UNIVERSITAT DE BARCELONA
Doctorat de Matemàtiques i Informàtica

DOCTORAL DISSERTATION

Restricted Weak Type Extrapolation of Multi-Variable Operators and Related Topics

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In loving memory of my grandfather

Josep Roure Escuer†

1931 – 2019

“*S’han de fer de tripas corazones.*”

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Resum

TESI DOCTORAL

Restricted Weak Type Extrapolation of Multi-Variable Operators and Related Topics

per Eduard ROURE PERDICES

En el camp de la Teoria de pesos, un resultat que ha atret l'atenció de molts investigadors és l'anomenat Teorema d'extrapolació de Rubio de Francia. En la seva forma més simple, diu que si tenim un operador T que està acotat a l'espai de Lebesgue $L^p(v)$, per algun $p \geq 1$ i cada pes v en A_p, llavors T està acotat a l'espai de Lebesgue $L^q(w)$, per cada $q > 1$ i cada pes w en A_q.

L'extrapolació de Rubio de Francia proporciona un potent conjunt d'eines en l'Anàlisi Harmònica, però té un punt feble; no permet arribar a l'extrem $q = 1$. Els treballs de M. J. Carro, L. Grafakos, i J. Soria [9], i M. J. Carro i J. Soria [14] resolen aquest problema, obtenint esquemes d'extrapolació de tipus dèbil $(1, 1)$ amb pesos en A_1.

En aquest projecte de tesi vam començar a estudiar aquests articles per produir extensions multivariable dels resultats d'extrapolació que s'hi exposen. Hem tingut èxit en aquesta tasca, i ara posseïm esquemes d'extrapolació multivariable de tipus mixt i dèbil restringit que són de gran utilitat en l'obtenció d'acotacions d'operadors en múltiples variables pels quals no es coneixen resultats de dominació sparse, i també quan treballem en espais de Lorentz pels quals la dualitat no està disponible. Com a cas particular, hem estudiat operadors producte, commutadors en dos variables i multiplicadors bilineals.

Les desigualtats de tipus Sawyer han jugat un paper fonamental en les demostracions dels nostres teoremes d'extrapolació, així com en l'estudi del producte puntual d'operadors maximals de Hardy-Littlewood. Hem sigut capaços d'ampliar les desigualtats de Sawyer clàssiques de [27] al tipus dèbil restringit amb pesos en $A_p^{\mathcal{R}}$, i també hem demostrat les corresponents extensions multivariable.

Durant una estada de tres mesos a la Universitat d'Alabama, vam iniciar una col·laboració amb David V. Cruz-Uribe. El nostre objectiu era estudiar els operadors fraccionaris i de Calderón-Zygmund en múltiples variables, i obtenir-ne acotacions de tipus dèbil restringit amb pes. Combinant tècniques de dominació sparse i propietats dels espais de Lorentz, vam demostrar diverses estimacions per aquests operadors, i també pels seus commutadors.

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Abstract

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by Eduard ROURE PERDICES

A remarkable result in Harmonic Analysis is the so-called Rubio de Francia's extrapolation theorem. Roughly speaking, it says that if one has an operator T that is bounded on $L^p(v)$, for some $p \geq 1$ and every weight v in A_p, then T is bounded in $L^q(w)$, for every $q > 1$ and every weight w in A_q.

Rubio de Francia's extrapolation theory is very useful in practice, but there is an issue: it does not allow to produce estimates for $q = 1$. The works of M. J. Carro, L. Grafakos, and J. Soria [9], and M. J. Carro and J. Soria [14] give a solution to this problem, allowing to extrapolate down to the endpoint $q = 1$.

In this project, we started building upon these works to produce multi-variable extensions of the extrapolation results that they presented. We have succeeded in this endeavor, and now we possess extrapolation schemes in the setting of weighted Lorentz spaces that are of great use when trying to bound multi-variable operators for which no sparse domination is known, and also when working with Lorentz spaces outside the Banach-range. As a particular case, we have studied product-type operators, two-variable commutators, averaging operators, and bi-linear multipliers.

Sawyer-type inequalities play a fundamental role in the proof of our multi-variable extrapolation schemes and are essential to complete the characterization of the weighted restricted weak type bounds for the point-wise product of Hardy-Littlewood maximal operators. In this work, we have extended the classical weak $(1, 1)$ Sawyer-type inequalities proved in [27] to the general restricted weak type case, even in the multi-variable setting.

In 2017, at the University of Alabama, we started a collaboration with David V. Cruz-Uribe to produce restricted weak type bounds for fractional operators, Calderón-Zygmund operators, and commutators of these operators. We managed to obtain satisfactory results on this matter, even two-weight norm inequalities, applying a wide variety of techniques on sparse domination, function spaces, and weighted theory.

Acknowledgements

Axiom: people are troublesome. And yet, this project could not have been possible without the effort of quite a considerable amount of people. Some of them did a remarkable job; others should have been replaced long ago.

In what follows, you will find a list of names of those who got more involved with me in this enterprise. I hope I did not forget anyone.

I would like to express my sincere gratitude to my grandparents, Josep R.†, Avelino P., and Gloria E., my mother, Yolanda P., and my aunts, Rosa Mari R. and Anna B., for their unconditional support.

I would also like to thank Sonja S. for her support and invaluable advice during these years.

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I am indebted to David V. Cruz-Uribe for inviting me to stay at the University of Alabama in the fall of 2017 and collaborating with me during that period.

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Eduard Roure Perdices
October 9, 2019

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Chapter 1

Introduction

> *If my calculations are correct, when this baby hits 88 miles per hour, you're gonna see some serious shit.*
>
> Doc Brown, *Back to the Future*, 1985

This short chapter is intended to be a brief description of our project. In Section 1.1, we include general notation and conventions. In Section 1.2, we review existing works on Rubio de Francia's extrapolation and introduce the primary goal of our study. In Section 1.3, we present our results on multi-variable mixed and restricted weak type Rubio de Francia's extrapolation. In Section 1.4, we discuss some of the results that we have obtained for the operator $M^{\otimes}$. In Section 1.5, we summarize our results on Sawyer-type inequalities for Lorentz spaces. In Section 1.6, we expose restricted weak type estimates for classical operators obtained via sparse domination techniques. In Section 1.7, we propose possible projects to extend our research further.

1.1 Notation and Conventions

The following notation is standard:

$\mathbb{N}$	the set of all natural numbers, including 0
$\mathbb{Z}$	the set of all integers
$\mathbb{Z}^n$	the n-fold product of $\mathbb{Z}$
$\mathbb{R}$	the set of all real numbers
$\mathbb{R}^n$	the n-dimensional Euclidean space
$B(x,R)$	the ball of radius R centered at x in $\mathbb{R}^n$
χ_E	the characteristic function of a set E
dt, dx, dy, dz	the Lebesgue measure
$\lvert\mu\rvert$	the total variation of a finite Borel measure μ on $\mathbb{R}^n$
$\log$	the logarithm with base e
$\log_2$	the logarithm with base 2
$\log^+$	the function $\max\{0, \log\}$
$\inf$	the infimum
$\sup$	the supremum

supp	the support
ess inf	the essential infimum
ess sup	the essential supremum
$L^\infty_c(\mathbb{R}^n)$	the space of bounded functions on $\mathbb{R}^n$ with compact support
$L^1_{loc}(\mathbb{R}^n)$	the space of locally integrable functions on $\mathbb{R}^n$
$\mathscr{C}^\infty_c(\mathbb{R}^n)$	the space of smooth functions on $\mathbb{R}^n$ with compact support
Δ	the Laplacian
∇	the gradient

In general, we will work in $\mathbb{R}^n$, with $1 \leq n \in \mathbb{N}$. Unless otherwise specified, by a *function* f we mean a real or complex-valued function on $\mathbb{R}^n$. If we say that a function f is *measurable*, but we don't specify any measure, then it is with respect to the Lebesgue measure on $\mathbb{R}^n$. The same applies to measurable sets and also to the expression a.e.; that is, *almost everywhere*.

Given a measure ν, and a ν-measurable set E, we use the notation

$$\nu(E) := \int_E d\nu.$$

If ν is the Lebesgue measure, then we simply write $|E|$. Given a measurable function f, and a measurable set E, with $|E| \neq 0$, we use the notation

$$⨍_E f := \frac{1}{|E|} \int_E f(x)dx.$$

A *cube* Q is a subset of $\mathbb{R}^n$ that admits an expression as a Cartesian product of n intervals of the same length, the *side length of* Q, denoted by ℓ_Q. If these intervals are all open, then the cube is called *open*, and if they are all closed, then the cube is called *closed*.

Given non-negative quantities $\mathscr{A}$ and $\mathscr{B}$, we write $\mathscr{A} \lesssim \mathscr{B}$ if there exists a finite constant $C > 0$, independent of $\mathscr{A}$ and $\mathscr{B}$, such that $\mathscr{A} \leq C\mathscr{B}$. If $\mathscr{A} \lesssim \mathscr{B} \lesssim \mathscr{A}$, then we write $\mathscr{A} \approx \mathscr{B}$. The constant C is called the *implicit constant*. Usually, we will denote implicit constants by $c, \widetilde{c}, C$, or $\widetilde{C}$. In many cases, they will depend on some parameters $\alpha_1, \dots, \alpha_\ell$, and if we want to point out that dependence, we shall do it using subscripts, e.g. $\mathscr{A} \lesssim_{\alpha_1,\dots,\alpha_\ell} \mathscr{B}$, or $\mathscr{A} \approx_{\alpha_1,\dots,\alpha_\ell} \mathscr{B}$, or $\mathscr{A} \leq C_{\alpha_1,\dots,\alpha_\ell}\mathscr{B}$. We shall use numerical subscripts to label different implicit constants appearing in the same argument. We write $\mathscr{A} \leq C(\alpha_1, \dots, \alpha_\ell)\mathscr{B}$ when we want to interpret C as a function of the parameters $\alpha_1, \dots, \alpha_\ell$. In these cases, we may replace C by other symbols, like $\phi, \varphi, \Phi, \psi$, or Ψ, especially when the dependence on the parameters is monotonically increasing.

Given real or complex vector spaces $X_1, \dots, X_m$, and Y, endowed with quasi-norms $\|\cdot\|_{X_1}, \dots, \|\cdot\|_{X_m}$, and $\|\cdot\|_Y$, respectively, and an operator T defined on $X_1 \times \cdots \times X_m$ and taking values in Y, we use the notation

$$T : X_1 \times \cdots \times X_m \longrightarrow Y$$

to indicate that T is a *bounded operator from* $X_1 \times \cdots \times X_m$ *to* Y; that is, there exists a finite constant $C > 0$ such that for all $f_1 \in X_1, \dots, f_m \in X_m$,

$$\|T(f_1, \dots, f_m)\|_Y \leq C \prod_{i=1}^{m} \|f_i\|_{X_i}.$$

Among all such constants C, we shall denote by $\|T\|_{\prod_{i=1}^m X_i \to Y}$ the smallest one.

We adhere to the usual convention that the *empty sum* (the sum containing no terms) is equal to zero, and the *empty product* is equal to one.

1.2 Background and Motivation

In the topic of weighted theory, a result that has attracted the attention of many researchers in the field is the so-called *Rubio de Francia's extrapolation theorem* (see [42, 100, 101]), which provides a precious shortcut when trying to prove weighted strong bounds. In its simplest form, it says that if a sublinear operator T satisfies that

$$T : L^p(v) \longrightarrow L^p(v),$$

for some $p \geq 1$, and every Muckenhoupt weight v in A_p, then

$$T : L^q(w) \longrightarrow L^q(w),$$

for every $q > 1$, and every Muckenhoupt weight w in A_q (see Subsections 2.1.1 and 2.1.3 for definitions).

Many alternative proofs of this result are available in the literature (see [28, 38]), also tracking the sharp dependence of $\|T\|_{L^q(w) \to L^q(w)}$ in terms of $[w]_{A_q}$ (see [36]), and off-diagonal results where the domain and target Lebesgue spaces differ both in terms of exponents and weights (see [37, 50] for strong type results, and [88] for weak type ones).

Also, it was discovered that the operator T plays no role in the extrapolation arguments, and one can present all the results for families of pairs of non-negative measurable functions (see [26, 32, 37]).

Around the beginning of the current millennium, the topic of multi-variable operators started gathering interest, with the resolution of Calderón's conjecture (see [60, 62]) and the presentation of a systematic treatment of multi-linear Calderón-Zygmund operators (see [49]), and the first results on multi-variable Rubio de Francia's extrapolation appeared.

In [47], it was proved that if an m-variable operator T satisfies that

$$T : L^{p_1}(v_1) \times \cdots \times L^{p_m}(v_m) \longrightarrow L^p(v_1^{p/p_1} \dots v_m^{p/p_m}),$$

for some exponents $1 \leq p_1, \dots, p_m < \infty$, with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, and all weights $v_1 \in A_{p_1}, \dots, v_m \in A_{p_m}$, then

$$T : L^{q_1}(w_1) \times \cdots \times L^{q_m}(w_m) \longrightarrow L^q(w_1^{q/q_1} \dots w_m^{q/q_m}),$$

for all exponents $1 < q_1, \dots, q_m < \infty$, with $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, and all weights $w_1 \in A_{q_1}, \dots, w_m \in A_{q_m}$.

In [37], the sharp dependence of $\|T\|_{L^{q_1}(w_1)\times\cdots\times L^{q_m}(w_m)\to L^q(w_1^{q/q_1}\dots w_m^{q/q_m})}$ in terms of $[w_i]_{A_{q_i}}$, $i = 1, \dots, m$, was established, and analogous multi-variable weak type extrapolation results were studied in [15]. Once again, the operator T plays no role, and all the results can be presented for $(m+1)$-tuples of non-negative measurable functions.

It is worth mentioning that very recently, multi-variable strong type extrapolation theorems for $A_{\vec{P}}$ weights have been obtained in [72, 73] (see also [89]), solving in the affirmative a question that has been going around for about a decade, since the publication of [69], where such weights were introduced.

Rubio de Francia's extrapolation theory provides a potent set of tools in Harmonic Analysis, but it has a weak spot; namely, it does not allow to produce estimates in the endpoint $q_1 = \cdots = q_m = 1$, which can be easily seen by considering m-variable commutators (see [69]).

In the case of one-variable extrapolation, the works of M. J. Carro, L. Grafakos, and J. Soria (see [9]), and M. J. Carro and J. Soria (see [14]), give a solution to this problem, allowing to extrapolate down to the endpoint $q_1 = 1$ assuming a slightly stronger extrapolation hypothesis. In general terms, they proved that if a sub-linear operator T satisfies that

$$T : L^{p,1}(v) \longrightarrow L^{p,\infty}(v),$$

for some $p > 1$, and every weight v in $\widehat{A}_p$, then

$$T : L^{q,\min\{1,\frac{q}{p}\}}(w) \longrightarrow L^{q,\infty}(w),$$

for every $q \geq 1$, and every weight w in $\widehat{A}_q$.

Here, for $r \geq 1$, the class $\widehat{A}_r$ contains all the weights of the form $(Mh)^{1-r}u$, where $h \in L^1_{loc}(\mathbb{R}^n)$, and $u \in A_1$. If $r = 1$, then $\widehat{A}_1 = A_1$, but for $r > 1$, $A_r \subsetneq \widehat{A}_r \subseteq A_r^{\mathcal{R}}$.

In general, the classical strong and weak type Rubio de Francia's extrapolation theorems rely on three fundamental ingredients: factorization results for A_r weights, construction of A_1 weights via the Rubio de Francia's iteration algorithm, and sharp weighted bounds for the Hardy-Littlewood maximal operator M. However, in the setting of restricted weak type Rubio de Francia's extrapolation, many technical difficulties appear. For instance, no factorization result is known for $A_r^{\mathcal{R}}$ weights, which justifies the need for the class $\widehat{A}_r$. Also, in this setting, the Rubio de Francia's iteration algorithm can not be defined and has to be carefully replaced by the Hardy-Littlewood maximal operator M in the construction of weights. Fortunately, we do have sharp weighted restricted weak type bounds for M.

The main purpose of this project is to build upon the work in [9, 14] and extend to the multi-variable setting the restricted weak type Rubio de Francia's extrapolation results presented there.

1.3 Our Extrapolation and its Applications

The first result that we were able to prove, presented in Theorem 3.2.1, allows us to extrapolate down to the endpoint $(1, 1, \frac{1}{2})$ from a diagonal estimate. In general terms, if a two-variable operator T satisfies that

$$T : L^{r,1}(v_1) \times L^{r,1}(v_2) \longrightarrow L^{\frac{r}{2},\infty}(v_1^{1/2}v_2^{1/2}),$$

for some exponent $1 < r < \infty$, and all weights $v_1, v_2 \in \widehat{A}_r$, then

$$T : L^{1,\frac{1}{r}}(w_1) \times L^{1,\frac{1}{r}}(w_2) \longrightarrow L^{\frac{1}{2},\infty}(w_1^{1/2}w_2^{1/2}),$$

for all weights $w_1, w_2 \in A_1$. The crucial point in the proof of this theorem is the endpoint estimate

$$M^{\otimes} : L^1(w_1) \times L^1(w_2) \longrightarrow L^{\frac{1}{2},\infty}(w_1^{1/2}w_2^{1/2}), \tag{1.3.1}$$

proved in [69], and refined in Theorem 2.4.1.

Here, the operator $M^{\otimes}$ is defined for locally integrable functions f_1 and f_2 by

$$M^{\otimes}(f_1, f_2)(x) := Mf_1(x)Mf_2(x), \quad x \in \mathbb{R}^n,$$

where M is the *Hardy-Littlewood maximal operator*, defined for functions $f \in L^1_{loc}(\mathbb{R}^n)$ by

$$Mf(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|dy, \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all cubes $Q \subseteq \mathbb{R}^n$ containing x.

The approach to establishing general downwards extrapolation results is now evident: find some auxiliary operator $\mathscr{X}$ for which we can prove mixed and restricted weak type inequalities, and use the extrapolation hypotheses to transfer such bounds to the generic operator T. The operator $\mathscr{X}$ plays the same role as the Hardy-Littlewood maximal operator plays in the one-variable restricted weak type extrapolation theory of Rubio de Francia.

As it turns out, sometimes we can take $M^{\otimes}$ to be our auxiliary operator (see Theorem 3.2.4, Lemma 3.2.6, Theorem 3.2.7, Theorem 3.3.1, and Theorem 3.3.2). Therefore, the study of restricted weak type bounds for $M^{\otimes}$ becomes a fundamental and interesting question in this project. Moreover, our preliminary mixed type inequalities for $M^{\otimes}$ in Theorem 2.2.10 encouraged us to develop the multi-variable mixed type extrapolation theory of Rubio de Francia presented in Section 3.3.

After a detailed analysis of the proof of (1.3.1) in [69], and taking into account Lemma 3.2.6, we conclude that the complete solution to multi-variable mixed and restricted weak type extrapolation, along with the corresponding bounds for $M^{\otimes}$, relies on the development of weighted inequalities for operators of the form

$$\mathscr{Z} f = \frac{Mf}{W}$$

on Lorentz spaces, being W some nice weight. This discovery forced us into developing our theory of Sawyer-type inequalities for Lorentz spaces, displayed in Sections 2.3 and 2.4.

As a consequence of such results, we can obtain our main mixed type extrapolation schemes, discussed in Theorems 3.3.27, 3.3.31 and 3.3.35. Ignoring some technicalities, what we have is that if a two-variable operator T satisfies that

$$T : L^{p_1}(v_1) \times L^{p_2,1}(v_2) \longrightarrow L^{p,\infty}(v_1^{p/p_1} v_2^{p/p_2}),$$

for some exponents $1 < p_1, p_2 < \infty$, with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and all weights $v_1 \in A_{p_1}$ and $v_2 \in A_{p_2}^{\mathcal{R}}$, then

$$T : L^{q_1,\min\{p_1,q_1\}}(w_1) \times L^{q_2,\min\{1,\frac{q_2}{p_2}\}}(w_2) \longrightarrow L^{q,\infty}(w_1^{q/q_1} w_2^{q/q_2}),$$

for all exponents $q_1 > 1$ and $q_2 \geq 1$, with $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, and all weights $w_1 \in A_{q_1}$ and $w_2 \in \widehat{A}_{q_2}$.

Let us point out that in the mixed type setting, and when working with A_r weights, we can either follow the classical approach, using the Rubio de Francia's iteration algorithm, or our new strategy, with Sawyer-type inequalities, to run the extrapolation arguments, with the first option leading to better constants than the second one, but in the restricted weak type setting, the first option is not available, and we have no choice but to use Sawyer-type inequalities.

