UNIVERSITAT DE BARCELONA

Gröbner's problem and the geometry of GT–varieties

Tesi doctoral de Liena Colarte Gómez

$$\begin{tikzcd}
\mathbb{P}^n \arrow[r, "\nu_{n,d}"] \arrow[d, "\cong"'] & X_{n,d} \subset \mathbb{P}^{N_{n,d}-1} \arrow[d, "\pi_{n,d}"] \\
\mathbb{P}^n \arrow[r, "\varphi_{\Omega_{n,d}}"'] & Y_{n,d} \subset \mathbb{P}^{M_{n,d}-1}
\end{tikzcd}$$

UNIVERSITAT DE BARCELONA

Gröbner's problem and the geometry of GT–varieties

TESI DE DOCTORAT

Autora: Liena Colarte Gómez

Directora i tutora de tesi: Dra. Rosa María Miró Roig

Programa de Doctorat en Matemàtiques i Informàtica

Barcelona, maig de 2021

Gröbner's problem and the geometry of GT–varieties

Programa de Doctorat en Matemàtiques i Informàtica de l'Escola de Doctorat de la Universitat de Barcelona, Facultat de Matemàtiques i Informàtica, Departament de Matemàtiques i Informàtica.

Memòria presentada per aspirar al grau de Doctor en Matemàtiques per la Universitat de Barcelona.

Liena Colarte Gómez

Certifico que la present memòria ha estat desenvolupada per Liena Colarte Gómez i dirigida per mi.

Dra. Rosa María Miró Roig

Maig de 2021

Abstract

Within the framework of algebraic geometry and commutative algebra, this thesis makes advances in the Gröbner's longstanding problem of determining whether a monomial projection of the Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ is an aCM variety, where $N_{n,d} = \binom{n+d}{n}$; and it contributes to the fundamental problem of describing the internal structure of the ring of invariants of a finite subgroup of $\mathrm{GL}(n+1, \mathbb{K})$. Our approach towards these subjects involves combinatorics with an application to the Lefschetz properties of artinian ideals. The heart of this dissertation is expounded in four chapters with an introductory Chapter 1 collecting all the basic notions and results needed onwards; and an Appendix A containing two algorithms and implementations with the software Wolfram *Mathematica* [91].

In Chapter 2, we treat Gröbner's problem and we study the invariants of the cyclic extension $\overline{G} \subset \mathrm{GL}(n+1, \mathbb{K})$ of a finite diagonal abelian group $G \subset \mathrm{GL}(n+1, \mathbb{K})$ of order d. We prove that the set $\mathcal{B}_1$ of monomial invariants of G of degree d minimally generates the ring $R^{\overline{G}}$ of invariants. We establish that $\mathcal{B}_1$ parameterizes an aCM monomial projection X_d of $X_{n,d}$, we call to X_d a $\overline{G}-$variety with group G. They form a family of aCM monomial projections of $X_{n,d}$ blending commutative algebra, algebraic geometry, combinatorics and the Lefschetz properties.

In Chapter 3, we study the geometry of $\overline{G}-$varieties X_d with group G. We investigate their Hilbert function and series from the perspectives of invariant theory and combinatorics. We prove that their homogeneous ideals $\mathrm{I}(X_d)$ are generated by binomials of degree at most 3 and we exhibit examples reaching this bound. We identify the canonical module ω_{X_d} of X_d with an ideal $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}})) \subset R^{\overline{G}}$ and we prove that it is generated by monomial of degree d and $2d$. We characterize the Castelnuovo–Mumford regularity of X_d in terms of ω_{X_d}.

In Chapter 4, we investigate the invariants of finite supgroups of $\mathrm{SL}(3, \mathbb{K})$ and we relate them to the weak Lefschetz property. We consider the cyclic extension $\overline{D_{2d}}$ of a representation in $\mathrm{SL}(n+1, \mathbb{K})$ of the dihedral group D_{2d} of order $2d$. We prove that $R^{\overline{D_{2d}}}$ is minimally generated by a set of monomials and binomials of degree $2d$ which generates a non monomial $GT-$system with group D_{2d} and parameterizes an aCM projection $S_{D_{2d}}$ of $X_{2,d}$. We describe a minimal graded free resolution of $S_{D_{2d}}$ and we compute a minimal set of generators of $\mathrm{I}(S_{D_{2d}})$ of degree 2.

In Chapter 5, we introduce $RL-$varieties $\mathcal{X}_d$: a family of smooth rational non aCM monomial projections of $X_{n,d}$ related to $\overline{G}-$varieties X_d with group G. They are parameterized by a set of monomials of degree d determined by ω_{X_d} which defines an embedding of $\mathbb{P}^n$. These properties allow us to describe their normal bundles $\mathcal{N}_{\mathcal{X}_d}$ and to contribute to the classical problem of computing the dimension of the cohomology of the normal bundle of projective varieties.

Para mamá

Per en Martí

Acknowledgments

My deepest gratitude to my advisor Dra. Rosa Maria Miró Roig, who has accompanied and guided me through all these years. I am very grateful to Dra. Emilia Mezzetti for her invaluable collaborations and her hospitality during my stay at *Università degli Studi di Trieste, Dipartimento di Matematica e Geoscienze*. I would like to thank Dra. Roberta di Gennaro for introducing me to the *theory of multiarrangements* and her hospitality during my stay at *Università degli Studi di Napoli "Federico II", Dipartimento di Matematica e Applicazioni "Renato Cacciopoli".* I want to express my gratitude to the *Lefschetz congresses*, participants and organizers, for providing motivation and a wealth environment where to learn and share interesting ideas which have play a central role in the development of this thesis.

La realización de la presente disertación ha sido posible gracias a la ayuda para contratos predoctorales para la formación de doctores *FPI 2017* del ministerio de economía, industria y competitividad del gobierno de España con referencia **BES-2017-082865** asociada al proyecto *Fibrados vectoriales en geometría algebraica* con referencia **MTM2016-78623-P**. Así mismo, dicha ayuda ha permitido llevar a cabo una estada de tres meses en la *Università degli Studi di Trieste, Dipartimento di Matematica e Geoscienze.*

The research stay at *Università degli Studi di Napoli "Federico II", Dipartimento di Matematica e Applicazioni "Renato Cacciopoli",* was supported by *Young Investigator Training Program 2018* supported by *Associazione di Fondazioni e di Casse di Risparmio Spa (ACRI)* and *Università degli Studi di Urbino Carlo Bo.*

In an ideal world, ideas would magically happen while one is comfortably sat in his office at regular hours. In the real world, comfortable chairs do not exist and ideas come to us in the most curious and sometimes annoying ways at the most unearthly hours one can imagine. It is my personal belief that each single event, as irrelevant as it might be, contributes at some level to our thinking. I feel compelled to thank every person and every happening I had interacted with so far. My gratitude is only compared to the number of cups of tea and coffee I have drunk in the last years, which is certainly countable but ridiculously high. The most part of this thesis has been carried out while sitting at my desk in front of my computer in my moderately comfortable chair in the good company of my beloved cats doing their best: sleeping besides the heat of the screen; and my beloved Martí, *mormeu*; anything can't beat it.

Agraeixo profundament a la meva família matemàtica de la Universitat de Barcelona. Un fortíssim gràcies a la Dra. Laura Costa per les nostres discussions matemàtiques que han enriquit la meva formació com a doctora, i per la seva invaluable ajuda i amabilitat. Al nostre *grup de recerca en geometria algebraica (GREGA)*. Als meus companys de despatx, a l'Andrés Rojas i al mio carissimo Dr. Vincenzo Antonelli, grazie mille. Un especial agraïment a l'Helena Gaset Carretero de la secció de Beques de Personal Investigador en Formació, unitat de Beques i Ajuts a l'Estudiant de la Universitat de Barcelona per la seva ajuda en els tràmits administratius i la seva inestimable amabilitat.

Para mi familia, no existen suficientes palabras, y las que están al abasto son injustamente insulsas, para expresarle mi más profundo agradecimiento. La presente tesis no hubiera sido posible sin el constante apoyo y esfuerzo de Iliana, *mamá*, a la cual le dedico todas y cada una de estas páginas. Al meu estimat pare, Jordi, a mis queridas abuelas Maria Elena y Isabel, mis tíos Atito y Michel, a la meva segona mare Isabel i el meu mentomentodo i estimat segon pare Jordi, a tots, gràcies y gracias.

Contents

Notation

$\mathbb{K}$	an algebraically closed field of characteristic zero
R	the polynomial ring $\mathbb{K}[x_0, \ldots, x_n]$
$\mathbb{P}^n$	the $n-$dimensional projective space over $\mathbb{K}$
$\mathcal{M}_{n,d}$	the set of all monomials of degree d in R
$N_{n,d}$	the cardinality $\binom{n+d}{n}$ of $\mathcal{M}_{n,d}$, equivalently the dimension of the $\mathbb{K}-$vector space R_d
$\Omega_{n,d}$	a subset of $\mathcal{M}_{n,d}$
$\mu_{n,d}$	the cardinality of $\Omega_{n,d}$
$X_{n,d}$	the Veronese variety in $\mathbb{P}^{N_{n,d}-1}$ parameterized by $\mathcal{M}_{n,d}$
$\nu_{n,d}$	the dth Veronese embedding of $\mathbb{P}^n$.
$Y_{n,d}$	the variety in $\mathbb{P}^{\mu_{n,d}-1}$ parameterized by $\Omega_{n,d}$
CM	Cohen–Macaulay
aCM	arithmetically Cohen–Macaulay
pdim	projective dimension
h.s.o.p	homogeneous system of parameters
GL$(n+1, \mathbb{K})$	the group of invertible $(n+1) \times (n+1)$ matrices with coefficients in $\mathbb{K}$

$\mathrm{SL}(n+1,\mathbb{K})$	the subgroup of $\mathrm{GL}(n+1,\mathbb{K})$ of matrices with determinant $\pm 1 \in \mathbb{K}$.
$\mathrm{diag}(\beta_0,\dots,\beta_n)$	a diagonal matrix of $\mathrm{GL}(n+1,\mathbb{K})$ with $\beta_0,\dots,\beta_n$ in the main diagonal
$\mathcal{S}_{n+1}$	the group of permutations of $n+1$ elements
$M_{d;\alpha_0,\dots,\alpha_n}$	$\mathrm{diag}(e^{\alpha_0},\dots,e^{\alpha_n})$ with $d \in \mathbb{Z}_{\geq 0}$ and e a dth primitive root of $1 \in \mathbb{K}$
Λ	a finite subgroup of $\mathrm{GL}(n+1,\mathbb{K})$ of order $\lvert\Lambda\rvert$
$\overline{\Lambda}$	the cyclic extension of Λ
R^{Λ}	the ring of invariants of Λ
G	a finite diagonal abelian subgroup of $\mathrm{GL}(n+1,\mathbb{K})$
$\overline{G}$	the cyclic extension of G
$\mathbb{K}[H]$	the semigroup ring of an affine semigroup H
X_d	a $\overline{G}-$variety with group G
$A(X_d)$	the homogeneous coordinate ring of X_d
$\mathrm{I}(X_d)$	the homogenous ideal of X_d
WLP	weak Lefschetz property
GT	Galois–Togliatti
J^{-1}	the inverse system of an ideal $J \subset R$
HF	Hilbert function
HS	Hilbert series
relint	relative interior
ω_{X_d}	the canonical module of $A(X_d)$

reg	Castelnuovo–Mumford regularity
$\mathcal{X}_d$	an $RL-$variety
$\mathcal{N}_{\mathcal{X}_d}$	the normal bundle of $\mathcal{X}_d$
$\mathrm{H}^i(X,\mathcal{E})$	the ith cohomology of a coherent sheaf $\mathcal{E}$ on X
$\mathrm{h}^i(X,\mathcal{E})$	the dimension of $\mathrm{H}^i(X,\mathcal{E})$

Introduction

This thesis presents progress of research on two problems of relevant significance that thrive unsolved and encompass the imposing edifices of commutative algebra and algebraic geometry. First, it contributes to the remarkable Gröbner's longstanding problem regarding the arithmetic Cohen–Macaulayness of projections of Veronese varieties. Second, it makes advances in the fundamental problem of determining the internal structure of the algebra of invariants of finite groups. We work under a determined effort to evince the symbiosis between these two subjects and to understand their connection, a priori unexpected, with Lefschetz properties of artinian ideals.

The third main ingredient of this dissertation is, undoubtedly, combinatorics. It provides a vantage point from which to tackle these fascinating topics. On one hand, a unified and alternative treatment of the abovementioned questions that not only stresses close connections between them and other branches of mathematics, but appeals to a broad audience. On the other hand, it provides a wealth amount of machinery that has leaded us to investigate geometric aspects of arithmetically Cohen–Macaulay projections of Veronese varieties. In this direction, our utmost is towards finding explicitly the minimal free resolution of arithmetically Cohen–Macaulay projection of the Veronese variety.

Without further ado, let us contextualise and introduce the aims of this thesis.

A *monomial projection* of the Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ parameterized by the set $\mathcal{M}_{n,d} \subset R$ of all monomials of degree d is a variety $Y_{n,d}$ parameterized by a subset $\Omega_{n,d} \subset \mathcal{M}_{n,d}$ of $1 \leq \mu_{n,d} \leq N_{n,d}$ monomials. In 1967, Gröbner [39] showed that the Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ is an arithmetically Cohen–Macaulay (shortly aCM) variety and exhibited examples of aCM and non aCM monomial projections of $X_{n,d}$. Motivated by this phenomenon, he posed the problem of determining whether a monomial projection $Y_{n,d} \subset \mathbb{P}^{\mu_{n,d}-1}$ of the Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ is an aCM variety. Since then, *Gröbner's problem* (Problem 2.1.1) has been the center of attention of many works and it has been tackled from different perspectives as geometry, algebra or combinatorics.

One point of view consists of determining the aCM property of a monomial projection $Y_{n,d} \subset \mathbb{P}^{\mu_{n,d}-1}$ of the Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ in terms of either the set of monomials $\Omega_{n,d}$ parameterizing $Y_{n,d}$ or the deleted monomials $\mathcal{M}_{n,d} \setminus \Omega_{n,d}$. In this setting, $Y_{n,d}$ is called a *simple* monomial projection if $\Omega_{n,d}$ is obtained from $\mathcal{M}_{n,d}$ by deleting one monomial. Analogously, *double* and *triple* monomial projections of $X_{n,d}$ are defined if two or three monomials are deleted, respectively. Otherwise, $Y_{n,d}$ is called a *multiple* monomial projection of $X_{n,d}$. This standpoint is based on the fact that the homogeneous coordinate ring of $Y_{n,d}$ is the semigroup ring $\mathbb{K}[\Omega_{n,d}]$, i.e. the $\mathbb{K}-$subalgebra of R generated by $\Omega_{n,d}$. Thus, Gröbner's problem can be regarded as determining whether a semigroup ring is a Cohen–Macaulay (shortly CM) ring, in addition, it provides algebraic and combinatoric techniques to tackle it. This insight was first applied by Schenzel [72] to positively answer Gröbner's problem for simple monomial projections of $X_{n,d}$ and, subsequently, by Trung [84] and Hoa [48] for double and triple monomial projections of $X_{n,d}$, respectively. To the same extent, monomial projections of the *rational normal curve* $X_{1,d} \subset \mathbb{P}^d$ (Example 1.3.13(ii)) were treated by Cavaliere and Niese [11] and by Trung [85]. Notwithstanding, Gröbner's problem for multiple monomial projections of $X_{n,d}$, with the exception of the rational normal curve $X_{1,d}$, remains open and barely known.

Our purpose in this thesis is fourfold. First, we contribute to Gröbner's problem for multiple monomial projections of the Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ in any dimension $n \geq 2$ (Chapter 2). Our approach blends algebra, combinatorics and invariant theory of finite groups with an application to an active area of research: the weak Lefschetz property of artinian ideals. Second, we study the geometry of the family of aCM monomial projections of the Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ parameterized by monomial invariants of degree d of a finite abelian group G of order d linearly represented in $\mathrm{GL}(n+1,\mathbb{K})$, we call them $\overline{G}-$varieties with group G (Chapter 3). Our investigation addresses their Hilbert function and series, a minimal set of generators of their homogeneous ideal and the canonical module of their homogeneous coordinate ring, all three are key ingredients to describe how looks like their minimal free graded resolution. Third, we investigate projections $S_{D_{2d}}$ of the Veronese surface $X_{2,d} \subset \mathbb{P}^{N_{2,d}-1}$ parameterized by invariants of a dihedral group $D_{2d} \subset \mathrm{SL}(3, \mathbb{K})$ of order $2d$. We prove that S_{2d} is an aCM surface and, as in Chapter 3, we concern about the geometry of $S_{D_{2d}}$. We compute their minimal graded free resolution and a minimal set of generators of their homogeneous ideal. Fourth, we introduce a family of smooth rational monomial projections of the Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ related to $\overline{G}-$varieties with group $G \subset \mathrm{GL}(n+1, \mathbb{K})$, we call them $RL-$varieties and we present their normal bundles $\mathcal{N}_{\mathcal{X}_d}$. To determine the cohomology of the normal sheaf of an arbitrary variety $X \subset \mathbb{P}^N$ is a very difficult problem and it is out of reach in most cases. $RL-$varieties $\mathcal{X}_d$ are achievable for this matter since they are smooth rational projections of $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ parameterized by a subset $\Omega_{n,d} \subset \mathcal{M}_{n,d}$ determined by the action of G which induces an embedding. These facts allow us to compute the dimension of the cohomology of $\mathcal{N}_{\mathcal{X}_d}$ (Chapter 5).

Let $2 \leq n < d$ be integers and e a dth primitive root of $1 \in \mathbb{K}$. We consider an abelian group $G = \Gamma_1 \oplus \cdots \oplus \Gamma_s \subset \mathrm{GL}(n+1, \mathbb{K})$ of order $d = d_1 \cdots d_s$, where each $\Gamma_i \subset \mathrm{GL}(n+1, \mathbb{K})$ is a cyclic group of order d_i generated by a diagonal matrix

$$M_{d_i;\alpha^i_{\sigma_i(0)},\ldots,\alpha^i_{\sigma_i(n)}} := \mathrm{diag}(e_i^{\alpha^i_{\sigma_i(0)}}, \ldots, e_i^{\alpha^i_{\sigma_i(n)}})$$

where $\sigma_i \in \mathcal{S}_{n+1}$, $e_i = e^{d/d_i}$ is a d_ith primitive root of $1 \in \mathbb{K}$ and $0 \leq \alpha^i_0 \leq \cdots \leq \alpha^i_n < d_i$ are integers such that $\mathrm{GCD}(d_i, \alpha^i_0, \ldots, \alpha^i_n) = 1$ (Notation 2.2.1).

The *cyclic extension* of G is defined as the finite abelian group $\overline{G} \subset \mathrm{GL}(n+1, \mathbb{K})$ generated by G and $M_{d;1,\ldots,1} = \mathrm{diag}(e, \ldots, e)$ (Definition 1.3.2). The *ring of invariants* of G is $R^G = \{p \in R \mid g(p) = p, \forall g \in G\}$ and it inherits a natural grading from R

$$R^G = \bigoplus_{t \geq 0} R^G_t, \ R^G_t := R_t \cap R^G.$$

The graded $\mathbb{K}-$subalgebra $R^{\overline{G}} := \bigoplus_{t\geq 0} R^G_{td} \subset R^G \subset R$ is the ring of invariants of its cyclic extension $\overline{G} \subset \mathrm{GL}(n+1, \mathbb{K})$, it is called the dth *Veronese* subalgebra of R^G. Since $\overline{G}$ acts diagonally on R, each graded component $R_t^{\overline{G}} = R_{td}^G$ has a monomial $\mathbb{K}-$basis, we denote it by $\mathcal{B}_t$.

The first main result of this thesis concerns the problem of determining a minimal set of generators of $R^{\overline{G}}$ (see, for instance, [77] and [81]). We prove that the set $\mathcal{B}_1 = \{m_1, \ldots, m_{\mu_d}\}$ of all monomial invariants of G of degree d minimally generates the ring of invariants of $\overline{G}$, i.e $R^{\overline{G}} = \mathbb{K}[\mathcal{B}_1]$ (Theorem 2.2.11). The set $\mathcal{B}_1$ is called a minimal set of *fundamental* monomial invariants of $\overline{G}$. The proof is based on showing that any monomial of degree td in $\mathcal{B}_t$ can be factored as a product of t monomials in $\mathcal{B}_1$. It is developed in a purely combinatoric way and the main tools we use are a combination of results on *normal affine semigroups* $H \subset \mathbb{Z}_{\geq 0}^{n+1}$ (Definition 1.2.12) appearing as the $\mathbb{Z}_{\geq 0}^{n+1}-$solutions of linear systems of congruences (see Subsection 1.2.1) and *zero–sums* over finite abelian groups (see Section 2.2).

In [52], Eagon and Hochster proved that the ring of invariants of any finite group acting linearly on R is a CM ring (Theorem 1.3.10). This result provides the motivation for our perspective to contribute to Gröbner's problem as well as further developments in this thesis. The minimal set $\mathcal{B}_1$ of fundamental monomial invariants of $\overline{G} \subset \mathrm{GL}(n+1, \mathbb{K})$ parameterizes a monomial projection $X_d \subset \mathbb{P}^{\mu_d - 1}$ of the Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$. We call a $\overline{G}-$*variety with group* $G \subset \mathrm{GL}(n+1, \mathbb{K})$ to X_d (Definition 2.2.17). As a consequence of Theorem 2.2.11, we establish that the homogeneous coordinate ring $A(X_d)$ of X_d is isomorphic to $R^{\overline{G}}$ (Theorem 2.2.18). Thus, $\overline{G}-$varieties X_d with group $G \subset \mathrm{GL}(n+1, \mathbb{K})$ are aCM monomial projections of $X_{n,d}$ parameterized by the set $\mathcal{B}_1$ of all monomial invariants of G of degree d.

From a combinatoric point of view, the ring $R^{\overline{G}}$ is the semigroup ring $\mathbb{K}[H_{\mathcal{A}}]$ of the normal affine semigroup $H_{\mathcal{A}} \subset \mathbb{Z}_{\geq 0}^{n+1}$ of the $\mathbb{Z}_{\geq 0}^{n+1}-$solutions of the linear system of congruences:

$$(*)_{\mathcal{A}}: \begin{cases} y_0 & + & y_1 & + & \cdots & + & y_n & \equiv & 0 \bmod d \\ \alpha^1_{\sigma_1(0)} y_0 & + & \alpha^1_{\sigma_1(1)} y_1 & + & \cdots & + & \alpha^1_{\sigma_1(n)} y_n & \equiv & 0 \bmod d_1 \\ & & & & & & & \vdots & \\ \alpha^s_{\sigma_s(0)} y_0 & + & \alpha^s_{\sigma_s(1)} y_1 & + & \cdots & + & \alpha^s_{\sigma_s(n)} y_n & \equiv & 0 \bmod d_s. \end{cases}$$

This strategy gives an alternative way to show that any $\overline{G}-$variety X_d with group $G \subset \mathrm{GL}(n+1, \mathbb{K})$ is an aCM variety, since Hochster in [51] proved that the semigroup ring associated to a normal affine semigroup is a CM ring. In addition, this approach is computationally friendly and we have implemented it with the software Wolfram *Mathematica* [91] to compute the examples related to the set $\mathcal{B}_1$ of fundamental monomial invariants of $\overline{G}$. Both points of view were studied by Stanley [78] and they play a central role in this thesis.

In addition to Gröbner's problem, the second line of motivation of this thesis concerns the weak Lefschetz property of artinian ideals. This area of research has made considerable progress in recent years, in part due to its interplay with, among other, algebra, algebraic and differential geometry, combinatorics and representation theory. In Section 1.4, we introduce this notion and we review recent developments in this area. Given an integer i_0 and an artinian ideal $J \subset R$, we say that J *fails the WLP in degree* i_0 if for any linear form $L \in (R/J)_1$ the multiplication map

$$\times(L) : (R/J)_{i_0} \longrightarrow (R/J)_{i_0+1}$$

does not have maximal rank, i.e. it is neither injective nor surjective.

In [59], Mezzetti, Miró–Roig and Ottaviani established a connection between the failure of the WLP and the existence of varieties satisfying at least one Laplace equation. The precise result is known as the Tea Theorem (Theorem 1.4.6). It shows: let $J \subset R$ be an artinian ideal generated by $r \leq N_{n-1,d}$ forms $F_1, \ldots, F_r$ of degree d and J^{-1} its inverse system (Definition 1.4.4). Then, the artinian ideal J fails the WLP in degree $d-1$ if and only if the variety $Y = \overline{\varphi_{J_d^{-1}}(\mathbb{P}^n)} \subset \mathbb{P}^{N_{n,d}-r-1}$, where $\varphi_{J_d^{-1}} : \mathbb{P}^n \dashrightarrow \mathbb{P}^{N_{n,d}-r-1}$ is the rational map defined by J_d^{-1}, satisfies at least one Laplace equation of order $d-1$. They called a *Togliatti system* to such an ideal J, a *Togliatti variety* to the variety $Y \subset \mathbb{P}^{N_{n,d}-r-1}$ associated to J_d^{-1} (Definition 1.4.7) and a *monomial* Togliatti system to any Togliatti system which can be generated by monomials. The name is in honour of the italian mathematician Togliatti, who proved that for $n = 2$ the only *smooth* monomial Togliatti system (i.e. its associated Togliatti variety Y is smooth) of cubics is the monomial ideal

$$T = (x_0^3, x_1^3, x_2^3, x_0x_1x_2) \subset \mathbb{K}[x_0, x_1, x_2],$$

known also as *Togliatti's example.* In addition to the Togliatti variety $Y \subset \mathbb{P}^{N_{n,d}-r-1}$, to a Togliatti system J we associate the variety $X = \varphi_J(\mathbb{P}^n) \subset$

$\mathbb{P}^{r-1}$ image of the morphism $\varphi_J : \mathbb{P}^n \longrightarrow \mathbb{P}^{r-1}$ defined by $(F_1 : \cdots : F_r)$. In [17], the authors called X the variety *parameterized by the Togliatti system* J.

In [57], it was introduced the notion of a *Galois–Togliatti* system (shortly $GT-$system) $I_d \subset R$ with a finite cyclic group $\mathbb{Z}/d\mathbb{Z}$ (Definition 1.4.10). A Togliatti system I_d generated by μ_d forms of degree d is called a $GT-$system with group $\mathbb{Z}/d\mathbb{Z}$ if the associated morphism $\varphi_{I_d} : \mathbb{P}^n \longrightarrow \mathbb{P}^{\mu_d-1}$ is a *Galois covering* with group $\mathbb{Z}/d\mathbb{Z}$ (Definition 1.4.11). This geometric condition on φ_{I_d} translates into the variety X_d parameterized by I_d. The subgroup Λ of $\mathrm{Aut}(\mathbb{P}^n)$ commuting with φ_{I_d} is isomorphic to $\mathbb{Z}/d\mathbb{Z}$ and Λ acts transitively on any fibre $\varphi_{I_d}^{-1}(p)$, $p \in X_d$. For instance, the quotient variety $\mathbb{P}^n/\Lambda$ of $\mathbb{P}^n$ by the action of $\Lambda \subset \mathrm{Aut}(\mathbb{P}^n)$ gives rise to a Galois covering with group $\Lambda \subset \mathrm{GL}(n+1, \mathbb{K})$. Moreover, the coordinate ring of $\mathbb{P}^n/\Lambda$ is the ring of invariants of Λ which makes the study of this variety appealing.

Motivated by these facts, the authors of [17] considered a finite cyclic group $\Gamma = \langle M_{d;\alpha_0,\dots,\alpha_n} \rangle \subset \mathrm{GL}(n+1, \mathbb{K})$ of order d and they proved that the ideal I_d generated by all monomial invariants of Γ of degree d is a $GT-$system with group Γ, provided $\mu_d \leq N_{n-1,d}$ (Theorem 1.4.6). For instance, Togliatti's example $T = (x_0^3, x_1^3, x_2^3, x_0x_1x_2) \subset \mathbb{K}[x_0, x_1, x_2]$ is a $GT-$system with group $\Gamma = \langle M_{3;0,1,2} \rangle \subset \mathrm{GL}(3, \mathbb{K})$ since it is generated by a minimal set of fundamental monomial invariants of $\overline{\Gamma}$. If $\mu_d \leq N_{n-1,d}$, they call X_d a *$GT-$variety with group* Γ. From the perspective of this dissertation, $GT-$systems and $GT-$varieties with cyclic group $\Gamma \subset \mathrm{GL}(n+1, \mathbb{K})$ are studied in [57], [18], [20] and [17]. Later in [19], the notions of a $GT-$system and a $GT-$variety with cyclic group $\Gamma \subset \mathrm{GL}(n+1, \mathbb{K})$ have been extended by considering any finite subgroup $\Lambda \subset \mathrm{GL}(n+1, \mathbb{K})$, even not abelian, and the authors investigated $GT-$systems and $GT-$surfaces with a dihedral group linearly represented in $\mathrm{SL}(3, \mathbb{K})$.

Resuming our previous discussion, the ideal $I_d \subset R$ generated by the minimal set $\mathcal{B}_1$ of fundamental invariants of $\overline{G}$ is an artinian ideal inducing a Galois covering $\varphi_{I_d} : \mathbb{P}^n \longrightarrow \mathbb{P}^{\mu_d-1}$ with group G. Actually, the terminology *$\overline{G}-$variety with group* $G \subset \mathrm{GL}(n+1, \mathbb{K})$ has been conceived not only to emphasize the roles of the abelian group G and the ring $R^{\overline{G}}$, but the Galois covering as well. Furthermore, if the condition $\mu_d \leq N_{n-1,d}$ is satisfied (Theorem 1.4.6), we prove that I_d is automatically a $GT-$system with group $G \subset \mathrm{GL}(n+1, \mathbb{K})$ (Proposition 2.3.1). Thus, $GT-$varieties with group

$G \subset \mathrm{GL}(n+1,\mathbb{K})$ are aCM monomial projections of the Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ parameterized by a minimal set $\mathcal{B}_1$ of fundamental monomial invariants of $\overline{G}$ which generates an ideal I_d failing the WLP in degree $d-1$ and such that the monomial projection of $X_{n,d}$ induced by $\langle I_d^{-1}\rangle_d$ is a Togliatti variety satisfying at least one Laplace equation of order $d-1$.

In contrast to $\overline{G}-$varieties X_d with group $G \subset \mathrm{GL}(n+1,\mathbb{K})$, there are examples of non aCM varieties X parameterized by a *monomial* Togliatti system (see, for instance, Section 2.4). In view of these facts, in [17] we posed the analogous of Gröbner's problem for this kind of monomial projections of the Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$. This thesis contributes to this question in Section 2.4 with a family of aCM monomial projections of the Veronese surface $X_{2,d} \subset \mathbb{P}^{N_{2,d}-1}$ parameterized by Togliatti systems arising from the $GT-$system $T = (x_0^3, x_1^3, x_2^3, x_0x_1x_2)$ with cyclic group $\Gamma = \langle M_{3;0,1,2}\rangle \subset \mathrm{GL}(3,\mathbb{K})$, but their coordinate rings are neither the ring of invariants of a finite group $\Lambda \subset \mathrm{GL}(3,\mathbb{K})$ nor the semigroup ring associated to a normal affine semigroup (Theorem 2.4.10).

The heart of this thesis (Chapter 3) deals with the geometry of any $\overline{G}-$variety X_d with group $G \subset \mathrm{GL}(n+1,\mathbb{K})$. The frame of reference of this objective is, first, [57] where the authors computed a minimal graded free resolution of $GT-$surfaces with cyclic group $\Gamma = \langle M_{d;0,1,2}\rangle \subset \mathrm{GL}(3,\mathbb{K})$ of order $d \geq 3$ and, second, [20] where we computed a minimal set of binomial generators of the homogeneous ideal of any $GT-$threefold with cyclic group $\Gamma = \langle M_{d;0,1,2,3}\rangle \subset \mathrm{GL}(4,\mathbb{K})$ of order $d \geq 4$. The results obtained in both works agree that the homogeneous ideal of these varieties are minimally generated by binomials of degree 2 and 3 and they rose the interest of these varieties.

As we have pointed out before, any $\overline{G}-$variety X_d with group G is an aCM variety whose homogeneous coordinate ring $A(X_d)$ is isomorphic to $R^{\overline{G}} = \mathbb{K}[\mathcal{B}_1] = \mathbb{K}[H_{\mathcal{A}}]$ (Theorem 2.2.18). These facts provide, on one hand, a motivation for determining a minimal free graded resolution of $A(X_d)$ and, one the other hand, techniques from invariant theory and combinatorics to explore this problem and to tackle the geometry of X_d. Along Chapter 3, we work on these subjects which generalize and extend the results obtained so far for $\overline{G}-$varieties with a finite cyclic group $\Gamma = \langle M_{d;\alpha_0,\dots,\alpha_n}\rangle \subset \mathrm{GL}(n+1,\mathbb{K})$ in [57], [20], [17] and [21].

We take new variables $w_1,\dots,w_{\mu_d}$ and we set $S = \mathbb{K}[w_1,\dots,w_{\mu_d}]$. Since

X_d is an aCM variety, a minimal graded free $S-$resolution of $A(X_d)$ is an exact sequence of length $c := \operatorname{codim}(X_d) = \mu_d - n - 1$:

$$F_\bullet : \quad 0 \longrightarrow F_c \longrightarrow \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow S \longrightarrow A(X_d) \longrightarrow 0,$$

where

$$F_i \cong \bigoplus_{j\geq 1}^{f_i} S(-j-i)^{\beta_{i,j}}$$

with $\beta_{i,f_i} > 0$ and $\beta_i = \beta_{i,1}, \ldots, \beta_{i,f_i}$ is the ith graded Betti number of $A(X_d)$, $1 \leq i \leq c$ (see Section 1.1). The minimal free resolution encodes a large quantity of geometric information of X_d, including for instance its Hilbert function and series. The first graded Betti number $\beta_1 = \beta_{1,1}, \ldots, \beta_{1,f_1}$ collects the cardinality $\beta_{1,j}$ of generators of degree $1+j$, $j = 1, \ldots, f_1$, in a minimal set of generators of the homogeneous ideal $\mathrm{I}(X_d) \subset S$ of X_d. Similarly, the last Betti number $\beta_c = \beta_{c,1}, \ldots, \beta_{c,f_c}$ defines the CM–type $\dim(F_c)$ of $A(X_d)$ and collects the cardinality $\beta_{c,j}$ of generators of degree $c+j$ in a minimal set of generators of $\omega_{X_d}(\mu_d)$, $j = 1, \ldots, f_c$, where ω_{X_d} is the canonical module of $A(X_d)$. On the other hand, f_c+1 is the *Castelnuovo–Mumford regularity* $\operatorname{reg}(A(X_d))$ of $A(X_d)$ (Definition 3.1.13). In the opposite direction, the knowledge of the Hilbert function and series of X_d, the ideal $\mathrm{I}(X_d)$, the canonical module ω_{X_d} and $\operatorname{reg}(A(X_d))$ plays an important role in determining a minimal graded free $S-$resolution $F_\bullet$ of $A(X_d)$, as well as how complex $F_\bullet$ could be (see Section 3.1).

Our first concern is the Hilbert function and series of $A(X_d)$ (Section 3.1). We interpret both numerical functions from the invariant theory point of view which allows us to describe them in terms of the monomial invariants of $\overline{G}$ and to conclude that X_d is a variety of degree $\frac{d^{n+1}}{|\overline{G}|}$ (Proposition 3.1.2). The Hilbert series of $A(X_d)$ is

$$\mathrm{HS}(A(X_d), t) = \frac{\delta_n z^n + \cdots + \delta_1 z + 1}{(1-z)^{n+1}},$$

where $\delta_1, \ldots, \delta_n$, the so called $h-$vector of $A(X_d)$, is the sequence of multiplicities of the degrees of the monomials $x_0^{a_0} \cdots x_n^{a_n} \in \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_n$ satisfying $a_0 < d, \ldots, a_n < d$. Exploring different strategies, we compute explicitly the Hilbert function and series of different families of $\overline{G}-$varieties, for instance: for any $\overline{G}-$variety with cyclic group $G = \langle M_{d;\alpha_0,\ldots,\alpha_n} \rangle \subset \mathrm{GL}(n+1, \mathbb{K})$

of prime order d and $\alpha_0 < \cdots < \alpha_n$ in any dimension $n \geq 2$; for any $GT-$surface with cyclic group $G = \langle M_{d;0,\alpha_1,\alpha_2} \rangle \subset \mathrm{GL}(3, \mathbb{K})$ of order $d \geq 3$ and $\alpha_1 < \alpha_2$ (Theorem 3.1.21); and for any $GT-$threefold with group $G = \langle M_{d;0,1,2,3} \rangle \subset \mathrm{GL}(4, \mathbb{K})$ of order $d \geq 4$ (Theorem 3.1.26 and Corollary 3.1.27).

Afterwards, we focus our attention on the structure of the homogeneous ideal $\mathrm{I}(X_d)$ of X_d and we determine a minimal set of generators of $\mathrm{I}(X_d)$ (Section 3.2). The ideal $\mathrm{I}(X_d)$ is the kernel of the morphism

$$\rho : S \longrightarrow \mathbb{K}[m_1, \ldots, m_{\mu_d}]$$

defined by $\rho(w_i) = m_i$, it is called the *ideal of syzygies* among the invariants of $\overline{G}$. $\mathrm{I}(X_d)$ is the homogeneous binomial prime ideal generated by the set of binomials:

$$\{w_{i_1} \cdots w_{i_k} - w_{j_1} \cdots w_{j_k} \in S \mid m_{i_1} \cdots m_{i_k} = m_{j_1} \cdots m_{j_k},\ k \geq 2\}.$$

Our main result in this direction proves that $\mathrm{I}(X_d)$ is generated by binomials of degree at most 3 (Theorem 3.2.6). The proof is inspired by the theory of Markov basis of lattice ideals, which we have previously used in [20], and the main technique is to use zero-sums over abelian groups. Moreover, by means of families of examples in any dimension $n \geq 2$, we show that this bound is sharp and it depends strongly on the group G. To explore the minimal generation of $\mathrm{I}(X_d)$, we focus on $GT-$threefolds with group $G = \langle M_{d;0,1,2,3} \rangle \subset \mathrm{GL}(4, \mathbb{K})$ (Theorem 3.2.24 and Corollary 3.2.25).

Lastly, we study the canonical module ω_{X_d} of $\overline{G}-$varieties X_d with group $G \subset \mathrm{GL}(n+1, \mathbb{K})$ (Section 3.3). Stanley [78] and Danilov [22] proved independently that the canonical module of the semigroup ring $\mathbb{K}[H]$ associated to a normal affine semigroup $H \subset \mathbb{Z}^{n+1}_{\geq 0}$ is the ideal of $\mathbb{K}[H]$ induced by the *relative interior* $\mathrm{relint}(H)$ of H (Definition 1.2.7). Thus, we can identify the canonical module ω_{X_d} of $A(X_d) \cong \mathbb{K}[H_{\mathcal{A}}]$ with the ideal

$$\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}})) = (x_0^{a_0} \cdots x_n^{a_n} \in R^{\overline{G}} \mid a_0 \cdots a_n \neq 0).$$

Our main result regarding ω_{X_d} shows that $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))$ is generated by monomials of degree at most $2d$ (Theorem 3.3.3). The approach we develop is similar to the one performed in the proof of Theorem 2.2.11. Moreover, it allows us to characterize the Castelnuovo–Mumford regularity $\mathrm{reg}(A(X_d))$

in terms of the set $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1$ of generators of $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))$ of degree d. We establish that

$$n \leq \mathrm{reg}(A(X_d)) \leq n+1$$

with equality $\mathrm{reg}(A(X_d)) = n+1$ if and only if $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1 \neq \emptyset$ (Theorem 3.3.5). For the sake of completeness, we end this block discussing how a minimal graded free $S-$resolution of $A(X_d)$ looks like in view of the developments on the Hilbert series $\mathrm{HS}(A(X_d), z)$, the ideal $\mathrm{I}(X_d)$, the canonical module ω_{X_d} and $\mathrm{reg}(A(X_d))$. Among others, we gather the results obtained so far for $GT-$surfaces with group $G = \langle M_{d;0,\alpha_1,\alpha_2}\rangle \subset \mathrm{GL}(3,\mathbb{K})$ in [17].

One of the advantages of the strategy built to study $\overline{G}-$varieties X_d with group $G \subset \mathrm{GL}(n+1,\mathbb{K})$ and, in particular, Togliatti systems, $GT-$systems and $GT-$varieties, is that it applies for any finite subgroup Λ of $\mathrm{GL}(n+1,\mathbb{K})$ of degree $|\Lambda|$ (Proposition 1.4.17). In addition, when $\Lambda \subset \mathrm{GL}(n+1,\mathbb{K})$ is a non abelian group, the generators of R_d^{Λ} are not necessarily monomials. Hence, if they generate a non monomial artinian ideal J, to prove that J is a Togliatti system, i.e. it fails the WLP in degree $|\Lambda|-1$, we only need to check that $\dim(R_d^{\Lambda}) \leq N_{n-1,|\Lambda|}$ (Proposition 4.1.1). Any development in this direction shed new light on non monomial Togliatti (GT) systems, while the majority of works towards these notions deal with the monomial case. Furthermore, if R^{Λ} is minimally generated by forms of degree $|\Lambda|$, then they parameterize an aCM projection of the Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ from the linear system $\langle J^{-1}\rangle_{|\Lambda|}$. In view of these facts, we investigate the invariants of non abelian finite groups $\Lambda \subset \mathrm{GL}(n+1,\mathbb{K})$. This thesis presents our progresses (Chapter 4) in this area.

It is worth to mention that, when dealing with invariants of finite groups, there are certain points that make difficult to work with a non abelian group $\Lambda \subset \mathrm{GL}(n+1,\mathbb{K})$ of order $|\Lambda|$. The landmark of our approach is finding a minimal set of fundamental invariants of the cyclic extension $\overline{\Lambda}$ of Λ. However, elements in such a set of generators could be very cumbersome to describe or manipulate (see, for instance, Section 4.1). In addition, when taking the cyclic extension $\overline{\Lambda}$, the computational complexity increases considerably.

Thus far, we have considered the classification of finite subgroups of $\mathrm{SL}(3,\mathbb{K})$ given in [6] and [90], which we include in Section 4.1. They are classified in types A–L, only A being abelian. Among them, groups

$\Lambda \subset \mathrm{SL}(3,\mathbb{K})$ of types B,C,D,H and I give rise to a Togliatti system, i.e the condition $\dim(R^{\Lambda}_{|\Lambda|}) \leq |\Lambda| + 1$ is satisfied. However, showing that they are $GT-$systems with group $\Lambda \subset \mathrm{SL}(3,\mathbb{K})$ is, in general, out of reach. In this context, our main contribution positively answers this question for a representation in $\mathrm{SL}(3,\mathbb{K})$ of the dihedral group D_{2d} of order $2d$, $d \geq 3$ (Section 4.2). We take a cyclic group $\Gamma = \langle M_{d;0,1,d-1}\rangle \subset \mathrm{SL}(3,\mathbb{K})$ of order $d \geq 3$ and the linear transformation

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

which fixes the variable x_0 and permutes x_1 and x_2. They generate a dihedral group $D_{2d} = \langle M_{d;0,1,d-1}, \sigma\rangle \subset \mathrm{SL}(3,\mathbb{K})$ of order $2d$ and we consider its cyclic extension $\overline{D_{2d}} \subset \mathrm{GL}(3,\mathbb{K})$. Our main result proves that the ring $R^{\overline{D_{2d}}}$ is minimally generated by monomials and binomials of degree $2d$ which we completely describe (Theorem 4.2.6). Our approach is based on the knowledge of the rings $R^{\overline{\Gamma}}$ and $R^{D_{2d}}$ and the natural structure of $R^{\overline{D_{2d}}}$ as a subalgebra of $R^{D_{2d}}$. Thus, the ideal generated by a minimal set of fundamental invariants of $\overline{D_{2d}}$ is a $GT-$system with group D_{2d} (Proposition 4.2.9). The associated $GT-$surfaces $S_{D_{2d}}$ with group D_{2d} are treated subsequently (Subsection 4.2.1). We establish that $S_{D_{2d}}$ is an aCM surface whose coordinate ring is isomorphic to $R^{\overline{D_{2d}}}$ (Theorem 4.2.12). We compute its Hilbert function and series and the CM–type of its homogenous coordinate ring $A(S_{D_{2d}})$ in terms of the Hilbert function and series and the CM–type of the ring $R^{\overline{\Gamma}}$. In addition to that $\mathrm{reg}(A(S_{D_{2d}})) = 3$, we determine how a minimal free graded resolution of $A(S_{D_{2d}})$ looks like (Theorem 4.2.14). Finally, we address the problem of finding an explicit minimal set of generators of the homogeneous ideal $\mathrm{I}(S_{D_{2d}})$ of $S_{D_{2d}}$. We show that $\mathrm{I}(S_{D_{2d}})$ is minimally generated by quadrics and we describe them (Theorem 4.2.17).

The last objective of this thesis appears lately, motivated by the results obtained so far from the study of the canonical module of $\overline{G}-$varieties with group $G \subset \mathrm{GL}(n+1,\mathbb{K})$ (Section 3.3) and the recent methods developed by Alzati and Re [4] to compute the cohomology of the normal bundle of smooth rational varieties embedded in $\mathbb{P}^N$. As seen before, the canonical module ω_{X_d} of a $\overline{G}-$variety X_d with group $G \subset \mathrm{GL}(n+1,\mathbb{K})$ is identified with the ideal $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))$ induced by the relative interior of the normal affine semigroup

$H_{\mathcal{A}} \subset \mathbb{Z}^{n+1}_{\geq 0}$ associated to $R^{\overline{G}} \cong \mathbb{K}[H_{\mathcal{A}}]$. We call X_d a *level* $\overline{G}-$variety with an *enough general* group $G \subset \mathrm{GL}(n+1,\mathbb{K})$ (Definitions 5.1.1 and 5.1.6) if the following two conditions are satisfied: $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))$ is generated by monomials of degree d, i.e. $A(X_d)$ is a level ring with $\mathrm{reg}(A(X_d)) = n+1$; and the abelian group $G \subset \mathrm{GL}(n+1,\mathbb{K})$ contains at least one matrix $\mathrm{diag}(e^{j\lambda_0},\dots,e^{j\lambda_n})$ with at least three entries two by two distinct, where e^j is a d'th primitive root of $1 \in \mathbb{K}$. If so, we associate to X_d a smooth rational variety $\mathcal{X}_d$ embedded in $\mathbb{P}^{N_d}$ by a monomial parametrization $f_d : \mathbb{P}^n \longrightarrow \mathbb{P}^{N_d}$ defined by $\mathcal{M}_{n,d} \setminus \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1$ where $N_d = N_{n,d} - |\,\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))| - 1$ (Proposition 5.1.11). We call $\mathcal{X}_d \subset \mathbb{P}^{N_d}$ an *RL−variety associated to* X_d and we give examples in any dimension $n \geq 2$ (Definition 5.1.7). The name has been conceived to stress the link with the **r**elative interior and the **l**evelness.

In contrast to its associated $\overline{G}-$variety X_d, the $RL-$variety $\mathcal{X}_d$ is a non aCM monomial projection of the Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ parameterized by a set of monomials $\mathcal{M}_{n,d} \setminus \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))$ generating an artinian ideal J_d which has the WLP (Definition 1.4.1 and Proposition 5.1.10). The interest of $RL-$varieties $\mathcal{X}_d \subset \mathbb{P}^{N_d}$ resides in as being a rational smooth projection of $X_{n,d}$ from the linear system $\langle \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1 \rangle$ and to deduce, if any, which role plays the action of the group $G \subset \mathrm{GL}(n+1,\mathbb{K})$. This thesis contributes to this point. We compute the dimension of the cohomology of the normal bundle $\mathcal{N}_{\mathcal{X}_d}$ of an $RL-$variety $\mathcal{X}_d$ associated to a level $\overline{G}-$variety with an enough general group $G \subset \mathrm{GL}(n+1,\mathbb{K})$ (Section 5.2). We presented the normal bundle $\mathcal{N}_{\mathcal{X}_d}$ of $\mathcal{X}_d$ by an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{n+1}(1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{N_d+1}(d) \longrightarrow \mathcal{N}_{\mathcal{X}_d} \longrightarrow 0,$$

leading to $\mathrm{h}^i(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k))$ with the exception of $i = n-1, n$ and $k \geq d+n+1$ (Proposition 5.2.3). For the remaining cases, we determine $\mathrm{h}^i(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k))$ applying the results of [4] where we strongly use the action of the abelian group $G \subset \mathrm{GL}(n+1,\mathbb{K})$. Theorem 5.2.6 gathers the cohomology table of $\mathcal{N}_{\mathcal{X}_d}$.

Part of the results of this thesis have been published in:

1. L. Colarte-Gómez, E. Mezzetti and R. M. Miró-Roig, *On the arithmetic Cohen–Macaulayness of varieties parameterized by monomial Togliatti systems.* Annali di Matematica Pura ed Applicata. (2021). https://doi.org/10.1007/s10231-020-01058-2

2. L. Colarte-Gómez, E. Mezzetti, R. M. Miró-Roig and M. Salat, *On the coefficients of the permanent and the determinant of a circulant matrix. Applications.* Proceedings of the American Mathematical Society. **147:2** (2019), 547–558.

3. L. Colarte-Gómez, E. Mezzetti, R. M. Miró-Roig and M. Salat, *Togliatti systems associated to the dihedral group and the weak Lefschetz property.* Israel Journal of Mathematics, to appear.

4. L. Colarte-Gómez and R. M. Miró-Roig, *Minimal set of binomial generators for certain Veronese* $3-$*fold projections.* Journal of Pure and Applied Algebra. **224:2** (2020), 768–788.

5. L. Colarte-Gómez and R. M. Miró-Roig, *The canonical module of GT-varieties and the normal bundle of RL-varieties.* Mediterranean Journal of Mathematics, to appear. (2022).

This thesis is structured in five chapters, each of them is accompanied by an introduction we refer for further details; and an Appendix A.

Chapter 1 is a compilation of all the basic notations, definitions and tools needed in the main body of this dissertation. Each section is illustrated with examples which have been prepared to familiarize the reader with subsequent chapters. Section 1.1 gives an introduction to CM rings and modules towards algebraic geometry. Section 1.2 is devoted to affine semigroups and their associated semigroup rings. The definition of a normal affine semigroup is given along to examples coming from combinatorics and the Cohen–Macaulayness of their associated semigroup rings. In Section 1.3, we give an introduction to the theory of invariant rings of finite groups. Lastly, Section 1.4 deals with the WLP of artinian ideals. In particular, the notions of having/failing the WLP (Definition 1.4.1), Togliatti systems (Definition 1.4.7), Galois coverings (Definition 1.4.11), $GT-$system and $GT-$varieties (Definition 1.4.18) are defined and exemplified.

Chapter 2 treats the Gröbner's problem and presents our main contribution to this area based on invariant theory of finite groups and combinatorics with an application to the WLP. The results of this chapter are illustrated with examples, to this end we have implemented a routine to solve linear systems of congruences with the software Wolfram *Mathematica* [91], which is collected in Appendix A. 2.1 presents and reviews historically the Gröbner's problem (Problem 2.1.1). A criterion to determine the Cohen–Macaulayness of semigroup rings associated to a simplicial affine semigroup is introduced (Theorem 2.1.4), which plays an important role in the last section of this chapter. In Section 2, we study the invariants of finite abelian groups $G \subset \mathrm{GL}(n+1,\mathbb{K})$ and their cyclic extensions $\overline{G} \subset \mathrm{GL}(n+1,\mathbb{K})$. This section contains an introduction to zero–sums over abelian groups. In the main result of this Chapter we prove that a minimal set $\mathcal{B}_1$ of fundamental monomial invariants of $\overline{G}$ is the set of monomial invariants of G of degree d (Theorem 2.2.11). In Subsection 2.2.1, we define the notion of a $\overline{G}-$variety X_d with group $G \subset \mathrm{GL}(n+1,\mathbb{K})$ (Definition 2.2.17) and we show that X_d is an aCM monomial projection of the Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ parameterized by $\mathcal{B}_1$ (Theorem 2.2.18). In Section 2.3, we study the connection of the monomial artinian ideal I_d generated by $\mathcal{B}_1$ and the WLP (Proposition 2.3.1). In section 2.4, we investigate Gröbner's problem for surfaces parameterized by Togliatti systems. We prove using Theorem 2.1.4 the arithmetic Cohen–Macaulayness of a family of monomial projections of the Veronese surface $X_{2,d}$ parameterized by monomial Togliatti systems which are not connected neither to invariant theory nor to normal affine semigroups (Theorem 2.4.10).

The results of subsequent chapters are illustrated with examples computed or/and checked with the software Macaulay2 [36], as much as the computational complexity has allowed it.

In Chapter 3, we study $\overline{G}-$varieties X_d with group $G \subset \mathrm{GL}(n+1,\mathbb{K})$ from a geometric point of view. Section 3.1 is devoted to the Hilbert function $\mathrm{HF}(A(X_d),t)$ and series $\mathrm{HS}(A(X_d),z)$ of $A(X_d)$. The notions Castelnuovo–Mumford regularity and the Betti diagram of $A(X_d)$ are given (Definition 3.1.13 and Definition 3.1.10). We interpret $\mathrm{HF}(A(X_d),t)$ and $\mathrm{HS}(A(X_d),z)$ from the invariant theory standpoint (Proposition 3.1.2) and we explore different strategies to compute them. In Subsections 3.1.1 and 3.1.2, we we deal with the Hilbert function and series of $GT-$surfaces with group $G = \langle M_{d;0,\alpha_1,\alpha_2}\rangle \subset \mathrm{GL}(3,\mathbb{K})$ (Theorem 3.1.21) and $GT-$threefolds with group $G = \langle M_{d;0,1,2,3}\rangle \subset \mathrm{GL}(4,\mathbb{K})$ (Theorem 3.1.26), respectively. In Section 3.2, we look at a set of generators of the homogeneous ideal $\mathrm{I}(X_d)$ of X_d. In our main result we prove that $\mathrm{I}(X_d)$ is generated by binomials of degree at most 3 (Theorem 3.2.6) and we show that this bound is sharp, i.e. it cannot be improved. It is enhanced by Subsection 3.2.1, where we compute a minimal set of binomial generators of the homogeneous ideal of any $GT-$threefold with group $G = \langle M_{d;0,1,2,3}\rangle \subset \mathrm{GL}(4,\mathbb{K})$ (Theorem 3.2.24 and Corollary 3.2.25). In Section 3.3, we study the canonical module ω_{X_d} of X_d. We identify it with the ideal $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}})) \subset R^{\overline{G}}$ and we prove that $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))$ is generated by monomials of degree d and $2d$ (Theorem 3.3.3). We characterize the Castelnuovo–Mumford regularity of $A(X_d)$ in terms of the degree of generators of $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))$ (Theorem 3.3.5). The examples of this section have been computed or/and checked with routines implemented with the software Wolfram *Mathematica* [91], which we explain in Appendix A. In Subsection 3.3.1, we gather all the main results of this chapter for sake of previous and further investigations on the minimal graded free $S-$resolution of $A(X_d)$.

Chapter 4 deals with the invariants of non abelian finite subgroups of $\mathrm{SL}(3,\mathbb{K})$ and enlarges the family of non monomial Togliatti systems, as well as $GT-$systems, considering a representation in $\mathrm{SL}(3,\mathbb{K})$ of the dihedral group D_{2d} or order $2d$, $d \geq 3$. Section 2.3 contains the classification of finite subgroups of $\mathrm{SL}(3,\mathbb{K})$ given in [6] and [90]; as well as new examples of Togliatti systems coming from invariant theory (Table 4.1.1 and Example 4.1.2). In Section 4.2, we study the invariants of the cyclic extension $\overline{D_{2d}} \subset \mathrm{GL}(3,\mathbb{K})$. In our main results we prove that $R^{\overline{D_{2d}}}$ is minimally generated by monomials and binomials of degree $2d$ (Theorem 4.2.6) and we show that they generate a $GT-$system with group D_{2d} (Proposition 4.2.9). In Subsection 4.2.1, we focus our attention on the geometry of the associated $GT-$surface $S_{D_{2d}}$ with group D_{2d} (Theorems 4.2.12, 4.2.14 and 4.2.17).

In Chapter 5, we introduce and study $RL-$varieties: a family of smooth rational monomial projections of the Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ related to $\overline{G}-$varieties with group $G \subset \mathrm{GL}(n+1,\mathbb{K})$. In Section 5.1, we define and investigate the notions of level $\overline{G}-$variety with an enough general group (Definitions 5.1.1 and 5.1.6) and its associated $RL-$variety $\mathcal{X}_d$ (Definition 5.1.7 and Proposition 5.1.11). In Section 5.2, we introduce the normal bundle $\mathcal{N}_{\mathcal{X}_d}$ of an $RL-$variety $\mathcal{X}_d$ and we compute the dimension of the cohomology of $\mathcal{N}_{\mathcal{X}_d}$ (Proposition 5.2.3 and Theorem 5.2.6).

In Appendix A, we collect two algorithms and their implementation with the software Wolfram *Mathematica* [91]. Algorithm 1 computes the monomial invariants of G of degree td, $t \geq 0$. Algorithm 2 gives a minimal set of monomial generators of the ideal $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))$. They are based on the results obtained so far in Chapters 2 and 3. We provide functions in Wolfram *Mathematica*'s language which illustrate them in addition to concrete examples.

Chapter 1

Preliminaries

This introductory chapter contains the main objects, results and tools that we shall use in the forthcoming chapters. We do not claim any originality on this chapter, which is divided in four sections. Section 1.1 is devoted to introduce Cohen–Macaulay rings and modules as well as their basic properties. In Section 1.2, we define affine semigroups and semigroup rings, and we relate them to algebraic varieties. In Subsection 1.2.1, we introduce affine normal semigroups and we recall the Cohen–Macaulay property of their associated semigroup ring. A large family of affine normal semigroups appears, for instance, as the set of $\mathbb{Z}_{\geq 0}^{n+1}-$solutions of linear systems of congruences. In Section 1.3, we present the algebra of invariants of a linear finite group acting on R. We gather the classical results on its Hilbert function and series, as well as on the Cohen–Macaulay property. In Section 1.4, we focus on artinian algebras failing the weak Lefschetz property. In this context, we introduce Togliatti and Galois–Togliatti systems. Lastly, we relate both notions to invariant theory of finite groups.

1.1 Cohen–Macaulay rings and modules

We begin with a quick overview of dimension theory. After introducing the notions height of an ideal and dimension of rings and modules, we present Noether's normalization theorem for affine algebras. The content of this section is addressed towards algebraic geometry, thus in the last part we focus on graded $R-$modules. We recall the graded version of Noether's normalization theorem, the notions of perfect modules, minimal graded free $R-$resolutions, projective dimension and give characterizations of the

Cohen–Macaulay property of graded R−modules. We mainly follow [9], [25] and [69].

Through this section, A denotes a commutative ring with unit.

Definition 1.1.1. Let $\mathfrak{p} \subset A$ be a prime ideal. The *height* of $\mathfrak{p}$ is the supremum of lengths of strictly descending chains of prime ideals contained in $\mathfrak{p}$. The *height* of an arbitrary ideal I of A is defined as

$$\mathrm{height}(I) = \inf\{\mathrm{height}(\mathfrak{p}) \mid \mathfrak{p} \text{ is a prime ideal containing } I\}.$$

The *Krull dimension* of A is the supremum of the heights of its prime ideals

$$\dim(A) = \sup\{\mathrm{height}(\mathfrak{p}) \mid \mathfrak{p} \subset A \text{ is a prime ideal}\}.$$

If $I \subset A$ is a proper ideal, we have the inequality

$$\mathrm{height}(I) + \dim(A/I) \leq \dim(A).$$

The *codimension* of I if defined as

$$\mathrm{codim}(I) = \dim(A) - \dim(A/I).$$

For noetherian rings, we have the following classical result and a characterization for the Krull dimension in the local case.

Theorem 1.1.2 (Krull's principal ideal theorem)**.** *Let A be a noetherian ring and $I = (y_1, \ldots, y_r) \subsetneq A$ an ideal. Then,* $\mathrm{height}(\mathfrak{p}) \leq r$ *for every prime ideal $\mathfrak{p}$ which is minimal among the prime ideals of A containing I.*

Proof. See [25, Theorem 10.2]. □

Let A be a noetherian ring and $I \subsetneq A$ an ideal. We say that I is an *artinian* ideal if $\dim(A/I) = 0$.

Theorem 1.1.3. *Let $(A, \mathfrak{m})$ be a noetherian local ring. Then, the following conditions are equivalent.*

(i) $\dim(A) = r$*.*

(ii) $\mathrm{height}(\mathfrak{m}) = r$*.*

(iii) *r is the infimum of all $m \in \mathbb{Z}_{\geq 0}$ for which there are $y_1, \ldots, y_m \in A$ such that* $\operatorname{rad}(y_1, \ldots, y_m) = \mathfrak{m}$.

(iv) *r is the infimum of all $m \in \mathbb{Z}_{\geq 0}$ for which there are $y_1, \ldots, y_m \in \mathfrak{m}$ such that $(y_1, \ldots, y_m)$ is an artinian ideal.*

In particular, if $\dim(A) = r$, *then elements $y_1, \ldots, y_r$ as in (iii) and (iv) are called a* system of parameters *of A.*

Proof. See [69, VIII §9 Theorem 20]. □

Example 1.1.4. $\dim(\mathbb{K}) = 0$ and $\dim(R) = n + 1$.

Let M be a finitely generated $A-$module and $\operatorname{Ann}(M) = (0 : M)_A$ its *annihilator*. We denote by $\operatorname{Supp}(M)$ the *support* of M defined as the set of all prime ideals of A containing $\operatorname{Ann}(M)$. The *dimension* of M is defined as $\dim(M) = \dim(A/\operatorname{Ann}(M))$. If $(A, \mathfrak{m})$ is a noetherian local ring and $0 \neq M$ is a finitely generated $A-$module with $\dim(M) = n$, a *system of parameters for M* is a sequence of elements $y_1, \ldots, y_n$ such that $\dim(M/(y_1, \ldots, y_n)M) = 0$.

Proposition 1.1.5. *Let $(A, \mathfrak{m})$ be a noetherian local ring and M a finitely generated $A-$module. Then, for any $y_1, \ldots, y_r \in \mathfrak{m}$*

$$\dim(M/(y_1, \ldots, y_r)) \geq \dim(M) - r,$$

and the equality holds if and only if $y_1, \ldots, y_r$ is part of a system of parameters of M.

Proof. See [25, Proposition 10.8 and Corollary 10.9]. □

Next, we recall the Noether's normalization theorem. As usual, a finitely generated $\mathbb{K}-$algebra will be called an *affine* $\mathbb{K}-$algebra.

Theorem 1.1.6. *Let A be an affine $\mathbb{K}-$algebra, $I \subsetneq A$ an ideal and set* $r = \dim(A)$. *Then, there exist $y_1, \ldots, y_r \in A$ such that*

(i) *$y_1, \ldots, y_r$ are algebraically independent over $\mathbb{K}$.*

(ii) *A is integral over $\mathbb{K}[y_1, \ldots, y_r]$.*

(iii) There exists an integer $0 \le t \le r$ *such that*

$$I \cap \mathbb{K}[y_1, \dots, y_r] = \sum_{i=t+1}^{r} y_i \mathbb{K}[y_1, \dots, y_r] = (y_{t+1}, \dots, y_r).$$

If $y_1, \dots, y_r$ *satisfy (i) and (ii), then* $\mathbb{K}[y_1, \dots, y_r]$ *is called a* Noether normalization of A.

Proof. See [69, VII §7 Theorem 25] and [25, Theorem 13.3]. □

Cohen–Macaulay modules. Our next goal is to introduce, in terms of *regular sequences*, the notions of *grade* and *depth*, and some of the results needed in the sequel.

Definition 1.1.7. Let M be an $A-$module. A sequence $y_1, \dots, y_r$ of elements of A is called an *$M-$regular sequence* if the following two conditions are satisfied:

(i) for each $i = 2, \dots, r$, y_i is not a zero divisor in $M/(y_1, \dots, y_{i-1})M$ and y_1 is not a zero divisor in M.

(ii) $(y_1, \dots, y_r)M \neq M$.

An $M-$regular sequence $y_1, \dots, y_r$ contained in an ideal $I \subset A$ is said *maximal in* I if for any $y_{r+1} \in I$, $y_1, \dots, y_r, y_{r+1}$ is not an $M-$regular sequence in I.

Example 1.1.8. $x_0, \dots, x_n$ is a maximal $R-$regular sequence.

If A is noetherian, any $M-$regular sequence can be extended to a maximal one. In this setting, we have:

Theorem 1.1.9. *Let A be a noetherian ring, M a finitely generated A–module and let $I \subset A$ be an ideal such that $IM \neq M$. Then, all maximal $M-$regular sequences in I have the same length* $n := \operatorname{grade}(I, M)$*, called the* grade of I on M.

Proof. See [9, Theorem 1.2.5]. □

For any ideal I in a noetherian ring A, the bound $\operatorname{height}(I) \leq \operatorname{grade}(I)$ is always satisfied. The *grade* of M is defined as

$$\operatorname{grade}(M) = \operatorname{grade}(\operatorname{Ann}(M), A).$$

We focus now on local rings $(A, \mathfrak{m})$. For a finitely generated $A-$module M, the grade of $\mathfrak{m}$ on M is called the *depth* of M, denoted $\operatorname{depth} M$. We have:

Proposition 1.1.10. *Let $(A, \mathfrak{m})$ be a noetherian local ring and $0 \neq M$ a finitely generated $A-$module.*

(i) Every permutation of an $M-$regular sequence is again an $M-$regular sequence.

(ii) Every $M-$regular sequence is part of a system of parameters of M.

Proof. See [9, Propositions 1.1.6 and 1.2.12]. □

As a consequence, $\operatorname{depth}(M) \leq \dim(M)$. This inequality motivates the following definition.

Definition 1.1.11. Let $(A, \mathfrak{m})$ be a noetherian local ring and M a finitely generated $A-$module. We say that M is a *Cohen–Macaulay* (shortly CM) module if $\operatorname{depth}(M) = \dim(M)$. A is called a *CM ring* if A itself is a CM module.

In general, for an arbitrary noetherian ring A, a finitely generated A–module M is a CM module if the localization $M_{\mathfrak{m}}$ is a CM module for any maximal ideal $\mathfrak{m} \in \operatorname{Supp}(M)$. Next, we see how CM rings and modules interact with grade, regular sequences and height.

Theorem 1.1.12. *Let $(A, \mathfrak{m})$ be a noetherian local ring and $0 \neq M$ a finitely generated CM $A-$module. Then*

(i) $\operatorname{grade}(I, M) = \dim(M) - \dim(M/IM)$ *for any ideal $I \subset \mathfrak{m}$.*

(ii) $y_1, \dots, y_r$ *is an $M-$regular sequence if and only if*

$$\dim(M/(y_1, \dots, y_r)M) = \dim(M) - r.$$

(iii) *$y_1, \dots, y_r$ is an $M-$regular sequence if and only it it is part of a system of parameters of M.*

Proof. See [9, Theorem 2.1.2]. □

The grade and the height of a proper ideal I in a CM noetherian ring coincide, i.e. $\mathrm{height}(I) = \mathrm{grade}(I, A)$. If, in addition, A is local, from the above theorem it follows that

$$\mathrm{height}(I) = \mathrm{codim}(I) = \mathrm{grade}(I, A).$$

Regular rings. Regular local rings are examples of CM noetherian local rings. Let us recall their definition.

Definition 1.1.13. A noetherian local ring $(A, \mathfrak{m})$ is *regular* if the maximal ideal $\mathfrak{m}$ is generated by a system of parameters, called a *regular system of parameters.*

Proposition 1.1.14. *Let $(A, \mathfrak{m})$ be a noetherian local ring and $y_1, \dots, y_r$ a minimal systems of generators of $\mathfrak{m}$.*

(i) *A is regular if and only if $y_1, \dots, y_r$ is an $A-$regular sequence or equivalently $r = \dim(A)$.*

(ii) *If A is regular and $I \subset A$ is an ideal, then A/I is regular if and only if I is generated by a subset of a regular system of parameters.*

Proof. See [9, Propositions 2.2.4 and 2.2.5]. □

As a consequence, we have that any regular local ring $(A, \mathfrak{m})$ is a CM ring. Furthermore, the following proposition characterizes the CM property of local rings in terms of their regular subrings.

Proposition 1.1.15. *Let $(A, \mathfrak{m})$ be a noetherian local ring and A' a regular local subring such that A is a finitely generated $A'-$module. Then, A is a CM ring if and only if it is a free $A'-$module.*

Proof. See [9, Proposition 2.2.11]. □

Example 1.1.16. (i) $\mathbb{K}$ is regular and a CM ring.

(ii) The localization $R_{\mathfrak{m}}$ at $\mathfrak{m} = (x_0, \dots, x_n)$ is a regular noetherian local ring, so it is a CM ring.

In general, a noetherian ring A is called *regular* if for any maximal ideal $\mathfrak{m} \subset A$ the localization $A_{\mathfrak{m}}$ is regular. We have the following.

Theorem 1.1.17. *Set $x_0, \dots, x_n$ indeterminates. A noetherian ring A is regular if and only if $A[x_0, \dots, x_n]$ is regular. In particular, R is regular.*

Proof. See [9, Theorem 2.2.13]. □

Graded** $R-$**modules. Through the rest of this section we deal only with finitely generated graded modules over R. We denote $\mathfrak{m} = (x_0, \dots, x_n)$. We see R as a graded ring with the standard grading $R = \oplus_{t\geq 0} R_t$, where R_t denotes the $\mathbb{K}-$vector space generated by all monomials of degree t. We investigate the CM property of finitely generated graded $R-$modules. In this setting, we introduce the notions of minimal graded free $R-$resolutions, projective dimension and perfect rings.

Theorem 1.1.18. *Let M be a finitely generated graded $R-$module. Then, M is a CM module if and only if $M_{\mathfrak{m}}$ is a CM module. In particular, R is a CM ring.*

Proof. See [9, Corollary 2.2.15]. □

The most important graded $R-$modules arise in algebraic geometry as the coordinate rings of projective varieties. They are positively graded affine $\mathbb{K}-$algebras of the form R/I where $I \subset R$ is an homogeneous ideal. A projective variety $X \subset \mathbb{P}^n$ is an *arithmetically Cohen-Macaulay* (shortly aCM) variety if its homogeneous coordinate ring $R/\operatorname{I}(X)$ is a CM ring. In this context, we have the graded version of Noether's normalization theorem (Theorem 1.1.6).

Theorem 1.1.19. *Let M be a positively graded affine $\mathbb{K}-$algebra and $r := \dim(M)$. Then, there exist $y_1, \dots, y_r \in R$ satisfying the following equivalent conditions:*

(i) $y_1, \ldots, y_r$ *is a h.s.o.p.*

(ii) M *is integral over* $\mathbb{K}[y_1, \ldots, y_r]$*.*

(iii) M *is a finitely generated* $\mathbb{K}[y_1, \ldots, y_r]-$*module.*

In particular, $y_1, \ldots, y_r$ *are algebraically independent over* $\mathbb{K}$*. Moreover,* $y_1, \ldots, y_r$ *can be chosen to be of degree* 1*.*

Proof. See [9, Theorem 1.5.17]. □

Theorems 1.1.18 and 1.1.17 also apply to non standard graded polynomial rings, i.e. $\mathbb{K}[y_0, \ldots, y_n]$ with $\deg(y_i) \geq 1$, $i = 0, \ldots, n$. We have:

Theorem 1.1.20. *Let* M *be a positively graded affine* $\mathbb{K}-$*algebra and* $r := \dim(M)$*. Then the following conditions are equivalent.*

(i) M *is CM.*

(ii) There is a h.s.o.p $y_1, \ldots, y_r$ *of* M *such that* M *is a free* $\mathbb{K}[y_1, \ldots, y_r]-$*module.*

(iii) For any h.s.o.p $y_1, \ldots, y_r$ *of* M*, it holds that* M *is a free* $\mathbb{K}[y_1, \ldots, y_r]-$*module.*

Proof. It follows from Theorems 1.1.17 and 1.1.18 and Proposition 1.1.15. □

Keeping the above notation, if $y_1, \ldots, y_r$ is a h.s.o.p of M and M is CM, there exist homogeneous $\eta_1, \ldots, \eta_t \in M$ such that

$$M = \oplus_{i=1}^{t} \eta_i \mathbb{K}[y_1, \ldots, y_r]. \tag{1.1.1}$$

Such a decomposition is called a *Hironaka decomposition of* M.

A *minimal graded free* $R-$*resolution* of a finitely generated graded $R-$module M is an exact sequence

$$F_\bullet : \quad \cdots \longrightarrow F_r \xrightarrow{\delta_r} F_{r-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\delta_1} F_0 \longrightarrow M \longrightarrow 0,$$

where each F_i is a finite graded free $R-$module and for each $i \geq 1$, $\delta_i(F_i) \subset \mathfrak{m}F_{i-1}$. The free $R-$module F_i is called the *ith syzygy module* of M and its rank β_i the *ith Betti number* of M. The *Hilbert syzygy theorem* assures that every finitely generated graded $R-$module M has a minimal graded free $R-$resolution of length smaller or equal to $n+1$. The *projective dimension* of M is the length of a minimal graded free $R-$resolution of M, denoted $\operatorname{pdim}(M)$. A finitely generated graded $R-$module $0 \neq M$ is called *perfect* if $\operatorname{pdim}(M) = \operatorname{grade}(M)$. In particular, a graded ideal $I \subset R$ is called *perfect* if R/I is perfect. We have the following.

Proposition 1.1.21. *Let M be a finitely generated graded $R-$module. Then M is a CM module if and only if M is perfect.*

Proof. See [9, Corollary 2.2.15]. □

Corollary 1.1.22. *Let $I \subset R$ be an homogeneous ideal. Then, R/I is a CM ring (or I is a CM ideal) if and only if* $\operatorname{codim}(I) = \operatorname{pdim}(R/I)$.

Proof. Since R is a CM ring, $\operatorname{grade}(I, R) = \operatorname{codim}(I)$. Now, the result follows from Proposition 1.1.21. □

We end this section introducing the canonical module of a CM ring R/I. We set $\dim(R/I) = d$, $\mathfrak{m}_I = \mathfrak{m}/I$ and $\mathbb{K}(R/I) = (R/I)/\mathfrak{m}_I$.

Definition 1.1.23. A finitely generated graded R/I–module C is the canonical module of R/I if there exist homogeneous isomorphisms

$$\operatorname{Ext}^i_{R/I}(\mathbb{K}(R/I), C) \cong \begin{cases} 0 & \text{for } i \neq d \\ \mathbb{K}(R/I) & \text{for } i = d. \end{cases}$$

The canonical module of R is $R(-n-1)$.

If R/I is a CM ring with canonical module C, then there is an isomorphism $C \cong \operatorname{Ext}^{n+1-d}_R(R/I, R(-n-1))$. Therefore, we have:

Remark 1.1.24. Set $r = \operatorname{codim}(I)$ and

$$F_\bullet : \quad 0 \longrightarrow F_r \longrightarrow F_{r-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

a minimal graded free $R-$resolution of R/I. Dualizing $F_\bullet$ we obtain a minimal graded free $R-$resolution of $C(n+1)$:

$$0 \longrightarrow R \longrightarrow F_1^\vee \longrightarrow \cdots \longrightarrow F_{r-1}^\vee \longrightarrow F_r^\vee \longrightarrow C(n+1) \longrightarrow 0.$$

The rth Betti number of R/I corresponds to the cardinality of a minimal set of generators of C.

Definition 1.1.25. Let R/I be a CM ring. We define the *Cohen–Macaulay type* (shortly CM–type) of R/I as the rth Betti number β_r of R/I. If $\beta_r = 1$, then R/I is called a *Gorenstein* ring, I a *Gorenstein* ideal and the associated variety an *arithmetically Gorenstein* variety. If the canonical module of R/I is generated in only one degree, R/I is called a *level* ring, the CM–type of R/I is β_r.

Example 1.1.26. (i) A minimal graded free R–resolution of $R/\mathfrak{m} = \mathbb{K}$ is given by the Koszul complex:

$$\begin{gathered} 0 \longrightarrow R(-n-1) \longrightarrow R(-n)^{\binom{n+1}{n}} \longrightarrow \cdots \longrightarrow \\ \longrightarrow R(-i)^{\binom{n+1}{i}} \longrightarrow R(-(i-1))^{\binom{n+1}{i-1}} \longrightarrow \cdots \longrightarrow \\ \longrightarrow R(-2)^{\binom{n+1}{2}} \longrightarrow R(-1)^{n+1} \longrightarrow R \longrightarrow R/\mathfrak{m} \longrightarrow 0. \end{gathered}$$

The CM–type of $R/\mathfrak{m}$ is 1, so $R/\mathfrak{m}$ is a Gorenstein ring and, in particular, it is a level ring of CM–type 1.

(ii) Any Gorenstein ring R/I is a level ring of CM–type 1.

(iii) A ring R/I, or an ideal I, are said to be a *complete intersection (CI)* if I is generated by an R–regular sequence. Any CI R/I is a Gorenstein ring, hence a level ring of CM–type 1. A minimal graded free R–resolution of a CI R/I is given by the Koszul complex. As a first example of a CI we have the maximal ideal $\mathfrak{m}$. As another example: let $h_1 \in R_{d_1}$ and $h_2 \in R_{d_2}$ be two irreducible forms of degrees $0 < d_1 < d_2$. Then $R/(h_1, h_2)$ is a CI and

$$0 \longrightarrow R(-d_1-d_2) \xrightarrow{\begin{pmatrix} h_1 \\ -h_2 \end{pmatrix}} R(-d_2) \oplus R(-d_1) \xrightarrow{\begin{pmatrix} h_2 & h_1 \end{pmatrix}} R \longrightarrow R/(h_1, h_2) \longrightarrow 0$$

is a minimal graded free R–resolution of $R/(h_1, h_2)$.

(iv) The *twisted cubic* $X \subset \mathbb{P}^3$ is the rational curve image of the morphism

$$\varphi : \mathbb{P}^1 \longrightarrow \mathbb{P}^3, \quad \varphi((y_0 : y_1)) = (y_0^3 : y_0^2 y_1 : y_0 y_1^2 : y_1^3).$$

The homogeneous ideal $\mathrm{I}(X) \subset R = \mathbb{K}[x_0, x_1, x_2, x_3]$ of X is generated by three quadrics $\mathrm{I}(X) = (x_0x_2 - x_1^2, x_1x_3 - x_2^2, x_0x_3 - x_1x_2)$. A minimal graded free R–resolution of $R/\mathrm{I}(X)$ is

$$0 \longrightarrow R(-3)^2 \longrightarrow R(-2)^3 \longrightarrow R \longrightarrow R/\mathrm{I}(X) \longrightarrow 0.$$

The equality $\operatorname{codim}(\mathrm{I}(X)) = \operatorname{pdim}(R/\mathrm{I}(X)) = 2$ is satisfied, so $R/\mathrm{I}(X)$ is a CM ring and X is an aCM curve in $\mathbb{P}^3$. Moreover, $R/\mathrm{I}(X)$ is a level ring of CM–type 2.

(v) Set $R = \mathbb{K}[x_0, x_1, x_2]$. In $\mathbb{P}^2$, we consider the Fermat cubic $S_1 = V(x_0^3 + x_1^3 + x_2^3)$ and the three lines $L = V(x_0 + x_1 + x_2)$, $L_1 = V(x_0 - x_1)$, $L_2 = (ax_0 + x_2)$ where $a = -\sqrt[3]{2}$. The Fermat cubic S_1 and the two conics $C_1 = L \cup L_1$ and $C_2 = L \cup L_2$ intersect in four non colinear points

$$X = \{(1 : -1 : 0), (1 : 0 : -1), (0 : 1 : -1),\ (1 : 1 : a)\} \subset \mathbb{P}^2.$$

The homogenous ideal $\mathrm{I}(X)$ of X is generated by one cubic and two quadrics. $R/\mathrm{I}(X)$ has a minimal graded free $R-$resolution:

$$0 \longrightarrow R(-4) \oplus R(-3) \longrightarrow R(-3) \oplus R(-2)^2 \longrightarrow R \longrightarrow R/\mathrm{I}(X) \longrightarrow 0.$$

The equality $\operatorname{codim}(\mathrm{I}(X)) = \operatorname{pdim}(R/\mathrm{I}(X)) = 2$ is satisfied. We have that $R/\mathrm{I}(X)$ is a non level CM ring of CM–type 2.

1.2 Affine semigroups and semigroup rings

An *affine semigroup* H of $\mathbb{Z}^{n+1}$ is a finitely generated semigroup of $\mathbb{Z}^{n+1}$. H is called a *positive* affine semigroup if the group H_0 of invertible elements of H is $H_0 = \{0\}$. Through this thesis we deal only with positive affine semigroups H of $\mathbb{Z}^{n+1}$, which we refer as *affine semigroups* $H \subseteq \mathbb{Z}^{n+1}_{\geq 0}$. In this section, we present the basic definitions, properties and results on this topic needed in the sequel. We define the semigroup ring associated to an affine semigroup $H \subset \mathbb{Z}^{n+1}_{\geq 0}$ and we show that it is the coordinate ring of an affine variety. We define affine normal semigroups, we see that their semigroup rings are CM rings and we introduce two families of affine normal semigroups which play a central role in subsequently chapters. For this section, we mainly refer to [9] and [63].

Let $H \subseteq \mathbb{Z}^{n+1}_{\geq 0}$ be an affine semigroup. We denote by $\mathbb{Z}(H)$ the subgroup of $\mathbb{Z}^{n+1}$ generated by H. The *rank* of H is defined as the rank of $\mathbb{Z}(H)$ as the $\mathbb{Z}-$module generated by H. We often refer to the elements of $\mathbb{Z}^{n+1}$ as *lattice points*. Any element $l = (a_0, \ldots, a_n) \in \mathbb{Z}^{n+1}_{\geq 0}$ defines a monomial

$m_l := x_0^{a_0} \cdots x_n^{a_n} \in R$ and, conversely, any monomial $m = x_0^{a_0} \cdots x_n^{a_n} \in R$ defines an element $l_m := (a_0, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^{n+1}$. The assignation $H \longrightarrow \mathcal{M}(H) := \{m_h \in R \mid h \in H\}$ gives a bijection between the set of affine semigroups $H \subset \mathbb{Z}_{\geq 0}^{n+1}$ and the set of monomial semigroups $M \subset R$, with inverse $M \longrightarrow \mathrm{H}(M) := \{h_m \in \mathbb{Z}_{\geq 0}^{n+1} \mid m \in M\}$.

Definition 1.2.1. Let $H \subset \mathbb{Z}_{\geq 0}^{n+1}$ be an affine semigroup. The *semigroup ring* $\mathbb{K}[H]$ of H is the $\mathbb{K}-$vector subspace of R with a monomial $\mathbb{K}-$basis $\{m_h \in R \mid h \in H\}$. Endowed with the multiplication defined by $m_{h_1} \cdot m_{h_2} = m_{h_1+h_2}$, $h_1, h_2 \in H$, the semigroup ring $\mathbb{K}[H]$ is the $\mathbb{K}-$subalgebra of R generated by $\{m_h \mid h \in H\}$, that is $\mathbb{K}[H] = \mathbb{K}[\mathcal{M}(H)]$.

Example 1.2.2. (i) $\mathbb{Z}_{\geq 0}^{n+1}$ is a positive affine semigroup and R is its associated semigroup ring.

(ii) Let $1 < d$ be an integer. The set $H_1 = \{(t_1 d, t_2 d, t_3 d) \mid t_1, t_2, t_3 \in \mathbb{Z}_{\geq 0}\}$ is the affine semigroup of $\mathbb{Z}_{\geq 0}^3$ generated by $(d, 0, 0), (0, d, 0)$ and $(0, 0, d)$. The associated semigroup ring is the polynomial ring $\mathbb{K}[x_0^d, x_1^d, x_2^d]$.

