and only if $u \in \bigcap_{q>1} A_q$ (see [58, Corollary 3.3.4 and Theorem 3.3.7]). In particular, for every $p \geqslant 1$, if $w \in B_p(u)$ then $\Lambda^p(w)$ is a Banach function space. (We refer the reader to [1, 58] for more details on this class of weights.)

To end this section, we should point out that if we are able to see when M'_u is bounded over $(\Lambda^p_u(w))'$, by means of Theorem 7.1.1 we will have that, in particular, if T satisfies (7.1.1) for every $v \in A_{p_0}$, then $T : \Lambda^p_u(w) \to \Lambda^p_u(w)$ whenever $p \geqslant 1$ and $w \in B_p(u)$.

The next section will be devoted to study the boundedness of M'_u over $(\Lambda^p_u(w))'$.

7.2 Boundedness on the associate space of $\Lambda^p_u(w)$

Our first goal is to see for which conditions on p and the weights u, w,

$$M'_u : (\Lambda^p_u(w))' \to (\Lambda^p_u(w))' \tag{7.2.1}$$

holds, where M'_u is the often-called dual Hardy-Littlewood maximal function induced by the weight u (see (7.1.2)).

(I) If $u = 1$, it is known that when $w \in \Delta_2$ (that is, when $\Lambda^p(w)$ is a r.i. quasi-Banach function space) then (7.2.1) is equivalent to $w \in B^*_\infty$ (see Section 2.2.3).

(II) If $w = 1$ and $1 < p < \infty$, then (7.2.1) is equivalent to $M'_u : L^{p'}(u) \to L^{p'}(u)$, which in turn remains true whenever $u \in A_p$. On the other side, for instance, if $p < 1$ then $(\Lambda^p_u(1))' = \{0\}$ (see (2.1.7)) so for this case there is nothing to prove.

To see the general case, we need first a couple of results. Both are related with the Fefferman-Stein maximal operator (see (5.2.1)).

Proposition 7.2.1 ([86]). *For every $f \in L^1_{loc}(\mathbb{R}^n)$, there exists a linear operator L_f such that*

$$Mf(x) \approx L_f(|f|)(x), \qquad a.e. \;\; x \in \mathbb{R}^n.$$

Moreover, the adjoint of L_f, $\widetilde{L_f}$, satisfies that for every $g \in L^1_{loc}(\mathbb{R}^n)$,

$$M^{\#}\left(\widetilde{L_f}(|g|)\right)(x) \lesssim Mg(x), \qquad a.e. \;\; x \in \mathbb{R}^n. \tag{7.2.2}$$

Proposition 7.2.2. *Given $u \in A_q$, $1 < q < \infty$, and $w \in B^*_\infty$. If $\lim_{t\to\infty} f^*_u(t) = 0$, then*

$$\|f\|_{\Lambda^p_u(w)} \leqslant c_{n,p}\varphi_q\left(\|u\|_{A_q}\right)\|w\|_{B^*_\infty}\left\|M^{\#}f\right\|_{\Lambda^p_u(w)},$$

where φ_q is an nondecreasing function on $[1,\infty)$.

Proof. First, by means of [19, Corollary 4.3 (a)], there exists a nondecreasing function φ_q on $[1,\infty)$ such that

$$f^*_u(t) \leqslant \varphi_q\left(\|u\|_{A_q}\right)Q\left(\left(M^{\#}f\right)^*_u\right)(t) = \varphi_q\left(\|u\|_{A_q}\right)\int_t^\infty \left(M^{\#}f\right)^*_u(s)\,\frac{ds}{s}, \qquad \forall t > 0.$$

Therefore, due to (2.2.24),

$$\|f\|_{\Lambda^p_u(w)} \leqslant \varphi_q\left(\|u\|_{A_q}\right)\left\|Q\left(\left(M^{\#}f\right)^*_u\right)\right\|_{L^p(w)} \leqslant c_{n,p}\varphi_q\left(\|u\|_{A_q}\right)\|w\|_{B^*_\infty}\left\|M^{\#}f\right\|_{\Lambda^p_u(w)}.$$

□

With the previous result at hand, we are able to find conditions so that (7.2.1) holds.

Theorem 7.2.3. *Given $u \in A_\infty$. For every $0 < p < \infty$,*

$$M'_u : \left(\Lambda^p_u(w)\right)' \to \left(\Lambda^p_u(w)\right)', \qquad \forall w \in B_p(u) \cap B^*_\infty.$$

Proof. First, since $w \in B_p(u) \subseteq B_p$, we have that $(\Lambda^p_u(w))' \neq \{0\}$. Then, by definition of associate space,

$$\left\|\frac{M(fu)}{u}\right\|_{\left(\Lambda^p_u(w)\right)'} = \sup_{\|h\|_{\Lambda^p_u(w)}\leqslant 1}\int_{\mathbb{R}^n} M(fu)(x)h(x)\,dx, \tag{7.2.3}$$

where the supremum is taken over all nonnegative functions h satisfying $\|h\|_{\Lambda^p_u(w)} \leqslant 1$.