Further exploiting the Sawyer-type inequality in Theorem 2.3.8, its dual version in Theorem 2.4.12, and the ideas introduced in Section 3.2, in Chapter 4 we manage to produce the general multi-variable restricted weak type extrapolation scheme that we were seeking, fulfilling the original goal of our project. In general terms, and combining Theorems 4.2.2 and 4.2.7, we get that if an m-variable operator T satisfies that

$$T : L^{p_1,1}(v_1) \times \cdots \times L^{p_m,1}(v_m) \longrightarrow L^{p,\infty}(v_1^{p/p_1} \dots v_m^{p/p_m}),$$

for some exponents $1 < p_1, \dots, p_m < \infty$, with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, and all weights $v_1 \in \widehat{A}_{p_1,\infty}, \dots, v_m \in \widehat{A}_{p_m,\infty}$, then

$$T : L^{q_1,\min\{1,\frac{q_1}{p_1}\}}(w_1) \times \cdots \times L^{q_m,\min\{1,\frac{q_m}{p_m}\}}(w_m) \longrightarrow L^{q,\infty}(w_1^{q/q_1} \dots w_m^{q/q_m}),$$

for all exponents $1 \leq q_1, \dots, q_m < \infty$, with $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, and all weights $w_1 \in \widehat{A}_{q_1,\infty}, \dots, w_m \in \widehat{A}_{q_m,\infty}$ satisfying certain technical hypotheses.

Here, for $r \geq 1$, the class $\widehat{A}_{r,\infty}$ is an extension of $\widehat{A}_r$, satisfying that $\widehat{A}_r \subseteq \widehat{A}_{r,\infty} \subseteq A_r^{\mathcal{R}}$. In practice, its use allows us to apply our extrapolation schemes iteratively under suitable conditions.

Inspired by [37], in Subsection 4.2.3 we also produce the corresponding one-variable off-diagonal restricted weak type extrapolation results that can be used to derive our multi-variable extrapolation theorems.

For simplicity, in Chapter 3 we decided to work on two-variable extrapolation results. In the end, the extension from the two-variable setting to the multi-variable one is just a matter of notation, as we see in Chapter 4.

As usual, the operator T plays no role, and we also present our extrapolation schemes for tuples of measurable functions.

For technical reasons, in all our extrapolation theorems, we require the constants in each bound to depend increasingly on the constants of the weights involved. This hypothesis may seem restrictive at first, but as it was pointed out in [36, Footnote 3], it turns out that it is not, since sharp constants are this way.

Note that when studying mixed and restricted weak type bounds for multi-variable operators, the Lorentz spaces that we consider have first exponents of the form $1 \leq r_1, \dots, r_m < \infty$, and r such that $\frac{1}{r} = \frac{1}{r_1} + \cdots + \frac{1}{r_m}$. Hence, we can identify each choice of exponents $r_1, \dots, r_m$ with the point $(\frac{1}{r_1}, \dots, \frac{1}{r_m})$ in the *space of parameters* $(0,1]^m$.

A relevant region inside this m-cube is the so-called *Banach-range* (see Figure 1.1),

$$\mathfrak{B}_m := \{(x_1, \dots, x_m) \in (0,1]^m : x_1 + \cdots + x_m < 1\},$$

where the corresponding values of the exponent r are strictly bigger than one, and hence, $L^{r,\infty}(v)$ is a Banach space, being v a weight. In particular, duality is available (see Subsection 2.1.2).

Duality has proved to be a powerful tool in the study of weighted inequalities for classical operators, especially when combined with sparse domination techniques, so working with Lorentz spaces where duality is not available is a problem in practice. This problem gets worse as we increase the number of variables m, since the Banach-range shrinks fast. In fact, one can check that

$$|\mathfrak{B}_m| = \frac{1}{m!}.$$

This lack of duality can sometimes be circumvented by wisely using Kolmogorov's inequalities (see Chapter 5), but this is not always the case, and that's when our extrapolation techniques kick in. We can prove bounds in the Banach-range by hand, and then effectively extend them outside this range via a multi-variable mixed or restricted weak type extrapolation argument.

In particular, our extrapolation schemes are handy for overcoming two fundamental problems of weak Lebesgue spaces $L^{r,\infty}(v)$ with $0 < r \leq 1$, strongly related with the lack of duality: the lack of Hölder-type inequalities with the change of measures, and the lack of Minkowski's integral inequality.

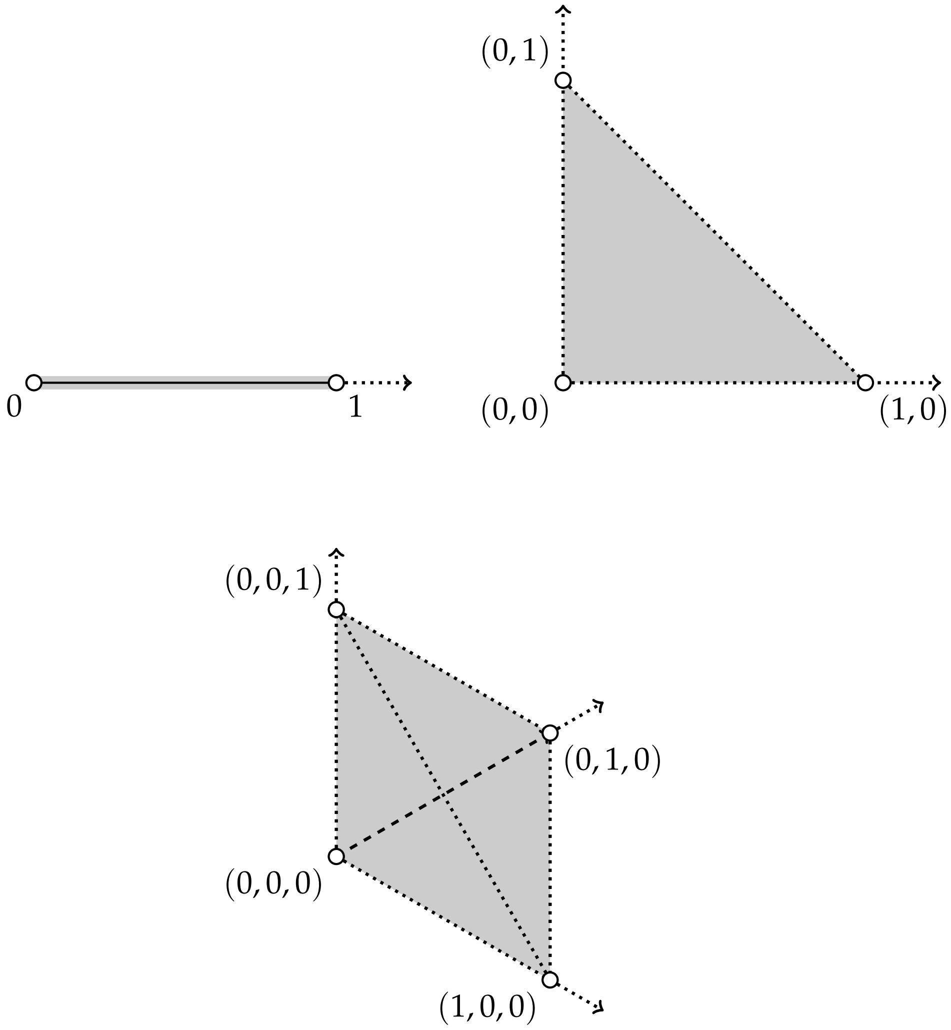

FIGURE 1.1: Pictorial representation of the Banach-range for one, two, and three variables.

The first problem becomes an obstacle when working with product-type operators. Nevertheless, using our Hölder-type inequalities from Subsection 2.2.1, we can obtain bounds for such operators in the Banach-range, and then apply an extrapolation argument to extend them past such range of exponents. For the exact details, see Proposition 3.4.1, Theorem 3.4.2, and Theorem 4.3.1. These arguments also apply to some two-variable commutators, as shown in Theorem 3.4.11.

The second problem is an impediment when trying to produce bounds for averaging operators. In this case, the strategy is to prove bounds in the Banach-range using Minkowski's integral inequality and then extrapolate outside this range, as we see in Theorem 3.4.4 and Theorem 4.3.2.

As a particular case, we started working with bi-linear multipliers of the form

$$T_m(f,g)(x) := \int_{\mathbb{R}} \int_{\mathbb{R}} m(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x(\xi+\eta)}d\xi d\eta,$$

initially defined for Schwartz functions f and g, and $x \in \mathbb{R}$. The study of such bi-linear multiplier operators was initiated by R. Coifman and Y. Meyer (see [18, 19]). In recent years, the interest in this area has increased, following the works by M. Lacey and C. Thiele on the bi-linear Hilbert transform and Calderón's conjecture (see [61, 62, 63]). For more information and results on bi-linear multipliers and related topics, see [41, 45, 46, 49, 57, 78, 87].

We found that, for nice symbols m, it is possible to write T_m as an averaging operator of products of modulated and translated Hilbert transforms, and hence, we are able to deduce mixed type bounds for these operators using our multi-variable extrapolation tools, combined with bounds on weighted Lorentz spaces for the point-wise product of two Hilbert transforms. See Theorem 3.4.7 for the details.

We are currently preparing manuscripts to publish our extrapolation material (see [11, 12]).

1.4 The Operator $M^{\otimes}$

Due to its close relation with multi-variable extrapolation, one of our goals in this investigation is to study weighted estimates for the *m-fold product of Hardy-Littlewood maximal operators*, defined for locally integrable functions $f_1, \dots, f_m$ by

$$M^{\otimes}(f_1, \dots, f_m)(x) := Mf_1(x) \cdots Mf_m(x), \quad x \in \mathbb{R}^n.$$

This operator is classical and has been of great use to obtain weighted bounds for several types of multi-variable operators, like the *bi-linear Hardy-Littlewood maximal operator*, which was introduced by A. Calderón in 1964, and it is defined by

$$\overline{M}(f,g)(x) := \sup_{r>0} \frac{1}{|B(0,r)|} \int_{B(0,r)} |f(x-y)g(x+y)|dy, \quad x \in \mathbb{R}^n.$$

In virtue of Hölder's inequality, we have that

$$\overline{M}(f,g) \lesssim M(f^{1/\theta})^{\theta} M(g^{\frac{1}{1-\theta}})^{1-\theta},$$

for every $0 < \theta < 1$, and hence,

$$\overline{M} : L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \longrightarrow L^{p}(\mathbb{R}^n), \tag{1.4.1}$$

whenever $p_1 > \frac{1}{\theta}, p_2 > \frac{1}{1-\theta}$, and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and A. Calderón conjectured that for $\theta = \frac{1}{2}$,

$$\overline{M} : L^{2}(\mathbb{R}^n) \times L^{2}(\mathbb{R}^n) \longrightarrow L^{1}(\mathbb{R}^n).$$

This conjecture was proved by M. Lacey in [60], establishing the unexpected fact that (1.4.1) also holds if $p_1, p_2 > 1$ are such that $\frac{2}{3} < p \leq 1$.

Similarly, weighted estimates for the operator $M^{\otimes}$ will imply weighted estimates for $\overline{M}$. Using Hölder's inequality, one can obtain that

$$M^{\otimes} : L^{p_1}(w_1) \times L^{p_2}(w_2) \longrightarrow L^p(w_1^{p/p_1} w_2^{p/p_2}), \tag{1.4.2}$$

for $p_1, p_2 > 1$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $w_1 \in A_{p_1}$, and $w_2 \in A_{p_2}$. Moreover,

$$\overline{M} : L^{p_1}(w_1) \times L^{p_2}(w_2) \longrightarrow L^p(w_1^{p/p_1} w_2^{p/p_2}),$$

whenever $p_1 > \frac{1}{\theta}$, $p_2 > \frac{1}{1-\theta}$, $w_1 \in A_{\theta p_1}$, and $w_2 \in A_{(1-\theta)p_2}$. It is worth mentioning that much more delicate weighted estimates for the bi-linear Hilbert transform have been recently obtained in [34].

Consider now a multi-linear Calderón-Zygmund operator T (see Subsection 2.1.6), and let T_* be its *maximal truncated operator*, defined by

$$\begin{aligned} &T_*(f_1, \dots, f_m)(x) \\ &= \sup_{\delta > 0} \left| \int_{\{|x-y_1|^2 + \cdots + |x-y_m|^2 > \delta^2\}} K(x, y_1, \dots, y_m) f_1(y_1) \dots f_m(y_m) dy_1 \dots dy_m \right|. \end{aligned}$$

Then, L. Grafakos and R. H. Torres proved in [48] the following *multi-variable Cotlar's inequality*: for every $\eta > 0$, there exist constants $C_\eta, C_T > 0$ such that for every $\vec{f} = (f_1, \dots, f_m)$ in any product of Lebesgue spaces $L^{q_i}(\mathbb{R}^n)$, with $1 \le q_i < \infty$, the inequality

$$T_*(f_1, \dots, f_m)(x) \le C_\eta \left(M(|T(f_1, \dots, f_m)|^\eta)(x)^{1/\eta} + C_T M^{\otimes}(f_1, \dots, f_m)(x) \right)$$

holds for every $x \in \mathbb{R}^n$. As a consequence, one can deduce weighted estimates for T_* by proving weighted bounds for the easier operators T and $M^{\otimes}$.

In this setting of multi-linear Carderón-Zygmund operators, many other results have been proved where the role of the operator $M^{\otimes}$ is fundamental (see, for example, [78, 94]).

Concerning weighted bounds for $M^{\otimes}$, the easy estimate in (1.4.2) becomes much more difficult when we want to characterize the weights for which

$$M^{\otimes} : L^{p_1,1}(w_1) \times L^{p_2,1}(w_2) \longrightarrow L^{p,\infty}(w_1^{p/p_1} w_2^{p/p_2}). \tag{1.4.3}$$

Obviously, if Hölder's inequality for Lorentz spaces with the change of measures holds; that is, if for $0 < p_1, p_2 < \infty$, and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and all weights w_1 and w_2, there exists a constant $C > 0$ such that for all measurable functions f and g,

$$\|fg\|_{L^{p,\infty}(w_1^{p/p_1} w_2^{p/p_2})} \le C \|f\|_{L^{p_1,\infty}(w_1)} \|g\|_{L^{p_2,\infty}(w_2)}, \tag{1.4.4}$$

as it happens with the Lebesgue spaces, then for all weights $w_1 \in A_{p_1}^{\mathcal{R}}$ and $w_2 \in A_{p_2}^{\mathcal{R}}$,

$$\begin{aligned}\|M^{\otimes}(f,g)\|_{L^{p,\infty}(w_1^{p/p_1} w_2^{p/p_2})} &\lesssim \|Mf\|_{L^{p_1,\infty}(w_1)} \|Mg\|_{L^{p_2,\infty}(w_2)} \\ &\lesssim \|f\|_{L^{p_1,1}(w_1)} \|g\|_{L^{p_2,1}(w_2)},\end{aligned}$$

as we expect. This is what happens in the particular case when all the weights are equal. Note that (1.4.4) is trivially true if $p_1 = \infty > p_2$, or $p_2 = \infty > p_1$.

However, we will see in Subsection 2.2.1 that (1.4.4) does not hold for arbitrary weights. Nevertheless, we will be able to prove new Hölder-type inequalities powerful enough to produce alternative characterizations of the classes of weights A_p and $A_p^{\mathcal{R}}$, adapted to the operator $M^{\otimes}$ (see Subsection 2.2.2), yielding necessary conditions to have strong, weak, mixed, and restricted weak type bounds of $M^{\otimes}$ for A_∞ weights (see Subsection 2.2.3). In particular, given $1 \leq p_1, \dots, p_m < \infty$, and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, and weights $w_1, \dots, w_m \in A_\infty$, and $w = w_1^{p/p_1} \dots w_m^{p/p_m}$, if

$$M^{\otimes} : L^{p_1}(w_1) \times \cdots \times L^{p_\ell}(w_\ell) \times L^{p_{\ell+1},1}(w_{\ell+1}) \times \cdots \times L^{p_m,1}(w_m) \longrightarrow L^{p,\infty}(w),$$

with $0 \leq \ell \leq m$, then $w_i \in A_{p_i}$, for $i = 1, \dots, \ell$, and $w_i \in A_{p_i}^{\mathcal{R}}$, for $i = \ell + 1, \dots, m$, and if

$$M^{\otimes} : L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m) \longrightarrow L^{p}(w),$$

then $w_i \in A_{p_i}$, for $i = 1, \dots, m$.

Surprisingly, we couldn't find in the literature any reference to these natural questions about necessary conditions to establish weighted bounds for $M^{\otimes}$, apart from the case when $m = 1$, which corresponds to the Hardy-Littlewood maximal operator M (see [17, 58, 85]). We published our work on $M^{\otimes}$ in [13].

The study of the converse of such results for $M^{\otimes}$ relies on the development of new Sawyer-type inequalities for Lorentz spaces and weights in $A_p^{\mathcal{R}}$ (see Theorem 2.4.1).

1.5 Sawyer-Type Inequalities

"Sawyer-type inequalities" is a terminology coined in the paper [27], where their authors prove that if $u \in A_1$, and $v \in A_1$ or $uv \in A_\infty$, then

$$uv\left(\left\{x \in \mathbb{R}^n : \frac{|T(fv)(x)|}{v(x)} > t\right\}\right) \leq \frac{C}{t} \int_{\mathbb{R}^n} |f(x)| u(x) v(x) dx, \quad t > 0, \tag{1.5.1}$$

where T is either the Hardy-Littlewood maximal operator or a linear Calderón-Zygmund operator. This result extends some questions previously considered by B. Muckenhoupt and R. Wheeden in [86], and solves in the affirmative a conjecture formulated by E. Sawyer in [103], concerning the Hilbert transform.

These problems were advertised by B. Muckenhoupt in [84], where the terminology "mixed type norm inequalities" was introduced and was also used since then in other papers like [2] or [80]. In general, this terminology refers to certain weighted estimates for some classical operators T, where a weight v is included in their level sets; that is,

$$\left\{x \in \mathbb{R}^n : \frac{|Tf(x)|}{v(x)} > t\right\}, \quad t > 0. \tag{1.5.2}$$

The structure of such sets makes impossible, or very difficult, to use classical tools to measure them, such as the Vitali's covering lemma or interpolation theorems.

In Chapter 2, we consider *mixed restricted weak type norm inequalities*, or *Sawyer-type inequalities for Lorentz spaces*; that is, we study estimates of the form

$$w\left(\left\{x \in \mathbb{R}^n : \frac{|Tf(x)|}{v(x)} > t\right\}\right)^{1/p} \leq \frac{C}{t}\|f\|_{L^{p,1}(u)}, \quad t > 0, \tag{1.5.3}$$

where $p \geq 1$, T is a classical operator, and u, v, w are weights. We also consider extensions of such inequalities to the multi-variable setting. Our goal is to prove estimates like (1.5.3) for sub-linear and multi-sub-linear maximal operators, and multi-linear Calderón-Zygmund operators.

Observe that in the classical situation, namely when $u = w$, and $v \approx 1$, and T is either the Hardy-Littlewood maximal operator or a linear Calderón-Zygmund operator, the inequality (1.5.3) holds if $w \in A_p^{\mathcal{R}}$ (some authors use the notation $A_{p,1}$ for this class of weights, as in [17]). The case when $v \not\approx 1$ is much more difficult, and in this work, we will study it in great detail.

Our primary motivation to consider Sawyer-type inequalities for Lorentz spaces comes from the study of the m-fold product of Hardy-Littlewood maximal operators, $M^{\otimes}$. As we will see in Theorem 2.2.8, published in [13], given exponents $1 \leq p_1, \dots, p_m < \infty$, and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, and weights $w_1, \dots, w_m$ in A_∞, and $w = \nu_{\vec{w}} = w_1^{p/p_1} \dots w_m^{p/p_m}$, a necessary condition to have

$$M^{\otimes} : L^{p_1,1}(w_1) \times \cdots \times L^{p_m,1}(w_m) \longrightarrow L^{p,\infty}(w) \tag{1.5.4}$$

is that $w_i \in A_{p_i}^{\mathcal{R}}$, for $i = 1, \dots, m$. It is reasonable to think that this last condition is also sufficient for (1.5.4) to hold, since the endpoint case was proved in [69]; that is, for weights $w_1, \dots, w_m \in A_1$, we have that

$$M^{\otimes} : L^1(w_1) \times \cdots \times L^1(w_m) \longrightarrow L^{\frac{1}{m},\infty}(w_1^{1/m} \dots w_m^{1/m}). \tag{1.5.5}$$

To prove this result, one has to control the following quantity for $t > 0$, which is related to the level sets in (1.5.2):

$$w\left(\left\{M^{\otimes}(\vec{f}) > t\right\}\right) = w\left(\left\{Mf_i > \frac{t}{\prod_{j \neq i} Mf_j}\right\}\right).$$

This is achieved by applying the classical Sawyer-type inequality (1.5.1) for the Hardy-Littlewood maximal operator M in combination with the observation that for locally integrable functions $h_1, \dots, h_k$, $\prod_{j=1}^k (Mh_j)^{-1} \in RH_\infty$, with constant depending only on k and the dimension n.