(iii) Set $e_1 = (3, 0, 0), e_2 = (0, 3, 0), e_3 = (0, 0, 3), e_4 = (1, 1, 1)$ and H_2 be the affine semigroup of $\mathbb{Z}_{\geq 0}^3$ generated by them. Then $H_2 = \{(a, b, c) \in \mathbb{Z}_{\geq 0}^{n+1} \mid a \equiv b \equiv c \mod (3)\}$ and H_1 is a subsemigroup of H_2. The semigroup ring of H_2 is the subring $\mathbb{K}[x_0^3, x_1^3, x_2^3, x_0 x_1 x_2] \subset \mathbb{K}[x_0, x_1, x_3]$.

An affine semigroup $H \subseteq \mathbb{Z}_{\geq 0}^{n+1}$ and its associated semigroup ring $\mathbb{K}[H]$ are intrinsically related. If $I \subseteq \mathbb{K}[H]$ is a subset, we denote $\mathrm{H}(I) := \{h \in H \mid m_h \in I\}$. Naturally, if I is a $\mathbb{K}-$vector subspace of $\mathbb{K}[H]$, then $\mathrm{H}(I)$ is a subsemigroup of H.

Definition 1.2.3. Let $H \subset \mathbb{Z}_{\geq 0}^{n+1}$ be an affine semigroup. A subset $\mathcal{H} \subset H \setminus \{0\}$ is called an *ideal* if for all $h_1, h_2 \in \mathcal{H}$, $\quad h_1 + h_2 \in \mathcal{H}$. The radical $\mathrm{rad}(\mathcal{H})$ of an ideal $\mathcal{H} \subset H$ is $\{h \in H \mid zh \in \mathcal{H}$ for some integer $0 < z\}$. An ideal $\mathcal{H} \subset H$ is said *radical* if $\mathcal{H} = \mathrm{rad}(\mathcal{H})$. An ideal $H \neq \mathcal{H} \subset H$ is said *prime* if given $h_1 + h_2 \in \mathcal{H}$, then $h_1 \in \mathcal{H}$ or $h_2 \in \mathcal{H}$.

Proposition 1.2.4. *Let $H \subseteq \mathbb{Z}_{\geq 0}^{n+1}$ be an affine semigroup and $I, I' \subset \mathbb{K}[H]$ $\mathbb{K}-$vector subspaces.*

(i) $I \subseteq I'$ if and only if $\mathrm{H}(I) \subseteq \mathrm{H}(I')$.

(ii) $\mathrm{H}(I \cap I') = \mathrm{H}(I) \cap \mathrm{H}(I')$ *and* $\mathrm{H}(I + I') = \mathrm{H}(I) \cup \mathrm{H}(I')$.

(iii) I *is a prime (respectively radical) ideal if and only if* $\mathrm{H}(I)$ *is a prime (respectively radical) ideal.*

(iv) *If* I *is an ideal, then* $\mathrm{H}(\mathrm{rad}(I)) = \mathrm{rad}(\mathrm{H}(I))$.

Proof. See [9, Proposition 6.1.1]. □

Example 1.2.5. We take $e_1 = (3,0,0), e_2 = (0,3,0), e_3 = (0,0,3), e_4 = (1,1,1) \in \mathbb{Z}^3_{\geq 0}$ and H_1 and H_2 the affine semigroups generated by $\{e_1, e_2, e_3\}$ and $\{e_1, e_2, e_3, e_4\}$, respectively (Example 1.2.2). By construction $H_1 \setminus \{0\}$ is an ideal of H_2, but not prime. Indeed, $h_1 = (4,4,4), h_2 = (5,5,5) \in H_2 \setminus H_1$ and $(9,9,9) \in H_1$. Now let H' be the affine semigroup generated by $\{e_1, e_2\}$. $H' \setminus \{0\}$ is a prime ideal of H_1 and H_2. Let $h_1 = (a,b,c), h_2 = (d,e,f) \in H_1 \setminus \{0\}$ (respectively $h_2 \in H_2 \setminus \{0\}$) such that $h_1 + h_2 = (a+d, b+e, c+f) \in H'$. Therefore $c = f = 0$ and we obtain that $h_1, h_2 \in H' \setminus \{0\}$.

A subset F of an affine semigroup H is called a *face* if the complement $H \setminus F$ is an ideal of H. There is a correspondence between the set of prime ideals of the semigroup ring $\mathbb{K}[H]$ and its faces.

Theorem 1.2.6. *Let* $H \subset \mathbb{Z}^{n+1}_{\geq 0}$ *be an affine semigroup.* $F \subset H$ *is a face of* H *if and only if the ideal* $I \subset \mathbb{K}[H]$ *generated by the monomials* m_h *with* $h \in H \setminus F$ *is a prime ideal.*

Proof. See [63, Lemma 7.10]. □

A non-empty subset of the euclidean space $\mathbb{R}^{n+1}$ is called a *cone* if it is closed under $\mathbb{R}-$linear combinations with non negative coefficients. For an affine semigroup $H \subset \mathbb{Z}^{n+1}_{\geq 0}$, we define the *cone generated by* H as

$$\mathrm{cone}(H) := \{\sum_{i=1}^{t} r_i h_i \mid h_i \in H,\, r_i \in \mathbb{R}_{\geq 0}\} \subset \mathbb{R}^{n+1}_{\geq 0}.$$

It is the smallest cone of $\mathbb{R}^{n+1}$ containing H. We denote by $\mathrm{relint}(\mathrm{cone}(H))$ the *relative interior* of $\mathrm{cone}(H)$ defined as the interior of the $\mathbb{R}-$vector space $\langle H \rangle \subset \mathbb{R}^{n+1}$ relative to $\mathrm{cone}(H)$, with the induced topology.

Definition 1.2.7. Let $H \subset \mathbb{Z}^{n+1}_{\geq 0}$ be an affine semigroup. The relative interior of H is $\operatorname{relint}(H) := H \cap \operatorname{relint}(\operatorname{cone}(H))$.

We denote by $\mathrm{I}(\operatorname{relint}(H))$ the ideal of $\mathbb{K}[H]$ generated by the monomials m_h with $h \in \operatorname{relint}(H)$. We have:

Proposition 1.2.8. *Let $H \subset \mathbb{Z}^{n+1}_{\geq 0}$ be an affine semigroup.* $\mathrm{I}(\operatorname{relint}(H))$ *is a radical ideal of* $\mathbb{K}[H]$.

Proof. See [9, Lemma 6.1.6]. □

Example 1.2.9. Set $e_1 = (3,0,0), e_2 = (0,3,0), e_3 = (0,0,3)$ and $e_4 = (1,1,1) \in \mathbb{Z}^3_{\geq 0}$, H', H'' and H the affine semigroups generated by $\{e_1, e_2\}$, $\{e_3, e_4\}$ and $\{e_1, e_2, e_3, e_4\}$, respectively (Examples 1.2.2). It holds $H \setminus H'' = H' \setminus \{0\}$. So H'' is a face of H and (x_0^3, x_1^3) is a prime ideal of $\mathbb{K}[H]$ (Example 1.2.5). We have that $\operatorname{relint}(H) = \{(a,b,c) \in H_2 \mid abc \neq 0\}$. From this description, it follows that $\operatorname{relint}(H) \subset H$ is a radical ideal and $\mathrm{I}(\operatorname{relint}(H)) = (x_0x_1x_2) \subset \mathbb{K}[H]$ is a principal ideal.

Next, we present the geometrical interpretation of the semigroup ring of an affine semigroup. Let $H \subseteq \mathbb{Z}^{n+1}_{\geq 0}$ be an affine semigroup minimally generated by $h_1, \ldots, h_r$. We write $h_i = (a_0^i, \ldots, a_n^i) \in \mathbb{Z}^{n+1}_{\geq 0}$ and we set $m_{h_i} = x_0^{a_0^i} \cdots x_n^{a_n^i}$, $i = 1, \ldots, r$. We define the morphism $\varphi_H : \mathbb{A}^{n+1} \longrightarrow \mathbb{A}^r$ by sending an affine point $p = (y_0, \ldots, y_n)$ to

$$\varphi_H(y_0, \ldots, y_n) = (m_{h_1}(p), \ldots, m_{h_r}(p)) := ((y_0^{a_0^1} \cdots y_n^{a_n^1}), \ldots, (y_0^{a_0^r} \cdots y_n^{a_n^r})).$$

We take new variables $w_1, \ldots, w_r$ and $S = \mathbb{K}[w_1, \ldots, w_r]$. We define an epimorphism of rings

$$\rho : S \longrightarrow \mathbb{K}[H] \text{ by } \rho(w_i) := m_{h_i},\ i = 1, \ldots, r.$$

Theorem 1.2.10. *Let $H \subset \mathbb{Z}^{n+1}_{\geq 0}$ be an affine semigroup minimally generated by $h_1, \ldots, h_r$. Then $\mathbb{K}[H]$ is the coordinate ring of the affine variety $X := \varphi_H(\mathbb{A}^{n+1}) \subseteq \mathbb{A}^r$. Moreover,* $\mathrm{I}(X) \subset S$ *is a binomial prime ideal.*

Proof. The kernel $\ker(\rho)$ of $\rho : S \longrightarrow \mathbb{K}[H]$ is a prime ideal of S and we have $\mathbb{K}[H] \cong S/\ker(\rho)$. In particular, ρ structures $\mathbb{K}[H]$ as a $\mathbb{K}-$subalgebra of S. The ideal $\ker(\rho)$ is a binomial prime ideal generated by all binomials of the form

$$\prod_{i=1}^{r} w_i^{\alpha_i} - \prod_{i=1}^{r} w_i^{\beta_i} \text{ such that } \prod_{i=1}^{r} m_{h_i}^{\alpha_i} = \prod_{i=1}^{r} m_{h_i}^{\beta_i}. \tag{1.2.1}$$

By construction, they belong to $\ker(\rho)$. To see the converse, let $f = \sum_{i=1}^{t} \alpha_i w_1^{\alpha_{1,i}} \cdots w_r^{\alpha_{r,i}} \in \ker(\rho)$ with $\alpha_i \in \mathbb{K}^*$ and $(\alpha_{1,i}, \dots, \alpha_{r,i}) \in \mathbb{Z}_{\geq 0}^{r+1}$, we have $\rho(f) = \sum_{i=1}^{t} \alpha_i m_{h_1}^{\alpha_{1,i}} \cdots m_{h_r}^{\alpha_{r,i}} = 0$. First, we assume that all $m_{h_1}^{\alpha_{1,i}} \cdots m_{h_r}^{\alpha_{r,i}}$ are of the same degree, namely d. Since $\rho(f) = 0$, we have that $\sum_{i=1}^{t} \alpha_i m_{h_1}^{\alpha_{1,i}} \cdots m_{h_r}^{\alpha_{r,i}} = 0$ is a trivial linear combination of the monomial $\mathbb{K}-$basis of R_d. Therefore, for $m_{h_1}^{\alpha_{1,1}} \cdots m_{h_r}^{\alpha_{r,1}}$ there exists $2 \leq j \leq t$ such that $m_{h_1}^{\alpha_{1,j}} \cdots m_{h_r}^{\alpha_{r,j}} = m_{h_1}^{\alpha_{1,1}} \cdots m_{h_r}^{\alpha_{r,1}}$. Redefining α_j if needed, we obtain $0 = \rho(f) = \alpha_1(m_{h_1}^{\alpha_{1,1}} \cdots m_{h_r}^{\alpha_{r,1}} - m_{h_1}^{\alpha_{1,j}} \cdots m_{h_r}^{\alpha_{r,j}}) + \sum_{i=2}^{t} \alpha_i m_{h_1}^{\alpha_{1,i}} \cdots m_{h_r}^{\alpha_{r,i}} = \sum_{i=2}^{t} \alpha_i m_{h_1}^{\alpha_{1,i}} \cdots m_{h_r}^{\alpha_{r,i}}$. We define $b_1 = w_1^{\alpha_{1,1}} \cdots w_r^{\alpha_{r,1}} - w_1^{\alpha_{1,j}} \cdots w_r^{\alpha_{r,j}}$ and we iterate the same process for $\sum_{i=2}^{t} \alpha_i m_{h_1}^{\alpha_{1,i}} \cdots m_{h_r}^{\alpha_{r,i}}$. It stops at step $s \leq t$ and we get binomials $b_1, \dots, b_s$ such that $f = \sum_{i=1}^{s} \gamma_i b_i$ for certain $\gamma_i \in \mathbb{K}^*$. If $\rho(f)$ is not homogeneous, we apply the same argument to each homogeneous component of $\rho(f)$. □

The semigroup ring $\mathbb{K}[H]$ of an affine semigroup $H \subseteq \mathbb{Z}_{\geq 0}^{n+1}$ inherits from $R = \bigoplus_{t \geq 0} R_t$ a natural grading $\mathbb{K}[H] = \oplus_{t \geq 0} \mathbb{K}[H]_t$, where $\mathbb{K}[H]_t := \mathbb{K}[H] \cap R_t$. If $h = (a_0, \dots, a_n) \in H$, we denote $\deg(h) := a_0 + \cdots + a_n$. With this notation, each component $\mathbb{K}[H]_t$ has a monomial $\mathbb{K}-$basis $\{m_h \mid \deg(m_h) = \deg(h) = t\}$. If H is minimally generated by $h_1, \dots, h_r$, then $\mathbb{K}[H]$ is minimally generated by $m_{h_1}, \dots, m_{h_r}$ as a graded $\mathbb{K}-$algebra and they generate the homogeneous maximal ideal of $\mathbb{K}[H]$. In addition, if $\deg(m_{h_1}) = \cdots = \deg(m_{h_r})$, then the projective version of Theorem 1.2.10 is true. In this setting, $\mathbb{K}[H]$ is the homogeneous coordinate ring of the projective variety $X := \overline{\varphi_H(\mathbb{P}^n)}$, where $\varphi_H : \mathbb{P}^n \dashrightarrow \mathbb{P}^{r-1}$ is the rational map sending a projective point $p = (y_0 : \dots : y_n) \notin V(m_{h_1}, \dots, m_{h_r})$ to

$$\varphi_H(y_0 : \dots : y_n) = (m_{h_1}(p) : \cdots : m_{h_r}(p)) = ((y_0^{a_0^1} \cdots y_n^{a_n^1}) : \dots : (y_0^{a_0^r} \cdots y_n^{a_n^r})).$$

The homogeneous ideal $\mathrm{I}(X) \subset S$ of X is the homogeneous binomial prime ideal generated by all binomials of the same form as in (1.2.1).

Example 1.2.11. Take $\mathbb{K}[H] = \mathbb{K}[x_0^3, x_1^3, x_2^3, x_0x_1x_2]$ (Example 1.2.2(iii)), $S = \mathbb{K}[w_1, w_2, w_3, w_4]$ and $\rho : S \longrightarrow \mathbb{K}[H]$ given by $\rho(w_1) = x_0^3, \rho(w_2) = x_1^3, \rho(w_3) = x_2^3$ and $\rho(w_4) = x_0x_1x_2$. The semigroup ring $\mathbb{K}[H]$ is the homogeneous coordinate ring of the cubic surface $X = V(w_4^3 - w_1w_2w_3)$ in $\mathbb{P}^3$.

1.2.1 Normal affine semigroups

In this subsection, we introduce normal affine semigroups, we recall the CM property of their associated semigroup rings and we see a combinatorial application involving linear systems of congruences.

Definition 1.2.12. An affine semigroup $H \subseteq \mathbb{Z}^{n+1}_{\geq 0}$ is said *normal* if the following condition holds: if $h_1, h_2, h_3 \in H$ and $zh_1 = zh_2 + h_3$ for some $z \in \mathbb{Z}_{\geq 0}$, then there exists $h \in H$ such that $h_3 = zh$.

The *normalization* of H is defined as

$$\overline{H} := \{h \in \mathbb{Z}(H) \mid zh \in H \text{ for some integer } 0 < z\}.$$

Since by assumption H is positive, $\overline{H} \subset \mathbb{Z}^{n+1}_{\geq 0}$ is a positive affine semigroup containing H. We have that H is normal if and only if $H = \overline{H}$.

Example 1.2.13. (i) Let $H \subset \mathbb{Z}^3_{\geq 0}$ be the affine semigroup generated by $e_1 = (3,0,0), e_2 = (0,3,0), e_3 = (0,0,3), e_4 = (1,1,1)$ (Example 1.2.2(iii)). We have that $H = \{(a,b,c) \in \mathbb{Z}^{n+1}_{\geq 0} \mid a \equiv b \equiv c \mod (3)\}$ is normal. Indeed, $\mathbb{Z}(H) = \{(a,b,c) \in \mathbb{Z}^{n+1} \mid a \equiv b \equiv c \mod (3)\}$ and if $h = (a,b,c) \in \mathbb{Z}(H)$ is such that $zh \in H$ for integer $z > 0$, then $(a,b,c) \in \mathbb{Z}^{n+1}_{\geq 0}$ and hence $h \in H$.
(ii) Let $d \geq 1$ be an integer and $H = \{(t_1d, t_2d, t_3d) \mid t_1, t_2, t_3 \in \mathbb{Z}_{\geq 0}\}$ (Example 1.2.2(ii)). For $d = 1$, $H = \mathbb{Z}^3_{\geq 0}$ is normal. However for $d > 1$, $(1,1,1) \in \overline{H}$ but $(1,1,1) \notin H$, so H is not normal.

In [51], Hochster proved that the semigroup ring of any normal semigroup is a CM ring. In particular,

Theorem 1.2.14. *Let $H \subset \mathbb{Z}^{n+1}_{\geq 0}$ be an affine normal semigroup. Then $\mathbb{K}[H]$ is a CM ring.*

Proof. See [51, Theorem 1] or [9, Theorem 6.3.5]. □

For instance, since the semigroup H of Example 1.2.13(i) is normal, the associated semigroup $\mathbb{K}[H] = \mathbb{K}[x_0^3, x_1^3, x_2^3, x_0x_1x_2]$ is a CM ring. Geometrically, $\mathbb{K}[H]$ is the homogeneous coordinate ring of the cubic surface $V(w_4^3 - w_1w_2w_3)$ in $\mathbb{P}^3$ (Example 1.2.11). Actually, H is a member of a large family of affine normal semigroups of $\mathbb{Z}_{\geq 0}^{n+1}$ which can be translated combinatorially as the $\mathbb{Z}_{\geq 0}^{n+1}$−solutions of linear integral systems. In [78], this topic and related questions, including the CM property, were treated from the point of view of invariant theory of finite groups. Next section contains an exposition of invariant theory of finite groups and we will see examples pointing out this perspective.

Let $0 < r$ be an integer and $\mathcal{A} = (\alpha_{i,j})$ a $r \times (n+1)$ matrix of integers. We denote by $(*)_{\mathcal{A}}$ the homogeneous linear systems of integral equations:

$$\left\{\begin{array}{ccccccc} \alpha_{1,0}y_0 & + & \cdots & + & \alpha_{1,n}y_n & = & 0 \\ \vdots & & & & & & \\ \alpha_{r,0}y_0 & + & \cdots & + & \alpha_{r,n}y_n & = & 0 \end{array}\right.$$

The set of $\mathbb{Z}_{\geq 0}^{n+1}$−solutions of $(*)_{\mathcal{A}}$ defines a positive affine semigroup $H_{\mathcal{A}} \subset \mathbb{Z}_{\geq 0}^{n+1}$. Keeping this notation, we have:

Proposition 1.2.15. *$H_{\mathcal{A}} \subseteq \mathbb{Z}_{\geq 0}^{n+1}$ is an affine normal semigroup.*

Proof. Assume that $h = (y_0, \ldots, y_n) \in \mathbb{Z}(H_{\mathcal{A}})$ and $z > 0$ is an integer such that $zh \in H_{\mathcal{A}}$. We have that $(zy_0, \ldots, zy_n)$ is a $\mathbb{Z}_{\geq 0}^{n+1}$−solution of $(*)_{\mathcal{A}}$, so $h \in \mathbb{Z}_{\geq 0}^{n+1}$ and since $(*)_{\mathcal{A}}$ is homogenous, we obtain $h \in H_{\mathcal{A}}$. $\square$

A consequence of Theorem 1.2.14 is that the semigroup ring $\mathbb{K}[H_{\mathcal{A}}]$ is a CM ring. Similar arguments apply to homogeneous linear systems of congruences. This standpoint plays a crucial role in this thesis. In this setting, we fix integers $0 < d_1, \ldots, d_r$, $\mathcal{A}$ is an $r \times (n+1)$ matrix of positive integers and $H_{\mathcal{A}}$ is the set of $\mathbb{Z}_{\geq 0}^{n+1}$−solutions of the systems:

$$(*)_{\mathcal{A};t_1,\ldots,t_r} : \left\{\begin{array}{ccccccc} \alpha_{1,0}y_0 & + & \cdots & + & \alpha_{1,n}y_n & = & t_1d_1 \\ \vdots & & & & & & \\ \alpha_{r,0}y_0 & + & \cdots & + & \alpha_{r,n}y_n & = & t_rd_r \end{array}\right.$$

where $0 < t_1, \ldots, t_r$. With this notation, we have:

Proposition 1.2.16. *$H_{\mathcal{A}} \subset \mathbb{Z}^{n+1}_{\geq 0}$ is an affine normal semigroup.*

Proof. Assume $h = (y_0, \dots, y_n) \in \mathbb{Z}(H_{\mathcal{A}})$ and $z > 0$ is an integer such that $zh \in H_{\mathcal{A}}$. The first hypothesis implies that h is a $\mathbb{Z}^{n+1}-$solution of some system $(*)_{\mathcal{A};t_1,\dots,t_r}$. From the second we obtain that $h \in \mathbb{Z}^{n+1}_{\geq 0}$. □

By Theorem 1.2.14, $\mathbb{K}[H_{\mathcal{A}}]$ is a CM ring. For $i = 0, \dots, n$, we set $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ with 1 in position ith and $M := \mathrm{LCM}(d_1, \dots, d_r)$. The semigroup $H_{\mathcal{A}}$ contains all points Me_i, $i = 1, \dots, r$ and $i = 0, \dots, n$. Therefore, $\mathrm{cone}(H_{\mathcal{A}}) = \mathbb{R}^{n+1}_{\geq 0}$ and, hence, $\mathrm{relint}(H_{\mathcal{A}})$ is the set of points of $H_{\mathcal{A}}$ which are outside any coordinate hyperplane of $\mathbb{R}^{n+1}$. In other words, $\mathrm{relint}(H_{\mathcal{A}}) = \{(a_0, \dots, a_n) \in H_{\mathcal{A}} \mid a_0 \cdots a_n \neq 0\}$, and since $(M, \dots, M)$ is a $\mathbb{Z}^{n+1}_{\geq 0}-$solution of $(*)_{\mathcal{A};t_1,\dots,t_r}$ for some integers $0 < t_1, \dots, t_r$, we have that $\mathrm{relint}(H_{\mathcal{A}}) \neq \emptyset$.

Example 1.2.17. (i) Let $d > 0$ be an integer and set

$$(*)_{\mathcal{A};t} : y_0 + \cdots + y_n = td$$

with $t \geq 0$. The semigroup $H_{\mathcal{A}} = \{(a_0, \dots, a_n) \in \mathbb{Z}^{n+1}_{\geq 0} \mid a_0 + \cdots + a_n = td, \text{ for some integer } t \geq 0\}$ is normal. $\mathbb{K}[H_{\mathcal{A}}]$ is the dth Veronese subalgebra $\oplus_{t\geq 0} R_{td}$ of R, which is minimally generated by the set of monomials of degree d and it is a CM ring. Let $t_0 = \min\{t \in \mathbb{Z}_{>0} \mid td \geq n+1\}$. The ideal $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}})) \subset \mathbb{K}[H_{\mathcal{A}}]$ is generated by the set $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_{t_0} := \{x_0^{a_0} \cdots x_n^{a_n} \mid a_0 + \cdots + a_n = t_0 d \text{ and } a_0 \cdots a_n \neq 0\}$. Indeed, let $h = (b_0, \dots, b_n) \in \mathrm{relint}(H_{\mathcal{A}})$. We have $b_0 + \cdots + b_n \geq t_0 d$ and $b_0 \cdots b_n \neq 0$, so $h_1 := (b_0 - 1, \dots, b_n - 1) \in \mathbb{Z}^{n+1}_{\geq 0}$. If $\deg(h) > t_0 d$, then $\deg(h_1) > t_0 d - (n+1)$. We define i as the smallest integer $0 \leq j \leq n$ such that

$$\sum_{l=0}^{j-1} (b_l - 1) < t_0 d - (n+1) \quad \text{and} \quad \sum_{l=0}^{j} (b_l - 1) \geq t_0 d - (n+1)$$

and we set $B := \sum_{l=0}^{i-1} (b_l - 1)$. Therefore, $h_2 := (b_0 - 1, \dots, b_{i-1} - 1, t_0 d - (n+1) - B, 0, \dots, 0) \in \mathbb{Z}^{n+1}_{\geq 0}$ has degree $\deg(h_2) = t_0 d - (n+1)$ and so $h' = h_2 + (1, \dots, 1) \in \mathrm{relint}(H_{\mathcal{A}})$. Hence, we obtain that $m_{h_2} \in \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_{t_0}$ divides m_h.

(ii) Set $e_1 = (3, 0, 0)$, $e_2 = (0, 3, 0)$, $e_3 = (0, 0, 3)$, $e_4 = (1, 1, 1)$ and $H \subset \mathbb{Z}^3_{\geq 0}$ the affine semigroup generated by $\{e_1, e_2, e_3, e_4\}$ (Examples 1.2.2(iii) and

1.2.13(i)). We have that H is a normal semigroup and it coincides with the set of $\mathbb{Z}^3_{\geq 0}-$solutions of the systems of congruences:

$$(*)_{\mathcal{A};t_1,t_2} : \begin{cases} y_0 & + & y_1 & + & y_2 & = & 3t_1 \\ & & y_1 & + & 2y_2 & = & 3t_2 \end{cases}$$

with $t_1, t_2 \in \mathbb{Z}_{\geq 0}$. $\mathbb{K}[H]$ is a CM ring.

1.3 Rings of invariants of finite groups

Invariant theory of finite groups and affine semigroups and semigroup rings are the main tools we use in this thesis. In this section, we gather the basic ideas and results needed onwards. We define the ring of invariants of a finite subgroup of $\mathrm{GL}(n+1,\mathbb{K})$ acting on R. We address the internal structure of the ring of invariants by means of describing a minimal set of generators and determining the Hilbert function and series. On the other hand, we discuss the CM property of these rings and a geometric interpretation, which plays a central role in next chapter. We mainly follow [77], [81] and [9].

Let $\Lambda \subset \mathrm{GL}(n+1,\mathbb{K})$ be a finite group of order $|\Lambda|$. For any pair $(\lambda, f) \in \Lambda \times R$ the assignation $\lambda(f) = \lambda \circ f$ defines a natural action of Λ on R. A polynomial $f \in R$ satisfying $\lambda(f) = f$ for all $\lambda \in \Lambda$ is called an *invariant of* Λ. Any finite group Λ admits invariants.

Theorem 1.3.1. *Let $\Lambda \subset \mathrm{GL}(n+1,\mathbb{K})$ be a finite group. Then there exist $n+1$, but not $n+2$, algebraically independent invariants of Λ.*

Proof. See [81, Proposition 2.1.1]. □

$R^\Lambda = \{f \in R \mid \lambda(f) = f,\ \forall \lambda \in \Lambda\}$ is called *the ring of invariants of* Λ or *the algebra of invariants of* Λ. It has the structure of a $\mathbb{K}-$subalgebra of R. The above theorem is equivalent to say that R^Λ has Krull dimension $n+1$. R^Λ inherits a natural grading $R^\Lambda = \oplus_{t\geq 0} R^\Lambda_t,\ R^\Lambda_t = R_t \cap R^\Lambda$.

Definition 1.3.2. Let $\Lambda \subset \mathrm{GL}(n+1,\mathbb{K})$ be a finite group of order $|\Lambda|$ and e a $|\Lambda|$th primitive root of $1 \in \mathbb{K}$. The *cyclic extension* of Λ is the finite group $\overline{\Lambda} \subset \mathrm{GL}(n+1,\mathbb{K})$ generated by Λ and the diagonal matrix $\mathrm{diag}(e,\ldots,e)$.

The ring of invariants of the cyclic extension $\overline{\Lambda}$ of Λ is the graded $\mathbb{K}-$subalgebra $R^{\overline{\Lambda}} = \oplus_{t\geq 0} R^{\overline{\Lambda}}_t$, $\quad R^{\overline{\Lambda}} = R^{\Lambda}_{t\,|\Lambda|}$, called the $|\Lambda|$th Veronese subalgebra of R^{Λ}.

Definition 1.3.3. Let $\Lambda \subset \mathrm{GL}(n+1,\mathbb{K})$ be a finite group of order $|\Lambda|$. A finite set $\{f_1,\ldots,f_r\} \subset R^{\Lambda}$ is called *a set of fundamental invariants of* Λ if $f_1,\ldots,f_r$ generate R^{Λ} as a $\mathbb{K}-$algebra, i.e. $R^{\Lambda} = \mathbb{K}[f_1,\ldots,f_r]$.

The existence of a finite set of fundamental invariants of R^{Λ} is granted by Noether's degree bound:

Theorem 1.3.4. *Let* $\Lambda \subset \mathrm{GL}(n+1,\mathbb{K})$ *be a finite group of order* $|\Lambda|$. *Then* R^{Λ} *is generated by no more than* $N_{n+1,|\Lambda|}$ *homogenous invariants of degree not exceeding* $|\Lambda|$.

Proof. See [77, Theorem 1.2] and [87, VIII §15]. □

Thus, if $\mathcal{B}_i$ is a $\mathbb{K}-$basis of the vector space R^{Λ}_i, $i = 1,\ldots,|\Lambda|$, then $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_{|\Lambda|}$ generates R^{Λ}. However, $\mathcal{B}$ is not necessarily a minimal set of fundamental invariants and a complete description of such a set of generators is, in general, unknown.

Example 1.3.5. (i) Let $d > 0$ be an integer, e a dth primitive root of $1 \in \mathbb{K}$ and we denote by $G_V \subset \mathrm{GL}(n+1,\mathbb{K})$ the finite cyclic group of order d generated by $\mathrm{diag}(e,\ldots,e)$. R^{G_V} is the dth Veronese subalgebra of R and a minimal set of fundamental invariants of G_V is the set of $N_{n,d} < N_{n+1,d}$ monomials of degree exactly d (see also Example 1.2.17(i)).

(ii) Let e be a $3rd-$primitive root of $1 \in \mathbb{K}$ and $\Lambda = \langle \mathrm{diag}(1,e,e^2)\rangle \subset \mathrm{GL}(3,\mathbb{K})$ a cyclic group of order 3. The set of all monomial invariants of $\mathrm{diag}(1,e,e^2)$ is a $\mathbb{K}-$basis of R^{Λ}_t. By Noether's degree bound, the following $\{x_0, x_0^2, x_1x_2, x_0^3, x_0x_1x_2, x_2^3, x_3^3\}$ is a set of fundamental invariants of Λ and R^{Λ} is minimally generated by $\{x_0, x_1x_2, x_1^3, x_2^3\}$.

(iii) Let $\overline{\Lambda}$ be the cyclic extension of Λ in (ii), i.e. the finite abelian group of order 9 generated by $\mathrm{diag}(1,e,e^2)$ and $\mathrm{diag}(e,e,e)$. $R^{\overline{\Lambda}} = \bigoplus_{t\geq 0} R^{\Lambda}_{3t}$ is the 3rd Veronese subalgebra of R^{Λ}. We have that $\{x_0^3, x_1^3, x_2^3, x_0x_1x_2\} \subset R^{\overline{\Lambda}}$ and a monomial $m = x_0^{a_0}x_1^{a_1}x_2^{a_2}$ is an invariant $\overline{\Lambda}$ if and only if (a_0,a_1,a_2) is a

$\mathbb{Z}^3_{\geq 0}-$solution of the linear system of congruences:

$$(*)_{\mathcal{A};t_1,t_2} : \left\{ \begin{array}{ccccccc} y_0 & + & y_1 & + & y_2 & = & 3t_1 \\ & & y_1 & + & 2y_2 & = & 3t_2 \end{array} \right.$$

for some $t_1, t_2 \in \mathbb{Z}_{\geq 0}$ (Example 1.2.17(ii)). $R^{\overline{\Lambda}} = \mathbb{K}[H_{\mathcal{A}}]$, where $H_{\mathcal{A}} \subset \mathbb{Z}^3_{\geq 0}$ is the normal affine semigroup generated by $(3,0,0),(0,3,0),(0,0,3)$ and $(1,1,1)$.

(iv) The dihedral group $D_{2\cdot 3}$ of order 6 can be represented in $\mathrm{GL}(2,\mathbb{K})$ as the linear group of order 6 generated by the matrices:

$$M = \begin{pmatrix} e & 0 \\ 0 & e^2 \end{pmatrix} \qquad \text{and} \qquad \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where e is a 3rd primitive root of $1 \in \mathbb{K}$. A minimal set of fundamental invariants of $D_{2\cdot 3}$ is $\{x_0, x_1x_2, x_1^3 + x_2^3\}$ and $R^{D_{2\cdot 3}} = \mathbb{K}[x_0, x_1x_2, x_1^3 + x_2^3]$ is a non standard grading polynomial ring.

Hilbert function and Hilbert series. The Hilbert function

$$\mathrm{HF}(R^{\Lambda}, \bullet) : \mathbb{Z}_{\geq 0} \longrightarrow \mathbb{Z}_{\geq 0}$$

of the ring of invariants R^{Λ} is defined by $\mathrm{HF}(R^{\Lambda}, t) := \dim_{\mathbb{K}} R^{\Lambda}_t$. Fix $t \geq 1$ and a $\mathbb{K}-$basis $\mathcal{C}_t = (f_1, \ldots, f_{N_t})$ of the $N_{n,t}-$dimensional vector space R_t. If $\lambda \in \Lambda$, we denote by $\lambda^{(t)} : R_t \longrightarrow R_t$ the induce linear transformation on R_t, i.e. the $N_{n,t} \times N_{n,t}$ matrix whose $ith-$column is the coordinate vector of $\lambda(f_i)$. In this linear setting, R^{Λ}_t appears as the invariant subspace of R_t :

$$\{v \in \mathbb{K}^{N_t} \mid \lambda^{(t)} v = v, \ \forall \lambda \in \Lambda\},$$

which provides the following expression:

$$\mathrm{HF}(R^{\Lambda}, t) = \frac{1}{|\Lambda|} \sum_{\lambda \in \Lambda} \mathrm{trace}(\lambda^{(t)}).$$

The Hilbert series of R^{Λ}, also called the *Molien series of* Λ, is the formal series:

$$\mathrm{HS}(R^{\Lambda}, z) = \sum_{t \geq 0} \mathrm{HF}(R^{\Lambda}, t) z^t.$$

In [65], Molien gave an explicit formula for $\mathrm{HS}(R^{\Lambda}, z)$ in terms of the elements of $\Lambda \subset \mathrm{GL}(n+1, \mathbb{K})$. More precisely,

Theorem 1.3.6. *Let $\Lambda \subset \mathrm{GL}(n+1,\mathbb{K})$ be a finite group of order $|\Lambda|$. Then the Molien series of Λ is given by*

$$\mathrm{HS}(R^{\Lambda}, z) = \frac{1}{|\Lambda|}\sum_{\lambda\in\Lambda}\frac{1}{\det(\mathrm{Id} - z\lambda)},$$

where Id *denotes the identity matrix.*

Proof. See [9, Theorem 6.4.8] or [77, Theorem 2.1]. □

Example 1.3.7. (i) Let $d > 0$ be an integer and $R^{G_V} \subset \mathrm{GL}(n+1,\mathbb{K})$ the dth Veronese subalgebra of R (Example 1.3.5(i)). The Hilbert function of R^{G_V} is $\mathrm{HF}(R^{G_V}, t) = \dim(R_t) = N_{n,t}$ if t is a multiple of d and it is $\mathrm{HF}(R^{G_V}, t) = 0$ otherwise. By Theorem 1.3.6, we have that

$$\mathrm{HS}(R^{G_V}, z) = \sum_{t\geq 0} N_{n,td}z^{td} = \frac{1}{d}\sum_{k=0}^{d-1}\frac{1}{(1-e^k z)^{n+1}}.$$

(ii) Let e be a $3rd$ primitive root of $1 \in \mathbb{K}$ and $\Lambda = \langle \mathrm{diag}(1, e, e^2)\rangle \subset \mathrm{GL}(3,\mathbb{K})$ a cyclic group of order 3. We have $\Lambda = \{\mathrm{Id}, \mathrm{diag}(1, e, e^2), \mathrm{diag}(1, e^2, e)\}$. So, by Theorem 1.3.6,

$$\mathrm{HS}(R^{\Lambda}, z) = \frac{1}{3}\left(\frac{1}{(1-z)^3} + \frac{2}{1-z^3}\right) = \frac{z^4+z^2+1}{(1-z)(1-z^3)^2}.$$

Expanding $\mathrm{HS}(R^{\Lambda}, z)$, we obtain:

$$\mathrm{HF}(R^{\Lambda}, t) = \begin{cases} \frac{t^2+3t+6}{2} & \text{if} \quad t \equiv 0 \bmod (3) \\ \\ \frac{t^2+3t+2}{6} & \text{otherwise.} \end{cases}$$

(iii) Take the cyclic extension $\overline{\Lambda}$ of Λ (Example 1.3.5(iii)). As a direct consequence of (ii), the Hilbert function and series of $R^{\overline{\Lambda}}$ are, respectively:

$$\mathrm{HF}(R^{\overline{\Lambda}}, t) = \begin{cases} \frac{t^2+3t+6}{2} & \text{if} \quad t \equiv 0 \bmod (3) \\ \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathrm{HS}(R^{\overline{\Lambda}}, t) = \frac{z^6+z^3+1}{(1-z^3)^3}.$$

The Cohen–Macaulay property. For any finite group $\Lambda \subset \mathrm{GL}(n+1, \mathbb{K})$, R^Λ is a CM ring. This fundamental result was proved by Hochster and Eagon [52] as a consequence of the existence of a Reynolds operator for the pair (R, R^Λ) and the fact that R is integral over R^Λ. In general, if A' is a subring of a ring A, a *Reynolds operators for* (A, A') is an $A'-$linear map $\phi : A \longrightarrow A'$ such that the restriction of ϕ to A' is the identity. The existence of such a map is equivalent to the fact that A' is a direct summand of A as an $A'-$module.

Proposition 1.3.8. *Assume that there exists a Reynolds operators for the pair* (A, A')*. Then,*

(i) for every ideal I of A' one has $IA \cap A' = I$.

(ii) If A is a noetherian ring, then A' is a noetherian ring.

(ii) If $y_1, \ldots, y_r$ is an $A-$regular sequence in A', then it is an $A'-$regular sequence.

Proof. See [9, Proposition 6.4.4]. □

Theorem 1.3.9. *Let A be a CM ring and A' a subring of A such that there exists a Reynolds operator for (A, A'). If A is integral over A', then A' is a CM ring.*

Proof. See [52, Proposition 12]. □

For the pair (R, R^Λ), the assignation

$$\phi(f) = \frac{1}{|\Lambda|} \sum_{\lambda \in \Lambda} \lambda(f), \quad f \in R,$$

defines a Reynolds operator. On the other hand, set z an indeterminate, let $f \in R$ and consider the following polynomial

$$p_f(z) := \prod_{\lambda \in \Lambda} (z - \lambda(f)) \in R[z].$$

$p_f(z)$ is a monic polynomial in z of degree $|\Lambda|$ with coefficients in R^Λ. Furthermore, since $\mathrm{Id} \in \Lambda$, we have that f is a solution of the equation $p_f(z) = 0$. So, R is integral over R^Λ.

Proposition 1.3.10. *For any finite group $\Lambda \subset \mathrm{GL}(n+1,\mathbb{K})$, R^Λ is a CM ring.*

Proof. See [52, Proposition 13]. $\square$

By Noether's graded normalization theorem (Theorem 1.1.19) and Theorem 1.1.20, there exists a h.s.o.p $\theta_0,\dots,\theta_n$ of R^Λ such that R^Λ is a free $\mathbb{K}[\theta_0,\dots,\theta_n]-$module. This approach appears as an standard alternative for proving R^Λ is a CM ring (see, for instance, [77] and [81]). In subsequently chapters, we will take advantage of this strategy, which is useful to understand the structure of R^Λ when Λ acts diagonally on R, as well as its Hilbert series.

We end this section with the geometrical interpretation of the ring R^Λ. Let $\mathcal{B} = \{f_1,\dots,f_r\}$ be a minimal set of fundamental invariants of Λ, take variables $w_1,\dots,w_r$ and $S = \mathbb{K}[w_1,\dots,w_r]$. The assignation $\rho(f_i) = w_i$, $i = 1,\dots,r$, defines an epimorphism of rings $\rho : S \longrightarrow R^\Lambda$.

Theorem 1.3.11. *Let $\Lambda \subset \mathrm{GL}(n+1,\mathbb{K})$ be a finite group, $\mathcal{B} = \{f_1,\dots,f_r\}$ a minimal set of fundamental invariants of Λ and $\varphi_{\mathcal{B}} : \mathbb{A}^{n+1} \longrightarrow \mathbb{A}^r$ be the morphism defined by $(f_1,\dots,f_r)$. Then, R^Λ is the coordinate ring of the affine variety $X := \varphi_{\mathcal{B}}(\mathbb{A}^{n+1}) \subseteq \mathbb{A}^r$. Moreover, X is an aCM variety.* $\square$

The ideal $\mathrm{I}(X) = \ker(\varphi_{\mathcal{B}})$ of X is often called the *ideal of syzygies among* $f_1,\dots,f_r$, denoted $\mathrm{syz}(f_1,\dots,f_r)$ (or $\mathrm{syz}(\mathcal{B})$ for simplicity). The projective version of Theorem 1.3.11 is true when $\mathcal{B}$ is a minimal set of fundamental homogeneous invariants of Λ all of the same degree. Furthermore, since R^Λ contains an h.s.o.p of R, $\mathcal{B}$ induces a morphism of projective varieties $\varphi_{\mathcal{B}} : \mathbb{P}^n \longrightarrow \mathbb{P}^{r-1}$. Hence, R^Λ is the homogeneous coordinate ring of an aCM projective variety $X = \varphi_{\mathcal{B}}(\mathbb{P}^n)$.

Since Veronese varieties play a central role through this thesis, we introduce the following notation.

Notation 1.3.12. Let $n, d \geq 1$ be integers and $\mathcal{M}_{n,d} = \{m_1,\dots,m_{N_{n,d}}\} \subset R$ the set of monomials of degree d, ordered lexicographically. The *Veronese variety* $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ is the image of the Veronese embedding of $\mathbb{P}^n$

$$\nu_{n,d} : \mathbb{P}^n \longrightarrow \mathbb{P}^{N_{n,d}-1}$$

which sends a point $p = (y_0 : \dots : y_n) \in \mathbb{P}^n$ to $\nu_{n,d}(p) = (m_1(p) : \cdots : m_{N_{n,d}}(p)) \in \mathbb{P}^{N_{n,d}-1}$.

Example 1.3.13. (i) Let $n, d \geq 1$ be integers and R^{G_V} the dth Veronese subalgebra of R (Example 1.3.5(i)). The set $\mathcal{M}_{n,d} \subset R$ of all monomials of degree d is a minimal set of fundamental invariants of G_V and R^{G_V} is the homogeneous coordinate ring of the aCM Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$.

(ii) Let $d \geq 1$ be an integer. We set $S = \mathbb{K}[y_0, \ldots, y_d]$. The *rational normal curve of degree* d is the Veronese curve $X_{1,d} \subset \mathbb{P}^d$. It is the image of the morphism

$$\nu_{1,d} : \mathbb{P}^1 \longrightarrow \mathbb{P}^d, \quad \nu_{1,d}(x_0 : x_1) = (x_0^d : x_0^{d-1}x_1 : \cdots : x_0x_1^{d-1} : x_1^d).$$

The homogeneous coordinate ring $A(X_{1,d})$ of the rational normal curve $X_{1,d} \subset \mathbb{P}^d$ is isomorphic to the dth Veronese subalgebra of $\mathbb{K}[x_0, x_1]$. The homogeneous ideal $\mathrm{I}(X_{1,d}) \subset S$ is generated by the $\binom{d}{2}$ quadrics obtained from the 2×2 minors of the matrix

$$\begin{pmatrix} y_0 & y_1 & \cdots & y_{d-1} \\ y_1 & y_2 & \cdots & y_d \end{pmatrix}.$$

A minimal graded free $S-$resolution of $A(X_{1,d})$ is given by the *Eagon–Northcott complex*:

$$0 \longrightarrow S(-d)^{d-1} \longrightarrow S(1-d)^{d(d-2)} \longrightarrow \cdots \longrightarrow S(i-d)^{(d-i-1)\binom{d}{i}} \longrightarrow$$

$$\longrightarrow \cdots \longrightarrow S(-3)^{2\binom{d}{3}} \longrightarrow S(-2)^{\binom{d}{2}} \longrightarrow S \longrightarrow S/\,\mathrm{I}(X_{1,d}) \longrightarrow 0.$$

$A(X_{1,d})$ is a level ring of CM–type $d-1$.

(iii) Set $S = \mathbb{K}[y_0, y_1, y_2, y_3, y_4, y_5]$. The Veronese surface $X_{2,5} \subset \mathbb{P}^5$ is the image of the morphism

$$\nu_{2,5} : \mathbb{P}^2 \longrightarrow \mathbb{P}^5, \quad \nu_{2,5}(x_0 : x_1 : x_2) = (x_0^2 : x_0x_1 : x_0x_2 : x_1^2 : x_1x_2 : x_2^2).$$

The coordinate ring $A(X_{2,5})$ of $X_{2,5} \subset \mathbb{P}^5$ is the 5th Veronese subalgebra of $\mathbb{K}[x_0, x_1, x_2]$. The homogeneous ideal $\mathrm{I}(X_{2,5}) \subset S$ of $X_{2,5}$ is generated by the quadrics obtained from the 2×2 minors of the symmetric matrix

$$\begin{pmatrix} y_0 & y_3 & y_4 \\ y_3 & y_1 & y_5 \\ y_4 & y_5 & y_2 \end{pmatrix}.$$

A minimal graded free $S-$resolution of $A(X_{2,5})$ is:

$$0 \longrightarrow S(-4)^3 \longrightarrow S(-3)^8 \longrightarrow S(-2)^6 \longrightarrow S \longrightarrow S/\operatorname{I}(X_{2,5}) \longrightarrow 0.$$

$A(X_{2,5})$ is a CM level ring of CM–type 3.

(iv) Take $\Lambda = \langle \operatorname{diag}(1, e, e^2)\rangle \subset \operatorname{GL}(3, \mathbb{K})$ a cyclic group of order 3, where e is a 3rd primitive root of $1 \in \mathbb{K}$ (Example 1.3.5(iii)). $\{x_0^3, x_1^3, x_2^3, x_0x_1x_2\}$ is a set of fundamental invariants of its cyclic extension $\overline{\Lambda}$ and $R^{\overline{\Lambda}}$ is the homogeneous coordinate ring of the aCM cubic surface $X = V(w_4^3 - w_1w_2w_3)$ in $\mathbb{P}^3$ (Example 1.2.11).

1.4 Artinian ideals and the weak Lefschetz property

We finish this preliminary chapter presenting the weak Lefschetz properties of artinian algebras. It provides context and motivation for developing the results of this dissertation. We begin with the following definition.

Definition 1.4.1. Let $J \subset R$ be an homogeneous artinian ideal and $A = R/J =: \bigoplus_{i\geq 0} A_i$ the associated artinian graded $\mathbb{K}-$algebra. We say that A (or J) has *the weak Lefschetz property (WLP)* if there exists a homogeneous linear form $L \in A_1$ such that for all $i \geq 0$, the multiplication map

$$\times L : A_i \longrightarrow A_{i+1}$$

has maximal rank, i.e. it is injective or surjective.

In [80], Stanley proved that any monomial complete intersection $J = (x_0^{a_0}, \ldots, x_n^{a_n})$ of R has the WLP. Since then, the weak Lefschetz property of artinian graded $\mathbb{K}-$algebras have been extensively studied, from many different perspectives, as one can see in [86, 67, 61, 62, 41, 40]. The natural problem of determining which artinian $\mathbb{K}-$algebras hold or fail the WLP remains open and a deeper research is needed to understand which conditions prevents such algebras from having the WLP. We say that A *fails de WLP in degree* i_0 if for any linear form $L \in A_1$, the multiplication map $\times L : A_{i_0} \longrightarrow A_{i_0+1}$ does not have maximal rank. By abuse of notation, we say that the ideal J has the WLP (respectively fails the WLP in degree i_0).

Example 1.4.2. (i) The ideal $J = (x_0^3, x_1^3, x_2^3, (x_0 + x_1 + x_2)^3) \subset R = \mathbb{K}[x_0, x_1, x_2]$ has the WLP. Take $L = x_0 + 2x_1 + 3x_2$, the multiplication map $\times L : (R/J)_i \longrightarrow (R/J)_{i+1}$ has maximal rank for all $i \geq 0$.

(ii) The ideal $J = (x_0^4, x_1^4, x_2^4, x_3^4, x_0x_1x_2x_3) \subset R = \mathbb{K}[x_0, x_1, x_2, x_3]$ fails the WLP in degree 5. J is a monomial ideal and, as we will see in Proposition 1.4.3, it suffices to show that the multiplication by $x_0 + x_1 + x_2 + x_3 \in R_1$ does not have maximal rank. Indeed,

$$\times(x_0 + x_1 + x_2 + x_3) : (R/J)_5 \longrightarrow (R/J)_6$$

is neither injective nor surjective.

In this thesis, we mainly work with monomial artinian ideals $J \subset R$. For a monomial artinian ideal $J \subset R$ to check if J has or fails the WLP it suffices to prove the behaviour of the multiplication map by the linear form $x_0 + x_1 + \cdots + x_n \in R_1$. Indeed, we have:

Proposition 1.4.3. *Let $J \subset R$ be a monomial artinian ideal and set $L = x_0+x_1+\cdots+x_n \in R_1$. Then, J has the WLP if and only if the multiplication map $\times L : (R/J)_i \longrightarrow (R/J)_{i+1}$ has maximal rank for all $i \geq 0$.*

Proof. See [62, Proposition 2.2]. □

In [7], Brenner and Kaid showed that any ideal of the form

$$J = (x_0^3, x_1^3, x_2^3, f(x_0, x_1, x_2)) \subset \mathbb{K}[x_0, x_1, x_2],$$

where $f(x_0, x_1, x_2)$ is a homogeneous polynomial of degree 3, fails the WLP in degree 2 if and only if $f(x_0, x_1, x_2)$ belongs to the monomial ideal

$$T = (x_0^3, x_1^3, x_2^3, x_0x_1x_2) \subset \mathbb{K}[x_0, x_1, x_2]. \tag{1.4.1}$$

On the other hand, in [82] and [83], Togliatti proved that the only smooth projection of the Veronese surface $X_{2,3} \subset \mathbb{P}^9$ in $\mathbb{P}^5$ satisfying a Laplace equation of order 2 and parameterized by an ideal I generated by forms of degree 3 such that I_3^{-1} is an artinian ideal, where I^{-1} denotes the inverse system of I (Definition 1.4.4), is the smooth rational surface parameterized by

$$\overline{T} = (x_0^2x_1, x_0^2x_2, x_0x_1^2, x_0x_2^2, x_1^2x_2, x_1x_2^2) \subset \mathbb{K}[x_0, x_1, x_2]. \tag{1.4.2}$$

If we look at the systems T and $\overline{T}$, we realize that $\overline{T}$ is the Macaulay's inverse system of T, and vice versa. Motivated by these facts, in [59] Mezzetti, Miró-Roig and Ottaviani established a connection between artinian ideals failing the WLP and the existence of rational varieties satisfying at least one Laplace equation. To state the result we need first to recall the notions of Macaulay's inverse system and Laplace equations.

Take new variables $z_0, \dots, z_n$ and $\overline{R}=\mathbb{K}[z_0, \dots, z_n]$. Let $m = x_0^{a_0} \cdots x_n^{a_n} \in R$ and $\delta = z_0^{b_0} \cdots z_n^{b_n} \in \overline{R}$ be two monomials and we define $m \circ \delta = z_0^{b_0-a_0} \cdots z_n^{b_n-a_n}$ if $b_i \geq a_i$ for all $i = 0, \dots, n$, and $m \circ \delta = 0$ otherwise. Extended by linearity, $\circ$ induces an operation on $\overline{R}$ that structures it as an $R-$module.

Definition 1.4.4. Let $J \subset R$ be an artinian ideal. The *Macaulay's inverse system* of J, denoted by J^{-1}, is the $R-$module $\{f \in \overline{R} \mid J \circ f = 0\}$.

J^{-1} inherits a natural grading $\oplus_{t \geq 0} J_t^{-1}$ from $\overline{R}$, where $J_t^{-1} := J^{-1} \cap \overline{R}_t$. In the monomial case, by abuse of notation, we regard J^{-1} in R by the linear change of variables which sends z_i to x_i, $i = 0, \dots, n$. In particular, if J is a homogeneous artinian ideal generated by monomials $F_1, \dots, F_r$ of degree d, then J_d^{-1} is the $\mathbb{K}-$vector space with monomial basis $\mathcal{M}_{n,d} \setminus \{F_1, \dots, F_r\}$.

On the other hand, let $f_1, \dots, f_r \in R$ be homogeneous polynomials of degree d and $\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^{r-1}$ be a rational map defined as $(f_1 : \ldots : f_r)$. We denote $Y = \overline{\varphi(\mathbb{P}^n)}$ and let $p = \varphi(p_0)$ with $p_0 = (y_0 : \ldots : y_n) \notin V(f_1, \dots, f_r)$. The sth *osculating space of* Y *at* p is the $\mathbb{K}-$vector space $T_p^{(s)}Y$ spanned by all partial derivatives of φ of degree s evaluated at p_0. We denote by $\mathbb{T}_p^{(s)}(Y)$ its projectivization. The *expected dimension of* $\mathbb{T}_p^{(s)}Y$ is $N_{n,s} - 1$. Nevertheless, linear dependences among the derivatives of φ at p_0 may occur, leading to the following definition.

Definition 1.4.5. Let $Y \subset \mathbb{P}^r$ be as above. We say that Y satisfies $0 < \chi \in \mathbb{Z}$ *Laplace equations of order* s if the following two conditions are satisfied.

(i) For all smooth points $p \in Y$, $\dim(\mathbb{T}_p^{(s)}Y) < N_{n,s} - 1$.

(ii) There is a general point $q \in Y$ such that $\dim(\mathbb{T}_q^{(s)}Y) = N_{n,s} - 1 - \chi$.

The result relating artinian ideals failing the WLP and varieties satisfying at least one Laplace equation, also known in the literature as the Tea theorem, is the following:

Theorem 1.4.6. *Let $J \subset R$ be an artinian ideal generated by r forms $F_1, \dots, F_r$ of degree d and let J^{-1} be its Macaulay's inverse system. If $r \leq N_{n-1,d}$, then the following conditions are equivalent.*

(i) J fails the WLP in degree $d-1$.

(ii) $F_1, \dots, F_r$ become $\mathbb{K}-$linearly dependent on a general hyperplane $H \subset \mathbb{P}^n$.

(iii) The $n-$dimensional variety $Y := \overline{\varphi(\mathbb{P}^n)}$ satisfies at least one Laplace equation of order $d-1$, where $\varphi = \mathbb{P}^n \dashrightarrow \mathbb{P}^{N_{n,d}-r-1}$ is the rational map associated to J_d^{-1}.

Proof. See [59, Theorem 3.2]. □

As a consequence of the above theorem and in honour to Togliatti, the authors of [59] introduced the following definition:

Definition 1.4.7. A *Togliatti system* is a homogeneous artinian ideal $J = (F_1, \dots, F_r) \subset R$ generated by $r \leq N_{n-1,d}$ forms of degree d satisfying the three equivalent conditions in Theorem 1.4.6.

Example 1.4.8. (i) $J = (x_0^4, x_1^4, x_2^4, x_0^2x_2^2, x_0x_1^2x_2) \subset R = \mathbb{K}[x_0, x_1, x_2]$ is a monomial Togliatti system generated by $r = 5$ monomials of degree $d = 4$. The inequality $r \leq N_{n-1,d}$ is satisfied. Take $L = x_0 + x_1 + x_2 \in R_1$. The multiplication map $\times L : (R/J)_3 \longrightarrow (R/J)_4$ is not injective. By Proposition 1.4.3, J fails the WLP in degree 3. By Theorem 1.4.6, J is a Togliatti system.
(ii) $J = (x_0^6, x_0^4x_1x_2, x_0^2x_1^2x_2^2, x_1^3x_2^3, x_1^6+x_2^6, x_0^3(x_1^3+x_2^3), x_0(x_1^4x_2+x_1x_2^4)) \subset R = \mathbb{K}[x_0, x_1, x_2]$ is a non monomial Togliatti system generated by $r = 7$ forms of degree $d = 6$. The inequality $r \leq N_{n-1,d}$ is satisfied. As a consequence of Proposition 4.2.9, J fails the WLP in degree 5. Indeed, J is generated by a minimal set of fundamental invariants of the dihedral group $D_{2\cdot 3} \subset \mathrm{GL}(3, \mathbb{K})$ of order 6 generated by the matrices:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e^2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

where e is a 3rd primitive root of $1 \in \mathbb{K}$.

To any Togliatti system J, we can associate, in a natural way, two projective varieties which will play a central role in this thesis. In fact,

(i) Since J is an artinian ideal, it induces a morphism $\varphi_J : \mathbb{P}^n \longrightarrow \mathbb{P}^{r-1}$ defined by $(F_1 : \cdots : F_r)$. Its image $X := \varphi_J(\mathbb{P}^n)$ is an $n-$dimensional projective variety called a *variety parameterized by the Togliatti system J.*

(ii) The $n-$dimensional variety Y parameterized by a $\mathbb{K}-$basis of J_d^{-1} is called a *Togliatti variety.*

We say that X is *apolar* to Y, and vice versa.

Definition 1.4.9. (i) A Togliatti system J is called a *monomial Togliatti system* if it can be generated by monomials.

(ii) A Togliatti system J is called a *smooth Togliatti system* if the associated Togliatti variety Y is smooth.

Let $J = (F_1, \ldots, F_r)$ be a Togliatti system generated by $r \leq N_{n-1,d}2$ forms of degree d. Its Togliatti variety Y exhibits a non expected behaviour: it satisfies at least one Laplace equation of order $d-1$. Since [59], many works have focused on the study of Togliatti systems, Togliatti varieties and the varieties parameterized by them, as for example [58, 57, 60, 64, 20, 17, 18, 19, 21, 1, 23]. The earliest ones deal with the problem of classifying them in terms of number of generators or minimality. Nevertheless, the failure of the WLP does not impose, in general, other conditions on the structure of a Togliatti system, neither statements (ii) and (iii) of Theorem 1.4.6. In [57], a new family of Togliatti systems was introduced, the so called Galois–Togliatti systems (shortly $GT-$system).

Definition 1.4.10. A $GT-$system is a Togliatti system $J = (F_1, \ldots, F_r) \subset R$ generated by $r \leq N_{n-1,d}$ forms of degree d whose associated morphism $\varphi_J : \mathbb{P}^n \longrightarrow \mathbb{P}^{r-1}$ is a Galois covering with a finite cyclic group $\mathbb{Z}/d\mathbb{Z}$.

Before continuing, we present the notion of Galois covering and gather some related results needed in the sequel.

Galois coverings. We recall that a *covering* of a variety X is a pair (Y, f), where Y is a variety and $f : Y \longrightarrow X$ is a finite morphism. The *group of deck transformations* $\mathrm{Aut}(f)$ is defined as the subgroup of $\mathrm{Aut}(Y)$

commuting with f. We say that $f : Y \longrightarrow X$ is a *covering with group* $\mathrm{Aut}(f)$.

Definition 1.4.11. A covering $f : Y \longrightarrow X$ with group $\mathrm{Aut}(f)$ is a *Galois* covering if $\mathrm{Aut}(f)$ acts transitively on a fibre $f^{-1}(x)$ for some $x \in X$.

When a group Λ acts on a variety X, there is a natural way to construct Galois coverings.

Definition 1.4.12. Let Λ be a group acting on a variety X. A *quotient of* X *by* Λ is a variety Y and a surjective morphism $p : X \longrightarrow Y$ such that any morphism $\rho : X \longrightarrow Z$ to a variety Z factors through p if and only if $\rho(x) = \rho(\lambda(x))$, for all $x \in X$ and $\lambda \in \Lambda$.

Remark 1.4.13. If it exists, the quotient variety is unique up to isomorphism and it is denoted by X/Λ. In particular, the morphism $p : X \longrightarrow X/\Lambda$ verifies that if $x, y \in X$, then $p(x) = p(y)$ if and only if $\lambda(x) = y$ for some $\lambda \in \Lambda$.

Proposition 1.4.14. *Let Λ be a finite group acting on an affine variety X. Then, X/Λ is the affine variety whose coordinate ring $A(X/\Lambda)$ is the ring of regular functions on X invariants of Λ, and $\pi : X \longrightarrow X/\Lambda$ is the quotient of X by Λ.*

Proof. See [74, §12, Proposition 18]. □

Proposition 1.4.15. *Let Λ be a finite group acting on a projective variety X and X/Λ its quotient space. If the orbit of any point $x \in X$ is contained in an affine open subset of X, then X/Λ is a projective variety and $\pi : X \longrightarrow X/\Lambda$ is the quotient of X by Λ.*

Proof. See [74, §12, Proposition 19]. □

Proposition 1.4.16. *Let X be an irreducible projective variety and $\Lambda \subset \mathrm{Aut}(X)$ be a finite group. If the quotient variety X/Λ exists, then $\pi : X \longrightarrow X/\Lambda$ is a Galois covering with group Λ.*

Proof. Set $\Lambda = \{\lambda_1, \ldots, \lambda_n, \mathrm{Id}\}$. The group $\mathrm{Aut}(\pi)$ consists of all automorphisms of X commuting with π. If $f : X \longrightarrow X$ belongs to $\mathrm{Aut}(\pi)$, then for all $x \in X$ we have $\pi(f(x)) = \pi(x)$. For any $x \in X$, there exists $\lambda_i \in \Lambda$

such that $f(x) = \lambda_i(x)$, and hence $X = V(f-\lambda_1) \cup \cdots \cup V(f-\lambda_n)$. The irreducibility of X allows us to conclude that $f = \lambda_i$, for some $\lambda_i \in \Lambda$. Therefore, $\operatorname{Aut}(\pi) = \Lambda$ and it is clear that given $\pi(x) \in X/\Lambda$, the fibre $\pi^{-1}(\pi(x)) = \Lambda_x$, so $\operatorname{Aut}(\pi) = \Lambda$ acts transitively on $\pi^{-1}(\pi(x))$. $\square$

When X is the $n-$dimensional projective space $\mathbb{P}^n$, a finite group Λ of automorphisms of X can be regarded as a finite subgroup of $\mathrm{GL}(n+1,\mathbb{K})$. If $\mathcal{B} = \{g_1,\dots,g_r\}$ is a minimal set of homogeneous fundamental invariants of Λ of the same degree $\deg(g_i) = d$, $i = 1,\dots,r$, then the quotient variety $\mathbb{P}^n/\Lambda$ is the projective variety of $\mathbb{P}^{r-1}$ whose homogeneous coordinate ring is the ring R^Λ of invariants. In particular, we have the following.

Proposition 1.4.17. *Let $\Lambda \subset \mathrm{GL}(n+1,\mathbb{K})$ be a finite group of order $|\Lambda|$ and $\mathcal{B} = \{g_1,\dots,g_r\}$ a minimal set of homogeneous fundamental invariants of Λ with $\deg(g_1) = \cdots = \deg(g_r) =: d$. Let $\varphi_{\mathcal{B}} : \mathbb{P}^n \longrightarrow \mathbb{P}^{r-1}$ be the morphism defined by $(g_1 : \cdots : g_r)$. It holds:*

(i) R^Λ is the homogeneous coordinate ring of the projective variety $X := \varphi_{\mathcal{B}}(\mathbb{P}^n) \subset \mathbb{P}^{r-1}$. Thus X is the quotient variety $\mathbb{P}^n/\Lambda$ and it is an aCM variety.

(ii) $\varphi_{\mathcal{B}} : \mathbb{P}^n \longrightarrow \mathbb{P}^{r-1}$ is a Galois covering of X with group Λ.

(iii) The homogeneous ideal of X is the ideal $\mathrm{syz}(\mathcal{B})$ of syzygies among the invariants $g_1,\dots,g_r$.

Proof. (i) and (iii) They follow from the projective version of Theorem 1.3.11.

(ii) It is a consequence of Proposition 1.4.16. $\square$

The above result evinces a closed connection between $GT-$systems and the theory of invariants of finite groups and rouses attention for the varieties parameterized by them. Recently in [19], Definition 1.4.10 has been generalized as follows.

Definition 1.4.18. Let $\Lambda \subset \mathrm{GL}(n+1,\mathbb{K})$ be a finite group of order d with $2 \le n < d$. A Togliatti system $J \subset R$ generated by $r \le N_{n-1,d}$ forms $F_1,\dots,F_r$ of degree d is said to be a *$GT-$system with group* Λ if the associated morphism $\varphi_J : \mathbb{P}^n \longrightarrow \mathbb{P}^{r-1}$ is a Galois covering with group Λ. In this case, $X = \varphi_J(\mathbb{P}^n)$ is called a *$GT-$variety with group* Λ.

Monomial $GT-$systems with a finite cyclic group have been treated subsequently in [57], [18] and [17]; while in [19] the authors studied $GT-$systems with the dihedral group acting on $\mathbb{K}[x_0, x_1, x_2]$. In fact, $GT-$systems with a dihedral group form the first known family of non-monomial $GT-$systems, which we will study in detail in Chapter 4. In [17] and [19], the authors applied invariant theory techniques to tackle $GT-$systems with group a finite cyclic group or a dihedral group, and their associated varieties.

Example 1.4.19. (i) Take $n = 2$, $d = 3$, e a 3rd primitive root of $1 \in \mathbb{K}$ and $\Lambda = \langle \operatorname{diag}(1, e, e^2) \rangle \subset \operatorname{GL}(3, \mathbb{K})$ a cyclic group of order 3. Its cyclic extension $\overline{\Lambda} = \langle \operatorname{diag}(1, e, e^2), \operatorname{diag}(e, e, e) \rangle \subset \operatorname{GL}(3, \mathbb{K})$ is an abelian group of order 9. A minimal set of fundamental invariants of $\overline{\Lambda}$ is $\{x_0^3, x_1^3, x_2^3, x_0x_1x_2\}$ (Example 1.3.5(iii)). The ideal generated by them is the Togliatti system T (see (1.4.1)) described by Brenner and Kaid and, as we pointed out before, it is the first Togliatti system that appears in the literature. It follows from Proposition 1.4.17 that T is a $GT-$system with group Λ (see also [17, Corollary 3.4]).

(ii) As another example of $GT-$system with finite cyclic group have: take $n = 3, d = 4$, e a 4th primitive root of $1 \in \mathbb{K}$ and $\Lambda = \langle \operatorname{diag}(1, e, e^2, e^3) \rangle \subset \operatorname{GL}(4, \mathbb{K})$ a cyclic group of order 4. There are $r = 10$ monomial invariants of Λ of degree 4 ([20, Example 3.2]):

$$x_0^4, x_1^4, x_0x_1^2x_2, x_0^2x_2^2, x_0^2x_1x_3, x_2^4, x_1x_2^2x_3, x_1^2x_3^2, x_0x_2x_3^2, x_3^4.$$

They form a minimal set of fundamental invariants of the cyclic extension $\overline{\Lambda} = \langle \operatorname{diag}(1, e, e^2, e^3), \operatorname{diag}(e, e, e, e) \rangle \subset \operatorname{GL}(4, \mathbb{K})$ of Λ ([17, Theorem 3.1]). The ideal J generated by them fails the WLP in degree 3. By Propositions 1.4.17 and 1.4.3, J is a $GT-$system with group Λ (see also [20, Proposition 3.3] or [17, Corollary 3.4]).

(iii) The dihedral group $D_{2\cdot 4}$ of order 8 can be represented in $\operatorname{GL}(3, \mathbb{K})$ as the group generated by the matrices $M = \operatorname{diag}(1, e, e^3)$ and $\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, where e is a 4rd primitive root of $1 \in \mathbb{K}$. Its cyclic extension $\overline{D_{2\cdot 4}} = \langle M, \sigma, \operatorname{diag}(e, e, e) \rangle \subset \operatorname{GL}(3, \mathbb{K})$ is a non abelian group of order 64. A minimal set of fundamental invariants of $\overline{D_{2\cdot 4}}$ is the following set of $r = 9$ monomials and binomials of degree 8: $\{x_0^8, x_0^6x_1x_2, x_0^4x_1^2x_2^2, x_0^2x_1^3x_2^3, x_1^4x_2^4, x_0^4(x_1^4 +$

$x_2^4), x_0^2(x_1^5x_2 + x_1x_2^5), x_1^6x_2^2 + x_1^2x_2^6, x_1^8 + x_2^8\}$ (Example 3.3.12(ii)). By Proposition 4.2.9, the ideal generated by them is a non monomial $GT-$system with group $D_{2\cdot 4}$.

Chapter 2

Invariants of finite abelian groups and aCM projections of Veronese varieties. Applications

In this chapter, we relate invariant theory of finite groups to the longstanding problem, posed by Gröbner in [39], of determining when a monomial projection $Y_{n,d}$ of the Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ is an aCM variety. $Y_{n,d} \subset \mathbb{P}^{\mu_{n,d}-1}$ is a projective variety parameterized by a subset $\Omega_{n,d} \subset \mathcal{M}_{n,d}$ of $\mu_{n,d} \leq N_{n,d}$ monomials of degree d. As a nice family of examples we prove: the set $\mathcal{B}_1$ of monomial invariants of degree d of a finite diagonal abelian group $G \subset \mathrm{GL}(n+1, \mathbb{K})$ of order d parameterizes an aCM monomial projection X_d of the Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ (Theorems 2.2.11 and 2.2.18). We call X_d a $\overline{G}-$*variety with group* G and we show that the homogeneous coordinate ring of X_d is isomorphic to the dth Veronese subalgebra of R^G, i.e. the ring $R^{\overline{G}}$ of invariants of the cyclic extension $\overline{G} \subset \mathrm{GL}(n+1, \mathbb{K})$ of G. Set $|\mathcal{B}_1| = \mu_d$ and $\varphi_{\mathcal{B}_1} : \mathbb{P}^n \longrightarrow \mathbb{P}^{\mu_d-1}$ the morphism defined by $\mathcal{B}_1$. We show that $\varphi_{\mathcal{B}_1}$ is a Galois covering with group G and that the ideal $I_d \subset R$ generated by $\mathcal{B}_1$ is a $GT-$system with group G (Definition 1.4.18), provided $\mu_d \leq N_{n-1,d}$ (Proposition 2.3.1).

This chapter is structured as follows. In Section 2.1, we give an outlook on the state of the art of the Gröbner's problem from a historical point of view. We gather the main results, techniques and contributions towards this subject from the standpoint of deciding when a monomial projection $Y_{n,d} \subset \mathbb{P}^{\mu_{n,d}-1}$ of the Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ is an aCM variety in terms of the deleted monomials $\mathcal{M}_{n,d} \setminus \Omega_{n,d}$.

In Section 2.2, we study the ring of invariants of a finite diagonal abelian group $G \subset \mathrm{GL}(n+1, \mathbb{K})$ of order d. We focus on determining a minimal set of fundamental invariants of its cyclic extension $\overline{G} \subset \mathrm{GL}(n+1, \mathbb{K})$ (Definition

1.3.2). Our main result proves that the set $\mathcal{B}_1$ of monomial invariants of G of degree d generates the algebra $R^{\overline{G}}$ (Theorem 2.2.11). The arguments we develop to achieve our goal are combinatorics and involve the notions of *zero-sum sequences* and the *Davenport constant*, which are introduced along some results on this topic. In Subsection 2.2.1, we give a concrete h.s.o.p of $R^{\overline{G}}$ and its corresponding Hironaka decomposition. We introduce $\overline{G}-$varieties X_d with group G and we established that they are aCM monomial projections of the Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$. The main results of this section for finite cyclic groups $G \subset \mathrm{GL}(n+1, \mathbb{K})$ have been published in [17].

In Section 2.3, we analyse the WLP of the monomial artinian ideal I_d generated by the minimal set $\mathcal{B}_1$ of fundamental invariants of $\overline{G}$. We show that I_d is a $GT-$system with group G, provided $\mu_d \leq N_{n-1,d}$ (Proposition 2.3.1) and we exhibit examples showing that $GT-$varieties with group G are a wealth subfamily of $\overline{G}-$varieties. Moreover, they are monomial projections of Veronese varieties such that their apolar variety (Theorem 1.4.6) satisfies at least one Laplace equation of order $d-1$.

In Section 2.4, we introduce a new family of monomial projections of the Veronese surface $X_{2,d}$ which are aCM surfaces (Theorem 2.4.10 and Corollary 2.4.12). They are parameterized by Togliatti systems which naturally arise from $GT-$systems with a finite cyclic group. Nevertheless, their coordinate rings are neither the ring of invariants of any finite group nor they correspond to the semigroup ring of a normal affine semigroup. The content of this last section has been published in [17].

2.1 Monomial projections of Veronese varieties

The Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ is the variety parameterized by set $\mathcal{M}_{n,d} \subset R$ of monomials of degree d. As we observed before, the Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ is the image of the Veronese embedding $\nu_{n,d} : \mathbb{P}^n \longrightarrow \mathbb{P}^{N_{n,d}-1}$ defined by $\mathcal{M}_{n,d}$. Given a subset $\Omega_{n,d} \subseteq \mathcal{M}_{n,d}$ of $\mu_{n,d} \leq N_{n,d}$ monomials, we denote by $\varphi_{\Omega_{n,d}} : \mathbb{P}^n \longrightarrow \mathbb{P}^{\mu_{n,d}-1}$ the rational map defined by $\Omega_{n,d}$ and we say that $Y_{n,d} := \overline{\varphi_{\Omega_{n,d}}(\mathbb{P}^n)} \subset \mathbb{P}^{\mu_{n,d}-1}$ is the *monomial projection of the Veronese variety $X_{n,d}$ parameterized by $\Omega_{n,d}$*. Set $\mathbb{P}^{N_{n,d}-1} = \operatorname{Proj} \mathbb{K}[w_{m_i}]_{m_i \in \mathcal{M}_{n,d}}$. So, we have the commutative diagram:

$$\begin{tikzcd}
\mathbb{P}^n \arrow[r, "\nu_{n,d}"] \arrow[dr, "\varphi_{\Omega_{n,d}}"'] & X_{n,d} \arrow[d, "\pi"] \\
& Y_{n,d}
\end{tikzcd}$$

where π is the projection of the Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ from the linear subspace generated by the coordinate points $(0 : \cdots : 0 : 1 : 0 : \cdots : 0) \in \mathbb{P}^{N_{n,d}-1}$ with 1 in position i such that $m_i \in \Omega_{n,d}$ to the linear subspace $V(w_{m_i},\ m_i \in \Omega_{n,d}) \subset \mathbb{P}^{N_{n,d}-1}$. In particular, $Y_{n,d} \subset \mathbb{P}^{\mu_{n,d}-1}$ is called a *simple* monomial projection if $\Omega_{n,d}$ is obtained from $\mathcal{M}_{n,d}$ by deleting only one monomial.

The homogenous coordinate ring of $Y_{n,d} \subset \mathbb{P}^{\mu_{n,d}-1}$ is isomorphic to the semigroup ring $\mathbb{K}[\Omega_{n,d}] \subset R$ associated to the monomial semigroup generated by $\Omega_{n,d}$. Thus, $\mathbb{K}[\Omega_{n,d}]$ is the semigroup ring of the affine semigroup $\mathrm{H}(\Omega_{n,d}) \subset \mathbb{Z}^{n+1}_{\geq 0}$ (Section 1.2). For the Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$, $\mathbb{K}[\mathcal{M}_{n,d}]$ is called the *dth Veronese subalgebra of R*. In [39], Gröbner proved that for any integers $n, d \geq 1$, $\mathbb{K}[\mathcal{M}_{n,d}]$ is a CM ring and showed a family of simple monomial projections of $X_{n,d}$ whose homogeneous coordinate rings are not CM rings. Precisely, fix integers $2 \leq n$, $4 \leq d$ and let $\Omega_{n,d} = \mathcal{M}_{n,d} \setminus \{x_0^{d-2}x_1^2\}$. The author proved that $\mathbb{K}[\Omega_{n,d}]$ is not a CM ring. Actually, this family generalizes the first known example of a non CM domain $\mathbb{K}[x_0^4, x_0^3x_1, x_0x_1^3, x_1^4]$, given by Macaulay in [55]. Geometrically, it corresponds to the homogeneous coordinate ring of the rational quartic in $\mathbb{P}^3$ obtained as the monomial projection of the rational normal curve $X_{1,4}$ of degree 4 in $\mathbb{P}^4 = \mathrm{Proj}(\mathbb{K}[w_0, w_1, w_2, w_3, w_4])$ from the coordinate point $(0 : 0 : 1 : 0 : 0)$ to the hyperplane $V(w_2) \subset \mathbb{P}^4$. Observe that the monomial projection of the rational normal curve $X_{1,4} \subset \mathbb{P}^4$ from the coordinate point $(1 : 0 : 0 : 0 : 0)$ to the hyperplane $V(w_0)$ is a rational twisted cubic in $\mathbb{P}^3$ and, hence, it is an aCM curve. Motivated by this behaviour, at the end of [39] Gröbner posed the following problem.

Problem 2.1.1. *To determine which monomial projections $Y_{n,d} \subset \mathbb{P}^{\mu_{n,d}-1}$ of the Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ are aCM varieties.*

From the point of view of semigroup rings, this formulation can be regarded as the problem of characterizing when a semigroup ring is a CM ring in terms of its associated semigroup. The first fundamental contribution towards this topic is due to Hochster [51], who proved that the semigroup ring of any normal semigroup is a CM ring (Theorem 1.2.14). As an example of a normal affine semigroup we have $\mathrm{H}(\mathcal{M}_{n,d})$ (Example 1.2.17(i)). Thus, Hochster's result provides a nice way of reproving that any Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ is an aCM variety. Actually, $\mathrm{H}(\mathcal{M}_{n,d})$ belongs to a bigger family of normal affine semigroups which we have studied in Section 1.2. Precisely, the $\mathbb{Z}_{\geq 0}^{n+1}$−solutions of the linear systems of congruences:

$$(*)_{\mathcal{A};t_1,\dots,t_s} : \left\{ \begin{array}{ccccccc} \alpha_{1,0}y_0 & + & \cdots & + & \alpha_{1,n}y_n & = & t_1d_1 \\ \vdots & & & & & & \\ \alpha_{r,0}y_0 & + & \cdots & + & \alpha_{r,n}y_n & = & t_rd_r \end{array} \right.$$

where $d_1, \dots, d_r \in \mathbb{Z}_{\geq 0}$ and $\mathcal{A} = (\alpha_{i,j})$ is a $r \times (n+1)$ matrix with coefficients $\alpha_{i,j} \in \mathbb{Z}_{\geq 0}$. In [78], Stanley related these kind of semigroups to rings of invariants of finite groups and studied the Cohen–Macaulay property. Nevertheless, there are non normal semigroup rings which are CM rings. In Section 2.4, we will introduce a new family of non normal CM semigroup rings.

In [72], Schenzel positively answer Gröbner's problem (Problem 2.1.1) for simple monomial projections of the Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$. Using Hochster's result, the author proved the following:

Theorem 2.1.2. *Let $n \geq 1$ and $d \geq 2$ be integers, $m := x_0^{a_0} \cdots x_n^{a_n} \in \mathcal{M}_{n,d}$, $\Omega_{n,d} := \mathcal{M}_{n,d} \setminus \{m\}$ and $Y_{n,d}$ the simple monomial projection of $X_{n,d}$ parameterized by $\Omega_{n,d}$. Then $Y_{n,d}$ is an aCM variety if and only if one of the following cases holds:*

(i) $m = x_i^d$, *for some* $0 \leq i \leq n$ *and all* $n \geq 1$,

(ii) $n = 1$ *and* $m \in \{x_0^{d-1}x_1, x_0x_1^{d-1}\}$,

(iii) $n = d = 2$ *and* $m \in \mathcal{M}_{2,2}$.

Proof. See [72, Theorem 2, Proposition 2 and §4]. □

Notwithstanding, to determine whether a semigroup is a normal semigroup could not always be an easy task, especially if its generators have been chosen arbitrarily. The second fundamental contribution to the Gröbner's problem, or in far greater generality, to determine whether a semigroup ring is a CM ring is a criterion due to Goto, Suzuki and Watabane [35] and Hoa and Trung [47] for simplicial semigroup rings. This result has been of key importance in most of the subsequently attempts to solve Problem 2.1.1. It will play a central role in Section 2.4.

Definition 2.1.3. An affine semigroup $H \subset \mathbb{Z}_{\geq 0}^{n+1}$ is called *simplicial* if there are $\mathbb{Q}-$linearly independent elements $e_0, \ldots, e_n \in H$ verifying the following condition: for any $h \in H$ there exist $z, z_0, z_1, \ldots, z_n \in \mathbb{Z}_{\geq 0}$ with $z > 0$ such that $zh = z_0 e_0 + \cdots + z_n e_n$.

For instance, if $\Omega_{n,d} \subseteq \mathcal{M}_{n,d}$ contains $x_0^d, \ldots, x_n^d$, then $\mathrm{H}(\Omega_{n,d})$ is simplicial.

Theorem 2.1.4. *Let $H \subseteq \mathbb{Z}_{\geq 0}^{n+1}$ be an affine semigroup and set $H_1 := \{h \in \overline{H} \mid h + e_i \in H$ and $h + e_j \in H$ for some $0 \leq i \neq j \leq n\}$. Then, $\mathbb{K}[H]$ is a CM ring if and only if $H = H_1$.*

Proof. See [35, Theorem 2.6] and [47, Corollary 4.4]. □

Remark 2.1.5. For simplicial normal affine semigroups $H \subset \mathbb{Z}_{\geq 0}^{n+1}$, Theorem 2.1.4 holds automatically since $H \subseteq H_1 \subseteq \overline{H} = H$.

In [11], Cavaliere and Niese characterized aCM monomial projections of the rational normal curve $X_{1,d} \subset \mathbb{P}^d$ whose coordinate rings are semigroup rings of simplicial affine semigroups. In our notation, they are monomial projections of $X_{1,d} \subset \mathbb{P}^d$ parameterized by a subset of monomials

$$\Omega_{1,d} := \{x_0^d, x_0^{d_1} x_1^{d-d_1}, \ldots, x_0^{d_r} x_1^{d-d_r}, x_1^d\},$$

where $1 \leq r \leq d-1$ and $1 \leq d_1 < d_2 < \cdots < d_r < d$ are integers. The authors combine the theory of numerical semigroups, i.e. semigroups $\langle z_1, \ldots, z_r \rangle$ of $\mathbb{Z}_{\geq 0}$ with $\mathrm{GCD}(z_1, \ldots, z_r) = 1$, and Theorem 2.1.4 to give a specific criterion ([11, Theorem 4.6]) of the CM property of such monomial projections $Y_{1,d} \subset \mathbb{P}^{r+1}$. They applied their result to study the CM–type of the homogeneous coordinate ring of $Y_{1,d}$.

Shortly after in [85], Trung dealt with monomial projections $Y_{1,d}$ of the rational normal curve $X_{1,d} \subset \mathbb{P}^d$ in general. They are defined by non-decreasing sequence of integers $d_0, \dots, d_{2r+1}$ such that $0 = d_0 \leq d_1 < d_2 \leq d_3 < \cdots < d_{2r} \leq d_{2r+1} = d$. Precisely, $Y_{1,d}$ is parameterized by a subset of monomials

$$\Omega_{1,d} := \{x_0^{d-a} x_1^a \mid a \in \cup_{i=0}^r [d_{2i}, d_{2i+1}]\},$$

where $[d_{2i}, d_{2i+1}]$ denotes the set of integers z with $d_{2i} \leq z \leq d_{2i+1}$. The approach consisted of determining whether the curve $Y_{1,d} \subset \mathbb{P}^{d_{2r}}$ is an aCM curve in terms of arithmetical relations between $d_0, d_1, \dots, d_{2r}, d_{2r+1}$. The author distinguished the following three cases.