Now, we observe that h can be chosen to be in $L^1(\mathbb{R}^n)$. Otherwise, we can take $h_k = \chi_{\mathbb{B}(0,k)}h \in L^1(\mathbb{R}^n)$ (with $\mathbb{B}(0,k)$ being the ball of center 0 and radius k) so by the monotone convergence theorem,

$$\int_{\mathbb{R}^n} M(fu)(x)h(x)\,dx = \lim_{k\to\infty}\int_{\mathbb{R}^n} M(fu)(x)h_k(x)\,dx,$$

and, as well as that $\|h_k\|_{\Lambda^p_u(w)} \leqslant \|h\|_{\Lambda^p_u(w)}$.

Hence, take such a nonnegative function h in (7.2.3) satisfying $h \in L^1(\mathbb{R}^n)$. Further, assume that u is bounded and take $q > 1$ such that $u \in A_q$. Then, by Proposition 7.2.1,

$$\int_{\mathbb{R}^n} M(fu)(x)h(x)\,dx \leqslant C_n \int_{\mathbb{R}^n} |f(x)|\widetilde{L_{fu}}(h)(x)u(x)\,dx \leqslant C_n \|f\|_{(\Lambda^p_u(w))'} \left\|\widetilde{L_{fu}}(h)\right\|_{\Lambda^p_u(w)},$$

and if we are able to see that

$$\lim_{t\to\infty}(\widetilde{L_{fu}}(h))^*_u(t) = 0, \tag{7.2.4}$$

by virtue of Proposition 7.2.2 and estimate (7.2.2) we will deduce that

$$\begin{aligned}\left\|\widetilde{L_{fu}}(h)\right\|_{\Lambda^p_u(w)} &\leqslant c_{n,p}\varphi_q\left(\|u\|_{A_q}\right)\|w\|_{B^*_\infty}\left\|M^\#\left(\widetilde{L_{fu}}(h)\right)\right\|_{\Lambda^p_u(w)}\\ &\leqslant \tilde{c}_{n,p}\varphi_q\left(\|u\|_{A_q}\right)\|w\|_{B^*_\infty}\|Mh\|_{\Lambda^p_u(w)}\\ &\leqslant \tilde{c}_{n,p}\varphi_q\left(\|u\|_{A_q}\right)\|w\|_{B^*_\infty}\|M\|_{\Lambda^p_u(w)},\end{aligned}$$

where $\|M\|_{\Lambda^p_u(w)} < \infty$ since $w \in B_p(u)$.

Thus, we have to show that (7.2.4) holds. However, this is just a consequence of that, by construction, it is known that $\widetilde{L_{fu}}(h)$ is bounded in $L^1(\mathbb{R}^n)$, so for every $y > 0$,

$$\begin{aligned}u\left(\left\{x \in \mathbb{R}^n : |\widetilde{L_{fu}}(h)| > y\right\}\right) &\leqslant C_u\left|\left\{x \in \mathbb{R}^n : |\widetilde{L_{fu}}(h)| > y\right\}\right|\\ &\leqslant \frac{C_u}{y}\left\|\widetilde{L_{fu}}\right\|_{L^1(\mathbb{R}^n)\to L^{1,\infty}(\mathbb{R}^n)}\|h\|_{L^1(\mathbb{R}^n)},\end{aligned}$$

and hence,

$$(\widetilde{L_{fu}}(h))^*_u(t) \leqslant \frac{C_u}{t}\left\|\widetilde{L_{fu}}\right\|_{L^1(\mathbb{R}^n)\to L^{1,\infty}(\mathbb{R}^n)}\|h\|_{L^1(\mathbb{R}^n)} \xrightarrow[t\to\infty]{} 0.$$

Finally, if u is not bounded, taking $N \in \mathbb{N}$, we just have to observe that $u_N = \min(u, N) \in A_q$ (where q depends on u but not on N) is a bounded weight that satisfy

$$\|u_N\|_{A_q} \leqslant 2^q\|u\|_{A_q}, \qquad \|M\|_{\Lambda^p_{u_N}(w)} \leqslant \|M\|_{\Lambda^p_u(w)} \qquad \text{and} \qquad \|h\|_{\Lambda^p_{u_N}(w)} \leqslant 1,$$

so that

$$\int_{\mathbb{R}^n} M(fu)(x)h(x)\,dx = \lim_{N\to\infty}\int_{\mathbb{R}^n} M(fu_N)(x)h(x)\,dx \leqslant \tilde{C}_{n,p}\varphi_q\left(\|u\|_{A_q}\right)\|w\|_{B^*_\infty}\|M\|_{\Lambda^p_u(w)}.$$