As we will show in Theorem 2.4.1, it turns out that the bound (1.5.4) holds if $w_i \in A_{p_i}^{\mathcal{R}}$, for $i = 1, \dots, m$, solving in the affirmative the open question in [13] and completing the characterization of the restricted weak type bounds of $M^{\otimes}$ for A_∞ weights. We also obtain the analogous characterizations of strong, weak, and mixed type bounds of $M^{\otimes}$. The strategy that we follow is similar to the one in [69] for the endpoint case (1.5.5), but we have to replace the classical Sawyer-type inequality (1.5.1) by the estimate obtained in Theorem 2.3.8, which is a new restricted weak Sawyer-type inequality involving the class of weights $A_p^{\mathcal{R}}$; that is,

$$\left\| \frac{Mf}{v} \right\|_{L^{p,\infty}(uv^p)} \leq C_{u,v} \|f\|_{L^{p,1}(u)}, \tag{1.5.6}$$

for $p > 1$, $u \in A_p^{\mathcal{R}}$, and $uv^p \in A_\infty$. The $A_p^{\mathcal{R}}$ condition on the weight u is a natural assumption since it is necessary when $v \approx 1$. In Lemma 2.3.10 we also manage to track the dependence of the constant $C_{u,v}$ on the weights u and uv^p, even in the endpoint case $p = 1$, refining the bound (1.5.1) in [27].

Quite recently, the bound (1.5.1) has been extended to the multi-variable setting in [75]. More precisely, for weights $w_1, \dots, w_m \in A_1$, and $v \in A_\infty$,

$$\left\| \frac{\mathcal{M}(\vec{f})}{v} \right\|_{L^{\frac{1}{m},\infty}(\nu_{\vec{w}} v^{1/m})} \leq \left\| \frac{\prod_{i=1}^m Mf_i}{v} \right\|_{L^{\frac{1}{m},\infty}(\nu_{\vec{w}} v^{1/m})} \lesssim \prod_{i=1}^m \|f_i\|_{L^1(w_i)}. \tag{1.5.7}$$

Inspired by this result, we follow a similar approach to extend our Sawyer-type inequality (1.5.6) to the multi-variable setting, obtaining a generalization of (1.5.7) in Theorem 2.4.6. That is, for weights $w_1, \dots, w_m$ and v such that for $i = 1, \dots, m$, $w_i \in A_{p_i}^{\mathcal{R}}$ and $w_i v^{p_i} \in A_\infty$,

$$\left\| \frac{\mathcal{M}(\vec{f})}{v} \right\|_{L^{p,\infty}(\nu_{\vec{w}} v^p)} \leq \left\| \frac{\prod_{i=1}^m Mf_i}{v} \right\|_{L^{p,\infty}(\nu_{\vec{w}} v^p)} \lesssim \prod_{i=1}^m \|f_i\|_{L^{p_i,1}(w_i)}. \tag{1.5.8}$$

Observe that this result is an extension of (1.5.4). To our knowledge, this multi-variable mixed restricted weak type inequalities for maximal operators involving the $A_p^{\mathcal{R}}$ condition on the weights have not been previously studied, and we found no record of them being conjectured in the literature.

Motivated by the conjecture of E. Sawyer in [103], we can ask ourselves if it is possible to obtain bounds like (1.5.8) for multi-linear Calderón-Zygmund operators T. Once again, the endpoint case $p_1 = \cdots = p_m = 1$ has already been considered and extensively investigated in [75]. There, it was shown that for weights $w_1, \dots, w_m \in A_1$, and $\nu_{\vec{w}} v^{1/m} \in A_\infty$,

$$\left\| \frac{T(\vec{f})}{v} \right\|_{L^{\frac{1}{m},\infty}(\nu_{\vec{w}} v^{1/m})} \lesssim \prod_{i=1}^{m} \|f_i\|_{L^1(w_i)}, \tag{1.5.9}$$

as a corollary of (1.5.7), combined with a result in [90], that allows replacing $\mathcal{M}$ by T using an extrapolation type argument based on the A_∞ extrapolation theorem obtained in [32, 35].

We succeed in our goal and manage to obtain an extension of (1.5.9) to the general restricted weak type setting. In Theorem 2.4.10 we prove, among other things, that for weights $w_1, \dots, w_m$ and v such that for $i = 1, \dots, m$, $w_i \in A_{p_i}^{\mathcal{R}}$ and $w_i v^{p_i} \in A_\infty$, and some other technical hypotheses on the weights,

$$\left\| \frac{T(\vec{f})}{v} \right\|_{L^{p,\infty}(\nu_{\vec{w}} v^{p})} \lesssim \prod_{i=1}^{m} \|f_i\|_{L^{p_i,1}(w_i)}. \tag{1.5.10}$$

To achieve this, we build upon (1.5.8), but unlike in [75], we manage to avoid the use of extrapolation arguments like the ones in [90]. Instead, we present in Theorem 2.4.8 a novel technique that allows us to replace $\mathcal{M}$ by T exploiting the fine structure of the Lorentz space $L^{p,\infty}(\nu_{\vec{w}} v^p)$, the $A_p^{\mathcal{R}}$ condition, and the recent advances in sparse domination. It is worth mentioning that we couldn't find in the literature any trace of results like (1.5.10), involving multi-linear Calderón-Zygmund operators, $A_p^{\mathcal{R}}$ weights, and mixed restricted weak type inequalities.

It is curious that we didn't find much about Sawyer-type inequalities for Lorentz spaces apart from the endpoint results studied in [27, 74, 75, 86, 90, 103], some results for commutators in [6, 7], and some endpoint estimates for multi-variable fractional operators in [95]. As we have seen before, these inequalities are fundamental to understand the behavior of the operator $M^{\otimes}$, but they appear naturally in the study of other classical operators, even in the one-variable case. Consider, for example, the case of the Hilbert transform H. Indeed, if $p > 1$ and $w \in A_p^{\mathcal{R}}$, it is well known that $H : L^{p,1}(w) \longrightarrow L^{p,\infty}(w)$. Hence, duality, linearity and self-adjointness of H yield

$$\left\| \frac{H(fw)}{w} \right\|_{L^{p',\infty}(w)} \leq C_w \|f\|_{L^{p',1}(w)}.$$

This is an example of an estimate like (1.5.3) involving the $A_p^{\mathcal{R}}$ condition on the weights and obtained almost without effort. The same inequality holds for the Hardy-Littlewood maximal operator M, but we cannot use the same argument, as shown in [14]. In Theorem 2.4.12 we will generalize such result for M, obtaining as a particular case an alternative proof of the result in [14]. In [54, 68], one can find similar endpoint estimates for Calderón-Zygmund operators, with $p' = 1$ and $w \in A_1$.

As we will see in Chapters 3 and 4, Sawyer-type inequalities for Lorentz spaces play a fundamental role in the proofs of our multi-variable extrapolation schemes.

For convenience, we made our work on Sawyer-type inequalities available online (see [93]).

1.6 Sparse Domination and Restricted Weak Type Bounds

During a stay at the University of Alabama in 2017, we started a collaboration with David V. Cruz-Uribe. Our goal was to produce bounds for fractional integral operators, Calderón-Zygmund singular integral operators, and commutators of these operators in the context of Lorentz spaces with restricted Muckenhoupt $A_p^{\mathcal{R}}$ weights, exploiting recent sparse domination techniques presented in [21, 23, 65, 66, 70, 71].

We got satisfactory results on this matter, presented in Chapter 5. We highlight the characterization of the tuples of weights $(w_1, \dots, w_m, v)$ for which the multi-variable fractional operators $\mathcal{M}_\alpha$ and $\mathcal{I}_\alpha$ satisfy the bounds

$$\mathcal{M}_\alpha, \mathcal{I}_\alpha : L^{p_1,1}(w_1) \times \cdots \times L^{p_m,1}(w_m) \longrightarrow L^{q,\infty}(v),$$

with $0 \leq \alpha < nm$, $1 \leq p_1, \dots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, and $p \leq q$. For more details, see Theorem 5.2.2, Theorem 5.2.6, and Corollary 5.2.10. For more information about these operators, and bounds for them, see [59, 82].

In particular, in Theorem 5.2.7 we obtain a complete characterization of the restricted weak type bounds for the multi-sub-linear maximal operator $\mathcal{M}$ introduced in [69], along with the corresponding estimates for multi-variable sparse operators and multi-linear Calderón-Zygmund operators.

In the case of linear commutators of fractional integrals I_α, and linear Calderón-Zygmund operators T, we establish two-weight restricted weak type bounds

$$[b, T], [b, I_\alpha] : L^{p,1}(w) \longrightarrow L^{q,\infty}(v),$$

with $1 < q$, $1 \leq p \leq q$, $0 < \alpha < n$, and $b \in BMO$, working with pairs of weights (w, v) satisfying some logarithmic bump conditions, as shown in Theorem 5.3.9 and Theorem 5.3.11. For strong and weak type bounds for these commutators, see [21, 23].

From the bounds for the operator $\mathcal{I}_\alpha$, in Theorem 5.4.1 and Theorem 5.4.2, we can obtain Poincaré and Sobolev-type inequalities for products of functions, following the approach in [82].

Motivated by novel works of K. Moen [81], and C. Hoang and K. Moen [53], recently we started extending our results for commutators to the multi-variable setting, with very promising expectations (see [33]).

1.7 Further Research

In what follows, we briefly describe some potential future research projects that one could investigate, apart from the various questions that we have left open in the following chapters of this document.

(a) Let $T_1, \dots, T_m$ be one-variable operators defined for measurable functions. Fix exponents $1 \leq p_1, \dots, p_m < \infty$, with $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$, and weights $w_1, \dots, w_m$, with $w = w_1^{p/p_1} \dots w_m^{p/p_m}$, and suppose that for $i = 1, \dots, m$,

$$T_i : L^{p_i,1}(w_i) \longrightarrow L^{p_i,\infty}(w_i).$$

Study the existence of a non-trivial constant $C = C(w_1, \dots, w_m) > 0$ such that for all measurable functions $f_1, \dots, f_m$,

$$\|T_1 f_1 \dots T_m f_m\|_{L^{p,\infty}(w)} \leq C \prod_{i=1}^{m} \|f_i\|_{L^{p_i,1}(w_i)}.$$

A particular case of interest is when $T_1 = \dots = T_m = H$, the Hilbert transform on $\mathbb{R}$.

(b) Fix $m \geq 1$. Given measurable functions $f_1, \dots, f_m$, and g, suppose that for all weights $u_1, \dots, u_m \in A_1$,

$$\|g\|_{L^{\frac{1}{m},\infty}(u_1^{1/m} \dots u_m^{1/m})} \leq \varphi([u_1]_{A_1}, \dots, [u_m]_{A_1}) \prod_{i=1}^{m} \|f_i\|_{L^1(u_i)},$$

where $\varphi : [1,\infty)^m \longrightarrow [0,\infty)$ is a function increasing in each variable. Given exponents $1 \leq q_1, \dots, q_m < \infty$, and $\frac{1}{q} = \frac{1}{q_1} + \dots + \frac{1}{q_m}$, and weights $w_i \in \widehat{A}_{q_i,\infty}$, $i = 1, \dots, m$, study the existence of a non-trivial constant $C_\varphi = C_\varphi(w_1, \dots, w_m) > 0$ such that

$$\|g\|_{L^{q,\infty}(w_1^{q/q_1} \dots w_m^{q/q_m})} \leq C_\varphi \prod_{i=1}^{m} \|f_i\|_{L^{q_i,1}(w_i)}.$$

(c) Prove multi-variable Sawyer-type inequalities for Lorentz spaces with tuples of weights in $A_{\vec{P}}^{\mathcal{R}}$, as it was done in [75] for weights in $A_{\vec{1}}$.

(d) Establish multi-variable restricted weak type extrapolation results for tuples of weights in $A_{\vec{P}}^{\mathcal{R}}$, analogous to the results obtained in [72, 73, 89] for weights in $A_{\vec{P}}$.

Chapter 2

Hardy, Littlewood, Lorentz, Hölder, and Sawyer

> *No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world.*
>
> Godfrey Harold Hardy, *A Mathematician's Apology*, 1941

We devote this chapter to the study of new Hölder-type and Sawyer-type inequalities for Lorentz spaces with weights in $A_p^{\mathcal{R}}$. In Section 2.1, we provide general information about Lebesgue and Lorentz spaces, classical Hölder's inequalities, common classes of weights, types of bounds, dyadic grids and sparse collections of cubes, multi-linear Calderón-Zygmund operators, and commutators. In Section 2.2, we present our Hölder-type inequalities for Lorentz spaces, along with alternative characterizations of A_p and $A_p^{\mathcal{R}}$, and necessary conditions to obtain strong, weak, mixed, and restricted weak type bounds of $M^{\otimes}$ for A_∞ weights. In Section 2.3, we discuss our Sawyer-type inequalities involving the Hardy-Littlewood maximal operator. In Section 2.4, we give applications of our Sawyer-type results for M, including weak, mixed, and restricted weak type bounds for $M^{\otimes}$, multi-variable Sawyer-type inequalities for classical operators, and a dual Sawyer-type inequality for M.

2.1 General Preliminaries

In this section, we introduce some basic concepts that we will use throughout this document. This introduction is not intended to be exhaustive.

2.1.1 Lebesgue and Lorentz spaces

We include a brief exposition about Lebesgue and Lorentz spaces, containing definitions and well-known properties. For a detailed discussion, see [4, 44].

Given $0 < p < \infty$, and a σ-finite measure space (X, ν), $L^p(X, \nu)$ is the set of ν-measurable functions f on X such that

$$\|f\|_{L^p(X,\nu)} := \left(\int_X |f|^p d\nu \right)^{1/p} < \infty,$$

and $L^\infty(X, \nu)$ is the set of ν-measurable functions f on X such that

$$\|f\|_{L^\infty(X,\nu)} := \nu\text{-}\operatorname*{ess\,sup}_{x \in X} |f(x)| = \inf\{C > 0 : \nu(\{x \in X : |f(x)| > C\}) = 0\} < \infty.$$

If $1 \le p \le \infty$, then we have Minkowski's inequality, or triangle inequality; that is, for all ν-measurable functions $f, g \in L^p(X, \nu)$,

$$\|f + g\|_{L^p(X,\nu)} \le \|f\|_{L^p(X,\nu)} + \|g\|_{L^p(X,\nu)}.$$

In general, such inequality fails for $0 < p < 1$, but one can fix this issue by multiplying the right-hand side by a suitable constant, like $2^{\frac{1-p}{p}}$.

The *Lebesgue space* $L^p(X, \nu)$ is a Banach space for $1 \le p \le \infty$, and a quasi-Banach space for $0 < p < 1$.

Given $0 < p, q < \infty$, and a ν-measurable function f on X, define

$$\|f\|_{L^{p,q}(X,\nu)} := \left(p \int_0^\infty y^q \lambda_f^\nu(y)^{q/p} \frac{dy}{y} \right)^{1/q},$$

and for $q = \infty$, define

$$\|f\|_{L^{p,\infty}(X,\nu)} := \sup_{y>0} y \lambda_f^\nu(y)^{1/p},$$

where λ_f^ν is the *distribution function of f with respect to ν*, defined on $[0, \infty)$ by

$$\lambda_f^\nu(y) := \nu(\{x \in X : |f(x)| > y\}).$$

The set of all ν-measurable functions f on X with $\|f\|_{L^{p,q}(X,\nu)} < \infty$ is denoted by $L^{p,q}(X, \nu)$, and it is called the *Lorentz space with indices p and q*. The space $L^{\infty,\infty}(X, \nu)$ is $L^\infty(X, \nu)$ by definition.

For $0 < p \le \infty$, $L^{p,p}(X, \nu) = L^p(X, \nu)$, and hence, Lebesgue spaces are particular examples of Lorentz spaces. The space $L^{p,\infty}(X, \nu)$ is usually called *weak* $L^p(X, \nu)$.

Some Lorentz spaces that will be of great interest for us are $L^{p,1}(\mathbb{R}^n, \nu)$, $L^{p,p}(\mathbb{R}^n, \nu)$, and $L^{p,\infty}(\mathbb{R}^n, \nu)$, where $d\nu(x) = w(x)dx$, and $0 < w \in L^1_{loc}(\mathbb{R}^n)$. For such measures on $\mathbb{R}^n$, we shall use the notation $L^{p,q}(w)$, or $L^{p,q}(\mathbb{R}^n)$ if $w = 1$.

If $1 \le q \le p < \infty$, or $p = q = \infty$, then we have the triangular inequality for the functional $\| \cdot \|_{L^{p,q}(X,\nu)}$, but for other choices of indices, such inequality may fail. However, for all ν-measurable functions $f, g \in L^{p,q}(X, \nu)$, we have the estimate

$$\|f+g\|_{L^{p,q}(X,\nu)} \leq 2^{1/p} \max\{1, 2^{\frac{1-q}{q}}\}(\|f\|_{L^{p,q}(X,\nu)} + \|g\|_{L^{p,q}(X,\nu)}).$$

In general, $L^{p,q}(X,\nu)$ is a quasi-Banach space, but if $1 < p < \infty$ and $1 \leq q \leq \infty$, or if $p = q = 1$, or if $p = q = \infty$, then it can be normed to become a Banach space.

Lorentz spaces are nested. More precisely, if $0 < p < \infty$, and $0 < q < r \leq \infty$, then

$$L^{p,q}(X,\nu) \longrightarrow L^{p,r}(X,\nu),$$

and for every $f \in L^{p,q}(X,\nu)$,

$$\|f\|_{L^{p,r}(X,\nu)} \leq \left(\frac{q}{p}\right)^{\frac{r-q}{rq}} \|f\|_{L^{p,q}(X,\nu)}. \tag{2.1.1}$$

Given parameters $0 < r < p < \infty$, consider the quantity

$$|||f|||_{L^{p,\infty}(X,\nu)} := \sup_{0<\nu(E)<\infty} \nu(E)^{\frac{1}{p}-\frac{1}{r}} \left(\int_E |f|^r d\nu\right)^{1/r},$$

where the supremum is taken over all ν-measurable sets $E \subseteq X$ such that $0 < \nu(E) < \infty$. We have that

$$\|f\|_{L^{p,\infty}(X,\nu)} \leq |||f|||_{L^{p,\infty}(X,\nu)} \leq \left(\frac{p}{p-r}\right)^{1/r} \|f\|_{L^{p,\infty}(X,\nu)}.$$

This result is classical (see [43, Chapter V, Lemma 2.8] or [44, Exercise 1.1.12]), and we will refer to these inequalities as *Kolmogorov's inequalities*.

2.1.2 Classical Hölder's Inequalities

Given $0 < p \leq \infty$, the *conjugate exponent* p' is defined by the relation

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Let (X,ν) be a σ-finite measure space. The *classical Hölder's inequality* asserts that for $1 \leq p \leq \infty$, and all functions $f \in L^p(X,\nu)$ and $g \in L^{p'}(X,\nu)$,

$$\int_X |fg| d\nu \leq \|f\|_{L^p(X,\nu)} \|g\|_{L^{p'}(X,\nu)}.$$

This inequality is sharp in the sense that

$$\|g\|_{L^{p'}(X,\nu)} = \sup\left\{\int_X |fg| d\nu : \|f\|_{L^p(X,\nu)} \leq 1\right\}.$$

We will refer to this as the *duality between $L^p(X,\nu)$ and $L^{p'}(X,\nu)$*, or just duality.

Similarly, for $1 \le p < \infty$, and all functions $f \in L^{p,1}(X,\nu)$ and $g \in L^{p',\infty}(X,\nu)$,

$$\int_X |fg| d\nu \le \|f\|_{L^{p,1}(X,\nu)} \|g\|_{L^{p',\infty}(X,\nu)}.$$

Once again, this inequality is sharp in the sense that

$$\frac{1}{p}\|g\|_{L^{p',\infty}(X,\nu)} \le \sup\left\{\int_X |fg| d\nu : \|f\|_{L^{p,1}(X,\nu)} \le 1\right\} \le \|g\|_{L^{p',\infty}(X,\nu)}.$$

We will also refer to this as *duality*.