(a) $d_1 = 0$ and $d_{2r} = d$,

(b) $d_1 = 0$ and $d_{2r} < d$,

(c) $d_1 > 0$ and $d_{2r} < d$.

For (a) and (b) only partial solutions were found ([85, Theorem 2.1 and Theorem 3.5]), while case (c) is completely settled in [85, Theorem 4.1 and Remark 4.2]. Concretely, if $d_1 > 0$ and $d_{2r} < d$, then $Y_{1,d}$ is an aCM curve if and only if it is the rational normal curve $X_{1,d} \subset \mathbb{P}^d$. Similarly, many other works have tackled the Gröbner's problem and other related topics for monomial projections of the rational normal curve $X_{1,d} \subset \mathbb{P}^d$ (see, for instance, [53, 44, 5, 46]).

The perspective of Schenzel in [72] was continued in a very natural way as follows. In [84] (respectively [48]) a monomial projection $Y_{n,d}$ of the Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ parameterized by $\Omega_{n,d}$ is called a *double* (respectively *triple*) monomial projection of $X_{n,d}$ if $\Omega_{n,d}$ is obtained from $\mathcal{M}_{n,d}$ by deleting two (respectively three) monomials. Gröbner's problem was successfully answered for double monomial projections of $X_{n,d}$ by Trung [84] and, shortly after, for triple monomial projections of $X_{n,d}$ by Hoa [48]. We remark that the techniques used in [72] do not apply for double and triple monomials projections of $X_{n,d}$ and the authors of [48] and [84] developed different strategies for tackling these cases. In both works, they divided the monomial projections in several types according to the classes of sum representations of the elements of $\mathrm{H}(\mathcal{M}_{n,d})$ and they checked case by case when these monomial projections of $X_{n,d}$ are aCM varieties. The precise result for double monomial projections of $X_{n,d}$ is collected in [84, Table II, pag 576-577]. Triples monomial projections of $X_{n,d}$ are rather complicated, they are split in nine different types and each one of them in several subtypes. The result is gathered in [48, Tables A–E and §6].

Outcomes for simple, double and triple monomial projections of the Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ agree in the following *extremal* cases. Monomial projections of $X_{n,d}$ are aCM varieties when the deleted monomials belong to $\{x_0^d, \dots, x_n^d\}$. For $n \geq 2$, they are not aCM varieties when all the coefficients of the deleted monomials belong to the relative interior $\operatorname{relint}(\mathrm{H}(\mathcal{M}_{n,d})) = \{(a_0, \dots, a_n) \in \mathrm{H}(\mathcal{M}_{n,d}) \mid a_0 \cdots a_n \neq 0\}$ (Definition 1.2.7). We present generalizations of both statements.

Proposition 2.1.6. *Let $0 \leq k \leq n$ be an integer and $\Omega_{n,d} = \mathcal{M}_{n,d} \setminus \{x_0^d, \dots, x_k^d\}$. Then, $\mathbb{K}[\Omega_{n,d}]$ is a CM ring.*

Proof. We prove that the affine semigroup $\mathrm{H}(\Omega_{n,d}) \subset \mathbb{Z}_{\geq 0}^{n+1}$ is normal and then the result follows from Theorem 1.2.14. We proceed by induction on $k \geq 0$. The initial case $k = 0$ is $\mathrm{H}(\mathcal{M}_{n,d} \setminus \{x_0^d\})$ and it is a CM ring by Theorem 2.1.2. We fix $0 < k \leq n$, we write $\Omega'_{n,d} = \Omega_{n,d} \cup \{x_k^d\}$ and we assume by induction that $\mathrm{H}(\Omega'_{n,d})$ is normal. We have that $\overline{\mathrm{H}(\Omega_{n,d})} \subset \mathrm{H}(\Omega'_{n,d})$ and $\overline{\mathrm{H}(\Omega_{n,d})}$ is normal. Let $l \in \overline{\mathrm{H}(\Omega_{n,d})}$ with $zl \in \mathrm{H}(\Omega_{n,d})$, therefore $l \in \mathrm{H}(\Omega'_{n,d})$ so $l = z_k e_k + l'$ with $l' \in \mathrm{H}(\Omega_{n,d})$ and $zl = zz_k e_k + zl' \in \mathrm{H}(\Omega_{n,d})$. Since $\overline{\mathrm{H}(\Omega_{n,d})}$ is normal, we obtain that $z_k e_k \in \overline{\mathrm{H}(\Omega_{n,d})}$, which is a contradiction unless $z_k = 0$. So, $l = l'$ and then $l \in \mathrm{H}(\Omega_{n,d})$. □

We set $\mathcal{M}_{n,d} = \{m_1, \dots, m_{N_{n,d}}\}$ and let $\nu_{n,d} : \mathbb{P}^n \longrightarrow \mathbb{P}^{N_{n,d}-1}$ be the Veronese embedding given by $(m_1, \dots, m_{N_{n,d}})$. We take variables $w_1, \dots, w_{N_{n,d}}$ and $S = \mathbb{K}[w_1, \dots, w_{N_{n,d}}]$. The homogeneous ideal $\mathrm{I}(X_{n,d})$ of the Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ is the homogenous binomial prime ideal generated by all binomials of degree 2 of the form:

$$\prod_{i=1}^{N_{n,d}} w_i^{\alpha_i} - \prod_{i=1}^{N_{n,d}} w_i^{\beta_i} \quad \text{such that} \quad \prod_{i=1}^{N_{n,d}} m_i^{\alpha_i} = \prod_{i=1}^{N_{n,d}} m_i^{\beta_i}, \tag{2.1.1}$$

(see [39]). We denote by p_{m_i} the coordinate point $(0 : \ldots : 1 : 0 : \ldots : 0) \in \mathbb{P}^{N_{n,d}-1}$ with 1 in position i. From (2.1.1), it follows that $p_{x_0^d}, \dots, p_{x_n^d} \in X_{n,d}$. Moreover, if $m_i \notin \{x_0^d, \dots, x_n^d\}$, then $p_{m_i} \notin X_{n,d}$. Indeed, we write $m_i = x_0^{a_0} \cdots x_n^{a_n}$ with at least $a_j, a_k > 0$ for some $j, k \in \{0, \dots, n\}$, $j \neq k$. We have

$m_i^d = (x_0^d)^{a_0} \cdots (x_n^d)^{a_n}$. As a consequence $0 \neq w_i^d - w_{x_0^d}^{a_0} \cdots w_{x_n^d}^{a_0} \in \mathrm{I}(X_{n,d})$ and it does not vanish at p_{m_i}. Geometrically, we have that monomial projections of $X_{n,d}$ from the linear space spanned by a set of coordinate points lying on $X_{n,d}$ are aCM varieties.

We consider now the other *extremal* case. Given a monomial $m = x_0^{a_0} \cdots x_n^{a_n} \in R$, we denote by $l_m = (a_0, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^{n+1}$ its associated lattice point, it holds:

Proposition 2.1.7. *Let $2 \leq n < d$ be integers and $m_{i_1}, \ldots, m_{i_k}$ monomials such that $l_{m_{i_1}}, \ldots, l_{m_{i_k}} \in \operatorname{relint}(\mathrm{H}(\mathcal{M}_{n,d}))$. If $\Omega_{n,d} = \mathcal{M}_{n,d} \setminus \{m_{i_1}, \ldots, m_{i_k}\}$, then $\mathbb{K}[\Omega_{n,d}]$ is a non CM ring.*

Proof. For simplicity, we denote $\mathrm{H}(\Omega_{n,d})$ by H and we set $m := m_{i_1}$. For each $i = 0, \ldots, n$, we denote $e_i = (0, \ldots, d, \ldots)$ with d in position i. We consider $H_1 := \{h \in \overline{H} \mid h+ \in H \text{ and } h + e_j \in H \text{ for some } 0 \leq i \neq j \leq n\}$. We write $m = x_0^{a_0} \cdots x_n^{a_n}$, $l := (a_0, \ldots, a_n) \in \mathbb{Z}(H)$ and we prove that $l \in \overline{H} \cap H_1$, by Theorem 2.1.4 it follows that $\mathbb{K}[\Omega_{n,d}]$ is a non CM ring.

Let $M = \mathrm{LCM}(a_0, \ldots, a_n, d)$ and we set $l' := Ml$. Then, we obtain $(Ma_0, \ldots, Ma_n) = (dk_0, \ldots, dk_n) = \sum_{j=0}^{n} k_j (0, \ldots, d, \ldots, 0)$, for certain integers $1 \leq k_0, \ldots, k_n$. So, $l' \in H$ and hence $l \in \overline{H}$. Since $n \geq 2$, we set $l^1 := l + (d, 0, \ldots, 0)$ and $l^2 := l + (0, d, 0, \ldots, 0)$. We see that $l^1 \in H$. Indeed, by hypothesis $a_0 + \cdots + a_n = d$ and each $a_i > 0$. Thus, $l \neq (a_0 + a_1, 0, a_2, \ldots, a_n) \in H$ and it follows that

$$l^1 = (a_0 + a_1, 0, a_2, \ldots, a_n) + (d - a_1, a_1, 0, \ldots, 0) \in H.$$

Analogously, $l^2 \in H(\Omega_{n,d})$, so $l \in H_1$ and the proof is complete. $\square$

In Proposition 2.1.6 (respectively Proposition 2.1.7), $\Omega_{n,d}$ parameterizes a monomial projection of the Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ from the linear space spanned by a set of coordinate points on $X_{n,d}$ (respectively outside $X_{n,d}$). However, this geometric conditions alone are not enough to conclude whether a monomial projection of $X_{n,d}$ is an aCM variety. For instance, for $n = 2$ and $d = 3$, the monomial projection $Y_{2,3} \subset \mathbb{P}^3$ of the Veronese surface $X_{2,3} \subset \mathbb{P}^9$ parameterized by $\Omega_{2,3} = \{x_0^3, x_1^3, x_2^3, x_0x_1x_2\}$ is an aCM surface and $Y_{2,3}$ is a monomial projection of $X_{2,3}$ from the linear space spanned by a set of coordinate points outside $X_{2,3}$.

In [49], Hoa considered the complexity of solving the Gröbner's problem by applying Theorem 2.1.4. It is established that only a finite number of operations are required to check the aCM property of an arbitrary projection of the Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$. However, as the author pointed out, this number is very large and with the exception of projections of the rational normal curves $X_{1,d} \subset \mathbb{P}^d$, the arithmetical conditions involved in the criterion are very cumbersome. Computationally and algorithmically approaches to Problem 2.1.1 can be found as well in [32, 33, 31]. Since then, monomial projections of Veronese varieties $X_{n,d}$ have been the focus of many other works from various perspectives either directly related to Gröbner's problem (see, for instance, [70, 10, 54, 38, 37]) or indirectly (see, for instance, [8, 50, 12, 44, 13]). Nevertheless, the Gröbner problem of determining the aCM property of monomial projections of Veronese varieties $X_{n,d}$ in terms of the deleting monomials $\mathcal{M}_{n,d} \setminus \Omega_{n,d}$ remains open [72, 85, 84, 48]. In this thesis, we contribute to Problem 2.1.1 from this perspective with new families of aCM monomial projections of the Veronese variety $X_{n,d}$ and a new family of non monomial projection of the Veronese surface $X_{2,n}$, which blends invariant theory of finite groups, combinatorics and the weak Lefschetz property of artinian ideals.

2.2 Invariants of finite abelian groups

In this section, we study the algebra of invariants of finite abelian groups acting linearly on R. Precisely, let $G \subset \mathrm{GL}(n+1, \mathbb{K})$ be a finite abelian group of order d and $\overline{G} \subset \mathrm{GL}(n+1, \mathbb{K})$ its cyclic extension (Definition 1.3.2). We consider the natural action of G on $R = \mathbb{K}[x_0, \ldots, x_n]$ which sends $(g, f) \in G \times R$ to $g(f) = f \circ g \in R$. Our main interest relies on the internal structure of the dth Veronese subalgebra $R^{\overline{G}}$ of the ring of invariants $R^G = \{f \in R \mid g(f) = f, \ \forall g \in G\}$ of G. The goal is to determine a minimal set of generators of $R^{\overline{G}}$ and to introduce a new family of aCM monomial projections of Veronese varieties $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ naturally related with $R^{\overline{G}}$.

For this purpose, we first observe that there always exists a linear change of variables

$$(*)_{\mathcal{A}} : \begin{cases} y_0 &= a_{0,0}x_0 + \cdots + a_{0,n}x_n \\ &\vdots \\ y_n &= a_{n,0}x_0 + \cdots + a_{n,n}x_n \end{cases}$$

with associated matrix $\mathcal{A} = (a_{i,j})$, such that all matrices $\mathcal{A}^{-1}g\mathcal{A}$, $g \in G$, are simultaneously diagonal ([6, Theorem 8, Chapter IX]). Thus, the groups $G_{\mathcal{A}} := \mathcal{A}^{-1}G\mathcal{A} = \{\mathcal{A}^{-1}g\mathcal{A} \mid g \in G\} \subset \mathrm{GL}(n+1, \mathbb{K})$ and G are isomorphic by $\mathcal{A}$. Since the change of variables $(*)_{\mathcal{A}}$ induces a natural isomorphism of rings $R^G \cong \mathbb{K}[y_1, \ldots, y_n]^{G_{\mathcal{A}}}$, we may assume that G is diagonal.

As a finite abelian group, G is a direct sum of cyclic groups

$$G = \Gamma_1 \oplus \cdots \oplus \Gamma_s \subset \mathrm{GL}(n+1, \mathbb{K})$$

of order $d_1, \ldots, d_s$, respectively, such that $d = d_1 \cdots d_s$. We write $\Gamma_j = \langle g_{i_j} \rangle \subset \mathrm{GL}(n+1, \mathbb{K})$, $j = 1, \ldots, s$. It follows that for any element $g \in G$, there are integers $0 \le p_j \le d_j$, $j = 1, \ldots, s$, such that $g = g_{i_1}^{p_1} \cdots g_{i_s}^{p_s}$. In the diagonal setting, any matrix g_{i_j} is a diagonal matrix $\mathrm{diag}(\lambda_0, \ldots, \lambda_n)$, where all λ_k are d_jth root of $1 \in \mathbb{K}$. So, it is natural to consider the following notation.

Notation 2.2.1. Fix integers $1 \le n < d$, $\sigma \in \mathcal{S}_{n+1}$ and e a dth primitive root of $1 \in \mathbb{K}$. We denote by $M_{d;\alpha_{\sigma(0)},\ldots,\alpha_{\sigma(n)}}$ the diagonal matrix $\mathrm{diag}(e^{\alpha_{\sigma(0)}}, \ldots, e^{\alpha_{\sigma(n)}})$ where $0 \le \alpha_0 \le \cdots \le \alpha_n < d$ are integers such that $\mathrm{GCD}(d, \alpha_0, \ldots, \alpha_n) = 1$. In particular, for $\sigma = \mathrm{Id}$ we just write $M_{d;\alpha_0,\ldots,\alpha_n}$.

From now onwards, we fix integers $1 \le n < d$ and a finite abelian group $G = \Gamma_1 \oplus \cdots \oplus \Gamma_s \subset \mathrm{GL}(n+1, \mathbb{K})$ of order $d = d_1 \cdots d_s$, where each $\Gamma_i \subset \mathrm{GL}(n+1, \mathbb{K})$, $i = 1, \ldots, s$, is a cyclic subgroup of G of order $1 < d_i$ generated by a diagonal matrix

$$M_{d_i;\alpha^i_{\sigma_i(0)},\ldots,\alpha^i_{\sigma_i(n)}}.$$

We consider $\overline{G} \subset \mathrm{GL}(n+1, \mathbb{K})$ the cyclic extension of G, i.e. the diagonal abelian group generated by $M_{d_1;\alpha^1_{\sigma_1(0)},\ldots,\alpha^1_{\sigma_1(n)}}, \ldots, M_{d_s;\alpha^s_{\sigma_s(0)},\ldots,\alpha^s_{\sigma_s(n)}}$ and the diagonal matrix $\mathrm{diag}(e, \ldots, e)$, where e is a primitive dth root of $1 \in \mathbb{K}$.

Remark 2.2.2. With the above notation, let $G_1 = \langle M_{d;\alpha_0,\dots,\alpha_n}\rangle \subset \mathrm{GL}(n+1,\mathbb{K})$ be a cyclic group of order d and $\sigma \in \mathcal{S}_{n+1}$. The actions of G_1 and $G_2 := \langle M_{d;\alpha_{\sigma(0)},\dots,\alpha_{\sigma(n)}}\rangle$ on R are isomorphic by the linear change of variables $y_i = x_{\sigma(i)}$. Therefore, we have an isomorphism $R^{G_1} \cong R^{G_2}$ of rings. The analogous assertion is not true in general for an arbitrary non cyclic abelian group (see, for instance, Example 2.2.5(iii)). Nevertheless, for simplicity we usually exemplify our results with finite abelian groups generated by matrices $M_{d;\alpha_0,\dots,\alpha_n}$ with $\alpha_0 \leq \cdots \leq \alpha_n$, i.e. we assume that $\sigma = \mathrm{Id} \in \mathcal{S}_{n+1}$.

The ring R^G inherits the natural grading of R, that is R^G is the positively graded $\mathbb{K}-$subalgebra

$$R^G = \bigoplus_{t\geq 0} R^G_t, \quad R^G_t := R_t \cap R^G.$$

We will focus on the subring $R^{\overline{G}} \subset R^G$, which is the positively graded $\mathbb{K}-$subalgebra

$$R^{\overline{G}} = \bigoplus_{t\geq 0} R^{\overline{G}}_t, \quad R^{\overline{G}}_t := R^G_{td}.$$

In other words, each component $R^{\overline{G}}_t$ is the $\mathbb{K}-$vector space of all homogeneous invariants of G of degree td.

One of the fundamental problems of invariant theory of finite groups is to determine a minimal set of generators of the ring of invariants, also called a minimal set of *fundamental invariants* (Section 1.3). Precisely, for a given finite group $\Lambda \subset \mathrm{GL}(n+1,\mathbb{K})$ of order $|\Lambda|$, one wants to find a minimal set of invariants $\{f_1,\dots,f_k\}$ of Λ such that $R^\Lambda = \mathbb{K}[f_1,\dots,f_k]$. We recall that one positive answer is Noether's degree bound (Theorem 1.3.4). In a non constructive way, it establishes that R^Λ is generated as a $\mathbb{K}-$algebra by at most $N_{n+1,|\Lambda|}$ invariants of Λ, of total degree not exceeding $|\Lambda|$. See Example 1.3.5(i) for a simple but relevant example.

However, a precise description of a minimal set of fundamental invariants of any finite group $\Lambda \subset \mathrm{GL}(n+1,\mathbb{K})$ is, in general, unknown. Using combinatorial techniques, we provide a concrete answer for the ring $R^{\overline{G}}$ introduced above. In order to study the algebra $R^{\overline{G}}$, it is useful to determine first each $\mathbb{K}-$vector space $R^{\overline{G}}_t$. For seek of completeness we include a simple proof.

Remark 2.2.3. All the result we present in this section can be rewritten in a suitable way for any finite abelian group $G \subset \mathrm{GL}(n+1, \mathbb{K})$ by undoing the change of variable $(*)_{\mathcal{A}}$.

Lemma 2.2.4. *For all $t \geq 1$, the set of all monomial invariants of G of degree t is a $\mathbb{K}-$basis of R_t^G.*

Proof. We fix an integer $t \geq 1$ and a polynomial $p \in R_t^G$. We write $p = \beta_1 m_1 + \cdots + \beta_l m_l$, where $\beta_i \in \mathbb{K}^*$ and $m_i \in R$ is a monomial of degree t, $i = 1, \ldots, l$. It suffices to prove that each monomial $m_i \in R^G$, $i = 1, \ldots, l$. So we fix $g \in G$ and we check that $g(m_i) = m_i$, $i = 1, \ldots, l$. Since p is an invariant of G and g is a diagonal matrix, we have $g(p) = \beta_1 g(m_1) + \cdots + \beta_l g(m_l)$, where each $g(m_i)$ is sent to a multiple of m_i, namely $\lambda_{i,g} m_i$, $i = 1, \ldots, l$. Therefore $\beta_1 m_1 + \cdots + \beta_l m_l = \beta_1 \lambda_{1,g} m_1 + \cdots + \beta_l \lambda_{l,g} m_l$ or, equivalently,

$$(\beta_1 - \beta_1 \lambda_{1,g}) m_1 + \cdots + (\beta_l - \beta_l \lambda_{l,g}) m_l = 0,$$

which implies $\lambda_{i,g} = 1$, $i = 1, \ldots, l$. This proves that $m_1, \ldots, m_l \in R^G$ and the result follows. $\square$

As a corollary, for each $t \in \mathbb{Z}_{\geq 0}$ the set of all monomial invariants of G of degree td is a $\mathbb{K}-$basis of $R_t^{\overline{G}}$, we denote it by $\mathcal{B}_t$. Even further, we obtain good information of how a minimal set of fundamental invariants of $\overline{G}$ looks like. Let $\{m_1, \ldots, m_k\}$ be a set of monomial invariants of $\overline{G}$ satisfying the following two conditions: any invariant monomial m of $\overline{G}$ of degree td, $t \geq 1$, can be factored as a product of t monomials in $\{m_1, \ldots, m_k\}$, not necessarily different; and if $m_i = m_1^{p_1} \cdots m_k^{p_k}$, then all $p_j = 0$ except for $p_i = 1$. The set $\{m_1, \ldots, m_k\}$ is a minimal set of fundamental monomial invariants of $\overline{G}$.

On the other hand, the problem of determining the algebra $R^{\overline{G}}$ of invariants becomes equivalent to study linear systems of congruences (Section 1.2). Indeed, for each $t \geq 1$ the set $\mathcal{B}_t$ is uniquely determined by the $\mathbb{Z}_{\geq 0}^{n+1}-$solutions of the systems:

$$(*)_{\mathcal{A};t,r_1,\ldots,r_s} : \begin{cases} y_0 + y_1 + \cdots + y_n = td \\ \alpha^1_{\sigma_1(0)} y_0 + \alpha^1_{\sigma_1(1)} y_1 + \cdots + \alpha^1_{\sigma_1(n)} y_n = r_1 d_1 \\ \qquad\vdots \\ \alpha^s_{\sigma_s(0)} y_0 + \alpha^s_{\sigma_s(1)} y_1 + \cdots + \alpha^s_{\sigma_s(n)} y_n = r_s d_s \end{cases} \tag{2.2.1}$$

with $0 \leq r_i \leq \frac{\alpha_n^i td}{d_i}$, $i = 1, \ldots, s$. This point of view is useful for computing invariants of $\overline{G}$. In some particular cases, it provides a complete description of any $\mathbb{K}-$basis $\mathcal{B}_t$.

Example 2.2.5. (i) Take $G = \langle M_{5;0,1,4} \rangle \subset \mathrm{GL}(3, \mathbb{K})$ a cyclic group of order 5. We fix $t \geq 1$, a monomial $x_0^{a_0} x_1^{a_1} x_2^{a_2} \in R_t^{\overline{G}}$ if and only if there is an integer $0 \leq r \leq 4t$ such that $(a_0, a_1, a_2) \in \mathbb{Z}^3_{\geq 0}$ is a solution of one of the systems:

$$(*)_{\mathcal{A};t,r} = \left\{ \begin{array}{ccccccc} y_0 & + & y_1 & + & y_2 & = & 5t \\ & & y_1 & + & 4y_2 & = & rt. \end{array} \right.$$

Solving these systems, we obtain the $\mathbb{Z}^3_{\geq 0}-$solutions: $\{(5t, 0, 0), (0, 0, 5t)\} \cup \Big\{(5(t-r) + 3a_2, 5r - 4a_2, a_2) \mid \max \Big\{0, \lceil \frac{5(r-t)}{3} \rceil \Big\} \leq a_2 \leq \lfloor \frac{5r}{4} \rfloor, \ r = 1, 2, 3 \Big\}$. We list $\mathcal{B}_t$ for $t = 1, 2$ and 3.

$$\begin{array}{rl} \mathcal{B}_1 = & \{x_0^5, x_1^5, x_0^3 x_1 x_2, x_0 x_1^2 x_2^2, x_2^5\} \\ \mathcal{B}_2 = & \{x_0^{10}, x_0^5 x_1^5, x_0^8 x_1 x_2, x_1^{10}, x_0^3 x_1^6 x_2, x_0^6 x_1^2 x_2^2, x_0 x_1^7 x_2^2, x_0^4 x_1^3 x_2^3, x_0^2 x_1^4 x_2^4, x_0^5 x_2^5, \\ & x_1^5 x_2^5, x_0^3 x_1 x_2^6, x_0 x_1^2 x_2^7, x_2^{10}\} \\ \mathcal{B}_3 = & \{x_0^{15}, x_0^{10} x_1^5, x_0^{13} x_1 x_2, x_0^5 x_1^{10}, x_0^8 x_1^6 x_2, x_0^{11} x_1^2 x_2^2, x_1^{15}, x_0^3 x_1^{11} x_2, x_0^6 x_1^7 x_2^2, \\ & x_0^9 x_1^3, x_2^3, x_0 x_1^{12} x_2^2, x_0^4 x_1^8 x_2^3, x_0^7 x_1^4 x_2^4, x_0^{10} x_2^5, x_0^2 x_1^9 x_2^4, x_0^5 x_1^5 x_2^5, x_0^8 x_1 x_2^6, \\ & x_1^{10} x_2^5, x_0^3 x_1^6 x_2^6, x_0^6 x_1^2 x_2^7, x_0 x_1^7 x_2^7, x_0^4 x_1^3 x_2^8, x_0^2 x_1^4 x_2^9, x_0^5 x_2^{10}, x_1^5 x_2^{10}, x_0^3 x_1 x_2^{11}, \\ & x_0 x_1^2 x_2^{12}, x_2^{15}\}. \end{array}$$

(ii) Take $G = \langle M_{3;0,1,2}, M_{6;0,2,3} \rangle \subset \mathrm{GL}(3, \mathbb{K})$ an abelian group of order 18. We fix $t \geq 1$, a monomial $x_0^{a_0} x_1^{a_1} x_2^{a_2} \in R_t^{\overline{G}}$ if and only if there are integers $0 \leq r_1 \leq 2t$ and $0 \leq r_2 \leq 3t$ such that (a_0, a_1, a_2) is a $\mathbb{Z}^3_{\geq 0}-$solution of the system:

$$(*)_{\mathcal{A};t,r_1,r_2} = \left\{ \begin{array}{ccccccc} y_0 & + & y_1 & + & y_2 & = & 18t \\ & & y_1 & + & 2y_2 & = & 3r_1 \\ & & 2y_1 & + & 3y_2 & = & 6r_2. \end{array} \right.$$

We list $\mathcal{B}_t$ for $t=1$ and 2.

$$\begin{aligned}\mathcal{B}_1 = \; & \{x_2^{18}, x_1^6x_2^{12}, x_1^{12}x_2^6, x_1^{18}, x_0^3x_1^3x_2^{12}, x_0^3x_1^9x_2^6, x_0^3x_1^{15}, x_0^6x_2^{12}, x_0^6x_1^6x_2^6, x_0^6x_1^{12},\\ & x_0^9x_1^3x_2^6, x_0^9x_1^9, x_0^{12}x_2^6, x_0^{12}x_1^6, x_0^{15}x_1^3, x_0^{18}\}\\ \mathcal{B}_2 = \; & \{x_2^{36}, x_1^6x_2^{30}, x_1^{12}x_2^{24}, x_1^{18}x_2^{18}, x_1^{24}x_2^{12}, x_1^{30}x_2^6, x_1^{36}, x_0^3x_1^3x_2^{30}, x_0^3x_1^9x_2^{24},\\ & x_0^3x_1^{15}x_2^{18}, x_0^3x_1^{27}x_2^6, x_0^3x_1^{33}, x_0^6x_2^{30}, x_0^6x_1^6x_2^{24}, x_0^6x_1^{12}x_2^{18}, x_0^6x_1^{18}x_2^{12}, x_0^6x_1^{24}x_2^6,\\ & x_0^6x_1^{30}, x_0^9x_1^3x_2^{24}, x_0^9x_1^9x_2^{18}, x_0^9x_1^{15}x_2^{12}, x_0^9x_1^{21}x_2^6, x_0^9x_1^{27}, x_0^{12}x_2^{24}, x_0^{12}x_1^6x_2^{18},\\ & x_0^{12}x_1^{12}x_2^{12}, x_0^{12}x_1^{18}x_2^6, x_0^{12}x_1^{24}, x_0^{15}x_1^3x_2^{18}, x_0^{15}x_1^9x_2^{12}, x_0^{15}x_1^{15}x_2^6, x_0^{15}x_1^{21},\\ & x_0^{18}x_2^{18}, x_0^{18}x_1^6x_2^{12}, x_0^{18}x_1^{12}x_2^6, x_0^{18}x_1^{18}, x_0^{21}x_1^3x_2^{12}, x_0^{21}x_1^9x_2^6, x_0^{21}x_1^{15}, x_0^{24}x_2^{12},\\ & x_0^{24}x_1^6x_2^6, x_0^{24}x_1^{12}, x_0^{27}x_1^3x_2^6, x_0^{27}x_1^9, x_0^{30}x_2^6, x_0^{30}x_1^6, x_0^{33}x_1^3, x_0^{36}\}.\end{aligned}$$

(iii) Take $G_1 = \langle M_{4;0,1,2}, M_{4;0,1,3}\rangle$ and $G_2 = \langle M_{4;0,1,2}, M_{4;1,0,3}\rangle$ abelian subgroups of $\mathrm{GL}(3,\mathbb{K})$ of order 16. Notice that the generators of G_2 are obtained from the generators of G_1 with the following permutations: $\sigma_1 = \mathrm{Id}$ and σ_2 is the transposition defined as $(0,1,2) \longrightarrow (1,0,2)$ (Notation 2.2.1 and Remark 2.2.2). We can check that $G_2 = \langle M_{4;0,1,2}, M_{4;1,0,3}\rangle = \langle M_{4;1,0,3}, M_{4;1,1,1}\rangle$. The rings $R^{\overline{G}_1}$ and $R^{\overline{G}_2}$ are not isomorphic, as we have pointed out in Remark 2.2.2). Indeed, $R_1^{\overline{G}_1}$ and $R_2^{\overline{G}_2}$ have 15 and 41 monomials of degree 16, respectively:

$$\begin{aligned}& \{x_2^{16}, x_1^4x_2^{12}, x_1^8x_2^8, x_1^{12}x_2^4, x_1^{16}, x_0^4x_2^{12}, x_0^4x_1^4x_2^8, x_0^4x_1^8x_2^4, x_0^4x_1^{12}, x_0^8x_2^8, x_0^8x_1^4x_2^4, x_0^8x_1^8,\\ & x_0^{12}x_2^4, x_0^{12}x_1^4, x_0^{16}\}\end{aligned}$$

$$\begin{aligned}& \{x_2^{16}, x_1^4x_2^{12}, x_1^8x_2^8, x_1^{12}x_2^4, x_1^{16}, x_0x_1^2x_2^{13}, x_0x_1^6x_2^9, x_0x_1^{10}x_2^5, x_0x_1^{14}x_2, x_0^2x_2^{14},\\ & x_0^2x_1^4x_2^{10}, x_0^2x_1^8x_2^6, x_0^2x_1^{12}x_2^2, x_0^3x_1^2x_2^{11}, x_0^3x_1^6x_2^7, x_0^3x_1^{10}x_2^3, x_0^4x_2^{12}, x_0^4x_1^4x_2^8, x_0^4x_1^8x_2^4,\\ & x_0^4x_1^{12}, x_0^5x_1^2x_2^9, x_0^5x_1^6x_2^5, x_0^5x_1^{10}x_2, x_0^6x_2^{10}, x_0^6x_1^4x_2^6, x_0^6x_1^8x_2^2, x_0^7x_1^2x_2^7, x_0^7x_1^6x_2^3, x_0^8x_2^8,\\ & x_0^8x_1^4x_2^4, x_0^8x_1^8, x_0^9x_1^2x_2^5, x_0^9x_1^6x_2, x_0^{10}x_2^6, x_0^{10}x_1^4x_2^2, x_0^{11}x_1^2x_2^3, x_0^{12}x_2^4, x_0^{12}x_1^4, x_0^{13}x_1^2x_2,\\ & x_0^{14}x_2^2, x_0^{16}\}.\end{aligned}$$

The rest of this section is devoted to prove our main result: $\mathcal{B}_1$ is a minimal set of fundamental invariants of $\overline{G}$ (Theorem 2.2.11). In next subsection, we will prove that $R^{\overline{G}}$ is the coordinate ring of an aCM monomial projection of the Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ (Theorems 2.2.14 and 2.2.18). To begin with, we introduce the combinatorial objects and techniques needed in the sequel.

Zero-sum sequences over abelian groups. Let H be an additive finite abelian group of order $|H|$. A *sequence over H* is a finite sequence $L =$

$(h_1, \ldots, h_l)$ of elements of H, where the repetition is allowed and the order is disregard. The *length* of any sequence L over H is defined to be the number of elements appearing in L counted with multiplicity, we denote it by $l(L)$. We define the sum of the sequence L as $\Sigma(L) := h_1 + \cdots + h_l \in H$. Given a sequence L over H, a *subsequence* L' of L is a sequence over H contained in L. In this case, we naturally define the *residue subsequence* $L \setminus L'$ *of* L *by* L'. If $L = (h_1, \ldots, h_l)$ and $L' = (h'_1, \ldots, h'_{l'})$ are two sequences over H, we define the union $L \cup L' := (h_1, \ldots, h_l, h'_1, \ldots, h'_{l'})$, which is also a sequence over H.

Definition 2.2.6. A sequence L over H is said to be a *zero-sum* if $\Sigma(L) = 0$. The *Davenport constant* $D(H)$ *of* H is defined as the minimum positive integer s such that any sequence L over H of length $l(L) = s$ has a zero-sum.

Let us see an example.

Example 2.2.7. Set $H = \mathbb{Z}/5\mathbb{Z}$, a cyclic group of order 5. Explicitly, $H = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$. $L_1 = (\bar{1}, \bar{1}, \bar{2}, \bar{4}, \bar{4})$ and $L_2 = (\bar{0}, \bar{2}, \bar{3})$ are sequences over $\mathbb{Z}/5\mathbb{Z}$ of length $l(L_1) = 5$ and $l(L_2) = 3$, respectively. We have $\Sigma(L_1) = \bar{2}$ and $\Sigma(L_2) = \bar{0} + \bar{2} + \bar{3} = \bar{0}$. In particular, L_2 is a zero-sum. $(\bar{1}, \bar{2}, \bar{4})$ is subsequence of L with residue subsequence $(\bar{1}, \bar{4})$. The Davenport constant of H is $D(H) = 9$.

By the fundamental theorem of finite abelian groups, we have that any additive finite abelian group H is a direct sum of cyclic groups

$$H = C_1 \oplus \cdots \oplus C_k, \tag{2.2.2}$$

of orders $n_1, \ldots, n_k$, respectively, where n_k is the exponent $e(H)$ of the group H, $|H| = n_1 \cdots n_k$ and $n_1 \mid n_2 \mid \cdots \mid n_k$. By [29, Theorem], if H is cyclic, then $D(H) \leq 2|H| - 1$ and every sequence S over H of length $l(S) \geq 2|H| - 1$ has a zero-sum subsequence of length $|H|$. In general, we have:

Proposition 2.2.8. *For any finite abelian group H of order $|H|$ and exponent $e(H)$,*

$$D(H) \leq |H| + e(H) - 1,$$

and any sequence L over H of length $l(L) \geq |H| + e(H) - 1$ has a zero-sum subsequence over H of length $e(H)$.

Proof. See [30, Proposition 4.5]. □

In particular, we have the following key lemma.

Lemma 2.2.9. *Let H be a finite abelian group of order $|H|$ and exponent $e(H)$. Then, any sequence L over H of length $l(L) \geq 2|H|-1$ has a zero-sum subsequence over H of length $|H|$.*

Proof. We consider H with the decomposition (2.2.2). So, we have $|H| = n_1 \cdots n_k$ and $n_k = e(H)$. Let L be a sequence over H of length $l(L) \geq 2|H|-1 = |H|+n_1 \cdots n_k - 1$. Applying Proposition 2.2.8, we obtain a zero-sum subsequence L_1 of L of length $l(L_1) = n_k$. Now we define $L^1 = L \setminus L_1$, which is the residue subsequence of L by L_1 and it has length $l(L^1) \geq |H| + (n_1 \cdots n_{k-1} - 1)n_k - 1$. If $l(L^1) \geq |H| + n_k - 1$, we apply again Proposition 2.2.8 to L^1, as before we obtain a zero-sum subsequence L_2 of L^1 of length $l(L_2) = n_k$. We consider $L^1 \setminus L_2$, we repeat the same argument; and so on. We stop the process at step $n_1 \cdots n_{k-1}$ and we obtain $n_1 \cdots n_{k-1}$ zero-sum subsequences of L of length n_k. The union of all these zero-sum subsequences is, by construction, a zero-sum subsequence of L of length $|H|$, as required. □

Before resume our initial discussion, we give an example.

Example 2.2.10. Set $H = \mathbb{Z}/2\mathbb{Z}$, a cyclic group of order 2 and we write $H = \{\bar{0}, \bar{1}\}$. We have a total of 4 ordered sequences over H of length $3 = 2|H| - 1$,

$$L_1 = (\bar{0}, \bar{0}, \bar{0}),\ L_2 = (\bar{0}, \bar{0}, \bar{1})$$
$$L_3 = (\bar{0}, \bar{1}, \bar{1}),\ L_4 = (\bar{1}, \bar{1}, \bar{1}).$$

$(\bar{0}, \bar{0})$ is a zero-sum subsequence of L_1 and L_2, while $(\bar{1}, \bar{1})$ is a zero-sum subsequence of L_3 and L_4. Furthermore, $(0, \bar{1})$ is a sequence over H of length 2 which does not admit a zero-sum subsequence. So, the Davenport constant $D(H)$ of H is 3.

The set $\mathcal{B}_t$ of all monomial invariants of G of degree td is a $\mathbb{K}-$basis of $R_t^{\overline{G}}$. Now we ask for a subset of these monomials that minimally generates $R^{\overline{G}}$ as a $\mathbb{K}-$algebra, or equivalently, we want to find a minimal set of fundamental monomial invariants of $\overline{G}$. We have regarded G as a direct sum of diagonal cyclic groups Γ_i of order d_i, $i = 1, \ldots, s$. Precisely,

$\Gamma_i = \langle M_{(d_i;\alpha^i_{\sigma_i(0)},\dots,\alpha^i_{\sigma_i(n)})}\rangle \subset \mathrm{GL}(n+1,\mathbb{K})$, with $\sigma_i \in \mathcal{S}_{n+1}$ and integers $0 \le \alpha_0 \le \cdots \le \alpha_n < d_i$ such that $\mathrm{GCD}(\alpha_0,\dots,\alpha_n,d_i) = 1$. In this setting, we define the finite abelian group $H := \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_s\mathbb{Z}$ of order d and we denote $H = \{a_0 \oplus \cdots \oplus a_s \mid a_i \in \mathbb{Z}/d_i\mathbb{Z},\ i = 1,\dots,s\}$. For all $j = 0,\dots,n$, we have that

$$\alpha^1_{\sigma_1(j)} \oplus \cdots \oplus \alpha^s_{\sigma_s(j)} \in H.$$

With this notation, it follows that a monomial $m = x_0^{b_0}\cdots x_n^{b_n}$ of degree td is an invariant of $\overline{G}$ if and only if

$$b_0(\alpha^1_{\sigma_1(0)} \oplus \cdots \oplus \alpha^s_{\sigma_s(0)}) + \cdots + b_n(\alpha^1_{\sigma_1(n)} \oplus \cdots \oplus \alpha^s_{\sigma_s(n)}) = 0 \in H.$$

In [17, Theorem 3.1], the counterpart of the following theorem is proved for finite cyclic groups.

Theorem 2.2.11. $\mathcal{B}_1$ *is a minimal set of fundamental invariants of* $R^{\overline{G}}$.

Proof. It is enough to prove that any monomial of $R_t^{\overline{G}}$, $t \ge 1$, can be factored as a product of t monomials of $\mathcal{B}_1$. We proceed by induction on t. For $t = 1$ the result is true. So, we fix $t > 1$ and let $m = x_0^{b_0}\cdots x_n^{b_n} \in R_t^{\overline{G}}$ be a monomial. For simplicity, we set:

$$\alpha_0 := \alpha^1_{\sigma_1(0)} \oplus \cdots \oplus \alpha^s_{\sigma_s(0)}, \dots, \alpha_n := \alpha^1_{\sigma_1(n)} \oplus \cdots \oplus \alpha^s_{\sigma_s(n)},$$

which are elements of the abelian group $H = \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_s}$ of order d associated to G. We define the sequence $L = (\alpha_0, \overset{b_0}{\dots}, \alpha_0, \dots, \alpha_n, \overset{b_n}{\dots}, \alpha_n)$ over H, where the notation means that each α_i is repeated b_i times in L. Notice that L is a zero-sum sequence over H. Indeed, $m \in R^{\overline{G}}$ is equivalent to $\Sigma(L) = b_0\alpha_0 + \cdots + b_n\alpha_n = 0$. By Lemma 2.2.9, there exists a zero-sum subsequence $L' = (\alpha_0, \overset{c_0}{\dots}, \alpha_0, \dots, \alpha_n, \overset{c_n}{\dots}, \alpha_n)$ of L of length d, i.e. $\Sigma(L') = c_0\alpha_0 + \cdots + c_n\alpha_n = 0$ and $c_0 + \cdots + c_n = d$. We denote by $L'' = (\alpha_0, \overset{b_0-c_0}{\dots}, \alpha_0, \dots, \alpha_n, \overset{b_n-c_n}{\dots}, \alpha_n)$ the residual subsequence of L by L', it has length $(t-1)d$ and it is a zero-sum subsequence of L, i.e. $\Sigma(L'') = (c_0 - b_0)\alpha_0 + \cdots + (c_n - b_n)\alpha_n = 0$. By construction, we have inequalities $0 \le c_0 \le a_0, \dots, 0 \le c_n \le a_n$. Therefore, the monomial $m' = x_0^{c_0}\cdots x_n^{c_n} \in \mathcal{B}_1$ divides m and the monomial $m/m' = x_0^{b_0-c_0}\cdots x_n^{b_n-c_n} \in R_{t-1}^{\overline{G}}$. By induction, m/m' can be expressed a product of $t-1$ monomials of $\mathcal{B}_1$, and the result follows. $\square$

We illustrate Theorem 2.2.11 with an example.

Example 2.2.12. Take $G = \langle M_{(5;0,1,4)} \rangle \subset \mathrm{GL}(3,\mathbb{K})$ a cyclic group of order 5 (Example 2.2.5(i)). We write $m_1 = x_0^5$, $m_2 = x_1^5$, $m_3 = x_0^3x_1x_2$, $m_4 = x_0x_1^2x_2^2$, $m_5 = x_2^5$. By Theorem 2.2.11, $\mathcal{B}_1 = \{m_1, m_2, m_3, m_4, m_5\}$ is a set of fundamental invariants of $\overline{G}$ or, equivalently, $R^{\overline{G}} = \mathbb{K}[m_1, \ldots, m_5]$.

$$\begin{aligned}\mathcal{B}_2 = \ & \{x_0^{10}, x_0^5x_1^5, x_0^8x_1x_2, x_1^{10}, x_0^3x_1^6x_2, x_0^6x_1^2x_2^2, x_0x_1^7x_2^2, x_0^4x_1^3x_2^3, x_0^2x_1^4x_2^4, x_0^5x_2^5,\\ & x_1^5x_2^5, x_0^3x_1x_2^6, x_0x_1^2x_2^7, x_2^{10}\},\end{aligned}$$

and we have factorizations:

$$\begin{array}{lcrlcr}
x_0^{10} & = & m_1^2 & x_0^4x_1^3x_2^3 & = & m_3m_4\\
x_0^5x_1^5 & = & m_1m_2 & x_0^2x_1^4x_2^4 & = & m_4^2\\
x_0^8x_1x_2 & = & m_1m_3 & x_0^5x_2^5 & = & m_1m_5\\
x_1^{10} & = & m_1^2 & x_1^5x_2^5 & = & m_2m_5\\
x_0^3x_1^6x_2 & = & m_2m_3 & x_0^3x_1x_2^6 & = & m_3m_5\\
x_0^6x_1^2x_2^2 & = & m_1m_4 & x_0x_1^2x_2^7 & = & m_4m_4\\
x_0x_1^7x_2^2 & = & m_2m_4 & x_2^{10} & = & m_5^2.
\end{array}$$

However, these factorizations are not unique. For instance, the monomial $x_0^6x_1^2x_2^2 = m_1m_4 = m_3^2$.

2.2.1 Varieties parametrized by invariants of finite abelian groups

In this subsection, we study the CM property of the ring $R^{\overline{G}}$. We introduce a new family of monomial projection of the Veronese varieties $X_{n,d} \subset \mathbb{P}^{N_{n,d}}$, we call them $\overline{G}-$varieties. We relate their homogeneous coordinate ring with the ring $R^{\overline{G}}$ and we conclude that they are aCM varieties (Theorem 2.2.18).

For any finite group $\Lambda \subset \mathrm{GL}(n+1,\mathbb{K})$, Noether's graded normalization theorem (Theorem 1.1.19) assures the existence of a h.s.o.p. $y_0, \ldots, y_n$ of the ring R^Λ. We have that R^Λ is a CM ring if and only if R^Λ is a free $\mathbb{K}[y_0, \ldots, y_n]-$module (Section 1.3). For sake of completeness, we particularize this discussion for the cyclic extension $\overline{G} \subset \mathrm{GL}(n+1,\mathbb{K})$ of an arbitrary finite abelian group $G \subset \mathrm{GL}(n+1,\mathbb{K})$. We give a particular h.s.o.p. of $R^{\overline{G}}$ and we include a proof of the fact that $R^{\overline{G}}$ is a CM ring.

Proposition 2.2.13. *$x_0^d, \ldots, x_n^d$ is a h.s.o.p. of $R^{\overline{G}}$.*

Proof. $x_0^d, \ldots, x_n^d$ are invariants of $\overline{G}$ (see (2.2.1)). We consider the quotient algebra $A := R^{\overline{G}}/\langle x_0^d, \ldots, x_n^d\rangle$. For $t \geq n+1$, we have that $A_t = \langle 0\rangle$ and for $1 \leq t \leq n$, a $\mathbb{K}-$basis of A_t is formed by the set of all monomials $m = x_0^{a_0} \cdots x_n^{a_n} \in R^{\overline{G}}$ of degree td such that $a_0 < d, \ldots, a_n < d$ (Lemma 2.2.4). We write $\theta_1, \ldots, \theta_D$ the set of all such monomials and $\theta_0 = 1$. Then, $R^{\overline{G}} = \langle \theta_0, \theta_1, \ldots, \theta_D\rangle$ as a $\mathbb{K}[x_0^d, \ldots, x_n^d]-$module. $\square$

Theorem 2.2.14. *$R^{\overline{G}}$ is a CM ring.*

Proof. Let $\phi : R \longrightarrow R^{\overline{G}}$ be the so called Reynolds operator, which sends any $p \in R$ to $\phi(p) = \frac{1}{d}\sum_{g\in\overline{G}} g(p) \in R^{\overline{G}}$. We define $U = \{p - \phi(p) \mid p \in R\}$. Since the restriction of ϕ to $R^{\overline{G}}$ is the identity, it follows that $\phi^2 = \phi$. So $U \subseteq \{p \in R \mid \phi(p) = 0\}$. Conversely, if $\phi(p) = 0$, then $p = p - \phi(p)$ is an element of U. Therefore $U = \{p \in R \mid \phi(p) = 0\}$ is an $R^{\overline{G}}-$module and we have a direct sum decomposition $R = R^{\overline{G}} \oplus U$. Since $x_0^d, \ldots, x_n^d$ is also an h.s.o.p. for R and R is a CM ring, R is a free $\mathbb{K}[x_0^d, \ldots, x_n^d]-$module and we have $R/(x_0^d, \ldots, x_n^d) = R^{\overline{G}}/(x_0^d, \ldots, x_n^d) \oplus U/(x_0^d, \ldots, x_n^d)U$. We consider the monomials $\theta_0, \ldots, \theta_D$ described in the proof of Proposition 2.2.13 and we complete it to a basis of R, namely $\{\theta_0, \ldots, \theta_D, \overline{\theta_{D+1}}, \ldots, \overline{\theta_E}\}$. We lift $\overline{\theta_{D+1}}, \ldots, \overline{\theta_E}$ to a homogeneous elements $\theta_{D+1}, \ldots, \theta_E$ of U. Since R is a CM ring, we have that $R = \bigoplus_{i=0}^{E} \theta_i \mathbb{K}[x_0^d, \ldots, x_n^d]$ and from the decomposition $R = R^G \oplus U$ we obtain

$$R^G = \bigoplus_{i=0}^{D} \theta_i \mathbb{K}[x_0^d, \ldots, x_n^d]. \tag{2.2.3}$$

Hence $R^{\overline{G}}$ is a free $\mathbb{K}[x_0^d, \ldots, x_n^d]-$module which is equivalent to say that $R^{\overline{G}}$ is a CM ring. $\square$

Decomposition (2.2.3) is called a *Hironaka decomposition* of the CM ring $R^{\overline{G}}$. It will play an important role when we compute the Hilbert series of $R^{\overline{G}}$ (Section 3.1). So far, we have established that $R^{\overline{G}}$ is a CM ring and a free $\mathbb{K}[x_0^d, \ldots, x_n^d]-$module of rank $D+1$, where D is the number of monomial invariants $x_0^{a_0} \cdots x_n^{a_n}$ of $\overline{G}$ of degree at most nd such that $a_0 < d, \ldots, a_n < d$. Let us see a couple of examples.

Example 2.2.15. (i) Take $G = \langle M_{5;0,1,4}\rangle \subset \mathrm{GL}(3,\mathbb{K})$ a cyclic group of order 5 (Example 2.2.5(i)). A minimal set of fundamental monomial invariants of $\overline{G}$ is $\mathcal{B}_1 = \{x_0^5, x_1^5, x_0^3x_1x_2, x_0x_1^2x_2^2, x_2^5\}$. $R^{\overline{G}}$ is a CM ring, it is a free $\mathbb{K}[x_0^5, x_1^5, x_2^5]-$module of rank 4 with a Hironaka decomposition:

$$\begin{aligned} R^{\overline{G}} &= (x_0^3x_1x_2)\mathbb{K}[x_0^5,x_1^5,x_2^5] \;\oplus\; (x_0x_1^2x_2^2)\mathbb{K}[x_0^5,x_1^5,x_2^5] \;\oplus \\ &\quad (x_0^4x_1^3x_2^3)\mathbb{K}[x_0^5,x_1^5,x_2^5] \;\oplus\; (x_0^2x_1^4x_2^4)\mathbb{K}[x_0^5,x_1^5,x_2^5]. \end{aligned}$$

(ii) Take $G = \langle M_{(3;0,1,2)}, M_{(6;0,2,3)}\rangle \subset \mathrm{GL}(3,\mathbb{K})$ an abelian group of order 18 (Example 2.2.5(ii)). A minimal set of fundamental invariants of $\overline{G}$ is

$$\begin{aligned} \mathcal{B}_1 = \{&x_2^{18}, x_1^6x_2^{12}, x_1^{12}x_2^6, x_1^{18}, x_0^3x_1^3x_2^{12}, x_0^3x_1^9x_2^6, x_0^3x_1^{15}, x_0^6x_2^{12}, \\ &x_0^6x_1^6x_2^6, x_0^6x_1^{12}, x_0^9x_1^3x_2^6, x_0^9x_1^9, x_0^{12}x_2^6, x_0^{12}x_1^6, x_0^{15}x_1^3, x_0^{18}\}. \end{aligned}$$

$R^{\overline{G}}$ is a CM algebra and it is a free $\mathbb{K}[x_0^{12}, x_1^{12}, x_2^{12}]-$module of rank 17 with a Hironaka decomposition:

$$\begin{aligned} R^{\overline{G}} = \;& (x_1^6x_2^{12})\mathbb{K}[x_0^{12},x_1^{12},x_2^{12}] \oplus (x_1^{12}x_2^6)\mathbb{K}[x_0^{12},x_1^{12},x_2^{12}] \oplus \\ & (x_0^3x_1^3x_2^{12})\mathbb{K}[x_0^{12},x_1^{12},x_2^{12}] \oplus (x_0^3x_1^9x_2^6)\mathbb{K}[x_0^{12},x_1^{12},x_2^{12}] \oplus \\ & (x_0^3x_1^{15})\mathbb{K}[x_0^{12},x_1^{12},x_2^{12}] \oplus (x_0^6x_2^{12})\mathbb{K}[x_0^{12},x_1^{12},x_2^{12}] \oplus \\ & (x_0^6x_1^6x_2^6)\mathbb{K}[x_0^{12},x_1^{12},x_2^{12}] \oplus (x_0^6x_1^{12})\mathbb{K}[x_0^{12},x_1^{12},x_2^{12}] \oplus \\ & (x_0^9x_1^3x_2^6)\mathbb{K}[x_0^{12},x_1^{12},x_2^{12}] \oplus (x_0^9x_1^9)\mathbb{K}[x_0^{12},x_1^{12},x_2^{12}] \oplus \\ & (x_0^{12}x_2^6)\mathbb{K}[x_0^{12},x_1^{12},x_2^{12}] \oplus (x_0^{12}x_1^6)\mathbb{K}[x_0^{12},x_1^{12},x_2^{12}] \oplus \\ & (x_0^{15}x_1^3)\mathbb{K}[x_0^{12},x_1^{12},x_2^{12}] \oplus (x_0^9x_1^{15}x_2^{12})\mathbb{K}[x_0^{12},x_1^{12},x_2^{12}] \oplus \\ & (x_0^{12}x_1^{12}x_2^{12})\mathbb{K}[x_0^{12},x_1^{12},x_2^{12}] \oplus (x_0^{15}x_1^9x_2^{12})\mathbb{K}[x_0^{12},x_1^{12},x_2^{12}] \oplus \\ & \oplus (x_0^{15}x_1^{15}x_2^6,)\mathbb{K}[x_0^{12},x_1^{12},x_2^{12}]. \end{aligned}$$

We ask under what circumstances we simply has $R^{\overline{G}} = \mathbb{K}[x_0^d, \dots, x_n^d]$, that is when $R^{\overline{G}}$ is a polynomial ring.

Proposition 2.2.16. *(i) If $n = 1$, then $R^{\overline{G}} = \mathbb{K}[x_0^d, x_1^d]$.*

(ii) For $n \geq 2$, $R^{\overline{G}}$ is minimally generated by at least $n + 2$ monomial invariants of $\overline{G}$.

Proof. (i) It follows directly from [77, Corollary 4.3].

(ii) It is enough to show that there exists at least one monomial $x_0^{a_0}\cdots x_n^{a_n}\in R^{\overline{G}}$ of degree d such that $a_0<d,\dots,a_n<d$. We set $m:=x_0^d\cdots x_n^d$ and $m':=x_0\cdots x_n$ and we have that $m/m'=x_0^{a_0-1}\cdots x_n^{a_n-1}$ is a monomial of degree $nd+d-n+1\geq 2d-1$. Lemma 2.2.9 assures the existence of an monomial invariant $x_0^{b_0}\cdots x_n^{b_n}$ of $\overline{G}$ of degree d dividing m/m', so $b_0<d,\dots,b_n<d$. □

In the sequel, we write $\mathcal{B}_1=\{m_1,\dots,m_{\mu_d}\}$ the set of fundamental monomial invariants of $\overline{G}$ (Theorem 2.2.11).

Definition 2.2.17. A $\overline{G}-$variety with group $G\subset \mathrm{GL}(n+1,\mathbb{K})$ is a monomial projection $X_d\subset\mathbb{P}^{\mu_d-1}$ of the Veronese variety $X_{n,d}\subset\mathbb{P}^{N_{n,d}-1}$ parameterized by $\mathcal{B}_1$.

We denote by $I_d:=(m_1,\dots,m_{\mu_d})\subset R$ the monomial artinian ideal generated by $\mathcal{B}_1$ and $\varphi_{I_d}:\mathbb{P}^n\longrightarrow\mathbb{P}^{\mu_d-1}$ the morphism defined by $\mathcal{B}_1$. The $\overline{G}-$variety $X_d\subset\mathbb{P}^{\mu_d-1}$ with group $G\subset\mathrm{GL}(n+1,\mathbb{K})$ is the image $X_d=\varphi_{I_d}(\mathbb{P}^n)\subset\mathbb{P}^{\mu_d-1}$. The $\overline{G}-$variety X_d with group G is also called *the variety parameterized by* I_d. The following particularizes for $\overline{G}$ the projective version of Theorem 1.3.11 and it generalizes [17, Corollary 3.8].

We take new variables $w_1,\dots,w_{\mu_d}$ and $S=\mathbb{K}[w_1,\dots,w_{\mu_d}]$. We denote by $\mathrm{I}(X_d)\subset S$ the homogeneous ideal of X_d.

Theorem 2.2.18. *Let X_d be a $\overline{G}-$variety with group $G\subset\mathrm{GL}(n+1,\mathbb{K})$. Then, X_d is an aCM monomial projection of the Veronese variety $X_{n,d}\subset\mathbb{P}^{N_{n,d}-1}$.*

Proof. We denote by $A(X_d)=S/\,\mathrm{I}(X_d)$ the homogeneous coordinate ring of X_d. By Theorem 2.2.11, $R^{\overline{G}}=\mathbb{K}[m_1,\dots,m_{\mu_d}]$. We will see that $A(X_d)\cong\mathbb{K}[m_1,\dots,m_{\mu_d}]$ and the result follows from Theorem 2.2.14. To this end, we consider the morphism $\rho:S\longrightarrow\mathbb{K}[m_1,\dots,m_{\mu_d}]$ given by $\rho(w_i)=m_i$. We have that $\mathbb{K}[m_1,\dots,m_{\mu_d}]\cong S/\ker(\rho)$ and that $\ker(\rho)\subset S$ is the homogeneous prime binomial ideal generated by the set of binomials:

$$\{w_{i_1}\cdots w_{i_k}-w_{j_1}\cdots w_{j_k}\in S\ \mid\ m_{i_1}\cdots m_{i_k}=m_{j_1}\cdots m_{j_k},\ k\geq 2\}$$

(it follows from the projective version of Theorem 1.2.10). Hence $\mathrm{I}(X_d)=\ker(\rho)$. □

Remark 2.2.19. (i) Theorem 2.2.18 is a new contribution to the Gröbner's problem [39] (Problem 2.1.1).

(ii) In [57, Theorem 7.3], using another approach it is proved that any $\overline{G}-$surfaces with cyclic group $\langle M_{d;0,1,2}\rangle \subset \mathrm{GL}(3,\mathbb{K})$ of order $3 \leq d$ is an aCM surface.

In view of Proposition 2.2.16, Theorem 2.2.14 provides an extensive new family of aCM monomial projections of the Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ in any dimension $n \geq 2$. The interest of $\overline{G}-$varieties X_d with group $G \subset \mathrm{GL}(n+1,\mathbb{K})$ relies on two facts. On one hand, the coordinate ring of X_d is isomorphic to $R^{\overline{G}}$. Combinatorics and invariant theory of finite groups give enough techniques to tackle the geometry of X_d, as we will see in Chapter 3. On the other hand, we will show in next section that the associated ideal $I_d = (m_1, \ldots, m_{\mu_d})$ fails the weak Lefschetz property provided $\mu_d \leq N_{n-1,d}$. In this case, X_d is apolar to a Togliatti variety Y parameterized by $\mathcal{M}_{n,d} \setminus \{m_1, \ldots, m_{\mu_d}\}$ which satisfies at least one Laplace equation of order $d-1$ (Theorem 1.4.6).

2.3 GT-systems and GT-varieties with a finite abelian group

In this section, we analyse whether the ideal generated by a minimal set of fundamental invariants of a finite abelian group $G \subset \mathrm{GL}(n+1,\mathbb{K})$ fails the WLP (Definition 1.4.1). We fix integers $2 \leq n < d$ and $G = \Gamma_1 \oplus \cdots \oplus \Gamma_s \subset \mathrm{GL}(n+1,\mathbb{K})$ a finite abelian group of order d (Notation 2.2.1) and we consider its cyclic extension $\overline{G} \subset \mathrm{GL}(n+1,\mathbb{K})$. In Theorem 2.2.11, we have proved that the set $\mathcal{B}_1$ of all monomial invariants of G of degree d is a set of fundamental invariants of $\overline{G}$. As usual, we write $\mathcal{B}_1 = \{m_1, \ldots, m_{\mu_d}\}$ and we denote by $I_d = (m_1, \ldots, m_{\mu_d}) \subset R$ the ideal generated by $\mathcal{B}_1$. Since $\{x_0^d, \ldots, x_n^d\} \subset I_d$ (Proposition 2.2.13), I_d is a monomial artinian ideal. We ask whether I_d is a $GT-$system with group G (Definition 1.4.18). We obtain that it depends only on the cardinality $|\mathcal{B}_1| = \mu_d$.

Proposition 2.3.1. *If $\mu_d \leq N_{n-1,d}$, then I_d is a $GT-$system with group $G \subset \mathrm{GL}(n+1,\mathbb{K})$.*

Proof. We want prove that I_d is a Togliatti system whose associated morphism is a Galois covering with group $G \subset \mathrm{GL}(n+1,\mathbb{K})$. The second condition follows from Proposition 1.4.17. To prove that I_d is a Togliatti system it is enough to see that it fails the WLP in degree $d-1$ (Theorem 1.4.6). Let $L \in R_1$ be a linear form and we consider the homogenous polynomial $f = \prod_{\mathrm{Id}_G \neq g \in G} g(L)$ of degree $d-1$. We will see that the multiplication map $\times L : (R/I_d)_{d-1} \longrightarrow (R/I_d)_d$ is not injective. $\times L(f) = L \cdot f = \prod_{g \in G} g(f)$ is an invariant of G of degree d. By Theorem 2.2.11, $L(f) \in I_d$ and $\times L$ is not injective. Since we are assuming that $\mu_d \leq N_{n-1,d}$, we can apply Theorem 1.4.6 and conclude that R/I_d fails the WLP in degree $d-1$. $\square$

Let us see a few examples of $GT-$systems with a finite abelian group.

Example 2.3.2. (i) Take $G = \langle M_{3;0,1,2} \rangle \subset \mathrm{GL}(3,\mathbb{K})$ a cyclic group of order 3. A minimal set of fundamental monomial invariants of $\overline{G}$ is $\mathcal{B}_1 = \{x_0^3, x_1^3, x_2^3, x_0x_1x_2\}$. The condition $\mu_3 \leq 4$ is satisfies, so Proposition 2.3.1 implies that Togliatti's example $T = (x_0^3, x_1^3, x_2^3, x_0x_1x_2)$ (see (1.4.1)) is a $GT-$system with group G.

(ii) Take $G_1 = \langle M_{5;0,1,4} \rangle$ and $G_2 = \langle M_{3;0,1,2}, M_{6;0,2,3} \rangle$ cyclic groups of orders 5 and 18, respectively (Example 2.2.15). $R^{\overline{G}_1}$ is generated by $5 \leq 6$ invariant monomials and $R^{\overline{G}_2}$ is generated by $16 \leq 19$ invariant monomials. The condition $\mu_d \leq N_{n-1,d}$ on the number of generators is satisfied, so by Proposition 2.3.1, both finite abelian groups give rise to $GT-$systems with groups G_1 and G_2, respectively.

(iii) Take $G = \langle M_{7;0,1,1,2}, M_{7;0,1,1,3} \rangle \subset \mathrm{GL}(4,\mathbb{K})$ an abelian group of order 49. The ideal $I_d \subset \mathbb{K}[x_0,x_1,x_2,x_3]$ is generated by 624 monomials of degree 49. The condition $624 \leq \binom{2+49}{2} = 1275$ is satisfied, so by Proposition 2.3.1, I_d is a $GT-$system with group G.

Remark 2.3.3. Let $2 \leq n < d$ be integers and take $G = \langle M_{d;0,1,\dots,1} \rangle \subset \mathrm{GL}(n+1,\mathbb{K})$ a cyclic group of order d. A minimal set of fundamental invariants of $\overline{G}$ is

$$\mathcal{B}_1 = \{x_0^d\} \cup \{x_1^{a_0} \cdots x_n^{a_n} \mid a_1 + \cdots + a_n = d\}$$

and set $I_d = (\mathcal{B}_1)$. The cardinality of $\mathcal{B}_1$ is $\mu_d = N_{n-1,d} + 1$, so the bound in Theorem 1.4.6 is not satisfied. Hence, I_d is not a Togliatti system. Even though, the morphism $\varphi_{I_d} : \mathbb{P}^n \longrightarrow \mathbb{P}^{\mu_d - 1}$ is a Galois covering with group G. In [2, Theorem 7.8] it is proved that I_d has the WLP. For $L = x_0 + \cdots + x_n$, the multiplication map

$$\times(L) : (R/I_d)_i \longrightarrow (R/I_d)_{i+1}$$

is injective for any $0 \leq i \leq d-1$ and it is surjective for any $i \geq d$.

Proposition 2.3.4. *Let $2 \leq n < d$ be integers and $G = \langle M_{d;\alpha_0,\ldots,\alpha_n} \rangle \subset \mathrm{GL}(n+1, \mathbb{K})$ a cyclic group of order d. If $M_{d;\alpha_0,\ldots,\alpha_n} \neq M_{d;0,1,\ldots,1}, M_{d;0,\ldots,0,1}$, then I_d is GT$-$system with group G.*

Proof. We denote by Γ the cyclic group of order d generated by $M_{d;0,1,\ldots,1}$. Let $\mathcal{B}_1$ (respectively $\mathcal{A}_1$) be the set of all μ_d (respectively $N_{n-1,d}+1$) monomial invariants of G (respectively Γ) of degree d. By Proposition 2.3.1, it is enough to show that $\mu_d \leq N_{n-1,d}$. To see it, we define a monomorphism $f : R_1^{\overline{G}} \longrightarrow R_1^{\overline{\Gamma}}$ of $\mathbb{K}-$vector spaces such that $f(\mathcal{B}_1) \subsetneq \mathcal{A}_1$. This implies that $\mu_d \leq N_{n-1,d}$ and the result follows. We distinguish the following cases.

Case 1: there are $0 < \alpha_i < \alpha_j$ with $\mathrm{GCD}(\alpha_i, d) = 1$. For simplicity we assume $i = 1$, the remaining cases follow analogously. Given $m = x_0^{a_0} \cdots x_n^{a_n} \in \mathcal{B}_1$ we define:

$$f(m) := \begin{cases} m & \text{if } \ a_0 = d \\ x_1^{a_0+a_1} x_2^{a_2} \cdots x_n^{a_n} & \text{otherwise.} \end{cases}$$

Let $m_1 = x_0^{b_0} \cdots x_n^{b_n}, m_2 = x_0^{c_0} \cdots x_n^{c_n} \in \mathcal{B}_1$ be such that $f(m_1) = f(m_2)$. Since $\mathrm{GCD}(\alpha_1, d) = 1$, we have that $f(m) \in \mathcal{B}_1$ if and only if $a_0 \neq 0, d$ and so we can assume that $f(m_1) \notin \mathcal{B}_1$ and $f(m_2) \notin \mathcal{B}_2$, or equivalently $0 < b_0, c_0 < d$. We have:

$$b_0 + b_1 = c_0 + c_1 \quad \text{and} \quad b_i = c_i, \ i = 2, \ldots, n. \tag{2.3.1}$$

with $\alpha_1 b_1 + \sum_{i=2}^n \alpha_i b_i$ and $\alpha_1 c_1 + \sum_{i=2}^n \alpha_i c_i$ both multiples of d. Combining this with (2.3.1), we obtain that $\alpha_1(b_0 - c_0)$ is a multiple of d and hence $b_0 - c_0 = 0$. It follows from (2.3.1) that $m_1 = m_2$.

We set $m' := x_1 x_j^{d-1}$ and we assume that $m' = f(m)$ for some $m \in \mathcal{B}_1$. Since $\alpha_1 - \alpha_j \neq 0$ is not a multiple of d, we have that $m' \notin \mathcal{B}_1$. Thus $m = x_0 x_j^{d-1}$, but this is a contradiction since $m \notin \mathcal{B}_1$. Therefore, the $\mathbb{K}-$linear extension of f to $R_1^{\overline{G}}$ defines a monomorphism such that $f(\mathcal{B}_1) \subsetneq \mathcal{A}_1$ as required.

Case 2: for all $\alpha_i > 0$, $\mathrm{GCD}(\alpha_i, d) > 1$. For simplicity we assume $0 < \alpha_1$, the remaining cases follow analogously. Notice that there is $\alpha_1 < \alpha_i$ such that $\mathrm{GCD}(\alpha_1, \alpha_i) = 1$, otherwise G would be a cyclic group of order strictly smaller than d. Let p be the integer such that $\mathrm{GCD}(\alpha_1, d)^p \mid d$ and $\mathrm{GCD}(\alpha_1, d)^{p+1} \nmid d$ and we set $h = \frac{d}{\mathrm{GCD}(\alpha_1,d)^p}$. In particular, $1 < \mathrm{GCD}(\alpha_i, d) \leq h$. Given $m \in \mathcal{B}_1$, we define:

$$f(m) := \begin{cases} m & \text{if } \ a_0 = d \\ x_1^{a_0+a_1} x_2^{a_2} \cdots x_n^{a_n} & \text{if } \ \ kh \neq a_0 < d \\ x_1^{a_1} x_2^{a_2} \cdots x_i^{a_i+a_0} \cdots x_n^{a_n} & \text{if } \ \ a_0 = kh < d \end{cases}$$

The arguments are similar to those of Case 1. Let $m_1 = x_0^{b_0} \cdots x_n^{b_n}, m_2 = x_0^{c_0} \cdots x_n^{c_n} \in \mathcal{B}_1$ be such that $f(m_1) = f(m_2)$. We have $f(m) \in \mathcal{B}_1$ if and only if $a_0 \neq 0, d$. Indeed, if $0 < a_0 < d$ and $a_0 \neq kh$, then $f(m) \in \mathcal{B}_1$ implies that $\alpha_1 a_0$ is a multiple of d and we obtain that a_0 is a multiple of h, which is a contradiction. If $0 < a_0 < d$ and $a_0 = kh$, then $f(m) \in \mathcal{B}_1$ implies that $\alpha_i a_0 = \alpha_i kh$ is a multiple of d and we obtain that k is a multiple of $\mathrm{GCD}(\alpha_1, d)^p$ and so $kh \geq d$, which is a contradiction.

We assume that $kh \neq b_0 < d$ and that $0 < c_0 = kh < d$, the remaining cases follow as in Case 1. We have

$$b_1 + b_0 = c_1, \quad b_i = c_i + kh \quad \text{and} \quad b_j = c_j,\ j \in \{1, \ldots, n\} - \{1, i\}. \quad (2.3.2)$$

Therefore $\alpha_1(b_1 + b_0) + \sum_{j=2}^{n} \alpha_j b_j - \alpha_i kh$ is a multiple of d and we obtain that $\alpha_1 b_0$ is a multiple of h. Since $\mathrm{GCD}(\alpha_1, h) = 1$, we have that b_0 is a multiple of h and we arrive to a contradiction.

Finally, we consider $m' = x_1 x_i^{d-1} \in \mathcal{A}_1$. We have that $m' \notin \mathcal{B}_1$, otherwise $|\alpha_i - \alpha_1| < d$ would be a multiple of d. If $m' = f(m)$ for some $m \in \mathcal{B}_1$, then $m = x_0 x_i^{d-1}$ or $m = x_0^{kh} x_1 x_i^{d-1-kh}$. We have that $x_0 x_i^{d-1} \notin R^G$, otherwise α_i would be a multiple of d. For $x_0^{kh} x_1 x_i^{d-1-kh}$ we have that $\alpha_1 + d\alpha_i - \alpha_i - \alpha_i kh > 0$ is multiple of d. We obtain that $\mathrm{GCD}(\alpha_i, d)$ divides α_1, which is a contradiction.

The number of monomials for the remaining cases $G = \langle M_{d;0,\overset{k}{\cdots},1,0,1,\ldots,1} \rangle \subset \mathrm{GL}(n+1, \mathbb{K})$ with $2 \leq k \leq n-2$, can be bounded as follows. We may assume that $k \leq \lfloor \frac{n}{2} \rfloor$. If $m \in \mathcal{B}_1$, then either m is a monomial of degree d in the variables $x_0, \ldots, x_{k-1}$ or it is a monomial of degree d in the variables $x_k, \ldots, x_n$. Thus $\mu_d \leq N_{k-1,d} + N_{n-k,d}$. Applying iteratively the identity $N_{n-l,d} = N_{n-l-1,d} + N_{n-l,d-1}, \quad n-l \geq 1$, we obtain that $N_{n-1,d} > N_{n-k,d} + N_{k,d-1} \geq N_{n-k,d} + N_{k-1,d}$. □

Remark 2.3.5. (i) For any integer $2 < d$ and any cyclic group $G = \langle M_{d;0,\alpha_1,\alpha_2} \rangle \subset \mathrm{GL}(3, \mathbb{K})$ with $\alpha_1 < \alpha_2 < d$, in [57] the authors showed that the number μ_d of monomial invariants of G of degree d is bounded by $d+1$ and in [17] we computed a closed expression for μ_d, which we will explain in Subsection 3.1.1.

(ii) Let $2 \leq n < d$ be integers and $G = \langle M_{d;0,1,2,\ldots,n} \rangle \subset \mathrm{GL}(n+1, \mathbb{K})$ a cyclic group of order d. Using different techniques, in [18, Theorem 4.8] we compute the number of monomial invariants of G of degree $n+1$ and we prove that it does not exceed $N_{n-1,n+1}$.

(iii) Let $d \geq 4$ be an integer and $G = \langle M_{d;0,1,2,3} \rangle \subset \mathrm{GL}(4, \mathbb{K})$ a cyclic group of order d. In [20, Proposition 3.3], we counted the exact number μ_d of monomial invariants of G of degree d and we checked that $\mu_d \leq N_{2,d}$, which we will explain in Subsection 3.1.2.

In Chapter 3, we will study the Hilbert function and series of the rings $R^{\overline{G}}$. This will lead us to different techniques for counting the number of invariants of the finite abelian groups $\overline{G}$. It is worthwhile to point out that Proposition 1.4.18 provides examples of $GT-$systems with finite abelian groups $G \subset \mathrm{GL}(n+1, \mathbb{K})$ for any $n \geq 2$ and partially motivates the following definition.

Definition 2.3.6. If I_d is a $GT-$system, we call a $GT-$variety with group G to the $\overline{G}-$variety parameterized by I_d.

By Theorem 1.4.6, any $GT-$variety X_d with finite abelian group $G \subset \mathrm{GL}(n+1, \mathbb{K})$ is an aCM monomial projection of the Veronese varieties $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$. They are apolar to rational varieties satisfying at least one Laplace equation. $GT-$varieties with group G form a subfamily of $\overline{G}-$varieties which blends commutative algebra, combinatorics, invariant theory of finite groups, geometry and the WLP.

2.4 A new family of aCM surfaces parametrized by monomial Togliatti systems

In Section 2.2, we have proved that all $\overline{G}-$varieties with group $G \subset \mathrm{GL}(n+1, \mathbb{K})$ are aCM varieties (Theorem 2.2.18) and we have seen under which conditions the ideal generated by a minimal set $\mathcal{B}_1$ of fundamental invariants of $\overline{G}$ is a $GT-$system with group G (Proposition 2.3.1). However, being an aCM variety could fail for varieties parameterized by an arbitrary monomial Togliatti system. For instance, the ideal

$$I = \{x_0^5, x_1^5, x_2^5, x_0^3x_1x_2, x_0^2x_1^2x_2, x_0x_1^3x_2\} \subset \mathbb{K}[x_0, x_1, x_2]$$

is a monomial Togliatti system, since the multiplication map

$$\times(x_0 + x_1 + x_2) : (\mathbb{K}[x_0, x_1, x_2]/I)_4 \longrightarrow (\mathbb{K}[x_0, x_1, x_2]/I)_5$$

is not injective (Theorem 1.4.6), but the surface $X := \varphi_I(\mathbb{P}^2) \subset \mathbb{P}^5$ is not an aCM surface. Indeed, we have checked that $\mathrm{codim}(X) = 3 < \mathrm{pdim}(S/\mathrm{I}(X)) = 4$. In this section, we prove the aCM property of a new family of surfaces parameterized by monomial Togliatti systems: their coordinate rings are neither the ring of invariants of a finite group nor the semigroup ring associated to a normal affine semigroup. Our result is based on the criterion Theorem 2.1.4.

Through this section R denotes the polynomial ring $\mathbb{K}[x_0, x_1, x_2]$.

Definition 2.4.1. We define the affine semigroup

$$H_3 := \langle (3, 0, 0), (0, 3, 0), (0, 0, 3), (1, 1, 1) \rangle \subset \mathbb{Z}^3_{\geq 0}.$$

Set $m = (1, 1, 1)$. Inductively for $t \geq 2$, we define

$$H_{3t} := \langle (3t, 0, 0), (0, 3t, 0), (0, 0, 3t), m + H_{3(t-1)} \rangle,$$

where $m + H_{3(t-1)} = \{m + h \mid h \in H_{3(t-1)}\}$.

Let us illustrate the above definition with the following examples.

Example 2.4.2.

$$
\begin{aligned}
H_6 &= \langle (6,0,0),(0,6,0),(0,0,6),(4,1,1),(1,4,1),(1,1,4),(2,2,2)\rangle \\
H_9 &= \langle (9,0,0),(0,9,0),(0,0,9),(7,1,1),(1,7,1),(1,1,7),(5,2,2), \\
&\quad (2,5,2),(2,2,5),(3,3,3)\rangle \\
H_{12} &= \langle (12,0,0),(0,12,0),(0,0,12),(10,1,1),(1,10,1),(1,1,10), \\
&\quad (8,2,2),(2,8,2),(2,2,8),(6,3,3),(3,6,3),(3,3,6),(4,4,4)\rangle.
\end{aligned}
$$

We denote by $J_{3t} \subset R$ the monomial artinian ideal associated to H_{3t}.

Proposition 2.4.3. *For any $t \geq 1$, the ideal J_{3t} is a monomial Togliatti system.*

Proof. For any $t \geq 1$, we have that

$$J_{3t} = (x_0^{3t}, x_1^{3t}, x_2^{3t}, x_0x_1x_2J_{3(t-1)})$$

is an artinian ideal minimally generated by $\mu_{3t} = 3t+1$ monomials of degree $3t$ and μ_{3t} verifies the bound in Theorem 1.4.6. We want to prove that J_{3t} fails the WLP in degree $3t-1$, i.e. for any linear form $L \in (R/J_{3t})_1$, the multiplication map $\times L : (R/J_{3t})_{3t-1} \longrightarrow (R/J_{3t})_{3t}$ is not injective. We proceed by induction on t. The first ideal J_3 is Togliatti's example $T = (x_0^3, x_1^3, x_2^3, x_0x_1x_2)$ (see (1.4.1)), which is the first $GT-$system with cyclic group known in the literature. Let $t > 1$ and we assume that $J_{3(t-1)}$ fails the WLP in degree $3(t-1)-1$. Let L be a homogenous linear form of $(R/J_{3t})_1 = (R/J_{3(t-1)})_1$. By induction, there is a homogenous form f of degree $3(t-1)-1$ such that $f' = L \cdot f \in J_{3(t-1)}$. We define $f'' = x_0x_1x_2f$. The multiplication map $\times L : (R/J_{3t})_{3t-1} \longrightarrow (R/J_{3t})_{3t}$ sends f'' to $L(f'') = x_0x_1x_2f' \in x_0x_1x_2J_{3(t-1)} \subset J_{3t}$, so $\times L$ is not injective. □

By Theorems 2.2.11 and 2.2.14, we have that $\mathbb{K}[H_3]$ is a CM ring. Notwithstanding, for any $t > 1$ the semigroup H_{3t} is not a normal semigroup and $\mathbb{K}[H_{3t}]$ is not the ring of invariants of a finite group $\Lambda \subset \mathrm{GL}(3,\mathbb{K})$. Indeed, H_{3t} is not normal since m belongs to the normalization $\overline{H_{3t}}$ of H_{3t} but $m \notin H_{3t}$. To check the second assertion, assume by contradiction that $\mathbb{K}[H_{3t}]$ is the ring of invariants of a finite group $\Lambda \subset \mathrm{GL}(3,\mathbb{K})$, and let

$\phi : R \longrightarrow R^\Lambda$ be the Reynolds operator for the pair (R, R^Λ) (Section 1.3). We have that for all $t > 1$, $(3, 3(t-1), 0) \notin H_{3t}$ (Lemma 2.4.7), or equivalently $x_0^3 x_1^{3(t-1)} \notin R^\Lambda$. We observe that $(3, 3(t-1), 0) + tm$ can be written as $[(t-1)m + (3,0,0)] + [m + (0, 3(t-1), 0)] \in H_{3t}$. So, $x_0^t x_1^t x_2^t \cdot x_0^3 x_1^{3(t-1)} \in R^\Lambda$ and we have $\phi(x_0^t x_1^t x_2^t \cdot x_0^3 x_1^{3(t-1)}) = x_0^t x_1^t x_2^t \cdot \phi(x_0^3 x_1^{3(t-1)}) = x_0^t x_1^t x_2^t \cdot x_0^3 x_1^{3(t-1)}$. Therefore, $\phi(x_0^3 x_1^{3(t-1)}) = x_0^3 x_1^{3(t-1)}$ and we arrive to a contradiction.

Our goal is to prove that all rings $\mathbb{K}[H_{3t}]$ are CM rings. To this end, we will apply Theorem 2.1.4. But first we need some technical lemmas. We fix $t > 1$ and we set $f_1 = (3t, 0, 0), f_2 = (0, 3t, 0), f_3 = (0, 0, 3t)$.

Remark 2.4.4. (i) Notice that f_1, f_2 and f_3 are $\mathbb{Q}-$linearly independent and $(3t)H_{3t} \subset \langle f_1, f_2, f_3 \rangle$.

(ii) By construction $H_{3t} \subset H_3$, so $\overline{H_{3t}} \subset H_3$. This means that for all $u = (a_1, a_2, a_3) \in H_{3t}$ there exist $f \geq 1$ and $r \in \{0, \ldots, 2tf\}$ such that u is a solution of the system:

$$(*)_{\mathcal{A};ft,r} = \begin{cases} a_1 + a_2 + a_3 = 3ft \\ a_2 + 2a_3 = 3r. \end{cases}$$

The converse is not true: $(3, 3(t-1), 0) \in H_3 \setminus H_{3t}$.

(iii) All generators of H_{3t} different from f_1, f_2, f_3 have all three components different from 0.

Remark 2.4.5. By construction, we can describe

$$H_{3t} = \left\{ u = A_1 f_1 + A_2 f_2 + A_3 f_3 + \sum_{j=1}^{3(t-1)+1} A_{j+3}(m + h_j) \right\} \subset \mathbb{Z}^3_{\geq 0},$$

where $A_i \in \mathbb{Z}_{\geq 0}$ for $i = 1, \ldots, 3t+1$ and h_j is a generator of $H_{3(t-1)}$, for $j = 1, \ldots, 3(t-1)+1$. Notice that a generator $h = (a_1, a_2, a_3)$ of H_{3t} different from f_1, f_2, f_3 can be expressed as $sm + h'$, where $0 < s = \min\{a_1, a_2, a_3\} \leq t$ and $h' \in \{(3(t-s), 0, 0), (0, 3(t-s), 0), (0, 0, 3(t-s))\}$.

We give a couple of examples.

Example 2.4.6. (i) Take H_6. We have $(4,1,1) = m + (3,0,0)$, $(1,4,1) = m + (0,3,0)$, $(1,1,4) = (1,1,1) + (0,0,3)$ and $(2,2,2) = 2m$.

(ii) Take H_9. We have: $(7,1,1) = m + (6,0,0)$, $(1,7,1) = m + (0,6,0)$, $(1,1,7) = m + (0,0,6)$, $(5,2,2) = 2m + (3,0,0)$, $(2,5,2) = 2m + (0,3,0)$, $(2,2,5) = 2m + (0,0,3)$ and $(3,3,3) = 3m$.