□

Remark 7.2.4. *When $u = 1$ we have already pointed out in (I) (at the beginning of this section) that the only condition imposed on w is to belong into $B^*_\infty \cap \Delta_2$. Nevertheless, in Theorem 7.2.3 it has appeared also the condition $B_p(u)$, which becomes B_p when $u = 1$, and is quite smaller than Δ_2. This makes us to think that this condition could be improved, but by the time we are writing this thesis is unknown to us how to do it. However, this is not a big deal since for our purposes we will also need to assume (7.1.5).*

7.3 Weighted strong-type extrapolation on $\Lambda^p_u(w)$

In this section, we want to study the operators for which for some $p_0 \geqslant 1$,

$$T : L^{p_0}(v) \to L^{p_0}(v), \qquad \forall v \in A_{p_0},$$

and see for which conditions on p, u and w the weighted strong-type boundedness

$$T : \Lambda^p_u(w) \to \Lambda^p_u(w) \tag{7.3.1}$$

is satisfied. In particular, by means of Theorem 7.1.1, whenever $\Lambda^p(w)$ is a Banach function space and if (7.1.4) holds, then we obtain (7.3.1).

Corollary 7.3.1. *Assume that for some pair of nonnegative functions (f, g) and for some $1 \leqslant p_0 < \infty$, (7.1.3) holds. Let $1 \leqslant p < \infty$ and $u \in A_\infty$. Then,*

$$\|g\|_{\Lambda^p_u(w)} \leqslant C_1 \varphi\left(C_2 \|M'_u\|_{\left(\Lambda^p_u(w)\right)'} \|M\|^{p_0-1}_{\Lambda^p_u(w)}\right) \|f\|_{\Lambda^p_u(w)}, \qquad \forall w \in B_p(u) \cap B^*_\infty.$$

Proof. Since $w \in B_p(u) \subseteq B_p$, then $\Lambda^p(w)$ is a Banach function space. Therefore, the result follows by means of Theorems 7.1.1 and 7.2.3, together with (7.1.5). □

Remark 7.3.2. *(i) If $w = 1$, then necessarily $p > 1$ since $1 \notin B_1(u)$ whenever $u \in A_\infty$. Besides, for $u \in A_\infty$, $1 \in B_p(u)$ is equivalent to $u \in A_p$. Thus, we recover the Rubio de Francia extrapolation theorem (see Theorem 2.3.1) and, as expected, we can not extrapolate till the exponent $p = 1$.*

(ii) If $u = 1$, by virtue of Propositions 2.2.11 and 2.2.18, the constant of Corollary 7.3.1 can be estimated by $C_1 \varphi(C_2 \|w\|_{B^*_\infty} \|w\|^{p_0-1}_{B_p})$.

Remark 7.3.3. *At least for $p > 1$, by means of the Hilbert transform (see (7.1.6)) the condition $B_p(u) \cap B^*_\infty$ on the weight w of Corollary 7.3.1 is sharp in the sense that it can not be found a greater class for w.*

Remark 7.3.4. *If $0 < p < 1$, arguing as in the proof of Theorem 7.1.1 for $\mathbb{X}(u) = \Lambda^p_u(w)$, and using that $\Lambda^p_u(w) = (\Lambda^1_u(w))^p$, in addition to that, by means of Theorem 2.3.1, we can consider p_0 as big as we want, it can be seen that Corollary 7.3.1 also holds for this range of p (although with a different constant) since Theorem 7.2.3 is also true for these exponents.*

Now, as a consequence of Corollary 7.3.1 and Remark 7.3.4 we have the next result.

Corollary 7.3.5. *Let T be an operator satisfying that, for some $1 \leqslant p_0 < \infty$,*

$$T : L^{p_0}(v) \to L^{p_0}(v), \qquad \forall v \in A_{p_0},$$

with constants less than or equal to $\varphi(\|v\|_{A_{p_0}})$, where φ is a positive nondecreasing function on $[1, \infty)$. Let $0 < p, q < \infty$ and $u \in A_\infty$. Then,

$$T : \Lambda^{p,q}_u(w) \to \Lambda^{p,q}_u(w), \qquad \forall w \in B_p(u) \cap B^*_\infty. \tag{7.3.2}$$

Further, if T is sublinear,

$$T : \Lambda^{p,\infty}_u(w) \to \Lambda^{p,\infty}_u(w), \qquad \forall w \in B_p(u) \cap B^*_\infty. \tag{7.3.3}$$

Proof. First, from the definition of the $B_p(u)$ class of weights and by Lemma 2.2.19,

$$w \in B_p(u) \cap B^*_\infty \qquad \Longleftrightarrow \qquad \tilde{w} = W^{\frac{q}{p}-1} w \in B_q(u) \cap B^*_\infty.$$

Therefore, since $\|\cdot\|_{\Lambda^{p,q}_u(w)} = \|\cdot\|_{\Lambda^q_u(\tilde{w})}$, we obtain (7.3.2) by means of Corollary 7.3.1 and Remark 7.3.4.