Given $m \ge 2$, and exponents $0 < p_1, \ldots, p_m \le \infty$, with

$$\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m},$$

the *classical multi-variable Hölder's inequality* asserts that for all functions $f_1 \in L^{p_1}(X,\nu), \ldots, f_m \in L^{p_m}(X,\nu)$,

$$\|f_1 \ldots f_m\|_{L^p(X,\nu)} \le \prod_{i=1}^m \|f_i\|_{L^{p_i}(X,\nu)}.$$

In the particular case when $X = \mathbb{N}$, and ν is the counting measure on $\mathbb{N}$, we obtain the *discrete Hölder's inequality*, which asserts that for sequences of real or complex numbers $\{x_j^1\}_{j\in\mathbb{N}}, \ldots, \{x_j^m\}_{j\in\mathbb{N}}$,

$$\left(\sum_{j=0}^{\infty} |x_j^1 \ldots x_j^m|^p\right)^{1/p} \le \prod_{i=1}^m \left(\sum_{j=0}^{\infty} |x_j^i|^{p_i}\right)^{1/p_i},$$

provided that $0 < p_1, \ldots, p_m < \infty$.

We also have a version of *Hölder's inequality for weak $L^p(X,\nu)$ spaces*, which asserts that for all functions $f_1 \in L^{p_1,\infty}(X,\nu), \ldots, f_m \in L^{p_m,\infty}(X,\nu)$,

$$\|f_1 \ldots f_m\|_{L^{p,\infty}(X,\nu)} \le p^{-\frac{1}{p}} \left(\prod_{i=1}^m p_i^{1/p_i}\right) \prod_{i=1}^m \|f_i\|_{L^{p_i,\infty}(X,\nu)},$$

provided that $0 < p_1, \ldots, p_m < \infty$.

For more information about such inequalities, see [3, 4, 44, 52, 79].

2.1.3 Common Classes of Weights

A positive, and locally integrable function w on $\mathbb{R}^n$ is called *weight*.

Given $f \in L^1_{loc}(\mathbb{R}^n)$, the *Hardy-Littlewood maximal operator* M, introduced in [51], is defined by

$$Mf(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy, \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all cubes $Q \subseteq \mathbb{R}^n$ containing x. Given $f_1, \dots, f_m \in L^1_{loc}(\mathbb{R}^n)$, we also define

$$M^{\otimes}(f_1, \dots, f_m)(x) := Mf_1(x) \dots Mf_m(x), \quad x \in \mathbb{R}^n.$$

In [85], Muckenhoupt studied the boundedness of M on Lebesgue spaces $L^p(w)$, obtaining that for $1 < p < \infty$,

$$M : L^p(w) \longrightarrow L^p(w)$$

if, and only if $w \in A_p$; that is, if

$$[w]_{A_p} := \sup_Q \left(\fint_Q w \right) \left(\fint_Q w^{1-p'} \right)^{p-1} < \infty.$$

Moreover, if $1 \leq p < \infty$,

$$M : L^p(w) \longrightarrow L^{p,\infty}(w)$$

if, and only if $w \in A_p$, where a weight $w \in A_1$ if

$$[w]_{A_1} := \sup_Q \left(\fint_Q w \right) \|\chi_Q w^{-1}\|_{L^\infty(w)} = \sup_Q \left(\fint_Q w \right) (\operatorname*{ess\,inf}_{x \in Q} w(x))^{-1} < \infty.$$

Buckley proved in [8] that for $1 \leq p < \infty$,

$$\|M\|_{L^p(w) \to L^{p,\infty}(w)} \lesssim_n [w]_{A_p}^{1/p},$$

and if $p > 1$, then

$$\|M\|_{L^p(w) \to L^p(w)} \lesssim_{n,p} [w]_{A_p}^{\frac{1}{p-1}}.$$

In [17, 58], Chung, Hunt, and Kurtz, and Kerman, and Torchinsky proved that for $1 \leq p < \infty$,

$$M : L^{p,1}(w) \longrightarrow L^{p,\infty}(w)$$

if, and only if $w \in A_p^{\mathcal{R}}$, where a weight w is in $A_p^{\mathcal{R}}$ (also denoted by $A_{p,1}$) if

$$[w]_{A_p^{\mathcal{R}}} := \sup_Q w(Q)^{1/p} \frac{\|\chi_Q w^{-1}\|_{L^{p',\infty}(w)}}{|Q|} < \infty,$$

or equivalently, if

$$\|w\|_{A_p^{\mathcal{R}}} := \sup_Q \sup_{E\subseteq Q} \frac{|E|}{|Q|}\left(\frac{w(Q)}{w(E)}\right)^{1/p} < \infty.$$

Also,

$$\|M\|_{L^{p,1}(w)\to L^{p,\infty}(w)} \approx_{n,p} [w]_{A_p^{\mathcal{R}}}.$$

We have that $[w]_{A_p^{\mathcal{R}}} \leq \|w\|_{A_p^{\mathcal{R}}} \leq p[w]_{A_p^{\mathcal{R}}}$. Moreover, $A_1 = A_1^{\mathcal{R}}$, and in virtue of [44, Exercise 1.1.11], for $1 < p < q$, $A_p \subseteq A_p^{\mathcal{R}} \subseteq A_q$, with

$$[w]_{A_p^{\mathcal{R}}} \leq [w]_{A_p}^{1/p}, \quad \text{and} \quad [w]_{A_q} \leq \left(\frac{p'}{p'-q'}\right)^{q-1} [w]_{A_p^{\mathcal{R}}}^q. \tag{2.1.2}$$

For a complete study of the boundedness of M on Lorentz spaces, see [10].

A remarkable subclass of $A_p^{\mathcal{R}}$ is $\widehat{A}_p$, introduced in [9]. Given $1 \leq p < \infty$, a weight w belongs to the class $\widehat{A}_p$ if there exist a function $f \in L^1_{loc}(\mathbb{R}^n)$, and a weight $u \in A_1$ such that $w = (Mf)^{1-p}u$. It is possible to associate a constant to this class of weights, given by

$$\|w\|_{\widehat{A}_p} := \inf\, [u]_{A_1}^{1/p},$$

where the infimum is taken over all weights $u \in A_1$ such that $w = (Mf)^{1-p}u$. If $w \in \widehat{A}_p$, then $\|w\|_{A_p^{\mathcal{R}}} \lesssim_{n,p} \|w\|_{\widehat{A}_p}$, and $\widehat{A}_p \subseteq A_p^{\mathcal{R}}$, but it is not known if such inclusion is strict for $p > 1$. Note that $\widehat{A}_1 = A_1$, and for $p > 1$, $A_p \subsetneq \widehat{A}_p$.

We now introduce some other classes of weights that will appear later. For more information about them, see [27, 31, 39, 43].

Define the class of weights

$$A_\infty := \bigcup_{p\geq 1} A_p = \bigcup_{p\geq 1} A_p^{\mathcal{R}}.$$

It is known that a weight $w \in A_\infty$ if, and only if

$$[w]_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q) < \infty.$$

This quantity is usually referred to as the *Fujii-Wilson A_∞ constant* (see [40]).

More generally, given a weight u, and $p > 1$, we say that $w \in A_p(u)$ if

$$[w]_{A_p(u)} := \sup_Q \left(\frac{1}{u(Q)}\int_Q wu\right)\left(\frac{1}{u(Q)}\int_Q w^{1-p'}u\right)^{p-1} < \infty,$$

and $w \in A_1(u)$ if

$$[w]_{A_1(u)} := \sup_Q \left(\frac{1}{u(Q)} \int_Q wu \right) \|\chi_Q w^{-1}\|_{L^\infty(wu)}$$
$$= \sup_Q \left(\frac{1}{u(Q)} \int_Q wu \right) (\operatorname*{ess\,inf}_{x\in Q} w(x))^{-1} < \infty,$$

and as before, we define

$$A_\infty(u) := \bigcup_{p\geq 1} A_p(u).$$

A weight u is said to be *doubling* if there exists a constant $D_u > 0$ such that for every cube $Q \subseteq \mathbb{R}^n$, $u(2Q) \leq D_u u(Q)$, where $2Q$ denotes the cube with the same center as Q but with twice its side length. If $u \in A_\infty$, then u is doubling.

Given a doubling weight u, and $w \in A_\infty(u)$, then

$$[w]_{A_\infty(u)} := \sup_Q \frac{1}{wu(Q)} \int_Q M_u(w\chi_Q)u < \infty,$$

where

$$M_u f(x) := \sup_{Q\ni x} \frac{1}{u(Q)} \int_Q |f(y)|u(y)dy, \quad x \in \mathbb{R}^n,$$

is the *weighted Hardy-Littlewood maximal operator*. Its centered version, defined via a supremum over cubes centered at x, will be denoted by M_u^c. If $p > 1$, then M_u is bounded on $L^p(wu)$ if, and only if $w \in A_p(u)$, provided that u is doubling.

Given $s > 1$, we say that a weight $w \in RH_s$ if

$$[w]_{RH_s} := \sup_Q \frac{|Q|}{w(Q)} \left(\fint_Q w^s \right)^{1/s} < \infty,$$

and $w \in RH_\infty$ if

$$[w]_{RH_\infty} := \sup_Q \frac{|Q|}{w(Q)} \|\chi_Q w\|_{L^\infty(\mathbb{R}^n)} = \sup_Q \frac{|Q|}{w(Q)} \operatorname*{ess\,sup}_{x\in Q} w(x) < \infty.$$

We have that

$$A_\infty = \bigcup_{1<s\leq\infty} RH_s.$$

In [69], the following multi-variable extension of the Hardy-Littlewood maximal operator was introduced in connection with the theory of multilinear Calderón-Zygmund operators:

$$\mathcal{M}(\vec{f}) := \sup_Q \left(\prod_{i=1}^m \fint_Q |f_i| \right) \chi_Q,$$

for $\vec{f} = (f_1, \dots, f_m)$, with $f_i \in L^1_{loc}(\mathbb{R}^n)$, $i = 1, \dots, m$. Commonly, this operator is referred to as the *curly operator*.

For $1 \le p_1, \dots, p_m < \infty$, $\vec{P} = (p_1, \dots, p_m)$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, and weights $w_1, \dots, w_m$, with $\vec{w} = (w_1, \dots, w_m)$, and $\nu_{\vec{w}} := w_1^{p/p_1} \dots w_m^{p/p_m}$,

$$\mathcal{M} : L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m) \longrightarrow L^{p,\infty}(\nu_{\vec{w}})$$

if, and only if $\vec{w} \in A_{\vec{P}}$; that is, if

$$[\vec{w}]_{A_{\vec{P}}} := \sup_Q \left(\fint_Q \nu_{\vec{w}} \right)^{1/p} \prod_{i=1}^m \left(\fint_Q w_i^{1-p_i'} \right)^{1/p_i'} < \infty,$$

where $\left(\fint_Q w_i^{1-p_i'} \right)^{1/p_i'}$ is replaced by $(\operatorname{ess\,inf}_{x \in Q} w_i(x))^{-1}$ if $p_i = 1$. Moreover, if $1 < p_1, \dots, p_m < \infty$, then

$$\mathcal{M} : L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m) \longrightarrow L^{p}(\nu_{\vec{w}})$$

if, and only if $\vec{w} \in A_{\vec{P}}$.

2.1.4 Types of Operators

Let $m \ge 1$, and let T be an m-variable operator defined for measurable functions on $\mathbb{R}^n$. Given exponents $0 < p_1, q_1, \dots, p_m, q_m, p, q < \infty$, and weights $w_1, \dots, w_m, w$, suppose that

$$T : L^{p_1,q_1}(w_1) \times \cdots \times L^{p_m,q_m}(w_m) \longrightarrow L^{p,q}(w).$$

(a) We say that T is of *strong type* $(p_1, \dots, p_m, p)$ if $q_1 = p_1, \dots, q_m = p_m$, and $q = p$.

(b) We say that T is of *weak type* $(p_1, \dots, p_m, p)$ if $q_1 = p_1, \dots, q_m = p_m$, and $q = \infty$.

(c) We say that T is of *restricted weak type* $(p_1, \dots, p_m, p)$ if $q_1 = \cdots = q_m = 1$, and $q = \infty$. We may also use this terminology in the case when $0 < q_i \le 1$, $i = 1, \dots, m$.

(d) We say that T is of *mixed type* $(p_1, \dots, p_\ell, p_{\ell+1}, \dots, p_m, p)$, with $1 \le \ell < m$, if $q_1 = p_1, \dots, q_\ell = p_\ell$, and $q_{\ell+1} \le 1, \dots, q_m \le 1$, and $q = \infty$. We may also use this terminology if $1 \le p_1, \dots, p_m < \infty$, and $w_i \in A_{p_i}$, for $i = 1, \dots, \ell$, and $w_i \in A^{\mathcal{R}}_{p_i}$, for $i = \ell + 1, \dots, m$, independently of the choice of the other exponents.

Analogously, we will talk about strong, weak, mixed, and restricted weak type inequalities.

The definitions of strong and weak types are standard (see [4]), but the ones of mixed and restricted weak types may vary depending on the source.

2.1.5 Dyadic Grids and Sparse Collections of Cubes

A *general dyadic grid* $\mathscr{D}$ is a collection of cubes in $\mathbb{R}^n$ with the following properties:

(a) For any $Q \in \mathscr{D}$, its side length l_Q is of the form 2^k, for some $k \in \mathbb{Z}$.

(b) For all $Q, R \in \mathscr{D}$, $Q \cap R \in \{\emptyset, Q, R\}$.

(c) The cubes of a fixed side length 2^k form a partition of $\mathbb{R}^n$.

The *standard dyadic grid in* $\mathbb{R}^n$ consists of the cubes $2^{-k}([0,1)^n + j)$, with $k \in \mathbb{Z}$ and $j \in \mathbb{Z}^n$. It is well known (see [54]) that if one considers the perturbed dyadic grids

$$\mathscr{D}_\alpha := \{2^{-k}([0,1)^n + j + \alpha) : k \in \mathbb{Z}, j \in \mathbb{Z}^n\},$$

with $\alpha \in \{0, \frac{1}{3}\}^n$, then for any cube $Q \subseteq \mathbb{R}^n$, there exist α, and a cube $Q_\alpha \in \mathscr{D}_\alpha$ such that $Q \subseteq Q_\alpha$ and $l_{Q_\alpha} \leq 6 l_Q$.

A collection of cubes $\mathcal{S}$ is said to be η*-sparse* if there exists $0 < \eta < 1$ such that for every cube $Q \in \mathcal{S}$, there exists a set $E_Q \subseteq Q$ with $\eta|Q| \leq |E_Q|$, and for every $Q \neq R \in \mathcal{S}$, $E_R \cap E_Q = \emptyset$.

Given an η-sparse collection of dyadic cubes $\mathcal{S}$, we define the *sparse operator* $\mathcal{A}_\mathcal{S}$ by

$$\mathcal{A}_\mathcal{S}(\vec{f}) := \sum_{Q \in \mathcal{S}} \left(\prod_{i=1}^m \fint_Q f_i \right) \chi_Q.$$

For more information about these topics, see [66].

2.1.6 Calderón-Zygmund Operators

We say that a function $\omega : [0,\infty) \longrightarrow [0,\infty)$ is a *modulus of continuity* if it is continuous, increasing, sub-additive and such that $\omega(0) = 0$. We say that ω satisfies the *Dini condition* if

$$\|\omega\|_{\text{Dini}} := \int_0^1 \frac{\omega(t)}{t} dt < \infty.$$

We give the definition of the multi-linear ω-Calderón-Zygmund operators. We denote by $\mathscr{S}(\mathbb{R}^n)$ the *space of all Schwartz functions on* $\mathbb{R}^n$ and by $\mathscr{S}'(\mathbb{R}^n)$ its dual space, the *set of all tempered distributions on* $\mathbb{R}^n$.

Definition 2.1.1. An *m-linear ω-Calderón-Zygmund operator* is an m-linear and continuous operator $T : \mathscr{S}(\mathbb{R}^n) \times \cdots \times \mathscr{S}(\mathbb{R}^n) \longrightarrow \mathscr{S}'(\mathbb{R}^n)$ that extends to a bounded m-linear operator from $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, for some $1 \leq q_1, \ldots, q_m < \infty$, with $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, and for which there exists a locally integrable function $K(y_0, y_1, \ldots, y_m)$, defined away from the diagonal $y_0 = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, satisfying, for some constant $C_K > 0$:

(a) the size estimate

$$|K(y_0, y_1, \dots, y_m)| \le \frac{C_K}{(|y_0 - y_1| + \dots + |y_0 - y_m|)^{nm}},$$

for all $(y_0, y_1, \dots, y_m) \in (\mathbb{R}^n)^{m+1}$ with $y_0 \neq y_j$ for some $j \in \{1, \dots, m\}$,

(b) the smoothness estimate

$$\begin{aligned} &|K(y_0, y_1, \dots, y_i, \dots, y_m) - K(y_0, y_1, \dots, y_i', \dots, y_m)| \\ &\le \frac{C_K}{(|y_0 - y_1| + \dots + |y_0 - y_m|)^{nm}} \omega\left(\frac{|y_i - y_i'|}{(|y_0 - y_1| + \dots + |y_0 - y_m|)^{nm}}\right), \end{aligned}$$

for $i = 0, \dots, m$, and whenever $|y_i - y_i'| \le \frac{1}{2} \max_{0 \le j \le m}\{|y_i - y_j|\}$,

and such that

$$T(f_1, \dots, f_m)(x) = \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} K(x, y_1, \dots, y_m) f_1(y_1) \dots f_m(y_m) dy_1 \dots dy_m,$$

whenever $f_1, \dots, f_m \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ and $x \in \mathbb{R}^n \setminus \bigcap_{j=1}^m \operatorname{supp} f_j$.

If we take $\omega(t) = t^\varepsilon$ for some $\varepsilon > 0$, we recover the *classical multi-linear Calderón-Zygmund operators*. In general, an m-linear ω-Calderón-Zygmund operator with ω satisfying the Dini condition can be extended to a bounded operator from $L^1(\mathbb{R}^n) \times \dots \times L^1(\mathbb{R}^n)$ to $L^{\frac{1}{m},\infty}(\mathbb{R}^n)$.

The theory of Calderon-Zygmund operators has been investigated by many authors. For more information on this matter, see [49, 69, 77] and the publications cited there.

2.1.7 BMO and Commutators

Given $f \in L^1_{loc}(\mathbb{R}^n)$, the *sharp maximal operator* $M^\#$ is defined by

$$M^\# f(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q \left| f - \fint_Q f \right|, \quad x \in \mathbb{R}^n.$$

If $b \in L^1_{loc}(\mathbb{R}^n)$ is such that $M^\# b \in L^\infty(\mathbb{R}^n)$, we say that b is a function of *bounded mean oscillation*, and we denote by BMO the set of all these functions. For $b \in BMO$, we write

$$\|b\|_{BMO} := \|M^\# b\|_{L^\infty(\mathbb{R}^n)}.$$

Note that $L^\infty(\mathbb{R}^n) \subseteq BMO$, and for $b \in L^\infty(\mathbb{R}^n)$, $\|b\|_{BMO} \le 2\|b\|_{L^\infty(\mathbb{R}^n)}$.

For further information about these concepts, see [38, 43].

Given a one-variable operator T defined for measurable functions on $\mathbb{R}^n$, and a measurable function b, the *commutator* $[b, T]$ is formally defined for a measurable function f by

$$[b, T]f(x) := b(x)Tf(x) - T(bf)(x), \quad x \in \mathbb{R}^n.$$

The first results on these commutators where obtained in [20], where it was proved that if T is a classical singular integral operator with smooth kernel, and $b \in BMO$, then $[b,T]$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Moreover, if T is one of the Riesz transforms on $\mathbb{R}^n$, then the condition $b \in BMO$ is necessary. Such results were further extended in [1], establishing boundedness properties of commutators of general linear operators on weighted Lebesgue spaces. Endpoint estimates for commutators were studied in [91]. For further results on commutators, see [16, 70].

Similarly, given an m-variable operator T defined for measurable functions on $\mathbb{R}^n$, and measurable functions $b_1, \dots, b_m$, with $\vec{b} = (b_1, \dots, b_m)$, the *m-variable commutators* $[\vec{b}, T]_i$, $i = 1, \dots, m$, are formally defined for measurable functions $f_1, \dots, f_m$ by

$$\begin{aligned}[\vec{b}, T]_i(f_1, \dots, f_m)(x) := b_i(x) T(f_1, \dots, f_m)(x) \\ - T(f_1, \dots, f_{i-1}, b_i f_i, f_{i+1}, \dots, f_m)(x), \quad x \in \mathbb{R}^n.\end{aligned}$$

Multi-variable commutators of multi-variable Calderón-Zygmund operators were considered and studied in [69, 92, 94].