Any $u \in H_{3t}$ represents a monomial of degree a multiple of $3t$, namely $(3t)f$. For any representation

$$u = A_1 f_1 + A_2 f_2 + A_3 f_3 + \sum_{j=1}^{3(t-1)+1} A_{j+3}(m + h_j)$$

in H_{3t}, it holds that $\sum_{i=1}^{3t+1} A_i = f$.

Lemma 2.4.7. *Let $w = (a_1, a_2, a_3) \in H_3$ be such that $a_i, a_j \neq 0$ and $a_k = 0$, for $\{i,j,k\} = \{1,2,3\}$. Then $w \in H_{3t}$ if and only if a_i and a_j are multiples of $3t$.*

Proof. We can assume $(i,j,k) = (1,2,3)$. If $w = (a_1, a_2, 0) \in H_{3t}$, then w cannot be generated in H_{3t} by any element belonging to $m + H_{3(t-1)}$. So we obtain $w = A_1 f_1 + A_2 f_2$ with $a_1 = 3tA_1$ and $a_2 = 3tA_2$. Conversely, $w = (3tA_1, 3tA_2, 0) \in H_{3t}$ for all integers $A_1, A_2 \geq 0$. □

Corollary 2.4.8. *If $w \in H_3$ is as in Lemma 2.4.7, then either $w \in H_{3t}$ or $w + f_i, w + f_j \notin H_{3t}$.*

Remark 2.4.9. If $w = (a_1, a_2, a_3) \in H_{3t}$ only has one nonzero component, namely a_i, then $w = A_i f_i$, where $a_i = 3tA_i$.

We are now ready to prove the main theorem of this section.

Theorem 2.4.10. *For any $t \geq 1$, $\mathbb{K}[H_{3t}]$ is a CM ring.*

Proof. By Theorem 2.1.4, it is enough to prove that $H^1 = \{w \in \overline{H_{3t}} \mid w + f_i, w + f_j \in H_{3t}$ for some $i, j \in \{1,2,3\}, i \neq j\} \subseteq H_{3t}$. We *claim* that this inclusion is a consequence of the following condition:

Condition $(*)$*:* if $w = (a_1, a_2, a_3) \in H_3$ is such that $a_1 a_2 a_3 \neq 0$ and $w + f_i \in H_{3t}$ for some $i \in \{1,2,3\}$, then either $w \in H_{3t}$ or $w + f_j, w + f_k \notin H_{3t}$ for $\{i,j,k\} = \{1,2,3\}$.

Proof of the claim. We have already shown the same statement for elements w with $a_1a_2a_3 = 0$ in Corollary 2.4.8 and Remark 2.4.9. Since $H^1 \subset \overline{H_{3t}} \subset H_3$, an element $w \in H^1$ satisfying $w + f_j, w + f_k \in H_{3t}$, for some $j, k \in \{1, 2, 3\}$ such that $j \neq k$, belongs to H_{3t}. This proves the claim.

Proof of Condition $(*)$. We can assume $(i, j, k) = (1, 2, 3)$. Set $w + f_1 = A_1f_1 + A_2f_2 + A_3f_3 + \sum_j A_{j+3}(m + h_j) \in H_{3t}$. We may suppose that $A_1 = 0$, otherwise the result is trivial. We observe the following. Let $u = m + h_j = s_jm + (3(t - s_j), 0, 0)$ and $v = m + h_i = s_im + (3(t - s_i), 0, 0)$, with $s_j, s_i > 0$, be two generators of H_{3t}. Therefore, we can write $u + v = [(s_j - 1)m + (3(t - s_j + 1), 0, 0)] + [(s_i + 1)m + [(3(t - s_i - 1), 0, 0)]$. Similarly, if we replace h_j, h_i by $(0, 3(t - s_j), 0), (0, 3(t - s_i), 0)$ or $(0, 0, 3(t - s_j))$, $(0, 0, 3(t - s_i))$, respectively. So, after doing suitable transformations on the summands of $w + f_1$, we reduce it to one of the following forms.

<u>Case 1</u>: $w + f_1 = A_2f_2 + A_3f_3 + [s_1m + (3(t - s_1), 0, 0)] + [s_2m + (0, 3(t - s_2), 0)] + [s_3m + (0, 0, 3(t - s_3))]$ with $0 < s_1 < t$. Since $s_1 + s_2 + s_3 + 3(t - s_1) = 3t + a_1$, we have $0 \le s_2, s_3 < t$, where $s_2 > 0$ or $s_3 > 0$. Let us assume that $s_2, s_3 > 0$, the other cases follow in the same way up to minor modifications. By hypothesis, $w + f_1$ can be written as a sum of $A_2 + A_3 + 3$ generators of H_{3t}. The first component of $w + f_1$ corresponds to $a_1 + 3t = s_1 + 3(t - s_1) + s_2 + s_3$, so $a_1 = s_2 + s_3 - 2s_1$. Notice that $w = (s_2 + s_3 - 2s_1, s_1 + s_2 + s_3 + A_23t + 3(t - s_2), s_1 + s_2 + s_3 + A_33t + 3(t - s_3))$. If $s_2, s_3 \ge s_1$, we have $w = A_2f_2 + A_3f_3 + [(s_2 - s_1)m + (0, 3(t - s_2 + s_1), 0)] + [(s_3 - s_1)m + (0, 0, 3(t - s_3 + s_1))]$. Indeed, $s_1 + s_2 + s_3 = s_2 - s_1 + s_3 - s_1 + 3s_1$, hence $w \in H_{3t}$. Otherwise, suppose for instance that $s_2 < s_1$ and write

$$w = (s_2 + s_3 - 2s_1)m + (0, A_23t + 3t - 3s_2 + 3s_1, A_33t + 3t - 3s_3 + 3s_1). \quad (2.4.1)$$

If $w \in H_{3t}$, then w is a sum of $A_2 + A_3 + 2$ generators of H_{3t}. We observe that $A_23t + 3t - 3s_2 + 3s_1 > (A_2 + 1)3t$, $A_33t + 3t - 3s_3 + 3s_1 > A_33t$ and $s_2 + s_3 - 2s_1 < s_3 < t$. This means that we can write w as a sum of at least $A_2 + 2$ generators of type $sm + (0, 3(t - s), 0)$ plus at least $A_3 + 1$ generators of type $sm + (0, 0, 3(t - s))$, where all $s < t$. Indeed, since $a_1 = s_2 + s_3 - 2s_1 < t$, a generator in w cannot be of the form tm, otherwise $w + f_1$ does. If this was the case, such generator would be either f_2, or f_3, or it would correspond to $sm + (0, 3(t - s), 0)$ or $sm + (0, 0, 3(t - s))$ with $0 < s < t$. But this is a contradiction, because that would give rise to an expression of w with at least $A_2 + A_3 + 3$ summands (Remark 2.4.4(iii)). Performing the same

kind of arguments, we see that $w+f_2, w+f_3 \notin H_{3t}$. The case $s_3 < s_1$ is analogous.
Case 2: $w+f_1 = A_2f_2+A_3f_3+tm+[s_1m+(3(t-s_1),0,0)]+[s_2m+(0,3(t-s_2),0)]+[s_3m+(0,0,3(t-s_3))]$, where $s_1>0$ and some $s_i>0,\ i=2,3$. We assume $s_2,s_3>0$ for simplicity. By hypothesis, $w+f_2$ is a sum of A_2+A_3+4 generators of H_{3t}. If $s_2>s_1$ (respectively $s_3>s_1$),

$$w = A_2f_2+A_3f_3+(t-s_1)m+(0,3s_1,0)+s_2m+(0,3(t-s_2),0)+$$

$$(s_3-s_1)m+(0,0,3(t-s_3+s_1)),$$

hence $w\in H_{3t}$. We see that if $s_2,s_3<s_1$, then $w\notin H_{3t}$. If not, w can be written as a sum of A_2+A_3+3 generators and we have:

$$w = m(t+s_2+s_3-2s_1)+(0,3tA_2+3t-3s_2+3s_1,3tA_3+3t-3s_3+3s_1).$$

Notice that $t+s_2+s_3-2s_1<t$, $3tA_2+3t-3s_2+3s_1>(A_2+1)3t$ and $3tA_3+3t-3s_3+3s_1>(A_3+1)3t$. So, w is a sum of at least A_2+A_3+4 generators of H_{3t}. Arguing in a similar way, we also obtain that $w+f_2, w+f_3\notin H_{3t}$.
Case 3: $w+f_1 = A_2f_2+A_3f_3+2tm+[s_1m+(3(t-s_1),0,0)]+[s_2m+(0,3(t-s_2),0)+s_3m+(0,0,3(t-s_3))]$. Here the situation is slightly different. If $s_1>0$, then $w\in H_{3t}$. Indeed, $w=A_2f_2+A_3f_3+[(t-s_1)m+(0,3(t-s_1),0)]+[(t-s_1)m+(0,0,3(t-s_1))]+[s_2m+(0,3(t-s_2),0)]+[s_3m+(0,0,3(t-s_3))]$. So we suppose $s_1=0$, in which case $s_2,s_3>0$ and we have:

$$w=(s_2+s_3-t)m+(0,3tA_2+3t+3t-3s_2,3tA_3+3t+3t-3s_3),$$

with $s_2+s_3-t<t$, $3tA_2+3t+3t-3s_2>(A_2+1)3t$ and $3tA_3+3t+3t-3s_3>(A_3+1)3t$. If $w\in H_{3t}$, then it should be written as a sum of at least A_2+A_3+4 generators, which is a contradiction. Performing the same arguments we also obtain $w+f_2, w+f_3\notin H_{3t}$.

Case 4: $w+f_1=A_2f_2+A_3f_3+K(tm)+[s_1m+(3(t-s_1),0,0)]+[s_2m+(0,3(t-s_2),0)]+[s_3m+(0,0,3(t-s_3))]$, with $K\geq 3$. We always have $w\in H_{3t}$, indeed $tm+tm+tm=f_1+f_2+f_3$.

This proves *Condition* $(*)$ and the theorem follows. □

Let us see how Theorem 2.4.10 is applied to $\mathbb{K}[H_6]$.

Example 2.4.11. Case 1. The only possibility is $w + f_1 = A_2(0,6,0) + A_3(0,0,6) + [(1,1,1) + (3,0,0)] + [(1,1,1) + (0,3,0)] + [(1,1,1) + (0,0,3)]$, where necessarily $a_1 = 0$. For simplicity, we set $A_2 = A_3 = 0$. If $s_1, s_2 > 0$, then $w = (0, 1+4+1, 1+1+4) = f_2 + f_3 \in H_6$.

Case 2. We consider $w + f_1 = (2,2,2) + [(1,1,1) + (3,0,0)] + [(1,1,1) + (0,3,0)] + [(1,1,1) + (0,0,3)]$, with $s_1 = s_2 = s_3 = 1$. Then, we have: $w = (2,2,2) + (0, 2+4, 2+4) = [m + (0,3,0)] + [m + (0,0,3)] \in H_6$.

Case 3. We consider $w + f_1 = (2,2,2) + (2,2,2) + [(1,1,1) + (0,3,0)] + [(1,1,1) + (0,0,3)]$, with $a_1 = 0$. Then, we have: $w = (0,9,9), w + (0,6,0) = (0,15,9), w + (0,0,6) = (0,9,15) \notin H_6$.

Fix an integer $k \geq 1$. For each integer $t' \geq 0$, we define $H^k_{3(1+t'k)} := \langle (3(1+t'k),0,0), (0,3(1+t'k),0), (0,0,3(1+t'k)), km + H^k_{3(1+(t'-1)k)} \rangle \subset \mathbb{Z}^3_{\geq 0}$. We have:

Corollary 2.4.12. *$\mathbb{K}[H^k_{3(1+kt')}]$ is a CM ring for all integers $k \geq 1$ and $t' \geq 0$.*

Proof. It follows from the same proof as Theorem 2.4.10 replacing m by km. □

Remark 2.4.13. (i) $H^k_{3(1+t'k)}$ is generated by $3(t'+1)+1$ elements in $\mathbb{Z}^3$.

(ii) Our initial family H_{3t} can be rewritten as $H^1_{3(1+t')}$ for $t' \geq 0$.

To prove Theorem 2.4.10, we have strongly used the particular shape of the generators of H_3. The same arguments do not apply, in general, if we replace H_{3t} by an arbitrary $GT-$system. This evinces the complexity of checking the arithmetical condition $H_1 = H_{3t}$ of Theorem 2.1.4, as it is remarked in [49].

Chapter 3

The geometry of $\overline{G}-$varieties

As we have seen in Subsection 2.2.1, any $\overline{G}-$variety X_d with a finite abelian group $G \subset \mathrm{GL}(n+1, \mathbb{K})$ of order d is an aCM monomial projection of the Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ related to invariant theory of finite groups and to the theory of semigroup rings. The homogeneous coordinate ring $A(X_d)$ of X_d is a graded CM ring isomorphic to the ring $R^{\overline{G}}$ (Theorem 2.2.18). Combinatorially, $A(X_d)$ is isomorphic to the semigroup ring of the normal affine semigroup $H_{\mathcal{A}} \subset \mathbb{Z}_{\geq 0}^{n+1}$ associated to $R^{\overline{G}}$. These features endow the homogeneous coordinate ring $A(X_d)$ with a rich structure. Along this chapter, we take advantage of these connections to study the geometry behind a $\overline{G}-$variety X_d with group $G \subset \mathrm{GL}(n+1, \mathbb{K})$. We pursue to determine the Hilbert function and series of $A(X_d)$ (Propositions 3.1.2 and 3.1.15), to understand the structure of a minimal set of binomial generators of the homogeneous ideal $\mathrm{I}(X_d)$ of X_d (Theorem 2.4.10), to investigate the canonical module of $A(X_d)$ (Theorem 3.3.3), to characterize the Castelnuovo–Mumford regularity of $A(X_d)$ (Theorem 3.3.5) and to describe the Betti diagram (Definition 3.1.10) of a minimal graded free resolution of $A(X_d)$ (Subsection 3.3.1).

This chapter is organized as follows. In Section 3.1, we deal with the Hilbert function and series of $A(X_d)$. We interpret both functions from invariant theory which allows us to describe them in terms of the Molien series of $\overline{G}$ and the monomial basis of $R^{\overline{G}}$ (Proposition 3.1.2). Moreover, this provides us with a range of strategies to compute the Hilbert function and series. In particular, we give an explicit combinatorial description of the Hilbert function and series of $GT-$varieties with a cyclic group of prime order (Proposition 3.1.4). In Subsections 3.1.1 and 3.1.2, we give a closed formula for the Hilbert function and series of any $GT-$surface with finite cyclic group (Theorem 3.1.21) and for $GT-$threefolds with cyclic group $\langle M_{d;0,1,2,3}\rangle \subset \mathrm{GL}(4,\mathbb{K})$ of order $3 < d$ (Theorem 3.1.26 and Corollary 3.1.27). The results of Subsection 3.1.1 has been published in [17].

Section 3.2 is devoted to study the homogeneous ideal $\mathrm{I}(X_d)$ of any $\overline{G}-$variety with group $G \subset \mathrm{GL}(n+1,\mathbb{K})$. Our main result proves that $\mathrm{I}(X_d)$ is generated by binomials of degree at most 3 (Theorem 2.4.10). Furthermore, we give examples of $\overline{G}-$varieties X_d with group $G \subset \mathrm{GL}(n+1,\mathbb{K})$ whose ideal $\mathrm{I}(X_d)$ is minimally generated by binomials of degree 2 and 3. We characterize combinatorially when a binomial of $\mathrm{I}(X_d)$ belongs to a minimal set of binomial generators of $\mathrm{I}(X_d)$ (Proposition 3.2.4). In Subsection 3.2.1, we deal with a minimal set of binomial generators of any $GT-$threefold with cyclic group $G = \langle M_{d;0,1,2,3}\rangle \subset \mathrm{GL}(4,\mathbb{K})$ of order $3 < d$. We prove that their homogeneous ideals are generated by binomials of degree 2 if d is even and by binomials of degree 2 and 3 if d is odd (Theorem 3.2.24 and Corollary 3.2.25). For d odd, we provide a complete description of the binomials of degree 3 belonging to a minimal set of generators of $\mathrm{I}(X_d)$. The results of Section 3.2 for finite cyclic groups have been published in [21]. The results of Subsection 3.2.1 have been published in [20].

In Section 3.3, we investigate the canonical module ω_{X_d} of any $\overline{G}-$variety X_d with group $G \subset \mathrm{GL}(n+1,\mathbb{K})$. We identify ω_{X_d} with $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}})) = (x_0^{a_0}\cdots x_n^{a_n} \in R^{\overline{G}} \mid a_0\cdots a_n \neq 0) \subset R^{\overline{G}}$ (Theorem 3.3.1). Our main result shows that ω_{X_d} is generated by monomials of degree d and $2d$ (Theorem 3.3.3). This leads us to characterize the Castelnuovo–Mumford regularity $\mathrm{reg}(A(X_d))$ of $A(X_d)$ in terms of the set of monomials $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1$ of $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))$ of degree d (Theorem 3.3.5). Finally in Subsection 3.3.1, we gather all the results obtained along this chapter to tackle the Betti diagram of X_d. We introduce families of $\overline{G}-$varieties with group $G \subset \mathrm{GL}(n+1,\mathbb{K})$ whose homogeneous coordinate rings are level rings (Proposition 3.3.8). We focus on $\overline{G}-$surfaces with group $G \subset \mathrm{GL}(3,\mathbb{K})$. Using the knowledge of the Hilbert series, Castelnuovo–Mumford regularity and the structure of their homogeneous ideal (Corollary 3.1.24 and Corollary 3.2.8), we describe the Betti diagram of their homogeneous coordinate ring and we explicitly compute it for $GT-$surfaces with cyclic group $\langle M_{d;0,\alpha_1,\alpha_2}\rangle \subset \mathrm{GL}(3,\mathbb{K})$ of order $2 < d$ with $0 < \alpha_1 < \alpha_2$ (Theorem 3.3.14). We end this chapter discussing the complexity of finding the Betti diagram of any $\overline{G}-$variety with group $G \subset \mathrm{GL}(n+1,\mathbb{K})$. The results of Section 3.3 for finite cyclic groups have been published in [21]. The results of Subsection 3.3.1 for $GT-$surfaces with a finite cyclic group have been published in [17].

3.1 Hilbert function and Hilbert series

We consider a finite abelian group

$$G = \langle M_{(d_1;\alpha^1_{\sigma_1(0)},\dots,\alpha^1_{\sigma_1(n)})}, \dots, M_{(d_s;\alpha^s_{\sigma_s(0)},\dots,\alpha^s_{\sigma_s(n)})}\rangle \subset \mathrm{GL}(n+1,\mathbb{K})$$

of order $d = d_1 \cdots d_s$ and its cyclic extension $\overline{G} \subset \mathrm{GL}(n+1,\mathbb{K})$ (see Notation 2.2.1). As usual, we denote by I_d the ideal generated by the set $\mathcal{B}_1 = \{m_1,\dots,m_{\mu_d}\}$ of all monomial invariants of $\overline{G}$ of degree d and by $X_d \subset \mathbb{P}^{\mu_d-1}$ the associated $\overline{G}-$variety with group $G \subset \mathrm{GL}(n+1,\mathbb{K})$.

The *Hilbert function of* X_d is defined as the Hilbert function of its coordinate ring $A(X_d)$:

$$\begin{array}{rccl} \mathrm{HF}(A(X_d),\bullet): & \mathbb{Z}_{\geq 0} & \longrightarrow & \mathbb{Z}_{\geq 0} \\ & t & \longrightarrow & \mathrm{HF}(A(X_d),t) := \dim_{\mathbb{K}} A(X_d)_t \end{array}$$

and, analogously, the Hilbert series of X_d is the formal series:

$$\mathrm{HS}(A(X_d),z) = \sum_{t\geq 0} \mathrm{HF}(A(X_d),t)z^t.$$

Both numerical invariants codify geometrical information of X_d. For instance, for t large enough, the Hilbert function $\mathrm{HF}(A(X_d),t)$ is a polynomial in $\mathbb{Q}[t]$ of degree $\dim(X_d) = n$, called the *Hilbert polynomial* of X_d and denoted $\mathrm{HP}(A(X_d),t)$. The *degree* $\deg(X_d)$ of X_d is defined algebraically as $n!$ times the leading coefficient of $\mathrm{HP}(A(X_d),t)$. It corresponds geometrically to the number of points of intersection of X_d with a sufficiently general linear subspace of $\mathbb{P}^{\mu_d-1}$ of dimension $\mu_d - n - 1$ (see, for instance, [43, Chapter I §7]). In general terms, we have:

Proposition 3.1.1. *There exists a polynomial* $Q_{A(X_d)}(z) = \sum_{i=0}^{s} h_i z^i \in \mathbb{Z}[z]$ *of finite degree* s *satisfying the following conditions:*

(i) $h_0 = 1$, $h_j \geq 0$ *and* $\sum_{i=0}^{j} h_i \leq \sum_{i=0}^{j} h_{s-i}$ *for all* $j = 0,\dots,s$.

(ii) $Q_{A(X_d)}(1) = \deg(X_d)$.

(iii) The Hilbert series of $A(X_d)$ *is*

$$\mathrm{HS}(A(X_d), z) = \frac{Q_{A(X_d)}(z)}{(1-z)^{n+1}}.$$

The sequence $(h_0, \ldots, h_s)$ *is called the* $h-$vector *of* $A(X_d)$.

Proof. (i) Since $A(X_d)$ is a CM domain (Theorem 2.2.18), we apply [9, Corollary 4.1.10 and Theorem 4.4.9] obtaining the first assertion.

(ii) and (iii) They follow from [9, Corollaries 4.1.8 and 4.1.9]. □

Since $A(X_d) \cong R^{\overline{G}}$ (Theorem 2.2.18), $\mathrm{HF}(A(X_d), t)$ and $\mathrm{HS}(A(X_d), z)$ can be interpreted from the invariant theory point of view. On one hand, $\mathrm{HS}(A(X_d), z)$ can be described by means of the Molien series of $R^{\overline{G}}$, which is expressed only in terms of the finite abelian group $\overline{G} \subset \mathrm{GL}(n+1, \mathbb{K})$. On the other hand, in Proposition 2.2.13, we have seen that $x_0^d, \ldots, x_n^d$ is a h.s.o.p. for $R^{\overline{G}}$, also called a *set of primary invariants* of $\overline{G} \subset \mathrm{GL}(n+1, \mathbb{K})$. By Theorem 2.2.14, $R^{\overline{G}}$ is a free $\mathbb{K}[x_0^d, \ldots, x_n^d]-$module with Hironaka decomposition

$$R^{\overline{G}} = \bigoplus_{i=0}^{D} \theta_i \mathbb{K}[x_0^d, \ldots, x_n^d],$$

where $\theta_0, \theta_1, \ldots, \theta_D$, called a *set of secondary invariants of* $\overline{G}$, are the monomial invariants of degree at most nd representing the monomial $\mathbb{K}-$basis of the quotient algebra

$$R^{\overline{G}}/(x_0^d, \ldots, x_n^d)R^{\overline{G}}.$$

Any set of primary and secondary invariants of $\overline{G}$ determines the Hilbert series of $A(X_d)$.

Proposition 3.1.2. *(i)* $\mathrm{HF}(A(X_d), t) = \dim_{\mathbb{K}} R_t^{\overline{G}} = |\mathcal{B}_t|$.

(ii) $\mathrm{HS}(A(X_d), z^d) = \mathrm{HS}(R^{\overline{G}}, z)$, *where*

$$\mathrm{HS}(R^{\overline{G}}, z) = \frac{1}{|\overline{G}|} \sum_{g \in \overline{G}} \frac{1}{\det(\mathrm{Id} - zg)}$$

is the Molien series of $\overline{G}$.

(iii) *Let* $\delta_1, \ldots, \delta_n$ *be the multiplicities of the sequence* $(\deg(\theta_1), \deg(\theta_2), \ldots, \deg(\theta_D))$ *of degrees of the secondary invariants* $\theta_1, \ldots, \theta_D$ *of* $\overline{G}$. *Then,*

$$\mathrm{HS}(A(X_d), z) = \frac{\delta_n z^n + \delta_{n-1} z^{n-1} + \cdots + \delta_1 z + 1}{(1-z)^{n+1}}.$$

In particular, $\deg(Q_{A(X_d)}(z)) \leq n$.

(iv) $\deg(X_d) = D + 1 = \frac{d^{n+1}}{|\overline{G}|}$.

Proof. (i) and (ii) They follow from Lemma 2.2.4 and Theorem 1.3.6.
(iii) We denote $A := \mathbb{K}[x_0^d, \ldots, x_n^d]$. Then, A is a polynomial ring and

$$R^{\overline{G}} = \bigoplus_{i=0}^{D} \theta_i A.$$

The Hilbert series of A and $\theta_i A$ equal to, respectively,

$$\mathrm{HS}(A, z) = \frac{1}{(1-z^d)^{n+1}} \text{ and } \mathrm{HS}(\theta_i A, z) = \frac{z^{\deg(\theta_i)}}{(1-z^d)^n}.$$

Since the Hilbert series is additive with respect to direct sums, we obtain

$$\mathrm{HS}(R^{\overline{G}}, z) = \sum_{i=1}^{D} \frac{z^{\deg(\theta_i)}}{(1-z^d)} = \frac{\delta_n z^{nd} + \cdots + \delta_1 z^d + 1}{(1-z^d)^{n+1}}.$$

The statement now follows from the fact $\mathrm{HS}(A(X_d), z^d) = \mathrm{HS}(R^{\overline{G}}, z)$.
(iv) With the notation of Proposition 3.1.1, by (iii) we have

$$\deg(X_d) = Q_{A(X_d)}(1) = \sum_{i=0}^{n} \delta_i.$$

From (ii) and (iii) we obtain the equality

$$\frac{\delta_n z^{nd} + \cdots + \delta_1 z^d + 1}{(1-z^d)^{n+1}} = \frac{1}{|\overline{G}|} \sum_{g \in \overline{G}} \frac{1}{\det(\mathrm{Id} - zg)},$$

and, hence,

$$Q_{A(X_d)}(z^d) = \frac{1}{|\overline{G}|}\sum_{g\in\overline{G}} \frac{(1-z^d)^{n+1}}{\det(\mathrm{Id}-zg)} = \frac{1}{|\overline{G}|}(\sum_{i=0}^{d-1} z^i)^{n+1} + \frac{1}{d^2}\sum_{\mathrm{Id}\neq g\in\overline{G}} \frac{(1-z^d)^{n+1}}{\det(\mathrm{Id}-zg)}.$$

Taking the limit $z\to 1$ in both sides, we get $Q_{A(X_d)}(1) = \frac{d^{n+1}}{|\overline{G}|}$, so $\deg(X_d) = \frac{d^{n+1}}{|\overline{G}|}$ as required. □

The degree $\deg(X_d)$ of X_d equals to the rank of $R^{\overline{G}}$ as a free $\mathbb{K}[x_0^d,\dots,x_n^d]-$module. Proposition 3.1.2 provides us with new methods to determine $\mathrm{HS}(A(X_d),z)$. For instance, it can be deduced from the set of secondary invariants $\{\theta_1,\dots,\theta_D\}$ of $\overline{G}$. The complexity of this strategy relies on the fact that one needs first to compute $\mathcal{B}_1,\dots,\mathcal{B}_n$. We illustrate it with a few examples.

Example 3.1.3. (i) Take $G = \langle M_{3;0,1,2}\rangle \subset \mathrm{GL}(3,\mathbb{K})$ a cyclic group of order 3. The $\overline{G}-$variety $X_3 \subset \mathbb{P}^3$ is the cubic associated to the $GT-$system $I_3 = (x_0^3, x_1^3, x_2^3, x_0x_1x_2)$ with group G, known as Togliatti's example (see (1.4.1)). We have

$$\begin{aligned} \mathcal{B}_1 &= \{x_0^3, x_1^3, x_2^3, x_0x_1x_2\} \\ \mathcal{B}_2 &= \{x_0^6, x_0^3x_1^3, x_0^4x_1x_2, x_1^6, x_0x_1^4x_2, x_0^2x_1^2x_2^2, x_0^3x_2^3, x_1^3x_2^3, x_0x_1x_2^4, x_2^6\}. \end{aligned}$$

x_0^3, x_1^3, x_2^3 is a h.s.o.p of $R^{\overline{G}}$ and $\{x_0x_1x_2, x_0^2x_1^2x_2^2\}$ is a set of secondary invariants of $\overline{G}$. By Proposition 3.1.2(iii), the $h-$vector of $A(X_3)$ is $(1,1,1)$, as in Example 1.3.7(iii), we get:

$$\mathrm{HS}(A(X_3),z) = \frac{z^2+z+1}{(1-z)^3}.$$

(ii) Take $G = \langle M_{3;0,1,1}, M_{3;0,1,2}\rangle \subset \mathrm{GL}(3,\mathbb{K})$ an abelian group of order 9. A minimal set of fundamental invariants of $\overline{G}$ is

$$\mathcal{B}_1 = \{x_2^9, x_1^3x_2^6, x_1^6x_2^3, x_1^9, x_0^3x_2^6, x_0^3x_1^3x_2^3, x_0^3x_1^6, x_0^6x_2^3, x_0^6x_1^3, x_0^9\}.$$

The $\overline{G}-$variety $X_9 \subset \mathbb{P}^9$ is a $GT-$surface with group G (Proposition 2.3.1). We have

$$\begin{aligned} \mathcal{B}_2 = \{ & x_2^{18}, x_1^3x_2^{15}, x_1^6x_2^{12}, x_1^9x_2^9, x_1^{12}x_2^6, x_1^{15}x_2^3, x_1^{18}, x_0^3x_2^{15}, x_0^3x_1^3x_2^{12}, x_0^3x_1^6x_2^9, \\ & x_0^3x_1^9x_2^6, x_0^3x_1^{12}x_2^3, x_0^3x_1^{15}, x_0^6x_2^{12}, x_0^6x_1^3x_2^9, x_0^6x_1^6x_2^6, x_0^6x_1^9x_2^3, x_0^6x_1^{12}, x_0^9x_2^9, \\ & x_0^9x_1^3x_2^6, x_0^9x_1^6x_2^3, x_0^9x_1^9, x_0^{12}x_2^6, x_0^{12}x_1^3x_2^3, x_0^{12}x_1^6, x_0^{15}x_2^3, x_0^{15}x_1^3, x_0^{18}\}. \end{aligned}$$

x_0^9, x_1^9, x_2^9 is a h.s.o.p and $\{x_1^3x_2^6, x_1^6x_2^3, x_0^3x_2^6, x_0^3x_1^3x_2^3, x_0^3x_1^6, x_0^6x_2^3, x_0^6x_1^3, x_0^6x_1^6x_2^6\}$ is a set of secondary invariants of $\overline{G}$. By Proposition 3.1.2(iii), the $h-$vector of X_9 is $(1,7,1)$ and

$$\mathrm{HS}(A(X_9), z) = \frac{z^2+7z+1}{(1-z)^3}.$$

(iii) Take $G = \langle M_{6;0,1,2,3}\rangle \subset \mathrm{GL}(4,\mathbb{K})$ a cyclic group of order 6. A minimal set of fundamental invariants of $\overline{G}$ is:

$$\begin{aligned}\mathcal{B}_1 \;=\; & \{x_0^6, x_1^6, x_0x_1^4x_2, x_0^2x_1^2x_2^2, x_3x_0^2x_1^3, x_0^3x_2^3, x_3x_0^3x_1x_2, x_3^2x_0^4, x_2^6, x_3x_1x_2^4,\\ & x_3^2x_1^2x_2^2, x_3^3x_1^3, x_3^2x_0x_2^3, x_3^3x_0x_1x_2, x_3^4x_0^2, x_3^6\}.\end{aligned}$$

The $\overline{G}-$threefold $X_6 \subset \mathbb{P}^{15}$ is a $GT-$threefold with group G (Proposition 2.3.1). We have

$$\begin{aligned}\mathcal{B}_2 = \{ & x_0^{12}, x_0^6x_1^6, x_0^7x_1^4x_2, x_0^8x_1^2x_2^2, x_3x_0^8x_1^3, x_0^9x_2^3, x_3x_0^9x_1x_2, x_3^2x_0^{10}, x_1^{12}, x_0x_1^{10}x_2,\\ & x_0^2x_1^8x_2^2, x_3x_0^2x_1^9, x_0^3x_1^6x_2^3, x_3x_0^3x_1^7x_2, \underline{x_0^4x_1^4x_2^4}, \underline{x_3x_0^4x_1^5x_2^2}, x_3^2x_0^4x_1^6, \underline{x_0^5x_1^2x_2^5},\\ & \underline{x_3x_0^5x_1^3x_2^3}, \underline{x_3^2x_0^5x_1^4x_2}, x_0^6x_2^6, x_3x_0^6x_1x_2^4, x_3^2x_0^6x_1^2x_2^2, x_3^3x_0^6x_1^3, x_3^2x_0^7x_2^3,\\ & x_3^3x_0^7x_1x_2, x_3^4x_0^8, x_1^6x_2^6, x_3x_1^7x_2^4, x_3^2x_1^8x_2^2, x_3^3x_1^9, x_0x_1^4x_2^7, \underline{x_3x_0x_1^5x_2^5}, x_3^2x_0x_1^6x_2^3,\\ & x_3^3x_0x_1^7x_2, x_0^2x_1^2x_2^8, x_3x_0^2x_1^3x_2^6, \underline{x_3^2x_0^2x_1^4x_2^4}, \underline{x_3^3x_0^2x_1^5x_2^2}, x_3^4x_0^2x_1^6, x_0^3x_2^9,\\ & x_3x_0^3x_1x_2^7, \underline{x_3^2x_0^3x_1^2x_2^5}, \underline{x_3^3x_0^3x_1^3x_2^3}, \underline{x_3^4x_0^3x_1^4x_2}, x_3^2x_0^4x_2^6, \underline{x_3^3x_0^4x_1x_2^4}, \underline{x_3^4x_0^4x_1^2x_2^2},\\ & \underline{x_3^5x_0^4x_1^3}, \underline{x_3^4x_0^5x_2^3}, \underline{x_3^5x_0^5x_1x_2}, x_3^6x_0^6, x_2^{12}, x_3x_1x_2^{10}, x_3^2x_1^2x_2^8, x_3^3x_1^3x_2^6, \underline{x_3^4x_1^4x_2^4},\\ & \underline{x_3^5x_1^5x_2^2}, x_3^6x_1^6, x_3^2x_0x_2^9, x_3^3x_0x_1x_2^7, \underline{x_3^4x_0x_1^2x_2^5}, \underline{x_3^5x_0x_1^3x_2^3}, x_3^6x_0x_1^4x_2, x_3^4x_0^2x_2^6,\\ & \underline{x_3^5x_0^2x_1x_2^4}, x_3^6x_0^2x_1^2x_2^2, x_3^7x_0^2x_1^3, x_3^6x_0^3x_2^3, x_3^7x_0^3x_1x_2, x_3^8x_0^4, x_3^6x_2^6, x_3^7x_1x_2^4,\\ & x_3^8x_1^2x_2^2, x_3^9x_1^3, x_3^8x_0x_2^3, x_3^9x_0x_1x_2, x_3^{10}x_0^2, x_3^{12}\}.\end{aligned}$$

There are $\delta_1 = 12$ secondary invariants of degree 6 and there are $\delta_2 = 21$ secondary invariants of degree 12, which correspond to the underlined monomials in $\mathcal{B}_2$. By Proposition 3.1.2(iv), the sum of the sequence $h-$vector equals to $\deg(X_6) = 36$. Since $1+\delta_1+\delta_2 = 34 < 36$, applying Proposition 3.1.2(iii) we have that the $h-$vector of $A(X_6)$ is $(1,12,21,2)$ and

$$\mathrm{HS}(A(X_6), z) = \frac{2z^3+21z^2+12z+1}{(1-z)^4}.$$

Alternatively, one can proceed as in (i) and (ii), and determine the set $\mathcal{B}_3$. But it contains 226 monomial of degree 18.

The Molien series of $\overline{G}$ gives an expression of the Hilbert series of $A(X_d)$ in terms of the elements of the group $\overline{G}$. Precisely,

$$\mathrm{HS}(A(X_d), z^d) = \frac{1}{|\overline{G}|}\sum_{g\in\overline{G}} \frac{1}{\det(\mathrm{Id} - zg)}.$$

This formula is, however, far from the reduced form in Proposition 3.1.2:

$$\frac{h_n z^n + \cdots + h_1 z + 1}{(1-z)^{n+1}}.$$

To transform the Molien series as above could not always be an easy task depending on the group $\overline{G}$. This strategy appears more tractable when we deal with cyclic groups of prime order.

Proposition 3.1.4. *Let $2 < n \leq d$ be integers with d prime and $G = \langle M_{d;\alpha_0,\dots,\alpha_n}\rangle \subset \mathrm{GL}(n+1,\mathbb{K})$ a cyclic group of order d and $0 \leq \alpha_0 < \cdots < \alpha_n < d$. Then, for any $t \in \mathbb{Z}_{\geq 0}$*

$$\mathrm{HF}(A(X_d), t) = \frac{1}{d}\binom{td+n}{n} + \frac{d-1}{d}.$$

Proof. Since $\mathrm{HF}(A(X_d), t) = \mathrm{HF}(R^G, td)$, we can determined $\mathrm{HF}(A(X_d), t)$ from the expansion of the Molien series of R^G:

$$\frac{1}{d}\sum_{g\in G} \frac{1}{\det(Id - zg)} = \frac{1}{d}\sum_{k=0}^{d-1} \frac{1}{\det(\mathrm{Id} - zM_{d;k\alpha_0,\dots,k\alpha_n})}.$$

Fix $1 \leq k \leq d-1$, therefore

$$\frac{1}{\det(\mathrm{Id} - zM_{d;k\alpha_0,\dots,k\alpha_n})} = \frac{1}{(1-z)(1-e^{k\alpha_0}z)\cdots(1-e^{k\alpha_n}z)}.$$

Since d is prime and the exponents $0 \leq \alpha_0 < \cdots < \alpha_n < d$, the classes of $k\alpha_0, \dots, k\alpha_n \bmod d$ are represented by two by two distinct integers in the set $\{0,\dots,d-1\}$. Using the factorization $(1-z^d) = \prod_{j=0}^{d-1}(1-e^j z)$, we can write it as:

$$\frac{1}{(1-z^d)} \prod_{\substack{j\neq k\alpha_i \bmod d\\ i=0,\dots,n}} (1-e^j z),$$

which gives us the following expression:

$$\mathrm{HS}(R^G,z) = \frac{1}{d}\left(\frac{1}{(1-z)^{n+1}} + \frac{1}{1-z^d}\sum_{k=1}^{d-1}\prod_{\substack{j\neq k\alpha_i \bmod d\\ i=0,\dots,n}}(1-e^j z)\right)$$

$$= \frac{1}{d}\left(\sum_{i=0}^{\infty}(-1)^i\binom{-(n+1)}{i}z^i + \sum_{i=0}^{\infty}\Big(\sum_{k=1}^{d-1}\prod_{\substack{j\neq k\alpha_i \bmod d\\ i=0,\dots,n}}(1-e^j z)\Big)z^{id}\right).$$

Therefore, $\mathrm{HF}(A(X_d),t)$ is the coefficient of the tdth term of the expansion of $\mathrm{HS}(R^G,z)$. The expansion of the first summand at z^{td} provides $\binom{td+n}{n}$. For each $1\leq k\leq d-1$, $\prod_{\substack{j\neq k\alpha_i \bmod d\\ i=0,\dots,n}}(1-e^j z))$ is a polynomial in z of degree strictly smaller than d, so the second provides $d-1$ at z^{td}. Thus,

$$\mathrm{HF}(A(X_d),t) = \frac{1}{d}\binom{td+n}{n} + \frac{d-1}{d}.$$

□

Remark 3.1.5. Notice that any $\overline{G}-$variety with group G as in Proposition 3.1.4 is a $GT-$variety with group G (Proposition 2.3.4).

As an examples, we analyse some particular cases of Proposition 3.1.4 for small values of n. For $n=2$, the Hilbert function of any $GT-$surface X_d with group G is

$$\mathrm{HF}(A(X_d),t) = \frac{dt^2+3t+2}{2}.$$

For $n=3$ and any $GT-$threefold X_d with group G we have:

$$\mathrm{HF}(A(X_d),t) = \frac{d^2t^3+6dt^2+11t+6}{3!}$$

and, analogously, for $n=4$ and any $GT-$fourfold X_d with group G:

$$\mathrm{HF}(A(X_d),t) = \frac{d^3t^4+10d^2t^3+35dt^2+50t+24}{4!}, \quad \forall t\geq 0.$$

Next, we continue with the general case. For $n, x \in \mathbb{Z}$, we write

$$\binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!} = \sum_{i=0}^{n} s_{n,i} x^i.$$

The $s_{n,i}$ are called *Stirling numbers of the first kind.* In particular, $s_{n,0} = 0$ and $s_{n,n} = \frac{1}{n!}$. However, a closed formula for an arbitrary $s_{n,i}$ is not available. In terms of the Stirling numbers of the first kind, the Hilbert function of any $GT-$variety X_d with cyclic group G as in Proposition 3.1.4 is

$$\mathrm{HF}(A(X_d), t) = 1 + \sum_{m=1}^{n-1} \left(\sum_{k=m}^{n} s_{n,k} \binom{k}{m} n^{k-m} d^{m-1} \right) t^m + \frac{d^{n-1} t^n}{n!}.$$

From this expression and by means of the so called Eulerian numbers, which we next introduce, we can give a similar expression for the Hilbert series of X_d. Let $0 \leq k \leq m$ be integers, the *Eulerian number* $A_{m,k}$ is defined as

$$A_{m,k} = \sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} (k-j+1)^m.$$

In particular, $A_{m,0} = 1$, $A_{m,m} = 0$ and it holds that $\sum_{k=0}^{m} A_{m,k} = m!$. Moreover, we have

$$\sum_{t=0}^{\infty} t^m z^t = \frac{\displaystyle\sum_{k=0}^{m} A_{m,k} z^{m-k}}{(1-z)^{m+1}}.$$

For simplicity, we denote $\mathrm{HF}(A(X_d), t) = \sum_{m=0}^{n} C_m t^m$ where

$$C_0 = 1,\ C_n = \frac{d^{n-1}}{n!} \text{ and } C_m = \sum_{k=m}^{n} s_{n,k} \binom{k}{m} n^{k-m} d^{m-1},\ m = 1, \ldots, n-1.$$

With this notation,

$$\mathrm{HS}(A(X_d), z) = \frac{\sum_{m=0}^{n} \sum_{k=0}^{m} C_m A_{m,k} \sum_{j=0}^{n-m} (-1)^j \binom{n-m}{j} z^{m-k+j}}{(1-z)^{n+1}}.$$

We obtain the following formula for the number of secondary invariants of $\overline{G}$ of degree nd

$$\delta_n = 1 + \sum_{m=1}^{n-1} (-1)^{n-m} \left(\sum_{k=m}^{n} s_{n,k} \binom{k}{m} n^{k-m} d^{m-1} \right) + \frac{d^{n-1}}{n!}.$$

For $GT-$surfaces X_d with group G:

$$\mathrm{HS}(A(X_d), z) = \frac{\frac{d-1}{2} z^2 + \frac{d-1}{2} z + 1}{(1-z)^3},$$

for $GT-$threefolds X_d:

$$\mathrm{HS}(A(X_d), z) = \frac{\frac{d^2-6d+5}{6} z^3 + \frac{2d^2-2}{3} z^2 + \frac{d^2+6d-7}{6} z + 1}{(1-z)^4},$$

and for $GT-$fourfolds X_d, $\mathrm{HS}(A(X_d), z)$ equals to

$$\frac{\frac{d^3-10d^2+35d-26}{24} z^4 + \frac{11d^3-30d^2-35d+54}{24} z^3 + \frac{11d^3+30d^2-35d-6}{24} z^2 + \frac{d^3+10d^2+35d-46}{24} z + 1}{(1-z)^5}.$$

Let us see some particular examples, which show that Proposition 3.1.4 is not longer true for cyclic groups $G = \langle M_{d;\alpha_0,\ldots,\alpha_n} \rangle \subset \mathrm{GL}(n+1, \mathbb{K})$ of prime order d with $\alpha_i = \alpha_j$ for some $i, j \in \{0, \ldots, n\}$ with $i \neq j$.

Example 3.1.6. (i) Take $G = \langle M_{3;0,1,2} \rangle \subset \mathrm{GL}(3, \mathbb{K})$ a cyclic group of order 3. The Hilbert series of the $GT-$surface X_3 with group G is

$$\mathrm{HS}(A(X_3), z) = \frac{z^2 + z + 1}{(1-z)^3},$$

verifying as well Example 3.1.3(i).

(ii) Take $G = \langle M_{5;0,1,2,3} \rangle \subset \mathrm{GL}(4, \mathbb{K})$ a cyclic group of order 5. The Hilbert series of the $GT-$threefold X_5 with group G is

$$\mathrm{HS}(A(X_5), z) = \frac{16z^2 + 8z + 1}{(1-z)^4}.$$

We have checked that $R^{\overline{G}}$ has 8 secondary invariants of degree 5, 16 secondary invariants of degree 10 and none secondary invariants of degree 15, verifying Proposition 3.1.2.

(iii) Take $G = \langle M_{11;0,1,4,7,10} \rangle \subset \mathrm{GL}(5,\mathbb{K})$ a cyclic group of order 11. We have:

$$\mathrm{HS}(A(X_{11}),z) = \frac{20z^4+445z^3+745z^2+120z+1}{(1-z)^5}.$$

We have checked that $R^{\overline{G}}$ has a total of 1331 secondary invariants: 120 of degree 11, 745 of degree 22, 445 of degree 33 and 20 of degree 44, verifying Proposition 3.1.2.

(iv) Proposition 3.1.4 is not longer true if we drop the hypothesis on α_i. For instance, let $G = \langle M_{5;0,1,1,2} \rangle \subset \mathrm{GL}(4,\mathbb{K})$ be a cyclic group of order 5. We have checked that $R^{\overline{G}}$ has 10 secondary invariants of degree 5, 12 of degree 10 and 2 of degree 15. By Proposition 3.1.2,

$$\mathrm{HS}(R^{\overline{G}},z) = \frac{2z^3+12z^2+10z+1}{(1-z)^4},$$

which does not agree with the formula obtained for a cyclic group $G \subset \mathrm{GL}(4,\mathbb{K})$ of order 5 satisfying the hypothesis of Proposition 3.1.4:

$$\frac{16z^2+8z+1}{(1-z)^4}.$$

Combinatorics is a third perspective from which the Hilbert function and series of $\overline{G}-$varieties X_d with group G can be examined. We take

$$G = \langle M_{d_1;\alpha^1_{\sigma_1(0)},\dots,\alpha^1_{\sigma_1(n)}},\dots,M_{d_s;\alpha^s_{\sigma_s(0)},\dots,\alpha^s_{\sigma_s(n)}} \rangle \subset \mathrm{GL}(n+1,\mathbb{K}).$$

For any $t \in \mathbb{Z}_{\geq 0}$, $\mathrm{HF}(A(X_d),t)$ equals to the number of $\mathbb{Z}^{n+1}_{\geq 0}-$solutions of the associated linear systems of congruences

$$(*)_{\mathcal{A};t,r_1,\dots,r_s}: \left\{ \begin{array}{ccccccccc} y_0 & + & y_1 & + & \cdots & + & y_n & = & td \\ \alpha^1_{\sigma_1(0)}y_0 & + & \alpha^1_{\sigma_1(1)}y_1 & + & \cdots & + & \alpha^1_{\sigma_1(n)}y_n & = & r_1d_1 \\ & & & & & & & \vdots & \\ \alpha^s_{\sigma_s(0)}y_0 & + & \alpha^s_{\sigma_s(1)}y_1 & + & \cdots & + & \alpha^s_{\sigma_s(n)}y_n & = & r_sd_s \end{array} \right.$$

$$0 \leq r_i \leq \frac{\alpha_n^i td}{d_i},\ i = 1, \ldots, s \text{ (Section 2.2)}.$$

In Subsection 3.1.1, we will use this strategy to compute the Hilbert function and series of any GT−surface with finite cyclic group and, in Subsection 3.1.2, of any GT−threefold with cyclic group $G = \langle M_{d;0,1,2,3} \rangle \subset \mathrm{GL}(4, \mathbb{K})$ of order $d \geq 4$. Now, we study this point of view and we present a result due to Elashvili and Jibladze [28]. We focus on the following family of linear systems of congruences. Let $2 \leq n < d$ be integers and

$$(*)_{\mathcal{A};t,r} : \begin{cases} y_0 + y_1 + y_2 + \cdots + y_n = td \\ \quad\ \ y_1 + 2y_2 + \cdots + ny_n = rd \end{cases}$$

$$t \geq 0,\ r = 0, \ldots, nt.$$

Remark 3.1.7. $(*)_{\mathcal{A};t,r}$ is the linear system of congruences associated to any GT−variety X_d with cyclic group $G = \langle M_{d;0,1,2,\ldots,n} \rangle \subset \mathrm{GL}(n+1, \mathbb{K})$ of order $d > n$.

The systems $(*)_{\mathcal{A};t,r}$ are distinguished from the following perspective. Take $G_1 = \langle M_{d;\alpha_0,\ldots,\alpha_n} \rangle \subset \mathrm{GL}(n+1, \mathbb{K})$ a cyclic group of the same order d with $\alpha_0 < \cdots < \alpha_n$. Set $N = \alpha_n$, take new variables $z_0, \ldots, z_N$, $R' := \mathbb{K}[z_0, \ldots, z_N]$ and consider $G_2 = \langle M_{d;0,1,2,\ldots,N} \rangle \subset \mathrm{GL}(N+1, \mathbb{K})$. Therefore, any monomial $x_0^{a_0} \cdots x_n^{a_n} \in R^{G_1}$ is identified with a monomial

$$z_{\alpha_0}^{a_0} z_{\alpha_1}^{a_1} \cdots z_{\alpha_n}^{a_n} \in (R')^{G_2}.$$

In other words, monomial invariants of G_1 are monomial invariants of G_2 in the variables $z_{\alpha_0}, \ldots, z_{\alpha_n}$. Thus, if one is interested, for instance, in describing the monomial invariants of any cyclic group G_1, it suffices to focus on the family $\{\langle M_{d;0,1,2,\ldots,N} \rangle \mid 2 \leq N < d\}$ (see [42]). Moreover, equations in $(*)_{\mathcal{A};t,r}$ are more tractable than those associated to G_1.

Theorem 3.1.8. *Let $n \geq 2$ and $t \geq 0$ be integers and $G = \langle M_{n+1;0,1,\ldots,n} \rangle \subset \mathrm{GL}(n+1, \mathbb{K})$ a cyclic group of order $n+1$. Then,*

$$\mathrm{HF}(A(X_{n+1}), t) = \frac{1}{(t+1)(n+1)} \sum_{k \mid n+1} \varphi(k) \binom{\frac{n+1}{k}(t+1)}{\frac{n+1}{k}t},$$

where $\varphi(k)$ is the Euler function evaluated at k.

Proof. See [28, Theorem 1]. $\square$

With Theorem 3.1.8, we recover Proposition 3.1.4 when $d = n+1$ is prime. Indeed, making use of the basic properties of binomial coefficients, we have:

$$\begin{aligned}
\mathrm{HF}(A(X_d),t) =& \frac{1}{(t+1)(n+1)}\binom{(t+1)(n+1)}{t(n+1)} + \frac{1}{(t+1)(n+1)}n(t+1) \\
=& \frac{1}{(t+1)(n+1)}\frac{(t+1)(n+1)}{n+1}\binom{(t+1)(n+1)-1}{n} + \frac{n}{n+1} \\
=& \frac{1}{n+1}\binom{t(n+1)+n}{n} + \frac{n}{n+1}.
\end{aligned}$$

Nevertheless, Theorem 3.1.8 and the discussion accompany Proposition 3.1.4 expose the complexity of determining an explicit formula for the Hilbert function or series of a $\overline{G}-$variety X_d with group $G \subset \mathrm{GL}(n+1,\mathbb{K})$ from which more information could be inferred. For instance, manipulating the systems $(*)_{\mathcal{A};t,r}$, we obtain the Hilbert function $\mathrm{HF}(A(X_d),t)$ of a $\overline{G}-$variety X_d with group $G = \langle M_{d;0,1,2,\dots,n}\rangle \subset \mathrm{GL}(n+1,\mathbb{K})$ as a result of summing the following series:

$$2 + \sum_{r=1}^{nt-1}\left(\sum_{a_n=\max\{0,(r-(n-1)t)d\}}^{\lfloor\frac{rd}{n}\rfloor}\left(\sum_{\substack{a_{n-1}=\max\{0,(r-(n-2)t)d\\ -2a_n\}}}^{\lfloor\frac{rd-na_n}{n-1}\rfloor}\cdots\left(\sum_{\substack{a_i=\max\{0,(r-(i-1)t)d-\\ 2a_{i+1}-\cdots-(n-i+1)a_n\}}}^{\lfloor\frac{rd-(i+1)a_{i+1}-\cdots-na_n}{i}\rfloor}\right.\right.\right.$$
$$\left.\left.\left.\left(\sum_{\substack{a_3=\max\{0,(r-2t)d\\ -2a_4-\cdots-(n-2)a_n\}}}^{\lfloor\frac{rd-4a_4-\cdots-na_n}{3}\rfloor}\left(1+\lfloor\tfrac{rd-3a_3-\cdots-na_n}{2}\rfloor-\max\{0,(r-t)d-2a_3-\cdots-(n-1)a_n\}\right)\right)\right)\cdots\right)\right),$$

Dealing with the above expression in high dimensions is out of reach. We will resume this discussion in the following two subsections.

The Hilbert function and series of a $\overline{G}-$variety X_d with group $G \subset \mathrm{GL}(n+1,\mathbb{K})$ can be computed from the graded Betti numbers of $A(X_d)$. Even though the converse is not true, both numerical invariants play an important role in finding the Betti numbers or, even further, a minimal graded free resolution of $A(X_d)$ (Proposition 3.1.12). The graded Betti numbers contain significantly more information of X_d than the Hilbert function and series (see, for instance, [26]). To determine the graded Betti numbers or a minimal graded free resolution of a variety $Y \subset \mathbb{P}^r$ is a classical and difficult problem. Next, we study how these notions are related and we introduce the *Castelnuovo–Mumford regularity* of $A(X_d)$.

In Theorem 2.2.11, we have proved that the set $\mathcal{B}_1 = \{m_1, \dots, m_{\mu_d}\}$ of all monomial invariants of G of degree d minimally generates $R^{\overline{G}}$, i.e. $R^{\overline{G}} = \mathbb{K}[\mathcal{B}_1]$. Take $w_1, \dots, w_{\mu_d}$ new variables and $S = \mathbb{K}[w_1, \dots, w_{\mu_d}]$. The homogeneous ideal $\mathrm{I}(X_d)$ of X_d is the kernel of the morphism $\rho : S \longrightarrow \mathbb{K}[\mathcal{B}_1]$ defined by $\rho(w_i) = m_i,\ i = 1, \dots, \mu_d$. Concretely, $\mathrm{I}(X_d)$ is the homogeneous binomial prime ideal generated by the set of binomials:

$$\{w_{i_1} \cdots w_{i_k} - w_{j_1} \cdots w_{j_k} \in S \mid m_{i_1} \cdots m_{i_k} = m_{j_1} \cdots m_{j_k},\ k \geq 2\}$$

(see the proof of Theorem 2.2.18).

Remark 3.1.9. $\mathrm{I}(X_d)$ does not contain any linear form. Indeed, a linear form $l = \sum_{i=1}^{\mu_d} \alpha_i w_i \in \mathrm{I}(X_d)$ if and only if $\sum_{i=1}^{\mu_d} \alpha_i \rho(w_i) = \sum_{i=1}^{\mu_d} \alpha_i m_i = 0$. Since it is a trivial combination of elements in the monomial $\mathbb{K}$−basis of R_d, it follows that $\alpha_1 = \cdots = \alpha_{\mu_d} = 0$.

$A(X_d) \cong S/\,\mathrm{I}(X_d)$ is a CM ring (Theorem 2.2.18) and, hence,

$$\mathrm{pdim}(A(X_d)) = \mathrm{codim}(X_d) = \mu_d - n - 1.$$

To simplify the notation, from now on we set

$$c := \mathrm{codim}(X_d) = \mu_d - n - 1. \tag{3.1.1}$$

We consider a minimal graded free S−resolution $F_\bullet$ of $A(X_d)$:

$$F_\bullet : \quad 0 \longrightarrow F_c \longrightarrow \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow S \longrightarrow A(X_d) \longrightarrow 0,$$

where

$$F_i \cong \bigoplus_{j\geq 1}^{f_i} S(-j-i)^{\beta_{i,j}}$$

and $\beta_{i,f_i} > 0$, $1 \leq i \leq c$. The ith graded Betti number of $A(X_d)$ is $\beta_i = \beta_{i,1}, \ldots, \beta_{i,f_i}$ and it does not depend on the choice of $F_\bullet$. $A(X_d)$ is a *level* ring if

$$F_c \cong S(-f_c - c)^{\beta_{c,f_c}}.$$

If so, $\beta_c = 0, \ldots, 0, \beta_{c,f_c}$ and β_{c,f_c} is the CM–type of $A(X_d)$.

Definition 3.1.10. The *Betti diagram or Betti table* of $A(X_d)$ is a labelled table of $1 + c$ columns and $r + 1 = 1 + \max_{1 \leq i \leq c}\{f_i\}$ rows whose entries are the graded Betti numbers of $A(X_d)$:

	0	1	2	$\cdots$	c
0	1	$-$	$-$	$\cdots$	$-$
1	$-$	$\beta_{1,1}$	$\beta_{2,1}$	$\cdots$	$\beta_{c,1}$
2	$-$	$\beta_{1,2}$	$\beta_{2,2}$	$\cdots$	$\beta_{c,2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
r	$-$	$\beta_{1,r}$	$\beta_{2,r}$	$\cdots$	$\beta_{c,r}$

where '$-$' symbolises 0.

Remark 3.1.11. The ith column of the Betti table of $A(X_d)$ describes the free graded $S-$module F_i. The jth row of the Betti table of $A(X_d)$ gives partial information of the $S-$linear maps $(\delta_l)_{1 \leq l \leq c}$.

Proposition 3.1.12. *Let $F_\bullet$ be a minimal graded free S-resolution of $A(X_d)$. For each $1 \leq k \leq u := \max_{1 \leq i \leq c}\{f_i\} + c$, we set $B_k := \sum_{i+j=k}(-1)^i \beta_{i,j}$. Then,*

(i) for each $0 \leq t$,

$$\mathrm{HF}(A(X_d), t) = \sum_{k=0}^{u} B_k \binom{\mu_d - 1 + t - k}{\mu_d - 1}$$

where $\binom{\mu_d - 1 + t - k}{\mu_d - 1} = 0$ if $t < k$.

(ii) Conversely, the alternate sums B_k can be deduced inductively from $\mathrm{HF}(A(X_d), t)$ as

$$B_k = \mathrm{HF}(A(X_d), k) - \sum_{l<k} B_k \binom{\mu_d - 1 + k - l}{\mu_d - 1}.$$

Proof. See [26, Corollaries 1.2 and 1.10]. □

Definition 3.1.13. Let $F_\bullet$ be a minimal graded $S-$resolution of $A(X_d)$.

(i) $\beta_{1,1} + \cdots + \beta_{1,f_1}$ is the *minimal number of generators* of $\mathrm{I}(X_d)$.

(ii) $\min_{j \leq f_1}\{\beta_{1,j} \neq 0\}$ is the *initial degree* of $\mathrm{I}(X_d)$.

(iii) $f_1 + 1$ is the maximum of the degrees of elements in a minimal set of generators of $\mathrm{I}(X_d)$.

(iv) The *Castelnuovo–Mumford regularity* of $A(X_d)$ is defined as

$$\mathrm{reg}(A(X_d)) := \max_{1 \leq i \leq c}\{f_i\} + 1.$$

Graphically, $\mathrm{reg}(A(X_d))$ coincides with the total number of rows of the Betti diagram of $A(X_d)$. Since $A(X_d) \cong S/\,\mathrm{I}(X_d)$ is a CM ring, $\mathrm{reg}(A(X_d)) = f_c + 1$, i.e. it is measured at the end of the resolution [71, Theorem 3.11].

Example 3.1.14. (i) Take $G = \langle M_{3;0,1,2}\rangle$ a cyclic group of order 3. $\mathcal{B}_1 = \{x_0^3, x_1^3, x_2^3, x_0x_1x_2\}$ is a minimal set of fundamental invariants of $\overline{G}$. The homogeneous ideal of the cubic surface $X_3 \subset \mathbb{P}^3$ is $\mathrm{I}(X_3) = (w_4^3 - w_1w_2w_3)$. $\mathrm{reg}(A(X_3)) = 3$ and the Betti diagram of $A(X_3)$ is

	0	1
0	1	–
1	–	–
2	–	1

(ii) Take $G = \langle M_{4;0,1,2,3}\rangle \subset \mathrm{GL}(4,\mathbb{K})$ a cyclic group of order 4. A minimal set of fundamental invariants of $\overline{G}$ is

$$\mathcal{B}_1 = \{x_0^4, x_1^4, x_0x_1^2x_2, x_0^2x_2^2, x_3x_0^2x_1, x_2^4, x_3x_1x_2^2, x_3^2x_1^2, x_3^2x_0x_2, x_3^4\}.$$

The homogeneous ideal $\mathrm{I}(X_4)$ of the $GT-$threefold $X_4 \subset \mathbb{P}^9$ with group G is minimally generated by the following 12 quadrics:

$$\begin{array}{rcl@{\qquad\qquad}rcl}
w_9^2 & - & w_4w_{10} & w_5w_7 & - & w_3w_9 \\
w_8w_9 & - & w_3w_{10} & w_5w_6 & - & w_4w_7 \\
w_8^2 & - & w_2w_{10} & w_5^2 & - & w_1w_8 \\
w_4w_8 & - & w_3w_9 & w_4w_5 & - & w_1w_7 \\
w_3w_8 & - & w_2w_9 & w_4^2 & - & w_1w_6 \\
w_7^2 & - & w_6w_8 & w_3^2 & - & w_2w_4.
\end{array}$$

$\operatorname{reg}(A(X_4)) = 3$ and the Betti diagram of $A(X_4)$ is

	0	1	2	3	4	5	6
0	1	–	–	–	–	–	–
1	–	12	16	6	–	–	–
2	–	–	36	96	100	48	9

Proposition 3.1.15. *Let $F_\bullet$ be a minimal graded $S-$resolution of $A(X_d)$ with Hilbert series:*

$$\operatorname{HS}(X(A_d), z) = \frac{Q_{A(X_d)}(z)}{(1-z)^{n+1}}.$$

(i) $f_1 + 1 \leq \operatorname{reg}(A(X_d))$.

(ii) $\operatorname{reg}(A(X_d)) = \deg(Q_{A(X_d)}) + 1$. *In particular,* $\operatorname{reg}(A(X_d)) \leq n+1$ *and the $h-$vector of $A(X_d)$ is of the form* $(1, c, h_2, \ldots, h_{\operatorname{reg}(A(X_d)-1)})$.

(iii) If $2 \leq h_{\operatorname{reg}(A(X_d))-1} < h_1$, *then* $\mathrm{I}(X_d)$ *is generated by binomials of degree at most* $\operatorname{reg}(A(X_d)) - 1$.

Proof. (i) From the definition of the Castelnuovo–Mumford regularity, the inequality $f_1 + 1 \leq \operatorname{reg}(A(X_d))$ holds automatically.

(ii) For $j \in \mathbb{Z}_{\geq 0}$, $\operatorname{HS}(S(-j), z) = \frac{t^j}{(1-z)^{\mu_d}}$. Since the Hilbert series is additive on exact sequences, we obtain

$$\operatorname{HS}(A(X_d), z) = \frac{P(z)}{(1-z)^{\mu_d}},$$

where $P(z) = \sum_{i=0}^{c} \sum_{j=1}^{f_i} \beta_{i,j} t^j$ is a polynomial of degree $c + f_c$. Therefore,

$$\frac{P(z)}{(1-z)^{\mu_d}} = \frac{Q_{A(X_d)}(z)}{(1-z)^{n+1}}.$$

Hence, $Q_{A(X_d)}(z) = \frac{P(z)}{(1-z)^c}$ is a polynomial of degree f_c and we get

$$\operatorname{reg}(A(X_d)) = \deg(Q_{A(X_d)}(z)) + 1.$$

The second part of the assertion is Proposition 3.1.2(iii).

(iii) It follows from [88, Proposition 3]. □

Example 3.1.16. (i) Take $G = \langle M_{3;0,1,2}\rangle \subset \mathrm{GL}(3,\mathbb{K})$ a cyclic group of order 3 (Examples 3.1.3(i) and 3.1.14(i)). The Hilbert series of cubic surface $X_3 \subset \mathbb{P}^3$ is

$$\mathrm{HS}(A(X_3), z) = \frac{z^2+z+1}{(1-z)^3},$$

$\mathrm{reg}(A(X_3)) = 3$ and $\mathrm{I}(X_3)$ is generated by one binomial of degree 3.

(ii) Take $G = \langle M_{4;0,1,2,3}\rangle \subset \mathrm{GL}(4,\mathbb{K})$ a cyclic group of order 4 (Example 3.1.14(ii)). The Hilbert series of the GT−threefold $X_4 \subset \mathbb{P}^9$ with group G is

$$\mathrm{HS}(A(X_4), z) = \frac{9z^2+6z+1}{(1-z)^4},$$

$\mathrm{reg}(A(X_4)) = 3$ and $\mathrm{I}(X_4)$ is generated by 12 binomials of degree 2.

Later in Section 3.3, we will investigate the canonical module ω_{X_d} of the coordinate ring $A(X_d)$ of any $\overline{G}$−variety X_d with group $G \subset \mathrm{GL}(n+1,\mathbb{K})$. We will see a combinatoric interpretation of the last component of the h−vector of $A(X_d)$ and we will characterize its Castelnuovo–Mumford regularity. Both results will allow us to say more about the Hilbert series and the minimal graded free S−resolution of $A(X_d)$.

3.1.1 The Hilbert function of GT-surfaces

Through this subsection $R = \mathbb{K}[x_0, x_1, x_2]$. We restrict our attention to GT−surfaces with cyclic group $G = \langle M_{d;0,\alpha_1,\alpha_2}\rangle \subset \mathrm{GL}(3,\mathbb{K})$ of order $d \geq 2$ and $0 < \alpha_1 < \alpha_2 < d$. The ideal $I_d \subset R$ generated by the minimal set $\mathcal{B}_1$ of fundamental monomial invariants of $\overline{G}$ is a GT−system with group G (Theorem 2.2.11 and Proposition 2.3.4). Our main goal is to compute the Hilbert function and series of $A(X_d)$ in terms of α_1, α_2 and d (Theorem 3.1.21). This leads us to prove that $A(X_d)$ is a level ring and to determine the CM–type and the Castelnuovo–Mumford regularity of $A(X_d)$.

To begin with, we recall that $\mathrm{HF}(A(X_d), t)$ coincides with the number of $\mathbb{Z}^3_{\geq 0}$−solutions of the systems of congruences:

$$(*)_{\mathcal{A};t,r} = \left\{ \begin{array}{ccccccc} y_0 & + & y_1 & + & y_2 & = & td \\ & & \alpha_1 y_1 & + & \alpha_2 y_2 & = & rd \end{array} \right.$$

with $r = 0, \ldots, \alpha_2 t$.

In Propositions 3.1.2 and 3.1.15, we have seen that the Hilbert series of X_d is of the form:

$$\mathrm{HS}(A(X_d), z) = \frac{\delta_2 z^2 + \delta_1 z + 1}{(1-z)^3},$$

where $\delta_1 = \mathrm{codim}(X_d) = \mu_d - 3$ and δ_2 is the number of monomial invariants $x_0^{a_0} x_1^{a_1} x_2^{a_2}$ of G of degree $2d$ such that $a_0, a_1, a_2 < d$.

Through this subsection, we will use the following notation.

Notation 3.1.17. If $z, z' \in \mathbb{Z}$, we write $\mathrm{GCD}(z, z')$ simply by (z, z'). We denote

$$\alpha_1' = \frac{\alpha_1}{(\alpha_1, d)},\ \alpha_2' = \frac{\alpha_2}{(\alpha_2, d)},\ d' = \frac{d}{(\alpha_1, d)},\ d'' = \frac{d}{(\alpha_2, d)}.$$

From now onwards, λ and μ are the uniquely determined integers such that $0 < \lambda \leq d'$ and $\alpha_2' = \lambda \alpha_1' + \mu d'$.

We have the following.

Lemma 3.1.18. $\mathrm{HF}(X_d, t)$ *equals the number of* $\mathbb{Z}^3_{\geq 0}-$*solutions* (y_0, y_1, y_2) *of the systems:*

$$(**)_{t,r} = \left\{ \begin{array}{ccccccc} y_0 & + & y_1 & + & \frac{y_2}{(\alpha_1,d)} & = & td \\ & & y_1 & + & \lambda\frac{y_2}{(\alpha_1,d)} & = & rd' \end{array} \right. \quad , r = 0, \ldots, t\lambda.$$

satisfying $y_1 + y_2 \leq td$.

Proof. Let $(y_0, y_1, y_2) \in \mathbb{Z}^3_{\geq 0}$ be a solution of a system $(*)_{\mathcal{A};t,r}$ for some $r \in \{0, \ldots, \alpha_2 t\}$. We observe that (α_1, d) divides y_2. We have

$$\alpha_1 y_1 + \alpha_2 y_2 = \alpha_1 y_1 + \alpha_1' \lambda y_2 + \mu d' y_2 = rd.$$

We write $y_2' = \frac{y_2}{(\alpha_1,d)}$, hence,

$$\alpha_1' y_1 + \alpha_1' \lambda y_2' = (r - \mu y_2') d'.$$

This implies that α_1' divides $(r - \mu y_2')$. Then, we obtain $y_1 + \lambda y_2' = r'd'$, where necessarily $0 \leq r' \leq \lambda t$. Thus, (y_0, y_1, y_2) induces a unique solution of the systems $(**)_{t,r}$ such that $y_1 + y_2 \leq td$.

Conversely, let $(y_0, y_1, y_2') \in \mathbb{Z}^3_{\geq 0}$ be a solution of $(**)_{t,r}$ for some $r \in \{0, \dots, t\lambda\}$ such that $y_1 + (\alpha_1, d)y_2' \leq td$. Since $y_1 + \lambda y_2' = rd'$, $\alpha_1 y_1 + \alpha_1 \lambda y_2' = r\alpha_1' d$. Using that $\alpha_1'\lambda = \alpha_2 - \mu d'$, we obtain

$$\alpha_1 y_1 + \alpha_1 \lambda y_2' = \alpha_1 y_1 + \alpha_2(\alpha_1, d)y_2' - \mu d'(\alpha_1, d)y_2' = r\alpha_1' d,$$

hence, $\alpha_1 y_1 + \alpha_2(\alpha_1, d)y_2' = (r\alpha_1' + \mu y_2')d$. We set $y_2 := (\alpha_1, d)y_2'$. Then, (y_0, y_1, y_2) verifies $\alpha_1 y_1 + \alpha_2 y_2 = r'd$ for some $0 \leq r' \leq tb$. So, (y_0, y_1, y_2) induces a unique solution of some system $(*)_{\mathcal{A};t,r}$ if and only if $y_1 + y_2 \leq td$. $\square$

Example 3.1.19. (i) Take $G = \langle M_{8;0,3,5}\rangle \subset \mathrm{GL}(3, \mathbb{K})$ a cyclic group of order 8. In this case, $\lambda = 7$ and $\mu = -2$. We observe that $M_{8;0,3,5} = M^3_{8;0,1,7}$ and $M_{8;0,1,7} = M^3_{8;0,3,5}$, so $G = \langle M_{8;0,1,7}\rangle$. The systems $(*)_{\mathcal{A};1,r}$ and $(**)_{1,r}$ give the same minimal set of fundamental invariants:

$$\mathcal{B}_1 \;=\; \{x_0^8, x_0^6x_1x_2, x_0^4x_1^2x_2^2, x_1^8, x_0^2x_1^3x_2^3, x_1^4x_2^4, x_2^8\}.$$

(ii) Take $G = \langle M_{6;0,2,3}\rangle \subset \mathrm{GL}(3, \mathbb{K})$ a cyclic group of order 6. A minimal set of fundamental invariants of $\overline{G}$ is:

$$\mathcal{B}_1 \;=\; \{x_0^6, x_0^3x_1^3, x_0^4x_2^2, x_1^6, x_0x_1^3x_2^2, x_0^2x_2^4, x_2^6\}.$$

In this case, $\lambda = 2$ and $d' = 3$. The $\mathbb{Z}^3_{\geq 0}$–solutions (y_0, y_1, y_2) of the systems

$$(**)_{1,r} = \left\{ \begin{array}{ccccccc} y_0 & + & y_1 & + & y_2 & = & 6 \\ & & y_1 & + & 3y_2 & = & 3r \end{array} \right. , \qquad r = 0, 1, 2, 3,$$

are: $(6,0,0)$, $(3,3,0)$, $(5,0,1)$, $(0,6,0)$, $(2,3,1)$, $(4,0,2)$, $(1,3,2)$, $(3,0,3)$, $(0,3,3)$, $(2,0,4)$, $(1,0,5)$ and $(0,0,6)$. Among them, only the following seven vectors $(6,0,0)$, $(3,3,0)$, $(5,0,1)$, $(0,6,0)$, $(2,3,1)$, $(4,0,2)$, $(3,0,3)$ satisfy the condition $y_1 + 2y_2 \leq 6$.

Remark 3.1.20. (i) Assume $(\alpha_1, d) = 1$ and write $\xi = e^{\alpha_1}$. Therefore, Lemma 3.1.18 says that $\mathrm{HF}(A(X_d), t)$ coincides with the Hilbert function of the GT–surface with group $\langle M_{d;0,1,\lambda}\rangle \subset \mathrm{GL}(3, \mathbb{K})$. From this, we can suppose either $(\alpha_1, d) = 1$ or $(\alpha_1, d), (\alpha_2, d) > 1$. In both cases $1 \neq \lambda$.

(ii) If $(\alpha_1, d), (\alpha_2, d) > 1$ with $(\alpha_1, d) < (\alpha_2, d)$, then we can choose integers μ and λ with $(\alpha_1, d) < \lambda \leq d'$ and such that $\alpha_2 = \lambda\alpha_1' + \mu d'$.

Our main result is the following.

Theorem 3.1.21. *Set* $\theta(\alpha_1,\alpha_2,d) := (\alpha_1,d)+(\lambda,d')+(\lambda-(\alpha_1,d),d')$. *Then,*

(i) $\mathrm{HF}(X_d,t) = \frac{d}{2}t^2 + \frac{1}{2}\theta(\alpha_1,\alpha_2,d)t + 1.$

(ii) $$\mathrm{HS}(S_d,z) = \frac{\frac{d-\theta(\alpha_1,\alpha_2,d)+2}{2}z^2 + \frac{d+\theta(\alpha_1,\alpha_2,d)-4}{2}z + 1}{(1-z)^3}.$$

Proof. (i) By Lemma 3.1.18, it suffices to count the number of $\mathbb{Z}^3_{\geq 0}$−solutions (y_0,y_1,y_2) of the systems

$$(**)_{t,r} : \left\{ \begin{array}{ccccccc} y_0 & + & y_1 & + & \frac{y_2}{(\alpha_1,d)} & = & td \\ & & y_1 & + & \lambda\frac{y_2}{(\alpha_1,d)} & = & rd' \end{array} \right. \qquad r = 0,\ldots,t\lambda$$

satisfying $y_1+y_2 \leq td$. Without loss of generality, we may assume that $(\alpha_1,d) < (\alpha_2,d)$. Fix $r \in \{0,\ldots,t\lambda\}$. The $\mathbb{Z}^3_{\geq 0}$−solutions of $(**)_{t,r}$ are determined by r and

$$\max\left\{0, \lceil\frac{(r-t(\alpha_1,d))d'}{\lambda-1}\rceil\right\} \leq y_2 \leq \lfloor\frac{rd'}{\lambda}\rfloor,$$

and they are of the form $(td-rd'+(\lambda-1)y_2, rd'-\lambda y_2, y_2)$. Imposing $y_1 + (\alpha_1,d)y_2 \leq td$, we obtain that $rd'-\lambda y_2 \leq td-(\alpha_1,d)y_2$ if and only if $(\lambda-(\alpha_1,d))y_2 \geq rd'-td$. Fixed $0 \leq r \leq t\lambda$, counting the number of y_2 in the range $\max\left\{0, \lceil\frac{(r-(\alpha_1,d)t)d')}{\lambda-(\alpha_1,d)}\rceil\right\} \leq y_2 \leq \lfloor\frac{rd'}{\lambda}\rfloor$ we get:

$$\mathrm{HF}(A(X_d),t) = 2 + \sum_{r=1}^{t\lambda-1}\left(\lfloor\frac{rd'}{\lambda}\rfloor+1\right) - \sum_{r=t(\alpha_1,d)+1}^{t\lambda-1}\left(\lceil\frac{(r-(\alpha_1,d)t)d'}{\lambda-(\alpha_1,d)}\rceil+1\right).$$

Given two positive integers m and n, it holds that

$$\sum_{i=1}^{n-1}\lfloor\frac{im}{n}\rfloor = \frac{(m-1)(n-1)+(m,n)-1}{2}.$$

Therefore,

$$\mathrm{HF}(A(X_d),t) = 2 + t\lambda - 1 + \frac{(td'-1)(t\lambda-1)+t(d',\lambda)-1}{2}$$
$$- \left(\sum_{r=1}^{t(\lambda-(\alpha_1,d))-1} \lceil \frac{rd't}{(\lambda-(\alpha_1,d))t} \rceil \right) - (t(\lambda-(\alpha_1,d))-1).$$

We observe that $\lceil \frac{rd't}{(\lambda-(\alpha_1,d))t} \rceil = \lfloor \frac{rd't}{(\lambda-(\alpha_1,d))t} \rfloor$ if rd' is a multiple of $\lambda-(\alpha_1,d)$, otherwise $\lceil \frac{rd't}{(\lambda-(\alpha_1,d))t} \rceil = \lfloor \frac{rd't}{(\lambda-(\alpha_1,d))t} \rfloor + 1$. We set

$$\mathcal{S} = \{r \in \mathbb{Z} \mid 1 \le r \le t(\lambda-(\alpha_1,d)-1) \text{ and } t(\lambda-(\alpha_1,d)) \text{ divides } rd't\}.$$

An integer $r \in \mathcal{S}$ if and only if rd' is a multiple of $\mathrm{LCM}(d', \lambda-(\alpha_1,d)) = \frac{d'(\lambda-(\alpha_1,d))}{(\lambda-(\alpha_1,d),d')}$. We determine the multiples of $\frac{(\lambda-(\alpha_1,d))}{(\lambda-(\alpha_1,d),d')}$ in the set $\{1,\dots,t(\lambda-(\alpha_1,d))-1\}$. Hence, $|\mathcal{S}| = t(\lambda-(\alpha_1,d),d')-1$ and we have:

$$\sum_{r=1}^{t(\lambda-(\alpha_1,d))-1} \lceil \frac{rd't}{(\lambda-(\alpha_1,d))t} \rceil = \frac{(td'-1)(t\lambda-t(\alpha_1,d)-1)}{2} + t(\lambda-(\alpha_1,d))-1-t(d',\lambda-(\alpha_1,d)).$$

We check that

$$\mathrm{HF}(A(X_d),t) = \frac{d}{2}t^2 + \frac{((\alpha_1,d)+(d',\lambda)+(d',\lambda-(\alpha_1,d)))}{2}t + 1. \qquad (3.1.2)$$

(ii) By definition, $\mathrm{HS}(A(X_d),z) = \sum_{t\ge 0} \mathrm{HF}(A(X_d),t)z^t$. Thus,

$$\begin{aligned}
\mathrm{HS}(A(X_d),z) &= \sum_{t\ge0} \frac{d}{2}t^2z^t + \sum_{t\ge0} \frac{\theta(\alpha_1,\alpha_2,d)}{2}tz^t + \sum_{t\ge0} z^t \\
&= \frac{\frac{d}{2}z(z+1)}{(1-z)^3} + \frac{\frac{\theta(\alpha_1,\alpha_2,d)}{2}z}{(1-z)^2} + \frac{1}{1-z}2 \\
&= \frac{\frac{d-\theta(\alpha_1,\alpha_2,d)+2}{2}z^2 + \frac{d+\theta(\alpha_1,\alpha_2,d)-4}{2}z + 1}{(1-z)^3}.
\end{aligned}$$

□

Corollary 3.1.22. *(i) The Castelnuovo–Mumford regularity of X_d is* 3. *Hence,* $\mathrm{I}(X_d)$ *is minimally generated by binomials of degree at most* 3.
(ii) $A(X_d)$ *is a level ring of CM–type* $\frac{d-\theta(\alpha_1,\alpha_2,d)+2}{2}$.

Proof. (i) Since I_d is a $GT-$system with group $G = \langle M_{d;0,\alpha_1,\alpha_2}\rangle \subset \mathrm{GL}(3,\mathbb{K})$, it holds

$$\mu_d = \frac{d+\theta(\alpha_1,\alpha_2,d)+2}{2} \leq d+1$$

which implies that $\delta_1 = \mu_d - 3 \leq d-2$. On the other hand, $\deg(A(X_d)) = \delta_2+\delta_1+1 = d$, so we have that $\delta_2 = d-\delta_1-1 \geq d-1-d+2 = 1$.

(ii) Any CM homogeneous domain over $\mathbb{K}$ with Castelnuovo–Mumford regularity less or equal than 3 is a level ring [88, Corollary 3.11]. Then, the cth graded Betti number of $A(X_d)$ is the CM–type of $A(X_d)$. Let $(1,c,h_2)$ be the $h-$vector of $A(X_d)$. Computing $\mathrm{HS}(A(X_d),z)$ from a minimal graded free $S-$resolution of $A(X_d)$ as in the proof of Proposition 3.1.15, we obtain that h_2 is the CM–type of $A(X_d)$. □

We look at the function $\theta(\alpha_1,\alpha_2,d) = (\alpha_1,d)+(\lambda,d')+(\lambda-(\alpha_1,d),d')$. It follows from the definition itself that $\theta(\alpha_1,\alpha_2,d) = 3$ if and only if $1 = (\alpha_1,d) = (\alpha_2,d) = (\alpha_2-1,d)$. If $\theta(\alpha_1,\alpha_2,d)=3$, then we have $c = h_2 = \frac{d-1}{2}$. and the Hilbert function and series of $A(X_d)$ appears plainly as

$$\mathrm{HF}(A(X_d),t) = \frac{dt^2+3t+2}{2}$$

$$\mathrm{HS}(A(X_d),z) = \frac{\frac{d-1}{2}z^2+\frac{d-1}{2}z+1}{(1-z)^3}.$$

Example 3.1.23. Let $0<\alpha_1<\alpha_2<d$ be integers with d prime and $G = \langle M_{d;0,\alpha_1,\alpha_2}\rangle \subset \mathrm{GL}(3,\mathbb{K})$ a cyclic group of order d. Then $\theta(\alpha_1,\alpha_2,d)=3$ and we are in the hypothesis of Proposition 3.1.4. So we can check that

$$\mathrm{HF}(A(X_d),t) = \frac{1}{d}\binom{td+n}{n}+\frac{d-1}{d} = \frac{dt^2+3t+2}{2}$$

$$\mathrm{HS}(A(X_d),z) = \frac{\frac{d-1}{2}z^2+\frac{d-1}{2}z+1}{(1-z)^3}.$$

Otherwise $\theta(\alpha_1, \alpha_2, d) > 3$, then $\delta_2 < \delta_1$ and we obtain $2 \leq \delta_2$. Thus, we are in the hypothesis of Proposition 3.1.15(iii) and we obtain:

Corollary 3.1.24. *If $\theta(\alpha_1, \alpha_2, d) > 3$, then $\mathrm{I}(X_d)$ is minimally generated by homogeneous binomials of degree 2.*

Proof. Since $\mathrm{reg}(A(X_d)) = 3$ and $\mathrm{I}(X_d)$ does not contain any linear form, the results follows form Proposition 3.1.15(iii). $\square$

For instance, we always have $\theta(\alpha_1, \alpha_2, d) > 3$ when d is even. However, if d is odd but not prime, the casuistry increases and the fact $\theta(\alpha_1, \alpha_2, d) > 3$ depends further on the values of α_1, α_2.