On the other hand, assume now that T is sublinear an take $0 < p < \infty$. Recall that if $w \in B_p(u) \cap B^*_\infty$, there exists some $\varepsilon > 0$ such that $w \in B_{p-\varepsilon}(u) \cap B^*_\infty$ and $w \in B_{p+\varepsilon}(u) \cap B^*_\infty$. Hence, again due to Corollary 7.3.1 and Remark 7.3.4,

$$T : \Lambda^{p-\varepsilon}_u(w) \to \Lambda^{p-\varepsilon}_u(w) \qquad \text{and} \qquad T : \Lambda^{p+\varepsilon}_u(w) \to \Lambda^{p+\varepsilon}_u(w),$$

so that by interpolation on weighted classical Lorentz spaces (see [58, Theorem 2.6.5]) we obtain (7.3.3).

□

For instance, in [60] the authors consider as hypothesis weighted weak-type estimates; that is, for some $1 \leqslant p_0 < \infty$,

$$T : L^{p_0}(v) \to L^{p_0,\infty}(v), \qquad \forall v \in A_{p_0}, \tag{7.3.4}$$

and then try to find conditions on p and w for which

$$T : \Lambda^p(w) \to \Lambda^{p,\infty}(w).$$

However, this is just a consequence of the weighted strong-type extrapolation settled in Corollary 7.3.1 and Remark 7.3.4.

Corollary 7.3.6. *Let T be an operator satisfying (7.3.4) with constant less than or equal to $\varphi(\|v\|_{A_{p_0}})$, with φ being a positive nondecreasing function on $[1, \infty)$. Let $0 < p < \infty$ and $u \in A_\infty$. Then,*

$$T : \Lambda^p_u(w) \to \Lambda^p_u(w), \qquad \forall w \in B_p(u) \cap B^*_\infty.$$

Proof. Observe that

$$\|Tf\|_{L^{p_0,\infty}(v)} \leqslant \varphi\left(\|v\|_{A_{p_0}}\right) \|f\|_{L^{p_0}(v)}, \qquad \forall v \in A_{p_0},$$

implies that for every $y > 0$,

$$\left\|\chi_{\{|Tf|>y\}}\right\|_{L^{p_0}(v)} \leqslant \frac{\varphi\left(\|v\|_{A_{p_0}}\right)}{y} \|f\|_{L^{p_0}(v)}, \qquad \forall v \in A_{p_0}.$$

Fix $y > 0$. Hence, by means of Corollary 7.3.1 and Remark 7.3.4 we obtain that, for every $0 < p < \infty$ and $u \in A_\infty$,

$$\left\|\chi_{\{|Tf|>y\}}\right\|_{\Lambda^p_u(w)} \leqslant \frac{C_{n,p_0,p,u,w}}{y} \|f\|_{\Lambda^p_u(w)}, \qquad \forall w \in B_p(u) \cap B^*_\infty. \tag{7.3.5}$$

Therefore, moving y from the right-hand side to the left-hand side of (7.3.5) and taking the supremum over all $y > 0$, we obtain the desired result.

□

Again, for $p > 1$ and by means of the Hilbert transform (see (7.1.6)) the condition $B_p(u) \cap B_\infty^*$ on the weight w of Corollary 7.3.6 is sharp in the sense that it can not be found a greater class for w. However, in [3, Theorem 1.1] it was also characterized the weak-type boundedness of the Hilbert transform for the range $0 < p \leqslant 1$ where it was seen that the class $B_p(u) \cap B_\infty^*$ was sufficient but not necessary (i.e, it is needed a greater class of weights). Indeed, if $p_0 = 1$, in Section 4.2.1 we have shown that

$$T : \Lambda^1(w) \to \Lambda^{1,\infty}(w), \qquad \forall w \in B_1^R \cap B_\infty^*,$$

so it would be interesting to study for a given $u \in A_\infty$, if there exists some greater class of weights than $B_p(u)$, let us say $B_p^R(u)$, so that $B_p^R(1) = B_p^R$, the weighted weak-type boundedness of the Hilbert transform is characterized by $B_p^R(u) \cap B_\infty^*$ and, assuming that (7.3.4) holds for $p_0 = 1$, we can achieve

$$T : \Lambda_u^1(w) \to \Lambda_u^{1,\infty}(w), \qquad \forall w \in B_1^R(u) \cap B_\infty^*.$$

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