2.2 Hölder-Type Inequalities for Lorentz Spaces

In this section, we present new Hölder-type inequalities for Lorentz spaces $L^{p,\infty}(w)$, and we discuss their applications to the study of the classes of weights A_p and $A_p^{\mathcal{R}}$ via the operator $M^{\otimes}$. The contents of this section are partially available in [13].

2.2.1 Modern Hölder's Inequalities

Let us start by giving a counterexample that shows that (1.4.4) does not hold for arbitrary weights.

Fix a dimension $n \geq 1$, and for $m \geq 2$, take exponents $0 < p_1, \dots, p_m < \infty$, and $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$. For $i = 1, \dots, m-1$, take functions on $\mathbb{R}^n$

$$f_i(x) = \frac{1}{|x|^{n/p_i}} \chi_{\{y \in \mathbb{R}^n : |y| \geq 1\}}(x), \quad f_m(x) = |x|^{n\left(\frac{1}{p} - \frac{1}{p_m}\right)} \chi_{\{y \in \mathbb{R}^n : |y| \geq 1\}}(x),$$

and weights

$$w_i(x) = 1, \quad w_m(x) = \frac{1}{|x|^{n\frac{p_m}{p}}} \chi_{\{y \in \mathbb{R}^n : |y| \geq 1\}}(x) + \chi_{\{y \in \mathbb{R}^n : |y| < 1\}}(x).$$

Then, $f_1 \dots f_m = \chi_{\{y \in \mathbb{R}^n : |y| \geq 1\}}$, and

$$w(x) = w_1(x)^{p/p_1} \dots w_m(x)^{p/p_m} = \frac{1}{|x|^n} \chi_{\{y \in \mathbb{R}^n : |y| \geq 1\}}(x) + \chi_{\{y \in \mathbb{R}^n : |y| < 1\}}(x),$$

so

$$\|f_1 \dots f_m\|_{L^{p,\infty}(w)}^p = \int_{\{y\in\mathbb{R}^n : |y|\geq 1\}} \frac{1}{|x|^n} dx = \infty,$$

but for $t > 0$, and $i = 1, \dots, m-1$,

$$\lambda_{f_i}^{w_i}(t) = \nu_n(t^{-p_i} - 1)\chi_{(0,1)}(t), \quad \lambda_{f_m}^{w_m}(t) = \frac{\nu_n p}{p_m - p} \max\{1, t\}^{-p_m},$$

and

$$\|f_i\|_{L^{p_i,\infty}(w_i)} = \nu_n^{1/p_i}, \quad \|f_m\|_{L^{p_m,\infty}(w_m)} = \left(\frac{\nu_n p}{p_m - p}\right)^{1/p_m},$$

where ν_n denotes the *volume of the unit ball in* $\mathbb{R}^n$ (see [44, Appendix A.3] for its explicit expression in terms of n).

Due to this fact, and in order to prove estimates for product-type operators like $M^{\otimes}$, we need the following Hölder-type inequalities.

Lemma 2.2.1. *Fix exponents $0 < p_1, \dots, p_m < \infty$, and $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$, and weights $w_1, \dots, w_m$, and $w = w_1^{p/p_1} \dots w_m^{p/p_m}$. Given a measurable function g, and measurable sets $E_1, \dots, E_{m-1} \subseteq \mathbb{R}^n$,*

$$\|\chi_{E_1} \cdots \chi_{E_{m-1}} g\|_{L^{p,\infty}(w)} \leq \left(\prod_{i=1}^{m-1} \|\chi_{E_i}\|_{L^{p_i,\infty}(w_i)}\right) \|g\|_{L^{p_m,\infty}(w_m)}. \tag{2.2.1}$$

Moreover, if $p > 1$, then for all measurable functions $f_1, \dots, f_{m-1}$,

$$\|f_1 \dots f_{m-1} g\|_{L^{p,\infty}(w)} \leq C \left(\prod_{i=1}^{m-1} \|f_i\|_{L^{p_i,1}(w_i)}\right) \|g\|_{L^{p_m,\infty}(w_m)}. \tag{2.2.2}$$

Proof. Let us start by proving (2.2.1). If $m = 1$, then there is nothing to prove. We discuss the case when $m = 2$. As usual, we may assume that $w_1(E_1) < \infty$, and $\|g\|_{L^{p_2,\infty}(w_2)} < \infty$.

Now, for every $t > 0$, we have that $\{\chi_{E_1}|g| > t\} = E_1 \cap \{|g| > t\}$, and hence, by Hölder's inequality,

$$\begin{aligned} tw(\{\chi_{E_1}|g| > t\})^{1/p} &\leq tw_1(\{\chi_{E_1}|g| > t\})^{1/p_1} w_2(\{\chi_{E_1}|g| > t\})^{1/p_2} \\ &\leq tw_1(E_1)^{1/p_1} w_2(\{|g| > t\})^{1/p_2}, \end{aligned}$$

from which (2.2.1) follows taking the supremum over all $t > 0$.

Finally, if $m > 2$, we use the previous computations to get that

$$\|\chi_{E_1} \cdots \chi_{E_{m-1}} g\|_{L^{p,\infty}(w)} \leq \|\chi_{E_1 \cap \dots \cap E_{m-1}}\|_{L^{q,\infty}(W)} \|g\|_{L^{p_m,\infty}(w_m)},$$

with $\frac{1}{q} = \frac{1}{p_1} + \cdots + \frac{1}{p_{m-1}}$, and $W = w_1^{q/p_1} \dots w_{m-1}^{q/p_{m-1}}$, and by Hölder's inequality,

$$\begin{aligned}\|\chi_{E_1\cap\cdots\cap E_{m-1}}\|_{L^{q,\infty}(W)} &= W(E_1 \cap \cdots \cap E_{m-1})^{1/q} \\ &\leq \prod_{i=1}^{m-1} w_i(E_1 \cap \cdots \cap E_{m-1})^{1/p_i} \leq \prod_{i=1}^{m-1} \|\chi_{E_i}\|_{L^{p_i,\infty}(w_i)}.\end{aligned}$$

To prove (2.2.2), for $i = 1, \dots, m-1$, take a function $f_i \in L^{p_i,1}(w_i)$, and for every integer k, write $E_k^i := \{2^k < |f_i| \leq 2^{k+1}\}$. It is clear that $|f_i| \leq 2\sum_{k\in\mathbb{Z}} 2^k \chi_{E_k^i}$ a.e., and since $p > 1$, $L^{p,\infty}(w)$ is a Banach space, so

$$\begin{aligned}&\|f_1 \cdots f_{m-1} g\|_{L^{p,\infty}(w)} \\ &\leq 2^{m-1} p' \sum_{k_1,\dots,k_{m-1}\in\mathbb{Z}} 2^{k_1+\cdots+k_{m-1}} \|\chi_{E_{k_1}^1 \cap\cdots\cap E_{k_{m-1}}^{m-1}} g\|_{L^{p,\infty}(w)} \\ &\leq 2^{m-1} p' \left(\prod_{i=1}^{m-1} \sum_{k_i\in\mathbb{Z}} 2^{k_i} w_i(E_{k_i}^i)^{1/p_i} \right) \|g\|_{L^{p_m,\infty}(w_m)} \\ &\leq 2^{m-1} p' \left(\prod_{i=1}^{m-1} \sum_{k_i\in\mathbb{Z}} 2^{k_i} w_i(\{|f_i| > 2^{k_i}\})^{1/p_i} \right) \|g\|_{L^{p_m,\infty}(w_m)} \\ &\leq 2^{2m-2} p' \left(\prod_{i=1}^{m-1} \sum_{k_i\in\mathbb{Z}} \int_{2^{k_i}}^{2^{k_i+1}} w_i(\{|f_i| > t\})^{1/p_i} dt \right) \|g\|_{L^{p_m,\infty}(w_m)} \\ &\leq 2^{2m-2} p' \left(\prod_{i=1}^{m-1} \frac{1}{p_i} \|f_i\|_{L^{p_i,1}(w_i)} \right) \|g\|_{L^{p_m,\infty}(w_m)},\end{aligned}$$

where in the second inequality we have used (2.2.1). Hence, (2.2.2) holds, with $C = 2^{2m-2} \frac{p'}{p_1 \cdots p_{m-1}}$. □

The next result is a weaker version of (1.4.4).

Lemma 2.2.2. *Fix exponents $0 < p_1, \dots, p_m < \infty$, and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, and weights $w_1, \dots, w_m$, and $w = w_1^{p/p_1} \dots w_m^{p/p_m}$. Given a measurable function g, and measurable functions $f_1, \dots, f_{m-1}$, with $\|f_i\|_{L^\infty(\mathbb{R}^n)} \leq 1$, $i = 1, \dots, m-1$, and parameters $0 < \delta_1, \dots, \delta_{m-1} < 1$, we have that*

$$\|f_1 \dots f_{m-1} g\|_{L^{p,\infty}(w)} \leq C \left(\prod_{i=1}^{m-1} \||f_i|^{\delta_i}\|_{L^{p_i,\infty}(w_i)} \right) \|g\|_{L^{p_m,\infty}(w_m)}. \tag{2.2.3}$$

Proof. If $m = 1$, then there is nothing to prove. We first discuss the case when $m = 2$.

Note that for every measurable function F, by Lemma 2.2.1, we have that

$$\begin{aligned}\sup_{t>0} t \|\chi_{\{|F|>t\}} g\|_{L^{p,\infty}(w)} &\leq \sup_{t>0} t \|\chi_{\{|F|>t\}}\|_{L^{p_1,\infty}(w_1)} \|g\|_{L^{p_2,\infty}(w_2)} \\ &= \|F\|_{L^{p_1,\infty}(w_1)} \|g\|_{L^{p_2,\infty}(w_2)}.\end{aligned} \tag{2.2.4}$$

Fix $0 < q < p$. By Kolmogorov's inequality (see [44, Exercise 1.1.12]),

$$\|f_1 g\|_{L^{p,\infty}(w)} \leq \sup_{0<w(A)<\infty} \|f_1 g \chi_A\|_{L^q(w)} \, w(A)^{\frac{1}{p}-\frac{1}{q}},$$

where the supremum is taken over all measurable sets $A \subseteq \mathbb{R}^n$ with $0 < w(A) < \infty$. For one of such sets A, we have that

$$\begin{aligned}
\|f_1 g \chi_A\|_{L^q(w)}^q &= \sum_{k<0} \int_{A\cap\{2^k<|f_1|\leq 2^{k+1}\}} |f_1 g|^q w \leq 2^q \sum_{k<0} 2^{kq} \|\chi_{\{|f_1|>2^k\}} g \chi_A\|_{L^q(w)}^q \\
&= 2^q \sum_{k<0} 2^{kq(1-\delta_1)} \left(2^{k\delta_1} \|\chi_{\{|f_1|^{\delta_1}>2^{k\delta_1}\}} g\chi_A\|_{L^q(w)} \right)^q \\
&\leq \frac{2^q}{2^{q(1-\delta_1)}-1} \left(\sup_{0<t<1} t \|\chi_{\{|f_1|^{\delta_1}>t\}} g \chi_A\|_{L^q(w)} \right)^q,
\end{aligned}$$

and hence, applying Kolmogorov's inequality again, and (2.2.4) with $F = |f_1|^{\delta_1}$, we get that

$$\begin{aligned}
&\sup_{0<w(A)<\infty} \|f_1 g \chi_A\|_{L^q(w)} \, w(A)^{\frac{1}{p}-\frac{1}{q}} \\
&\leq 2(2^{q(1-\delta_1)}-1)^{-\frac{1}{q}} \sup_{0<t<1} t \sup_{0<w(A)<\infty} \|\chi_{\{|f_1|^{\delta_1}>t\}} g \chi_A\|_{L^q(w)} w(A)^{\frac{1}{p}-\frac{1}{q}} \\
&\leq 2(2^{q(1-\delta_1)}-1)^{-\frac{1}{q}} \left(\frac{p}{p-q}\right)^{1/q} \sup_{0<t<1} t \|\chi_{\{|f_1|^{\delta_1}>t\}} g\|_{L^{p,\infty}(w)} \\
&\leq 2(2^{q(1-\delta_1)}-1)^{-\frac{1}{q}} \left(\frac{p}{p-q}\right)^{1/q} \| |f_1|^{\delta_1} \|_{L^{p_1,\infty}(w_1)} \|g\|_{L^{p_2,\infty}(w_2)} \\
&\leq 2 \left(\frac{p}{(\log 2) q (1-\delta_1)(p-q)} \right)^{1/q} \| |f_1|^{\delta_1} \|_{L^{p_1,\infty}(w_1)} \|g\|_{L^{p_2,\infty}(w_2)},
\end{aligned}$$

and (2.2.3) follows, with

$$\begin{aligned}
C = C_{\delta_1,p} &:= \inf_{0<q<p} 2 \left(\frac{p}{(\log 2) q (1-\delta_1)(p-q)} \right)^{1/q} \\
&= 2 \left(\inf_{0<\theta<1} \left((\log 2)(1-\delta_1) p \theta (1-\theta) \right)^{-\frac{1}{\theta}} \right)^{1/p}.
\end{aligned}$$

Finally, if $m > 2$, we iterate the previous case $m-1$ times. For $i = 1, \dots, m$, write $\frac{1}{r_i} := \frac{1}{p_i} + \cdots + \frac{1}{p_m}$, and $W_i := w_i^{r_i/p_i} \dots w_m^{r_i/p_m}$. Note that $W_i = w_i^{r_i/p_i} W_{i+1}^{r_i/r_{i+1}}$. After the kth iteration, with $1 \leq k \leq m-1$, we obtain that

$$\|f_1 \dots f_{m-1}g\|_{L^{p,\infty}(w)} \le \left(\prod_{i=1}^{k} C_{\delta_i,r_i} \| |f_i|^{\delta_i} \|_{L^{p_i,\infty}(w_i)}\right) \times \|f_{k+1} \dots f_{m-1}g\|_{L^{r_{k+1},\infty}(W_{k+1})},$$

so for $k = m-1$, we conclude that (2.2.3) holds, with

$$C = \prod_{i=1}^{m-1} C_{\delta_i,r_i}.$$

□

Remark 2.2.3. Observe that for $0 < \delta < 1$, if $p > 1$, then choosing $\theta = \frac{1}{p}$, we get that

$$C_{\delta,p} \le \frac{2p'}{(\log 2)} \frac{1}{(1-\delta)},$$

and if $p \le 1$, then for every $\alpha > \frac{1}{p} - 1$, and choosing $\theta = \frac{1}{p(1+\alpha)}$, we get that

$$C_{\delta,p} \le \frac{2}{(\log 2)^{1+\alpha}} \left(\frac{p(\alpha+1)^2}{\alpha p + p - 1}\right)^{1+\alpha} \frac{1}{(1-\delta)^{1+\alpha}}.$$

Moreover, for $0 < p < \infty$,

$$\begin{aligned} C_{\delta,p} &\ge 2\left(\inf_{0<q<p} \left(\frac{p}{p-q}\right)^{1/q}\right) \inf_{0<q<p} \left(\frac{1}{(\log 2)(1-\delta)q}\right)^{1/q} \\ &= \begin{cases} 2\left(\frac{e}{(\log 2)p(1-\delta)}\right)^{1/p}, & \delta \ge 1 - \frac{e}{p\log 2}, \\ 2e^{\frac{1}{p} - \frac{1}{e}(\log 2)(1-\delta)}, & \delta < 1 - \frac{e}{p\log 2}, \end{cases} \end{aligned}$$

and hence, $\lim_{\delta\to 1^-} C_{\delta,p} = \infty$, as expected, since Lemma 2.2.2 is false for $m = 2$ and $\delta_1 = 1$.

Note also that we can't remove the assumption that $\|f_i\|_{L^\infty(\mathbb{R}^n)} \le 1$, $i = 1, \dots, m-1$, as can be seen by a standard homogeneity argument, or by choosing functions on $\mathbb{R}^n$ $f_1 = \dots = f_{m-2} = \chi_{\{y\in\mathbb{R}^n : |y|\ge 1\}}$,

$$f_{m-1}(x) = |x|^{\beta}\chi_{\{y\in\mathbb{R}^n : |y|\ge 1\}}(x), \quad \text{and} \quad g(x) = |x|^{-\beta}\chi_{\{y\in\mathbb{R}^n : |y|\ge 1\}}(x),$$

and weights

$$\begin{aligned} &w_1(x) = \dots = w_{m-2}(x) = |x|^{-n-1+\delta_{m-1}}\chi_{\{y\in\mathbb{R}^n : |y|\ge 1\}}(x) + \chi_{\{y\in\mathbb{R}^n : |y|<1\}}(x), \\ &w_{m-1}(x) = |x|^{-n-\beta\delta_{m-1}p_{m-1}}\chi_{\{y\in\mathbb{R}^n : |y|\ge 1\}}(x) + \chi_{\{y\in\mathbb{R}^n : |y|<1\}}(x), \quad \text{and} \\ &w_m(x) = |x|^{-n+\beta p_m}\chi_{\{y\in\mathbb{R}^n : |y|\ge 1\}}(x) + \chi_{\{y\in\mathbb{R}^n : |y|<1\}}(x), \end{aligned}$$

where for $m > 2$, we take $\beta = \frac{1}{p_1} + \cdots + \frac{1}{p_{m-2}}$, and for $m = 2$, we take any $\beta > 0$.

2.2.2 New Characterizations of A_p and $A_p^{\mathcal{R}}$

Let us start by proving a property of A_∞ weights that will be essential to produce our alternative characterizations of A_p and $A_p^{\mathcal{R}}$.

Lemma 2.2.4. *Fix exponents $0 < p_1, \dots, p_m < \infty$, and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, and weights $w_1, \dots, w_m \in A_\infty$, and write $w = w_1^{p/p_1} \dots w_m^{p/p_m}$. Then, for every cube $Q \subseteq \mathbb{R}^n$,*

$$w_1(Q)^{p/p_1} \dots w_m(Q)^{p/p_m} \approx w(Q). \tag{2.2.5}$$

Proof. Note that if $m = 1$, then there is nothing to prove, so we may assume that $m \geq 2$. In virtue of Hölder's inequality, we have that

$$w(Q) = \int_Q w_1^{p/p_1} \dots w_m^{p/p_m} \leq w_1(Q)^{p/p_1} \dots w_m(Q)^{p/p_m}.$$

To establish the equivalence in (2.2.5), let us first assume that $m = 2$. Since $w_1, w_2 \in A_\infty$, by Theorem 2.1 in [31], we have that $w_i^{p/p_i} \in RH_{\frac{p_i}{p}}$, $i = 1, 2$, and the desired result follows from Theorem 2.6 in [31], with an implicit constant C_{w_1,w_2} depending on w_1 and w_2.

Finally, if $m > 2$, we iterate the previous case $m - 1$ times. For $i = 1, \dots, m$, write $\frac{1}{r_i} := \frac{1}{p_i} + \cdots + \frac{1}{p_m}$, and $W_i := w_i^{r_i/p_i} \dots w_m^{r_i/p_m} \in A_\infty$. After the kth iteration, with $1 \leq k \leq m - 1$, we obtain that

$$w_1(Q)^{p/p_1} \dots w_m(Q)^{p/p_m} \leq C_{w_{m-k},\dots,w_m} \left(\prod_{i=1}^{m-k-1} w_i(Q)^{p/p_i} \right) W_{m-k}(Q)^{p/r_{m-k}},$$

so for $k = m - 1$, we conclude that (2.2.5) holds, with an implicit constant $C_{w_1,\dots,w_m}$ depending on $w_1, \dots, w_m$. □

Remark 2.2.5. A different proof of this result can be found in [29].

Now we present an alternative characterization of $A_p^{\mathcal{R}}$.

Proposition 2.2.6. *Given weights $w_1, \dots, w_m \in A_\infty$, and exponents $0 < p_1, \dots, p_m < \infty$, with $p_m \geq 1$, and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, and $w = w_1^{p/p_1} \dots w_m^{p/p_m}$, the following statements are equivalent:*

(a) $w_m \in A_{p_m}^{\mathcal{R}}$.