Example 3.1.25. (i) Take $G = \langle M_{6;0,2,3} \rangle \subset \mathrm{GL}(3, \mathbb{K})$ a cyclic group of order 6. We have that $\theta(2, 3, 6) = 4 > 3$ and the Hilbert series of $A(X_6)$ is

$$\mathrm{HS}(A(X_6), z) = \frac{z^2 + 4z + 1}{(1-z)^3}.$$

We have checked that the ideal $\mathrm{I}(X_6)$ is minimally generated by 9 binomials of degree 2.

(ii) Take $G = \langle M_{9;0,1,5} \rangle \subset \mathrm{GL}(3, \mathbb{K})$ a cyclic group of order 9. In this case, $\theta(1, 5, 9) = 3$ and the Hilbert series of $A(X_9)$ is

$$\mathrm{HS}(A(X_9), t) = \frac{4z^2 + 4z + 1}{(1-z)^3}.$$

We have checked that the ideal $\mathrm{I}(X_9)$ is minimally generated by 6 binomials of degree 2 and 4 binomials of degree 3.

3.1.2 Hilbert function of GT-threefolds

Here we extend the combinatoric approach applied in Subsection 3.1.1 to compute the Hilbert function and series of $GT-$threefolds X_d with cyclic group $G = \langle M_{d;0,1,2,3} \rangle \subset \mathrm{GL}(4, \mathbb{K})$ of order $d \geq 4$.

$\mathrm{HF}(A(X_d), t)$ is the number of $\mathbb{Z}^4_{\geq 0}-$solutions of the linear systems of congruences:

$$(*)_{\mathcal{A};t,r} : \begin{cases} y_0 + y_1 + y_2 + y_3 = td \\ \quad\quad y_1 + 2y_2 + 3y_3 = rd \end{cases} \qquad r = 0, \ldots, 3t.$$

For $r = 0$ (respectively $r = 3t$), there is only one solution $(td, 0, 0, 0)$ (respectively $(0, 0, 0, td)$). Fixed $0 < r < 3t$, we choose y_3 and y_2 as independent variables of the system $(*)_{\mathcal{A};t,r}$. Then, any $\mathbb{Z}^4_{\geq 0}$−solution of $(*)_{\mathcal{A};t,r}$ can be expressed in terms of y_3 and y_2 as a vector of the form $(f_0(r, y_3, y_2), f_1(r, y_3, y_2), y_2, y_3)$ where:

$$\begin{aligned} y_3 &\in \{\max\{0, d(r-t)\}, \ldots, \lfloor \tfrac{rd}{3} \rfloor\},\\ y_2 &\in \{\max\{0, d(r-t) - 2y_2\}, \ldots, \lfloor \tfrac{rd-3y_2}{2} \rfloor\},\\ f_1(r, y_3, y_2) &= rd - 3y_3 - 2y_2,\\ f_0(r, y_3, y_2) &= (t-r)d + y_2 + 2y_3. \end{aligned}$$

This produces the following expression for any $t \geq 0$:

$$\begin{aligned} \mathrm{HF}(A(X_d), t) \quad = \quad & 2 + \sum_{r=1}^{3t-1} \sum_{y_3=\max\{0,d(r-2t)\}}^{\lfloor \frac{rd}{3} \rfloor} \Bigg(1 + \left\lfloor \frac{rd - 3y_3}{2} \right\rfloor \\ & - \max\{0, d(r-t) - 2y_3\} \Bigg). \end{aligned} \tag{3.1.3}$$

Our result is the following.

Theorem 3.1.26. *Let $d \geq 4$ be an integer and set $\theta(1, 2, 3, d) := 21 + 12(2, d) + 12(3, d) - (3, d)^2$. Then,*

$$\mathrm{HF}(A(X_d), t) = \frac{d^2}{6} t^3 + dt^2 + \frac{\theta(1, 2, 3, d)}{24} t + 1.$$

In particular,

$$\mathrm{HF}(A(X_d), t) = \begin{cases} \dfrac{d^2}{6} t^3 + dt^2 + \dfrac{11}{6} t + 1 & \textit{if} \quad (2, d) = (3, d) = 1,\\ \dfrac{d^2}{6} t^3 + dt^2 + \dfrac{5}{2} t + 1 & \textit{if} \quad (2, d) = 1 \textit{ and } (3, d) = 3,\\ \dfrac{d^2}{6} t^3 + dt^2 + \dfrac{7}{3} t + 1 & \textit{if} \quad (2, d) = 2 \textit{ and } (3, d) = 1,\\ \dfrac{d^2}{6} t^3 + dt^2 + 3t + 1 & \textit{if} \quad (2, d) = 2 \textit{ and } (3, d) = 3. \end{cases}$$

The proof is based on summing the series (3.1.3) and it is developed in a purely combinatoric way. Let us first analyse which information regarding X_d can be inferred from the above expressions. For instance, we have the following corollaries.

Corollary 3.1.27. *The Hilbert series* $\mathrm{HS}(A(X_d), z)$ *of* $A(X_d)$ *is*

$$\frac{(\frac{d^2}{6}-d+\frac{\theta(1,2,3,d)}{24}-1)z^3+(\frac{2d^2}{3}-\frac{\theta(1,2,3,d)}{12}+3)z^2+(\frac{d^2}{6}+d+\frac{\theta(1,2,3,d)}{24}-3)z+1}{(1-z)^4}.$$

In particular,

$$\mathrm{HS}(A(X_d),z)=\begin{cases} \dfrac{\frac{d^2-6d+5}{6}z^3+\frac{2d^2-2}{3}z^2+\frac{d^2+6d-7}{6}z+1}{(1-z)^4} & (2,d)=(3,d)=1,\\[2ex] \dfrac{\frac{d^2-6d+9}{6}z^3+\frac{2d^2-6}{3}z^2+\frac{d^2+6d-3}{6}z+1}{(1-z)^4} & (2,d)=1,\ (3,d)=3,\\[2ex] \dfrac{\frac{d^2-6d+8}{6}z^3+\frac{2d^2-5}{3}z^2+\frac{d^2+6d-4}{6}z+1}{(1-z)^4} & (2,d)=2,\ (3,d)=1,\\[2ex] \dfrac{\frac{d^2-6d+12}{6}z^3+\frac{2d^2-9}{3}z^2+\frac{d^2+6d}{6}z+1}{(1-z)^4} & (2,d)=2\,(3,d)=3. \end{cases}$$

Proof. It follows directly from Theorem 3.1.26. □

Corollary 3.1.28.

$$\mathrm{reg}(A(X_d))=\begin{cases} 3 & \textit{if} \quad d=4,5 \\ 4 & \textit{if} \quad 6\le d. \end{cases}$$

In particular, $A(X_d)$ *is a level ring for* $d=4,5$.

Proof. We distinguish four cases depending on the values of $(2,d)$ and $(3,d)$. If $1=(2,d)=(3,d)$, then $\delta_3=\frac{d^2-6d+5}{6}\geq 1$ if and only if $d\geq 7$. If $(2,d)=1$ and $(3,d)=3$, then $\delta_3=\frac{d^2-6d+9}{6}\geq 1$ if and only if $d\geq 9$. If $(2,d)=2$ and $(3,d)=1$, then $\delta_3=\frac{d^2-6d+8}{6}\geq 1$ if and only if $d\geq 8$. And finally, if $(2,d)=2$ and $(3,d)=3$, then $\delta_3=\frac{d^2-6d+12}{6}\geq 1$ if and only if $d\geq 6$. This proves that $\mathrm{reg}(A(X_d))=4$ for all integer $d\geq 6$.

For $d=4$, we have that $\delta_2=\frac{2d^2-5}{3}=9$ and, for $d=5$, we have that $\delta_3=\frac{2d^2-2}{3}=16$. This shows that $\mathrm{reg}(A(X_d))=3$ for $d=4$ and 5, now the result follows from [88, Corollary 3.11]. □

The Castelnuovo–Mumford regularity $\mathrm{reg}(A(X_d))$ gives an upper bound for the degrees of a minimal set of binomial generators of $\mathrm{I}(X_d)$. Thus, we can assure that $\mathrm{I}(X_d)$ is generated by binomials of degree at most 4 for $d \geq 6$. However, this bound can be improved as follows. We denote by $G_1 = \langle M_{d;0,1,2}\rangle$, $G_2 = \langle M_{d;0,1,3}\rangle$, $G_3 = \langle M_{d;0,2,3}\rangle$ and $G_4 = \langle M_{d;1,2,3}\rangle$ cyclic subgroups of $\mathrm{GL}(3,\mathbb{K})$ of order d acting on $\mathbb{K}[x_0,x_1,x_2], \mathbb{K}[x_0,x_1,x_3], \mathbb{K}[x_0,x_2,x_3]$ and $\mathbb{K}[x_1,x_2,x_3]$, respectively. Hence, $R^{\overline{G}}$ contains all monomial invariants of $\overline{G}_i$, $i = 1,2,3,4$. This implies that for any $GT-$threefold X_d with group $G = \langle M_{d;0,1,2,3}\rangle \subset \mathrm{GL}(4,\mathbb{K})$ of order $d \geq 4$ and $h-$vector $h = (1,h_1,h_2,h_3)$, we have $h_1 = \mathrm{codim}(X_d) \geq 2$. Moreover,

Corollary 3.1.29. *Let $d \geq 4$ be an integer and X_d a $GT-$threefold with cyclic group $G = \langle M_{d;0,1,2,3}\rangle \subset \mathrm{GL}(4,\mathbb{K})$ of order $d \geq 4$. Then, $\mathrm{I}(X_d)$ is minimally generated by binomials of degree at most* 3.

Proof. By Proposition 3.1.15(i), $\mathrm{I}(X_d)$ is generated by binomials of degree at most $\mathrm{reg}(A(X_d))$. For $d = 4,5$ we have that $\mathrm{reg}(A(X_d)) = 3$.

Fix $d \geq 6$ and let $h = (1,h_1,h_2,h_3)$ be the $h-$vector of $A(X_d)$. By Corollary 3.1.27, $h_3 = \frac{d^2}{6} - d + \frac{\theta(1,2,3,d)}{24} - 1$. Since $d \geq 6$, we have $0 < h_3$. The inequality $h_3 < h_1 = \frac{d^2}{6} + d + \frac{\theta(1,2,3,d)}{24} - 3$ holds for all $d \geq 2$. Therefore, we are in the hypothesis of [88, Proposition 3], we can conclude that $\mathrm{I}(X_d)$ is minimally generated by forms of degree smaller or equal than $\mathrm{reg}(A(X_d)) - 1 = 3$. $\square$

Example 3.1.30. (i) Take $G = \langle M_{4;0,1,2,3}\rangle \subset \mathrm{GL}(4,\mathbb{K})$ a cyclic group of order 4. A minimal set of fundamental invariants of $\overline{G}$ is

$$\mathcal{B}_1 = \{x_0^4, x_1^4, x_0x_1^2x_2, x_0^2x_2^2, x_3x_0^2x_1, x_2^4, x_3x_1x_2^2, x_3^2x_1^2, x_3^2x_0x_2, x_3^4\}.$$

The Hilbert series of $A(X_4)$ is

$$\mathrm{HS}(A(X_4),z) = \frac{9z^2+6z+1}{(1-z)^4},$$

$\mathrm{reg}(A(X_4)) = 3$ and $\mathrm{I}(X_4)$ is generated by 12 binomials of degree 2.

(ii) Take $G = \langle M_{5;0,1,2,3}\rangle \subset \mathrm{GL}(4,\mathbb{K})$ a cyclic group of order 5. A minimal set of fundamental invariants of $\overline{G}$ is

$$\mathcal{B}_1 = \{x_0^5, x_1^5, x_0x_1^3x_2, x_0^2x_1x_2^2, x_3x_0^2x_1^2, x_3x_0^3x_2, x_2^5, x_3x_1x_2^3, x_3^2x_1^2x_2, x_3^2x_0x_2^2, x_3^3x_0x_1, x_3^5\}.$$

The Hilbert series of $A(X_5)$ is

$$\mathrm{HS}(A(X_5),z)=\frac{16z^2+8z+1}{(1-z)^4},$$

$\mathrm{reg}(A(X_5))=3$ and $\mathrm{I}(X_5)$ is generated by 20 binomials of degree 2 and 8 binomials of degree 3.

(iii) Take $G=\langle M_{6;0,1,2,3}\rangle\subset \mathrm{GL}(4,\mathbb{K})$ a cyclic group of order 6. A minimal set of fundamental invariants of $\overline{G}$ is

$$\begin{aligned}\mathcal{B}_1 \;=\; \{&x_0^6, x_1^6, x_0x_1^4x_2, x_0^2x_1^2x_2^2, x_3x_0^2x_1^3, x_0^3x_2^3, x_3x_0^3x_1x_2, x_3^2x_0^4, x_2^6, x_3x_1x_2^4,\\ &x_3^2x_1^2x_2^2, x_3^3x_1^3, x_3^2x_0x_2^3, x_3^3x_0x_1x_2, x_3^4x_0^2, x_3^6\}.\end{aligned}$$

The Hilbert series of $A(X_6)$ is

$$\mathrm{HS}(A(X_6),z)=\frac{2z^3+21z^2+12z+1}{(1-z)^4},$$

$\mathrm{reg}(A(X_6))=4$ and $\mathrm{I}(X_6)$ is generated by 57 binomials of degree 2.

In Subsection 3.2.1, we will describe a minimal set of binomial generators of $\mathrm{I}(X_d)$ for any $GT-$threefold X_d with group $G=\langle M_{d;0,1,2,3}\rangle$ of order $d\geq 4$. In particular, we will prove that $\mathrm{I}(X_d)$ is minimally generated by binomials of degree 2 if d is even; and $\mathrm{I}(X_d)$ is minimally generated by binomials of degree 2 and 3, if d is odd. Furthermore, in Section 3.2, we will demonstrate that the homogeneous ideal of any $\overline{G}-$variety with finite abelian group $G\subset \mathrm{GL}(n+1,\mathbb{K})$ is generated by binomials of degree at most 3. To achieve our goal, we will expound a combinatorial approach based on *zero-sums over abelian groups*, far from the strategies described in this section.

The rest of this subsection is devoted to prove Theorem 3.1.26. We fix an integer $d\geq 4$ and a $GT-$threefold X_d with group $G=\langle M_{d;0,1,2,3}\rangle\subset \mathrm{GL}(4,\mathbb{K})$ of order d. As we have seen at the beginning of this subsection (see (3.1.3)):

$$\begin{aligned}\mathrm{HF}(A(X_d),t) \;=\; 2+\sum_{r=1}^{3t-1}\sum_{y_3=\max\{0,d(r-2t)\}}^{\lfloor \frac{rd}{3}\rfloor}&\Bigg(1+\left\lfloor\frac{rd-3y_3}{2}\right\rfloor\\ &-\max\{0,d(r-t)-2y_3\}\Bigg).\end{aligned}$$

So, it suffices to see that the sum of the above series coincides with

$$\frac{d^2}{6}t^3 + dt^2 + \frac{\theta(1,2,3,d)}{24}t + 1,$$

where $\theta(1,2,3,d) = 21 + 12(2,d) + 12(3,d) - (3,d)^2$.

We first observe that the series (3.1.3) can be rewritten as

$2 + (A) - (B) - (C) :=$

$$2+\sum_{r=1}^{3t-1}\sum_{\gamma=0}^{\lfloor\frac{rd}{3}\rfloor}\left(\left\lfloor\frac{rd-3\gamma}{2}\right\rfloor+1\right)-\sum_{r=1}^{2t-1}\sum_{\gamma=0}^{\lfloor\frac{rd}{2}\rfloor}(dr-2\gamma)-\sum_{r=1}^{t-1}\sum_{\gamma=0}^{dr-1}\left(\left\lfloor\frac{\gamma-rd}{2}\right\rfloor+1\right).$$

We treat separately each series (A), (B) and (C). Let us start with some notation and a couple of technical lemmas needed in the sequel. For $a, b \in \mathbb{Z}$, we denote by $\overline{a}^b$ the unique integer in $\{0, \dots, b-1\}$ such that $a \equiv \overline{a}^b \bmod b$. In particular, it holds $\lfloor\frac{a}{b}\rfloor = \frac{1}{b}(a - \overline{a}^b)$.

Lemma 3.1.31. *Given $b, k, t, d \in \mathbb{Z}_{\geq 0}$, we have:*

$$\sum_{r=1}^{bt-1}\left(\overline{rd}^b\right)^k = t(b,d)^{k+1}\sum_{i=0}^{\frac{b}{(b,d)}-1} i^k.$$

For $k = 1$, we have:

$$\sum_{r=1}^{bt-1}\overline{rd}^b = \frac{tb(b-(b,d))}{2}, \quad \textit{and} \quad \sum_{r=1}^{bt-1}\lfloor\frac{rd}{b}\rfloor = \frac{(tb-1)(dt-1)+t(d,b)-1}{2}.$$

Proof. The first equality follows from

$$\sum_{r=1}^{bt}\left(\overline{rd}^b\right)^k = t\sum_{r=1}^{b}\left(\overline{rd}^b\right)^k = t(b,d)\sum_{i=0}^{\frac{b}{(b,d)}-1}((b,d)i)^k.$$

We observe that $\displaystyle\sum_{r=1}^{bt-1}\frac{rd}{b} = \frac{1}{b}\sum_{r=1}^{bt-1}\left(rd-\overline{rd}^b\right)$ and we get the second identity.

$\square$

Lemma 3.1.32. *Given $d, t \in \mathbb{Z}_{\geq 0}$, we have:*
(i) $\sum_{r=1}^{2t-1}(\overline{rd}^2)^2 = (2-(d,2))t$ *and* $\sum_{r=1}^{3t-1}(\overline{rd}^3)^2 = \frac{(3-(3,d))(6-(3,d))t}{2}$.
(ii) $\sum_{r=1}^{3t-1}\sum_{\gamma=0}^{\lfloor\frac{rd}{3}\rfloor}\overline{\gamma - rd}^2 = \frac{(3t-1)(3td+1)}{12} + \frac{(3t-1)}{3} - \frac{(d,2)(3t-1)}{4}$.

Proof. (i) It follows from Lemma 3.1.31 with $k = 2$ and taking $b = 2, 3$, respectively.

(ii) We assume that t is odd and so $3t-1$ is even. The other case follows analogously. We rewrite the sum as

$$\sum_{r=1}^{3t-1}\sum_{\gamma=0}^{\lfloor\frac{rd}{3}\rfloor}\overline{rd-\gamma}^2 = \sum_{r=1}^{3t-1}\sum_{\gamma=0}^{rd}\overline{\gamma}^2 - \sum_{r=1}^{3t-1}\sum_{\gamma=0}^{rd-\lfloor\frac{rd}{3}\rfloor-1}\overline{\gamma}^2.$$

Hence, it is enough to study each summand:

$$\begin{aligned}
\sum_{r=1}^{3t-1}\sum_{\gamma=0}^{rd}\overline{\gamma}^2 &= \sum_{r=1}^{3t-1}\lfloor\tfrac{rd+1}{2}\rfloor \\
&= \tfrac{1}{2}\sum_{r=1}^{3t-1}\left(rd+1-\overline{rd+1}^2\right) \\
&= \tfrac{(3t-1)(3td+1)}{4} - \tfrac{(d,2)(3t-1)}{4}.
\end{aligned}$$

$$\begin{aligned}
\sum_{r=1}^{3t-1}\sum_{\gamma=1}^{rd-\lfloor\frac{rd}{3}\rfloor-1}\overline{\gamma}^2 &= \sum_{r=1}^{3t-1}\lfloor\tfrac{rd-\lfloor\frac{rd}{3}\rfloor}{2}\rfloor = \sum_{r=1}^{3t-1}\lfloor\tfrac{rd+1}{3}\rfloor \\
&= \tfrac{1}{3}\sum_{r=1}^{3t-1} rd+1-\overline{rd+1}^3 \\
&= \tfrac{(3t-1)(3td+1)}{6} - \tfrac{1}{3}\sum_{r=1}^{3t-1}\overline{rd+1}^3 \\
&= \tfrac{(3t-1)(3td+1)}{6} - \tfrac{(3t-1)}{3}.
\end{aligned}$$

□

We have:

$$\begin{aligned}
(A) &= \sum_{r=1}^{3t-1}\sum_{\gamma=0}^{\lfloor\frac{rd}{3}\rfloor} 1 + \lfloor\tfrac{rd-3\gamma}{2}\rfloor \\
&= (3t-1) + \sum_{r=1}^{3t-1}\lfloor\tfrac{rd}{3}\rfloor + \tfrac{1}{2}\sum_{r=1}^{3t-1}\sum_{\gamma=0}^{\lfloor\frac{rd}{3}\rfloor}(rd-3\gamma-\overline{rd-\gamma}^2) \\
&= (3t-1) + \tfrac{3t(3t-1)d}{2} + \tfrac{1}{4}\sum_{r=1}^{3t-1}\lfloor\tfrac{rd}{3}\rfloor + \tfrac{3t(3t-1)(6t-1)d^2}{72} \\
&\quad -\tfrac{1}{12}\sum_{r=1}^{3t-1}(\overline{rd}^3)^2 - \tfrac{1}{2}\sum_{r=1}^{3t-1}\sum_{\gamma=0}^{\lfloor\frac{rd}{3}\rfloor}\overline{rd-\gamma}^2.
\end{aligned}$$

Applying Lemmas 3.1.31 and 3.1.32 to the last expression, it yields:

$$(A) \;=\; \tfrac{3d^2}{4}t^3 - \tfrac{3d(d-6)}{8}t^2 + \tfrac{d(d-18)+9(2,d)-(3,d)((3,d)-12)+27}{24}t - \tfrac{(2,d)+6}{8}.$$

Analogously, we expand the second summand (B) and we apply Lemmas 3.1.31 and 3.1.32 to obtain:

$$(B) \;=\; \frac{2d^2}{3}t^3 - \frac{d(d-1)}{2}t^2 + \frac{d(d-6)-3(2,d)+6}{12}t.$$

Finally, for the last summand we have:

$$\begin{aligned}(C) \;&=\; \textstyle\sum_{r=1}^{t-1}\sum_{\gamma=0}^{rd-1}\left(1+\lfloor\frac{\gamma-rd}{2}\rfloor\right)\\ &\quad \tfrac{3dt(t-1)}{8} - \tfrac{1}{4}\textstyle\sum_{r=1}^{t-1}(rd)^2 - \tfrac{1}{2}\sum_{r=1}^{t-1}\sum_{\gamma=0}^{rd-1}\overline{rd-\gamma}^2.\end{aligned}$$

Making use of the following fact:

$$\textstyle\sum_{r=1}^{t-1}\sum_{\gamma=0}^{rd-1}\overline{rd-\gamma}^2 \;=\; \tfrac{t-1}{2} + \tfrac{dt(t-1)}{4} - \tfrac{(d,2)(t-1)}{4},$$

it follows that

$$(C) = \frac{-d^2}{12}t^3 + \frac{d(d+2)}{8}t^2 - \frac{d(d+6)+3(2,d)-6}{24} + \frac{2-(2,d)}{8}.$$

Reconstructing the sum $2+(A)-(B)-(C)$ and setting $\theta(1,2,3,d) := 21+12(2,d)+12(3,d)-(3,d)^2$, we obtain the desired formula:

$$\mathrm{HF}(A(X_d),t) = \frac{d^2}{6}t^3 + dt^2 + \frac{\theta(1,2,3,d)}{24}t + 1.$$

3.2 The homogeneous ideal of $\overline{G}-$varieties

In this section, we look at the homogeneous ideal $\mathrm{I}(X_d)$ of $\overline{G}-$varieties X_d with finite abelian group $G \subset \mathrm{GL}(n+1,\mathbb{K})$ of order d. From the invariant theory point of view, $\mathrm{I}(X_d)$ is the ideal of syzygies among the minimal set $\mathcal{B}_1$ of fundamental monomial invariants of $\overline{G}$. We have established that $\mathrm{I}(X_d)$ is a binomial prime ideal and we have described a set of binomial generators of $\mathrm{I}(X_d)$ in terms of $\mathcal{B}_1$ (see the proof of Theorem 2.2.18). Using the information of the Hilbert series and Castelnuovo–Mumford regularity of $A(X_d)$, $\mathrm{I}(X_d)$ can be generated by binomials of degree at most $n+1$ (Proposition 3.1.15). We have improved this bound for the homogenous ideal of any $GT-$threefold with cyclic group $G = \langle M_{d;0,1,2,3}\rangle \subset \mathrm{GL}(4,\mathbb{K})$ of order $d \geq 4$; precisely it is minimally generated by binomials of degree at most 3. In view of these facts, we ask for a sharp bound of the degrees of binomial generators of $\mathrm{I}(X_d)$.

Using zero–sum sequences over finite abelian groups and the structure of $R^{\overline{G}}$, we prove that $\mathrm{I}(X_d)$ is generated by binomials of degree at most 3. We give examples of $\overline{G}-$varieties X_d with group $G \subset \mathrm{GL}(n+1,\mathbb{K})$ in any dimension $n \geq 2$ reaching this bound. We characterize the binomials in a minimal set of binomial generators of $\mathrm{I}(X_d)$. This criterion is combinatoric and non constructive. We devote Subsection 3.2.1 to describe a minimal set of binomial generators of the homogeneous ideal of any $GT-$threefold with cyclic group $G = \langle M_{d;0,1,2,3}\rangle \subset \mathrm{GL}(4,\mathbb{K})$ of order $d \geq 4$.

From now onwards, we fix integers $2 \leq n < d$ and a finite abelian group

$$G := \langle M_{d_1;\alpha^1_{\sigma_1(0)},\dots,\alpha^1_{\sigma_1(n)}}, \dots, M_{d_s;\alpha^s_{\sigma_s(0)},\dots,\alpha^s_{\sigma_s(n)}}\rangle \subset \mathrm{GL}(n+1,\mathbb{K})$$

of order $d = d_1 \cdots d_s$. We recall $R^{\overline{G}} = \mathbb{K}[\mathcal{B}_1]$ (Theorem 2.2.11), where $\mathcal{B}_1 = \{m_1, \dots, m_{\mu_d}\}$ is the set of monomial invariants of G of degree d. We take new variables $w_1, \dots, w_{\mu_d}$ and $S = \mathbb{K}[w_1, \dots, w_{\mu_d}]$. We have that $A(X_d) = S/\,\mathrm{I}(X_d) \cong R^{\overline{G}}$ and $\mathrm{I}(X_d)$ is the kernel of the morphism

$$\rho : S \longrightarrow \mathbb{K}[\mathcal{B}_1], \quad \rho(w_i) = m_i,\ i = 1, \dots, \mu_d.$$

It is the homogeneous binomial prime ideal generated by the set

$$\{w_{i_1}\cdots w_{i_k} - w_{j_1}\cdots w_{j_k} \in S \mid m_{i_1}\cdots m_{i_k} = m_{j_1}\cdots m_{j_k},\ k \geq 2\}.$$

For each integer $k \geq 2$, we denote by $\mathrm{I}(X_d)_k$ the set of all binomials of $\mathrm{I}(X_d)$ of degree k. With this notation, we have

$$\mathrm{I}(X_d) = \sum_{k\geq 2}(\mathrm{I}(X_d)_k).$$

Our main goal is to determine the integer $2 \leq K \leq n+1$ such that

$$\sum_{k=2}^{K-1}(\mathrm{I}(X_d)_k) \subsetneq \mathrm{I}(X_d) \text{ and } \sum_{k=2}^{K}(\mathrm{I}(X_d)_k) = \mathrm{I}(X_d).$$

We begin introducing some definitions and notation needed in the sequel.

Definition 3.2.1. Let $k \geq 2$ be an integer.

(i) We call a $k-$*binomial* to any non zero binomial $w^\alpha = w^{\alpha+} - w^{\alpha-} := \prod_{l=1}^{k} w_{i_l} - \prod_{l=1}^{k} w_{j_l} \in \mathrm{I}(X_d)$ of degree k, i.e. $\prod_{l=1}^{k} m_{i_l} = \prod_{l=1}^{k} m_{j_l}$.
(ii) For any $k-$binomial $w^\alpha = w^{\alpha+} - w^{\alpha-} \in \mathrm{I}(X_d)_k$, we denote $\mathrm{supp}_+(w^\alpha)$ (respectively $\mathrm{supp}_-(w^\alpha)$) the support of the monomial $w^{\alpha+}$ (respectively support of $w^{\alpha-}$). We say that w^α is a *non trivial* $k-$binomial if $\mathrm{supp}_+(w^\alpha) \cap \mathrm{supp}_-(w^\alpha) = \emptyset$. Otherwise, we call w^α a *trivial* $k-$binomial.

Definition 3.2.2. Let $k \geq 3$ be an integer and $w^\alpha = w^{\alpha+} - w^{\alpha-} \in \mathrm{I}(X_d)_k$ a non trivial $k-$binomial. An $\mathrm{I}(X_d)_k-$*sequence from* $w^{\alpha+}$ *to* $w^{\alpha-}$ is a finite sequence $(w^1, \ldots, w^t)$ of monomials of S of degree k satisfying the following two conditions:

(i) $w^1 = w^{\alpha+}$ and $w^t = w^{\alpha-}$,

(ii) for all $1 \leq j < t$, $w^j - w^{j+1}$ is a trivial $k-$binomial.

Example 3.2.3. Take $G = \langle M_{6;0,1,2,3} \rangle \subset \mathrm{GL}(4, \mathbb{K})$ a cyclic group of order 6. A minimal set of fundamental invariants of $\overline{G}$ is

$$\begin{aligned}\mathcal{B}_1 \;=\; \{&x_0^6, x_0^4x_3^2, x_0^3x_1x_2x_3, x_0^3x_2^3, x_0^2x_1^3x_3, x_0^2x_1^2x_2^2, x_0^2x_3^4, x_0x_1^4x_2, x_0x_1x_2x_3^3,\\ &x_0x_2^3x_3^2, x_1^6, x_1^3x_3^3, x_1^2x_2^2x_3^2, x_1x_2^4x_3, x_2^6, x_3^6\}\end{aligned}$$

(Example 3.1.30(iii)). We set $S := \mathbb{K}[w_1, \ldots, w_{16}]$ and we consider the morphism $\rho : S \longrightarrow R$ given by $\rho(w_1) = x_0^6, \ldots$. The following homogeneous binomials $w_1w_{15} - w_4^2$ and $w_3w_{12}w_{15} - w_6w_9w_{14}$ are 2 and $3-$binomials, respectively. On the other hand, $\{w_3w_{12}w_{15}, w_5w_9w_{15}, w_6w_9w_{14}\}$ is an $\mathrm{I}(X_6)_3-$sequence from $w_3w_{12}w_{15}$ to $w_6w_9w_{14}$.

Proposition 3.2.4. *Let* $k \geq 3$ *be an integer and* $w^\alpha = w^{\alpha+} - w^{\alpha-} \in \mathrm{I}(X_d)_k$ *a* $k-$*binomial. Then* $w^\alpha \in (\mathrm{I}(X_d)_{k-1})$ *if and only if there exists an* $\mathrm{I}(X_d)_k-$*sequence from* $w^{\alpha+}$ *to* $w^{\alpha-}$.

Proof. See [20, Proposition 5.4]. □

Remark 3.2.5. Let $k \geq 3$ be an integer. A trivial k−binomial of $\mathrm{I}(X_d)_k$ belongs to the ideal $(\mathrm{I}(X_d)_{k-1})$.

The main result of this section is the following.

Theorem 3.2.6. $\mathrm{I}(X_d) = (\mathrm{I}(X_d)_2, \mathrm{I}(X_d)_3)$.

Proof. First, we prove that for all $k \geq 4$, any non trivial k−binomial admits an $\mathrm{I}(X_d)_k$−sequence. By Proposition 3.2.4, this implies $(\mathrm{I}(X_d)_k) \subset (\mathrm{I}(X_d)_{k-1})$. Fix $k \geq 4$ and let $w^\alpha = w^{\alpha_+} - w^{\alpha_-} = w_{i_1} \cdots w_{i_k} - w_{j_1} \cdots w_{j_k}$ be a non trivial k−binomial. For each w_{i_l} (respectively w_{j_l}), let $m_{i_l} = x_0^{a_0^l} \cdots x_n^{a_n^l}$ be its associated monomial (respectively $m_{j_l} = x_0^{b_0^l} \cdots x_n^{b_n^l}$), $l = 1, \ldots, k$. We have that

$$\sum_{l=1}^{k} a_s^l = \sum_{l=1}^{k} b_s^l, \quad 0 \leq s \leq n. \tag{3.2.1}$$

We consider the monomials m_{i_1} and m_{j_1} and for each $0 \leq s \leq n$ we define:

$$c_s = \begin{cases} 0 & \text{if } a_s^1 > b_s^1 \\ b_s^1 - a_s^1 & \text{otherwise.} \end{cases}$$

This gives rise a non zero monomial $m = x_0^{c_0} \cdots x_n^{c_n} \in R$ of degree strictly smaller than d. Clearly, m divides $m_{i_2} \cdots m_{i_k}$ (see (3.2.1)). Thus, we consider $m' = (m_{i_2} \cdots m_{i_k})/m$, which is a monomial of degree at least $(k-2)d \geq 2d$. We write $m' = x_0^{f_0} \cdots x_n^{f_n}$ and we define the sequence of integers $L = (\alpha_0, \overset{f_0}{\ldots}, \alpha_0, \ldots, \alpha_n, \overset{f_n}{\ldots}, \alpha_n)$. Since L has length at least $(k-2)d \geq 2d$, by Lemma 2.2.9, there is a zero-sum subsequence

$$L' = (\alpha_0, \overset{g_0}{\ldots}, \alpha_0, \ldots, \alpha_n, \overset{g_n}{\ldots}, \alpha_n) \subset L.$$

So, the monomial $x_0^{g_0} \cdots x_n^{g_n} \in R^{\overline{G}}$. By Theorem 2.2.11, we can do a factorization:

$$m_{i_2} \cdots m_{i_k} = m_{i_2}^1 \cdots m_{i_k}^1,$$

where all $m_{i_l}^1 \in R^{\overline{G}}$, $2 \leq l \leq k$, are monomials of degree d and, in particular:

$$m_{i_k}^1 = x_0^{g_0} \cdots x_n^{g_n}.$$

Notice that we have $m_{i_1}\cdots m_{i_k} = m_{i_1}m^1_{i_2}\cdots m^1_{i_k}$. We define $w^2 \in S$ to be the monomial $\rho^{-1}(m_{i_1})\rho^{-1}(m^1_{i_2})\cdots\rho^{-1}(m^1_{i_k})$. By construction, $w^{\alpha_+} - w^2$ is a trivial $k-$binomial. Observe that $m = x_0^{c_0}\cdots x_n^{c_n}$ divides $m^1_{i_2}\cdots m^1_{i_{k-1}}$, thus m_{j_1} divides $m_{i_1}m^1_{i_2}\cdots m^1_{i_{k-1}}$. Applying the same argument as before, we do a factorization:

$$m_{i_1}m^1_{i_2}\cdots m^1_{i_{k-1}} = m^2_{i_1}m^2_{i_2}\cdots m^2_{i_{k-1}},$$

where $m^2_{i_1} = m_{j_1}$ and all $m^2_{i_l} \in R^{\overline{G}}$, $2 \leq l \leq k-1$, are monomials of degree d. We set $w^3 = \rho^{-1}(m^2_{i_1})\cdots\rho^{-1}(m^2_{i_{k-1}})\cdot\rho^{-1}(m^1_{i_k})$. Since $m_{i_1}m^1_{i_2}\cdots m^1_{i_{k-1}}m^1_{i_k} = m^2_{i_1}m^2_{i_2}\cdots m^2_{i_{k-1}}m^1_{i_k}$, $w^2 - w^3$ is a trivial $k-$binomial. Furthermore, since $m^2_{i_1} = m_{j_1}$, also $w^3 - w^{\alpha_-}$ is a trivial $k-$binomial. Therefore,

$$(w_{i_1}\cdots w_{i_n}, w^2, w^3, w_{j_1}\cdots w_{j_n})$$

is an $\mathrm{I}(X_d)_k-$sequence, from which it follows that $(\mathrm{I}(X_d)_k) \subset (\mathrm{I}(X_d)_{k-1})$. The argument we have developed only requires that $(k-2)d \geq 2d$, which it is satisfied for all $k \geq 4$. Thus, we obtain

$$\cdots \subset (\mathrm{I}(X_d)_k) \subset (\mathrm{I}(X_d)_{k-1}) \subset \cdots \subset (\mathrm{I}(X_d)_3).$$

$\square$

Example 3.2.7. Take $G = \langle M_{5;0,1,2,3,4}\rangle \subset \mathrm{GL}(5,\mathbb{K})$ a cyclic group of order 5. A minimal set of fundamental invariants of $\overline{G}$ is:

$$\begin{aligned}\mathcal{B}_1 = \{&x_0^5, x_1^5, x_0x_1^3x_2, x_0^2x_1x_2^2, x_3x_0^2x_1^2, x_3x_0^3x_2, x_4x_0^3x_1, x_2^5, x_3x_1x_2^3, x_3^2x_1^2x_2,\\ &x_4x_1^2x_2^2, x_4x_3x_1^3, x_3^2x_0x_2^2, x_4x_0x_2^3, x_3^3x_0x_1, x_4x_3x_0x_1x_2, x_4^2x_0x_1^2, x_4x_3^2x_0^2,\\ &x_4^2x_0^2x_2, x_3^5, x_4x_3^3x_2, x_4^2x_3x_2^2, x_4^2x_3^2x_1, x_4^3x_1x_2, x_4^3x_3x_0, x_4^5\}.\end{aligned}$$

$\mathrm{I}(X_5)$ is minimally generated by 150 $2-$binomials and 20 $3-$binomials.

The arguments in the proof of Theorem 3.2.6 are false, in general, for a binomial $w^\alpha = w^{\alpha_+} - w^{\alpha_-} \in \mathrm{I}(X_d)_3$. For instance, the homogeneous ideal of the $GT-$fourfold with group $G = \langle M_{5;0,1,2,3,4}\rangle \subset \mathrm{GL}(5,\mathbb{K})$ in Example 3.2.7 is minimally generated by binomials of degree 2 and 3. However, there are $\overline{G}-$varieties whose homogeneous ideal is generated only by binomials of degree 2 (Corollary 3.1.24).

Proposition 3.2.8. *Let* $2 < d$ *be an integer and* $G = \langle M_{d;0,\alpha_1,\alpha_2}\rangle \subset \mathrm{GL}(3,\mathbb{K})$ *a cyclic group of order* d *with* $0 < \alpha_1 < \alpha_2 < d$. *If* $\mathrm{GCD}(\alpha_1, d) = \mathrm{GCD}(\alpha_2, d) = \mathrm{GCD}(\alpha_2 - 1, d) = 1$, *then the homogenous ideal of any* $GT-$*surface with group* G *is generated by homogeneous binomials of degree* 2 *and* 3.

Proof. The ring $R^{\overline{G}}$ has $\mathrm{codim}(A(X_d)) = \mu_d - 3 \geq 1$ secondary invariants of degree $2d$. This means that there are $\mu_d - 3$ different monomial invariants of G of degree $2d$ of the form $x_0^{b_0}x_1^{b_1}x_2^{b_2}$ such that $b_0, b_1, b_2 < d$, namely $f_1, \dots, f_{\mu_d-3}$. Set $m = x_0^d x_1^d x_2^d$ and $\overline{m}_i = m/f_i$, $i = 1, \dots, \mu_d - 3$. This gives $\mu_d - 3$ different monomial invariants of G of degree d satisfying $\mathrm{supp}(\overline{m}_i) = \{x_0, x_1, x_2\}$. Indeed, let $f_i = x_0^{b_0}x_1^{b_1}x_2^{b_2}$ be a secondary invariant of degree $2d$. Then, $m/f_i = x_0^{d-b_0}x_1^{d-b_1}x_2^{d-b_2}$ has degree d and its exponents $0 < d - b_i < d$, $i = 0, 1, 2$. So, we obtain

$$\mathcal{B}_1 = \{x_0^d, x_1^d, x_2^d\} \cup \{\overline{m}_i,\ i = 1, \dots, \mu_d - 3\}.$$

Let $w^\alpha = w^{\alpha+} - w^{\alpha-} \in \mathrm{I}(X_d)_2$ be a $2-$binomial. We write $\rho(w^{\alpha+}) = m_1m_2$ and $\rho(w^{\alpha-}) = m_3m_4$, for $m_i \in \mathcal{B}_1$. The homogeneous ideal $\mathrm{I}(X_d)$ does not contain any linear form, so $\{m_1, m_2\} \cap \{m_3, m_4\} = \emptyset$. Since $w^\alpha \in \mathrm{I}(X_d)$, we have that $m_1m_2 = m_3m_4$. Thus, from the above description of $\mathcal{B}_1$, it follows that m_1m_2 can not be of the form $x_i^d x_j^d$, for $i, j \in \{0, 1, 2\}$, otherwise $w^\alpha = 0$. Reordering if necessary, we may assume that $\rho(w_1) = x_0^d, \rho(w_2) = x_1^d$ and $\rho(w_3) = x_2^d$. The last statement is equivalent to say that there is no a $2-$binomial $w^\alpha = w^{\alpha+} - w^{\alpha-}$ such that $w^{\alpha+} = w_iw_j$ or $w^{\alpha-} = w_iw_j$ for $i, j \in \{1, 2, 3\}$. We have that $\rho(w_1w_2w_3) = m$. Moreover, m is divisible by each monomial $\overline{m}_i$, $i = 1, \dots, \mu_d - 3$. Fix $\overline{m}_i$, by Theorem 2.2.11, for each monomial $\overline{m}_i$ we have a factorization $m = \overline{m}_i g_1^i g_2^i$ where $g_1, g_2 \in \mathcal{B}_1 \setminus \{x_0^d, x_1^d, x_2^d\}$. This factorization induces a non trivial $3-$binomial

$$w_1w_2w_3 - \rho^{-1}(\overline{m}_i)\rho^{-1}(g_1)\rho^{-1}(g_2)$$

which does not admit an $\mathrm{I}(X_d)_2-$sequence. Indeed, if $(w^1, \dots, w^k)$ is an $\mathrm{I}(X_d)_2-$sequence from $w_1w_2w_3$ to $\rho^{-1}(\overline{m}_i)\rho^{-1}(g_1)\rho^{-1}(g_2)$, then by definition w^1 is of the form $w_1w_jw_k$ and $w_2w_3 - w_jw_k \in \mathrm{I}(X_d)_2$. As we have argued before, this implies that $w^1 = w_1w_2w_3$. Continuing in this way, we obtain the same assertion for each monomial in the sequence $(w^1, \dots, w^k)$. So $w_1w_2w_3 - \rho^{-1}(\overline{m}_i)\rho^{-1}(g_1)\rho^{-1}(g_2)$ is necessarily trivial, which means that g_1 or g_2 belong to $\{x_0^d, x_1^d, x_2^d\}$, and we arrive to a contradiction. □

As the following proposition shows, we can find $\overline{G}-$varieties in any dimension $n \geq 2$ whose homogeneous ideal is generated by binomials of degree 2 and 3. In this sense, the bound established in Theorem 3.2.6 is sharp.

Corollary 3.2.9. *Let $3 \leq n < d$ be integers and $G = \langle M_{d;0,\alpha_1,\dots,\alpha_n}\rangle \subset \mathrm{GL}(n+1,\mathbb{K})$ a cyclic group of order d. If d is odd and there are $\alpha_i < \alpha_j$ such that $\mathrm{GCD}(\alpha_i, d) = \mathrm{GCD}(\alpha_j, d) = \mathrm{GCD}(\alpha_j - 1, d) = 1$, then the homogeneous ideal $\mathrm{I}(X_d)$ of a $\overline{G}-$variety X_d with group G is minimally generated by binomials of degree 2 and 3.*

Proof. We set $\Gamma = \langle M_{d;0,\alpha_i,\alpha_j}\rangle \subset \mathrm{GL}(3,\mathbb{K})$ a cyclic group of order d acting on $\mathbb{K}[x_0, x_i, x_j]$. We denote by V_d the $GT-$surface with group $\Gamma := \langle M_{d;0,\alpha_i,\alpha_i}\rangle \subset \mathrm{GL}(3,\mathbb{K})$. By Proposition 3.2.8, a minimal set of binomial generators of $\mathrm{I}(V_d)$ contains a binomial $w^\alpha = w^{\alpha_+} - w^{\alpha_-}$ of degree 3. Under a suitable identification of variables, we have

$$\mathrm{I}(V_d) = \mathrm{I}(X_d) \cap \mathbb{K}[\rho^{-1}(x_0), \rho^{-1}(x_i), \rho^{-1}(x_j)].$$

So, we can see any element of $\mathrm{I}(V_d)$ as an element of $\mathrm{I}(X_d)$. In particular, $w^\alpha \in \mathrm{I}(X_d)$. Now if $(w^1, \dots, w^k)$ is an $\mathrm{I}(X_d)_2-$sequence from w^{α_+} to w^{α_-}, then

$$\rho(w^1) = \rho(w^2) = \cdots = \rho(w^k) \in \mathbb{K}[x_0, x_i, x_j].$$

Hence, we can regard $(w^1, \dots, w^k)$ as an $\mathrm{I}(V_d)_2-$sequence from w^{α_+} to w^{α_-}, which is a contradiction. $\square$

Proposition 3.2.4 characterizes the 3−binomials in a minimal set of binomial generators of $\mathrm{I}(X_d)$ in terms of $\mathrm{I}(X_d)_3-$sequences. It is natural to ask when such a $\mathrm{I}(X_d)_3-$sequence exists, and in case to design a procedure to find them. These objectives require a precise description of the set of generators $\mathcal{B}_1$ of $R^{\overline{G}}$ and the binomial generators of $\mathrm{I}(X_d)$, which are out of reach for an arbitrary $\overline{G}-$variety X_d with group $G \subset \mathrm{GL}(n+1,\mathbb{K})$. Notwithstanding, there are examples of rings $R^{\overline{G}}$ which are achievable for that matter. In the rest of this section, we deal with a family of $GT-$threefolds which provide a wealth field to investigate these questions.

3.2.1 A minimal set of binomial generators of GT-threefolds

In this subsection, we compute a minimal set of binomial generators of any $GT-$threefold X_d with cyclic group $G = \langle M_{d;0,1,2,3}\rangle \subset \mathrm{GL}(4,\mathbb{K})$ of order

$d \geq 4$. We prove that $\mathrm{I}(X_d)$ is minimally generated by binomials of degree 2 if d is even and it is minimally generated by binomials of degree 2 and 3 if d is odd. To this end, we develop a procedure for constructing $\mathrm{I}(X_d)_k$−sequences, which lead us to determine, for d odd, the 3−binomials in a minimal set of binomial generators of $\mathrm{I}(X_d)$. The key ingredient is a complete description of the set $\mathcal{B}_1$ of fundamental invariants of $\overline{G}$ (Theorem 2.2.11).

The content of this subsection has been published in [20]. In this article, we study $\mathrm{I}(X_d)$ from a different perspective: lattice ideals and Markov basis, and it is inspired by [14, 15, 16, 24, 27, 45].

As we have seen in Subsection 3.1.2, a monomial $m = x_0^{\alpha}x_1^{\beta}x_2^{\delta}x_3^{\gamma} \in \mathcal{B}_1$ if and only if $(\alpha, \beta, \delta, \gamma)$ is a $\mathbb{Z}^4_{\geq 0}$−solution of one of the linear system of congruences:

$$(*)_{\mathcal{A};1,r} : \left\{ \begin{array}{ccccccccc} y_0 & + & y_1 & + & y_2 & + & y_3 & = & d \\ & & y_1 & + & 2y_2 & + & 3y_3 & = & rd \end{array} \right. \qquad r = 0, 1, 2, 3$$

For $r = 0$ (respectively $r = 3$), we obtain $(d, 0, 0, 0)$ (respectively $(0, 0, 0, d)$). For $r = 1, 2$, we obtain $\mathbb{Z}^4_{\geq 0}$−solutions $(f_0(r, y_3, y_2), f_1(r, y_3, y_2), y_2, y_3)$ where

$$\begin{aligned} y_3 &\in \left\{\max\{0, d(r-t)\}, \dots, \lfloor \tfrac{rd}{3} \rfloor\right\} \\ y_2 &\in \left\{\max\{0, d(r-t) - 2y_2\}, \dots, \lfloor \tfrac{rd-3y_2}{2} \rfloor\right\} \\ f_1(r, y_3, y_2) &= rd - 3y_3 - 2y_2 \\ f_0(r, y_3, y_2) &= (t - r)d + y_2 + 2y_3. \end{aligned}$$

We write $d = 2k + \varepsilon = 3k' + \rho$ where $\varepsilon \in \{0, 1\}$ and $\rho \in \{0, 1, 2\}$. Therefore, any monomial of $\mathcal{B}_1$ is uniquely determined by the following set:

$$\begin{aligned} \mathcal{W}_d := \big\{(r, \gamma, \delta) \in \mathbb{Z}^3_{\geq 0} \mid 0 \leq r \leq 3, 0 \leq \gamma \leq rk' + \lfloor \tfrac{r\rho}{3} \rfloor, \\ \max\{0, d - 2\gamma\} \leq \delta \leq \lfloor \tfrac{rd-3\gamma}{2} \rfloor\big\}. \end{aligned}$$

Let us see the some examples that illustrate the set $\mathcal{W}_d$ and a couple of minimal set of fundamental invariants $\mathcal{B}_1$ of $\overline{G}$.

Example 3.2.10. (i) We take $d = 4$. We compute the $\mathbb{Z}^4_{\geq 0}$−solutions of the systems $(*)_{\mathcal{A};1,r}$. For $r = 0$ (respectively $r = 3$) there is only one possible solution $(4, 0, 0, 0)$ (respectively $(0, 0, 0, 4)$). For $r = 1$, the solutions are:

$$\left\{(\delta + 2\gamma, 4 - 2\delta - 3\gamma, \delta, \gamma) \mid \gamma \in \{0, 1\},\, \delta \in \left\{0, \dots, \lfloor \frac{4 - 3\gamma}{2} \rfloor\right\}\right\}.$$

For $r=2$, we have:

$$\left\{(\gamma+2\delta-4, 8-2\delta-3\gamma, \delta, \gamma) \mid \gamma\in\{0,1,2\}, \delta\in\left\{\max\{0, 8-2\gamma\}, \ldots, \lfloor\tfrac{8-3\gamma}{2}\rfloor\right\}\right\}.$$

We obtain:

$$\begin{aligned}
\mathcal{B}_1 &= \{x_0^4, x_1^4, x_0x_1^2x_2, x_0^2x_2^2, x_0^2x_1x_3, x_2^4, x_1x_2^2x_3, x_1^2x_3^2, x_0x_2x_3^2, x_3^4\}\\
\mathcal{W}_4 &= \{(0,0,0),(1,0,0),(1,0,1),(1,0,2),(1,1,0),(2,0,4),(2,1,2),\\
&\quad (2,2,0),(2,2,1),(3,4,0)\}.
\end{aligned}$$

(ii) We take $d=5$. Arguing as in (i) we obtain:

$$\begin{aligned}
\mathcal{B}_1 &= \{x_0^5, x_1^5, x_0x_1^3x_2, x_0^2x_1x_2^2, x_0^2x_1^2x_3, x_0^3x_2x_3, x_2^5, x_1x_2^3x_3, x_1^2x_2x_3^2, x_0x_2^2x_3^2,\\
&\quad x_0x_1x_3^3, x_3^5\}\\
\mathcal{W}_5 &= \{(0,0,0),(1,0,0),(1,0,1),(1,0,2),(1,1,0),(1,1,1),(2,0,5),(2,1,3),\\
&\quad (2,2,1),(2,2,2),(2,3,0),(3,5,0)\}.
\end{aligned}$$

This motivates the following notation.

Notation 3.2.11. For each $(r,\gamma,\delta)\in\mathcal{W}_d$ we set a variable $w_{(r,\gamma,\delta)}$ and $S:=\mathbb{K}[w_{(r,\gamma,\delta)}]_{(r,\gamma,\delta)\in\mathcal{W}_d}$. The ideal $\mathrm{I}(X_d)$ is identified with the kernel of the morphism

$$\phi: S\longrightarrow \mathbb{K}[\mathcal{B}_1],\quad \phi(w_{(r,\gamma,\delta)}) = x_0^{\delta+2\gamma-rd}x_1^{rd-2\delta-3\gamma}x_2^{\delta}x_3^{\gamma} =: m_{(r,\gamma,\delta)}.$$

It is the binomial prime ideal generated by:

$$\left\{\prod_{i=1}^{k} w_{(r_{j_i},\gamma_{j_i},\delta_{j_i})} - \prod_{i=1}^{k} w_{(r_{h_i},\gamma_{h_i},\delta_{h_i})} \mid \prod_{i=1}^{k} m_{(r_{j_i},\gamma_{j_i},\delta_{j_i})} = \prod_{i=1}^{k} m_{(r_{h_i},\gamma_{h_i},\delta_{h_i})}, k\geq 2\right\}.$$

Definition 3.2.12. Let $w=\prod_{i=1}^{k} w_{(r_i,\gamma_i,\delta_i)}\in S$ be a monomial of degree $k\geq 2$.

(i) We say that w *admits a suitable k−binomial* if there exists a monomial $w'=\prod_{i=1}^{k} w_{(r'_i,\gamma'_i,\delta'_i)}\in S$ of degree k such that $w-w'$ is a non trivial k−binomial.

(ii) We say that the variable $w_{(r,\gamma,\delta)}\in S$ admits a *special* k−binomial if there exists a suitable k−binomial $w-w'\in\mathrm{I}(X_d)$ such that $(r,\gamma,\delta)=\min\{\mathrm{supp}(m-m')\}$.

Determining whether a monomial w admits a suitable binomial or a variable $w_{(r,\gamma,\delta)}$ admits a special binomial gives us a method to construct $\mathrm{I}(X_d)_k-$sequences. For instance, if $w = w_{(r_1,\gamma_1,\delta_1)} \cdots w_{(r_k,\gamma_k,\delta_k)}$ and $w_{(r_1,\gamma_1,\delta_1)}$ admits a special $k-$binomial $w - w^1$, then it is a trivial $k-$binomial. If $\mathrm{supp}(w^1) \setminus \{w_{(r_1,\gamma_1,\delta_1)}\}$ contains a variable admitting a special $k-$binomial w^2, then $w^1 - w^2$ is trivial and (w, w^1, w^2) is an $\mathrm{I}(X_d)_k-$sequence from w to w^2.

Example 3.2.13. (i) The variable $w_{(0,0,0)} \in S$ admits a special $2-$binomial. Indeed, $w_{(0,0,0)}w_{(2,2k',0)} - w_{(1,k',0)}w_{(1,k',0)}$ is a non trivial suitable $2-$binomial and $(0,0,0) = \min\{(0,0,0),\ (2,2k',0),(1,k',0)\}$. Unlike, $w_{(3,d,0)}$ does not admit a special $k-$binomial for any $k \geq 2$.

(ii) For $d = 4$, the set of variables admitting a special $2-$binomial are indexed by $\mathcal{W}_4\backslash\{(1,1,0),(2,1,2),(2,2,0),(2,2,1),(3,4,0)\}$. And for example, $w_{(1,1,0)}$ admits a special $3-$binomial: $w_{(1,1,0)}w_{(2,1,2)}w_{(3,4,0)} - w_{(2,2,0)}w^2_{(2,2,1)}$.

Consider sets:

$$\mathcal{W}_d \setminus \{(2,2k'-1,0),(2,2k'-1,1),(2,2k',0),(3,d,0)\} \text{ if } \rho = 0,$$

$$\mathcal{W}_d \setminus \{(2,2k'-1,2),(2,2k',0),(2,2k',1),(3,d,0)\} \text{ if } \rho = 1,$$

$$\mathcal{W}_d \setminus \{(2,2k',0),(2,2k',1),(2,2k',2),(3,d,0)\} \text{ if } \rho = 2.$$

We will see through a series of lemmas that a variable $w_{(r,\gamma,\delta)}$ admits a special 2 or $3-$binomial if and only if (r,γ,δ) belongs to one of the above sets. Moreover, it allows us to establish which monomials of degree 2 admit a special $2-$binomial.

Lemma 3.2.14. *Each monomial $w = w_{(1,\gamma,\delta)}w_{(3,d,0)} \in S$ admits a suitable $2-$binomial $w - w'$ except: $(\gamma,\delta) = \left(k', \lfloor\frac{\rho}{2}\rfloor\right)$ if $\rho \neq 0$, and $\gamma = \delta = 0$ if $\varepsilon = 1$.*

Proof. Fix $(1,\gamma,\delta) \in \mathcal{W}_d$. If there exists such a suitable $2-$binomial $w - w'$, then

$$w' = w_{(2,\gamma_1,\delta_1)}w_{(2,\gamma_2,\delta_2)}$$

for some $0 \leq \gamma_i \leq 2k' + \lfloor\frac{2\rho}{3}\rfloor$, $\max\{0, d - 2\gamma_i\} \leq \delta_i \leq \lfloor\frac{2d-3\gamma_i}{2}\rfloor$, $i = 1,2$, and the following equalities are satisfied:

$$\gamma + d = \gamma_1 + \gamma_2 \text{ and } \delta = \delta_1 + \delta_2.$$

From this it follows that for $\rho = 1$ and $\gamma = k'$ $(d = 3k' + \rho)$, there are no γ_1 and γ_2 such that $\gamma + d = 4k' + 1$. Analogously, for $\rho = 2$ we have $\gamma_1 = \gamma_2 = 2k' + 1$, which implies $\delta_1 = \delta_2 = 0$. Hence, the equality $\delta_1 + \delta_2 = \delta = 1$ is not satisfied.

For the rest of γ's, we set $\gamma_1 := \lfloor \frac{d+\gamma}{2} \rfloor$ and $\gamma_2 := \lceil \frac{d+\gamma}{2} \rceil$. Observe that we always have $k \leq \gamma_1, \gamma_2 \leq 2k' + \lfloor \frac{\rho}{2} \rfloor$. Using the basic properties of the floor and ceiling functions, we obtain

$$\lfloor \frac{2d - 3\gamma_1}{2} \rfloor + \lfloor \frac{2d - 3\gamma_2}{2} \rfloor \leq \lfloor \frac{4d - 3(d+\gamma)}{2} \rfloor = \lfloor \frac{d - 3\gamma}{2} \rfloor,$$

where the equality holds if and only if γ_1 and γ_2 are not both odd. If so, we can find values δ_1 and δ_2 such that $\delta_1 + \delta_2 = \delta$, as long as $\delta \geq \max\{0, d - 2\gamma_1\} + \max\{0, d - 2\gamma_2\}$. The last condition always happens except for $\gamma = \delta = 0$ and $\varepsilon = 1$.

Finally, if γ_1 and γ_2 are odd, hence $\gamma \geq 2$, the result follows taking $m' = w_{(2,\gamma_1+1,\lfloor \frac{2d-3(\gamma_1+1)}{2} \rfloor)} w_{(2,\gamma_2-1,\lfloor \frac{2d-3(\gamma_2-1)}{2} \rfloor)}$. $\square$

Proposition 3.2.15. *All $w_{(1,\gamma,\delta)} \in S$ admit a special 2 or 3−binomial.*

Proof. It is enough to treat the three exceptions of Lemma 3.2.14. For $\varepsilon = 1$ and $(1, \gamma, \delta) = (1, 0, 0)$, we observe that $w_{(1,0,0)}w_{(2,2k',0)} - w_{(1,1,0)}w_{(2,2k'-1,0)}$ if $\rho = 0$, $w_{(1,0,0)}w_{(2,2k',1)} - w_{(1,0,1)}w_{(2,2k',0)}$ if $\rho = 1$, and $w_{(1,0,0)}w_{(2,2k'+1,0)} - w_{(1,1,0)}w_{(2,2k',0)}$ if $\rho = 2$, are all three special 2−binomials.

For $(1, \gamma, \delta) = \left(1, k', \lfloor \frac{\rho}{2} \rfloor\right)$ and $\rho \neq 0$, the monomial $w_{(1,k',\lfloor \frac{\rho}{2} \rfloor)}$ does not admit a special 2−binomial. However, for $\rho = 1$ and $\rho = 2$,

$$w_{(1,k',0)}w_{(2,2k'-1,2)}w_{(3,d,0)} - w_{(2,2k',0)}w^2_{(2,2k',1)}$$

$$w_{(1,k',1)}w_{(2,2k',1)}w_{(3,d,0)} - w_{(2,2k',2)}w^2_{(2,2k'+1,0)}$$

are special 3−binomials, respectively. $\square$

Proposition 3.2.16. *All $w_{(2,\gamma,\delta)} \in S$ admit a special 2 or 3−binomial except:*

(i) $\{w_{(2,2k'-1,0)}, w_{(2,2k'-1,1,)}, w_{(2,2k',0)}\}$ *if* $\rho = 0$,

(ii) $\{w_{(2,2k'-1,2)}, w_{(2,2k',0)}, w_{(2,2k',1)}\}$ *if* $\rho = 1$,

(iii) $\{w_{(2,2k',1)}, w_{(2,2k',2)}, w_{(2,2k'+1,0)}\}$ *if* $\rho = 2$.

Proof. For any $(2,\gamma,\delta) \in \mathcal{W}_d$ different from the excluded cases, we consider the monomial

$$w = w_{(2,\gamma,\delta)} w_{(2,2k'+\lfloor\frac{\rho}{2}\rfloor,\lceil\frac{\rho}{2}\rceil-\lfloor\frac{\rho}{2}\rfloor)}.$$

For convenience we denote:

$$\gamma' = 2k' + \lfloor\frac{\rho}{2}\rfloor, \text{ and } \delta' = \lceil\frac{\rho}{2}\rceil - \lfloor\frac{\rho}{2}\rfloor,$$

and we set $\gamma_1 := \gamma + 1$ and $\gamma_2 := \gamma' - 1$. We distinguish the following cases:

Case 1: If γ or γ' are odd, and $\delta = (2d-3\gamma)/2$ (hence $\rho \neq 2$), there exists δ_i with $\max\{0, d-2\gamma_i\} \leq \delta_i \leq \lfloor\frac{2d-3\gamma_i}{2}\rfloor$ such that $\delta_1 + \delta_2 = \delta + \delta'$.

Case 2: If γ and γ' are even, $\delta = (2d-3\gamma)/2$ and $\gamma < 2k'-2$ we take $\gamma_1 := \gamma + 2$ and $\gamma_2 := 2k' - 2$. Then, there exists δ_i with $\max\{0, d-2\gamma_i\} \leq \delta_i \leq \lfloor\frac{2d-3\gamma_i}{2}\rfloor$ such that $\delta_1 + \delta_2 = \delta + \delta'$. If $\gamma = 2k'-2$ and $\rho = 1$,

$$w_{(2,2k'-2,4)} w_{(2,2k',0)} - w^2_{(2,2k'-1,2)}$$

is a special 2−binomial. Finally, if $\rho = 0$, $\gamma = 2k'-2$ and $\delta = 3$, the element $(2, 2k'-2, 3)$ does not admit a special 2−binomial but it admits a special 3-binomial:

$$w_{(2,2k'-2,3)} w_{(2,2k'-1,0)} w_{(2,2k',0)} - w^3_{(2,2k'-1,1)}.$$

□

Lemma 3.2.17. *Any monomial* $w = w_{(0,0,0)} w_{(2,\gamma,\delta)} \in S$ *admits a suitable* 2−*binomial* $w - w'$*, with the following exceptions:* $(\gamma,\delta) = \left(2k' + \lfloor\frac{\rho}{2}\rfloor, \lceil\frac{\rho}{2}\rceil - \lfloor\frac{\rho}{2}\rfloor\right)$ *if* $\rho \neq 0$*, and* $\gamma = 0$ *if* $\varepsilon = 1$.

Proof. If w' admits a suitable 2−binomial $w - w'$, then

$$w' = w_{(1,\gamma_1,\delta_1)} w_{(1,\gamma_2,\delta_2)}$$

such that $0 \leq \gamma_i \leq k'$, $0 \leq \delta_i \leq \lfloor\frac{d-3\gamma_i}{2}\rfloor$ for $i = 1, 2$, and the following equalities are satisfied:

$$\gamma_1 + \gamma_2 = \gamma \text{ and } \delta_1 + \delta_2 = \delta.$$

From this it follows that $(2,\gamma,\delta) \neq (2, 2k'+1, 0)$ if $\rho = 2$ and $(2,\gamma,\delta) \neq (2, 2k', 1)$ if $\rho = 1$ and $\gamma = 0$ if $\varepsilon = 1$.

Otherwise, we set $\gamma_1 := \lfloor \frac{\gamma}{2} \rfloor$ and $\gamma_2 := \lceil \frac{\gamma}{2} \rceil$. If d is even and γ_1 and γ_2 are odd or d is odd and γ_1 and γ_2 are even, we take

$$w' = w_{(1,\gamma_1+1,\lfloor \frac{d-3(\gamma_1+1)}{2} \rfloor)} w_{(1,\gamma_2-1,\lfloor \frac{d-3(\gamma_2-1)}{2} \rfloor)}.$$

Or else, we take

$$w' = w_{(1,\gamma_1,\lfloor \frac{d-3\gamma_1}{2} \rfloor)} w_{(1,\gamma_2,\lfloor \frac{d-3\gamma_2}{2} \rfloor)}.$$

□

Lemma 3.2.18. *Assume that d is odd. Then,*

(i) any monomial $w = w_{(1,0,0)} w_{(2,\gamma,\delta)}$ admits a suitable $2-$binomial $w - w'$ except for $\gamma = 0, \ldots, k+1$ and $\delta = \max\{0, d-2\gamma\}$.

(ii) Any monomial $w = w_{(1,\gamma,\delta)} w_{(2,0,d)}$ admits a suitable $2-$binomial $w - w'$ except for $\gamma = 0$ and $\delta = 0, \ldots, k$ or $\gamma = 1$ and $\delta = k-1$.

Proof. We write $w' = w_{(1,\gamma_1,\delta_1)} w_{(2,\gamma_2,\delta_2)}$. (i) If $\delta > \max\{0, d-2\gamma\}$, we take

$$(1,\gamma_1,\delta_1) = (1,0,1) \text{ and } (2,\gamma_2,\delta_2) = (2,\gamma,\delta-1),$$

which satisfies that $w - w' \in \mathrm{I}(X_d)$. The remainder cases are

$$(2,\gamma,\max\{0, d-2\gamma\}), \quad \gamma = 0, \ldots, 2k' + \lfloor \frac{\rho}{2} \rfloor.$$

If $\gamma > k+1$, we have $(2,\gamma,\max\{0,d-2\gamma\}) = (2,\gamma,0)$ and we take

$$(1,\gamma_1,\delta_1) = (1,1,0) \text{ and } (2,\gamma_2,\delta_2) = (2,\gamma-1,0).$$

If $0 \leq \gamma \leq k+1$, then the equalities $\gamma_1 + \gamma_2 = \gamma$ and $\delta_1 + \delta_2 = \delta$ imply $\gamma_1 = i$ and $\gamma_2 = \gamma - i$ for some $0 \leq i \leq \gamma$, $0 \leq \delta_1 \leq \lfloor \frac{d-3i}{2} \rfloor$ and $d - 2(\gamma - i) \leq \delta_2 \leq \lfloor \frac{2d-3\gamma+3i}{2} \rfloor$. Then, we obtain a contradiction $\delta < d - 2(\gamma - i) \leq \delta_1 + \delta_2$.

(ii) If $2 \leq \gamma$ is even or $\gamma = 1$ and $\delta < k-1$, we take

$$(1,\gamma_1,\delta_1) = (1,\gamma-1,\delta+2) \text{ and } (2,\gamma_2,\delta_2) = (2,1,d-2).$$

If $2 \leq \gamma$ is odd, we take

$$(1,\gamma_1,\delta_1) = (1,\gamma-2,\delta+4) \text{ and } (2,\gamma_2,\delta_2) = (2,2,d-4).$$

In all these cases, $w - w' \in \mathrm{I}(X_d)$. For $\gamma = 0$, $\gamma_1 = \gamma_2 = 0$ and, hence, $w' = w$. For $\gamma = 1$ and $\delta = k-1$, we have $\gamma_1 = 0$ and $\gamma_2 = 1$, so $\delta_1 \leq k$ and $\delta_2 = d-2$. We obtain $\delta_1 + \delta_2 \leq d - 2 + k < d + k - 1$. $\square$

Remark 3.2.19. (i) The monomial $w_{(0,0,0)}w_{(3,d,0)}$ admits a non trivial suitable $2-$binomial if and only if $\rho = 0$. Indeed, assume that $w_{(0,0,0)}w_{(3,d,0)} - w_{(1,\gamma_1,\delta_1)}w_{(2,\gamma_2,\delta_2)}$ is a suitable $2-$binomial. Then we have $\gamma_1 + \gamma_2 = 3k' + \rho = k' + 2k' + \rho$. So $\gamma_1 = k'$ and $\gamma_2 = 2k' + \rho = 2k' + \lfloor \frac{\rho}{2} \rfloor$ if and only if $\rho = 0$.

(ii) If $\rho = 1$, then any monomial $w_{(1,k',0)}w_{(2,\gamma,\delta)}$ admits a suitable $2-$binomial except when $\gamma = 2k'$. Indeed, if $\gamma < 2k'$ we take $(r_1, \gamma_1, \delta_1) = (1, k'-1, \delta_1)$ and $(r_2, \gamma_2, \delta_2) = (2, \gamma+1, \delta_2)$ with $\delta = \delta_1 + \delta_2$, $0 \leq \delta_1 \leq \lfloor \frac{d-3k'+3}{2} \rfloor$ and $\max\{0, d-2\gamma-2\} \leq \delta_2 \leq \lfloor \frac{2d-3\gamma-3}{2} \rfloor$. If $\gamma = 2k'$, since $\gamma_1 < k'$ and $\gamma_2 \leq 2k'$, $\gamma = \gamma_1 + \gamma_2$ will never occur.

(iii) If $\rho = 2$ and $\varepsilon = 0$, the monomials $w_{(1,k',1)}w_{(2,2k'+1,0)}$ and $w_{(1,k',1)}w_{(2,2k,2)}$ do not admit a suitable $2-$binomial. If $d - 3\gamma$ is even and $\delta = \frac{2d-3\gamma}{2}$, we take $w' = w_{(1,k'-2,\lfloor \frac{d-3(k'-2)}{2} \rfloor)}w_{(2,\gamma+2,\lfloor \frac{2d-3(\gamma+2)}{2} \rfloor)}$. In any other case we take $w' = w_{(1,k'-1,\lfloor \frac{d-3(k'-1)}{2} \rfloor)}w_{(2,\gamma+1,\lfloor \frac{2d-3(\gamma+1)}{2} \rfloor)}$. Any monomial $w_{(1,k',1)}w_{(2,\gamma,\delta)}$ admits a suitable $2-$binomial except: $\gamma = 2k'+1$ and $(\gamma,\delta) = (2k', 2)$ when $\varepsilon = 0$. In a similar way, we see that any monomial $w_{(1,\gamma,\delta)}w_{(2,2k'+1,0)}$ admits a suitable $2-$binomial except $\gamma = k'$.

Proposition 3.2.20. *Assume that d is odd. The following monomials admit a suitable* $3-$*binomial.*

(i) $w_{(0,0,0)}w_{(2,0,d)}w_{(1,0,\delta)}$, $\delta = 0, \ldots, k-1$;

(ii) $w_{(0,0,0)}w_{(2,0,d)}w_{(3,d,0)}$;

(iii) $w_{(0,0,0)}w_{(1,0,0)}w_{(3,d,0)}$;

(iv) $w_{(1,0,0)}w_{(2,\gamma,d-2\gamma)}w_{(3,d,0)}$, $\gamma = 0, \ldots, k-1$.

The following three monomials do not admit a suitable $3-$*binomial:*

$$w_{(0,0,0)}w_{(2,0,d)}w_{(1,0,k)}, w_{(1,0,0)}w_{(2,k,1)}w_{(3,d,0)}, w_{(1,0,0)}w_{(2,k+1,0)}w_{(3,d,0)}.$$

Proof. For (i) to (iv), it suffices to exhibit explicitly a $3-$binomial in each case.

(i) For any $\delta \in \{0, \dots, k-1\}$ we take

$$w_{(0,0,0)}w_{(2,0,d)}w_{(1,0,\delta)} - w_{(1,0,k)}w_{(1,0,k)}w_{(1,0,\delta+1)}.$$

(ii) We take $w_{(0,0,0)}w_{(2,0,d)}w_{(3,d,0)} - w_{(1,0,k)}w_{(2,k,\lceil\frac{k+1}{2}\rceil)}w_{(2,k+1,\lfloor\frac{k+1}{2}\rfloor)}$.

(iii) We take $w_{(0,0,0)}w_{(1,0,0)}w_{(3,d,0)} - w_{(1,\lfloor\frac{k'+\lceil\frac{\rho}{2}\rceil}{2}\rfloor,0)}w_{(1,\lceil\frac{k'+\lceil\frac{\rho}{2}\rceil}{2}\rceil,0)}w_{(2,2k'+\lfloor\frac{\rho}{2}\rfloor,0)}$.

(iv) For any $0 \leq \gamma \leq k-1$, we take

$$w_{(1,0,0)}w_{(2,\gamma,d-2\gamma)}w_{(3,d,0)} - w_{(2,\gamma+1,\max\{0,d-2\gamma-2\})}w_{(2,k,1)+(2,k,1)}.$$

Assume that $w_{(0,0,0)}w_{(2,0,d)}w_{(1,0,k)} - w_{(1,\gamma_1,\delta_1)}w_{(1,\gamma_2,\delta_2)}w_{(1,\gamma_3,\delta_3)}$ is a suitable 3-binomial. Therefore, $\gamma_1 = \gamma_2 = \gamma_3 = 0$. Hence, $\delta_i \leq k$, $i = 1, 2, 3$ and the equality $\delta_1 + \delta_2 + \delta_3 = 3k+1$ is not satisfied.

Now assume that $w_{(1,0,0)}w_{(2,k,1)}w_{(3,d,0)} - w_{(2,\gamma_1,\delta_1)}w_{(2,\gamma_2,\delta_2)}w_{(2,\gamma_3,\delta_3)}$ is a suitable 3−binomial. Then, we have $\delta_1 + \delta_2 + \delta_3 \in \{0, 1\}$ and $\gamma_1 + \gamma_2 + \gamma_3 = d + k = 3k+1$. The first condition implies $\gamma_i \geq k$, $i = 1, 2, 3$. So, there is some $\gamma_i = k$ which forces $(2, \gamma_i, \delta_i) = (2, k, 1)$. The arguments for the last monomial are analogous using that there is some $\gamma_j = k+1$. $\square$

Proposition 3.2.21. *Assume that d is odd. Let $w^\alpha = w^{\alpha+} - w^{\alpha-}$ be a non trivial 3−binomial. If $w^{\alpha+}$ or $w^{\alpha-}$ is one of the following monomials:*

(i) $w_{(0,0,0)}w_{(2,0,d)}w_{(1,0,\delta)}$, $\delta = 0, \dots, k$;

(ii) $w_{(0,0,0)}w_{(2,0,d)}w_{(3,d,0)}$ *and* $\rho \neq 0$;

(iii) $w_{(0,0,0)}w_{(1,0,0)}w_{(3,d,0)}$ *and* $\rho \neq 0$;

(iv) $w_{(1,0,0)}w_{(2,\gamma,d-2\gamma)}w_{(3,d,0)}$, $\gamma = 0, \dots, k$ *and* $w_{(1,0,0)}w_{(2,k+1,0)}w_{(3,d,0)}$;

then there is no an $\mathrm{I}(X_d)_3$*−sequence from* $w^{\alpha+}$ *to* $w^{\alpha-}$.

Proof. Let $\{w^1, \dots, w^t\}$ be an $\mathrm{I}(X_d)_3-$sequence from $w^{\alpha+}$ to $w^{\alpha-}$. By definition, it exists a variable $w_{(r,\gamma,\delta)} \in \mathrm{supp}(w^{\alpha+})$ and a suitable 2−binomial $w^{\alpha'}$ such that $w^{\alpha+} - w^2 = w_{(r,\gamma,\delta)}w^{\alpha'}$. In particular, $\mathrm{supp}(w^{\alpha+})$ and $\mathrm{supp}(w^{\alpha-})$ contain a variable admitting a special 3−binomial.

If $w^{\alpha+}$ belongs to the above list, then by Lemmas 3.2.17, 3.2.14 and 3.2.18, any monomial of degree 2 that we can form from $\mathrm{supp}(w^{u+})$ in (i),

(ii) and (iii) do not admit a non trivial suitable 2−binomial. Thus, for these cases w^α does not admit an $\mathrm{I}(X_d)_3$−sequence from w^{α_+} to w^{α_-}. In case (iv), it suffices to see that the monomials $w_{(1,0,0)}w_{(2,\gamma,d-2\gamma)}$, $\gamma = 0, \ldots, k$, and $w_{(1,0,0)}w_{(2,k+1,0)}$ do not admit a suitable 2−binomial.

We fix $\gamma \in \{0, \ldots, k+1\}$ and we assume that there are $(1, \gamma_1, \delta_1) \neq (1, 0, 0)$ and $(2, \gamma_2, \delta_2)$ in $\mathcal{W}_d$ such that

$$\gamma_1 + \gamma_2 = \gamma \text{ and } \delta_1 + \delta_2 = d - 2\gamma,\ \gamma = 0, \ldots, k$$

and such that $\gamma_1 + \gamma_2 = k + 1$ and $\delta_1 + \delta_2 = 0$. We write $\gamma_2 = \gamma - \gamma_1$, so $\delta_2 \geq \delta + 2\gamma_1$. From this we deduce that $\delta_1 + \delta_2 \geq \delta_1 + \delta + 2\gamma_1$. Hence, $\delta_1 + 2\gamma_1 = 0$ and we arrive to a contradiction. □

We see a couple of examples, which shows that the last two propositions are false if d is even.

Example 3.2.22. Take $d = 4$. We only have to check that all monomials as in Proposition 3.2.20(ii) contain a submonomial of degree 2 admitting a non trivial suitable 2−binomial. Indeed, $w_{(0,0,0)}w_{(2,0,4)} - w^2_{(1,0,2)}$ and $w_{(1,0,0)}w_{(3,4,0)} - w^2_{(2,2,0)}$ are suitable 2−binomials of $\mathrm{I}(X_4)$, from which the result follows.

A consequence of Propositions 3.2.20 and 3.2.21 is that, if d is odd, then a minimal set of binomial generators of $\mathrm{I}(X_d)$ always contains 3−binomials. Moreover, we will prove that Proposition 3.2.21 describes them. By Theorem 2.4.10, $\mathrm{I}(X_d)$ is minimally generated by binomials of degree at most 3. Applied to $\mathrm{I}(X_d)$, we have that for any integer $k \geq 4$ and any k−binomial $w^\alpha = w^{\alpha_+} - w^{\alpha_-}$, there exists an $\mathrm{I}(X_d)_k$−sequence from w^{α_+} to w^{α_-}. The proof of Theorem 2.4.10, however, does not show a systematic way to construct such $\mathrm{I}(X_d)_k$−sequences.

Notation 3.2.23. Let $d \geq 5$ be an odd integer. We denote

$$\begin{aligned}
\mathfrak{B}^0_3 \quad &:= \quad \{w_{(0,0,0)}w_{(2,0,d)}w_{(1,0,\delta)}\}_{\delta=0}^{k-1} \cup \{w_{(1,0,0)}w_{(2,\gamma,d-2\gamma)}w_{(3,d,0)}\}_{\gamma=0}^{k-1} \\
\mathfrak{B}^1_3 = \mathfrak{B}^2_3 \quad &:= \quad \{w_{(0,0,0)}w_{(2,0,d)}w_{(1,0,\delta)}\}_{\delta=0}^{k-1} \cup \{w_{(1,0,0)}w_{(2,\gamma,d-2\gamma)}w_{(3,d,0)}\}_{\gamma=0}^{k-1} \\
&\quad\ \cup \{w_{(0,0,0)}w_{(2,0,d)}w_{(3,d,0)}, w_{(0,0,0)}w_{(1,0,0)}w_{(3,d,0)}\}.
\end{aligned}$$

Our main result is the following.

Theorem 3.2.24. *Let $d \geq 4$ and $k \geq 3$ be integers and $w^\alpha = w^{\alpha_+} - w^{\alpha_-} \in \mathrm{I}(X_d)$ a $k-$binomial. Then,*

(i) if d is even, there exists a $\mathrm{I}(X_d)_k-$sequence from w^{α_+} to w^{α_-}.

(ii) If d is odd and $k \geq 4$, there exists a $\mathrm{I}(X_d)_k-$sequence from w^{α_+} to w^{α_-}.

(iii) If d is odd and $k = 3$, then there exists a $\mathrm{I}(X_d)_3-$sequence from w^{α_+} to w^{α_-} if and only if neither w^{α_+} nor w^{α_-} belong to $\mathfrak{B}_3^\rho$.

As a direct corollary, we have:

Corollary 3.2.25. *Let $d \geq 4$ be an integer.*
(i) If d is even, then $\mathrm{I}(X_d) = (\mathrm{I}(X_d)_2)$.
(ii) If d is odd,

$$\mathrm{I}(X_d) = (\mathrm{I}(X_d)_2) + (w^\alpha \in \mathrm{I}(X_d)_3 \mid w^{\alpha_+} \in \mathfrak{B}_3^\rho \ \textit{or} \ w^{\alpha_-} \in \mathfrak{B}_3^\rho).$$

The rest of this subsection is devoted to prove Theorem 3.2.24. In Proposition 3.2.21, we have shown the converse part of Theorem 3.2.24(iii), so it remains to see that for any $k \geq 3$ and a $k-$binomial w^α such that $w^{\alpha_+}, w^{\alpha_-} \notin \mathfrak{B}_3^\rho$, there exists an $\mathrm{I}(X_d)_k-$sequence from w^{α_+} to w^{α_-}. For simplicity, we often use the following notation for α_+ and α_-, respectively:

$$a(0,0,0) + \sum_{i=1}^{b}(1, \gamma_i^1, \delta_i^1) + \sum_{j=1}^{c}(2, \gamma_j^2, \delta_j^2) + e(3, d, 0)$$

$$A(0,0,0) + \sum_{s=1}^{B}(1, \gamma_s^1, \delta_s^1) + \sum_{r=1}^{C}(2, \gamma_r^2, \delta_r^2) - E(3, d, 0),$$

where $0 \leq a, b, c, e, A, B, C, E \leq n$ are integers and $aA = 0 = eE$. Since w^α is a suitable $k-$binomial, we have restrictions

$$a + b + c + e = A + B + C + E \text{ and } b + 2c + 3e = B + 2C + 3E.$$

The following proposition allows us to focus on k−binomials of the form

$$\prod_{j=1}^{k} w_{(r_{i_j},\gamma_{i_j},\delta_{i_j})} - \prod_{j=1}^{k} w_{(r_{h_j},\gamma_{h_j},\delta_{h_j})}$$

with all $r_{i,j}, r_{h_j} \in \{1,2\}$. With the above notation, $a = e = A = E = 0$.

Proposition 3.2.26. *Let* $w^{\alpha} = w^{\alpha+} - w^{\alpha-}$ *be a non trivial suitable* $k-$*binomial such that* $w_{(0,0,0)} \in supp(w^{\alpha})$ *or* $w_{(3,d,0)} \in supp(w^{\alpha})$*. If* $w^{\alpha+}, w^{\alpha-} \notin \mathcal{M}_3^{\rho}$*, then there exist* $\mathrm{I}(X_d)_k-$*sequences*

$$\{w^{\alpha+},\dots,w^{\alpha'_+}\} \text{ and } \{w^{\alpha'_-},\dots,w^{\alpha-}\}$$

such that $w_{(0,0,0)}, w_{(3,d,0)} \notin supp(w^{\alpha'_+}) \cup supp(w^{\alpha'_-})$*.*

Proof. We write $\alpha^+ = a(0,0,0)+\sum_{i=1}^{b}(1,\gamma_i^1,\delta_i^1)+\sum_{j=1}^{c}(2,\gamma_j^2,\delta_j^2)+e(3,d,0)$ and we assume that $a>0$ or $e>0$. Analogous we deal with α_-. It is enough to see that we can always decrease the value of $a+e$ until we reach 0. We analyse separately several cases according to the value of $d = 2k+\varepsilon = 3k'+\rho$, $\varepsilon \in \{0,1\}$ and $\rho \in \{0,1,2\}$.