(b) There exists a constant $C > 0$ such that for all measurable sets $E_1, \dots, E_{m-1}$, and every measurable function g,

$$\|\chi_{E_1} \cdots \chi_{E_{m-1}} M g\|_{L^{p,\infty}(w)} \leq C \left(\prod_{i=1}^{m-1} w_i(E_i)^{1/p_i} \right) \|g\|_{L^{p_m,1}(w_m)}.$$

(c) *There exists a constant $c > 0$ such that for every cube Q, and every measurable function g,*

$$\|\chi_Q Mg\|_{L^{p,\infty}(w)} \leq c \left(\prod_{i=1}^{m-1} w_i(Q)^{1/p_i}\right) \|g\|_{L^{p_m,1}(w_m)}.$$

Proof. It is clear that (c) follows from (b), with $c = C$, taking $E_1 = \cdots = E_{m-1} = Q$.

To see that (b) follows from (a), we apply Lemma 2.2.1 and Remark 5.2.3, obtaining that

$$\begin{aligned}\|\chi_{E_1}\cdots\chi_{E_{m-1}} Mg\|_{L^{p,\infty}(w)} &\leq \left(\prod_{i=1}^{m-1} w_i(E_i)^{1/p_i}\right) \|Mg\|_{L^{p_m,\infty}(w_m)} \\ &\leq 2^n 24^{n/p_m} [w_m]_{A_{p_m}^{\mathcal{R}}} \left(\prod_{i=1}^{m-1} w_i(E_i)^{1/p_i}\right) \|g\|_{L^{p_m,1}(w_m)},\end{aligned}$$

and we can take $C = 2^n 24^{n/p_m} [w_m]_{A_{p_m}^{\mathcal{R}}}$.

Let us show that (a) follows from (c). Fix a cube Q, and using duality, choose a non-negative function g such that $\|g\|_{L^{p_m,1}(w_m)} \leq 1$ and

$$\int_Q g = \int_{\mathbb{R}^n} g(\chi_Q w_m^{-1}) w_m \geq \frac{1}{p_m} \|\chi_Q w_m^{-1}\|_{L^{p_m',\infty}(w_m)}.$$

Since $Mg \geq \left(\fint_Q g\right) \chi_Q$, (c) implies that

$$\frac{w(Q)^{1/p}}{|Q|} \|\chi_Q w_m^{-1}\|_{L^{p_m',\infty}(w_m)} \leq p_m c \left(\prod_{i=1}^{m-1} w_i(Q)^{1/p_i}\right),$$

and applying Lemma 2.2.4, we get that

$$\frac{w_m(Q)^{1/p_m}}{|Q|} \|\chi_Q w_m^{-1}\|_{L^{p_m',\infty}(w_m)} \leq p_m c C_{w_1,\dots,w_m}^{1/p},$$

and taking the supremum over all cubes Q, we obtain that $w_m \in A_{p_m}^{\mathcal{R}}$, with $[w_m]_{A_{p_m}^{\mathcal{R}}} \leq p_m c C_{w_1,\dots,w_m}^{1/p}$. $\square$

Similarly, we can obtain new characterizations of A_p weights.

Proposition 2.2.7. *Given weights $w_1, \dots, w_m \in A_\infty$, and exponents $0 < p_1, \dots, p_m < \infty$, with $p_m \geq 1$, and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, and $w = w_1^{p/p_1} \dots w_m^{p/p_m}$, the following statements are equivalent:*

(a) $w_m \in A_{p_m}$.

(b) *There exists a constant $C > 0$ such that for all measurable sets $E_1, \dots, E_{m-1}$, and every measurable function g,*

$$\|\chi_{E_1} \cdots \chi_{E_{m-1}} Mg\|_{L^{p,\infty}(w)} \leq C \left(\prod_{i=1}^{m-1} w_i(E_i)^{1/p_i} \right) \|g\|_{L^{p_m}(w_m)}.$$

(c) *There exists a constant $c > 0$ such that for every cube Q, and every measurable function g,*

$$\|\chi_Q Mg\|_{L^{p,\infty}(w)} \leq c \left(\prod_{i=1}^{m-1} w_i(Q)^{1/p_i} \right) \|g\|_{L^{p_m}(w_m)}.$$

Moreover, if $p_m > 1$, then the following statements are also equivalent to the previous ones:

(d) *There exists a constant $\widetilde{C} > 0$ such that for all measurable sets $E_1, \dots, E_{m-1}$, and every measurable function g,*

$$\|\chi_{E_1} \cdots \chi_{E_{m-1}} Mg\|_{L^{p}(w)} \leq \widetilde{C} \left(\prod_{i=1}^{m-1} w_i(E_i)^{1/p_i} \right) \|g\|_{L^{p_m}(w_m)}.$$

(e) *There exists a constant $\widetilde{c} > 0$ such that for every cube Q, and every measurable function g,*

$$\|\chi_Q Mg\|_{L^{p}(w)} \leq \widetilde{c} \left(\prod_{i=1}^{m-1} w_i(Q)^{1/p_i} \right) \|g\|_{L^{p_m}(w_m)}.$$

Proof. Again, it is clear that (c) follows from (b), with $c = C$, taking $E_1 = \cdots = E_{m-1} = Q$. Similarly, (e) follows from (d), with $\widetilde{c} = \widetilde{C}$.

To see that (b) follows from (a), we apply Lemma 2.2.1 and the weak type bound for M in [8, (2.6)], obtaining that

$$\begin{aligned} \|\chi_{E_1} \cdots \chi_{E_{m-1}} Mg\|_{L^{p,\infty}(w)} &\leq \left(\prod_{i=1}^{m-1} w_i(E_i)^{1/p_i} \right) \|Mg\|_{L^{p_m,\infty}(w_m)} \\ &\leq C_{n,p_m} [w_m]_{A_{p_m}}^{1/p_m} \left(\prod_{i=1}^{m-1} w_i(E_i)^{1/p_i} \right) \|g\|_{L^{p_m}(w_m)}, \end{aligned}$$

and we can take $C = C_{n,p_m}[w_m]_{A_{p_m}}^{1/p_m}$. Similarly, if $p_m > 1$, then (d) follows from (a) applying Hölder's inequality and the strong type bound for M in [8, Theorem 2.5], and we can take $\widetilde{C} = \widetilde{C}_{n,p_m}[w_m]_{A_{p_m}}^{\frac{1}{p_m-1}}$.

Let us show that (a) follows from (c). Fix a cube Q, and using duality, choose a non-negative function g such that $\|g\|_{L^{p_m}(w_m)} \leq 1$ and

$$\int_Q g = \int_{\mathbb{R}^n} g(\chi_Q w_m^{-1}) w_m \geq \|\chi_Q w_m^{-1}\|_{L^{p'_m}(w_m)}.$$

Since $Mg \geq \left(⨍_Q g \right) \chi_Q$, (c) implies that

$$\frac{w(Q)^{1/p}}{|Q|} \|\chi_Q w_m^{-1}\|_{L^{p'_m}(w_m)} \leq c \left(\prod_{i=1}^{m-1} w_i(Q)^{1/p_i} \right),$$

and applying Lemma 2.2.4, we get that

$$\frac{w_m(Q)^{1/p_m}}{|Q|} \|\chi_Q w_m^{-1}\|_{L^{p'_m}(w_m)} \leq c C^{1/p}_{w_1,\dots,w_m},$$

and taking the supremum over all cubes Q, we obtain that $w_m \in A_{p_m}$, with $[w_m]_{A_{p_m}} \leq c C^{1/p}_{w_1,\dots,w_m}$. The same argument shows that (a) follows from (e) even if $p_m = 1$, with $[w_m]_{A_{p_m}} \leq \tilde{c} C^{1/p}_{w_1,\dots,w_m}$, and the proof is complete. □

2.2.3 First Results Involving $M^{\otimes}$

As a consequence of Proposition 2.2.6, we obtain necessary conditions on the weights to produce weighted restricted weak type bounds for $M^{\otimes}$.

Theorem 2.2.8. *Given exponents $1 \leq p_1, \dots, p_m < \infty$, and $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$, and weights $w_1, \dots, w_m \in A_\infty$, and $w = w_1^{p/p_1} \dots w_m^{p/p_m}$, if*

$$M^{\otimes} : L^{p_1,1}(w_1) \times \dots \times L^{p_m,1}(w_m) \longrightarrow L^{p,\infty}(w),$$

then $w_i \in A^{\mathcal{R}}_{p_i}$, for $i = 1, \dots, m$.

Proof. Given a cube Q, we have that $\chi_Q \leq M(\chi_Q)$, so for $j = 1, \dots, m$, and $f_j \in L^{p_j,1}(w_j)$, we get that

$$\begin{aligned}
\|\chi_Q M f_j\|_{L^{p,\infty}(w)} &\leq \left\| \left(\prod_{i \neq j} M(\chi_Q) \right) M f_j \right\|_{L^{p,\infty}(w)} \\
&= \|M^{\otimes}(\overbrace{\chi_Q, \dots, \chi_Q}^{j-1}, f_j, \overbrace{\chi_Q, \dots, \chi_Q}^{m-j})\|_{L^{p,\infty}(w)} \\
&\leq C \left(\prod_{i \neq j} w_i(Q)^{1/p_i} \right) \|f_j\|_{L^{p_j,1}(w_j)},
\end{aligned}$$

and the desired result follows from Proposition 2.2.6. □

Remark 2.2.9. Observe that in virtue of Hölder's inequality for weak Lebesgue spaces, and Remark 5.2.3, we have that if $w_1 = \dots = w_m = w \in$

$\bigcap_{i=1}^{m} A_{p_i}^{\mathcal{R}}$, then for all measurable functions $f_1, \dots, f_m$,

$$\|M^{\otimes}(f_1,\dots,f_m)\|_{L^{p,\infty}(w)} \leq C_{n,p_1,\dots,p_m} \left(\prod_{i=1}^{m}[w]_{A_{p_i}^{\mathcal{R}}}\right) \prod_{i=1}^{m} \|f_i\|_{L^{p_i,1}(w)},$$

which suggests that the converse of Theorem 2.2.8 may be true. As we will show in Theorem 2.4.1, this is the case.

Just for fun, let us present here our first attempt at proving the converse of Theorem 2.2.8, based on Lemma 2.2.2. Such result motivated our study of mixed type inequalities and was a fundamental piece of our first mixed type extrapolation schemes before we developed our theory of Sawyer-type inequalities.

Theorem 2.2.10. *Let $1 \leq p_1, \dots, p_m < \infty$, with $p_i > 1$, $i = 1, \dots, m-1$, and let $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$. Let $w_1, \dots, w_m$ be weights, with $w_i \in A_{p_i}$, $i = 1, \dots, m-1$, and $w_m \in A_{p_m}^{\mathcal{R}}$, and write $w = w_1^{p/p_1} \dots w_m^{p/p_m}$. Then, for every measurable function g, and all measurable sets $E_1, \dots, E_{m-1} \subseteq \mathbb{R}^n$,*

$$\|M^{\otimes}(\chi_{E_1},\dots,\chi_{E_{m-1}},g)\|_{L^{p,\infty}(w)} \lesssim \left(\prod_{i=1}^{m-1} \|\chi_{E_i}\|_{L^{p_i,1}(w_i)}\right) \|g\|_{L^{p_m,1}(w_m)}. \tag{2.2.6}$$

Proof. For $i = 1, \dots, m-1$, since $p_i > 1$, and $w_i \in A_{p_i}$, then in virtue of Lemma 3.1.7, there exists $0 < \delta_i < 1$ such that $w_i \in A_{\delta_i p_i}$. Applying Lemma 2.2.2, we obtain that

$$\begin{aligned}
\|M^{\otimes}(\chi_{E_1},\dots,\chi_{E_{m-1}},g)\|_{L^{p,\infty}(w)} &\lesssim \left(\prod_{i=1}^{m-1} \|M(\chi_{E_i})\|_{L^{\delta_i p_i,\infty}(w_i)}^{\delta_i}\right) \|Mg\|_{L^{p_m,\infty}(w_m)} \\
&\lesssim \left(\prod_{i=1}^{m-1} \|\chi_{E_i}\|_{L^{\delta_i p_i}(w_i)}^{\delta_i}\right) \|g\|_{L^{p_m,1}(w_m)} \\
&\lesssim \left(\prod_{i=1}^{m-1} \|\chi_{E_i}\|_{L^{p_i,1}(w_i)}\right) \|g\|_{L^{p_m,1}(w_m)}.
\end{aligned}$$

□

Remark 2.2.11. If $p > 1$, then arguing as in the proof of Lemma 2.2.1, we can extend (2.2.6) to arbitrary measurable functions $f_1, \dots, f_{m-1}$, and g.

As a consequence of Proposition 2.2.7, and arguing as in the proof of Theorem 2.2.8, we obtain necessary conditions on the weights to produce weighted strong, weak, and mixed type bounds for $M^{\otimes}$.

Theorem 2.2.12. *Given exponents $1 \leq p_1, \dots, p_m < \infty$, and $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$, and weights $w_1, \dots, w_m \in A_\infty$, and $w = w_1^{p/p_1} \dots w_m^{p/p_m}$, if*

$$M^{\otimes} : L^{p_1}(w_1) \times \dots \times L^{p_\ell}(w_\ell) \times L^{p_{\ell+1},1}(w_{\ell+1}) \times \dots \times L^{p_m,1}(w_m) \longrightarrow L^{p,\infty}(w), \tag{2.2.7}$$

with $1 \leq \ell \leq m$, then $w_i \in A_{p_i}$, for $i = 1, \dots, \ell$, and $w_i \in A_{p_i}^{\mathcal{R}}$, for $i = \ell + 1, \dots, m$. Similarly, if

$$M^{\otimes} : L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m) \longrightarrow L^p(w), \tag{2.2.8}$$

then $w_i \in A_{p_i}$, for $i = 1, \dots, m$.

Remark 2.2.13. In virtue of Hölder's inequality and [8, Theorem 2.5], we have that, under the hypotheses of Theorem 2.2.12, if $1 < p_1, \dots, p_m < \infty$, then (2.2.8) holds if, and only if $w_i \in A_{p_i}$, for $i = 1, \dots, m$. Similarly, this last condition is also equivalent to (2.2.7) when $\ell = m$. In Remark 2.4.2, we discuss an alternative proof of this fact for the full range of exponents $1 \leq p_1, \dots, p_m < \infty$. This alternative approach allows us to establish the converse of Theorem 2.2.12 for $1 \leq \ell \leq m$.

2.3 Sawyer-Type Inequalities for Maximal Operators

We devote this section to the study of a novel restricted weak type inequality that extends the classical Sawyer-type inequality (1.5.1) for the Hardy-Littlewood maximal operator. To this end, we will need some previous results.

The following lemma contains well-known results on weights (see [27, 31, 43, 75]), but we will give most of their proofs since we need to keep track of the constants of the weights involved.

Lemma 2.3.1. *Let u and w be weights.*

(a) If $u \in A_1$, then $u^{-1} \in RH_\infty$, and $[u^{-1}]_{RH_\infty} \leq [u]_{A_1}$.

(b) If $u \in RH_\infty$, and $q > 0$, then $u^q \in RH_\infty$. If $q \geq 1$, then $[u^q]_{RH_\infty} \leq [u]_{RH_\infty}^q$.

(c) If $u \in RH_\infty$, and $[u]_{RH_\infty} \leq \beta$, then there exists $r > 1$, depending only on n, β, such that $u \in A_r$ and $[u]_{A_r} \leq c_{n,\beta}$. In particular, $RH_\infty \subseteq A_\infty$.

(d) If $u \in A_\infty$, and $w \in RH_\infty$, then $uw \in A_\infty$.

(e) If $u \in A_1 \cap RH_\infty$, then $u \approx 1$.

Fix $p \geq 1$, and $f_1, \dots, f_m \in L^1_{loc}(\mathbb{R}^n)$, and let $v = \prod_{i=1}^m (Mf_i)^{-1}$.

(f) $v^p \in RH_\infty$, and $1 \leq [v^p]_{RH_\infty} \leq c_{m,n,p}$.

(g) If $u \in A_\infty$, then $uv^p \in A_\infty$, with constant independent of $\vec{f} = (f_1, \dots, f_m)$.

Proof. To prove (a), fix a cube $Q \subseteq \mathbb{R}^n$. By Hölder's inequality, we have that

$$|Q| = \int_Q u^{-\frac{1}{2}} u^{1/2} \leq \left(\int_Q u^{-1} \right)^{1/2} \left(\int_Q u \right)^{1/2},$$

and hence,

$$\operatorname*{ess\,sup}_{x\in Q} u(x)^{-1} = (\operatorname*{ess\,inf}_{x\in Q} u(x))^{-1} \le [u]_{A_1} \frac{|Q|}{u(Q)} \le [u]_{A_1} \fint_Q u^{-1},$$

and the desired result follows taking the supremum over all cubes Q.

The property (b) follows from [31, Theorem 4.2]. Let $q \ge 1$, and fix a cube $Q \subseteq \mathbb{R}^n$. Then,

$$\operatorname*{ess\,sup}_{x\in Q} u(x) \le [u]_{RH_\infty} \fint_Q u \le [u]_{RH_\infty} \left(\fint_Q u^q\right)^{1/q},$$

from which the desired result follows, as before.

To prove (c), fix a cube $Q \subseteq \mathbb{R}^n$, and a measurable set $E \subseteq Q$. Then,

$$\frac{u(E)}{u(Q)} = \frac{1}{u(Q)} \int_Q \chi_E u \le \frac{|E|}{u(Q)} \operatorname*{ess\,sup}_{x\in Q} u(x) \le [u]_{RH_\infty} \frac{|E|}{|Q|} \le \beta \frac{|E|}{|Q|}.$$

In particular, for every $\varepsilon > 0$, and $\delta := \frac{\varepsilon}{\beta}$, if $|E| < \delta|Q|$, then $u(E) < \varepsilon u(Q)$, and the desired result follows from this fact applying the last theorem in [83].

To prove (d), take $q, r > 1$ such that $u \in A_q$ and $w \in A_r$. We will show that $uw \in A_s$, for $s := q + r - 1$. Fix a cube $Q \subseteq \mathbb{R}^n$. Then,

$$\fint_Q uw \le [w]_{RH_\infty} \left(\fint_Q u\right)\left(\fint_Q w\right),$$

and in virtue of Hölder's inequality with exponent $\alpha := 1 + \frac{r-1}{q-1}$,

$$\begin{aligned}\left(\fint_Q (uw)^{1-s'}\right)^{s-1} &\le \left(\fint_Q u^{(1-s')\alpha}\right)^{\frac{s-1}{\alpha}} \left(\fint_Q w^{(1-s')\alpha'}\right)^{\frac{s-1}{\alpha'}} \\ &= \left(\fint_Q u^{1-q'}\right)^{q-1} \left(\fint_Q w^{1-r'}\right)^{r-1},\end{aligned}$$

so $[uw]_{A_s} \le [w]_{RH_\infty}[u]_{A_q}[w]_{A_r} < \infty$.

The property (e) follows immediately from Corollary 4.6 in [31].

To prove (f), observe that in virtue of [44, Theorem 7.2.7], we have that for $0 < \delta < 1$, $(Mf_i)^\delta \in A_1$, and $[(Mf_i)^\delta]_{A_1} \le \frac{c_n}{1-\delta}$, $i = 1, \dots, m$. In particular, $w := \prod_{i=1}^m (Mf_i)^{\delta/m} \in A_1$, and $[w]_{A_1} \le \prod_{i=1}^m [(Mf_i)^\delta]_{A_1}^{1/m} \le \frac{c_n}{1-\delta}$. Since $v^p = w^{-\frac{mp}{\delta}}$, it follows from (a) and (b) that

$$[v^p]_{RH_\infty} \le [w^{-1}]_{RH_\infty}^{\frac{mp}{\delta}} \le [w]_{A_1}^{\frac{mp}{\delta}} \le \left(\frac{c_n}{1-\delta}\right)^{\frac{mp}{\delta}},$$

so

$$1 \le [v^p]_{RH_\infty} \le c_{m,n,p} := \inf_{0<\delta<1} \left(\frac{c_n}{1-\delta}\right)^{\frac{mp}{\delta}}.$$

To prove (g), we already know by (f) that $v^p \in RH_\infty$, with constant bounded by $c_{m,n,p}$, so by (c), there exists $r > 1$, depending only on m, n, p, such that $[v^p]_{A_r} \leq C_{m,n,p}$. By (d), for $q > 1$ such that $u \in A_q$, and $s = q + r - 1$, $[uv^p]_{A_s} \leq \widetilde{C}_{m,n,p}[u]_{A_q} < \infty$. □

The next lemma gives a result on weights that will be handy later on.