Case 1: Assume $\varepsilon = 0$ and $\rho = 0$. The hypothesis w^{α} non-trivial implies $(b,c) \neq (0,0)$ or $(b,c) = (0,0)$ and $a = e$. If $(b,c) = (0,0)$ and $a = e$ we have

$$w^{a}_{(0,0,0)} w^{a}_{(3,d,0)} - w^{a}_{(1,k',0)} w^{a}_{(2,2k',0)}.$$

Otherwise, since $m = w_{(3,d,0)} w_{(1,\gamma_1^1,\delta_1^1)}$ (respectively $m = w_{(0,0,0)} w_{(2,\gamma_1^2,\delta_1^2)}$) admits a special suitable 2−binomial $m - m'$ with $m' = w_{(2,\gamma_{c+1}^2,\delta_{c+1}^2)} w_{(2,\gamma_{c+2}^2,\delta_{c+2}^2)}$ (respectively $m' = w_{(1,\gamma_{b+1}^1,\delta_{b+1}^1)} w_{(1,\gamma_{b+2}^1,\delta_{b+2}^1)}$), we can write

$$w^{a_1} := w^{a}_{(0,0,0)} \prod_{i=2}^{b} w_{(1,\gamma_i^1,\delta_i^1)} \prod_{j=1}^{c+2} w_{(2,\gamma_j^2,\delta_j^2)} w^{e-1}_{(3,d,0)}$$

$$\left(\text{respectively } w^{a_1} := w^{a-1}_{(0,0,0)} \prod_{i=1}^{b+2} w_{(1,\gamma_i^1,\delta_i^1)} \prod_{j=2}^{c} w_{(2,\gamma_j^2,\delta_j^2)} w^{e}_{(3,d,0)}\right)$$

and build an $\mathrm{I}(X_d)_k-$sequence $(w^{\alpha+}, w^{a_1})$ such that

$$\deg_{w(0,0,0)} w^{a_1} + deg_{w(3,d,0)} w^{a_1} < a+e = \deg_{w(0,0,0)} w^{\alpha+} + deg_{w(3,d,0)} w^{\alpha+}.$$

As a result, we have decreased by one the value of $a+e$.

<u>Case 2:</u> Assume $\varepsilon = 0$ and $1 \le \rho \le 2$. The hypothesis w^{α} non-trivial implies $(b,c) \neq (0,0)$ and we can argue as in <u>Case 1</u> unless

$$w^{\alpha+} = w^{a}_{(0,0,0)} w^{b}_{(1,k',0)} w^{c}_{(2,2k',1)} w^{e}_{(3,d,0)}$$

$$(\text{respectively } w^{\alpha+} = w^{a}_{(0,0,0)} w^{b}_{(1,k',1)} w^{c}_{(2,2k'+1,0)} w^{e}_{(3,d,0)})$$

which are monomials not admitting a suitable $k-$binomial $w^{\alpha+} - w^{\alpha-}$.

<u>Case 3:</u> Assume $\varepsilon = 1$ and $\rho = 0$. Since $w_{(0,0,0)} w_{3,d,0)} - w_{(1,k',0)} w_{(2,2k',0)} \in \mathrm{I}(X_d)$, we can argue as in <u>Case 1</u> unless

$$w^{\alpha+} = w^{a}_{(0,0,0)} w^{b}_{(1,0,0)} w^{c}_{(2,0,d)} \text{ or } w^{\alpha+} = w^{b}_{(1,0,0)} w^{c}_{(2,0,d)} w^{e}_{(3,d,0)}.$$

The fact that $w^{\alpha+} - w^{\alpha-}$ is non-trivial implies $b, c > 0$ and the hypothesis $w^{\alpha+} \notin \mathfrak{B}^{0}_{3}$ implies $a+b+c > 3$ (respectively $b+c+e > 3$). Set $m = w_{(0,0,0)} w_{(1,0,0)} w_{(2,0,,d)}$ (respectively $m = w_{(1,0,0)} w_{(2,0,d)} w_{(3,d,0)}$). By Proposition 3.2.20, we have

$$w_{(0,0,0)} w_{(2,0,d)} w_{(1,0,0)} - w_{(1,0,k)} w_{(1,0,k)} w_{(1,0,1)} \in \mathrm{I}(X_d)$$

$$(\text{respectively } w_{(1,0,0)} w_{(2,0,d)} w_{(3,d,0)} - w_{(2,1,d-2\})} w_{(2,k,1)+(2,k,1)}) \in \mathrm{I}(X_d)$$

and we apply the same game decreasing a (respectively e) by one.

<u>Case 4:</u> Assume $\varepsilon = 1$ and $1 \le \rho \le 2$. From the hypothesis w^{α} non trivial, we have $(b,c) \neq 0$. So we proceed as in <u>Case 1</u> unless

$$w^{\alpha^{+}} = w^{a}_{(0,0,0)} w^{b}_{(1,0,0)} w^{c}_{(1,k',0)} w^{f}_{(2,0,d)} w^{g}_{(2,2k',1)} w^{e}_{(3,d,0)}$$

$$(\text{respectively } w^{\alpha+} = w^{a}_{(0,0,0)} w^{b}_{(1,0,0)} w^{c}_{(1,k',1)} w^{f}_{(2,0,d)} w^{g}_{(2,2k'+1,0)} w^{e}_{(3,d,0)})$$

with $(b,c,f,g) \neq (0,0,0,0)$ (respectively $(b,c,f,g) \neq (0,0,0,0)$). Since $w^{\alpha+} \notin \mathfrak{B}^{1}_{3}$, it follows that $(c,g) \neq (0,0)$ or $a+b+f+g+e > 3$. By

Proposition 3.2.20, we have non trivial $3-$binomials:

$$\begin{aligned}
&w_{(0,0,0)}w_{(1,0,0)}w_{(2,0,d)} - w_{(1,0,k)}w_{(1,0,k)}w_{(1,0,1)}\\
&w_{(0,0,0)}w_{(2,0,d)}w_{(3,d,0)} - w_{(1,0,k)}w_{(2,k,\lfloor\frac{k+1}{2}\rfloor)}w_{(2,k+1,\lfloor\frac{k+1}{2}\rfloor)}\\
&w_{(0,0,0)}w_{(1,0,0)}w_{(3,d,0)} - w_{(1,1,0)}w_{(1,k',0)}w_{(2,2k',0)}\\
&w_{(1,0,0)}w_{(2,0,d)}w_{(3,d,0)} - w_{(2,1,d-2)}w_{(2,k,1)}w_{(2,k,1)}
\end{aligned}$$

$$\begin{aligned}
&(\text{respectively } w_{(0,0,0)}w_{(1,0,0)}w_{(2,0,d)} - w_{(1,0,k)}w_{(1,0,k)}w_{(1,0,2)}\\
&w_{(0,0,0)}w_{(2,0,d)}w_{(3,d,0)} - w_{(1,0,k)}w_{(2,k,\lfloor\frac{k+1}{2}\rfloor)}w_{(2,k+1,\lfloor\frac{k+1}{2}\rfloor)}\\
&w_{(0,0,0)}w_{(1,0,0)}w_{(3,d,0)} - w_{(1,1,0)}w_{(1,k',0)}w_{(2,2k'+1,0)}\\
&w_{(1,0,0)}w_{(2,0,d)}w_{(3,d,0)} - w_{(2,1,d-2)}w_{(2,k,1)}w_{(2,k,1)}).
\end{aligned}$$

Then, we argue as in Case 3 decreasing a and e by one unless

$$w^{\alpha_+} = w^a_{(0,0,0)}w^c_{(1,k',0)}w^g_{(2,2k',1)}w^e_{(3,d,0)}$$

$$(\text{respectively } w^a_{(0,0,0)}w^c_{(1,k',1)}w^g_{(2,2k'+1,0)}w^e_{(3,d,0)}),$$

but such monomials do not admit a non trivial suitable $k-$binomial. □

Remark 3.2.27. Any suitable $k-$binomial of the form

$$\prod_{i=1}^{b} w_{(1,\gamma_i^1,\delta_i^1)}\prod_{j=1}^{c} w_{(2,\gamma_j^2,\delta_j^2)} - \prod_{i=1}^{b'} w_{(1,\gamma_i^3,\delta_i^3)}\prod_{j=1}^{c'} w_{(2,\gamma_j^4,\delta_j^4)}$$

satisfies $b = b'$ and $c = c'$.

Example 3.2.28. (i) Take $d = 4$ and consider the non trivial $3-$binomial

$$w_{(0,0,0)}w_{(1,0,0)}w_{(2,0,4)} - w_{(1,0,1)}w_{(1,0,1)}w_{(1,0,2)}.$$

Since $w_{(0,0,0)}w_{(2,0,4)} - w^2_{(1,0,2)}$ is a non trivial $2-$binomial, we define $w^{a_1} = w_{(1,0,0)}w^2_{(1,0,2)}$ and we get an $\mathrm{I}(X_4)_3-$sequence

$$(w_{(0,0,0)}w_{(1,0,0)}w_{(2,0,4)}, w_{(1,0,0)}w^2_{(1,0,2)}, w_{(1,0,1)}w_{(1,0,1)}w_{(1,0,2)})$$

from $w_{(0,0,0)}w_{(1,0,0)}w_{(2,0,4)}$ to $w_{(1,0,1)}w_{(1,0,1)}w_{(1,0,2)}$ where $w^{\alpha_+} = w^{a_1}$.

(ii) Take $d = 5$ and consider the non trivial $4-$binomial

$$w_{(0,0,0)}w_{(1,0,0)}w_{(2,0,5)}w_{(3,5,0)} - w^2_{(1,1,0)}w_{(2,1,3)}w_{(2,2,2)}.$$

We take the 3−binomial $w_{(0,0,0)}w_{(1,0,0)}w_{(2,0,5)} - w_{(1,0,1)}w^2_{(1,0,2)}$ and we define $w^{a_1} := w_{(1,0,1)}w^2_{(1,0,2)}w_{(3,5,0)}$. Observe that $w_{(0,0,0)} \notin \mathrm{supp}(w^{a_1})$. $w_{(1,0,1)}w_{(3,5,0)}$ admits a suitable 2−binomial $w_{(1,0,1)}w_{(3,5,0)} - w_{(2,2,1)}w_{(2,3,0)}$. We define $w^{a_2} := w^2_{(1,0,2)}w_{(2,2,1)}w_{(2,3,0)}$ and we get an $\mathrm{I}(X_5)_4-$sequence

$$(w_{(0,0,0)}w_{(1,0,0)}w_{(2,0,5)}w_{(3,5,0)}, w_{(1,0,1)}w^2_{(1,0,2)}w_{(3,5,0)}, w^2_{(1,0,2)}w_{(2,2,1)}w_{(2,3,0)})$$

with $w_{(0,0,0)}, w_{(3,5,0)} \notin \mathrm{supp}(w^{a_2})$.

In view of Proposition 3.2.26, we analyze when a monomial w of the form $w_{(r_1,\gamma_1,\delta_1)}w_{(r_2,\gamma_2,\delta_2)}$ with $r_1, r_2 \in \{1,2\}$ admits a suitable 2−binomial $w - w'$ with $w' = w_{(r_3,\gamma_3,\delta_3)}w_{(r_4,\gamma_4,\delta_4)}$ and $r_3, r_4 \in \{1,2\}$. This problem can be reformulated as follows. For which integer $s \geq 0$, setting $\gamma_3 := \gamma_1 \pm s$ and $\gamma_4 := \gamma_2 \mp s$, there exist $\max\{0, (r_i - 1)d - 2\gamma_i\} \leq \delta_i \leq \lfloor \frac{r_i d - 3\gamma_i}{2} \rfloor$, $i = 3, 4$, such that $\delta_3 + \delta_4 = \delta_1 + \delta_2$.

Lemma 3.2.29. *With the above notation, there are δ_3 and δ_4 with the following exceptions:*

(i) for any $1 \leq r_1, r_2 \leq 2$, if $(r_1d_1 - 3\gamma_1)$ and $(r_2d_2 - 3\gamma_2)$ are even, s is odd, and δ_1 and δ_2 are the maximum ones. We call it the maximum bound problem (MBP).

(ii) Assume $r_2 = 2$.

1. *If $r_1 = 1$, when doing $\gamma_1 + s$ and $\gamma_2 - s$ we have $\gamma_2 - s < k + \varepsilon$ and $\delta_1 + \delta_2 < max\{0, d - 2\gamma_2 - 2s\}$.*

2. *If $r_1 = 2$, when doing $\gamma_1 + s$ and $\gamma_2 - s$ we have $\delta_1 + \delta_2 < \max\{0, d - 2\gamma_1 - 2s\} + \max\{0, d - 2\gamma_2 + 2s\}$ and we have one of the following cases:*

    (a) $\gamma_1 \geq k + \varepsilon$ and $\gamma_2 - s < k + \varepsilon$,

    (b) $\gamma_1 < k + \varepsilon, \gamma_1 + s \geq k + \varepsilon, \gamma_2 \geq k + \varepsilon$ and $\gamma_1 > \gamma_2 - s$,

    (c) $\gamma_1, \gamma_2 < k + \varepsilon, \gamma_1 + s > k + \varepsilon$.

We call it the minimum bound problem (mbp).

Proof. We have $\max\{0,(r_1-1)d-2\gamma_1\}+\max\{0,(r_2-1)d-2\gamma_2\}\leq\delta_1+\delta_2\leq\lfloor\frac{r_1d-3\gamma_1}{2}\rfloor+\lfloor\frac{r_2d-3\gamma_2}{2}\rfloor$ and $\max\{0,(r_1-1)d-2(\gamma_1+s)\}+\max\{0,(r_2-1)d-2(\gamma_2-s)\}\leq\delta_3+\delta_4\leq\lfloor\frac{r_1d-3(\gamma_1+s)}{2}\rfloor+\lfloor\frac{r_2d-3(\gamma_2-s)}{2}\rfloor$. Therefore, the result holds for values: $\max\{0,(r_1-1)d-2(\gamma_1+s)\}+\max\{0,(r_2-1)d-2(\gamma_2-s)\}\leq\delta_1+\delta_2\leq\lfloor\frac{r_1d-3(\gamma_1+s)}{2}\rfloor+\lfloor\frac{r_2d-3(\gamma_2-s)}{2}\rfloor$.

(i) Using the basic properties of the floor and ceiling functions, we obtain

$$\begin{aligned}\lfloor\frac{r_1d-3\gamma_1}{2}\rfloor+\lfloor\frac{r_2d-3\gamma_2}{2}\rfloor&\leq\lfloor\frac{r_1d-3(\gamma_1+s)+r_2d-3(\gamma_2-s)}{2}\rfloor\\&\leq\lfloor\frac{r_1d-3(\gamma_1+s)}{2}\rfloor+\lfloor\frac{r_2d-3(\gamma_2-s)}{2}\rfloor+1.\end{aligned}$$

Furthermore, $\lfloor\frac{r_1d-3(\gamma_1+s)}{2}\rfloor+\lfloor\frac{r_2d-3(\gamma_2-s)}{2}\rfloor<\lfloor\frac{r_1d-3\gamma_1}{2}\rfloor+\lfloor\frac{r_2d-3\gamma_2}{2}\rfloor$ if and only if $(r_1d-3\gamma_1)$ and $(r_2d-3\gamma_2)$ are even and s is odd.

(ii) It is obtained determining which values satisfy $\max\{0,(r_1-1)d-2\gamma_1\}+\max\{0,(r_2-1)d-2\gamma_2\}\leq\delta_1+\delta_2<\max\{0,(r_1-1)d-2(\gamma_1+s)\}+\max\{0,(r_2-1)d-2(\gamma_2-s)\}$. $\square$

Let $w^\alpha=w^{\alpha_+}-w^{\alpha_-}$ be a non trivial $k-$binomial such that $w^{\alpha_+},w^{\alpha_-}\notin\mathfrak{B}_3^\rho$. By Proposition 3.2.26, if $w_{(0,0,0)}\in\operatorname{supp}(w^\alpha)$ or $w_{(3,d,0)}\in\operatorname{supp}(w^\alpha)$, then there are $\mathrm{I}(X_d)_k-$sequences $(w^{\alpha_+},\ldots,w^{\alpha'_+})$ and $(w^{\alpha'_-},\ldots,w^{\alpha_-})$ such that $w_{(0,0,0)},w_{(3,d,0)}\notin\operatorname{supp}(w^{\alpha'_-})\cup\operatorname{supp}(w^{\alpha_-})$ and we have $w^{\alpha'}:=w^{\alpha'_+}-w^{\alpha'_-}\in\mathrm{I}(X_d)$. Now, $w^{\alpha'}$ could be trivial or zero. In the first case,

$$(w^{\alpha_+},\ldots,w^{\alpha'_+},w^{\alpha'_-},\ldots,w^{\alpha_-})$$

is an $\mathrm{I}(X_d)_k-$sequence. In the second case, let $t_+,t_-\geq0$ be the length of the respectively $\mathrm{I}(X_d)_k-$sequences. Since w^α is non trivial, $t_+>0$ or $t_->0$. Assume $t_+>0$ (analogously, for $t_+=0$ and $t_->0$). Therefore, $(w^{\alpha_+},\ldots,w^{a_{t_+-1}},w^{\alpha'_-},\ldots,w^{\alpha_-})$ is an $\mathrm{I}(X_d)_k-$sequence. Next, we deal with $w^{\alpha'_+}-w^{\alpha'_-}$ being neither trivial nor zero.

Proposition 3.2.30. *Let w^α be a non trivial suitable $k-$binomial of the form*

$$\prod_{i=1}^{t}w_{(1,\gamma_i,\delta_i)}\prod_{i=t+1}^{k}w_{(2,\gamma_i,\delta_i)}-\prod_{i=1}^{t}w_{(1,\gamma'_i,\delta'_i)}\prod_{i=t+1}^{k}w_{(2,\gamma'_i,\delta'_i)}.$$

There are $\mathrm{I}(X_d)_k-$*sequences* $(w^{\alpha+},\dots,w_r^{\alpha+})$ *and* $(w^{\alpha-},\dots,w_u^{\alpha-})$ *with*

$$w_r^{\alpha+}=\prod_{i=1}^{t}w_{(1,\gamma_i^1,\delta_i^1)}\prod_{i=t+1}^{k}w_{(2,\gamma_i^1,\delta_i^1)}\ \text{and}\ w_u^{\alpha-}=\prod_{i=1}^{t}w_{(1,\gamma_i^2,\delta_i^2)}\prod_{i=t+1}^{k}w_{(2,\gamma_i^2,\delta_i^2)}$$

and such that $\gamma_i^1=\gamma_i^2$ *for all* $i=1,\dots,k$.

Proof. We may suppose that $\gamma_1\geq\cdots\geq\gamma_t,\gamma_{t+1}\geq\cdots\geq\gamma_n$ (respectively γ_i'). Let γ_ℓ be the first such that $\gamma_j\neq\gamma_j'$. We may also suppose that $\gamma_\ell=\gamma_\ell'+s$ with $s>0$. Hence $\sum_{j\neq\ell}\gamma_j+s=\sum_{j\neq\ell}\gamma_j'$. Let γ_i be the first such that $\gamma_i<\gamma_i'$ with $i>\ell$ and let $s_i>0$ be such that $\gamma_i+s_i=\gamma_i'$. We distinguish two cases.

Case 1: $s\leq s_i$. According to Lemma 3.2.29, when doing $\gamma_\ell-s$ and γ_i+s the mbp does not occur and the MPB appears when $r_\ell d-3\gamma_\ell$, $r_id-3\gamma_i$ are even, s is odd, $\delta_\ell=\frac{r_\ell d-3\gamma_\ell}{2}$ and $\delta_i=\frac{r_id-3\gamma_i}{2}$. If MBP does not occur, we define:

$$w^{a_2}:=w_{(r_1,\gamma_1,\delta_1)}w_{(r_2,\gamma_2,\delta_2)}\cdots w_{(r_\ell,\gamma_\ell-s,\bar\delta_\ell)}\cdots w_{(r_i,\gamma_i+s,\bar\delta_i)}\cdots w_{(r_n,\gamma_n,\delta_n)}.$$

Then (w^{a_+},w^{a_2}) is an $\mathrm{I}(X_d)-k$sequence and w^{a_2},w^{a_-} share the same γ in position ℓ. Now we assume that the MBP occurs. We divide the discussion in several subcases based on the parity of d.

1.1 $\varepsilon=0$, γ_ℓ and γ_i even and s odd.

1.1 $\varepsilon=1$, $r_l=r_i=2$, γ_ℓ and γ_i even and s odd.

1.3 $\varepsilon=1$, $r_l=r_i=1$, γ_ℓ and γ_i odd and s odd.

1.4 $\varepsilon=1$, $r_l=1$, $r_i=2$, γ_ℓ odd, γ_i even and s odd.

We treat 1.1, the remaining cases follow analogously. We proceed by modifying both w^{a+} and w^{a-}. Doing $\gamma_\ell-(s+1)$ and $\gamma_i+(s+1)$, the MBP does not occur. Since γ_ℓ and γ_i are even and s is odd, we obtain that γ_ℓ' is odd. If $\gamma_i'<r_ik'+\lfloor\frac{r_i\rho}{3}\rfloor$, then we do $\gamma_\ell'-1$ and $\gamma_i'+1$, since the mbp does not occur. We set

$$w^{a_2}=w_{(r_1,\gamma_1,\delta_1)}\cdots w_{(r_\ell,\gamma_\ell-(s+1),\bar\delta_\ell)}\cdots w_{(r_i,\gamma_i+s+1,\bar\delta_i)}\cdots w_{(r_n,\gamma_n,\delta_n)}$$

$$w^{a_2'} = w_{(r_1,\gamma_1',\delta_1')} \cdots w_{(r_\ell,\gamma_\ell'-1,\bar{\delta}_\ell')} \cdots w_{(r_i,\gamma_i'+1,\bar{\delta}_i')} \cdots w_{(r_n,\gamma_n',\delta_n')}.$$

(w^{α_+}, w^{a_2}) and $(w^{a_2'}, w^{\alpha_-'})$ are $\mathrm{I}(X_d)_k$−sequences and $w^{a_2}, w^{a_2'}$ share the same γ in position ℓ.

If $\gamma_i' = r_i k' + \lfloor \frac{r_i \rho}{3} \rfloor$, we can consider $\gamma_\ell' + 1$, $\gamma_i' - 1$ and set

$$w^{a_2} = w_{(r_1,\gamma_1,\delta_1)} \cdots w_{(r_\ell,\gamma_{l\ell}(s-1),\bar{\delta}_\ell)} \cdots w_{(r_i,\gamma_i+(s-1),\bar{\delta}_i)} \cdots w_{(r_n,\gamma_n,\delta_n)}$$

$$w^{a_{2'}} = w_{(r_1,\gamma_1,\delta_1)} \cdots w_{(r_\ell,\gamma_\ell'+1,\bar{\delta}_\ell')} \cdots w_{(r_i,\gamma_i'-1,\bar{\delta}_i')} \cdots w_{(r_n,\gamma_n',\delta_n')}.$$

In any case, w^{a_2} and w^{α_-} (respectively $w^{a_2'}$) share the same γ in position ℓ.

Case 2: $s > s_i$. Arguing as in Case 1, we distinguish cases 1.1, 1.2, 1.3 and 1.4 and we treat the first one. Assume that γ_ℓ and γ_i are even, s is odd and

$$\delta_\ell = \frac{r_\ell d - 3\gamma_\ell}{2}, \quad \delta_i = \frac{r_i d - 3\gamma_i}{2}.$$

We have that γ_i' is odd and we can argue as in Case 1 if we do $\gamma_\ell'+1$ and $\gamma_i'-1$. Since $s > s_i$, w^{a_2} and w^{α_-} (respectively $w^{a_2'}$) verifies the same hypothesis as w^{α_+} and w^{α_-} with $\gamma_\ell - s_i$ and γ_ℓ' (respectively $\gamma_\ell - (s_i - 1)$ and $\gamma_\ell' + 1$) in position ℓ. We apply the same strategy to w^{a_2} and we continue until, in step $t > 1$, the resulting monomial w^{a_t} verifies Case 1.

The result follows from iterating the above argument. □

Remark 3.2.31. If $w_r^{\alpha_+} - w_u^{\alpha_-}$ is trivial, we obtain an $\mathrm{I}(X_d)_k$−sequence from w^{α_+} to w^{α_-}.

Example 3.2.32. In Example 3.2.28(ii), we had

$$w_{(0,0,0)} w_{(1,0,0)} w_{(2,0,5)} w_{(3,5,0)} - w_{(1,1,0)}^2 w_{(2,1,3)} w_{(2,2,2)}$$

and we have build the $\mathrm{I}(X_5)_4$−sequence

$$(w_{(0,0,0)} w_{(1,0,0)} w_{(2,0,5)} w_{(3,5,0)}, w_{(1,0,1)} w_{(1,0,2)}^2 w_{(3,5,0)}, w_{(1,0,2)}^2 w_{(2,2,1)} w_{(2,3,0)}).$$

Now we apply Proposition 3.2.30 to the non trivial 4−binomial

$$w_{(1,1,0)}^2 w_{(2,1,3)} w_{(2,2,2)} - w_{(1,0,2)}^2 w_{(2,2,1)} w_{(2,3,0)}.$$

We have $\gamma_1 = \gamma_2 = 0, \gamma_3 = 2, \gamma_4 = 3$ and $\gamma_1' = \gamma_2' = 1, \gamma_3 = 1, \gamma_4 = 2$, with $\gamma_1 = \gamma_1' + 1$. The first $\gamma_i < \gamma_i'$ corresponds to γ_3 with $s_3 = 1$.

Then we choose the suitable $2-$binomial $w_{(1,1,0)}w_{(2,1,3)} - w_{(1,0,1)}w_{(2,2,2)}$ and we define $w^{a_2} := w_{(1,0,1)}w_{(1,1,0)}w^2_{(2,2,2)}$. Note the γ's involved in w^{a_2} by $\tilde{\gamma}_i$, $i = 1, 2, 3, 4$. Now $\tilde{\gamma}_1 = \gamma'_1, \tilde{\gamma}_2 = 1$, $\tilde{\gamma}_3 = \tilde{\gamma}_4 = 2$. The first $\tilde{\gamma}_i > \gamma'_i$ is $\gamma_2 = \gamma'_2 + 1$ and the first $\gamma_j < \gamma'_j$ with $j \geq 3$ is $\gamma_4 = 2$ with $s_4 = 1$. Then we choose the suitable $2-$binomial $w_{(1,1,0)}w_{(2,2,2)} - w_{(1,0,2)}w_{(2,3,0)}$ and we define $w^{a_3} := w_{(1,0,1)}w_{(1,0,2)}w_{(2,2,2)}w_{(2,3,0)}$. We have obtained an $\mathrm{I}(X_5)_4-$sequence from $w_{(0,0,0)}w_{(1,0,0)}w_{(2,0,5)}w_{(3,5,0)}$ to $w^2_{(1,1,0)}w_{(2,1,3)}w_{(2,2,2)}$. Precisely,

$$(w_{(0,0,0)}w_{(1,0,0)}w_{(2,0,5)}w_{(3,5,0)}, w_{(1,0,1)}w^2_{(1,0,2)}w_{(3,5,0)}, w^2_{(1,0,2)}w_{(2,2,1)}w_{(2,3,0)},$$
$$w_{(1,0,1)}w_{(1,0,2)}w_{(2,2,2)}w_{(2,3,0)}, w_{(1,0,1)}w_{(1,1,0)}w^2_{(2,2,2)}, w^2_{(1,1,0)}w_{(2,1,3)}w_{(2,2,2)}).$$

Finally, as a consequence of Proposition 3.2.30, to prove Theorem 3.2.24 if suffices to show that any non trivial non zero $k-$binomial:

$$w_r^{\alpha+} - w_u^{\alpha-} = \prod_{i=1}^{t} w_{(1,\gamma_i^1,\delta_i^1)} \prod_{i=t+1}^{n} w_{(2,\gamma_i^1,\delta_i^1)} - \prod_{i=1}^{t} w_{(1,\gamma_i^2,\delta_i^2)} \prod_{i=t+1}^{n} w_{(2,\gamma_i^2,\delta_i^2)}$$

with $\gamma_i^1 = \gamma_i^2$ for all $1 \leq i \leq n$ admits an $\mathrm{I}(X_d)_k-$sequence. We proceed as follows.

Let $1 \leq i \leq k$ be an integer. If $\delta_i^1 < \delta_i^2$, we set $a_i = \delta_i^2 - \delta_i^1$ and $b_i = 0$, otherwise we set $a_i = 0$ and $b_i = \delta_i^1 - \delta_i^2$. Therefore,

$$\delta_1^1 + a_1 - b_1 + \cdots + \delta_n^1 + a_n - b_n = \delta_1^2 + \cdots + \delta_n^2,$$

and we have the equality $a_1 + \cdots + a_n = b_1 + \cdots + b_n$. We may assume that $a_1 > 0$. Hence, $\delta_2^1 + \cdots + \delta_n^1 > \delta_2^2 + \cdots + \delta_n^2$. Without loss of generality, we can suppose that for all $2 \leq i \leq n$, $\delta_i^1 > \delta_i^2$. So, $b_i > 0$ and $\delta_i^1 + b_i = \delta_i^2$. Thus, $a_1 \leq b_2 + \cdots + b_n$ and we can consider $c_i \leq b_i$ such that $a_1 = c_2 + \cdots + c_n$. We set

$$w^{a_2} = w_{(r_1,\gamma_1^1,\delta_1^1+c_2)}w_{(r_2,\gamma_2^1,\delta_2^1-c_2)}w_{(r_3,\gamma_3^1,\delta_3^1)}\cdots w_{(r_n,\gamma_n^1,\delta_n^1)}.$$

$(w_r^{\alpha+}, w^{a_2})$ is an $\mathrm{I}(X_d)_k-$sequence. If $\delta_2^1 - c_2 = \delta_2^2$, then $(w_r^{\alpha+}, w^{a_2}, w_u^{\alpha-})$ is an $\mathrm{I}(X_d)_k-$sequence and we finish. Else, inductively for $2 < i \leq k$, we define w^{a_i} as:

$$w_{(r_1,\gamma_1^1,\delta_1^1+c_2+\cdots+c_i)}w_{(r_2,\gamma_2^1,\delta_2^1-c_2)}\cdots w_{(r_i,\gamma_i^1,\delta_i^1-c_i)}w_{(r_{i+1},\gamma_{i+1}^1,\delta_{i+1}^1)}\cdots w_{(r_k,\gamma_k^1,\delta_k^1)}.$$

At some step $2 \leq i \leq k$, we achieve $w^{a_i} - w^{\alpha-}$ trivial. As a result, we construct an $\mathrm{I}(X_d)_k-$sequence $(w^{\alpha+}, w^{a_2}, \ldots, w^{a_i}, w^{\alpha-})$. The proof of Theorem 3.2.24 is now complete.

3.3 The canonical module of $\overline{G}-$varieties

In this section, we study the canonical module ω_{X_d} of any $\overline{G}-$variety X_d with finite abelian group $G \subset \mathrm{GL}(n+1,\mathbb{K})$. In Theorem 2.2.18, we have seen that $A(X_d) \cong R^{\overline{G}}$. Moreover, $R^{\overline{G}}$ is the semigroup ring associated to the normal affine semigroup $H_{\mathcal{A}} \subset \mathbb{Z}^{n+1}_{\geq 0}$, i.e. $R^{\overline{G}} = \mathbb{K}[H_{\mathcal{A}}]$. This connection allows us to identify ω_{X_d} with the ideal of $R^{\overline{G}}$ generated by all monomials $m = x_0^{a_0}\cdots x_n^{a_n} \in R^{\overline{G}}$ of degree d and $2d$ satisfying $a_0\cdots a_n \neq 0$ (Theorem 3.3.3), to derive information of the Hilbert series of $A(X_d)$, and to characterize the Castelnuovo–Mumford regularity $\mathrm{reg}(A(X_d))$ of $A(X_d)$ (Theorem 3.3.5). In Subsection 3.3.1, we focus the relation between ω_{X_d} and a minimal graded free $S-$resolution $F_\bullet$ of $A(X_d)$. We investigate the CM–type of $A(X_d)$ and we give families of examples of $\overline{G}-$varieties whose homogeneous coordinate ring is a level ring and, in particular, a Gorenstein ring, i.e. of CM–type one. For sake of completeness, we gather all the results we have obtained so far for the Hilbert function and series, the Castelnuovo–Mumford regularity, the homogenous ideal and the canonical module in the interest of the Betti diagram of $A(X_d)$.

We fix integers $2 \leq n < d$ and we consider an abelian group

$$G := \langle M_{d_1;\alpha^1_{\sigma_1(0)},\ldots,\alpha^1_{\sigma_1(n)}}, \ldots, M_{d_s;\alpha^s_{\sigma_s(0)},\ldots,\alpha^s_{\sigma_s(n)}} \rangle \subset \mathrm{GL}(n+1,\mathbb{K})$$

of order $d = d_1\cdots d_s$. As usual, we denote by $\mathcal{B}_1 = \{m_1,\ldots,m_{\mu_d}\}$ the minimal set of fundamental monomial invariants of $\overline{G}$ (Theorem 2.2.11). $\mathbb{K}[H_{\mathcal{A}}]$ is the semigroup ring associated to the normal affine semigroup $H_{\mathcal{A}} \subset \mathbb{Z}^{n+1}_{\geq 0}$ of all $\mathbb{Z}^{n+1}_{\geq 0}-$solution of the linear system of congruences:

$$(*)_{\mathcal{A};t,r_1,\ldots,r_s} : \left\{ \begin{array}{ccccccccc} y_0 & + & y_1 & + & \cdots & + & y_n & = & td \\ \alpha^1_{\sigma_1(0)}y_0 & + & \alpha^1_{\sigma_1(1)}y_1 & + & \cdots & + & \alpha^1_{\sigma_1(n)}y_n & = & r_1d_1 \\ & & & & & & & \vdots & \\ \alpha^s_{\sigma_s(0)}y_0 & + & \alpha^s_{\sigma_s(1)}y_1 & + & \cdots & + & \alpha^s_{\sigma_s(n)}y_n & = & r_sd_s \end{array} \right.$$

$$t \geq 0,\ 0 \leq r_i \leq \frac{\alpha^i_n td}{d_i},\ i = 1,\ldots,s.$$

We recall that the *relative interior* of $H_{\mathcal{A}}$ is the set $\operatorname{relint}(H_{\mathcal{A}})$ of all points $(a_0, \dots, a_n) \in H_{\mathcal{A}}$ such that $0 \neq a_0 \cdots a_n$. Given a subset $H \subset H_{\mathcal{A}}$, we denote by $\mathrm{I}(H) \subset \mathbb{K}[H_{\mathcal{A}}]$ the ideal generated by all monomials m_l with $l \in H$. With this notation, we have

$$\mathrm{I}(\operatorname{relint}(H_{\mathcal{A}})) = (m \in \mathcal{B}_t \mid l_m \in \operatorname{relint}(H_{\mathcal{A}})) \subset \mathbb{K}[H_{\mathcal{A}}],$$

and by Proposition 1.2.8, $\mathrm{I}(\operatorname{relint}(H_{\mathcal{A}}))$ is a radical ideal of $\mathbb{K}[H_{\mathcal{A}}]$.

Stanley [79] and Danilov [22] proved independently that $\mathrm{I}(\operatorname{relint}(H))$ is the canonical module of the semigroup $\mathbb{K}[H]$ of any normal affine semigroup $H \subset \mathbb{Z}^{n+1}_{\geq 0}$. Then, we have:

Theorem 3.3.1. $\mathrm{I}(\operatorname{relint}(H_{\mathcal{A}}))$ *is the canonical module of* $R^{\overline{G}}$.

Proof. See [79, Theorem 6.7] or [9, Theorem 6.4.5]. □

As usual, we take variables $w_1, \dots, w_{\mu_d}$ and $S = \mathbb{K}[w_1, \dots, w_{\mu_d}]$. The canonical module ω_{X_d} of $A(X_d) = S/\mathrm{I}(X_d)$ is the ideal of $A(X_d)$ generated by the classes mod $\mathrm{I}(X_d)$ of the monomials $w \in S$ such that $\rho(w) \in \mathrm{I}(\operatorname{relint}(H_{\mathcal{A}}))$, where $\rho : S \to \mathbb{K}[\mathcal{B}_1]$ is the morphism defined by $\rho(w_i) = m_i$, $i = 1, \dots, mu_d$. Let us see an example.

Example 3.3.2. Take $G = \langle M_{3;0,1,2} \rangle \subset \mathrm{GL}(3, \mathbb{K})$ a cyclic group of order 3. $\mathbb{K}[H_{\mathcal{A}}]$ is the semigroup ring associated to the normal affine semigroup $H_{\mathcal{A}} \subset \mathbb{Z}^3_{\geq 0}$ of the $\mathbb{Z}^3_{\geq 0}$−solutions of the linear system of congruences:

$$(*)_{\mathcal{A};t,r} : \left\{ \begin{array}{ccccccc} y_0 & + & y_1 & + & y_2 & = & 3t \\ & & y_1 & + & 2y_2 & = & 3r \end{array} \right. \quad t \geq 0,\ r = 0, \dots, 6t.$$

By Theorem 2.2.11, $H_{\mathcal{A}}$ is minimally generated by

$$\{(3,0,0), (0,3,0), (0,0,3), (1,1,1)\}.$$

We have that $\mathrm{I}(\operatorname{relint}(H_{\mathcal{A}}))$ is the ideal generated by all monomials $m = x_0^{a_0} x_1^{a_1} x_2^{a_2}$ such that $1 \leq a_i$, $i = 0,1,2$. Therefore, any monomial $m \in \mathrm{I}(\operatorname{relint}(H_{\mathcal{A}}))$ is divisible by $x_0 x_1 x_2 \in \mathbb{K}[H_{\mathcal{A}}]$ and we obtain $\mathrm{I}(\operatorname{relint}(H_{\mathcal{A}}) = (x_0 x_1 x_2)$. By Theorem 3.3.1, the canonical module of $R^{\overline{G}}$ is the principal ideal $(x_0 x_1 x_2) \subset R^{\overline{G}}$. Indeed, X_3 is a cubic surface in $\mathbb{P}^3$. The canonical module of any cubic surface $X \subset \mathbb{P}^3$ is $\omega_X \cong \mathcal{O}_X(1)$.

Thus, Theorem 3.3.1 provides a combinatoric interpretation of the canonical module of $A(X_d)$ in terms of the monomial invariants of $\overline{G}$. For each $1 \leq t$, we denote by $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_t \subset \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))$ the set of all monomials of degree td. With this notation,

$$\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}})) = \sum_{1 \leq t} (\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_t).$$

It is natural to ask what can be said about a minimal set of generators of $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))$.

Theorem 3.3.3. $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}})) = (\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1, \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_2)$.

Proof. We fix an integer $k \geq 3$, it is enough to show that any monomial $m \in \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_k$ is divisible by a monomial $m' \in \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_{k-1}$. Let $m = x_0^{a_0} \cdots x_n^{a_n} \in \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_k$ and set $m_1 = m/(x_0 \cdots x_n) = x_0^{a_0-1} \cdots x_n^{a_n-1}$. Since $d \geq n+1$ and $k \geq 3$, m' is a monomial of degree $kd - (n+1) \geq 2d$. We define the sequence of integers $L = (\alpha_0, \overset{a_0-1}{\ldots}, \alpha_0, \ldots, \alpha_n, \overset{a_n-1}{\ldots}, \alpha_n)$. By Lemma 2.2.9, there exists a zero–sum subsequence

$$L' = (\alpha_0, \overset{b_0}{\ldots}, \alpha_0, \ldots, \alpha_n, \overset{b_n}{\ldots}, \alpha_n) \subset L.$$

Therefore, L' gives rise a monomial $m_2 := x_0^{b_0} \cdots x_n^{b_n} \in R^{\overline{G}}$ of degree d which divides m_1. Hence, we can factorize $m = m_2 m'$ and, by construction, $m' \in \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_{k-1}$ is the required monomial. $\square$

We illustrate Theorem 3.3.3 with an example.

Example 3.3.4. Take $G = \langle M_{6;0,1,2,3} \rangle \subset \mathrm{GL}(4, \mathbb{K})$ a cyclic group of order 6. We have that

$$\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1 = \{x_0^3 x_1 x_2 x_3, x_0 x_1 x_2 x_3^3\}$$

and $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_2$ is the following set of monomials:

$$\begin{aligned}&\{x_3x_0^9x_1x_2, x_3x_0^3x_1^7x_2, x_3x_0^4x_1^5x_2^2, x_3x_0^5x_1^3x_2^3, x_3^2x_0^5x_1^4x_2, x_3x_0^6x_1x_2^4, x_3^2x_0^6x_1^2x_2^2,\\ &x_3^3x_0^7x_1x_2, x_3x_0x_1^5x_2^5, x_3^2x_0x_1^6x_2^3, x_3^3x_0x_1^7x_2, x_3x_0^2x_1^3x_2^6, x_3^2x_0^2x_1^4x_2^4, x_3^3x_0^2x_1^5x_2^2,\\ &x_3x_0^3x_1x_2^7, x_3^2x_0^3x_1^2x_2^5, x_3^3x_0^3x_1^3x_2^3, x_3^4x_0^3x_1^4x_2, x_3^3x_0^4x_1x_2^4, x_3^4x_0^4x_1^2x_2^2, x_3^5x_0^5x_1x_2,\\ &x_3^3x_0x_1x_2^7, x_3^4x_0x_1^2x_2^5, x_3^5x_0x_1^3x_2^3, x_3^6x_0x_1^4x_2, x_3^5x_0^2x_1x_2^4, x_3^6x_0^2x_1^2x_2^2, x_3^7x_0^3x_1x_2,\\ &x_3^9x_0x_1x_2\}.\end{aligned}$$

Only the following four monomials $x_0x_1^5x_2^5x_3, x_0x_1^6x_2^3x_3^2, x_0^2x_1^3x_2^6x_3, x_0^2x_1^4x_2^4x_3^2 \in$ $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_2$ do not belong to the ideal $(\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1)$. From this observation and Theorem 3.3.3, we obtain $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}})) = (x_0^3x_1x_2x_3, x_0x_1x_2x_3^3,$ $x_0x_1^5x_2^5x_3, x_0x_1^6x_2^3x_3^2, x_0^2x_1^3x_2^6x_3, x_0^2x_1^4x_2^4x_3^2) \subset R^{\overline{G}}$.

We concern about the Hilbert series and the Castelnuovo–Mumford regularity of $A(X_d)$. By Theorem 2.2.14, $R^{\overline{G}}$ is a free $\mathbb{K}[x_0^d, \dots, x_n^d]-$module with a Hironaka decomposition

$$R^{\overline{G}} = \bigoplus_{i=0}^{D} \theta_i \mathbb{K}[x_0^d, \dots, x_n^d].$$

We called $\theta_0, \theta_1, \dots, \theta_D$ a set of secondary invariants of $\overline{G}$. They are the monomial invariants of $\overline{G}$ of degree at most nd representing the monomial $\mathbb{K}-$basis of the quotient algebra $R^{\overline{G}}/(x_0^d, \dots, x_n^d)R^{\overline{G}}$. By Proposition 3.1.2, we have

$$\mathrm{HS}(A(X_d), z) = \frac{\delta_n z^n + \cdots + \delta_1 z + 1}{(1-z)^{n+1}},$$

where $(1, \delta_1, \dots, \delta_n)$ is the sequence of multiplicities of degrees of $\theta_0, \dots, \theta_D$. Moreover, from Proposition 3.1.15

$$\mathrm{reg}(A(X_d)) = 1 + \deg(\delta_n z^n + \cdots + \delta_1 z + 1).$$

A consequence of Theorem 3.3.3 is the following.

Theorem 3.3.5. *For the Castelnuovo–Mumford regularity of $A(X_d)$ the following inequality yields*

$$n \leq \mathrm{reg}(A(X_d)) \leq n + 1.$$

The equality $\mathrm{reg}(A(X_d)) = n + 1$ *holds if and only if* $\emptyset \neq \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1$.

Proof. For each $1 \leq i \leq n$, δ_i is the number of monomials $m = x_0^{a_0} \cdots x_n^{a_n} \in$ $R^{\overline{G}}$ of degree id such that $a_0 < d, \dots, a_n < d$. We have that $\mathrm{reg}(A(X_d)) =$ $n+1$ if and only if $\delta_n > 0$. By Theorem 3.3.3, $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))$ is generated by monomials of degree d and $2d$. Assume that $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1$ is not empty and let $m = x_0^{a_0} \cdots x_n^{a_n} \in \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1$. Since $\deg(m) = d$, it follows that $0 < a_0, \dots, a_n < d$. Therefore, the monomial $x_0^{b_0} \cdots x_n^{b_n} = (x_0^d \cdots x_n^d)/m$ is a secondary invariant of degree nd, so $0 < \delta_n$ and hence $\operatorname{reg}(A(X_d)) = n+1$. Conversely, if $\operatorname{reg}(A(X_d)) = n+1$, then there exists a secondary invariant $x_0^{b_0} \cdots x_n^{b_n}$ of degree nd. Since $b_0 < d, \ldots, b_n < d$, we directly obtain that

$$x_0^{d-b_0} \cdots x_n^{d-b_n} \in \mathrm{I}(\operatorname{relint}(H_{\mathcal{A}}))_1.$$

This proves that $\operatorname{reg}(A(X_d)) = n+1$ if and only if $\emptyset \neq \mathrm{I}(\operatorname{relint}(H_{\mathcal{A}}))_1$. Suppose that $\emptyset = \mathrm{I}(\operatorname{relint}(H_{\mathcal{A}}))_1$. By Theorem 3.3.3, $\mathrm{I}(\operatorname{relint}(H_{\mathcal{A}}))_2$ contains at least one monomial $m = x_0^{a_0} \cdots x_n^{a_n}$. We see that necessarily $a_0 < d, \ldots, a_n < d$. Notice that by hypothesis $0 < a_0, \ldots, 0 < a_n$. We distinguished the following two cases. If for some index $0 \leq i \leq n$, it holds $d < a_i$, then $m/x_i^d \in \mathrm{I}(\operatorname{relint}(H_{\mathcal{A}}))_1$ and we arrive to a contradiction. Thus, $a_i \leq d$, $i = 0, \ldots, n$. Since $\deg(m) = 2d$, if $a_i = d$, then for all $0 \leq i \neq j \leq n$, $a_j < d$. If $a_i = d$ occurs, then m/x_i is a monomial of degree $2d-1$. Hence, there exists $m' \in \mathcal{B}_1$ dividing m/x_i. By construction, the monomial $(x_0^d \cdots x_n^d)/m'$ is a secondary invariant of degree nd and so $\operatorname{reg}(A(X_d)) = n+1$, which contradicts our assumption $\emptyset = \mathrm{I}(\operatorname{relint}(H(\mathcal{A})))_1$. $\square$

3.3.1 On a minimal free resolution of $\overline{G}-$varieties

In this subsection, we focus our attention on a minimal graded free $S-$resolution of the coordinate ring of any $\overline{G}-$variety X_d with group $G \subset \mathrm{GL}(n+1, \mathbb{K})$. We gather the results obtained along this chapter to investigate the CM–type and the Betti diagram of $A(X_d)$. In particular, the results on the Hilbert series, the homogeneous ideal and the Castelnuovo–Mumford regularity (Theorems 2.4.10 and 3.3.5) allows us to complete the picture in the case of $\overline{G}-$surfaces with group $G \subset \mathrm{GL}(3, \mathbb{K})$.

As usual, we denote $c := \operatorname{codim}(X_d) = \mu_d - n - 1$. We have the following.

Proposition 3.3.6. *(i)* $\beta_1 = \beta_{1,1}, \beta_{1,2}, 0, \ldots, 0$.

(ii) *If* $\operatorname{reg}(A(X_d)) = n+1$, *then the cth Betti number of* $A(X_d)$ *is of the form*

$$\beta_c = 0, \ldots, 0, \beta_{c,n-1}, \delta_n.$$

Otherwise, $A(X_d)$ *is a level ring of CM–type* δ_{n-1} *and* $\operatorname{reg}(A(X_d)) = n$.

Proof. (i) It follows directly from Theorem 2.4.10.

(ii) The cth Betti number of $A(X_d)$ describes the degrees of a set of minimal generators of the canonical module of $A(X_d)$. Hence, the result follows from Theorems 3.3.3 and 3.3.5. $\square$

Any $GT-$surface with cyclic group $G = \langle M_{d;0,\alpha_1,\alpha_2}\rangle \subset \mathrm{GL}(3,\mathbb{K})$ of order $3 \leq d$ with $0 < \alpha_1 < \alpha_2 < d$ is an example of a $\overline{G}-$variety with level homogeneous coordinate ring and Castelnuovo–Mumford regularity 3 (Corollary 3.1.22). Next, we introduce a family of arithmetically Gorenstein $GT-$varieties X_d in dimensions $n > 2$ with $\mathrm{reg}(A(X_d)) = n+1$. We use the structure of their homogeneous coordinate rings to construct from them families of $GT-$varieties X_d with level coordinate rings $A(X_d)$ and $\mathrm{reg}(A(X_d)) = n+1$.

Proposition 3.3.7. *Let $n \geq 2$ be an even integer and $G = \langle M_{n+1;0,1,2,\dots,n}\rangle \subset \mathrm{GL}(n+1,\mathbb{K})$ a cyclic group of order $n+1$. Then $R^{\overline{G}}$ is a Gorenstein ring and* $\mathrm{reg}(R^{\overline{G}}) = n+1$.

Proof. Notice that $m = x_0\cdots x_n \in R^{\overline{G}}$. Indeed, m is of degree $n+1$ and its associated point $(1,\dots,1)$ verifies $1+2+\cdots+n = \frac{n}{2}(n+1)$. Since $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}})) = \{x_0^{a_0}\dots x_n^{a_n} \in R^{\overline{G}} \mid 0 \neq a_0\cdots a_n\}$, then any monomial of $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))$ is divisible by m and, hence, $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}})) = (m)$ is a principal ideal. Thus, $R^{\overline{G}}$ is Gorenstein and by Theorem 3.3.3, $\mathrm{reg}(R^{\overline{G}}) = n+1$. $\square$

Proposition 3.3.8. *Fix integers $k \geq 1$ and $2 \leq n < d$ with n even and $G_k = \langle M_{kd;\alpha_0,\dots,\alpha_n}\rangle \subset \mathrm{GL}(n+1,\mathbb{K})$ be a finite cyclic group kd. If $R^{\overline{G}_1}$ is a Gorenstein ring, then $R^{\overline{G}_k}$ is a level ring.*

Proof. We denote by $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}^k))$ the canonical module of $R^{\overline{G}_k}$ (Theorem 3.3.1). The hypothesis $R^{\overline{G}_1}$ is a Gorenstein ring implies $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}^1))_1 = (m)$, where $m \in R^{\overline{G}_1}$. By Theorem 3.3.3,

$$\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}^k)) = (\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}^k))_1, \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}^k))_2).$$

Thus, the result follows from proving that any monomial $m' \in \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}^k))_2$ is divisible by a monomial $\bar{m} \in \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}^k))_1$. We fix $m' = x_0^{a_0}\cdots x_n^{a_n} \in \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}^k))_2$. Notice that m' is an invariant of G_1, so $m' \in \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}^1))_k$. Then, m divides m'. We define $m_1 = \frac{m'}{m}$. Since $m_1 \in R^{\overline{G}_1}$ is a monomial of degree $(2k-1)d$, by Theorem 2.2.11, we can factorize m' as a product of $2k$ monomials of $R^{\overline{G}_1}$, namely

$$m_1 = m_2 \cdots m_{2k}.$$

Hence $m' = mm_2 \cdots m_{2k}$. For each monomial m_i, $1 \leq i \leq 2k$, there is a unique integer $r_i \geq 0$ such that the lattice point l_{m_i} is a solution of the linear system $(*)_{\mathcal{A};1,r_i}$ of congruences associated to $\overline{G}_1$. By Lemma 2.2.9, there is a zero-sum subsequence $(r_{i_1}, \dots, r_{i_k}) \subset (r_1, \dots, r_{2k})$ over $\mathbb{Z}/k\mathbb{Z}$. Therefore, we obtain that $m_{i_1} \cdots m_{i_k} \in R^{\overline{G}_k}$ and

$$\overline{m} = m'/(m_{i_1} \cdots m_{i_k}) \in \mathrm{I}(\mathrm{relint}(H^k_{\mathcal{A}}))_1$$

is the required monomial. $\square$

Corollary 3.3.9. *Fix integers $1 \leq k$ and $2 \leq n$ with n even. Let $G_k = \langle M_{k(n+1);0,1,2,\dots,n} \rangle \subset \mathrm{GL}(n+1, \mathbb{K})$ be a cyclic group of order $k(n+1)$. Then $R^{\overline{G}_k}$ is a level ring.*

Proof. It follows directly from Propositions 3.3.7 and 3.3.8. $\square$

One of the parameters that measures the *complexity* of a minimal graded free resolution if the Castelnuovo–Mumford regularity. In Theorem 3.3.3, we have proved that

$$n \leq \mathrm{reg}(A(X_d)) \leq n+1.$$

On the other hand, $\mathrm{reg}(A(X_d))$ is the number of rows of the Betti diagram of $A(X_d)$. This relates $\mathrm{reg}(A(X_d))$ with the number of linear strands in a minimal graded free $S-$ resolution $F_\bullet$ of $A(X_d)$.

Definition 3.3.10. Let $N_\bullet$ be a complex of graded free $S-$modules:

$$N_\bullet : \qquad \cdots \longrightarrow N_i \xrightarrow{d_i} N_{i-1} \longrightarrow \cdots$$

with $N_i = \bigoplus_l S(-\alpha_{i,l})^{\beta_{i,l}}$ generated in degrees $j \geq i$ and $d_i N_i \subset \mathfrak{m} N_{i-1}$. The *linear strand* of $N_\bullet$ is the free subcomplex $N^1_\bullet \subset N_\bullet$ of free $S-$modules $N^1_i = S(-i)^{\beta_{i,l}}$. The Betti diagram of $N^1_\bullet$ is the ith row of the Betti table of $N_\bullet$. The *second linear strand* of $N_\bullet$ is the linear strand of the free complex $N_\bullet/N^1_\bullet(1)$, denoted $N^2_\bullet$. Continuing in this way, one can define the *rth linear strand* of $N_\bullet$ whose Betti diagram is the rth row of the Betti table of $N_\bullet$.

Graphically, the non trivial rows of the Betti diagram of $A(X_d)$ shows us how many strands a minimal graded free S−resolution $F_\bullet$ of $A(X_d)$ is made of. One can consider these strands as the building blocks of the resolution (see, for instance, [26]). In Theorem 2.4.10, we have proved that the homogeneous ideal $\mathrm{I}(X_d)$ of X_d is generated by binomials of degree at most 3. Moreover, $\mathrm{I}(X_d)$ does not contain any homogeneous linear form, so for the the initial degree j_0 of $\mathrm{I}(X_d)$ we have $2 \leq j_0 \leq 3$. Hence, the number of non zero strands in a minimal graded free S−resolution $F_\bullet$ of $A(X_d)$ increases linearly with the dimension of X_d. Moreover, the last Betti number β_c shows that the lengths of these strands are, in general, strictly smaller that $c = \mathrm{codim}(X_d) = \mathrm{pdim}(X_d)$ (Proposition 3.3.6).

Let us focus now on $\overline{G}$−surfaces. We fix a finite abelian group $G \subset \mathrm{GL}(3, \mathbb{K})$ of order $d = d_1 \cdots d_s \geq 3$. We set $\overline{d} = d^3/|\overline{G}|$. By Theorem 3.3.5, a GT−surface X_d with group G has $\mathrm{reg}(A(X_d)) \leq 3$. Therefore, $A(X_d)$ is a level ring (see [88]). There are three main types of $\overline{G}$−surfaces X_d with group G depending on the Castelnuovo–Mumford regularity and the CM–type of their homogeneous coordinate rings:

(A) $\mathrm{reg}(A(X_d)) = 2$.

(B) $\mathrm{reg}(A(X_d)) = 3$ and $A(X_d)$ is a Gorenstein ring.

(C) $\mathrm{reg}(A(X_d)) = 3$ and $A(X_d)$ is level of CM–type strictly greater than 1.

(A) By Theorem 3.3.3, $\mathrm{reg}(A(X_d)) = 2$ if and only if the minimal set $\mathcal{B}_1$ of fundamental monomial invariants of $\overline{G}$ does not contain any monomial $m = x_0^{a_0} x_1^{a_1} x_2^{a_2}$ such that $0 \neq a_0 a_1 a_2$. By Proposition 3.1.15,

$$\mathrm{HS}(A(X_d), z) = \frac{cz + 1}{(1 - z)^3}.$$

$\mathrm{I}(X_d)$ is minimally generated by binomials of degree 2. The CM–type of $A(X_d)$ is $|G|$ and the Betti diagram of $A(X_d)$ is of the form:

	0	1	2	$\cdots$	c
0	1	−	−	$\cdots$	−
1	−	$\beta_{1,1}$	$\beta_{2,1}$	$\cdots$	c

By Proposition 3.1.12, the graded Betti numbers of $A(X_d)$ can be deduced inductively from the Hilbert function $\mathrm{HF}(A(X_d), t)$ of $A(X_d)$.

Example 3.3.11. Take $G = \langle M_{4;0,1,1}\rangle \subset \mathrm{GL}(3,\mathbb{K})$ a cyclic group of order 4. We have $\mathcal{B}_1 = \{x_0^4, x_1^4, x_1^3x_2, x_1^2x_2^2, x_1x_2^3, x_2^4\}$ and $c = \mathrm{codim}(X_4) = 3$. The Hilbert series of $A(X_4)$ is $\mathrm{HS}(A(X_4), z) = \frac{3z+1}{(1-z)^3}$. We set $S = \mathbb{K}[w_1, w_2, w_3, w_4, w_5, w_6]$. The ideal $\mathrm{I}(X_4)$ is generated by the maximal minors of the 1−generic matrix of linear forms:

$$M = \begin{pmatrix} w_2 & w_3 & w_4 & w_5 \\ w_3 & w_4 & w_5 & w_6 \end{pmatrix}$$

([26, Chapter 6, §6B, Theorem 6.4]). Therefore, $\mathrm{I}(X_4)$ is a standard determinantal ideal and a minimal graded free $S-$resolution of $\mathrm{I}(X_4)$ is the Eagon–Northcott complex. The Betti diagram of $A(X_4)$ is

	0	1	2	3
0	1	–	–	–
1	–	6	8	3

(B) By Theorem 3.3.3, $\mathrm{reg}(A(X_d)) = 3$ and $A(X_d)$ is a Gorenstein ring if and only if $\mathcal{B}_1$ contains only one monomials $m = x_0^{a_0}x_1^{a_1}x_2^{a_2}$ with $0 \neq a_0a_1a_2$. By Proposition 3.1.15,

$$\mathrm{HS}(A(X_d), z) = \frac{z^2 + cz + 1}{(1-z)^3}.$$

If $\mathrm{I}(X_d)$ is minimally generated by 2−binomials, the Betti diagram of $A(X_d)$ is of the form:

	0	1	2	3	$\cdots$	$c-3$	$c-2$	$c-1$	c
0	1	–	–	–	$\cdots$	–	–	–	–
1	–	$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$	$\cdots$	$\beta_{3,1}$	$\beta_{2,1}$	$\beta_{1,1}$	–
2	–	–	–	–	$\cdots$	–	–	–	1

Else, $\mathrm{I}(X_d)$ is a cubic surface of $\mathbb{P}^3$ (Example 3.1.14).

Example 3.3.12. (i) For any cyclic group $G = \langle M_{4;0,\alpha_1,\alpha_2}\rangle$ with $0 < \alpha_1 < \alpha_2 < 4$ of order 4, the associated $GT-$variety X_4 with group G is a CI of two quadrics. The Betti diagram of $\mathrm{I}(X_4)$ is

	0	1	2
0	1	–	–
1	–	2	–
2	–	–	1

(ii) Take $G = \langle M_{6;0,2,3}\rangle \subset \mathrm{GL}(3,\mathbb{K})$ a cyclic group of order 6. A minimal set of fundamental invariants of $\overline{G}$ is $\mathcal{B}_1 = \{x_0^6, x_0^3x_1^3, x_0^4x_2^2, x_1^6, x_0x_1^3x_2^2, x_0^2x_2^4, x_2^6\}$. We have $c = 4$, the Hilbert series of $A(X_6)$ is

$$\mathrm{HS}(A(X_6), z) = \frac{z^3 + 4z + 1}{(1-z)^3}$$

and the Betti diagram of $A(X_d)$ is

	0	1	2	3	4
0	1	–	–	–	–
1	–	9	16	9	–
2	–	–	–	–	1

For any cyclic group $G = \langle M_{6;0,\alpha_1,\alpha_2}\rangle \subset \mathrm{GL}(3,\mathbb{K})$ of order 6 with $0 < \alpha_1 < \alpha_2 < 6$ and such that $(\alpha_1, \alpha_2) \in \{(1,3),(1,4),(2,3),(2,5),(3,4),(3,5)\}$, the associated $GT-$surface with group G is arithmetically Gorenstein with the above Betti diagram.

(C) By Theorem 3.3.3, $\overline{G}-$surfaces of type (C) are characterized from the fact that $\mathcal{B}_1$ contains at least two monomials $m = x_0^{a_0}x_1^{a_1}x_2^{a_2}$ such that $0 \neq a_0a_1a_2$. We set $\overline{d} := d^3/|\overline{G}|$ and $\theta = 2\mu_d - \overline{d} - 2$. From Proposition 3.1.15 we have that

$$\mathrm{HF}(A(X_d), t) = \frac{\overline{d}t^2 + \theta t + 2}{2} \text{ and } \mathrm{HS}(A(X_d), z) = \frac{\frac{\overline{d}-\theta+2}{2}z^2 + \frac{\overline{d}+\theta-4}{2}z + 1}{(1-z)^3}.$$

Thus, $A(X_d)$ has CM–type $\beta_{c,2} = \delta_2 = \frac{\overline{d}-\theta+2}{2}$. If $\mathrm{I}(X_d)$ is minimally generated by 2−binomials, the Betti diagram of $A(X_d)$ looks like

	0	1	2	3	$\cdots$	$c-3$	$c-2$	$c-1$	c
0	1	–	–	–	$\cdots$	–	–	–	–
1	–	$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$	$\cdots$	$\beta_{c-3,1}$	$\beta_{c-2,1}$	$\beta_{c-1,1}$	–
2	–	–	$\beta_{2,2}$	$\beta_{3,2}$	$\cdots$	$\beta_{c-3,2}$	$\beta_{c-2,2}$	$\beta_{c-1,2}$	δ_2

Else, if $A(X_d)$ is minimally generated by binomials of degree 2 and 3, we have

	0	1	2	3	$\cdots$	$c-3$	$c-2$	$c-1$	c
0	1	–	–	–	$\cdots$	–	–	–	–
1	–	$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$	$\cdots$	$\beta_{c-3,1}$	$\beta_{c-2,1}$	$\beta_{c-1,1}$	–
2	–	$\beta_{1,2}$	$\beta_{2,2}$	$\beta_{3,2}$	$\cdots$	$\beta_{c-3,2}$	$\beta_{c-2,2}$	$\beta_{c-1,2}$	δ_2

The Betti numbers of $A(X_d)$ can be deduce inductively from $\mathrm{HF}(A(X_d), t)$ of $A(X_d)$ (Proposition 3.1.12).

Example 3.3.13. (i) For any cyclic group $G = \langle M_{6;0,\alpha_1,\alpha_2}\rangle \subset \mathrm{GL}(3,\mathbb{K})$ or order 6 with $0 < \alpha_1 < \alpha \leq 6$ and such that $(\alpha_1, \alpha_2) \in \{(1,2),(1,5),(4,5)\}$, the associated $GT-$surface X_6 with group G has Betti diagram:

	0	1	2	3
0	1	–	–	–
1	–	4	2	–
2	–	–	3	2

(ii) For any finite group $G = \langle M_{8;0,\alpha_1,\alpha_2}\rangle \subset \mathrm{GL}(3,\mathbb{K})$ or order 8 with $0 < \alpha_1 < \alpha_2 < 8$ and such that $(\alpha_1, \alpha_2) \in \{(1,4),(1,5),(3,4),(3,7),(4,5),(4,7)\}$, the associated $GT-$surface X_8 with group G has Betti diagram

	0	1	2	3	4	5
0	1	–	–	–	–	–
1	–	13	30	25	4	–
2	–	–	–	–	5	2

Otherwise, X_8 has Betti diagram

	0	1	2	3	4
0	1	–	–	–	–
1	–	7	8	3	–
2	–	–	6	8	3

For any cyclic group $G = \langle M_{d;0,1,2}\rangle \subset \mathrm{GL}(3,\mathbb{K})$ of order $d \geq 3$, the minimal graded free resolution of $A(X_d)$ is determined in [57, Theorem 7.3]. In particular, for d odd, the minimal graded free resolution is the Eagon–Northcott complex (see [26]). For any cyclic group $G = \langle M_{d;0,\alpha_1,\alpha_2}\rangle \subset$

$\mathrm{GL}(3,\mathbb{K})$ of order $d \geq 3$ with $\alpha_1 < \alpha_2$, we have the following result which appears in [17, Theorem 4.14].

Let $\theta(\alpha_1,\alpha_2,d)$ be as in Theorem 2.4.10. We set $h = d - c - 2 = \frac{d-\theta(a,b,d)+2}{2} - 1$.

Theorem 3.3.14. *(i) If $\theta(a,b,d) = 3$, then the Betti numbers of $A(X_d)$ are*

$$\beta_{i,j} = \begin{cases} i\binom{c}{i+1} & \text{if } 1 \leq i \leq c-1,\ j=1 \\ i\binom{c}{i} & \text{if } 1 \leq i \leq c,\ j=2. \end{cases}$$

(ii) If $\theta(a,b,d) \geq 4$, then the Betti numbers of $A(X_d)$ are

$$b_{i,j} = \begin{cases} i\binom{c}{i+1} + (c-h-i)\binom{c}{i-1} & \text{if } 1 \leq i \leq c-h-1,\ j=1 \\ i\binom{c}{i+1} & \text{if } c-h \leq i \leq c-1,\ j=1 \\ (i-c+h+1)\binom{c}{i} & \text{if } c-h \leq i \leq c,\ j=2. \end{cases}$$

Proof. If $\theta(\alpha_1,\alpha_2,d) = 3$, then $d = 2c+1$. Otherwise, $c+3 \leq \theta(\alpha_1,\alpha_2,d) \leq 2c$ for all $9 \leq d$. Indeed, the inequality $d \geq c+3$ is equivalent to

$$\theta(\alpha_1,\alpha_2,d)+2 = \mathrm{GCD}(\alpha_1,d)+\mathrm{GCD}(\lambda,d')+\mathrm{GCD}(\lambda-\mathrm{GCD}(\alpha_1,d),d')+2 \leq d,$$

where $d' = \frac{d}{\mathrm{GCD}(\alpha_1,d)}$ and $0 < \lambda$ and μ are the unique integers such that $\alpha_2 = \alpha_1\lambda + \mu d'$. We see that it holds for any integer $d \geq 9$. We have that

$$d = \mathrm{GCD}(\alpha_1,d)\,\mathrm{GCD}(\lambda,d')\,\mathrm{GCD}(\lambda-\mathrm{GCD}(\alpha_1,d),d')\widetilde{d}$$

with $\widetilde{d} \geq 1$. Now consider the system of inequalities $xyz\widetilde{d} - x - y - z - 2 < 0$ with $x,y,z \geq 1$. There are no integer solutions for $\widetilde{d} \geq 5$. For $1 \leq \widetilde{d} \leq 4$, it holds $d \leq 8$. The result follows from [89, Corollary 3.4(i)] and Examples 3.3.12 and 3.3.13. □

We see a couple of more examples.

Example 3.3.15. (i) Take $G = \langle M_{11;0,1,6}\rangle \subset \mathrm{GL}(3,\mathbb{K})$ a cyclic group of order 11. The Betti diagram of $A(X_{11})$ is

	0	1	2	3	4	5
0	1	–	–	–	–	–
1	–	10	20	15	4	–
2	–	5	20	30	20	5

(ii) Take $G = \langle M_{3;0,1,1}, M_{3;0,1,2} \rangle$ a finite abelian group of order 9. The associated $GT-$surface X_9 with group G is an arithmetically Gorenstein surface and $A(X_9)$ has Betti diagram

	0	1	2	3	4	5	6	7
0	1	–	–	–	–	–	–	–
1	–	27	105	189	189	105	27	–
2	–	–	–	–	–	–	–	1

Notwithstanding, in higher dimensions the situation becomes less clear. Let X_d be a $\overline{G}-$variety with finite abelian group $G \subset \mathrm{GL}(n+1, \mathbb{K})$. The Castelnuovo–Mumford regularity $n \leq \operatorname{reg}(A(X_d)) \leq n+1$ and $\mathrm{I}(X_d)$ is minimally generated by binomials of degree at most 3. Thus, the complexity of a minimal graded free $S-$resolution of $A(X_d)$ increases linearly with the dimension n. Even if we have a complete description of the Hilbert function and series of $A(X_d)$, only partial information of the Betti diagram of $A(X_d)$ can be inferred. The general picture of the Betti diagram of $A(X_d)$ is the following. If $\operatorname{reg}(A(X_d)) = n+1$,

	0	1	2	3	$\cdots$	$c-3$	$c-2$	$c-1$	c
0	1	–	–	–	$\cdots$	–	–	–	–
1	–	$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$	$\cdots$	$\beta_{c-3,1}$	$\beta_{c-2,1}$	$\beta_{c-1,1}$	–
2	–	$\beta_{1,2}$	$\beta_{2,2}$	$\beta_{3,2}$	$\cdots$	$\beta_{c-3,2}$	$\beta_{c-2,2}$	$\beta_{c-1,2}$	–
3	–	–	$\beta_{2,3}$	$\beta_{3,3}$	$\cdots$	$\beta_{c-3,3}$	$\beta_{c-2,3}$	$\beta_{c-1,3}$	–
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1$	–	–	$\beta_{2,n-1}$	$\beta_{3,n-1}$	$\cdots$	$\beta_{c-3,n-1}$	$\beta_{c-2,n-1}$	$\beta_{c-1,n-1}$	$\beta_{c,n-1}$
n	–	–	$\beta_{2,n}$	$\beta_{3,n}$	$\cdots$	$\beta_{c-3,n}$	$\beta_{c-2,n}$	$\beta_{c-1,n}$	$\beta_{c,n}$

Otherwise $\operatorname{reg}(A(X_d)) = n$ and $A(X_d)$ is a level ring,

	0	1	2	3	$\cdots$	$c-3$	$c-2$	$c-1$	c
0	1	–	–	–	$\cdots$	–	–	–	–
1	–	$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$	$\cdots$	$\beta_{c-3,1}$	$\beta_{c-2,1}$	$\beta_{c-1,1}$	–
2	–	$\beta_{1,2}$	$\beta_{2,2}$	$\beta_{3,2}$	$\cdots$	$\beta_{c-3,2}$	$\beta_{c-2,2}$	$\beta_{c-1,2}$	–
3	–	–	$\beta_{2,3}$	$\beta_{3,3}$	$\cdots$	$\beta_{c-3,3}$	$\beta_{c-2,3}$	$\beta_{c-1,3}$	–
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1$	–	–	$\beta_{2,n-1}$	$\beta_{3,n-1}$	$\cdots$	$\beta_{c-3,n-1}$	$\beta_{c-2,n-1}$	$\beta_{c-1,n-1}$	$\beta_{c,n-1}$

We end this chapter posing the following problem and we exhibit an example involving a $\overline{G}-$threefold.

Problem 3.3.16. *To determine a minimal graded free $S-$resolution of any $\overline{G}-$variety X_d with a finite abelian group $G \subset \mathrm{GL}(n+1, \mathbb{K})$.*

Example 3.3.17. Take $G = \langle M_{6;0,1,2,3} \rangle \subset \mathrm{GL}(4, \mathbb{K})$ a cyclic group of order 6. $\mathrm{I}(X_d)$ is minimally generated by binomials of degree 2. $A(X_6)$ is a non level ring and it has the following Betti diagram

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1	–	–	–	–	–	–	–	–	–	–	–	–
1	–	57	322	796	844	258	–	–	–	–	–	–	–
2	–	–	13	184	1638	5352	8811	9064	6237	2850	803	108	4
3	–	–	–	–	–	–	–	–	–	–	–	7	2

As we have pointed out in Remark 1.1.24, if we dualize a minimal graded free $S-$resolution of $A(X_6)$, we get a minimal graded free $S-$resolution of $w_{X_6}(16)$. In this case, we obtain that w_{X_6} is generated by two monomials of degree 1 and 4 monomials of degree 2, as we have computed in Example 3.3.4.

Chapter 4

GT-surfaces with a dihedral group

The aim of this chapter is to investigate and to provide examples of $GT-$systems and $GT-$surfaces with a non abelian finite group $\Lambda \subset \mathrm{SL}(3,\mathbb{K})$ of order $|\Lambda|$. In [6] and [90], a classification of finite subgroups of $\mathrm{SL}(3,\mathbb{K})$ is given and, following this classification, we determine which groups $\Lambda \subset \mathrm{SL}(3,\mathbb{K})$ give rise to a Togliatti system, i.e. the ideal generated by the invariants of Λ of degree $|\Lambda|$ is an artinian ideal an it fails the WLP in degree $|\Lambda|-1$. We gather new examples of non monomial Togliatti systems. To determine the algebra of invariants of the cyclic extension $\overline{\Lambda} \subset \mathrm{GL}(3,\mathbb{K})$ of these groups requires an individualized approach in each case. We focus on the dihedral group $D_{2d} \subset \mathrm{SL}(3,\mathbb{K})$ of order $2d$ generated by a cyclic group $\Gamma = \langle M_{d;0,a,d-a}\rangle \subset \mathrm{SL}(3,\mathbb{K})$ of order d and the linear transformation σ which fixes x_0 and permutes x_1 and x_2. We take the cyclic extension $\overline{D_{2d}} \subset \mathrm{GL}(3,\mathbb{K})$ and we prove that $R^{\overline{D_{2d}}}$ is minimally generated by monomial and binomial invariants of D_{2d} of degree $2d$ (Theorem 4.2.6). We establish that the ideal I_{2d} generated by a minimal set of fundamental invariants of $\overline{D_{2d}}$ is a $GT-$system with group D_{2d} (Proposition 4.2.9). $GT-$systems I_{2d} with group D_{2d} are the first large family of non monomial $GT-$systems appearing in the literature [19]. The approach developed in Chapters 2 and 3 allows us to tackle the geometry of any $GT-$surface $S_{D_{2d}}$ with group D_{2d}. We prove that $S_{D_{2d}}$ is an aCM surface and we compute its Hilbert function and series (Theorem 4.2.12). Furthermore, we compute a minimal graded free resolution of its homogeneous coordinate ring $A(S_{D_{2d}})$ (Theorem 4.2.14). We show that the homogeneous ideal $\mathrm{I}(S_{D_{2d}})$ of $S_{D_{2d}}$ is minimally generated by quadrics, which we completely determine afterwards (Corollary 4.2.15 and Theorem 4.2.17).

This chapter is organized as follows. In Section 4.1, we recall the classification of finite subgroups Λ of $\mathrm{SL}(3,\mathbb{K})$ given in [90]. There are twelve types (A)–(L), with only (A) being abelian. We determine that a group Λ of order $|\Lambda|$ of any type (E),(F),(G),(J),(K) and (L) does not induce a Togliatti system since the required bound $r \leq |\Lambda|+1$ on the number of generators r of $R^{\Lambda}_{|\Lambda|}$ is not satisfied (see Proposition 4.1.1 and table (4.1.1)). We compute new examples of non monomial Togliatti systems induced by non abelian groups of types (B),(C) and (D) (Example 4.1.2). In Section 4.2, we focus on the dihedral group $D_{2d} \subset \mathrm{SL}(3, \mathbb{K})$ of order $2d$ generated by $\Gamma = \langle M_{d;0,a,d-a} \rangle$ and σ. We determine the invariants of $\overline{D_{2d}}$ and we compute the Hilbert function and series of $R^{\overline{D_{2d}}}$ (Proposition 4.2.2 and Corollary 4.2.3). We take advantage of the algebraic structure of $R^{\overline{D_{2d}}}$ as a subring of $R^{D_{2d}}$ to prove that it is minimally generated by invariants of degree $2d$ (Theorem 4.2.6). This lead us to establish that the ideal generated by a minimal set of fundamental invariants of $\overline{D_{2d}}$ is a $GT-$system with group D_{2d} (Proposition 4.2.9). Subsection 4.2.1 deals with $GT-$surfaces $S_{D_{2d}}$ with group D_{2d}. After establishing that $S_{D_{2d}}$ is an aCM surface and that $A(S_{D_{2d}}) \cong R^{\overline{D}_{2d}}$ (Theorem 4.2.12), we compute a minimal graded free resolution of $A(S(D_{2d}))$ (Theorem 4.2.14) and a minimal set of generators of $\mathrm{I}(S_{D_{2d}})$ (Theorem 4.2.17). All the results of Section 4.2 have been published in [19].

4.1 GT-systems with a finite group

In Chapter 1 §1.4, we introduced $GT-$systems and $GT-$varieties with an arbitrary finite group $\Lambda \subset \mathrm{GL}(n+1, \mathbb{K})$. We recall that an artinian ideal $J \subset R$ generated by $r \leq N_{n-1,d}$ forms $F_1, \ldots, F_r$ of degree d is a $GT-$system with group $\Lambda \subset \mathrm{GL}(n+1, \mathbb{K})$ if J is a Togliatti system, i.e. it fails the WLP in degree $d-1$, and the morphism $\varphi_J : \mathbb{P}^n \longrightarrow \mathbb{P}^{r-1}$ defined by $(F_1 : \cdots : F_r)$ is a Galois covering with group Λ. If $J \subset R$ is a $GT-$system with group Λ, then the variety $X_d = \varphi_J(\mathbb{P}^n)$ parameterized by J is called a $GT-$variety with group Λ.

The strategy develop in Chapters 2 and 3 is suitable for any finite group $\Lambda \subset \mathrm{GL}(n+1, \mathbb{K})$. However, it relies on their representation in $\mathrm{GL}(n+1, \mathbb{K})$ and on finding a minimal set of fundamental invariants of the cyclic extension $\overline{\Lambda}$ of Λ. As we have pointed out in Section 1.3, the last requisite is out of reach in most cases. When Λ is not abelian, the invariants of $\overline{\Lambda}$ are not all monomials ([6, Theorem 8, Chapter IX]) and it could be cumbersome to describe or manipulate them (see, for instance, Example 4.1.2(i)). In addition, taking the cyclic extension $\overline{\Lambda}$ increases significantly the computational complexity making difficult to provide examples. Nevertheless, if J is the ideal generated by a $\mathbb{K}-$basis of $R^{\Lambda}_{|\Lambda|}$, we have:

Proposition 4.1.1. *If J is artinian and* $\dim(R^{\Lambda}_{|\Lambda|}) \leq N_{n-1,|\Lambda|}$*, then J is a Togliatti system.*

Proof. We prove that J fails the WLP in degree $|\Lambda|-1$. Let $L \in R_1$ and $F := \prod_{Id \neq \lambda \in \Lambda} \lambda(L)$. Therefore

$$L \cdot F = \prod_{\lambda \in \Lambda} \lambda(L) \in R^{\Lambda}_{|\Lambda|}.$$

Since J is generated by a $\mathbb{K}-$basis of $R^{\Lambda}_{|\Lambda|}$, we obtain that $L \cdot F \in J$. Thus, the multiplication map $\times L : (R/J)_{d-1} \longrightarrow (R/J)_d$ is not injective and the result follows (Theorem 1.4.6 and Definition 1.4.7). $\square$

So, let us fix $n = 2$ and, as an starting point to investigate $GT-$systems and $GT-$surfaces with a non abelian finite group, we focus on the classification of finite subgroups Λ of $\mathrm{SL}(3, \mathbb{K})$ as in [90]. Consider the matrices

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \psi & 0 \\ 0 & 0 & \psi^2 \end{pmatrix}, T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, U = \begin{pmatrix} \kappa & 0 & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & \kappa\psi \end{pmatrix},$$

$$V = \frac{1}{\psi - \psi^2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \psi & \psi^2 \\ 1 & \psi^2 & \psi \end{pmatrix}, P = \frac{1}{\psi - \psi^2} \begin{pmatrix} 1 & 1 & \psi^2 \\ 1 & \psi & \psi \\ \psi & 1 & \psi \end{pmatrix}, Q = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & c & 0 \end{pmatrix}$$

where κ and ψ are a 6th and a 3rd primitive roots of $1 \in \mathbb{K}$ such that $\kappa^3 = \psi^2$ and $abc = -1$. The list of finite subgroups of $\mathrm{SL}(3, \mathbb{K})$ is the following.

(A) Diagonal abelian groups.

(B) Groups isomorphic to transitive groups of $\mathrm{GL}(2, \mathbb{K})$, i.e. each element has the form of

$$\begin{pmatrix} f & 0 & 0 \\ 0 & g & h \\ 0 & j & k \end{pmatrix} \quad \text{with} \quad f(gk - hj) = 1.$$

(C) Groups generated by (A) and T.

(D) Groups generated by (C) and Q.

(E) A group of order 108 generated by S, T, V.

(F) A group of order 216 generated by (E) and $P = UVU^{-1}$.

(G) A group of order 648 generated by (E) and U.

(H) A group of order 60 isomorphic to the alternating group A_5 generated by Q with $a = b = c = -1$ and matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \epsilon^4 & 0 \\ 0 & 0 & \epsilon \end{pmatrix} \qquad \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 & 1 \\ 2 & s & t \\ 2 & t & s \end{pmatrix}$$

where ϵ is a 5th primitive root of $1 \in \mathbb{K}$, $s = \epsilon^2 + \epsilon^3$ and $t = \epsilon + \epsilon^4$.

(I) A simple group of order 168 isomorphic to a permutation group generated by T and matrices

$$\begin{pmatrix} \beta & 0 & 0 \\ 0 & \beta^2 & 0 \\ 0 & 0 & \beta^4 \end{pmatrix} \qquad \frac{1}{\sqrt{-7}} \begin{pmatrix} \beta^4 - \beta^3 & \beta^2 - \beta^5 & \beta - \beta^6 \\ \beta^2 - \beta^5 & \beta - \beta^6 & \beta^4 - \beta^3 \\ \beta - \beta^6 & \beta^4 - \beta^3 & \beta^2 - \beta^5 \end{pmatrix}$$

where β is a 7th primitive root of $1 \in \mathbb{K}$.

(J) A group of order 180 generated by (H) and $\operatorname{diag}(\psi, \psi, \psi)$.

(K) A group of order 504 generated by (I) and $\operatorname{diag}(\psi, \psi, \psi)$.

(L) A group of order 1080 generated by (H) and

$$\frac{1}{\sqrt{5}} \begin{pmatrix} 1 & \lambda_1 & \lambda_1 \\ 2\lambda_2 & s & t \\ 2\lambda_2 & t & s \end{pmatrix}$$

where $\lambda_1 = \frac{1}{4}(-1 + \sqrt{15i})$ and λ_2 is the conjugate of λ_1.

In [90], the authors computed a minimal set of fundamental invariants and the Molien series $\mathrm{HS}(R^\Lambda, z)$ of the finite groups $\Lambda \subset \mathrm{SL}(3, \mathbb{K})$ of any type from (E) to (L). However, a minimal set of fundamental invariants of their cyclic extension $\overline{\Lambda}$ is out of reach. Expanding $\mathrm{HS}(R^\Lambda, z)$, we obtain that the bound $\dim(R^\Lambda_{|\Lambda|}) \leq |\Lambda| + 1$ is only satisfied for the groups (H) and

(I). They are artinian idelas and, By Proposition 4.1.1, they determine a Togliatti system.

(E)	$\|\Lambda\|=108$	$\mathrm{HS}(R^\Lambda,z)=\dfrac{z^{21}+z^{12}+z^9+1}{(1-z^6)^2(1-z^{12})}$	$\mathrm{HF}(R^\Lambda,108)=181$
(F)	$\|\Lambda\|=216$	$\mathrm{HS}(R^\Lambda,z)=\dfrac{z^{24}+z^{12}+1}{(1-z^6)(1-z^9)(1-z^{12})}$	$\mathrm{HF}(R^\Lambda,216)=343$
(G)	$\|\Lambda\|=648$	$\mathrm{HS}(R^\Lambda,z)=\dfrac{z^{36}+z^{18}+1}{(1-z^9)(1-z^{12})(1-z^{18})}$	$\mathrm{HF}(R^\Lambda,648)=1027$
(H)	$\|\Lambda\|=60$	$\mathrm{HS}(R^\Lambda,z)=\dfrac{z^{15}+1}{(1-z^2)(1-z^6)(1-z^{10})}$	$\mathrm{HF}(R^\Lambda,60)=40$
(I)	$\|\Lambda\|=168$	$\mathrm{HS}(R^\Lambda,z)=\dfrac{z^{21}+1}{(1-z^4)(1-z^6)(1-z^{14})}$	$\mathrm{HF}(R^\Lambda,168)=97$
(J)	$\|\Lambda\|=180$	$\mathrm{HS}(R^\Lambda,z)=\dfrac{z^{24}+z^{12}+1}{(1-z^6)^2(1-z^{15})}$	$\mathrm{HF}(R^\Lambda,180)=298$
(K)	$\|\Lambda\|=504$	$\mathrm{HS}(R^\Lambda,z)=\dfrac{z^{36}+z^{18}+1}{(1-z^6)(1-z^{12})(1-z^{21})}$	$\mathrm{HF}(R^\Lambda,504)=793$
(L)	$\|\Lambda\|=1080$	$\mathrm{HS}(R^\Lambda,z)=\dfrac{z^{45}+1}{(1-z^6)(1-z^{12})(1-z^{30})}$	$\mathrm{HF}(R^\Lambda,1080)=1693$

(4.1.1)

Our next goal is to prove that the groups of type (B),(C) and (D) also give rise to examples of non monomial Togliatti systems and $GT-$systems. In next section, we will study the invariants of a representation in $\mathrm{SL}(3,\mathbb{K})$ of the dihedral group.

Example 4.1.2. (i) Type (B). We take $\Lambda \subset \mathrm{SL}(3,\mathbb{K})$ a tetrahedral group of order 24 generated by the three matrices:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix} \text{ and } \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & i+1 & i-1 \\ 0 & i+1 & 1-i \end{pmatrix}$$

(see [90]). The cyclic extension $\overline{\Lambda} \subset \mathrm{GL}(3,\mathbb{K})$ is a group of order 576. We have determined that the following 15 forms of degree 24 give a minimal set of fundamental invariants of $\overline{\Lambda}$:

$$\begin{aligned}
&x_0^{24}\\
&x_0^{18}x_1^5x_2 - x_0^{18}x_1x_2^5\\
&x_0^{16}x_1^8 + x_0^{16}x_2^8 + 14x_0^{16}x_1^4x_2^4\\
&x_0^{12}x_1^2x_2^{10} - 2x_0^{12}x_1^6x_2^6 + x_0^{12}x_1^{10}x_2^2\\
&x_0^{12}x_1^{12} + x_0^{12}x_2^{12} - 33x_0^{12}x_1^4x_2^8 - 33x_0^{12}x_1^8x_2^4\\
&x_0^6x_1^{15}x_2^3 - 3x_0^6x_1^11x_2^7 + 3x_0^6x_1^7x_2^{11} - x_0^6x_1^3x_2^{15}\\
&x_0^6x_1^{17}x_2 - 34x_0^6x_1^{13}x_2^5 + 34x_0^6x_1^5x_2^{13} - x_0^6x_1x_2^{17}\\
&x_0^{10}x_1^{13}x_2 + 13x_0^{10}x_1^9x_2^5 - 13x_0^{10}x_1^5x_2^9 - x_0^{10}x_1x_2^{13}\\
&x_0^8x_1^{16} + 28x_0^8x_1^{12}x_2^4 + 198x_0^8x_1^8x_2^8 + 28x_0^8x_1^4x_2^{12} + x_0^8x_2^{16}\\
&x_0^4x_1^{18}x_2^2 + 12x_0^4x_1^{14}x_2^6 - 26x_0^4x_1^{10}x_2^{10} + 12x_0^4x_1^6x_2^{14} + x_0^4x_1^2x_2^{18}\\
&x_1^{22}x_2^2 - 35x_1^{18}x_2^6 + 34x_1^{14}x_2^{10} + 34x_1^{10}x_2^{14} - 35x_1^6x_2^{18} + x_1^2x_2^{22}\\
&x_0^4x_1^{20} - 19x_0^4x_1^{16}x_2^4 - 494x_0^4x_1^{12}x_2^8 - 494x_0^4x_1^8x_2^{12} - 19x_0^4x_1^4x_2^{16} + x_0^4x_2^{20}\\
&x_0^2x_1^{21}x_2 + 27x_0^2x_1^{17}x_2^5 + 170x_0^2x_1^{13}x_2^9 - 170x_0^2x_1^9x_2^{13} - 27x_0^2x_1^5x_2^{17} - x_0^2x_1x_2^{21}\\
&x_1^{24}+2370x_1^{20}x_2^4-8721x_1^{16}x_2^8 + 16796x_1^{12}x_2^{12} - 8721x_1^8x_2^{16} + 2370x_1^4x_2^{20}+x_2^{24}\\
&x_1^{24}+\tfrac{10626}{1025}x_1^{20}x_2^4+\tfrac{735471}{1025}x_1^{16}x_2^8+\tfrac{2704156}{1025}x_1^{12}x_2^{12}+\tfrac{735471}{1025}x_1^8x_2^{16}+\tfrac{10626}{1025}x_1^4x_2^{20}+x_2^{24}.
\end{aligned}$$

The ideal I_{24} generated by them is a non monomial $GT-$system with group Λ and the variety X_{24} parameterized by I_{24} is a $GT-$surface with group Λ. Furthermore, X_{24} is an aCM projection of the Veronese surface $X_{2,24} \subset \mathbb{P}^{324}$ in $\mathbb{P}^{14}$ from the linear system $\langle I_{24}^{-1}\rangle_{24}$.

(ii) Type (C). Finite subgroups Λ of $\mathrm{SL}(3,\mathbb{K})$ of type (C) generated by T and a diagonal cyclic subgroup $\Gamma = \langle \mathrm{diag}(\gamma_1,\gamma_2,\gamma_3)\rangle$ of $\mathrm{SL}(3,\mathbb{K})$ were first studied by Maschke in [56]. The author showed that R_t^{Λ} is generated by trinomials of degree t of the form:

$$x_0^{a_0}x_1^{a_1}x_2^{a_2} + x_0^{a_2}x_1^{a_0}x_2^{a_1} + x_0^{a_1}x_1^{a_2}x_2^{a_0}$$

where $x_0^{a_0}x_1^{a_1}x_2^{a_2}, x_0^{a_2}x_1^{a_0}x_2^{a_1}, x_0^{a_1}x_1^{a_2}x_2^{a_0}$ are monomial invariants of Γ of degree t which he described in terms of the γ_i. As an example, we take $\Lambda_1 \subset \mathrm{SL}(3,\mathbb{K})$ a group of order 48 generated by the matrices $M_{4;0,1,3}$ and T (see [90]). $R_{48}^{\Lambda_1}$

is generated by $\mu_{48} = 31$ forms of degree 48:

$$
\begin{array}{ll}
x_0^{16}x_1^{16}x_2^{16} & x_1^4x_2^8x_0^{36} + x_1^{36}x_2^4x_0^8 + x_1^8x_2^{36}x_0^4 \\
x_0^{48} + x_1^{48} + x_2^{48} & x_1^4x_2^4x_0^{40} + x_1^4x_2^{40}x_0^4 + x_1^{40}x_2^4x_0^4 \\
x_1^4x_0^{44} + x_2^{44}x_0^4 + x_1^{44}x_2^4 & x_1^{12}x_2^8x_0^{28} + x_1^8x_2^{28}x_0^{12} + x_1^{28}x_2^{12}x_0^8 \\
x_1^8x_0^{40} + x_2^{40}x_0^8 + x_1^{40}x_2^8 & x_1^{16}x_2^8x_0^{24} + x_1^8x_2^{24}x_0^{16} + x_1^{24}x_2^{16}x_0^8 \\
x_2^8x_0^{40} + x_1^{40}x_0^8 + x_1^8x_2^{40} & x_0^8x_2^{20}x_1^{20} + x_0^{20}x_2^8x_1^{20} + x_0^{20}x_2^{20}x_1^8 \\
x_2^4x_0^{44} + x_1^{44}x_0^4 + x_1^4x_2^{44} & x_1^8x_2^{16}x_0^{24} + x_1^{24}x_2^8x_0^{16} + x_1^{16}x_2^{24}x_0^8 \\
x_1^{12}x_0^{36} + x_2^{36}x_0^{12} + x_1^{36}x_2^{12} & x_1^8x_2^{12}x_0^{28} + x_1^{28}x_2^8x_0^{12} + x_1^{12}x_2^{28}x_0^8 \\
x_1^{16}x_0^{32} + x_2^{32}x_0^{16} + x_1^{32}x_2^{16} & x_1^{12}x_2^4x_0^{32} + x_1^4x_2^{32}x_0^{12} + x_1^{32}x_2^{12}x_0^4 \\
x_1^{20}x_0^{28} + x_2^{28}x_0^{20} + x_1^{28}x_2^{20} & x_1^{16}x_2^4x_0^{28} + x_1^4x_2^{28}x_0^{16} + x_1^{28}x_2^{16}x_0^4 \\
x_0^{24}x_1^{24} + x_2^{24}x_1^{24} + x_0^{24}x_2^{24} & x_1^{20}x_2^4x_0^{24} + x_1^4x_2^{24}x_0^{20} + x_1^{24}x_2^{20}x_0^4 \\
x_2^{20}x_0^{28} + x_1^{28}x_0^{20} + x_1^{20}x_2^{28} & x_1^4x_2^{20}x_0^{24} + x_1^{24}x_2^4x_0^{20} + x_1^{20}x_2^{24}x_0^4 \\
x_2^{16}x_0^{32} + x_1^{32}x_0^{16} + x_1^{16}x_2^{32} & x_1^4x_2^{16}x_0^{28} + x_1^{28}x_2^4x_0^{16} + x_1^{16}x_2^{28}x_0^4 \\
x_2^{12}x_0^{36} + x_1^{36}x_0^{12} + x_1^{12}x_2^{36} & x_1^4x_2^{12}x_0^{32} + x_1^{32}x_2^4x_0^{12} + x_1^{12}x_2^{32}x_0^4 \\
x_1^8x_2^8x_0^{32} + x_1^8x_2^{32}x_0^8 + x_1^{32}x_2^8x_0^8 & x_1^{16}x_2^{12}x_0^{20} + x_1^{12}x_2^{20}x_0^{16} + x_1^{20}x_2^{16}x_0^{12} \\
x_1^8x_2^4x_0^{36} + x_1^4x_2^{36}x_0^8 + x_1^{36}x_2^8x_0^4 & x_1^{12}x_2^{16}x_0^{20} + x_1^{20}x_2^{12}x_0^{16} + x_1^{16}x_2^{20}x_0^{12} \\
 & x_1^{12}x_2^{12}x_0^{24} + x_1^{12}x_2^{24}x_0^{12} + x_1^{24}x_2^{12}x_0^{12}.
\end{array}
$$

The ideal I_{48} generated by them is a non monomial Togliatti system. Indeed, I_{48} is an artinian ideal since $V(x_0^{16}x_1^{16}x_2^{16}, x_0^{48} + x_1^{48} + x_2^{48}, x_1^4x_0^{44} + x_2^{44}x_0^4 + x_1^{44}x_2^4) = \emptyset$ and the condition $\mu_d \leq |\Lambda_1| + 1$ is satisfied. By Proposition 4.1.1, I_{48} is a Togliatti system.

(iii) Type (D). Take $\Lambda_2 \subset \mathrm{SL}(3, \mathbb{K})$ a group of order 96 generated by Λ_1 and Q with $a = b = c = -1$ (see [90]). Notice that the invariants of Λ_2 are in particular invariants of Λ_1. $R_{96}^{\Lambda_2}$ is generated by 61 forms of degree 96:

$$
\begin{array}{l}
x_0^{32}x_1^{32}x_2^{32} \\
x_0^{96} + x_1^{96} + x_2^{96} \\
x_0^{48}x_1^{48} + x_2^{48}x_1^{48} + x_0^{48}x_2^{48} \\
x_1^8x_2^8x_0^{80} + x_1^8x_2^{80}x_0^8 + x_1^{80}x_2^8x_0^8 \\
x_1^4x_2^4x_0^{88} + x_1^4x_2^{88}x_0^4 + x_1^{88}x_2^4x_0^4 \\
x_0^8x_2^{44}x_1^{44} + x_0^{44}x_2^8x_1^{44} + x_0^{44}x_2^{44}x_1^8 \\
x_1^{20}x_2^{20}x_0^{56} + x_1^{20}x_2^{56}x_0^{20} + x_1^{56}x_2^{20}x_0^{20} \\
x_0^{16}x_2^{40}x_1^{40} + x_0^{40}x_2^{16}x_1^{40} + x_0^{40}x_2^{40}x_1^{16} \\
x_1^{28}x_2^{28}x_0^{40} + x_1^{28}x_2^{40}x_0^{28} + x_1^{40}x_2^{28}x_0^{28} \\
x_0^{24}x_2^{36}x_1^{36} + x_0^{36}x_2^{24}x_1^{36} + x_0^{36}x_2^{36}x_1^{24} \\
x_1^{24}x_2^{24}x_0^{48} + x_1^{24}x_2^{48}x_0^{24} + x_1^{48}x_2^{24}x_0^{24}
\end{array}
$$

$$
\begin{array}{l}
x_1^{16}x_2^{16}x_0^{64} + x_1^{16}x_2^{64}x_0^{16} + x_1^{64}x_2^{16}x_0^{16} \\
x_1^{12}x_2^{12}x_0^{72} + x_1^{12}x_2^{72}x_0^{12} + x_1^{72}x_2^{12}x_0^{12} \\
x_1^{8}x_0^{88} + x_2^{8}x_0^{88} + x_1^{88}x_0^{8} + x_2^{88}x_0^{8} + x_1^{8}x_2^{88} + x_1^{88}x_2^{8} \\
x_1^{4}x_0^{92} + x_2^{4}x_0^{92} + x_1^{92}x_0^{4} + x_2^{92}x_0^{4} + x_1^{4}x_2^{92} + x_1^{92}x_2^{4} \\
x_1^{44}x_0^{52} + x_2^{44}x_0^{52} + x_1^{52}x_0^{44} + x_2^{52}x_0^{44} + x_1^{44}x_2^{52} + x_1^{52}x_2^{44} \\
x_1^{40}x_0^{56} + x_2^{40}x_0^{56} + x_1^{56}x_0^{40} + x_2^{56}x_0^{40} + x_1^{40}x_2^{56} + x_1^{56}x_2^{40} \\
x_1^{36}x_0^{60} + x_2^{36}x_0^{60} + x_1^{60}x_0^{36} + x_2^{60}x_0^{36} + x_1^{36}x_2^{60} + x_1^{60}x_2^{36} \\
x_1^{32}x_0^{64} + x_2^{32}x_0^{64} + x_1^{64}x_0^{32} + x_2^{64}x_0^{32} + x_1^{32}x_2^{64} + x_1^{64}x_2^{32} \\
x_1^{28}x_0^{68} + x_2^{28}x_0^{68} + x_1^{68}x_0^{28} + x_2^{68}x_0^{28} + x_1^{28}x_2^{68} + x_1^{68}x_2^{28} \\
x_1^{24}x_0^{72} + x_2^{24}x_0^{72} + x_1^{72}x_0^{24} + x_2^{72}x_0^{24} + x_1^{24}x_2^{72} + x_1^{72}x_2^{24} \\
x_1^{20}x_0^{76} + x_2^{20}x_0^{76} + x_1^{76}x_0^{20} + x_2^{76}x_0^{20} + x_1^{20}x_2^{76} + x_1^{76}x_2^{20} \\
x_1^{16}x_0^{80} + x_2^{16}x_0^{80} + x_1^{80}x_0^{16} + x_2^{80}x_0^{16} + x_1^{16}x_2^{80} + x_1^{80}x_2^{16} \\
x_1^{12}x_0^{84} + x_2^{12}x_0^{84} + x_1^{84}x_0^{12} + x_2^{84}x_0^{12} + x_1^{12}x_2^{84} + x_1^{84}x_2^{12} \\
x_1^{4}x_2^{8}x_0^{84} + x_1^{8}x_2^{4}x_0^{84} + x_1^{4}x_2^{84}x_0^{8} + x_1^{84}x_2^{4}x_0^{8} + x_1^{8}x_2^{84}x_0^{4} + x_1^{84}x_2^{8}x_0^{4} \\
x_1^{8}x_2^{40}x_0^{48} + x_1^{40}x_2^{8}x_0^{48} + x_1^{8}x_2^{48}x_0^{40} + x_1^{48}x_2^{8}x_0^{40} + x_1^{40}x_2^{48}x_0^{8} + x_1^{48}x_2^{40}x_0^{8} \\
x_1^{8}x_2^{36}x_0^{52} + x_1^{36}x_2^{8}x_0^{52} + x_1^{8}x_2^{52}x_0^{36} + x_1^{52}x_2^{8}x_0^{36} + x_1^{36}x_2^{52}x_0^{8} + x_1^{52}x_2^{36}x_0^{8} \\
x_1^{8}x_2^{32}x_0^{56} + x_1^{32}x_2^{8}x_0^{56} + x_1^{8}x_2^{56}x_0^{32} + x_1^{56}x_2^{8}x_0^{32} + x_1^{32}x_2^{56}x_0^{8} + x_1^{56}x_2^{32}x_0^{8} \\
x_1^{8}x_2^{28}x_0^{60} + x_1^{28}x_2^{8}x_0^{60} + x_1^{8}x_2^{60}x_0^{28} + x_1^{60}x_2^{8}x_0^{28} + x_1^{28}x_2^{60}x_0^{8} + x_1^{60}x_2^{28}x_0^{8} \\
x_1^{8}x_2^{24}x_0^{64} + x_1^{24}x_2^{8}x_0^{64} + x_1^{8}x_2^{64}x_0^{24} + x_1^{64}x_2^{8}x_0^{24} + x_1^{24}x_2^{64}x_0^{8} + x_1^{64}x_2^{24}x_0^{8} \\
x_1^{8}x_2^{20}x_0^{68} + x_1^{20}x_2^{8}x_0^{68} + x_1^{8}x_2^{68}x_0^{20} + x_1^{68}x_2^{8}x_0^{20} + x_1^{20}x_2^{68}x_0^{8} + x_1^{68}x_2^{20}x_0^{8} \\
x_1^{8}x_2^{16}x_0^{72} + x_1^{16}x_2^{8}x_0^{72} + x_1^{8}x_2^{72}x_0^{16} + x_1^{72}x_2^{8}x_0^{16} + x_1^{16}x_2^{72}x_0^{8} + x_1^{72}x_2^{16}x_0^{8} \\
x_1^{8}x_2^{12}x_0^{76} + x_1^{12}x_2^{8}x_0^{76} + x_1^{8}x_2^{76}x_0^{12} + x_1^{76}x_2^{8}x_0^{12} + x_1^{12}x_2^{76}x_0^{8} + x_1^{76}x_2^{12}x_0^{8} \\
x_1^{4}x_2^{44}x_0^{48} + x_1^{44}x_2^{4}x_0^{48} + x_1^{4}x_2^{48}x_0^{44} + x_1^{48}x_2^{4}x_0^{44} + x_1^{44}x_2^{48}x_0^{4} + x_1^{48}x_2^{44}x_0^{4} \\
x_1^{4}x_2^{40}x_0^{52} + x_1^{40}x_2^{4}x_0^{52} + x_1^{4}x_2^{52}x_0^{40} + x_1^{52}x_2^{4}x_0^{40} + x_1^{40}x_2^{52}x_0^{4} + x_1^{52}x_2^{40}x_0^{4} \\
x_1^{4}x_2^{36}x_0^{56} + x_1^{36}x_2^{4}x_0^{56} + x_1^{4}x_2^{56}x_0^{36} + x_1^{56}x_2^{4}x_0^{36} + x_1^{36}x_2^{56}x_0^{4} + x_1^{56}x_2^{36}x_0^{4} \\
x_1^{4}x_2^{32}x_0^{60} + x_1^{32}x_2^{4}x_0^{60} + x_1^{4}x_2^{60}x_0^{32} + x_1^{60}x_2^{4}x_0^{32} + x_1^{32}x_2^{60}x_0^{4} + x_1^{60}x_2^{32}x_0^{4} \\
x_1^{4}x_2^{28}x_0^{64} + x_1^{28}x_2^{4}x_0^{64} + x_1^{4}x_2^{64}x_0^{28} + x_1^{64}x_2^{4}x_0^{28} + x_1^{28}x_2^{64}x_0^{4} + x_1^{64}x_2^{28}x_0^{4} \\
x_1^{4}x_2^{24}x_0^{68} + x_1^{24}x_2^{4}x_0^{68} + x_1^{4}x_2^{68}x_0^{24} + x_1^{68}x_2^{4}x_0^{24} + x_1^{24}x_2^{68}x_0^{4} + x_1^{68}x_2^{24}x_0^{4} \\
x_1^{4}x_2^{20}x_0^{72} + x_1^{20}x_2^{4}x_0^{72} + x_1^{4}x_2^{72}x_0^{20} + x_1^{72}x_2^{4}x_0^{20} + x_1^{20}x_2^{72}x_0^{4} + x_1^{72}x_2^{20}x_0^{4} \\
x_1^{4}x_2^{16}x_0^{76} + x_1^{16}x_2^{4}x_0^{76} + x_1^{4}x_2^{76}x_0^{16} + x_1^{76}x_2^{4}x_0^{16} + x_1^{16}x_2^{76}x_0^{4} + x_1^{76}x_2^{16}x_0^{4} \\
x_1^{4}x_2^{12}x_0^{80} + x_1^{12}x_2^{4}x_0^{80} + x_1^{4}x_2^{80}x_0^{12} + x_1^{80}x_2^{4}x_0^{12} + x_1^{12}x_2^{80}x_0^{4} + x_1^{80}x_2^{12}x_0^{4} \\
x_1^{28}x_2^{32}x_0^{36} + x_1^{32}x_2^{28}x_0^{36} + x_1^{28}x_2^{36}x_0^{32} + x_1^{36}x_2^{28}x_0^{32} + x_1^{32}x_2^{36}x_0^{28} + x_1^{36}x_2^{32}x_0^{28} \\
x_1^{24}x_2^{32}x_0^{40} + x_1^{32}x_2^{24}x_0^{40} + x_1^{24}x_2^{40}x_0^{32} + x_1^{40}x_2^{24}x_0^{32} + x_1^{32}x_2^{40}x_0^{24} + x_1^{40}x_2^{32}x_0^{24} \\
x_1^{24}x_2^{28}x_0^{44} + x_1^{28}x_2^{24}x_0^{44} + x_1^{24}x_2^{44}x_0^{28} + x_1^{44}x_2^{24}x_0^{28} + x_1^{28}x_2^{44}x_0^{24} + x_1^{44}x_2^{28}x_0^{24} \\
x_1^{20}x_2^{36}x_0^{40} + x_1^{36}x_2^{20}x_0^{40} + x_1^{20}x_2^{40}x_0^{36} + x_1^{40}x_2^{20}x_0^{36} + x_1^{36}x_2^{40}x_0^{20} + x_1^{40}x_2^{36}x_0^{20} \\
x_1^{20}x_2^{32}x_0^{44} + x_1^{32}x_2^{20}x_0^{44} + x_1^{20}x_2^{44}x_0^{32} + x_1^{44}x_2^{20}x_0^{32} + x_1^{32}x_2^{44}x_0^{20} + x_1^{44}x_2^{32}x_0^{20} \\
x_1^{20}x_2^{28}x_0^{48} + x_1^{28}x_2^{20}x_0^{48} + x_1^{20}x_2^{48}x_0^{28} + x_1^{48}x_2^{20}x_0^{28} + x_1^{28}x_2^{48}x_0^{20} + x_1^{48}x_2^{28}x_0^{20}
\end{array}
$$

$$\begin{aligned}
&x_1^{20}x_2^{24}x_0^{52}+x_1^{24}x_2^{20}x_0^{52}+x_1^{20}x_2^{52}x_0^{24}+x_1^{52}x_2^{20}x_0^{24}+x_1^{24}x_2^{52}x_0^{20}+x_1^{52}x_2^{24}x_0^{20}\\
&x_1^{16}x_2^{36}x_0^{44}+x_1^{36}x_2^{16}x_0^{44}+x_1^{16}x_2^{44}x_0^{36}+x_1^{44}x_2^{16}x_0^{36}+x_1^{36}x_2^{44}x_0^{16}+x_1^{44}x_2^{36}x_0^{16}\\
&x_1^{16}x_2^{32}x_0^{48}+x_1^{32}x_2^{16}x_0^{48}+x_1^{16}x_2^{48}x_0^{32}+x_1^{48}x_2^{16}x_0^{32}+x_1^{32}x_2^{48}x_0^{16}+x_1^{48}x_2^{32}x_0^{16}\\
&x_1^{16}x_2^{28}x_0^{52}+x_1^{28}x_2^{16}x_0^{52}+x_1^{16}x_2^{52}x_0^{28}+x_1^{52}x_2^{16}x_0^{28}+x_1^{28}x_2^{52}x_0^{16}+x_1^{52}x_2^{28}x_0^{16}\\
&x_1^{16}x_2^{24}x_0^{56}+x_1^{24}x_2^{16}x_0^{56}+x_1^{16}x_2^{56}x_0^{24}+x_1^{56}x_2^{16}x_0^{24}+x_1^{24}x_2^{56}x_0^{16}+x_1^{56}x_2^{24}x_0^{16}\\
&x_1^{16}x_2^{20}x_0^{60}+x_1^{20}x_2^{16}x_0^{60}+x_1^{16}x_2^{60}x_0^{20}+x_1^{60}x_2^{16}x_0^{20}+x_1^{20}x_2^{60}x_0^{16}+x_1^{60}x_2^{20}x_0^{16}\\
&x_1^{12}x_2^{40}x_0^{44}+x_1^{40}x_2^{12}x_0^{44}+x_1^{12}x_2^{44}x_0^{40}+x_1^{44}x_2^{12}x_0^{40}+x_1^{40}x_2^{44}x_0^{12}+x_1^{44}x_2^{40}x_0^{12}\\
&x_1^{12}x_2^{36}x_0^{48}+x_1^{36}x_2^{12}x_0^{48}+x_1^{12}x_2^{48}x_0^{36}+x_1^{48}x_2^{12}x_0^{36}+x_1^{36}x_2^{48}x_0^{12}+x_1^{48}x_2^{36}x_0^{12}\\
&x_1^{12}x_2^{32}x_0^{52}+x_1^{32}x_2^{12}x_0^{52}+x_1^{12}x_2^{52}x_0^{32}+x_1^{52}x_2^{12}x_0^{32}+x_1^{32}x_2^{52}x_0^{12}+x_1^{52}x_2^{32}x_0^{12}\\
&x_1^{12}x_2^{28}x_0^{56}+x_1^{28}x_2^{12}x_0^{56}+x_1^{12}x_2^{56}x_0^{28}+x_1^{56}x_2^{12}x_0^{28}+x_1^{28}x_2^{56}x_0^{12}+x_1^{56}x_2^{28}x_0^{12}\\
&x_1^{12}x_2^{24}x_0^{60}+x_1^{24}x_2^{12}x_0^{60}+x_1^{12}x_2^{60}x_0^{24}+x_1^{60}x_2^{12}x_0^{24}+x_1^{24}x_2^{60}x_0^{12}+x_1^{60}x_2^{24}x_0^{12}\\
&x_1^{12}x_2^{20}x_0^{64}+x_1^{20}x_2^{12}x_0^{64}+x_1^{12}x_2^{64}x_0^{20}+x_1^{64}x_2^{12}x_0^{20}+x_1^{20}x_2^{64}x_0^{12}+x_1^{64}x_2^{20}x_0^{12}\\
&x_1^{12}x_2^{16}x_0^{68}+x_1^{16}x_2^{12}x_0^{68}+x_1^{12}x_2^{68}x_0^{16}+x_1^{68}x_2^{12}x_0^{16}+x_1^{16}x_2^{68}x_0^{12}+x_1^{68}x_2^{16}x_0^{12}.
\end{aligned}$$

Arguing as in (ii) we obtain that the ideal I_{96} generated by them is a non monomial Togliatti system.

4.2 GT-systems and GT-surfaces with a dihedral group

In this section, we study the invariants of the dihedral group D_{2d} of order $2d$ represented in $\mathrm{SL}(3,\mathbb{K})$ by a cyclic group $\Gamma = \langle M_{d;0,a,d-a}\rangle$ of order $d \geq 0$ with $0 < a < \frac{d}{2}$ and the linear transformation

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

We consider the cyclic extension $\overline{D_{2d}} \subset \mathrm{GL}(3,\mathbb{K})$ of D_{2d} and we describe the invariants of $\overline{D_{2d}}$. Our main result proves that $R^{\overline{D_{2d}}}$ is minimally generated by forms of degree $2d$ (Theorem 4.2.6). As a consequence, we obtain that the ideal $I_{2d} \subset R$ generated by them is a non monomial $GT-$system with group D_{2d} (Proposition 4.2.9). In Subsection 4.2.1, we study the geometry of the $GT-$surface with group D_{2d} as we did for $GT-$surfaces with a finite abelian group.

Since the action of $D_{2d} = \langle M_{d;0,a,d-a}, \sigma\rangle \subset \mathrm{SL}(3,\mathbb{K})$ on R fixes the variable x_0, we have that $R^{D_{2d}} = \mathbb{K}[x_0] \otimes \mathbb{K}[x_1,x_2]^{D'_{2d}}$ where

$$D'_{2d} = \left\langle M_{d;a,d-a}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle \subset \mathrm{SL}(2,\mathbb{K})$$

is a representation of the dihedral group of order $2d$ in $\mathrm{SL}(2,\mathbb{K})$. Moreover, $\mathbb{K}[x_1,x_2]^{D'_{2d}} = \mathbb{K}[x_1x_2, x_1^d + x_2^d]$. Thus, the ring of invariants of D_{2d}

$$R^{D_{2d}} = \mathbb{K}[x_0, x_1x_2, x_1^d + x_2^d]$$

is a non standard graded polynomial ring (see, for instance, [76] and [77]). In this setting, we see the ring $R^{\overline{D_{2d}}}$ as $\mathbb{K}-$graded subalgebra of R as well as of $R^{D_{2d}}$:

$$R^{\overline{D_{2d}}} = \bigoplus_{t\geq 0} R_t^{\overline{D_{2d}}}, \quad \text{where} \quad R_t^{\overline{D_{2d}}} := R_{2dt}^{D_{2d}} = R_{2dt} \cap R^{D_{2d}}.$$

On the other hand, the cyclic group $\langle M_{d;0,a,d-a}\rangle = \langle M_{d;1,d-1}\rangle$. Hence, from now on, we fix an integer $d \geq 3$ and a dihedral group $D_{2d} = \langle \Gamma, \sigma\rangle \subset \mathrm{SL}(3,\mathbb{K})$ of order $2d$ with $\Gamma = \langle M_{d;0,1,d-1}\rangle$. Our main objective is to describe the invariants of $R^{\overline{D_{2d}}}$. To do it, we analyse first each graded component $R_t^{\overline{D_{2d}}}$.

We start noticing that any invariant of $\overline{D_{2d}}$ is an invariant of the cyclic group Γ. So, if $f \in R^{\overline{D_{2d}}}$, then any monomial which occurs in f is a monomial invariant of Γ of degree a multiple of $2d$. We recall that a monomial $m = x_0^{a_0}x_1^{a_1}x_2^{a_2} \in R_{2t}^{\Gamma}$ if and only if (a_0,a_1,a_2) is a $\mathbb{Z}^3_{\geq 0}-$solution of the system of linear congruences:

$$(*)_{\mathcal{A};2t,r} \begin{cases} y_0 + y_1 + y_2 = 2dt \\ y_1 + (d-1)y_2 = rd. \end{cases}, \qquad r = 0,\dots,2(d-1)t.$$

The action of $\langle M^l_{d;1,d-1}\sigma\rangle$ on R is the same as the action of $\langle \mathrm{diag}(1, e^{d-1}, e)^l\rangle$ for any $0 \leq l \leq d-1$. Moreover, we have:

Lemma 4.2.1. *(i) A monomial $m \in R^{\overline{\Gamma}}$ if and only if m is an invariant of $\langle M_{d;0,d-1,1}\rangle \subset \mathrm{SL}(3,\mathbb{K})$.*

(ii) There are $td+1$ monomial invariants of $R^{\overline{D_{2d}}}$ of degree $2td$:

$$\{x_0^{2dt-2a_1}x_1^{a_1}x_2^{a_1} \mid a_1 = 0,\dots,td.\}$$

Proof. (i) Let $m = x_0^{a_0}x_1^{a_1}x_2^{a_2} \in R^\Gamma$ be a monomial of degree $2td$. Then, $a_1 + (d-1)a_2 = rd$ for some $r \in \mathbb{Z}_{\geq 0}$. Since $0 < (d-1)a_1 + a_2 = da_1 - (a_1 - a_2) = d(a_1 + a_2) - rd$ is a multiple of d, it follows that m is an invariant of $\langle M_{d;1,d-1,1}\rangle$.

(ii) The action of σ on m is $x_0^{a_0}x_1^{a_2}x_2^{a_1}$. So, $m \in R^{\overline{D_{2d}}}$ if and only if $m \in R^{\overline{\Gamma}}$ and $a_1 = a_2$. Solving the linear system of congruences $(*)_{\mathcal{A};2t,r}$ we obtain the listed monomials. □

Using the above lemma and the fact that $R_t^{\overline{D_{2d}}} = R_{2dt}^{D_{2d}}$, we compute the Hilbert function of $R^{\overline{D_{2d}}}$.

Proposition 4.2.2. *For any $t \geq 0$, we have:*

$$\mathrm{HF}(R^{\overline{D_{2d}}}, t) = \frac{2dt^2 + (d + \mathrm{GCD}(d,2) + 2)t + 2}{2}.$$

Proof. Since $\mathrm{HF}(R^{\overline{D_{2d}}}, t) = \mathrm{HF}(R^{D_{2d}}, 2dt)$, we obtain that

$$\mathrm{HF}(R^{\overline{D_{2d}}}, t) = \frac{1}{2d}\sum_{g\in D_{2d}} \mathrm{trace}\left(g^{(2dt)}\right) = \frac{1}{2d}\,\mathrm{trace}\left(\sum_{g\in D_{2d}} g^{(2dt)}\right),$$

where $g^{(2dt)}$ is the restriction of g to R_{2dt}. We choose the set $\mathcal{M}_{2,2td} = \{m_1, \ldots, m_{N_{2,2td}}\} \subset R$ of all $N_{2,2td}$ monomials of degree $2dt$ as a basis of R_{2dt}. We denote by M the $N_{2,2td} \times N_{2,2td}$ matrix which represents the linear map $\sum_{g\in\rho(D_{2d})} g^{(2dt)}$ in the chosen basis and by $M_{(i,i)}$ its diagonal entries. Fixed $m_i \in \mathcal{M}_{2,2dt}$, we distinguish two cases:

<u>Case 1:</u> if $m_i \in R^{\overline{\Gamma}}$, then by Lemma 4.2.1,

$$M_{(i,i)} = \begin{cases} 2d & \text{if } \sigma(m_i) = m_i \\ d & \text{if } \sigma(m_i) \neq m_i. \end{cases}$$

<u>Case 2:</u> if $m_i \notin R^\Gamma$, then by Lemma 4.2.1(ii), m_i is not an invariant of σ and we obtain $M_{i,i} = 1 + \xi + \cdots + \xi^{d-1} = 0$, where ξ a dth root of $1 \in \mathbb{K}$.

Let μ_{2dt}^c denote the number of monomials of degree $2dt$ in R^Γ. Thus,

$$(2d)\,\mathrm{HF}(R^{\overline{D_{2d}}}, t) = d(\mu_{2dt}^c + td + 1).$$

As we have shown in Theorem 3.1.21, $\mu_{2dt}^c = dt^2 + 2t + \mathrm{GCD}(2,d)t + 1$. Altogether we obtain the desired expression. □

As a direct corollary:

Corollary 4.2.3. *The Hilbert series of $R^{\overline{D_{2d}}}$ is*

$$\mathrm{HS}(R^{\overline{D_{2d}}}, z) = \frac{\left(\frac{d-\mathrm{GCD}(d,2)}{2} z^{4d} + \frac{3d+\mathrm{GCD}(d,2)-2}{2} z^{2d} + 1\right)}{(1-z^{2d})^3}.$$

Thus far, we know that $R_t^{\overline{D_{2d}}}$ is a $\mathbb{K}-$vector space of dimension

$$\frac{\mu_{2dt}^c + td + 1}{2}.$$

To determine a $\mathbb{K}-$basis of $R_t^{\overline{D_{2d}}}$, we complete the set $\{x_0^{2td-2a_1} x_1^{a_1} x_2^{a_2} \mid a_1 = 0, \dots, td\}$ of $td+1$ monomial invariants of $\overline{D}_{2d}$ to a basis of $R_t^{\overline{D_{2d}}}$ using its relation with R_{2t}^{Γ}.

Proposition 4.2.4. *$R_t^{\overline{D_{2d}}}$ is generated by*

(i) $x_0^{2dt}, x_0^{2dt-2} x_1 x_2, x_0^{2dt-4} x_1^2 x_2^2, \dots, x_1^{td} x_2^{td}$*; and*

(ii) all the binomials $x_0^{a_0} x_1^{a_1} x_2^{a_2} + x_0^{a_0} x_1^{a_2} x_2^{a_1}$ *of degree* $2dt$ *such that* $a_1 \neq a_2$ *and* $x_0^{a_0} x_1^{a_1} x_2^{a_2} \in R^{\Gamma}$*.*

Proof. By Lemma 4.2.1, there are $\frac{\mu_{2dt}^c - td - 1}{2}$ binomials of degree $2dt$ of the form $x_0^{a_0} x_1^{a_1} x_2^{a_2} + x_0^{a_0} x_1^{a_2} x_2^{a_1} \in R^{\Gamma}$ with $a_1 \neq a_2$. Since the listed forms are $\mathbb{K}-$linearly independent, the result follows from Proposition 4.2.2. $\square$

From now on, we denote by $\mathcal{B}_{2td}$ the set of generators of $R_t^{\overline{D_{2d}}}$ in Proposition 4.2.4. Let us see some illustrative examples.

Example 4.2.5. (i) Take $d = 3$ and $D_{2\cdot 3} = \langle M_{3;0,1,2}, \sigma \rangle \subset \mathrm{SL}(3, \mathbb{K})$. We have $\mathrm{HF}(R^{\overline{D_{2\cdot 3}}}, 1) = 7$ and $\mathrm{HF}(R^{\overline{D_{2\cdot 3}}}, 2) = 19$ with

$$\begin{aligned}
\mathcal{B}_{2\cdot 3} = \; & \{x_0^6, x_0^3 x_1^3 + x_0^3 x_2^3, x_0^4 x_1 x_2, x_1^6 + x_2^6, x_0 x_1^4 x_2 + x_0 x_1 x_2^4, x_0^2 x_1^2 x_2^2, x_1^3 x_2^3\} \\
\mathcal{B}_{4\cdot 3} = \; & \{x_0^{12}, x_0^9 x_1^3 + x_0^9 x_2^3, x_0^{10} x_1 x_2, x_0^6 x_1^6 + x_0^6 x_2^6, x_0^7 x_1^4 x_2 + x_0^7 x_1 x_2^4, x_0^8 x_1^2 x_2^2, \\
& x_0^3 x_1^9 + x_0^3 x_2^9, x_0^4 x_1^7 x_2 + x_0^4 x_1 x_2^7, x_0^5 x_1^5 x_2^2 + x_0^5 x_1^2 x_2^5, x_0^6 x_1^3 x_2^3, x_1^{12} + x_2^{12}, \\
& x_0 x_1^{10} x_2 + x_0 x_1 x_2^{10}, x_0^2 x_1^8 x_2^2 + x_0^2 x_1^2 x_2^8, x_0^3 x_1^6 x_2^3 + x_0^3 x_1^3 x_2^6, x_0^4 x_1^4 x_2^4, \\
& x_1^9 x_2^3 + x_1^3 x_2^9, x_0 x_1^7 x_2^4 + x_0 x_1^4 x_2^7, x_0^2 x_1^5 x_2^5, x_1^6 x_2^6\}.
\end{aligned}$$

(ii) Take $d=4$ and $D_{2\cdot 4}=\langle M_{4;0,1,3},\sigma\rangle \subset \mathrm{SL}(3,\mathbb{K})$, we have $\mathrm{HF}(R^{\overline{D_{2\cdot 4}}},1)=9$ and $\mathrm{HF}(R^{\overline{D_{2\cdot 4}}},2)=25$ with

$$\begin{aligned}\mathcal{B}_{2\cdot 4}=&\{x_0^8, x_0^4x_1^4+x_0^4x_1^4, x_0^6x_1x_2, x_1^8+x_2^8, x_0^2x_1^5x_2+x_0^2x_1x_2^5, x_0^4x_1^2x_2^2,\\ &x_1^6x_2^2+x_1^2x_2^6, x_0^2x_1^3x_2^3, x_1^4x_2^4\}\\ \mathcal{B}_{4\cdot 4}=&\{x_0^{16}, x_0^{12}x_1^4+x_0^{12}x_2^4, x_0^{14}x_1x_2, x_0^8x_1^8+x_0^8x_2^8, x_0^{10}x_1^5x_2+x_0^{10}x_1x_2^5,\\ &x_0^{12}x_1^2x_2^2, x_0^4x_1^{12}+x_0^4x_2^{12}, x_0^6x_1^9x_2+x_0^6x_1x_2^9, x_0^8x_1^6x_2^2+x_0^8x_1^2x_2^6, x_0^{10}x_1^3x_2^3,\\ &x_1^{16}+x_2^{16}, x_0^2x_1^{13}x_2+x_0^2x_1x_2^{13}, x_0^4x_1^{10}x_2^2+x_0^4x_1^2x_2^{10}, x_0^6x_1^7x_2^3+x_0^6x_1^3x_2^7,\\ &x_0^8x_1^4x_2^4, x_1^{14}x_2^2+x_1^2x_2^{14}, x_0^2x_1^{11}x_2^3+x_0^2x_1^3x_2^{11}, x_0^4x_1^8x_2^4+x_0^4x_1^4x_2^8, x_0^6x_1^5x_2^5,\\ &x_1^{12}x_2^4+x_1^4x_2^{12}, x_0^2x_1^9x_2^5+x_0^2x_1^5x_2^9, x_0^4x_1^6x_2^6, x_1^{10}x_2^6+x_1^6x_2^{10}, x_0^2x_1^7x_2^7, x_1^8x_2^8\}.\end{aligned}$$

Our main goal is to prove that $\mathcal{B}_{2d}$ is a minimal set of fundamental invariants of $\overline{D_{2d}}$. To achieve it, we use the natural structure of $R^{\overline{D_{2d}}}$ as a subring of $R^{D_{2d}}$. We set $y_0=x_0$, $y_1=x_1x_2$ and $y_2=x_1^d+x_2^d$. As we have pointed out at the beginning of this section, $R^{D_{2d}}=\mathbb{K}[y_0,y_1,y_2]$ is a non standard graded polynomial ring with $\deg(y_0)=1$, $\deg(y_1)=2$ and $\deg(y_2)=d$. From this standpoint, $R_t^{\overline{D_{2d}}}=\mathbb{K}[y_0,y_1,y_2]_{2td}$ is the $\mathbb{K}-$vector subspace with monomial basis

$$\mathcal{A}_{2dt}=\{y_0^{b_0}y_1^{b_1}y_2^{b_2}\ \mid\ b_0+2b_1+db_2=2td\}.$$

In particular, for $t=1$ we have a change of basis

$$\rho:\mathbb{K}[y_0,y_1,y_2]_{2d}\longrightarrow R_1^{\overline{D_{2d}}}\quad \text{given by}$$

$$\left\{\begin{array}{lll} y_0^{b_0}y_1^{b_1}y_2^{b_2} & \mapsto & x_0^{b_0}x_1^{b_1}x_2^{b_1}(x_1^d+x_2^d)^{b_2},\ \text{ if }\ 0\le b_2\le 1\\ y_2^2 & \mapsto & (x_1^{2d}+x_2^{2d})+2x_1^dx_2^d.\end{array}\right. \tag{4.2.1}$$

Theorem 4.2.6. *$\mathcal{B}_{2d}$ is a minimal set of fundamental invariants of $\overline{D_{2d}}$.*

Proof. We see that for any $t\ge 2$, any monomial $y_0^{b_0}y_1^{b_1}y_2^{b_2}\in\mathcal{A}_{2dt}$ is divisible by a monomial of $\mathcal{A}_{2d}$. Then by induction, it follows that $\mathcal{A}_{2d}$ is a set of generators of $R^{\overline{D_{2d}}}\subset R^{D_{2d}}$. Using (4.2.1), we obtain that $\mathcal{B}_{2d}$ is a minimal set of generators of $R^{\overline{D_{2d}}}$. Let $m=y_0^{b_0}y_1^{b_1}y_2^{b_2}\in\mathcal{A}_{2dt}$ be a monomial of degree $b_0+2b_1+db_2=2dt$, $t\ge 2$. On one hand, we may suppose that $b_0<2d$, $b_1<d$ and $b_2<2$. Otherwise, y_0^{2d}, y_1^d or y_2^2 divide m and the result follows. On the other hand, if $b_2=0$, $b_0<2d$ and $b_1<d$, then we have $\deg(m)=2d$

and $t = 1$. Therefore it only remains to prove the case $b_0 < 2d$, $b_1 < d$ and $b_2 = 1$ with $b_0 + 2b_1 + d = 4d$. Since $b_0 + 2b_1 = 3d$ and $b_1 < d$, this implies that $b_0 \geq d$ and then $y_0^d y_2 \in \mathcal{A}_{2d}$ divides m, as required. $\square$

Remark 4.2.7. The change of variables ρ induces an isomorphism of graded $\mathbb{K}-$algebras $\rho : \mathbb{K}[\mathcal{A}_{2d}] \longrightarrow \mathbb{K}[\mathcal{B}_{2d}]$.

Example 4.2.8. We take $d = 3$ and $D_{2\cdot 3} = \langle M_{3;0,1,2}, \sigma \rangle \subset \mathrm{SL}(3, \mathbb{K})$. We express the invariants of $\mathcal{B}_{4\cdot 3}$ in terms of $\mathcal{B}_{2d}$ (Example 4.2.5(i)) writing all monomials of $\mathcal{A}_{4\cdot 3}$ as products of monomials of $\mathcal{A}_{2\cdot 3}$:

$$\begin{aligned}
\mathcal{A}_{2\cdot 3} &= \{y_0^6, y_0^3 y_2, y_0^4 y_1, y_2^2, y_0 y_1 y_2, y_0^2 y_1^2, y_1^3\} \\
\mathcal{A}_{4\cdot 3} &= \{y_0^{12}, y_0^{10} y_1, y_0^8 y_1^2, y_0^6 y_1^3, y_0^4 y_1^4, y_0^2 y_1^5, y_1^6, y_0^9 y_2, y_0^7 y_1 y_2, y_0^5 y_1^2 y_2, y_0^3 y_1^3 y_2, \\
&\quad y_0 y_1^4 y_2, y_0^6 y_2^2, y_0^4 y_1 y_2^2, y_0^2 y_1^2 y_2^2, y_1^3 y_2^2, y_0^3 y_2^3, y_0 y_1 y_2^3, y_2^4\}.
\end{aligned}$$

Then by (4.2.1), we obtain the following factorizations of the monomials and binomials of $\mathcal{B}_{4\cdot 3}$:

$$\begin{aligned}
&x_0^{12} &&= (x_0^6)(x_0^6) \\
&x_0^{10} x_1 x_2 &&= (x_0^6)(x_0^4 x_1 x_2) \\
&x_0^8 x_1^2 x_1^2 &&= (x_0^6)(x_0^2 x_1^2 x_2^2) \\
&x_0^6 x_1^3 x_2^3 &&= (x_0^6)(x_1^3 x_2^3) \\
&x_0^4 x_1^4 x_2^4 &&= (x_0^4 x_1 x_2)(x_1^3 x_2^3) \\
&x_0^2 x_1^5 x_2^5 &&= (x_0^2 x_1^2 x_2^2)(x_1^3 x_2^3) \\
&x_1^6 x_2^6 &&= (x_1^3 x_2^3)^2 \\
&x_0^9 x_1^3 + x_0^9 x_2^3 &&= x_0^6 (x_0^3 x_1^3 + x_0^3 x_2^3) \\
&x_0^6 x_1^6 + x_0^6 x_2^6 &&= x_0^6 (x_1^6 + x_1^6) \\
&x_0^7 x_1^4 x_2 + x_0^7 x_1 x_2^4 &&= x_0^4 x_1 x_2 (x_0^3 x_1^3 + x_0^3 x_2^3) \\
&x_0^4 x_1^7 x_2 + x_0^4 x_1 x_2^7 &&= x_0^4 x_1 x_2 (x_1^6 + x_2^6) \\
&x_0^5 x_1^5 x_2^2 + x_0^5 x_1^2 x_2^5 &&= x_0^2 x_1^2 x_2^2 (x_0^3 x_1^3 + x_0^3 x_2^3) \\
&x_0^2 x_1^8 x_2^2 + x_0^2 x_1^2 x_2^8 &&= x_0^2 x_1^2 x_2^2 (x_1^6 + x_2^6) \\
&x_0^3 x_1^6 x_2^3 + x_0^3 x_1^3 x_2^6 &&= x_1^3 x_2^3 (x_0^3 x_1^3 + x_0^3 x_2^3) \\
&x_1^9 x_2^3 + x_1^3 x_2^9 &&= x_1^3 x_2^3 (x_1^6 + x_2^6) \\
&x_0 x_1^7 x_2^4 + x_0 x_1^4 x_2^7 &&= x_1^3 x_2^3 (x_0 x_1^4 x_2 + x_0 x_1 x_2^4) \\
&x_0^3 x_1^9 + x_0^3 x_2^9 &&= (x_1^6 + x_2^6)(x_0^3 x_1^3 + x_0^3 x_2^6) - x_1^3 x_2^3 (x_0^3 x_1^3 + x_0^3 x_2^3) \\
&x_1^{12} + x_2^{12} &&= (x_1^6 + x_2^6)^2 - 2(x_1^3 x_2^3)^2 \\
&x_0 x_1^{10} x_2 + x_0 x_1 x_2^{10} &&= (x_0 x_1^4 x_2 + x_0 x_1 x_2^4)(x_1^6 + x_2^6) - x_1^3 x_2^3 (x_0 x_1^4 x_2 + x_0 x_1 x_2^4).
\end{aligned}$$

Notice that these decompositions are not unique, for instance $x_0^8x_1^2x_2^2$ can also be factored as $(x_0^4x_1x_2)^2$.

Proposition 4.2.9. *Let $d \geq 3$ be an integer and $D_{2d} = \langle M_{d;0,1,d-1}, \sigma\rangle \subset \mathrm{SL}(3,\mathbb{K})$ a dihedral group of order $2d$. Then, the ideal $I_{2d} \subset R$ generated by a minimal set of fundamental invariants $\overline{D}_{2d}$ is a $GT-$system with group D_{2d}.*

Proof. The condition $|\mathcal{B}_{2d}| \leq 2d+1$ on the number of generators of I_{2d} is satisfied (Theorem 1.4.6). Indeed, applying Proposition 4.2.2 we obtain:

$$|\mathcal{B}_{2d}| = \frac{3d+4+\mathrm{GCD}(d,2)}{2}.$$

The right term of the equality is smaller or equal than $2d+1$ for all $d \geq 3$. By Theorem 4.2.6 and Propositions 1.4.17 and 4.1.1, we can conclude that I_{2d} is a $GT-$system with group D_{2d}. □

4.2.1 GT-surfaces with a dihedral group

In this subsection, we consider the $GT-$surface $S_{D_{2d}}$ parameterized by a $GT-$system I_{2d} with group $D_{2d} = \langle M_{d;0,1,d-1}, \sigma\rangle \subset \mathrm{SL}(3,\mathbb{K})$. Namely, we take $I_{2d} \subset R$ the ideal generated by the minimal set $\mathcal{B}_{2d}$ of fundamental invariants of $\overline{D_{2d}}$ determined in Theorem 4.2.6. We denote $|\mathcal{B}_{2d}| = \mu_{2d}$. Then, $S_{D_{2d}}$ is the image of the morphism $\varphi_{I_{2d}} : \mathbb{P}^n \longrightarrow \mathbb{P}^{\mu_{2d}-1}$ defined by $\mathcal{B}_{2d}$. We relate the homogeneous coordinate ring $A(S_{D_{2d}})$ of $S_{D_{2d}}$ to the ring $R^{\overline{D_{2d}}}$, we establish that $S_{D_{2d}}$ is an aCM surface in $\mathbb{P}^{\mu_{2d}-1}$ and we determine a minimal graded free resolution of $A(S_{D_{2d}})$ (Theorems 4.2.12 and 4.2.14). Lastly, we look at a set of generators of the homogeneous ideal $\mathrm{I}(S_{D_{2d}})$ of any $GT-$surface $S_{D_{2d}}$ with group D_{2d} and we prove that $\mathrm{I}(S_{D_{2d}})$ is minimally generated by quadrics (Theorem 4.2.17).

We begin with some notation needed in the sequel.

Notation 4.2.10. We introduce a new set of variables:

$$\mathcal{W}_d := \left\{ w_{(r,\gamma)} \ \middle|\ 0 \leq r \leq 2(d-1) \ \text{ and } \ \max\{0, \lceil \frac{(r-2)d}{d-2} \rceil\} \leq \gamma \leq r \right\}$$

ordered lexicographically and we set $S = \mathbb{K}[w_{(r,\gamma)}]_{w_{(r,\gamma)} \in \mathcal{W}_d}$.

Each pair (r,γ) in $\mathcal{W}_d$ uniquely determines the exponents of an element in $\mathcal{B}_{2d}$ (Lemma 4.2.1 and Proposition 4.2.4). Hence, the cardinality of $\mathcal{W}_d$ is $\mu_{2d} = d+2+\frac{d+\mathrm{GCD}(2,d)}{2}$. We exhibit a few examples.

Example 4.2.11. (i) Take $d=3$ and $D_{2\cdot 3} = \langle M_{3;0,1,2}, \sigma\rangle \subset \mathrm{SL}(3,\mathbb{K})$. We have

$$\begin{aligned}\mathcal{B}_{2\cdot 3} = \; & \{x_0^6, x_0^3x_1^3 + x_0^3x_2^3, x_0^4x_1x_2, x_1^6+x_2^6, x_0x_1^4x_2 + x_0x_1x_2^4, x_0^2x_1^2x_2^2, x_1^3x_2^3\}\\ \mathcal{W}_3 = \; & \{w_{(0,0)}, w_{(1,0)}, w_{(1,1)}, w_{(2,0)}, w_{(2,1)}, w_{(2,2)}, w_{(3,3)}\}.\end{aligned}$$

(ii) Take $d=4$ and $D_{2\cdot 4} = \langle M_{4;0,1,3}, \sigma\rangle \subset \mathrm{GL}(3,\mathbb{K})$. We have

$$\begin{aligned}\mathcal{B}_{2\cdot 4} = \; & \{x_0^8, x_0^4x_1^4 + x_0^4x_1^4, x_0^6x_1x_2, x_1^8+x_2^8, x_0^2x_1^5x_2 + x_0^2x_1x_2^5, x_0^4x_1^2x_2^2,\\ & x_1^6x_2^2 + x_1^2x_2^6, x_0^2x_1^3x_2^3,\ x_1^4x_2^4\}\\ \mathcal{W}_4 = \; & \{w_{(0,0)}, w_{(1,0)}, w_{(1,1)}, w_{(2,0)}, w_{(2,1)}, w_{(2,2)}, w_{(3,2)}, w_{(3,3)}, w_{(4,4)}\}.\end{aligned}$$

(iii) Take $d=5$ and $D_{2\cdot 5} = \langle M_{5;0,1,4}, \sigma\rangle \subset \mathrm{SL}(3,\mathbb{K})$. We have:

$$\begin{aligned}\mathcal{B}_{2\cdot 5} = \; & \{x_0^{10}, x_0^5x_1^5 + x_0^5x_2^5, x_0^8x_1x_2, x_1^{10}+x_2^{10}, x_0^3x_1^6x_2 + x_0^3x_1x_2^6, x_0^6x_1^2x_2^2,\\ & x_0x_1^7x_2^2 + x_0x_1^2x_2^7, x_0^4x_1^3x_2^3, x_0^2x_1^4x_2^4, x_1^5x_2^5\}\\ \mathcal{W}_5 = \; & \{w_{(0,0)}, w_{(1,0)}, w_{(1,1)}, w_{(2,0)}, w_{(2,1)}, w_{(2,2)}, w_{(3,2)}, w_{(3,3)}, w_{(4,4)}, w_{(5,5)}\}.\end{aligned}$$

In this setting, we have the following.

Theorem 4.2.12. *Let $S_{D_{2d}}$ be a GT$-$surface with group D_{2d}.*

(i) The homogeneous coordinate ring $A(S_{D_{2d}})$ of $S_{D_{2d}}$ is isomorphic to $R^{\overline{D_{2d}}}$. Thus, $S_{D_{2d}}$ is an aCM projection of the Veronese variety $X_{2,2d} \subset \mathbb{P}^{N_{2,2d}-1}$ from the linear system $\langle I_{2d}^{-1}\rangle_{2d}$.

(ii) The Hilbert function and series of $A(S_{D_{2d}})$ are

$$\mathrm{HF}(A(S_{D_{2d}}),t) = \frac{2dt^2 + (d+\mathrm{GCD}(d,2)+2)t+2}{2}$$

$$\mathrm{HS}(A(S_{D_{2d}}),z) = \frac{\left(\frac{d-\mathrm{GCD}(d,2)}{2}z^2 + \frac{3d+\mathrm{GCD}(d,2)-2}{2}z+1\right)}{(1-z)^3}.$$

(iii) $S_{D_{2d}}$ *is an aCM surface of degree* $2d$ *with Castelnuovo–Mumford regularity* $\operatorname{reg}(A(S_{D_{2d}})) = 3$ *and CM–type* $\frac{d-\mathrm{GCD}(d,2)}{2}$.

Proof. (i) The homogeneous ideal $\mathrm{I}(S_{D_{2d}}) \subset S$ of $S_{D_{2d}}$ is the prime ideal kernel of the ring homomorphism $\varphi_d : S \longrightarrow \mathbb{K}[\mathcal{B}_{2d}]$ sending $w_{(r,\gamma)}$ to

$$\begin{cases} x_0^{2d-2\gamma} x_1^{\gamma} x_2^{\gamma} =: m_{(r,\gamma)} & \text{if } r = \gamma \\ x_0^{(2-r)d+(d-2)\gamma}(x_1^{rd-(d-1)\gamma} x_2^{\gamma} + x_1^{\gamma} x_2^{rd-(d-1)\gamma}) =: m_{(r,\gamma)} + \overline{m_{(r,\gamma)}} & \text{otherwise.} \end{cases}$$

By Theorem 4.2.6, $A(S_{D_{2d}}) = S/\,\mathrm{I}(S_{D_{2d}}) \cong R^{\overline{D_{2d}}}$ and by Theorem 1.3.10, $A(S_{D_{2d}})$ is a CM ring.

(ii) It is a direct consequences of (i), Proposition 4.2.2 and Corollary 4.2.3.

(iii) The information of the Hilbert series of $S_{D_{2d}}$ and (i) give (iii). □

Remark 4.2.13. As a consequence of Theorem 4.2.12, a $GT-$surface $S_{D_{2d}}$ with group D_{2d} is an arithmetically Gorenstein surface if and only if $d = 3$ or 4.

We denote the codimension of $S_{D_{2d}}$ by

$$C := \operatorname{codim}(S_{D_{2d}}) = \frac{3d + \mathrm{GCD}(d,2) - 2}{2}.$$

Set $h := \deg(S_{D_{2d}}) - C - 2 = \frac{d-\mathrm{GCD}(d,2)-2}{2}$. We have:

Theorem 4.2.14. *A minimal graded free* $S-$*resolution of* $A(S_{D_{2d}})$ *looks like*

$$0 \longrightarrow S^{b_{C,2}}(-C-2) \longrightarrow \oplus_{l=1,2} S^{b_{C-1,l}}(-C+1-l) \longrightarrow \cdots \longrightarrow$$

$$\longrightarrow \oplus_{l=1,2} S^{b_{C-h,l}}(-C+h-l) \longrightarrow S^{b_{C-h-1,1}}(-C+h) \longrightarrow \cdots \longrightarrow$$

$$\longrightarrow S^{b_{1,1}}(-2) \longrightarrow S \longrightarrow S/\,\mathrm{I}(S_{2d}) \longrightarrow 0$$

where

$$b_{i,l} := \begin{cases} i\binom{C}{i+1} + (C-i-h)\binom{C}{i-1} & \text{if } 1 \le i \le C-h-1, l = 1 \\ i\binom{r}{i+1} & \text{if } C-h \le i \le C, l = 1 \\ (i-C+h+1)\binom{C}{i} & \text{if } C-h \le i \le C, l = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For $d = 3, 4$ we explicitly compute the resolutions of $S_{D_{2\cdot 3}}$ and $S_{D_{2\cdot 4}}$ in Example 4.2.16(i),(ii). For all $d \geq 5$ we check that $C + 3 \leq 2d \leq 2C$ and then we apply [88, Corollary 3.4(ii)]. It holds $2d \leq 3d + \mathrm{GCD}(d, 2) - 2$ for all $d \geq 3$. On the other hand,

$$C + 3 = \frac{3d + \mathrm{GCD}(d, 2) + 4}{2} \leq 2d$$

if and only if $3d + \mathrm{GCD}(d, 2) + 4 \leq 4d$ if and only if $\mathrm{GCD}(d, 2) + 4 \leq d$. The last inequality holds for all $d \geq 5$. $\square$

As a direct corollary, we obtain:

Corollary 4.2.15. $\mathrm{I}(S_{D_{2d}})$ *is minimally generated by* $\frac{9d^2+2d+8}{8}$ *quadrics if* d *is even and by* $\frac{9d^2-4d+3}{8}$ *quadrics if* d *is odd.*

Let us illustrate Theorem 4.2.14 with some examples.

Example 4.2.16. (i) For $d = 3$, $S_{D_{2d}}$ has codimension $C = 4$ and degree $\deg(S_{D_{2d}}) = 6$, so $h = 0$. A minimal free resolution of $A(S_{D_{2\cdot 3}})$ is

$$0 \longrightarrow S(-6) \longrightarrow S^9(-4) \longrightarrow S^{16}(-3) \longrightarrow$$

$$\longrightarrow S(-2)^9 \longrightarrow S \longrightarrow S/\,\mathrm{I}(S_{D_{2\cdot 3}}) \longrightarrow 0.$$

As we remarked before, $S_{D_{2\cdot 3}}$ is an arithmetically Gorenstein surface.

(ii) For $d = 4$, $S_{D_{2d}}$ has codimension $C = 6$ and degree $\deg(S_{D_{2d}}) = 8$, so $h = 0$. A minimal free resolution of $A(S_{D_{2\cdot 4}})$ is

$$0 \longrightarrow S(-8) \longrightarrow S^{20}(-6) \longrightarrow S^{64}(-5) \longrightarrow S^{90}(-4) \longrightarrow$$

$$S^{64}(-3) \longrightarrow S^{20}(-2) \longrightarrow S \longrightarrow S/\,\mathrm{I}(S_{D_{2\cdot 4}}) \longrightarrow 0.$$

As we remarked before, $S_{D_{2\cdot 4}}$ is an arithmetically Gorenstein surface.

(iii) For $d = 5$, $S_{D_{2d}}$ has codimension $C = 7$ and degree $\deg(S_{D_{2d}}) = 10$, so we have $h = 1$ and a minimal free resolution of $A(S_{D_{2\cdot 5}})$ is

$$0 \longrightarrow S^2(-9) \longrightarrow S^7(-8) \oplus S^6(-7) \longrightarrow S^{70}(-6) \longrightarrow S^{154}(-5) \longrightarrow$$

$$\longrightarrow S^{168}(-4) \longrightarrow S^{98}(-3) \longrightarrow S^{26}(-2) \longrightarrow S \longrightarrow S/\,\mathrm{I}(S_{D_{2\cdot 5}})) \longrightarrow 0.$$

Our next goal is to describe a minimal set of generators of $\mathrm{I}(S_{D_{2d}})$. To achieve this goal, we take advantage of the natural structure of $R^{\overline{D_{2d}}}$ as a subring of $R^{D_{2d}}$. We define new variables $z_{(r,\gamma)}$ and we set $S' = \mathbb{K}[z_{(r,\gamma)}]$. We consider the linear change of variables induced by ρ (see (4.2.1)):

$$\begin{cases} z_{(r,\gamma)} = w_{(r,\gamma)}, & \text{if} \quad w_{(r,\gamma)} \neq w_{(2,0)} \\ z_{(2,0)} = w_{(2,0)} + 2w_{(d,d)}, & \end{cases} \tag{4.2.2}$$

It gives an isomorphism $\tilde{\rho} : \mathbb{K}[z_{(r,\gamma)}] \longrightarrow S$ of polynomial rings. We have the following commutative diagram

$$\begin{array}{ccc} S' & \xrightarrow{\psi_d} & \mathbb{K}[\mathcal{A}_{2d}] \\ \tilde{\rho}\Big\downarrow & & \rho\Big\downarrow \\ S & \xrightarrow{\varphi_d} & \mathbb{K}[\mathcal{B}_{2d}] \end{array}$$

where

$$\psi_d(z_{(r,\gamma)}) = \begin{cases} \rho^{-1}(\varphi_d(w_{(r,\gamma)})) & \text{if } z_{(r,\gamma)} \neq z_{(2,0)} \\ y_2^2 & \text{otherwise} \end{cases}$$

(see (4.2.1)). In particular, ψ_d sends bijectively the variables $z_{(r,\gamma)}$ to the monomials of $\mathcal{A}_{2d} = \{y_0^{b_0}y_1^{b_1}y_2^{b_2} \mid b_0 + 2b_1 + db_2 = 2d\}$ by the formula $\psi_d(z_{(r,\gamma)}) = y_0^{d(2-r)+(d-2)\gamma}y_1^{\gamma}y_2^{r-\gamma}$.

Theorem 4.2.17. *(i)* $\ker(\psi_d)$ *is a binomial ideal of* S' *minimally generated by quadrics.*

(ii) $\mathrm{I}(S_{D_{2d}}) = \tilde{\rho}(\ker(\psi_d))$ *and a minimal set of generators of* $\mathrm{I}(S_{D_{2d}})$ *are the following binomials and trinomials:*

$$\{w_{(r_1,\gamma_1)}w_{(r_2,\gamma_2)} - w_{(r_3,\gamma_3)}w_{(r_4,\gamma_4)} \mid (r_i,\gamma_i) \neq (2,0),\ r_1 + r_2 = r_3 + r_4,\ \gamma_1 + \gamma_2 = \gamma_3 + \gamma_4\}$$

$$\{(w_{(2,0)} + 2w_{(d,d)})w_{(\gamma_1,\gamma_1)} - w_{(r_2,\gamma_2)}w_{(r_3,\gamma_3)} \mid (r_i,\gamma_i) \neq (2,0),\ \gamma_1 + 2 = r_2 + r_3,\ \gamma_1 = \gamma_2 + \gamma_3\}.$$

Proof. (i) $\ker(\psi_d)$ is generated by the set of binomials:

$$\left\{\prod_{i=1}^{l} z_{(r_{j_i},\gamma_{j_i})} - \prod_{i=1}^{l} z_{(r_{m_i},\gamma_{m_i})} \mid \prod_{i=1}^{l} \psi_d(z_{(r_{j_i},\gamma_{j_i})}) = \prod_{i=1}^{l} \psi_d(z_{(r_{m_i},\gamma_{m_i})}),\ l \geq 2\right\}.$$

From this and Corollary 4.2.15, it follows that $\ker(\psi_d)$ is minimally generated by binomials of degree 2. Using the formula $\psi_d(z_{(r,\gamma)}) = y_0^{d(2-r)+(d-2)\gamma} y_1^{\gamma} y_2^{r-\gamma}$, we obtain that these binomials are:

$$\{z_{(r_1,\gamma_1)} z_{(r_2,\gamma_2)} - z_{(r_3,\gamma_3)} z_{(r_4,\gamma_4)} \mid r_1+r_2 = r_3+r_4,\ \gamma_1+\gamma_2 = \gamma_3+\gamma_4\}. \quad (4.2.3)$$

(ii) Since $\tilde{\rho}$ and ρ are isomorphisms of $\mathbb{K}-$algebras, the commutative diagram (4.2.1) gives $\mathrm{I}(S_{D_{2d}}) = \tilde{\rho}(\ker(\psi_d))$. Applying $\tilde{\rho}$ to (4.2.3), we obtain the description of the minimal set of generators in (ii). □

We end this subsection showing a couple of examples.

Example 4.2.18. (i) Take $d = 3$ and $D_{2\cdot 3} = \langle M_{3;0,1,2}, \sigma\rangle \subset \mathrm{SL}(3, \mathbb{K})$. The ideal $\mathrm{I}(S_{D_{2\cdot 3}})$ of the $GT-$surface $S_{2\cdot 3}$ with group $D_{2\cdot 3}$ is minimally generated by the following 6 binomials and 3 trinomials of degree 2:

$$\begin{array}{lcl@{\qquad}lcl}
w_{(0,0)}w_{(2,1)} & - & w_{(1,0)}w_{(1,1)} & w_{(1,0)}w_{(3,3)} & - & w_{(2,1)}w_{(2,2)} \\
w_{(0,0)}w_{(2,2)} & - & w_{(1,1)}^2 & w_{(1,0)}w_{(2,2)} & - & w_{(1,1)}w_{(2,1)} \\
w_{(0,0)}w_{(3,3)} & - & w_{(1,1)}w_{(2,2)} & w_{(1,1)}w_{(3,3)} & - & w_{(2,2)}^2
\end{array}$$

$$\begin{array}{lclcl}
w_{(1,0)}^2 & - & w_{(0,0)}w_{(2,0)} & - & 2w_{(0,0)}w_{(3,3)} \\
w_{(1,0)}w_{(2,1)} & - & w_{(1,1)}w_{(2,0)} & - & 2w_{(2,2)}^2 \\
w_{(2,0)}^2 & - & w_{(2,0)}w_{(2,2)} & - & 2w_{(2,2)}w_{(3,3)}.
\end{array}$$

(ii) Take $d = 4$ and $D_{2\cdot 4} = \langle M_{4;0,1,3}, \sigma\rangle \subset \mathrm{SL}(3, \mathbb{K})$. The ideal $\mathrm{I}(S_{D_{2\cdot 4}})$ of the $GT-$surface $S_{2\cdot 4}$ with group $D_{2\cdot 4}$ is minimally generated by the following 15 binomials and 5 trinomials of degree 2:

$$\begin{array}{lcl@{\qquad}lcl}
w_{(0,0)}w_{(2,2)} & - & w_{(1,1)}^2 & w_{(1,0)}w_{(3,3)} & - & w_{(1,1)}w_{(3,2)} \\
w_{(0,0)}w_{(3,3)} & - & w_{(1,1)}w_{(2,2)} & w_{(1,0)}w_{(3,3)} & - & w_{(2,1)}w_{(2,2)} \\
w_{(0,0)}w_{(3,2)} & - & w_{(1,0)}w_{(2,2)} & w_{(1,0)}w_{(4,4)} & - & w_{(2,1)}w_{(3,3)} \\
w_{(0,0)}w_{(2,1)} & - & w_{(1,0)}w_{(1,1)} & w_{(1,0)}w_{(4,4)} & - & w_{(2,2)}w_{(3,2)} \\
w_{(0,0)}w_{(4,4)} & - & w_{(1,1)}w_{(3,3)} & w_{(1,1)}w_{(4,4)} & - & w_{(2,2)}w_{(3,3)} \\
w_{(0,0)}w_{(4,4)} & - & w_{(2,2)}^2 & w_{(2,1)}w_{(4,4)} & - & w_{(3,2)}w_{(3,3)} \\
w_{(1,0)}w_{(2,2)} & - & w_{(1,1)}w_{(2,1)} & w_{(2,2)}w_{(4,4)} & - & w_{(3,3)}^2 \\
w_{(1,0)}w_{(3,2)} & - & w_{(2,1)}^2 & & &
\end{array}$$

$$\begin{array}{lllll}
w_{(1,0)}^2 & - & w_{(0,0)}w_{(2,0)} & - & 2w_{(0,0)}w_{(4,4)} \\
w_{(1,0)}w_{(2,1)} & - & w_{(1,1)}w_{(2,0)} & - & 2w_{(1,1)}w_{(4,4)} \\
w_{(1,0)}w_{(3,2)} & - & w_{(2,0)}w_{(2,2)} & - & 2w_{(2,2)}w_{(4,4)} \\
w_{(2,1)}w_{(3,2)} & - & w_{(2,0)}w_{(3,3)} & - & 2w_{(3,3)}w_{(4,4)} \\
w_{(3,2)}^2 & - & w_{(2,0)}w_{(4,4)} & - & 2w_{(4,4)}^2.
\end{array}$$

Chapter 5

Normal bundle of RL-varieties

Our purpose in this chapter is to study the normal bundle of a family of smooth rational monomial projections $\mathcal{X}_d \subset \mathbb{P}^{N_d}$ of Veronese varieties $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ naturally related to $\overline{G}-$varieties $X_d \subset \mathbb{P}^{\mu_d-1}$ with a finite abelian group $G \subset \mathrm{GL}(n+1,\mathbb{K})$ and whose coordinate rings $A(X_d)$ are a level rings with $\mathrm{reg}(A(X_d)) = n+1$. We take the embedding $f_d : \mathbb{P}^n \longrightarrow \mathbb{P}^{N_d}$ defined by $\mathcal{M}_{n,d} \backslash \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1$ where $H_{\mathcal{A}} \subset \mathbb{Z}^{n+1}_{\geq 0}$ is the affine semigroup associated X_d and $N_d = N_{n,d} - |\,\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1| - 1$. We define the smooth rational variety $\mathcal{X}_d \subset \mathbb{P}^{N_d}$ as to the image of f_d and we call it the $RL-$*variety* associated to X_d. The name $RL-$variety is conceived to stress the link with the notions of the **r**elative interior and **l**evelness. We take advantage of the action of the group $\overline{G} \subset \mathrm{GL}(n+1,\mathbb{K})$ to compute the cohomology of the normal bundle $\mathcal{N}_{\mathcal{X}_d}$ of any $RL-$variety $\mathcal{X}_d$ (Theorem 5.2.6).

This chapter is structured as follows. In Section 5.1, we define level $\overline{G}-$varieties X_d with an enough general group $G \subset \mathrm{GL}(n+1,\mathbb{K})$ and the $RL-$variety $\mathcal{X}_d$ associated to them (Definition 5.1.7). The $RL-$variety $\mathcal{X}_d$ is the image of the morphism $f_d : \mathbb{P}^n \longrightarrow \mathbb{P}^{N_d}$ defined by $\mathcal{M}_{n,d} \setminus \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1$. We show that the ideal J_d generated by $\mathcal{M}_{n,d} \setminus \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1$ is a monomial artinian ideal having the WLP (Proposition 5.1.10) and that $\mathcal{X}_d$ is a non aCM monomial projection of the Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$. We give examples of $RL-$varieties $\mathcal{X}_d$ in any dimension $n \geq 2$. We prove that $\mathcal{X}_d$ is a smooth rational variety and that the morphism $f_d : \mathbb{P}^n \longrightarrow \mathbb{P}^{N_d}$ is an embedding (Proposition 5.1.11). Section 5.2 contains the main result of this chapter. We introduce the normal bundle $\mathcal{N}_{\mathcal{X}_d}$ of an $RL-$variety $\mathcal{X}_d$ and we present it as the cokernel of the differential map df_d of the embedding f_d (Proposition 5.2.1). The rest of the section is devoted to compute the cohomology of $\mathcal{N}_{\mathcal{X}_d}$.

5.1 RL-varieties: a new family of smooth rational monomial projections of Veronese varieties

In Chapter 3, we have proved that the canonical module ω_{X_d} of a $\overline{G}-$variety X_d with an abelian group

$$G := \langle M_{d_1;\alpha^1_{\sigma_1(0)},\dots,\alpha^1_{\sigma_1(n)}}, \dots, M_{d_s;\alpha^s_{\sigma_s(0)},\dots,\alpha^s_{\sigma_s(n)}} \rangle \subset \mathrm{GL}(n+1,\mathbb{K})$$

of order $d = d_1 \cdots d_s$ is identified with the ideal

$$\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}})) = (x_0^{a_0} \cdots x_n^{a_n} \in R^{\overline{G}} \mid 0 \neq a_0 \cdots a_n) \subset R^{\overline{G}}$$

(Theorem 3.3.1) and $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))$ is generated by monomials of degree d and $2d$ (Theorem 3.3.3). We have characterized the Castelnuovo–Mumford regularity $\mathrm{reg}(A(X_d))$ in terms of the generators of $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))$:

$$n \leq \mathrm{reg}(A(X_d)) \leq n+1$$

and $\mathrm{reg}(A(X_d)) = n+1$ if and only $\emptyset \neq \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1$ (Theorem 3.3.5). We have constructed families of examples of $GT-$varieties X_d with a finite cyclic group such that $A(X_d)$ is a level ring and $\mathrm{reg}(A(X_d)) = n+1$ (Proposition 3.3.7 and Corollary 3.3.9) This motivates the following definition.

Definition 5.1.1. Let X_d be a $\overline{G}-$variety with group $G \subset \mathrm{GL}(n+1,\mathbb{K})$. We say that X_d is a *level* $\overline{G}-$*variety* if $A(X_d)$ is a level ring and, in addition, $\mathrm{reg}(A(X_d)) = n+1$.

Equivalently, a $\overline{G}-$variety X_d with group $G \subset \mathrm{GL}(n+1,\mathbb{K})$ is a level $\overline{G}-$variety if and only if $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))$ is minimally generated by monomials of degree d.

Proposition 5.1.2. *Any arithmetically Gorenstein $\overline{G}-$variety with group $G \subset \mathrm{GL}(n+1,\mathbb{K})$ is a level $\overline{G}-$variety.*

Proof. Let X_d be a $\overline{G}-$variety with group G. We prove that if $\mathrm{reg}(A(X_d)) = n$, then $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_2$ contains at least two different monomials. Therefore, if $A(X_d)$ is a Gorenstein ring, $\mathrm{reg}(A(X_d)) = n+1$ and the result follows. Let $m = x_0^{a_0} \cdots x_n^{a_n} \in \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_2$. We construct a monomial

$m' \in \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_2$ from m such that $m \neq m'$. To do it, we distinguish the following cases.

Case 1: $a_0 \leq d - a_0$ and $a_0 - 1 - n \geq -1$. We define

$$\overline{m} = x_0^{a_0-1} x_1^{d-1} \cdots x_n^{d-1} \text{ and } m_1 = \frac{x_0^d \cdots x_n^d}{\overline{m}} = x_0^{d-a_0+1} x_1 \cdots x_n.$$

We have $\deg(\overline{m}) = nd + a_0 - 1 - n$. Since $a_0 - 1 - n \geq -1$, there are integers $0 \leq c_0 \leq a_0 - 1$, $0 \leq c_1, \dots, c_n \leq d-1$ such that $m_2 = x_0^{c_0} \cdots x_n^{c_n}$ is a monomial of degree $d + a_0 - 1 - n$ and $\frac{\overline{m}}{m_2} \in R^{\overline{G}}$. Moreover, set $(f_0, \dots, f_n) := (c_0 + d - a_0 + 1, c_1 + 1, \dots, c_n + 1)$. Therefore

$$m' = x_0^{f_0} \cdots x_n^{f_n} = m_1 m_2 \in \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_2$$

and it verifies $f_0 > d - a_0 + 1 > a_0$.

Case 2: $a_0 \leq d - a_0$ and $a_0 - n - 1 < -1$. Let k be the minimum of the indexes in $\{0, \dots, n\}$ such that $a_0 + \cdots + a_k - n - 1 \geq -1$. Since $1 \leq a_0, \dots, a_n$, we have $n - 1 \leq a_0 + \cdots + a_{n-2}$, hence $k \leq n-2$. We define monomials:

$$\begin{array}{rcl} \overline{m} & = & x_0^{a_0-1} x_1^{a_1-1} \cdots x_k^{a_k-1} x_{k+1}^{d-1} \cdots x_n^{d-1} \\ m_1 & = & \frac{x_0^d \cdots x_n^d}{\overline{m}} = x_0^{d-a_0+1} \cdots x_k^{d-a_k+1} x_{k+1} \cdots x_n. \end{array}$$

Notice that $\deg(\overline{m}) = (n-k)d + a_0 + \cdots + a_k - n - 1$. Since $n - k$ is at least 2, there are integers $0 \leq c_0 \leq a_0 - 1, \dots, 0 \leq c_k \leq a_k - 1$ and $0 \leq c_{k+1}, \dots, c_n \leq d-1$ such that $m_2 = x_0^{c_0} \cdots x_n^{c_n}$ is a monomial of degree $d + a_0 + \cdots + a_k - 1 - n$. As in Case 1 we set

$$(f_0, \dots, f_n) := (c_0 + d - a_0 + 1, c_1 + d - a_1, \dots, c_k + d - a_k, c_{k+1} + 1 \dots, c_n + 1).$$

Therefore, $m' = x_0^{f_0} \cdots x_n^{f_n} = m_1 m_2 \in \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))$ is a monomial of $2d$ and it verifies $f_0 > d - a_0 + 1 > a_0$.

Case 3 and 4. The case $a_0 \geq d - a_0$ and $d - a_0 - n - 1 \geq -1$ follows as in Case 1 changing the roles of a_0 and $d - a_0$. Analogously, the remaining case $a_0 \geq d - a_0$ and $d - a_0 - n - 1 < -1$ follows as Case 2. $\square$

Let $2 \leq n < d$ be integers and $\Gamma = \langle M_{d;\alpha_0,\dots,\alpha_n} \rangle \subset \mathrm{GL}(n+1, \mathbb{K})$ a cyclic group of order d. If $1 \leq k$ is an integer such that $\mathrm{GCD}(\alpha_0, \dots, \alpha_n, kd) = 1$, we denote $\Gamma_k = \langle M_{kd;\alpha_0,\dots,\alpha_n} \rangle \subset \mathrm{GL}(n+1, \mathbb{K})$. More general, let $1 \leq$

$k_1, \ldots, k_s$ be integers and $G = \Gamma_1 \oplus \cdots \oplus \Gamma_s \subset \mathrm{GL}(n+1, \mathbb{K})$ a finite abelian group of order $d = d_1 \cdots d_s$. If each $(\Gamma_i)_{k_i}$ is a cyclic group of order $k_i d_i$, we denote by $G_k = (\Gamma_1)_{k_1} \oplus \cdots \oplus (\Gamma_s)_{k_s} \subset \mathrm{GL}(n+1, \mathbb{K})$ the abelian group of order kd where $k = k_1 \cdots k_s$. With this notation, Proposition 5.1.2 provides a direct generalization of Proposition 3.3.8:

Corollary 5.1.3. *If X_d is an arithmetically Gorenstein $\overline{G}-$variety with group $G \subset \mathrm{GL}(n+1, \mathbb{K})$, then X_{kd} is a level $\overline{G}-$variety with group $G_k \subset \mathrm{GL}(n+1, \mathbb{K})$.* □

Let us see examples of level and non level $\overline{G}-$varieties.