Lemma 2.3.2. *Let u and v be weights, and suppose that $u \in A_\infty$. Then, $uv \in A_\infty$ if, and only if $v \in A_\infty(u)$.*

Proof. Let us first assume that $uv \in A_\infty$. Since $u \in A_\infty$, there exists $s > 1$ such that $u \in RH_s$, and since $uv \in A_\infty$, there exists $r > 1$ such that $uv \in A_r$. Take $q := \frac{rs}{s-1} > 1$. We will show that $v \in A_q(u)$. Fix a cube Q. Then,

$$\begin{aligned} I_Q &:= \left(\frac{1}{u(Q)}\int_Q vu\right)\left(\frac{1}{u(Q)}\int_Q v^{1-q'}u\right)^{q-1} \\ &= \left(\frac{|Q|}{u(Q)}\right)^q \left(\frac{1}{|Q|}\int_Q vu\right)\left(\frac{1}{|Q|}\int_Q (vu)^{1-q'}u^{q'}\right)^{q-1}. \end{aligned}$$

Take $\alpha := \frac{q-1}{r-1} = 1 + \frac{r}{(r-1)(s-1)} > 1$ and observe that $(1-q')\alpha = 1 - r'$, $\frac{q-1}{\alpha} = r - 1$, $q'\alpha' = s$, and $\frac{q-1}{\alpha'} = \frac{q}{s}$. Using Hölder's inequality with exponent α, we get that

$$\begin{aligned} \left(\frac{1}{|Q|}\int_Q (vu)^{1-q'}u^{q'}\right)^{q-1} &\leq \left(\frac{1}{|Q|}\int_Q (vu)^{(1-q')\alpha}\right)^{\frac{q-1}{\alpha}}\left(\frac{1}{|Q|}\int_Q u^{q'\alpha'}\right)^{\frac{q-1}{\alpha'}} \\ &= \left(\frac{1}{|Q|}\int_Q (vu)^{1-r'}\right)^{r-1}\left(\frac{1}{|Q|}\int_Q u^s\right)^{q/s} \\ &\leq [u]_{RH_s}^q \left(\frac{1}{|Q|}\int_Q (vu)^{1-r'}\right)^{r-1}\left(\frac{u(Q)}{|Q|}\right)^q. \end{aligned}$$

Hence,

$$I_Q \leq [u]_{RH_s}^q \left(\frac{1}{|Q|}\int_Q vu\right)\left(\frac{1}{|Q|}\int_Q (vu)^{1-r'}\right)^{r-1} \leq [u]_{RH_s}^q [uv]_{A_r},$$

and $[v]_{A_q(u)} = \sup_Q I_Q \leq [u]_{RH_s}^q [uv]_{A_r} < \infty$.

For the converse, let us assume that $v \in A_\infty(u)$. It follows from Theorem 3.1 in [39] that there exist $\delta, C > 0$ such that for every cube $Q \subseteq \mathbb{R}^n$ and every measurable set $E \subseteq Q$,

$$\frac{u(E)}{u(Q)} \leq C\left(\frac{uv(E)}{uv(Q)}\right)^\delta.$$

Similarly, since $u \in A_\infty$, there exist $\varepsilon, c > 0$ such that for every cube $Q \subseteq \mathbb{R}^n$ and every measurable set $E \subseteq Q$,

$$\frac{|E|}{|Q|} \leq c \left(\frac{u(E)}{u(Q)} \right)^{\varepsilon},$$

so for every cube $Q \subseteq \mathbb{R}^n$ and every measurable set $E \subseteq Q$,

$$\frac{|E|}{|Q|} \leq cC^{\varepsilon} \left(\frac{uv(E)}{uv(Q)} \right)^{\varepsilon\delta},$$

and hence, $uv \in A_\infty$. □

Remark 2.3.3. This result is an extension of Lemma 2.1 in [27], where it is shown that if $u \in A_1$ and $v \in A_\infty(u)$, then $uv \in A_\infty$.

We introduce a *weighted version of the dyadic Hardy-Littlewood maximal operator*.

Definition 2.3.4. Let $\mathscr{D}$ be a general dyadic grid in $\mathbb{R}^n$, and let u be a weight. For a measurable function f, we consider the function

$$M_u^{\mathscr{D}}(f)(x) := \sup_{\mathscr{D} \ni Q \ni x} \frac{1}{u(Q)} \int_Q |f(y)| u(y) dy, \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all cubes $Q \in \mathscr{D}$ that contain x. If $u = 1$, we simply write $M^{\mathscr{D}}(f)$.

The following bound for the operator $M_u^{\mathscr{D}}$ is essential.

Theorem 2.3.5. *Let $\mathscr{D}$ be a general dyadic grid in $\mathbb{R}^n$, and let u and v be weights. If $u \in A_\infty$ and $uv \in A_\infty$, then there exists a constant $C_{u,v}$, independent of $\mathscr{D}$, such that for every $t > 0$, and every measurable function f,*

$$\left\| \frac{M_u^{\mathscr{D}}(fv)}{v} \right\|_{L^{1,\infty}(uv)} \leq C_{u,v} \int_{\mathbb{R}^n} |f(x)| u(x) v(x) dx.$$

Proof. In virtue of Lemma 2.3.2, $v \in A_\infty(u)$ and hence, this theorem follows from the proof of Theorem 1.4 in [27]. □

Remark 2.3.6. If we examine the proof of Theorem 1.4 in [27], and we combine it with Appendix A in [28], we can take

$$C_{u,v} = 2^q (2^n r [uv]_{A_r^{\mathcal{R}}})^{r(q-1)} \| M_u \|^q_{L^q(uv^{1-q})},$$

where $r, q > 1$ are such that $uv \in A_r^{\mathcal{R}}$ and $v \in A_{q'}(u)$.

Remark 2.3.7. The bound of Theorem 2.3.5 also holds for the weighted Hardy-Littlewood maximal operator M_u, with constant

$$C := 2^n 6^{np} p^p [u]^p_{A_p^{\mathcal{R}}} C_{u,v},$$

where $p \geq 1$ is such that $u \in A_p^{\mathcal{R}}$.

We can now state and prove the main result of this section.

Theorem 2.3.8. *Fix $p \geq 1$, and let u and v be weights such that $u \in A_p^{\mathcal{R}}$ and $uv^p \in A_\infty$. Then, there exists a constant $C > 0$ such that for every measurable function f,*

$$\left\| \frac{Mf}{v} \right\|_{L^{p,\infty}(uv^p)} \leq C \|f\|_{L^{p,1}(u)}.$$

Proof. It is known (see [54, 64]) that there exists a collection $\{\mathscr{D}_\alpha\}_\alpha$ of 2^n general dyadic grids in $\mathbb{R}^n$ such that

$$Mf \leq 6^n \sum_{\alpha=1}^{2^n} M^{\mathscr{D}_\alpha}(f).$$

Hence,

$$\left\| \frac{Mf}{v} \right\|_{L^{p,\infty}(uv^p)} \leq 12^n \sum_{\alpha=1}^{2^n} \left\| \frac{M^{\mathscr{D}_\alpha}(f)}{v} \right\|_{L^{p,\infty}(uv^p)},$$

and it suffices to establish the result for the operator $M^{\mathscr{D}}$, with $\mathscr{D}$ a general dyadic grid in $\mathbb{R}^n$.

We first discuss the case $p = 1$, which was proved in [27]. We reproduce the proof here keeping track of the constants. Indeed, by the definition of the A_1 condition,

$$\frac{1}{|Q|} \int_Q |f| \leq [u]_{A_1} \frac{1}{u(Q)} \int_Q |f| u,$$

so we get that $M^{\mathscr{D}}(f) \leq [u]_{A_1} M_u^{\mathscr{D}}(f)$. This estimate combined with Theorem 2.3.5 gives that

$$\left\| \frac{M^{\mathscr{D}}(f)}{v} \right\|_{L^{1,\infty}(uv)} \leq [u]_{A_1} \left\| \frac{M_u^{\mathscr{D}}(vfv^{-1})}{v} \right\|_{L^{1,\infty}(uv)} \leq [u]_{A_1} C_{u,v} \int_{\mathbb{R}^n} |f| u,$$

and hence, the desired result follows, with $C = 24^n [u]_{A_1} C_{u,v}$.

Now, we discuss the case $p > 1$. Let us take $f = \chi_E$, with E a measurable set in $\mathbb{R}^n$, and fix a cube $Q \in \mathscr{D}$. As before, by the definition of the $A_p^{\mathcal{R}}$ condition,

$$\frac{1}{|Q|} \int_Q f \leq \|u\|_{A_p^{\mathcal{R}}} \left(\frac{u(E \cap Q)}{u(Q)} \right)^{1/p},$$

so we get that $M^{\mathscr{D}}(\chi_E) \leq p[u]_{A_p^{\mathcal{R}}} (M_u^{\mathscr{D}}(\chi_E))^{1/p}$. In particular,

$$\left\| \frac{M^{\mathscr{D}}(\chi_E)}{v} \right\|_{L^{p,\infty}(uv^p)} \leq p[u]_{A_p^{\mathcal{R}}} \left\| \frac{M_u^{\mathscr{D}}(\chi_E)}{v^p} \right\|_{L^{1,\infty}(uv^p)}^{1/p}.$$

We can now apply Theorem 2.3.5 to conclude that

$$\left\| \frac{M_u^{\mathscr{D}}(\chi_E)}{v^p} \right\|_{L^{1,\infty}(uv^p)} = \left\| \frac{M_u^{\mathscr{D}}(v^p \chi_E v^{-p})}{v^p} \right\|_{L^{1,\infty}(uv^p)} \leq C_{u,v^p} u(E).$$

Combining all the previous estimates, we have that

$$\left\| \frac{M(\chi_E)}{v} \right\|_{L^{p,\infty}(uv^p)} \leq 24^n [u]_{A_p^{\mathcal{R}}} C_{u,v^p}^{1/p} \|\chi_E\|_{L^{p,1}(u)}.$$

Since $p > 1$, $L^{p,\infty}(uv^p)$ is a Banach space, and arguing as in the proof of Lemma 2.2.1, we can extend the previous estimate to arbitrary measurable functions f, gaining a factor of $4p'$ in the constant. Hence, the desired result follows, with $C = 4 \cdot 24^n p' [u]_{A_p^{\mathcal{R}}} C_{u,v^p}^{1/p}$. □

Remark 2.3.9. For $p = 1$ and $u \in A_1$, a more general version of Theorem 2.3.8 was established in [74], replacing the hypothesis that $uv \in A_\infty$ by the weaker assumption that $v \in A_\infty$. It is unknown to us whether the hypothesis that $uv^p \in A_\infty$ can be replaced by $v \in A_\infty$ when $p > 1$.

In virtue of Lemma 2.3.1, if $u \in A_\infty$ and $v \in RH_\infty$, then for every $p \geq 1$, $uv^p \in A_\infty$, and we have a whole class of non-trivial examples of weights that satisfy the hypotheses of Theorem 2.3.8.

Observe that the conclusion of Theorem 2.3.8 is completely elementary if $p > 1$ and $u \in A_p$, since

$$\begin{aligned} \left\| \frac{Mf}{v} \right\|_{L^{p,\infty}(uv^p)} &\leq \left\| \frac{Mf}{v} \right\|_{L^p(uv^p)} = \|Mf\|_{L^p(u)} \\ &\leq c_1 [u]_{A_p}^{\frac{1}{p-1}} \|f\|_{L^p(u)} \leq c_2 [u]_{A_p}^{\frac{1}{p-1}} \|f\|_{L^{p,1}(u)}. \end{aligned} \tag{2.3.1}$$

However, this argument doesn't work in the general case, because the inequality

$$\left\| \frac{h}{v} \right\|_{L^{p,\infty}(uv^p)} \lesssim \|h\|_{L^{p,\infty}(u)}$$

may fail for some measurable functions h on $\mathbb{R}^n$, and arbitrary weights u and v, as can be seen by choosing $h(x) = |x|^{-\frac{n}{p}} \chi_{\{y \in \mathbb{R}^n : |y| \geq 1\}}(x)$, $u = 1$, and $v(x) = h(x) + \chi_{\{y \in \mathbb{R}^n : |y| < 1\}}(x)$, with $0 < p < \infty$.

To provide applications of Theorem 2.3.8 we need to give a more precise estimate of the constant C that appears there in terms of the corresponding constants of the weights involved. We achieve this in the following lemma.

Lemma 2.3.10. *In Theorem 2.3.8, if $r \geq 1$ is such that $uv^p \in A_r^{\mathcal{R}}$, then one can take*

$$C = \phi_{r,p}^n([u]_{A_p^{\mathcal{R}}}, [uv^p]_{A_r^{\mathcal{R}}}),$$

where $\phi_{r,p}^n : [1, \infty)^2 \longrightarrow (0, \infty)$ is a function that increases in each variable, and it depends only on r, p, and the dimension n.

Proof. We first discuss the case when $r > 1$. We already know that we can take

$$C = \begin{cases} 24^n [u]_{A_1} C_{u,v}, & p = 1, \\ 4 \cdot 24^n p' [u]_{A_p^{\mathcal{R}}} C_{u,v^p}^{1/p}, & p > 1, \end{cases}$$

and in virtue of Remark 2.3.6,

$$C_{u,v^p} = 2^q (2^n r [uv^p]_{A_r^{\mathcal{R}}})^{r(q-1)} \|M_u\|^q_{L^q(uv^{p(1-q)})},$$

where $r, q > 1$ are such that $uv^p \in A_r^{\mathcal{R}}$ and $v^p \in A_{q'}(u)$. For convenience, we write $V := v^p$. Let us first bound the factor $\|M_u\|^q_{L^q(uV^{1-q})}$. For the space of homogeneous type $(\mathbb{R}^n, d_\infty, u(x)dx)$, it follows from the proof of Theorem 1.3 in [55] that

$$\|M_u\|^q_{L^q(uV^{1-q})} \le 2^{q-1} q' 40^{qD_u} (1 + 6 \cdot 800^{D_u}) [V]_{A_\infty(u)} [V]^{q-1}_{A_{q'}(u)},$$

where $D_u := p \log_2(2^n p [u]_{A_p^{\mathcal{R}}})$. Now, given a cube $Q \subseteq \mathbb{R}^n$, and applying Hölder's inequality with exponent q, we have that

$$\begin{aligned}
\int_Q M_u(V\chi_Q) u &= \int_Q \frac{M_u(V\chi_Q)}{V} uV \le \left\| \frac{M_u(V\chi_Q)}{V} \right\|_{L^q(uV)} uV(Q)^{1/q'} \\
&= \|M_u(V\chi_Q)\|_{L^q(uV^{1-q})} uV(Q)^{1/q'} \\
&\le \|M_u\|_{L^q(uV^{1-q})} \|V\chi_Q\|_{L^q(uV^{1-q})} uV(Q)^{1/q'} \\
&= \|M_u\|_{L^q(uV^{1-q})} uV(Q),
\end{aligned}$$

and taking the supremum over all cubes Q, we get that

$$[V]_{A_\infty(u)} \le \|M_u\|_{L^q(uV^{1-q})}.$$

Combining the previous estimates, we obtain that

$$\|M_u\|^q_{L^q(uV^{1-q})} \le (2^{q-1} q' 40^{qD_u} (1 + 6 \cdot 800^{D_u}))^{q'} [V]^q_{A_{q'}(u)}.$$

Now, we will bound the factor $[V]^q_{A_{q'}(u)}$. In virtue of [54, Proposition 2.2], and using the definitions of $[u]_{A_{2p}}$ and $[u]_{A_p^{\mathcal{R}}}$, and Kolmogorov's inequalities, we can deduce that

$$[u]_{A_\infty} \le c_n [u]_{A_{2p}} \le (2p-1)^{2p-1} c_n [u]^{2p}_{A_p^{\mathcal{R}}} =: c_{p,n} [u]^{2p}_{A_p^{\mathcal{R}}},$$

and applying Theorem 2.3 in [55], $u \in RH_s$ for $s = 1 + \frac{1}{2^{n+1} c_{p,n} [u]^{2p}_{A_p^{\mathcal{R}}} - 1}$, and $[u]_{RH_s} \le 2$. Since $uV \in A_{2r}$, Lemma 2.3.2 tells us that if we choose $q' = 2rs'$, then

$$[V]^q_{A_{q'}(u)} \le [u]^{qq'}_{RH_s} [uV]^q_{A_{2r}} \le 2^{qq'} (2r-1)^{q(2r-1)} [uV]^{2rq}_{A_r^{\mathcal{R}}}.$$

Finally, observe that $q' = 2^{n+2} r c_{p,n} [u]_{A_p^{\mathcal{R}}}^{2p}$, and $1 < q \leq 2$, so

$$
\begin{aligned}
C_{u,V} &\leq 2^2 (2^n r [uV]_{A_r^{\mathcal{R}}})^r \\
&\quad \times (2q' 40^{2D_u}(1 + 6 \cdot 800^{D_u}))^{q'} \times 2^{2q'} (2r-1)^{4r-2} [uV]_{A_r^{\mathcal{R}}}^{4r} \\
&\leq 2^{2+nr} (2r-1)^{4r-2} r^r [uv^p]_{A_r^{\mathcal{R}}}^{5r} \\
&\quad \times \left(2^{n+5} r c_{p,n} [u]_{A_p^{\mathcal{R}}}^{2p} 40^{5p \log_2 (2^n p [u]_{A_p^{\mathcal{R}}})} \right)^{2^{n+2} r c_{p,n} [u]_{A_p^{\mathcal{R}}}^{2p}} \\
&=: C_{r,p}^n ([u]_{A_p^{\mathcal{R}}}, [uv^p]_{A_r^{\mathcal{R}}}),
\end{aligned}
$$

and the desired result follows, with

$$
\phi_{r,p}^n ([u]_{A_p^{\mathcal{R}}}, [uv^p]_{A_r^{\mathcal{R}}}) = \begin{cases} 24^n [u]_{A_1} C_{r,1}^n ([u]_{A_1}, [uv]_{A_r^{\mathcal{R}}}), & p = 1, \\ 4 \cdot 24^n p' [u]_{A_p^{\mathcal{R}}} C_{r,p}^n ([u]_{A_p^{\mathcal{R}}}, [uv^p]_{A_r^{\mathcal{R}}})^{1/p}, & p > 1. \end{cases}
$$

The case when $r = 1$ follows, for example, from the case when $r = 2$ and the fact that if $uv^p \in A_1$, then $[uv^p]_{A_2^{\mathcal{R}}} \leq [uv^p]_{A_2}^{1/2} \leq [uv^p]_{A_1}^{1/2}$. □

2.4 Applications

In this section, we will provide several applications of the Sawyer-type inequality established in Theorem 2.3.8, obtaining mixed restricted weak type estimates for multi-variable maximal operators, sparse operators and Calderón-Zygmund operators.

2.4.1 Restricted Weak Type Bounds for $M^{\otimes}$

The first result that we present is the converse of Theorem 2.2.8. Combining both theorems, we obtain the complete characterization of the restricted weak type bounds of the operator $M^{\otimes}$ for A_∞ weights.

Theorem 2.4.1. *Let $1 \leq p_1, \dots, p_m < \infty$, and let $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. Let $w_1, \dots, w_m$ be weights, with $w_i \in A_{p_i}^{\mathcal{R}}$, $i = 1, \dots, m$, and $w = w_1^{p/p_1} \dots w_m^{p/p_m}$. Then, for every vector of measurable functions $\vec{f} = (f_1, \dots, f_m)$,*

$$
\|M^{\otimes}(\vec{f})\|_{L^{p,\infty}(w)} \leq \phi([w_1]_{A_{p_1}^{\mathcal{R}}}, \dots, [w_m]_{A_{p_m}^{\mathcal{R}}}) \prod_{i=1}^m \|f_i\|_{L^{p_i,1}(w_i)},
$$

where $\phi : [1, \infty)^m \longrightarrow (0, \infty)$ is a function increasing in each variable.

Proof. The case when $p_1 = \cdots = p_m = 1$ was proved in [69], and we build upon that proof to demonstrate the remaining cases.