Example 5.1.4. (i) All $\overline{G}-$surfaces with group G of type (B) or (C) are level $\overline{G}-$varieties (Subsection 3.3.1).

(ii) Let $n \geq 2$ and $k \geq 1$ be integers with n even. For cyclic group $G = \langle M_{k(n+1);0,1,2,\ldots,n} \rangle \subset \mathrm{GL}(n+1, \mathbb{K})$ of order $k(n+1)$, the associated $GT-$variety $X_{k(n+1)}$ with group G is a level $GT-$variety (Corollary 3.3.9).

(iii) Let $n \geq 2, k \geq 1$ and $0 \leq i, j \leq n$ be integers such that $i + j = n$ and $G_k = \langle M_{k(n+1);0,\overset{i}{\ldots},0,1,\overset{j}{\ldots},1,n+1-j} \rangle \subset \mathrm{GL}(n+1, \mathbb{K})$ a cyclic group of order $k(n+1)$. Then, X_{n+1} is a Gorenstein $\overline{G}-$variety with group G_1 and for any $k > 1$, $X_{k(n+1)}$ is a level $\overline{G}-$variety with group G_k. Using that the monomial $m = x_0 \cdots x_n \in R^{\overline{G}_1}$, we obtain that $R^{\overline{G}_1}$ is a Gorenstein ring. The assertion now follows from Corollary 5.1.3.

(iv) Take $G_1 = \langle M_{4;0,1,1,2} \rangle$ and $G_2 = \langle M_{4;0,0,1,3} \rangle \subset \mathrm{GL}(4, \mathbb{K})$ cyclic groups of order 4. Any $\overline{G}-$threefold X_4 with group G_i, $i = 1, 2$, is an arithmetically Gorenstein $\overline{G}-$threefold. For any integer $k \geq 1$, X_{4k} is a level $GT-$variety with group $(G_i)_k$.

(v) Take $G_1 = \langle M_{4;0,1,2,3} \rangle$ and $G_2 = \langle M_{5;0,1,2,3} \rangle \subset \mathrm{GL}(4, \mathbb{K})$ cyclic groups of order 4 and 5, respectively. The associated $GT-$threefolds X_4 and X_5 with group G_1 and G_2, respectively, are examples of non level $\overline{G}-$threefolds. Indeed, we have $\mathrm{reg}(A(X_4)) = \mathrm{reg}(A(X_5)) = 3$.

Proposition 5.1.5. *Let $2 \leq n < d$ be integers and $G = \langle M_{d;\alpha_0,\ldots,\alpha_n} \rangle \subset \mathrm{GL}(n+1, \mathbb{K})$ a cyclic group of order d and $\alpha_i \neq \alpha_j$ for some $i, j \in \{0, \ldots, n\}$. If $\mathrm{reg}(A(X_d)) = n + 1$, then there are at least three indexes two by two distinct.*

Proof. By contradiction, we assume $(\alpha_0, \ldots, \alpha_n) = (0, \overset{l+1}{\ldots}, 0, a, \overset{n-l}{\ldots}, a)$ with $0 < a < d$ such that $\mathrm{GCD}(a, d) = 1$. Therefore, for any monomial $m \in R^G$ of degree d it holds that $\mathrm{supp}(m) \in \{x_0, \ldots, x_l\}$ or $\mathrm{supp}(m) \in \{x_{l+1}, \ldots, x_n\}$. Hence $\emptyset \neq \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1$. By Theorem 3.3.5 we obtain $\mathrm{reg}(A(X_d)) = n$ and we arrive to a contradiction. $\square$

In particular, for any integers $2 \leq n < d$, the cyclic groups

$$\langle M_{d;0,1,\ldots,1}\rangle, \langle M_{d;0,0,1,\ldots,1}\rangle, \ldots, \langle M_{0,0,\ldots,0,1}\rangle \subset \mathrm{GL}(n+1, \mathbb{K})$$

give rise to non level $\overline{G}-$varieties.

Any element of a diagonal abelian group $G \subset \mathrm{GL}(n+1, \mathbb{K})$ of order d is of the form $\mathrm{diag}(e^{\lambda_0}, \ldots, e^{\lambda_n})$, where e is a d'th primitive root of $1 \in \mathbb{K}$ for some integer d' dividing d. Keeping this notation, we introduce the following definition and, after, we present $RL-$varieties.

Definition 5.1.6. Let $2 \leq n < d$ be integers and $G \subset \mathrm{GL}(n+1, \mathbb{K})$ an abelian group of order d. We say that G is *enough general* if it contains at least one diagonal matrix $\mathrm{diag}(e^{\lambda_0}, \ldots, e^{\lambda_n})$ such that there are at least three exponents $\lambda_i, \lambda_j, \lambda_k$ two by two distinct.

Definition 5.1.7. Let X_d be a level $\overline{G}-$variety with an enough general group $G \subset \mathrm{GL}(n+1, \mathbb{K})$. We call an *$RL-$variety associated to X_d* to any monomial projection of the Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ parameterized by $\mathcal{M}_{n,d} \setminus \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1$.

Let us see some examples.

Example 5.1.8. (i) Take $G = \langle M_{5;0,1,2}\rangle \subset \mathrm{GL}(3, \mathbb{K})$ a cyclic group of order 5. A minimal set of fundamental invariants of $\overline{G}$ is

$$\mathcal{B}_1 = \{x_0^5, x_1^5, x_0x_1^3x_2, x_0^2x_1x_2^2, x_2^5\}.$$

The ideal $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}})) = (x_0x_1^3x_2, x_0^2x_1x_2^2)$. The $GT-$surface X_5 with group G has $\mathrm{reg}(A(X_5)) = 3$. So, X_5 is a level $GT-$surface with an enough general group G. The $RL-$surface $\mathcal{X}_5 \subset \mathbb{P}^{18}$ associated to X_5 is the double monomial projection of the Veronese surface $X_{2,5} \subset \mathbb{P}^{20}$ parameterized by

$$\begin{aligned}\mathcal{M}_{2,5} \setminus \mathrm{I}(\mathrm{relint}(H_A))_1 = \{&x_2^5, x_1x_2^4, x_1^2x_2^3, x_1^3x_2^2, x_1^4x_2, x_1^5, x_0x_2^4, x_0x_1x_2^3, x_0x_1^2x_2^2,\\ &x_0x_1^4, x_0^2x_2^3, x_0^2x_1^2x_2, x_0^2x_1^3, x_0^3x_2^2, x_0^3x_1x_2, x_0^3x_1^2, x_0^4x_2,\\ &x_0^4x_1, x_0^5\}.\end{aligned}$$

(ii) Take $G = \langle M_{3;0,1,1}, M_{3;0,1,2}\rangle \subset \mathrm{GL}(3,\mathbb{K})$ a cyclic group of order 9. A minimal set of fundamental invariants of $\overline{G}$ is

$$\mathcal{B}_1 = \{x_2^9, x_1^3x_2^6, x_1^6x_2^3, x_1^9, x_0^3x_2^6, x_0^3x_1^3x_2^3, x_0^3x_1^6, x_0^6x_2^3, x_0^6x_1^3, x_0^9\}.$$

The ideal $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}})) = (x_0^3x_1^3x_2^3)$ and the $GT-$surface X_9 with group G is an arithmetically Gorenstein surface with $\mathrm{reg}(A(X_9)) = 3$. So X_9 is a level $GT-$surface with an enough general group G. The $RL-$surface $\mathcal{X}_9 \subset \mathbb{P}^{53}$ associated to X_9 is the simple monomial projection of the Veronese surface $X_{2,9} \subset \mathbb{P}^{54}$ parameterized by

$$\begin{aligned}\mathcal{M}_{2,9}\backslash \mathrm{I}(\mathrm{relint}(H_A))_1 = \{&x_2^9, x_1x_2^8, x_1^2x_2^7, x_1^3x_2^6, x_1^4x_2^5, x_1^5x_2^4, x_1^6x_2^3, x_1^7x_2^2, x_1^8x_2,\\ &x_1^9, x_0x_2^8, x_0x_1x_2^7, x_0x_1^2x_2^6, x_0x_1^3x_2^5, x_0x_1^4x_2^4, x_0x_1^5x_2^3,\\ &x_0x_1^6x_2^2, x_0x_1^7x_2, x_0x_1^8, x_0^2x_2^7, x_0^2x_1x_2^6, x_0^2x_1^2x_2^5, x_0^2x_1^3x_2^4,\\ &x_0^2x_1^4x_2^3, x_0^2x_1^5x_2^2, x_0^2x_1^6x_2, x_0^2x_1^7, x_0^3x_2^6, x_0^3x_1x_2^5, x_0^3x_1^2x_2^4,\\ &x_0^3x_1^4x_2^2, x_0^3x_1^5x_2, x_0^3x_1^6, x_0^4x_2^5, x_0^4x_1x_2^4, x_0^4x_1^2x_2^3, x_0^4x_1^3x_2^2,\\ &x_0^4x_1^4x_2, x_0^4x_1^5, x_0^5x_2^4, x_0^5x_1x_2^3, x_0^5x_1^2x_2^2, x_0^5x_1^3x_2, x_0^5x_1^4,\\ &x_0^6x_2^3, x_0^6x_1x_2^2, x_0^6x_1^2x_2, x_0^6x_1^3, x_0^7x_2^2, x_0^7x_1x_2, x_0^7x_1^2, x_0^8x_2,\\ &x_0^8x_1, x_0^9\}.\end{aligned}$$

(iii) Take $G = \langle M_{4;0,1,1,2}\rangle \subset \mathrm{GL}(4,\mathbb{K})$ a cyclic group of order 4. A minimal set of fundamental invariants of $\overline{G}$ is

$$\mathcal{B}_1 = \{x_0^4, x_2^4, x_1x_2^3, x_1^2x_2^2, x_1^3x_2, x_1^4, x_3x_0x_2^2, x_3x_0x_1x_2, x_3x_0x_1^2, x_3^2x_0^2, x_3^4\}.$$

The ideal $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}})) = (x_0x_1x_2x_3)$ and the $GT-$threefold X_4 with group G is an arithmetically Gorenstein threefold with $\mathrm{reg}(A(X_4)) = 4$. So X_4 is a level $GT-$threefold with an enough general group G. The $RL-$threefold $\mathcal{X}_4 \subset \mathbb{P}^{33}$ associated to X_4 is the simple monomial projection of the Veronese threefold $X_{3,4} \subset \mathbb{P}^{34}$ parameterized by

$$\begin{aligned}\mathcal{M}_{3,4}\backslash \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1 = \{&x_3^4, x_3^3x_2, x_3^2x_2^2, x_3x_2^3, x_2^4, x_3^3x_1, x_3^2x_1x_2, x_3x_1x_2^2, x_1x_2^3,\\ &x_3^2x_1^2, x_3x_1^2x_2, x_1^2x_2^2, x_3x_1^3, x_1^3x_2, x_1^4, x_3^3x_0, x_3^2x_0x_2,\\ &x_3x_0x_2^2, x_0x_2^3, x_3^2x_0x_1, x_0x_1x_2^2, x_3x_0x_1^2, x_0x_1^2x_2, x_0x_1^3,\\ &x_3^2x_0^2, x_3x_0^2x_2, x_0^2x_2^2, x_3x_0^2x_1, x_0^2x_1x_2, x_0^2x_1^2, x_3x_0^3, x_0^3x_2,\\ &x_0^3x_1, x_0^4\}.\end{aligned}$$

From now onwards, we fix a level $\overline{G}-$variety X_d with an enough general abelian group $G \subset \mathrm{GL}(n+1,\mathbb{K})$ of order d. We set

$$\eta_d = |\,\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1| \text{ and } N_d := N_{n,d} - \eta_d - 1.$$

The associated $RL-$variety $\mathcal{X}_d \subset \mathbb{P}^{N_d}$ of dimension $n \geq 2$ is the image of the morphism

$$f_d : \mathbb{P}^n \longrightarrow \mathbb{P}^{N_d}$$

defined by $J_d := \langle \mathcal{M}_{n,d} \setminus \mathrm{I}(\mathrm{relint}(H(\mathcal{A})))_1 \rangle$.

Remark 5.1.9. J_d contains all monomials $x_0^{a_0} \cdots x_n^{a_n} \in R$ of degree d such that $a_0 \cdots a_n = 0$ and all monomials of R which are not invariants of $\overline{G}$, i.e. $\mathcal{M}_{n,d} \setminus \mathcal{B}_1 \subset J_d$. In particular, J_d is a monomial artinian ideal.

We ask how does the ideal J_d behave with respect to the WLP (Definition 1.4.1). For instance, $J_d \subset R$ is not a Togliatti system (Definition 1.4.7), since it is generated by $N_d \geq N_{n-1,d} + 1$ forms of degree d:

$$\{x_i^{d-1} x_j \mid 0 \leq i, j \leq n\} \cup \{x_1^{a_1} \cdots x_n^{a_n} \mid a_1 + \cdots + a_n = d\} \subset J_d.$$

Actually, we have:

Proposition 5.1.10. *J_d has the WLP.*

Proof. We first prove that $(R/J_d)_{d+1} = 0$. It suffices to see that if $x_0^{a_0} \cdots x_n^{a_n} \in \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1$, then $x_0^{a_0} \cdots x_j^{a_j+1} \cdots x_n^{a_n} \in J_d$ for any $0 \leq j \leq n$. Since G is enough general, it contains a diagonal matrix $M = \mathrm{diag}(e^{\lambda_0}, \ldots, e^{\lambda_n})$ with at least three $\lambda_a, \lambda_b, \lambda_c$ different pairwise. We may assume that M generates a cyclic group of order $d'|d$. For any $0 \leq j \leq n$, $\alpha_j \neq \alpha_k$ for some $k \in \{a, b, c\}$. Without loss of generality, we set $(a, b, c) = (0, 1, 2)$ and $j = 1$, the remaining cases follow analogously. Let $m = x_0^{a_0} x_1^{a_1+1} x_2^{a_2} \cdots x_n^{a_n} \in R_{d+1}$ such that $m/x_1 \in \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1$. In particular, $0 < a_0, \ldots, a_n$, $m/x_1 \in R_1^{\overline{G}}$ and $(a_0, \ldots, a_n)$ verifies the linear congruence $\lambda_0 a_0 + \cdots + \lambda_n a_n \equiv 0 \bmod d'$. Consider $m' = m/x_0 = x_0^{a_0-1} x_1^{a_1+1} x_2^{a_2} \cdots x_n^{a_n}$. We have $\deg(m') = d$ and

$$\lambda_0(a_0 - 1) + \lambda_1(a_1 + 1) + \lambda_2 a_2 + \cdots + \lambda_n a_n = \lambda_0 a_0 + \cdots + \lambda_n a_n + \lambda_1 - \lambda_0.$$

Since $\lambda_0, \lambda_1 < d'$, $0 \neq \lambda_1 - \lambda_0$ is not a multiple of d'. Therefore, $m' \notin R^{\overline{G}}$. Moreover, m is divisible by m' and then by Remark 5.1.9(ii) we obtain that $m \in J_d$. Since all generators of J_d has degree d and $(R/J_d)_{d+1} = 0$, to prove that J_d has the WLP it is enough to show that the multiplication map

$$\times(x_0) : R_{d-1} \longrightarrow (R/J_d)_d = \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1$$

is surjective. Let $x_0^{a_0} \cdots x_n^{a_n} \in \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1$. Hence $x_0^{a_0-1} x_1^{a_1} \cdots x_n^{a_n} \in R_{d-1}$ and $\times(x_0)(x_0^{a_0-1} x_1^{a_1} \cdots x_n^{a_n}) = x_0^{a_0} \cdots x_n^{a_n}$. $\square$

The $RL-$variety $\mathcal{X}_d \subset \mathbb{P}^{N_d}$ of dimension $n \geq 2$ is a non aCM monomial projection of the Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ (Proposition 2.1.7). The coordinate ring of $\mathcal{X}_d$ is isomorphic to the non CM semigroup ring $\mathbb{K}[\mathcal{M}_{n,d} \setminus \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1]$. $\mathcal{M}_{n,d} \setminus \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1$ is a non normal semigroup (Theorem 1.2.14) and $\mathbb{K}[\mathcal{M}_{n,d} \setminus \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1]$ is not the ring of invariants of any finite group acting on R (Theorem 1.3.10). Geometrically, we have the following.

Proposition 5.1.11. *$\mathcal{X}_d$ is a smooth rational variety and f_d is an embedding.*

Proof. $\mathcal{X}_d$ is a toric variety parametrized by all monomials of degree d in $\mathcal{M}_{n,d} \setminus \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1$. Since

$$\{x_0^{a_0} \cdots x_n^{a_n} \in R_d \mid 0 = a_0 \cdots a_n\} \subset \mathcal{M}_{n,d} \setminus \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1,$$

$\mathcal{X}_d$ satisfies the smoothness criterion for toric varieties [34, Chapter 5 - Corollary 3.2]. In particular, $\mathcal{M}_{n,d} \setminus \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1$ contains all monomials $x_i^{d-1}x_j$ for all $i, j \in \{0, \ldots, n\}$, which is a sufficient condition to f_d be an embedding. $\square$

Example 5.1.12. Take $G = \langle M_{3;0,1,2} \rangle \subset \mathrm{GL}(3, \mathbb{K})$ a cyclic group of order 3. A minimal set of fundamental invariants of $\overline{G}$ is $\mathcal{B}_1 = \{x_0^3, x_1^3, x_2^3, x_0x_1x_2\}$. $X_3 \subset \mathbb{P}^3$ is a cubic surface and the associated $RL-$surface $\mathcal{X}_3 \subset \mathbb{P}^8$ is the smooth rational simple monomial projection of the Veronese surface $X_{2,3} \subset \mathbb{P}^9$ parameterized by $\mathcal{M}_{2,3} \setminus \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1 = \{x_0^3, x_0^2x_1, x_0^2x_2, x_0x_1^2, x_0x_2^2, x_1^3, x_1^2x_2, x_1x_2^2, x_2^3\}$. The multiplication map $\times(x_0) : (R/J_3)_i \longrightarrow (R/J_3)_{i+1}$ is injective for $i = 0, 1$ and it is surjective for $i \geq 2$, i.e. J_3 has the WLP. The morphism $f_3 : \mathbb{P}^2 \longrightarrow \mathbb{P}^8$ defined by $\mathcal{M}_{2,3} \setminus \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1$ is an embedding of $\mathbb{P}^2$. We set $S = \mathbb{K}[w_1, \ldots, w_9]$. The ideal $\mathrm{I}(\mathcal{X}_3) \subset S$ has $\mathrm{codim}(\mathrm{I}(\mathcal{X}_3)) = 6 < 8 = \mathrm{pdim}(\mathrm{I}(\mathcal{X}_3))$, i.e. $\mathcal{X}_3$ is a non aCM surface. Indeed, a minimal graded free $S-$resolution of $S/\mathrm{I}(\mathcal{X}_3)$ looks like:

$$0 \longrightarrow S(-10) \longrightarrow S(-9)^9 \longrightarrow S(-8)^{37} \longrightarrow S(-7)^{83} \longrightarrow$$

$$\longrightarrow S(-6)^{100} \oplus S(-5)^8 \longrightarrow S(-5)^{55} \oplus S(-4)^{36} \longrightarrow S(-4)^{10} \oplus S(-3)^{43} \longrightarrow$$

$$\longrightarrow S(-2)^{17} \longrightarrow S \longrightarrow S/\mathrm{I}(\mathcal{X}_3) \longrightarrow 0.$$

5.2 Normal bundle of RL-varieties

Keeping the notation of Section 5.1, we consider an $RL-$variety $\mathcal{X}_d \subset \mathbb{P}^{N_d}$ of dimension $n \geq 2$ associated to a $\overline{G}-$variety X_d with an enough general group $G \subset \mathrm{GL}(n+1, \mathbb{K})$. Denote by $\mathcal{I}_{\mathcal{X}_d} \subset \mathcal{O}_{\mathbb{P}^{N_d}}$ the ideal sheaf of $\mathcal{X}_d \subset \mathbb{P}^{N_d}$. Since any $RL-$variety is smooth (Proposition 5.1.11), $\mathcal{I}_{\mathcal{X}_d}/\mathcal{I}^2_{\mathcal{X}_d}$ is a locally free sheaf of rank $\mathrm{codim}(\mathcal{I}_{\mathcal{X}_d}) = N_d - n$ [43, Theorem 8.17] and the *normal bundle of* $\mathcal{X}_d$ *in* $\mathbb{P}^{N_d}$ is defined as the locally free sheaf on $\mathcal{X}_d$ of rank $N_d - n$:

$$\mathcal{N}_{\mathcal{X}_d} := \mathcal{H}\mathrm{om}_{\mathcal{O}_{\mathcal{X}_d}}(\mathcal{I}_{\mathcal{X}_d}/\mathcal{I}^2_{\mathcal{X}_d}, \mathcal{O}_{\mathcal{X}_d}).$$

We have the following exact sequence of locally free sheaves on $\mathcal{X}_d$:

$$0 \longrightarrow \mathcal{T}_{\mathcal{X}_d} \longrightarrow \mathcal{T}_{\mathbb{P}^{N_d}} \otimes \mathcal{O}_{\mathcal{X}_d} \longrightarrow \mathcal{N}_{\mathcal{X}_d} \longrightarrow 0,$$

where $\mathcal{T}_{\mathcal{X}_d}$ is the *tangent bundle of* $\mathcal{X}_d$ (see [43, II §8]). Since f_d is an embedding, taking the inverse image f_d^* we obtain the exact sequence of locally free sheaves on $\mathbb{P}^n$:

$$0 \longrightarrow \mathcal{T}_{\mathbb{P}^n} \longrightarrow f_d^*(\mathcal{T}_{\mathbb{P}^{N_d}}) \longrightarrow f_d^*(\mathcal{N}_{\mathcal{X}_d}) \longrightarrow 0,$$

where the first map is given by the *differential map* df_d *of* f_d. The embedding f_d identifies the normal bundle $\mathcal{N}_{\mathcal{X}_d}$ of $\mathcal{X}_d$ in $\mathbb{P}^{N_d}$ with the inverse image $f_d^*(\mathcal{N}_{\mathcal{X}_d})$ of $\mathcal{N}_{\mathcal{X}_d}$ by f_d (see, for instance, [4] and [73]).

Proposition 5.2.1. *Let* $\mathcal{X}_d \subset \mathbb{P}^{N_d}$ *be an* $RL-$*variety of dimension* $n \geq 2$. *There is an exact sequence of locally free sheaves on* $\mathbb{P}^n$*:*

$$0 \longrightarrow \mathcal{O}^{n+1}_{\mathbb{P}^n}(1) \longrightarrow \mathcal{O}^{N_d+1}_{\mathbb{P}^n}(d) \longrightarrow f_d^*(\mathcal{N}_{\mathcal{X}_d}) \longrightarrow 0. \tag{5.2.1}$$

Proof. Consider the *Euler sequence* for $\mathbb{P}^{N_d}$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{N_d}} \longrightarrow \mathcal{O}^{n+1}_{\mathbb{P}^{N_d}}(1) \longrightarrow \mathcal{T}_{\mathbb{P}^{N_d}} \longrightarrow 0.$$

Taking the inverse image f_d^*, we obtain an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\begin{pmatrix} m_1 \\ \vdots \\ m_{N_d+1} \end{pmatrix}} \mathcal{O}^{N_d+1}_{\mathbb{P}^n}(d) \longrightarrow f_d^*(\mathcal{T}_{\mathbb{P}^{N_d}}) \longrightarrow 0.$$

Therefore, we have the following commutative diagram of exact rows and columns:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathcal{O}_{\mathbb{P}^n} \arrow[r, "\cong"] \arrow[d] & \mathcal{O}_{\mathbb{P}^n} \arrow[d] & & \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \arrow[r, "\delta"] \arrow[d] & \mathcal{O}_{\mathbb{P}^n}(d)^{N_d+1} \arrow[r] \arrow[d] & f_d^*(\mathcal{N}_{\mathcal{X}_d}) \arrow[r] \arrow[d, "\wr|"] & 0 \\
0 \arrow[r] & \mathcal{T}_{\mathbb{P}^n} \arrow[r, "df_d"] \arrow[d] & f_d^*(\mathcal{T}_{\mathbb{P}^{N_d}}) \arrow[r] \arrow[d] & f_d^*(\mathcal{N}_{\mathcal{X}_d}) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}
$$

where the first column is the Euler sequence of $\mathbb{P}^n$ and δ is given by the matrix

$$\begin{pmatrix} \partial_{x_0} m_1 & \cdots & \partial_{x_n} m_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_0} m_{N_d+1} & \cdots & \partial_{x_n} m_{N_d+1} \end{pmatrix}.$$

$\square$

Example 5.2.2. Take $G = \langle M_{3;0,1,2} \rangle \subset \mathrm{GL}(3, \mathbb{K})$ a cyclic group of order 3. As we have seen in Example 5.1.12, the $RL-$surface $\mathcal{X}_3 \subset \mathbb{P}^8$ associated to the $GT-$surface X_3 with group G is the smooth rational simple monomial projection of the Veronese surface $X_{2,3} \subset \mathbb{P}^9$ parameterized by $\mathcal{M}_{2,3} \setminus \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1 = \{x_0^3, x_0^2x_1, x_0^2x_2, x_0x_1^2, x_0x_2^2, x_1^3, x_1^2x_2, x_1x_2^2, x_2^3\}$. $f_3^*\mathcal{N}_{\mathcal{X}_3}$

is the locally free sheaf on $\mathbb{P}^2$ of rank 6 presented as the cokernel of the differential map $df_3 : \mathcal{O}^3_{\mathbb{P}^2}(1) \longrightarrow \mathcal{O}^9_{\mathbb{P}^2}(3)$ given by the matrix:

$$\begin{pmatrix} 3x_0^2 & 0 & 0 \\ 2x_0x_1 & x_0^2 & 0 \\ 2x_0x_2 & 0 & x_0^2 \\ x_1^2 & 2x_0x_1 & 0 \\ x_2^2 & 0 & 2x_0x_2 \\ 0 & 3x_1^2 & 0 \\ 0 & 2x_1x_2 & x_1^2 \\ 0 & x_2^2 & 2x_1x_2 \\ 0 & 0 & 3x_2^2 \end{pmatrix}.$$

The rest of this chapter is devoted to compute the cohomology of the normal bundle $\mathcal{N}_{\mathcal{X}_d}$ of any $RL-$variety $\mathcal{X}_d \subset \mathbb{P}^{N_d}$. After twisting (5.2.1) by $\mathcal{O}_{\mathbb{P}^n}(-k)$ with $k \in \mathbb{Z}$, the long exact sequence of cohomology for (5.2.1) appears as:

$$\longrightarrow \mathrm{H}^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^{N_d+1}(d-k)) \longrightarrow \mathrm{H}^i(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) \longrightarrow \mathrm{H}^{i+1}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^{n+1}(1-k)) \longrightarrow \tag{5.2.2}$$

As we establish next, the $\mathbb{K}-$vector spaces $\mathrm{H}^i(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k))$ can be determined directly in mostly cases from $\mathrm{H}^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^{N_d+1}(d-k))$, $\mathrm{H}^{i+1}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^{n+1}(1-k))$ and the *Bott formulas* (see [66]):

$$\mathrm{h}^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) = \begin{cases} \binom{n+k}{n} & i = 0 \text{ and } k \geq 0 \\ \binom{-k-1}{n} & i = n \text{ and } k \leq -n-1 \\ 0 & \text{otherwise.} \end{cases} \tag{5.2.3}$$

Proposition 5.2.3. *Let $\mathcal{X}_d \subset \mathbb{P}^{N_d}$ be an $RL-$variety of dimension $n \geq 2$. We have:*

(i) for all $0 < i < n-1$ and for all $k \in \mathbb{Z}$, $\mathrm{H}^i(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) = 0$.

(ii)

$$\mathrm{h}^0(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) = \begin{cases} (N_d+1)\binom{n+d-k}{n} - (n+1)\binom{n+1-k}{n} & k \leq 1 \\ (N_d+1)\binom{n+d-k}{n} & 1 < k \leq d \\ 0 & \textit{otherwise.} \end{cases}$$

(iii)

$$\mathrm{h}^{n-1}(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) = \begin{cases} (n+1)\binom{k-2}{n} & n+2 \leq k < d+n+1 \\ 0 & k \leq n+1. \end{cases}$$

(iv) For all $k < d+n+1$, $\mathrm{H}^n(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) = 0$.

Proof. (i) From (5.2.3) and the additivity of the cohomology, it follows that $\mathrm{H}^i(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) = 0$ for all $0 < i < n-1$ and $k \in \mathbb{Z}$.
(ii) From (i) we obtain for any $k \in \mathbb{Z}$:

$$0 \longrightarrow \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^{n+1}(1-k)) \longrightarrow \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^{N_d+1}(d-k)) \longrightarrow \mathrm{H}^0(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) \longrightarrow 0.$$

Using (5.2.3) and the above sequence, we get the second assertion.
(iii) and (iv) From (i) and (ii) we have for any $k \in \mathbb{Z}$:

$$0 \longrightarrow \mathrm{H}^{n-1}(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) \longrightarrow \mathrm{H}^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^{n+1}(1-k)) \longrightarrow$$

$$\longrightarrow \mathrm{H}^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^{N_d+1}(d-k)) \longrightarrow \mathrm{H}^n(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) \longrightarrow 0.$$

Applying (5.2.3), we conclude that for any $k < d+n+1$, $\mathrm{H}^n(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) = 0$ and $\mathrm{H}^{n-1}(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) \cong \mathrm{H}^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^{n+1}(1-k))$. □

Thus far, we have computed the cohomology of the normal bundle $\mathcal{N}_{\mathcal{X}_d}$ of an $RL-$variety $\mathcal{X}_d \subset \mathbb{P}^{N_d}$ with the exception of $\mathrm{H}^{n-1}(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k))$ and $\mathrm{H}^n(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k))$ with $k \geq d+n+1$. Since for any $k \in \mathbb{Z}$ we have the exact sequence

$$0 \longrightarrow \mathrm{H}^{n-1}(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) \longrightarrow \mathrm{H}^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^{n+1}(1-k)) \longrightarrow$$

$$\longrightarrow \mathrm{H}^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^{N_d+1}(d-k)) \longrightarrow \mathrm{H}^n(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) \longrightarrow 0,$$

(see (5.2.2)), to obtain $\mathrm{H}^n(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k))$, $k \geq n+d+1$, it suffices to determine $\mathrm{H}^{n-1}(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k))$. In order to do it, we need to introduce some notation and a technical lemma. Let $0 \leq i \neq j \leq n$, $l \geq 1$ and $t \geq 1$ be integers. Given a monomial $m = x_0^{a_0} \cdots x_n^{a_n} \in R_l$, we denote by ∂_m the composition of the linear operators $\partial_{x_0} \overset{a_0}{\cdots} \partial_{x_0} \cdots \partial_{x_n} \overset{a_n}{\cdots} \partial_{x_n}$.

Lemma 5.2.4. *Let $0 \leq i \neq j \leq n$ and $k \geq d+n+1$ be integers and let $m \in R_{k-d-n-1}$ and $q, q' \in R_{k-n-1}$ be monomials such that m divides both q and q'. Then $x_i\partial_m q$ and $x_j\partial_m q'$ are linearly independent if and only if $x_i\partial_{m'} q$ and $x_j\partial_{m'} q'$ are linearly independent for any monomial $m' \in R_{k-d-n-1}\setminus\{m\}$ dividing q and q'.*

Proof. We write $m' = x_0^{b_0}\cdots x_n^{b_n}$, $m = x_0^{c_0}\cdots x_n^{c_n}$, $q = x_0^{a_0}\cdots x_n^{a_n}$, $q' = x_0^{a'_0}\cdots x_n^{a'_n}$. Assume that $x_i\partial_m q$ and $x_j\partial_m q'$ are linearly independent and there is a monomial $m' \in R_{k-d-n-1} \setminus \{m\}$ dividing q and q' and such that $x_i\partial_{m'} q$ and $x_j\partial_{m'} q'$ are linearly dependent. Therefore, we have the equality $x_0^{a_0-b_0}\cdots x_i^{a_i-b_i+1}\cdots x_n^{a_n-b_n} = x_0^{a'_0-b_0}\cdots x_j^{a'_j-b_j+1}\cdots x_n^{a'_n-b_n}$. So $a_l = a'_l$, $0 \leq l \neq i,j \leq n$, $a_i = a'_i - 1$ and $a_j = a'_j + 1$. We obtain a contradiction:

$$\begin{aligned} x_i\partial_m q &= Ax_0^{a_0-c_0}\cdots x_j^{a_j-c_j}\cdots x_i^{a_i-c_i+1}\cdots x_n^{a_n-c_n} \\ x_j\partial_m q' &= Bx_0^{a_0-c_0}\cdots x_j^{a_j-1-c_j+1}\cdots x_i^{a_i+1-c_i}\cdots x_n^{a_n-c_n},\ A,B \in \mathbb{K}^*. \end{aligned}$$

□

An $RL-$variety $\mathcal{X}_d \subset \mathbb{P}^{N_d}$ is a smooth rational variety embedded in $\mathbb{P}^{N_d}$. In [4], the authors introduced a new method to compute the cohomology of the normal bundle of varieties of this kind. With the notation of [4], the embedding $f_d : \mathbb{P}(U) \longrightarrow \mathbb{P}^{N_d}$ with $U = R_1^\vee$. The $RL-$variety $\mathcal{X}_d = f_d(\mathbb{P}(U))$ is the projection in $\mathbb{P}^{N_d}$ of the Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ from the projective space $\mathbb{P}(T)$ of dimension $N_{n,d} - N_d$, where $T^\vee$ is identified with $\langle \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}})_1\rangle$. Let $0 \leq i \neq j \leq n$, $l \geq 1$ and $t \geq 1$ be integers. We denote $D_{i,j} : S^lU \otimes S^tU \longrightarrow S^{l-1}U \otimes S^{t-1}U$ the linear map $\partial_{x_i}\otimes\partial_{x_j} - \partial_{x_j}\otimes\partial_{x_i}$.

Proposition 5.2.5. *Let $\mathcal{X}_d \subset \mathbb{P}^{N_d}$ be an $RL-$variety of dimension $n \geq 2$ associated to a level $GT-$variety X_d with an enough general group $G \subset$ $\mathrm{GL}(n+1,\mathbb{K})$. Then,*

$$\mathrm{h}^{n-1}(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) = \begin{cases} \eta_d + \frac{n(d-1)}{d}\binom{n+d-1}{n} & k = d+n+1 \\ (n+1)\eta_d & k = d+n+2 \\ 0 & k \geq d+n+3, \end{cases}$$

where $\eta_d = |\,\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1|$.

Proof. By [4, Theorem 2], we have that $\mathrm{h}^{n-1}(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-d-n-1)) = \dim(\mu^{-1}(T))$, where $\mu : U \otimes S^{d-1}U \longrightarrow S^dU$ is the multiplication map, and for all $k \geq d+n+2$:

$$\mathrm{H}^{n-1}(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) = (S^{k-d-n-1}U \otimes T) \bigcap \left(\bigcap_{0 \leq i,j,r,s \leq n} (ker(D_{i,j} \circ D_{r,s}) \right).$$

In particular, for $k = d+n+1, d+n+2$ we can conclude that

$$\mathrm{h}^{n-1}(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-d-n-1)) = \eta_d + \frac{n(d-1)}{d}\binom{n+d-1}{n}$$

$$\mathrm{H}^{n-1}(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-d-n-2)) = U \otimes T.$$

Moreover, for $k \geq d+n+3$ we have the following description:

$$\begin{aligned}\mathrm{H}^{n-1}(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) \cong \{&x_0 \otimes q_0 + \cdots + x_n \otimes q_n \in R_1 \otimes R_{k-n-2} \mid \\ &x_0\partial_m(q_0) + \cdots + x_n\partial_m(q_n) \in \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}})) \text{ for all monomial } m \in R_{k-d-n-1}\}.\end{aligned}$$

We want to prove that $\mathrm{H}^{n-1}(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) = 0$ for all $k \geq d+n+3$. Assume that there exist $q_0, \ldots, q_n \in R_{k-n-2}$ and a monomial $m \in R_{k-d-n-1}$ such that $0 \neq u_m := x_0\partial_m(q_0) + \cdots + x_n\partial_m(q_n) \in \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))$. Therefore, any monomial appearing in u_m belongs to $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))$ and, hence, it is an invariant of $\overline{G}$. Let $q \in R_{k-n-2}$ be a monomial such that $0 \neq x_i\partial_m q$ is a monomial which occurs in u_m. Given that G is enough general (Definition 5.1.6), it contains a diagonal matrix of the form $M = \mathrm{diag}(e^{\lambda_0}, \ldots, e^{\lambda_n})$ with at least three $\lambda_a, \lambda_b, \lambda_c$ different pairwise. Assuming that $\langle M \rangle \subset \mathrm{GL}(n+1, \mathbb{K})$ is a cyclic group of order $0 < d'|d$, the associated point of $x_0^{f_0} \cdots x_n^{f_n} \in R^{\overline{G}}$ satisfies the linear congruence equation $\lambda_0 y_0 + \cdots + \lambda_n y_n \equiv 0 \bmod d'$. Now, from the description of $\mathrm{H}^{n-1}(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k))$ and Lemma 5.2.4, we have that if $x_0 \otimes q_0 + \cdots + x_n \otimes q_n \in \mathrm{H}^{n-1}(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k))$, then for any monomial $m' \in R_{k-d-n-1}$, $x_i\partial_{m'}q \in \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}})) \subset R_1^{\overline{G}}$. We will show that there always exists a monomial $m' \in R_{k-d-n-1}$ dividing q such that $x_i\partial_{m'}q$ is not an invariant of $\langle M \rangle$. Thus, it concludes $\mathrm{H}^{n-1}(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) = 0$ for all $k \geq d+n+3$. Furthermore, for the arguments we develop we can assume, without loss of generality, that $G = \langle M_{d;\alpha_0,\ldots,\alpha_n} \rangle \subset \mathrm{GL}(n+1, \mathbb{K})$ is an enough general cyclic group of order d with $\alpha_i, \alpha_j, \alpha_l$ different pair-wise.

Consider monomials $q = x_0^{a_0}\cdots x_n^{a_n}$ and $m = x_0^{b_0}\cdots x_n^{b_n}$ such that m divides q and $x_i\partial_m q \in \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))$. In particular, we have that $b_j < a_j$ for all $0 \le j \ne i \le n$ and $b_i \le a_i - 1$. By assumption,

$$x_i\partial_m q := x_0^{c_0}\cdots x_n^{c_n} = x_0^{a_0-b_0}\cdots x_i^{a_i-b_i+1}\cdots x_n^{a_n-b_n} \in R_1^{\overline{G}}.$$

We distinguish two cases.

<u>Case 1:</u> $0 < b_i$. If $\alpha_i = 0$ or $\alpha_i > 0$ and $2\alpha_i - \alpha_j \not\equiv 0 \bmod d$, we define $m' = x_0^{b_0}\cdots x_l^{b_j+1}\cdots x_i^{b_i-1}\cdots x_n^{b_n}$. Then $x_i\partial_{m'}q = x_0^{c_0}\cdots x_j^{c_j-1}\cdots x_i^{c_i+1}\cdots x_n^{c_n}$. Otherwise $2\alpha_i - \alpha_l \not\equiv 0 \bmod d$ and we define $m' = x_0^{b_0}\cdots x_l^{b_l+1}\cdots x_i^{b_i-1}\cdots x_n^{b_n}$. Then $x_i\partial_{m'}q = x_0^{c_0}\cdots x_l^{c_l-1}\cdots x_i^{c_i+1}\cdots x_n^{c_n}$. The point associated to $x_i\partial m'q$ does not verify the equation $\alpha_0 y_0 + \cdots + \alpha_n y_n \equiv 0 \bmod d$.

<u>Case 2:</u> $b_i = 0$. We take $0 < b_h$, and we can assume that α_h, α_j are different pair-wise. We define $m' = x_0^{b_0}\cdots x_j^{b_j+1}\cdots x_h^{b_h-1}\cdots x_n^{b_n}$. Then $x_i\partial_{m'}q = x_0^{c_0}\cdots x_j^{c_j-1}\cdots x_i^{c_i}\cdots x_h^{c_h+1}\cdots x_n^{c_n}$. The point associated $x_i\partial m'q$ does not verify the linear congruence equation $\alpha_0 y_0 + \cdots + \alpha_n y_n \equiv 0 \bmod d$. $\square$

The main result of this section is the following.

Theorem 5.2.6. *Let X_d be a level $\overline{G}-$variety with an enough general group $G \subset \mathrm{GL}(n+1,\mathbb{K})$ of order d. Set $\eta_d := |\,\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1|$ and $N_d := N_{n,d} - \eta_d - 1$. The cohomology of the normal bundle $\mathcal{N}_{\mathcal{X}_d}$ of the RL$-$variety $\mathcal{X}_d \subset \mathbb{P}^{N_d}$ of dimension $n \ge 2$ associated to X_d is given by*

(i) for $0 < i < n-1$ and for all $k \in \mathbb{Z}$, $\quad \mathrm{h}^i(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) = 0$. *(ii)*

$$\mathrm{h}^0(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) = \begin{cases} (N_d+1)\binom{n+d-k}{n} - (n+1)\binom{n+1-k}{n} & k \le 1 \\ (N_d+1)\binom{n+d-k}{n} & 1 < k \le d \\ 0 & \textit{otherwise.} \end{cases}$$

(iii)

$$\mathrm{h}^{n-1}(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) = \begin{cases} (n+1)\binom{k-2}{n} & n+2 \le k < d+n+1 \\ \eta_d + \frac{n(d-1)}{d}\binom{n+d-1}{n} & k = d+n+1 \\ (n+1)\eta_d & k = d+n+2 \\ 0 & k \le n+1 \ \textit{ or } \ k \ge d+n+3. \end{cases}$$

(iv)

$$\mathrm{h}^n(\mathcal{X}_d,\mathcal{N}_{\mathcal{X}_d}(-k))=\begin{cases}(N_d+1)\binom{k-d-1}{n}-(n+1)\binom{k-2}{n} & k\geq d+n+3\\ 0 & \textit{otherwise.}\end{cases}$$

Proof. (i), (ii) and (iii) are Propositions 5.2.3 and 5.2.5.

(v) For any $k\in\mathbb{Z}$ we have the exact sequence

$$0\longrightarrow \mathrm{H}^{n-1}(\mathcal{X}_d,\mathcal{N}_{\mathcal{X}_d}(-k))\longrightarrow \mathrm{H}^n(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}^{n+1}(1-k))\longrightarrow$$
$$\longrightarrow \mathrm{H}^n(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}^{N_d+1}(d-k))\longrightarrow \mathrm{H}^n(\mathcal{X}_d,\mathcal{N}_{\mathcal{X}_d}(-k))\longrightarrow 0.$$

Form this and the formulas (5.2.3) the result follows. □

We end this chapter with a couple of examples illustrating Theorem 5.2.6.

Example 5.2.7. (i) Take $G=\langle M_{5;0,1,2}\rangle\subset\mathrm{GL}(3,\mathbb{K})$ a cyclic group of order 5. $\mathcal{M}_{2,5}\setminus\mathrm{I}(\mathrm{relint}(H_A))_1=\{x_2^5,x_1x_2^4,x_1^2x_2^3,x_1^3x_2^2,x_1^4x_2,x_1^5,x_0x_2^4,x_0x_1x_2^3,x_0x_1^2x_2^2,$ $x_0x_1^4,x_0^2x_2^3,x_0^2x_1^2x_2,x_0^2x_1^3,x_0^3x_2^2,x_0^3x_1x_2,x_0^3x_1^2,x_0^4x_2,x_0^4x_1,x_0^5\}$. $\mathcal{N}_{\mathcal{X}_5}$ is the cokernel of the differential map $df_5:\mathcal{O}_{\mathbb{P}^2}^3(1)\longrightarrow\mathcal{O}_{\mathbb{P}^2}(5)^{19}$ given by the matrix:

$$\begin{pmatrix}0&0&5x_2^4\\0&x_2^4&4x_1x_2^3\\0&2x_1x_2^3&3x_1^2x_2^2\\0&3x_1^2x_2^2&2x_1^3x_2\\0&4x_1^3x_2&x_1^4\\0&5x_1^4&0\\x_2^4&0&4x_0x_2^3\\x_1x_2^3&x_0x_2^3&3x_0x_1x_2^2\\x_1^2x_2^2&2x_0x_1x_2^2&2x_0x_1^2x_2\\x_1^4&4x_0x_1^3&0\\2x_0x_2^3&0&3x_0^2x_2^2\\2x_0x_1^2x_2&2x_0^2x_1x_2&x_0^2x_1^2\\2x_0x_1^3&3x_0^2x_1^2&0\\3x_0^2x_2^2&0&2x_0^3x_2\\3x_0^2x_1x_2&x_0^3x_2&x_0^3x_1\\3x_0^2x_1^2&2x_0^3x_1&0\\4x_0^3x_2&0&x_0^4\\4x_0^3x_1&x_0^4&0\\5x_0^4&0&0\end{pmatrix}$$

(Proposition 5.2.1). The cohomology table from degree -10 to 0 of $\mathcal{N}_{\mathcal{X}_5}$ is

	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0
2 :	150	82	30	.	.	.	.	.	.	.	.
1 :	.	.	6	26	30	18	9	3	.	.	.
0 :	.	.	.	.	.	19	57	114	190	282	390

(ii) Take $G = \langle M_{4;0,1,1,2}\rangle \subset \mathrm{GL}(4,\mathbb{K})$ a cyclic group of order 4. We have $\mathcal{M}_{3,4} \setminus \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1 = \{x_3^4, x_3^3x_2, x_3^2x_2^2, x_3x_2^3, x_2^4, x_3^3x_1, x_3^2x_1x_2, x_3x_1x_2^2, x_1x_2^3, x_3^2x_1^2, x_3x_1^2x_2, x_1^2x_2^2, x_3x_1^3, x_1^3x_2, x_1^4, x_3^3x_0, x_3^2x_0x_2, x_3x_0x_2^2, x_0x_2^3, x_3^2x_0x_1, x_0x_1x_2^2, x_3x_0x_1^2, x_0x_1^2x_2, x_0x_1^3, x_3^2x_0^2, x_3x_0^2x_2, x_0^2x_2^2, x_3x_0^2x_1, x_0^2x_1x_2, x_0^2x_1^2, x_3x_0^3, x_0^3x_2, x_0^3x_1, x_0^4\}$. $\mathcal{N}_{\mathcal{X}_4}$ is the cokernel of the differential map $df_4 : \mathcal{O}_{\mathbb{P}^3}^4(1) \longrightarrow \mathcal{O}_{\mathbb{P}^3}^{34}(4)$ given by the matrix

$$\begin{pmatrix}
0 & 0 & 0 & 4x_3^3\\
0 & 0 & x_3^3 & 3x_2x_3^2\\
0 & 0 & 2x_2x_3^2 & 2x_2^2x_3\\
0 & 0 & 3x_2^2x_3 & x_2^3\\
0 & 0 & 4x_2^3 & 0\\
0 & x_3^3 & 0 & 3x_1x_3^2\\
0 & x_2x_3^2 & x_1x_3^2 & 2x_1x_2x_3\\
0 & x_2^2x_3 & 2x_1x_2x_3 & x_1x_2^2\\
0 & x_2^3 & 3x_1x_2^2 & 0\\
0 & 2x_1x_3^2 & 0 & 2x_1^2x_3\\
0 & 2x_1x_2x_3 & x_1^2x_3 & x_1^2x_2\\
0 & 2x_1x_2^2 & 2x_1^2x_2 & 0\\
0 & 3x_1^2x_3 & 0 & x_1^3\\
0 & 3x_1^2x_2 & x_1^3 & 0\\
0 & 4x_1^3 & 0 & 0\\
x_3^3 & 0 & 0 & 3x_0x_3^2\\
x_2x_3^2 & 0 & x_0x_3^2 & 2x_0x_2x_3\\
x_2^2x_3 & 0 & 2x_0x_2x_3 & x_0x_2^2\\
x_2^3 & 0 & 3x_0x_2^2 & 0\\
x_1x_3^2 & x_0x_3^2 & 0 & 2x_0x_1x_3\\
x_1x_2^2 & x_0x_2^2 & 2x_0x_1x_2 & 0\\
x_1^2x_3 & 2x_0x_1x_3 & 0 & x_0x_1^2\\
x_1^2x_2 & 2x_0x_1x_2 & x_0x_1^2 & 0\\
x_1^3 & 3x_0x_1^2 & 0 & 0\\
2x_0x_3^2 & 0 & 0 & 2x_0^2x_3\\
2x_0x_2x_3 & 0 & x_0^2x_3 & x_0^2x_2\\
2x_0x_2^2 & 0 & 2x_0^2x_2 & 0\\
2x_0x_1x_3 & x_0^2x_3 & 0 & x_0^2x_1\\
2x_0x_1x_2 & x_0^2x_2 & x_0^2x_1 & 0\\
2x_0x_1^2 & 2x_0^2x_1 & 0 & 0\\
3x_0^2x_3 & 0 & 0 & x_0^3\\
3x_0^2x_2 & 0 & x_0^3 & 0\\
3x_0^2x_1 & x_0^3 & 0 & 0\\
4x_0^3 & 0 & 0 & 0
\end{pmatrix}$$

The cohomology table from degree -9 to 0 of the normal bundle $\mathcal{N}_{\mathcal{X}_4}$ is

	-9	-8	-7	-6	-5	-4	-3	-2	-1	0
3 :	710	344	116	.	.	.	.	.	.	.
2 :	.	.	4	46	40	16	4	.	.	.
1 :	.	.	.	.	.	.	.	.	.	.
0 :	.	.	.	.	.	34	136	340	676	1174

Appendix

Routines in Wolfram *Mathematica*

This appendix contains two algorithms which compute a minimal set of monomial generators of a finite diagonal abelian group $G \subset \mathrm{GL}(n+1, \mathbb{K})$ and a minimal set of monomial generators of the canonical module of a $\overline{G}-$variety X_d with group G, respectively. These routines are illustrated with functions written in Wolfram *Mathematica*'s language in addition to particular examples in each case.

Let us fix the notation along this appendix. Let $2 \leq n < d$ be integers, e a dth primitive root of $1 \in \mathbb{K}$ and $G = \Gamma_1 \oplus \cdots \oplus \Gamma_s \subset \mathrm{GL}(n+1, \mathbb{K})$ an abelian group of order $d = d_1 \cdots d_s$ where each Γ_i is a cyclic group of order d_i generated by a diagonal matrix

$$M_{d_i;\alpha^i_{\sigma_i(0)},\dots,\alpha^i_{\sigma_i(n)}} = \mathrm{diag}(e_i^{\alpha^i_{\sigma_i(0)}}, \dots, e_i^{\alpha^i_{\sigma_i(n)}}),\ e_i = e^{d/d_i},\ \sigma_i \in \mathcal{S}_{n+1}$$

(Notation 2.2.1). The cyclic extension of G is the abelian group $\overline{G} \subset \mathrm{GL}(n+1, \mathbb{K})$ generated by G and $M_{d;1,\dots,1} = \mathrm{diag}(e, \dots, e)$ (Definition 1.3.2). We prove in Theorem 2.2.11 that a minimal set of fundamental invariants of $\overline{G}$ is the set $\mathcal{B}_1$ of monomial invariants of G of degree d, i.e. $R^{\overline{G}} = \mathbb{K}[\mathcal{B}_1]$.

On the other hand, a monomial $x_0^{a_0} \cdots x_n^{a_n} \in R^{\overline{G}}$ if and only if $(a_0, \dots, a_n)$ is a $\mathbb{Z}^{n+1}_{\geq 0}-$solution of one of the linear systems of congruences

$$(*)_{\mathcal{A};t,r_1,\dots,r_s}: \begin{cases} y_0 + y_1 + \cdots + y_n = td \\ \alpha^1_{\sigma_1(0)}y_0 + \alpha^1_{\sigma_1(1)}y_1 + \cdots + \alpha^1_{\sigma_1(n)}y_n = r_1 d_1 \\ \vdots \\ \alpha^s_{\sigma_s(0)}y_0 + \alpha^s_{\sigma_s(1)}y_1 + \cdots + \alpha^s_{\sigma_s(n)}y_n = r_s d_s \end{cases}$$

for some integers $t > 0$ and $0 \leq r_i \leq \frac{\alpha^i_n td}{d_i},\ i = 1, \dots, s$.

In view of these facts, the following algorithm computes the set $\mathcal{B}_t$ of monomial invariants of G of degree td.

Algorithm 1

Input : integers $d_1, \ldots, d_s \geq 1$, $t > 0$ and $\alpha^1_{\sigma_1(0)}, \ldots, \alpha^1_{\sigma_1(n)}, \ldots, \alpha^s_{\sigma_s(0)}, \ldots, \alpha^s_{\sigma_s(n)}$

Output : the list $\mathcal{B}_t$ of monomial invariants of G of degree td

Initialization: $d := d_1 \cdots d_s$; $M_i := \max\{\alpha^j_{\sigma_i(0)},\ j = 0, \ldots, n\} \cdot \frac{td}{d_i}$, $i = 1, \ldots, s$; $L = \{\}$; $\mathcal{B}_t = \{\}$; $Eq_0 = \{(a_0, \ldots, a_n) \in \mathbb{Z}^{n+1}_{\geq 0} \mid a_0 + \cdots + a_n = td\}$;

for $i = 1, \ldots, s$ **do**
  for $k = 0, \ldots, M_i$ **do**
    $Eq_i = Eq_i \cup \{(a_0, \ldots, a_n) \in Eq_0 \mid a_0\alpha^i_{\sigma_i(0)} + \cdots + a_n\alpha^i_{\sigma_i(n)} = kd_i\}$;
  end
end
$L = Eq_1 \cap \cdots \cap Eq_s$;
return $\mathcal{B}_t = \{x_0^{a_0} \cdots x_n^{a_n} \mid (a_0, \ldots, a_n) \in L\}$;

To exemplify Algorithm 1 in Wolfram *Mathematica*'s language, we provide a function which computes $\mathcal{B}_t$ for any cyclic group $G = \langle M_{d;0,\alpha_1,\alpha_2}\rangle \subset \mathrm{GL}(3, \mathbb{K})$ of order $d \geq 3$ with $\alpha_1 < \alpha_2$. For convenience, we express the monomials of $\mathcal{B}_t$ in the variables x, y, z and we write $a = \alpha_1, b = \alpha_2$.

```
InvPoly[d_, t_, a_, b_] := Module[{k, j, M, S, Eq1, Eq2, Saux},
  S = {}; Eq0 = {al + be + ga == t*d}; M = b*t;
  For[k = 0, k <= M, k++,
       Eq1 = {a*be + b*ga == k*d};
       Saux = Solve[Eq0[[1]] && Eq1[[1]] && al >= 0 && be >= 0
              && ga >= 0, {al, be, ga}, Integers];
       For[j = 1, j <= Length[Saux], j++,
            S = Append[S, Saux[[j]]];
        ];
   ];
  x^al*y^be*z^ga /. S
  ]
```

```
In[1]:= InvPoly[3, 1, 1, 2]
```

```
        InvPoly[6, 1, 2, 3]
        InvPoly[11, 1, 1, 6]
Out[1]= {x^3, y^3, x y z, z^3}
Out[2]= {x^6, x^3 y^3, x^4 z^2, y^6, x y^3 z^2, x^2 z^4, z^6}
Out[3]= {x^11, y^11, x^5 y^5 z, x^4 y^4 z^3, x^3 y^3 z^5,
 x^2 y^2 z^7, x y z^9, z^11}.
```

On the other hand, given a $\overline{G}-$variety X_d with group $G \subset \mathrm{GL}(n+1, \mathbb{K})$, the canonical module ω_{X_d} of its homogeneous coordinate ring $A(X_d) \cong R^{\overline{G}}$ is identified with the ideal

$$\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}})) = (x_0^{a_0} \cdots x_n^{a_n} \in R^{\overline{G}} \mid a_0 \cdots a_n \neq 0) \subset R^{\overline{G}}$$

(Theorem 3.3.1). In Theorem 3.3.3, we have proved that $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))$ is generated by the subsets $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1$ and $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_2$ of monomials of $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))$ of degree d and $2d$, respectively. Thus, a simple modification of Algorithm 1 provides a routine to compute such a set of generators. However, it could be non minimal as we have seen in Section 3.3. The following algorithm determines a minimal set of generators of the ideal $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))$.

Algorithm 2

Input : integers $d_1, \ldots, d_s \geq 1$ and $\alpha^1_{\sigma_1(0)}, \ldots, \alpha^1_{\sigma_1(n)}, \ldots, \alpha^s_{\sigma_s(0)}, \ldots, \alpha^s_{\sigma_s(n)}$

Output : a minimal set L of generators of $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))$

$L_1 = \{$**Call** Algorithm 1 with $t = 1\} \cap \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))$;
$L_2 = \{$**Call** Algorithm 1 with $t = 2\} \cap \mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))$;
$L = L_1$;
for $i = 1, \ldots, length(L_2)$ **do**
  for $k = 0, \ldots, length(L_1)$ **do**
    if $L_1[k] \mid L_2[i]$ **then**
      $k = length(L_1) + 1$;
    end
  end
  if $k = length(L_1)$ **then**
    $L = L \cup \{L_2[i]\}$;
  end
end
return L;

To illustrate Algorithm 2 in Wolfram *Mathematica*'s language, we provide the following implementation. It is based on two functions. The first one *IsDivisible* takes arguments the coefficients of two monomials m_1 and m_2 and it determines whether m_2 is divisible by m_1. The second one *SocleDegree* takes arguments two lists L_1 and L_2 containing, respectively, the coefficients of the monomials of $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_1$ and $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))_2$. Using *IsDivisible*, it returns the list L of the coefficients of a minimal set of monomial generators of $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))$.

```
IsDivisible[m1_, m2_] := Module[{n, ban, i},
  n = Length[m1];
  ban = 1;
  For[i = 1, i <= n && ban == 1, i++,
       If[m1[[i]] > m2[[i]], ban = 0];
  ];
  Return[ban];
]

SocleDegree[L1_, L2_] := Module[{n1, n2, i, j, ban, L},
  n1 = Length[L1];
  n2 = Length[L2];
  L = L1;
  For[i = 1, i <= n2 , i++,
       ban = 0;
       For[j = 1, j <= n1 && ban == 0, j++,
            ban = IsDivisible[L1[[j]], L2[[i]]];
       ];
       If[ban  == 0, L = Append[L, L2[[i]]]];
  ];
  Return[L];
]
```

Let us see how it works for $GT-$surfaces X_d with cyclic group $G = \langle M_{d;0,\alpha_1,\alpha_2}\rangle \subset \mathrm{GL}(3,\mathbb{K})$ of order $d \geq 3$. We end this appendix with the concrete examples of Algorithm 1, they verify that $A(X_d)$ is a level ring with Castelnuovo–Mumford regularity 3 (Corollary 3.1.22).

Remark A.0.1. The function *InvPolySurfCanMod* is a minor modification of the function *InvPoly*. It computes the monomials of I(relint($H_{\mathcal{A}}$)) of degree *td*.

```
InvPolySurfCanMod[d_, t_, a_, b_] :=  Module[{k, i, M, S, Eq1,
    Eq2, Saux},
    S = {}; M = t*b; Eq1 = {al + be + ga == t*d};
    For[k = 0, k <= M, k++,
           Eq2 = {a*be + b*ga  == k*d};
           Saux = Solve[Eq1[[1]] && Eq2[[1]] && al > 0 &&
               be > 0 && ga > 0, {al, be, ga}, Integers];
            For[i = 1, i <= Length[Saux], i++,
                S = Append[S, Saux[[i]]];
            ];
    ];
    Return[S];
]
SurfCanMod[d_, a_, b_] := Module[{L},
    L = SocleDegree[InvPolySurfCanMod1[d, 1, a, b],
        InvPolySurfCanMod1[d, 2, a, b]];
  x^al*y^be*z^ga /. L
]

In[1]:= InvPolySurfCanMod[3, 1, 1, 2]
        InvPolySurfCanMod[3, 2, 1, 2]
        SurfCanMod[3, 1, 2]
Out[1]= {{1, 1, 1}}
Out[2]= {{4,1,1},{1,4,1},{2,2,2},{1,1,4}}
Out[3]= {x y z}

In[1]:= InvPolySurfCanMod[6, 1, 2, 3]
        InvPolySurfCanMod[6, 2, 2, 3]
        SurfCanMod[6, 2, 3]
Out[1]= {{1,3,2}}
Out[2]= {{7,3,2},{4,6,2},{5,3,4},{1,9,2},{2,6,4},{3,3,6},
{1,3,8}}
Out[3]= {x y^3 z^2}
```

```
In[1]:= InvPolySurfCanMod1[11, 1, 1, 6]
        InvPolySurfCanMod1[11, 2, 1, 6]
        SurfCanMod[11, 1, 6]
Out[1]= {{5,5,1},{4,4,3},{3,3,5},{2,2,7},{1,1,9}}
Out[2]= {{16,5,1},{5,16,1},{10,10,2},{15,4,3},{4,15,3},{9,9,4},
{14,3,5},{3,14,5},{8,8,6},{13,2,7},{2,13,7},{7,7,8},{12,1,9},
{1,12,9},{6,6,10},{5,5,12},{4,4,14},{3,3,16},{2,2,18},{1,1,20}}
Out[3]= {x^5 y^5 z, x^4 y^4 z^3, x^3 y^3 z^5, x^2 y^2 z^7,
x y z^9}.
```

Resum en llengua catalana

La present tesi contribueix a dos remarcables problemes oberts que s'emmarquen tant en l'àlgebra commutativa com en la geometria algebraica. El primer fa referència al problema, plantejat per Gröbner el 1967, de determinar quan una projecció monomial de la varietat de Veronese és una varietat aCM. El segon apunta al problema clàssic i fonamental de determinar un sistema minimal de generadors de l'anell d'invariants d'un group finit. El nostre enfoc fa un ús extensiu de la combinatòria, relacionant d'aquesta manera ambdues qüestions entre si. Així mateix, estableix una connexió entre elles i les propietats de Lefschetz dels ideals artinians.

El contingut d'aquesta dissertació s'ha organitzat en cinc capítols i un apèndix. El Capítol 1 és introductori i recopila els conceptes i resultats bàsics utilitzats en el cor d'aquest text: Capítols 2, 3, 4 i 5. L'Apèndix A conté dos algoritmes i implementacions en el programari Wolfram *Mathematica* [91]; amb els quals hem computat i verificat la major part dels exemples que il·lustren els resultats obtinguts. A continuació expliquem els principals avenços i contribucions que es troben a la tesi. Al Capítol 2, adrecem el problema de Gröbner i estudiem els invariants d'un grup $G \subset \mathrm{GL}(n+1, \mathbb{K})$ abelià i finit actuant sobre R diagonalment. Després de presentar l'evolució i principals avenços del problema de Gröbner, provem que el conjunt $\mathcal{B}_1$ d'invariants monomials de $G \subset \mathrm{GL}(n+1, \mathbb{K})$ de grau d generen de forma minimal l'anell d'invariants $R^{\overline{G}}$ de l'extensió cíclica $\overline{G} \subset \mathrm{GL}(n+1, \mathbb{K})$ de G. Anomenen $\overline{G}-$varietat amb grup G a la projecció monomial X_d de la varietat de Veronese $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ parametritzada per $\mathcal{B}_1$. Aquest resultat ens permet establir una nova família de projeccions monomials aCM de $X_{n,d}$: les $\overline{G}-$varietats X_d amb group G. Demostrem que l'anell $A(X_d)$ de coordenades homogènies de X_d és isomorf a l'anell $R^{\overline{G}}$ d'invariants de $\overline{G}$, que és un anell CM. L'ideal $I_d \subset R$ generat per $\mathcal{B}_1$ és un ideal artinià i monomial.

Demostrem que I_d falla la WLP en grau $d-1$ si el cardinal μ_d de $\mathcal{B}_1$ verifica la condició $\mu_d \leq N_{n-1,d}$ i que, en aquest cas, I_d és un $GT-$sistema amb grup G i X_d una $GT-$varietat amb grup G. Per últim, estudiem el problema de Gröbner sobre projeccions monomials de la superfície de Veronese $X_{2,d} \subset \mathbb{P}^{N_{2,d}-1}$ parametritzades pels generadors d'un sistema monomial de Togliatti.

Al Capítol 3, considerem la geometria de les $\overline{G}-$varietats X_d amb grup $G \subset \mathrm{GL}(n+1, \mathbb{K})$. Donat que són varietats aCM, perseguim l'objectiu de determinar explícitament la resolució lliure i minimal de qualsevol $\overline{G}-$varietat X_d amb grup G. Per aquest motiu, ens centrem en descriure la funció i la sèrie de Hilbert de X_d; en estudiar un sistema de generadors de l'ideal homogeni $\mathrm{I}(X_d)$ de X_d; en investigar el mòdul canònic ω_{X_d} de $A(X_d)$ i en determinar la regularitat de Castelnuovo i Mumford de $A(X_d)$. En finalitzar, recopilem tots els resultats per tal d'estudiar el diagrama de Betti de $A(X_d)$. En primer lloc, interpretem la funció i la sèrie de Hilbert de X_d des de la teoria d'invariants i la combinatòria; i les calculem explícitament per diverses famílies d'exemples. En particular, trobem explícitament ambdues funcions numèriques per qualsevol $GT-$superfície amb group cíclic $G \subset \mathrm{GL}(3, \mathbb{K})$ i per $GT-$sòlids amb group $G = \langle M_{d;0,1,2,3} \rangle \subset \mathrm{GL}(4, \mathbb{K})$ i $d \geq 4$. A continuació, tractem l'ideal $\mathrm{I}(X_d)$ i demostrem que $\mathrm{I}(X_d)$ és un ideal binomial i primer que es pot generar per binomis de grau com a màxim 3. Determinem explícitament un sistema minimal de generadors binomials de $\mathrm{I}(X_d)$ per qualsevol $GT-$sòlid amb grup $G = \langle M_{d;0,1,2,3} \rangle \subset \mathrm{GL}(4, \mathbb{K})$ i $d \geq 4$. Respecte el mòdul canònic ω_{X_d} de $A(X_d)$, l'identifiquem amb l'ideal $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}})) \subset R^{\overline{G}}$ i provem que es pot generar per invariants monomials de G de grau d i $2d$. Aquest resultat ens permet caracteritzar la regularitat de Castelnuovo i Mumford $\mathrm{reg}(A(X_d))$: establim que $n \leq \mathrm{reg}(A(X_d)) \leq n+1$ i $\mathrm{reg}(A(X_d)) = n+1$ si i només si $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))$ conté almenys un monomi de grau d.

Al Capítol 4, investiguem els invariants d'un grup finit $\Lambda \subset \mathrm{SL}(3, \mathbb{K})$ no abelià i la seva relació amb la WLP. Això ens permet proporcionar nous exemples de sistemes de Togliatti no monomials, fins ara poc estudiats. Centrem la nostra atenció en l'anell d'invariants del grup diedral $D_{2d} \subset \mathrm{SL}(3, \mathbb{K})$ d'ordre $2d$. Demostrem que $R^{\overline{D_{2d}}}$ és mínimament generat per invariants monomials i binomials de grau $2d$, fet que ens permet establir que parametritzen una projecció aCM $S_{D_{2d}}$ de la surperfície de Veronese

$X_{2,d}$. A més a més, l'ideal I_{2d} que generen és un $GT-$sistema amb grup D_{2d}. L'última part d'aquest capítol es dedica a l'estudi geomètric de les $GT-$superfícies $S_{D_{2d}}$ amb grup D_{2d}. Determinem explícitament una resolució lliure i minimal de $A(S_{D_{2d}})$ i un sistema minimal de generadors de grau 2 de l'ideal $\mathrm{I}(S_{D_{2d}})$.

Al Capítol 5, introduïm una nova família de varietats racionals i llises $\mathcal{X}_d$ associades de forma natural a $\overline{G}-$varietats *level* amb grup $G \subset \mathrm{GL}(n+1, \mathbb{K})$, és a dir, $\mathrm{reg}(A(X_d)) = n+1$ i $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}}))$ és generat per monomis de grau d. Les anonenem $RL-$varietats per emfatitzar el paper de l'interior **r**elatiu relint i la propietat de ser **l**evel. Són projeccions monomials no aCM de la varietat de Veronese $X_{n,d} \subset \mathbb{P}^{N_{n,d}-1}$ parametritzades pels η_d monomis de grau d de $\mathrm{I}(\mathrm{relint}(H_{\mathcal{A}})) \subset R^{\overline{G}}$ i submergides en $\mathbb{P}^{N_d}$, $N_d = N_{n,d} - \eta_d - 1$. Aquestes propietats ens permeten descriure el fibrat vectorial normal $\mathcal{N}_{\mathcal{X}_d}$ de $\mathcal{X}_d$. Determinar la cohomologia del feix normal d'una varietat $X \subset \mathbb{P}^N$ arbitrària és un problema obert de gran complexitat. En aquesta tesi, contribuïm a aquest tòpic calculant la dimensió de la cohomologia del fibrat normal $\mathcal{N}_{\mathcal{X}_d}$ de qualsevol $RL-$varietat $\mathcal{X}_d$.

Bibliography

[1] C. Almeida, A. V. Andrade and R.M. Miró-Roig, *Gaps in the number of generators of monomial Togliatti systems*. Journal of Pure and Applied Algebra. **223:4** (2019), 1817–1831.

[2] N. Altafi and M. Boij, *The weak Lefschetz property of equigenerated monomial ideals*. Journal of Algebra. **556** (2020), 136–168.

[3] A. Alzati, R. Re and A. Tortora, *An algorithm for determining the normal bundle of rational monomial curves*. Rendiconti del Circolo Matematico di Palermo Series 2. **67:2** (2018), 291–306.

[4] A. Alzati and R. Re, *Cohomology of normal bundles of special rational varieties*. Communications in Algebra. **48:6** (2020), 2492–2516.

[5] I. Bermejo, E. García-Llorente and I. García-Marco, *Algebraic invariants of projective monomial curves associated to generalized arithmetic sequences*. Journal of Symbolic Computation. **81** (2017), 1–19.

[6] H. F. Blichfeldt, L. E. Dickson and G. A. Miller, *Theory and applications of finite groups*. Wiley, New York, 1916.

[7] H. Brenner and A. Kaid, *Syzygy bundle on $\mathbb{P}^2$ and the Weak Lefschetz property*. Illinois Journal of Mathematics. **51:4** (2007), 1299–1308.

[8] E. Briales, A. Campillo, C. Marijuan and P. Pisón, *Minimal system of generators for ideals of semigroups*. Journal of Pure and Applied Algebra. **124:1–3** (1998), 7–30.

[9] W. Bruns and J. Herzog, *Cohen–Macaulay rings*. Cambridge University Press, 1993.

[10] A. Campillo and P. Gimenez, *Syzygies of affine toric varieties.* Journal of Algebra. **225:1** (2000), 142–161.

[11] M. P. Cavaliere and G. Niese, *On monomial curves and Cohen-Macaulay type.* Manuscripta Mathematica. **42:2–3** (1983), 147–159.

[12] G. Caviglia, *The pinched Veronese is Koszul.* Journal of Algebraic Combinatorics. **30:4** (2009), 539–548.

[13] G. Caviglia and A. Conca, *Koszul property of projections of the Veronese cubic surface.* Advances in Mathematics. **234** (2013), 404–413.

[14] H. Charalambous, A. Katsabekis and A. Thoma, *Minimal systems of binomial generators and the indispensable complex of a toric ideal.* Proceedings of the American Mathematical Society. **135:11** (2007), 3443–3451.

[15] H. Charalambous, A. Thoma, M. Vladoiu, *Binomial fibers and indispensable binomials.* Journal of Symbolic Computation. **74** (2016), 578–591.

[16] H. Charalambous, A. Thoma and M. Vladoiu, *Minimal generating set of lattice ideals.* Collectanea Mathematica. **68:3** (2017), 377–400.

[17] L. Colarte-Gómez, E. Mezzetti and R. M. Miró-Roig, *On the arithmetic Cohen–Macaulayness of varieties parameterized by monomial Togliatti systems.* Annali di Matematica Pura ed Applicata. (2021). https://doi.org/10.1007/s10231-020-01058-2

[18] L. Colarte, E. Mezzetti, R. M. Miró-Roig and M. Salat, *On the coefficients of the permanent and the determinant of a circulant matrix. Applications.* Proceedings of the American Mathematical Society. **147:2** (2019), 547–558.

[19] L. Colarte-Gómez, E. Mezzetti, R. M. Miró-Roig and M. Salat, *Togliatti systems associated to the dihedral group and the weak Lefschetz property.* Israel Journal of Mathematics, to appear.

[20] L. Colarte and R. M. Miró-Roig, *Minimal set of binomial generators for certain Veronese* $3-$*fold projections.* Journal of Pure and Applied Algebra. **224:2** (2020), 768–788.

[21] L. Colarte-Gómez and R. M. Miró-Roig, *The canonical module of GT-varieties and the normal bundle of RL-varieties.* Mediterranean Journal of Mathematics, to appear.

[22] V. I. Danilov, *The geometry of toric varieties.* Russian Mathematical Surveys. **33:2** (1978), 97–154.

[23] P. De Poi, E. Mezzetti, M. Michałek, R.M. Miró-Roig and E. Nevo, *Circulant matrices and Galois-Togliatti systems.* Journal of Pure and Applied Algebra. **224:11** (2020), 1–14.

[24] P. Diaconis and B. Sturmfels, *Algebraic Algorithms for sampling from conditional distributions.* The Annals of Statistics. **26:1** (1998), 363–397.

[25] D. Eisenbud, *Commutative Algebra with a view towards algebraic geometry.* Springer-Verlag, New York, 1995.

[26] D. Eisenbud, *The geometry of syzygies. A second course in commutative algebra and algebraic geometry.* Springer-Verlag, New York, 2005.

[27] D. Eisenbud and B. Sturmfels, *Binomial ideals.* Duke Mathematical Journal. **84:1** (1996), 1–45.

[28] A. Elashvili and M. Jibladze, *Hermite reciprocity for the regular representations of cyclic groups.* Indagationes Mathematicae. **9:2** (1998), 233–238.

[29] P. Erdös, A. Ginsburg and A. Ziv, *Theorem in the additive number theory.* The Bulletin of the Research Council of Israel. **10:F1** (1961), 41–43.

[30] W. Gao and A. Geroldinger, On long minimal zero sequences in finite abelian groups. Periodica Mathematica Hungarica. **38:3** (1999), 179–211.

[31] J. L. García-García, D. Marín-Aragón and A. Vigneron-Tenorio, *A characterization of some families of Cohen-Macaulay, Gorenstein and/or Buchsbaum rings.* Discrete Applied Mathematics. **263** (2019), 166–176.

[32] P. A. García-Sánchez and J. C. Rosales, *On Cohen-Macaulay and Gorenstein simplicial affine semigroups.* Proceedings of the Edinburgh Mathematical Society. **41:3** (1998), 517–537.

[33] J. L. García-García and A. Vigneron-Tenorio, *Computing families of Cohen-Macaulay and Gorenstein rings.* Semigroup Forum. **88:3** (2014), 610–620.

[34] I.M. Gelfand, M.M. Kapranov and A. V. Zelevinsky, *Discriminants, Resultants and multidimensional Discriminants.* Springer Science+Business Media, New York, 1994.

[35] S. Goto, N. Suzuki and K. Watanabe, *On affine semigroup rings.* Japanese Journal of Mathematics. **2:1** (1976), 1–12.

[36] D.R. Grayson and M.E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, Available at http://www.math.uiuc.edu/Macaulay2/

[37] O. Greco and I. Martino, *Cohen-Macaulay property and linearity of pinched Veronese rings.* Journal of Commutative Algebra, to appear.

[38] O. Greco and I. Martino, *Syzygies of the Veronese modules.* Communications in Algebra. **44:9** (2016), 3890–3806.

[39] W. Gröbner, *Über Veronesesche Varietäten und deren Projektionen.* Archiv der Mathematik. **16** (1965), 257 –264.

[40] T. Harima, T. Maeno, H. Morita, Y. Numata, A. Wachi and J. Watanabe, *The Lefschetz properties.* Springer, Heidelberg, 2013.

[41] T. Harima, J. Migliore, U. Nagel and J. Watanabe, *The weak and strong Lefschetz properties for artinian K-algebras.* Journal of Algebra. **262:1** (2003), 99–126.

[42] J. C. Harris and D. L. Wehlau, *Non-negative integer linear congruences.* Indagationes Mathematicae. **17:1** (2006), 37–44.

[43] R. Hartshorne, *Algebraic Geometry.* Springer-Verlag, New York, 1977.

[44] M. Hellus, L. T. Hoa and J. Stückrad, *Castelnuovo–Mumford regularity and the reduction number of some monomial curves.* Proceedings of the American Mathematical Society. **138:1** (2010), 27–35.

[45] R. Hemmecke and P. Malkin, *Computing generating sets of lattice ideals and Markov basis of lattices.* Journal of Symbolic Computation. **44:10** (2009), 1463–1476.

[46] J. Herzog and D. I. Stamate, *Cohen-Macaulay criteria for projective monomial curves via Gröbner bases.* Acta Mathematica Vietnamica. **44:1** (2019), 51–64.

[47] L. T. Hoa and N. V. Trung, *Affine semigroups and Cohen-Macaulay rings generated by monomials.* Transactions of the American Mathematical Society. **298:1** (1985), 145–167.

[48] L. T. Hoa, *Classification of the triple projections of Veronese varieties.* Mathematische Nachrichten. **128:1** (1986), 185–197.

[49] L. T. Hoa, *Algorithmetical aspects of the problem of classifying muti-projections of Veronese varieties.* Manuscripta Mathematica. **63:3** (1989), 317–331.

[50] L. T. Hoa and J. Stückrad, *Castelnuovo–Mumford regularity of simplicial toric rings.* Journal of Algebra. **259:1** (2003), 127–146.

[51] M. Hochster, *Rings of Invariants of Tori, Cohen-Macaulay Rings Generated by Monomials, and Polytopes.* Annals of Mathematics. **96:2** (1972), 318-337.

[52] J. A. Eagon and M. Hochster, *Cohen-Macaulay Rings, Invariant Theory, and the Generic Perfection of Determinantal Loci.* American Journal of Mathematics. **93:4** (1971), 1020-1058.

[53] Y. Kamoi, *Defining ideals of Cohen-Macaulay semigroup rings.* Communications in Algebra. **20:11** (1992), 3163–3189.

[54] T. Kawasaki, *On Macaulayfication of noetherian schemes.* Transactions of the American Mathematical Society. **352:6** (2000), 2517–2552.

[55] F. S. Macaulay, *The algebraic theory of modular systems.* Cambridge University Press, 1916.

[56] H. Maschke, *On ternary substitutions-groups of finite order which leaves a triangle unchanged.* American Journal of Mathematics. **17:2** (1895), 168–184.

[57] E. Mezzetti and R. M. Miró-Roig. *Togliatti systems and Galois coverings.* Journal of Algebra. **509:1** (2018), 263–291.

[58] E. Mezzetti and R. M. Miró-Roig, *The minimal number of generators of a Togliatti system.* Annali di Matematical Pura ed Applicata. **195:6** (2016), 2077–2098.

[59] E. Mezzetti, R. M. Miró-Roig and G. Ottaviani, *Laplace Equations and the Weak Lefschetz Property.* Canadian Journal of Mathematics. **65:3** (2013), 634–654.

[60] M. Michałek and R. M. Miró-Roig, *Smooth monomial Togliatti systems of cubics.* Journal of Combinatorial Theory, Series A. **143** (2016), 67–87.

[61] J. Migliore and R. M. Miró-Roig, *Ideals of general forms and the ubiquity of the weak Lefschetz property.* Journal of Pure and Applied Algebra. **182:1** (2003), 79–107.

[62] J. Migliore, R. M. Miró-Roig and U. Nagel, *Monomial ideals, almost complete intersections and the weak Lefschetz property.* Transactions of the American Mathematical Society. **363:1** (2011), 229–257.

[63] E. Miller and B. Sturmfels, *Combinatorial Commutative Algebra.* Springer-Verlag, New York, 2005.

[64] R. M. Miró-Roig and M. Salat, *On the classification of Togliatti systems.* Communications in Algebra. **46:6** (2018), 2459–2475.

[65] T. Molien, *Über die Invarianten der linearen Substitutionsgruppe.* Königlich Preussische Akademie der Wissenschaften. (1897), 1152–1156.

[66] C. Okonek, M. Schneider and H. Spindler, *Vector Bundles on Complex Projective Spaces. With an Appendix by S. I. Gelfand.* Birkhäuser, Basel, 2011.

[67] L. Reid, L. Roberts and M. Roitman, *On complete intersections and their Hilbert functions.* Canadian Mathematical Bulletin. **34:4** (1991), 525–535.

[68] G. Sacchiero, *Fibrati normali di curve razionali dello spazio proiettivo.* Annali dell'Università di Ferrara. **26** (1981), 33–40.

[69] P. Samuel and O. Zariski, *Commutative Algebra, vol II.* Springer-Verlag, Berlin-Heidelberg, 1960.

[70] U. Schäfer and P. Schenzel, *Dualizing complexes of affine semigroup rings.* Transactions of the American Mathematical Society. **322:2** (1990), 561–582.

[71] P. Schenzel, *On the use of local cohomology in algebra and geometry.* In: Six lectures on commutative algebra (J. Elias, J. M. Giral, R. M. Miró-Roig and S. Zarzuela, Eds.) Birkhäuser, Basel, 1998.

[72] P. Schenzel, *On Veronesean embeddings and projections of Veronesean varieties.* Archiv der Mathematik. **30** (1978), 391–397.

[73] E. Sernesi, *Topics on families of projective schemes.* Queen's papers in pure and applied mathematics. **73**, 1986.

[74] J. P. Serre, *Groupes algébriques et corps de classes.* Hermann, Paris, 1957.

[75] J. P. Serre, *Linear Representations of Finite Groups.* Springer, New York, 1977.

[76] G. Shephard and J. Todd, *Finite unitary reflection groups.* Canadian Journal of Mathematics. **6** (1954), 274–304.

[77] R. P. Stanley, *Invariants of finite groups and their application to combinatorics.* Bulletin of the American Mathematical Society. **1:3** (1979), 475–511.

[78] R. P. Stanley, *Linear diophantine equations and local cohomology.* Inventiones mathematicae **68:2** (1982), 57–73.

[79] R. P. Stanley, *Hilbert functions of graded algebras.* Advances in Mathematics. **28:2** (1978), 57–83.

[80] R. P. Stanley, *Weyl groups, the hard Lefschetz theorem, and the Sperner property.* SIAM Journal on Algebraic Discrete Methods. **1:2** (1980), 168–184.

[81] B. Sturmfels, *Algorithms in Invariant Theory.* Springer-Verlag, Wien, 2008.

[82] E. Togliatti, *Alcuni esempî di superficie algebriche degli iperspazî che rappresentano un'equazione di Laplace.* Commentarii Mathematici Helvetici. **1** (1929), 255–272.

[83] E. Togliatti, *Alcune osservazioni sulle superficie razionali che rappresentano equazioni di Laplace.* Annali di Matematica Pura ed Applicata. **25** (1946) 325–339.

[84] N. V. Trung, *Classification of the double projections of Veronese varieties.* Journal of Mathematics of Kyoto University. **22:4** (1982), 567–581.

[85] N. V. Trung, *Projections of one-dimensional Veronese varieties.* Mathematische Nachrichten. **118** (1984), 47–67.

[86] J. Watanabe, *The Dilworth number of Artinian rings and finite posets with rank function.* Advanced Studies in Pure Mathematics. **11** (1987), 303–312.

[87] H. Weyl, *The classical groups.* Princeton University Press, New Jersey, 1953.

[88] K. Yanagawa, *Zero-dimensional schemes and Hilbert functions of Cohen-Macaulay homogeneous domains.* RIMS Kôkyûroku. **934** (1996), 1–14.

[89] K. Yanagawa, *Some generalizations of Castelnuovo's Lemma on zero-dimensional schemes.* Journal of Algebra. **170:2** (1994), 429–439.

[90] S. S–T. Yau and Y. Yu, *Gorenstein quotient singularities in dimension three.* Memoirs of the American Mathematical Society. **505**, 1993.

[91] Wolfram Research, Inc., *Mathematica, Version 12.2*. Champaign, IL (2020).