We can assume, without loss of generality, that $f_i \in L_c^\infty(\mathbb{R}^n)$, $i = 1, \dots, m$. Fix $t > 0$ and define

$$
E_t := \{x \in \mathbb{R}^n : t < M^{\otimes}(\vec{f})(x) \leq 2t\}.
$$

For $i = 1, \dots, m$, and taking $\tilde{v}_i := \prod_{j\neq i}(Mf_j)^{-1}$, we have that

$$E_t = \{x \in \mathbb{R}^n : t\tilde{v}_i(x) < Mf_i(x) \leq 2t\tilde{v}_i(x)\}.$$

Using the fact that $\tilde{v}_i \in RH_\infty$, with constant independent of $\vec{f}$ (see Lemma 2.3.1), Hölder's inequality, and Theorem 2.3.8, we obtain that

$$\begin{aligned}
\lambda^w_{M^{\otimes}(\vec{f})}(t) - \lambda^w_{M^{\otimes}(\vec{f})}(2t) &= \int_{E_t} w \leq \int_{E_t} \left(\frac{M^{\otimes}(\vec{f})}{t}\right)^p w \\
&\leq \frac{1}{t^p}\prod_{i=1}^m \left(\int_{E_t} (Mf_i)^{p_i} w_i\right)^{p/p_i} \\
&\leq 2^{mp} t^{(m-1)p} \prod_{i=1}^m \left(\int_{\left\{\frac{Mf_i}{\tilde{v}_i} > t\right\}} \tilde{v}_i^{p_i} w_i\right)^{p/p_i} \\
&\leq 2^{mp} C_1^p \dots C_m^p \frac{1}{t^p} \prod_{i=1}^m \|f_i\|^p_{L^{p_i,1}(w_i)}.
\end{aligned} \tag{2.4.1}$$

Iterating this result, we get that for each $t > 0$ and every natural number N,

$$\lambda^w_{M^{\otimes}(\vec{f})}(t) \leq 2^{mp} C_1^p \dots C_m^p \left(\sum_{j=0}^N \frac{1}{2^{jp}}\right) \frac{1}{t^p} \prod_{i=1}^m \|f_i\|^p_{L^{p_i,1}(w_i)} + \lambda^w_{M^{\otimes}(\vec{f})}(2^{N+1}t),$$

and letting N tend to infinity, the last term vanishes, and we conclude that

$$\lambda^w_{M^{\otimes}(\vec{f})}(t) \leq \frac{2^{(m+1)p}}{2^p - 1} C_1^p \dots C_m^p \frac{1}{t^p} \prod_{i=1}^m \|f_i\|^p_{L^{p_i,1}(w_i)}.$$

Observe that in virtue of Lemma 2.3.1, for $i = 1, \dots, m$, we have that $w_i \tilde{v}_i^{p_i} \in A_{s_i}$, where $s_i > 1$ depends only on m, n, p_i, and

$$[w_i \tilde{v}_i^{p_i}]^{s_i}_{A^{\mathcal{R}}_{s_i}} \leq [w_i \tilde{v}_i^{p_i}]_{A_{s_i}} \lesssim_{m,n,p_i} [w_i]_{A_{2p_i}} \lesssim_{m,n,p_i} [w_i]^{2p_i}_{A^{\mathcal{R}}_{p_i}},$$

so by Lemma 2.3.10, we have that $C_i \leq \phi^n_{s_i,p_i}([w_i]_{A^{\mathcal{R}}_{p_i}}, C_{m,n,p_i}[w_i]^{\frac{2p_i}{s_i}}_{A^{\mathcal{R}}_{p_i}})$, and hence, the desired result follows, with

$$\phi([w_1]_{A^{\mathcal{R}}_{p_1}}, \dots, [w_m]_{A^{\mathcal{R}}_{p_m}}) = \frac{2^{m+1}}{(2^p - 1)^{1/p}} \prod_{i=1}^m \phi^n_{s_i,p_i}([w_i]_{A^{\mathcal{R}}_{p_i}}, C_{m,n,p_i}[w_i]^{\frac{2p_i}{s_i}}_{A^{\mathcal{R}}_{p_i}}),$$

which depends on the constants of the weights w_i in an increasing way. □

Remark 2.4.2. Concerning weak, and mixed type bounds, the proof of Theorem 2.4.1 can be easily modified, applying (2.3.1) in (2.4.1), to show that for

$1 \leq \ell \leq m$, if $w_i \in A_{p_i}$, $i = 1, \dots, \ell$, and $w_i \in A_{p_i}^{\mathcal{R}}$, $i = \ell + 1, \dots, m$, then

$$M^{\otimes} : L^{p_1}(w_1) \times \cdots \times L^{p_\ell}(w_\ell) \times L^{p_{\ell+1},1}(w_{\ell+1}) \times \cdots \times L^{p_m,1}(w_m) \longrightarrow L^{p,\infty}(w),$$

with constant depending in an increasing way on the constants of the weights $w_1, \dots, w_m$. Combining this with Theorem 2.2.12, we obtain the complete characterizations of the weak, and mixed type bounds of the operator $M^{\otimes}$ for A_∞ weights.

As an immediate consequence of Theorem 2.4.1, we can produce restricted weak type bounds for operators that are point-wise dominated by $M^{\otimes}$, like the averages of products of convolutions that we now present.

Theorem 2.4.3. *For every $i = 1, \dots, m$, let $\psi^i : [0, \infty) \longrightarrow [0, \infty)$ be a decreasing function that is continuous except at a finite number of points, and suppose that $\Psi^i(x) = \psi^i(|x|)$ is an integrable function on $\mathbb{R}^n$. Given $t > 0$, write $\Psi_t^i(x) = t^{-n}\Psi^i(t^{-1}x)$. For a measure μ on $(0, \infty)^m$ such that $|\mu|((0, \infty)^m) < \infty$, consider the averaging operator*

$$\begin{aligned} T_{\vec{\Psi},\mu}(f_1, \dots, f_m)(x) &:= \int_{(0,\infty)^m} \left(\prod_{i=1}^m \int_{\mathbb{R}^n} |f_i(x-y)| \Psi_{t_i}^i(y) dy \right) d\mu(t_1, \dots, t_m) \\ &= \int_{(0,\infty)^m} (|f_1| * \Psi_{t_1}^1)(x) \dots (|f_m| * \Psi_{t_m}^m)(x) d\mu(t_1, \dots, t_m), \end{aligned}$$

defined for locally integrable functions $f_1, \dots, f_m$ on $\mathbb{R}^n$. Take exponents $1 \leq q_1, \dots, q_m$, and $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, and weights $w_i \in A_{q_i}^{\mathcal{R}}$, $i = 1, \dots, m$, and $w = w_1^{q/q_1} \dots w_m^{q/q_m}$. Then,

$$T_{\vec{\Psi},\mu} : L^{q_1,1}(w_1) \times \cdots \times L^{q_m,1}(w_m) \longrightarrow L^{q,\infty}(w), \tag{2.4.2}$$

with constant bounded by $\Phi([w_1]_{A_{q_1}^{\mathcal{R}}}, \dots, [w_m]_{A_{q_m}^{\mathcal{R}}})$, where $\Phi : [1, \infty)^m \longrightarrow [0, \infty)$ is a function increasing in each variable.

Proof. In virtue of [44, Theorem 2.1.10], we have that for $i = 1, \dots, m$,

$$(|f_i| * \Psi_{t_i}^i) \leq \|\Psi^i\|_{L^1(\mathbb{R}^n)} M f_i,$$

so

$$|T_{\vec{\Psi},\mu}(f_1, \dots, f_m)| \leq |\mu|((0, \infty)^m) \left(\prod_{i=1}^m \|\Psi^i\|_{L^1(\mathbb{R}^n)} \right) M f_1 \dots M f_m,$$

and (2.4.2) follows from Theorem 2.4.1. □

Remark 2.4.4. Note that from the argument in the proof of Theorem 2.4.3, and taking into account Remark 2.2.13 and Remark 2.4.2, we can also deduce strong, weak, and mixed type bounds for $T_{\vec{\Psi},\mu}$.

Remark 2.4.5. In virtue of [44, Remark 2.2.4], given a Schwartz function f on $\mathbb{R}^n$, we have that

$$|f(x)| \leq \frac{C_{f,n}}{(1+|x|)^{n+1}} \in L^1(\mathbb{R}^n),$$

and hence, we can use Theorem 2.4.3, Remark 2.2.13, and Remark 2.4.2 to obtain strong, weak, mixed, and restricted weak type bounds for some bilinear constant coefficient paraproducts on $\mathbb{R}^n$ (see [99, Section 5]).

2.4.2 Sawyer-Type Inequalities for $M^{\otimes}$ and $\mathcal{M}$

The next application that we provide is an extension of Theorem 2.3.8 to the multi-variable setting, which in turn, extends Theorem 2.4.1. The proof is based on the previous one, and is similar to that of Theorem 1.4 in [75].

Theorem 2.4.6. *Let $1 \leq p_1, \dots, p_m < \infty$, and let $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. Let $w_1, \dots, w_m$ be weights, with $w_i \in A_{p_i}^{\mathcal{R}}$, $i = 1, \dots, m$, and $\nu_{\vec{w}} = w_1^{p/p_1} \dots w_m^{p/p_m}$. Let v be a weight such that $\nu_{\vec{w}} v^p$ is a weight, and $w_i v^{p_i} \in A_\infty$, $i = 1, \dots, m$. Then, there exists a constant $C > 0$ such that the inequalities*

$$\left\| \frac{\mathcal{M}(\vec{f})}{v} \right\|_{L^{p,\infty}(\nu_{\vec{w}} v^p)} \leq \left\| \frac{M^{\otimes}(\vec{f})}{v} \right\|_{L^{p,\infty}(\nu_{\vec{w}} v^p)} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i,1}(w_i)}$$

hold for every vector of measurable functions $\vec{f} = (f_1, \dots, f_m)$.

Proof. The first inequality follows from the fact that $\mathcal{M}(\vec{f}) \leq M^{\otimes}(\vec{f})$. For the second one, we can assume, without loss of generality, that $f_i \in L_c^\infty(\mathbb{R}^n)$, $i = 1, \dots, m$. Fix $y, R > 0$ and define

$$E_y^R := \{x \in \mathbb{R}^n : |x| < R,\, yv(x) < M^{\otimes}(\vec{f})(x) \leq 2yv(x)\}.$$

For $i = 1, \dots, m$, and taking $\tilde{v}_i := \prod_{j \neq i} (Mf_j)^{-1}$, and $v_i := \tilde{v}_i v$, we have that

$$E_y^R = \{x \in \mathbb{R}^n : |x| < R,\, yv_i(x) < Mf_i(x) \leq 2yv_i(x)\}.$$

Since $\tilde{v}_i \in RH_\infty$, and $w_i v^{p_i} \in A_\infty$, we know that $w_i v_i^{p_i} \in A_\infty$, with constant independent of $\vec{f}$ (see Lemma 2.3.1). In virtue of Hölder's inequality and

Theorem 2.3.8, we get that

$$\begin{aligned}
&\nu_{\vec{w}}v^p\left(\left\{x\in\mathbb{R}^n : |x|<R,\ \frac{M^{\otimes}(\vec{f})(x)}{v(x)}>y\right\}\right)\\
&-\nu_{\vec{w}}v^p\left(\left\{x\in\mathbb{R}^n : |x|<R,\ \frac{M^{\otimes}(\vec{f})(x)}{v(x)}>2y\right\}\right)\\
&=\int_{E_y^R}\nu_{\vec{w}}v^p\leq\int_{E_y^R}\left(\frac{M^{\otimes}(\vec{f})}{y}\right)^p\nu_{\vec{w}}\leq\frac{1}{y^p}\prod_{i=1}^m\left(\int_{E_y^R}(Mf_i)^{p_i}w_i\right)^{p/p_i}\\
&\leq 2^{mp}y^{(m-1)p}\prod_{i=1}^m\left(\int_{\left\{\frac{Mf_i}{v_i}>y\right\}}v_i^{p_i}w_i\right)^{p/p_i}\leq 2^{mp}C_1^p\dots C_m^p\frac{1}{y^p}\prod_{i=1}^m\|f_i\|^p_{L^{p_i,1}(w_i)}.
\end{aligned}$$

Iterating this result, we deduce that for each $y>0$ and every natural number N,

$$\begin{aligned}
&\nu_{\vec{w}}v^p\left(\left\{x\in\mathbb{R}^n : |x|<R,\ \frac{M^{\otimes}(\vec{f})(x)}{v(x)}>y\right\}\right)\\
&\leq 2^{mp}C_1^p\dots C_m^p\left(\sum_{j=0}^N\frac{1}{2^{jp}}\right)\frac{1}{y^p}\prod_{i=1}^m\|f_i\|^p_{L^{p_i,1}(w_i)}\\
&+\nu_{\vec{w}}v^p\left(\left\{x\in\mathbb{R}^n : |x|<R,\ \frac{M^{\otimes}(\vec{f})(x)}{v(x)}>2^{N+1}y\right\}\right),
\end{aligned}$$

and letting first N tend to infinity, and then R, the last term vanishes, and we conclude that

$$\lambda^{\nu_{\vec{w}}v^p}_{\frac{M^{\otimes}(\vec{f})}{v}}(y)\leq\frac{2^{(m+1)p}}{2^p-1}C_1^p\dots C_m^p\frac{1}{y^p}\prod_{i=1}^m\|f_i\|^p_{L^{p_i,1}(w_i)}.$$

For $i=1,\dots,m$, if we take $q_i>1$ such that $w_iv^{p_i}\in A_{q_i}^{\mathcal{R}}$, in virtue of Lemma 2.3.1, we have that $w_iv_i^{p_i}\in A_{s_i}$, where $s_i>1$ depends only on m,n,p_i,q_i, and $[w_iv_i^{p_i}]^{s_i}_{A^{\mathcal{R}}_{s_i}}\leq[w_iv_i^{p_i}]_{A_{s_i}}\lesssim_{m,n,p_i,q_i}[w_iv^{p_i}]^{2q_i}_{A^{\mathcal{R}}_{q_i}}$, so by Lemma 2.3.10, we have that $C_i\leq\phi^n_{s_i,p_i}([w_i]_{A^{\mathcal{R}}_{p_i}},C_{m,n,p_i,q_i}[w_iv^{p_i}]^{\frac{2q_i}{s_i}}_{A^{\mathcal{R}}_{q_i}})$, and hence, the desired result follows, with

$$C=\frac{2^{m+1}}{(2^p-1)^{1/p}}\prod_{i=1}^m\phi^n_{s_i,p_i}([w_i]_{A^{\mathcal{R}}_{p_i}},C_{m,n,p_i,q_i}[w_iv^{p_i}]^{\frac{2q_i}{s_i}}_{A^{\mathcal{R}}_{q_i}})),$$

which depends on the constants of the weights w_i and $w_iv^{p_i}$ in an increasing way. □

Remark 2.4.7. In the case when $p_1=\dots=p_m=1$, the previous result is a corollary of Theorem 1.4 in [75].

Observe that if we take weights $w_i \in A^{\mathcal{R}}_{p_i}$, $i = 1, \dots, m$, and $v \in RH_\infty$, then the hypotheses of Theorem 2.4.6 are satisfied.

2.4.3 Sawyer-Type Inequalities for $\mathcal{A}_{\mathcal{S}}$ and Whatnot

The next result will be crucial to work with Calderón-Zygmund operators in the mixed restricted weak type setting.

Theorem 2.4.8. *Let $0 < p < \infty$, let $\mathcal{S}$ be an η-sparse collection of cubes, and let v, w be weights. Suppose that there exists $0 < \varepsilon \leq 1$ such that $\varepsilon < p$, $wv^{-\varepsilon} \in A_\infty$, and*

$$[v^{-\varepsilon}]_{RH_\infty(w)} := \sup_Q \frac{w(Q)}{wv^{-\varepsilon}(Q)} \|\chi_Q v^{-\varepsilon}\|_{L^\infty(w)} < \infty.$$

Then, there exists a constant $C > 0$, independent of $\mathcal{S}$, such that the inequality

$$\left\| \frac{\mathcal{A}_{\mathcal{S}}(\vec{f})}{v} \right\|_{L^{p,\infty}(w)} \leq C \left\| \frac{\mathcal{M}(\vec{f})}{v} \right\|_{L^{p,\infty}(w)}$$

holds for every vector of measurable functions $\vec{f} = (f_1, \dots, f_m)$.

Proof. In virtue of Kolmogorov's inequalities, we obtain that

$$\left\| \frac{\mathcal{A}_{\mathcal{S}}(\vec{f})}{v} \right\|_{L^{p,\infty}(w)} \leq \sup_{0<w(F)<\infty} \left\| \frac{\mathcal{A}_{\mathcal{S}}(\vec{f})}{v} \chi_F \right\|_{L^\varepsilon(w)} w(F)^{\frac{1}{p} - \frac{1}{\varepsilon}},$$

where the supremum is taken over all measurable sets F with $0 < w(F) < \infty$. For one of such sets F, and $W := wv^{-\varepsilon}$, we have that

$$\begin{aligned}
\left\| \frac{\mathcal{A}_{\mathcal{S}}(\vec{f})}{v} \chi_F \right\|^{\varepsilon}_{L^\varepsilon(w)} &\leq \int_{\mathbb{R}^n} \sum_{Q \in \mathcal{S}} \chi_Q \left(\frac{\prod_{i=1}^m \fint_Q |f_i|}{v} \right)^{\varepsilon} \chi_F w \\
&= \sum_{Q \in \mathcal{S}} \left(\prod_{i=1}^m \fint_Q |f_i| \right)^{\varepsilon} \left(\frac{1}{W(3Q)} \int_Q \chi_F W \right) W(3Q) =: I.
\end{aligned}$$

Since $W \in A_\infty$, there exists $r \geq 1$ such that $W \in A^{\mathcal{R}}_r$. Hence,

$$\sup_Q \sup_{E \subseteq Q} \frac{|E|}{|Q|} \left(\frac{W(Q)}{W(E)} \right)^{1/r} = \|W\|_{A^{\mathcal{R}}_r} < \infty.$$

By hypothesis, $\mathcal{S}$ is η-sparse, so for each $Q \in \mathcal{S}$,

$$W(3Q) \leq \left(\frac{3^n}{\eta} \|W\|_{A^{\mathcal{R}}_r} \right)^r W(E_Q).$$

Using this, we get that

$$
\begin{aligned}
I &\leq \left(\frac{3^n}{\eta}\|W\|_{A_r^{\mathcal{R}}}\right)^r \sum_{Q\in\mathcal{S}} \left(\prod_{i=1}^m \fint_Q |f_i|\right)^\varepsilon \left(\frac{1}{W(3Q)}\int_Q \chi_F W\right) W(E_Q) \\
&= \left(\frac{3^n}{\eta}\|W\|_{A_r^{\mathcal{R}}}\right)^r \sum_{Q\in\mathcal{S}} \int_{E_Q} \left(\prod_{i=1}^m \fint_Q |f_i|\right)^\varepsilon \left(\frac{1}{W(3Q)}\int_Q \chi_F W\right) W =: II.
\end{aligned}
$$

The sides of an n-dimensional cube have Lebesgue measure 0 in $\mathbb{R}^n$, so we can assume that the cubes in $\mathcal{S}$ are open. For $Q \in \mathcal{S}$ and $z \in E_Q$, we define $Q^z := Q(z, l_Q)$, the open cube of center z and side length twice the side length of Q. We have that $E_Q \subseteq Q \subseteq Q^z \subseteq 3Q$, so

$$
\left(\prod_{i=1}^m \fint_Q |f_i|\right) \chi_{E_Q}(z) \leq \mathcal{M}(\vec{f})(z),
$$

and

$$
\frac{1}{W(3Q)}\int_Q \chi_F W \leq \frac{1}{W(Q^z)}\int_{Q^z} \chi_F W \leq M_W^c(\chi_F)(z).
$$

Since the sets $\{E_Q\}_{Q\in\mathcal{S}}$ are pairwise disjoint, and using Hölder's inequality with parameter $\frac{p}{\varepsilon} > 1$,

$$
\begin{aligned}
II &\leq \left(\frac{3^n}{\eta}\|W\|_{A_r^{\mathcal{R}}}\right)^r \int_{\mathbb{R}^n} \left(\mathcal{M}(\vec{f})\right)^\varepsilon M_W^c(\chi_F) W \\
&\leq \left(\frac{3^n}{\eta}\|W\|_{A_r^{\mathcal{R}}}\right)^r \left\|\left(\frac{\mathcal{M}(\vec{f})}{v}\right)^\varepsilon\right\|_{L^{\frac{p}{\varepsilon},\infty}(w)} \|M_W^c(\chi_F)\|_{L^{(\frac{p}{\varepsilon})',1}(w)} \\
&\leq \frac{p}{p-\varepsilon}\left(\frac{3^n}{\eta}\|W\|_{A_r^{\mathcal{R}}}\right)^r \|M_W^c\|_{L^{(\frac{p}{\varepsilon})',1}(w)} w(F)^{1-\frac{\varepsilon}{p}} \left\|\frac{\mathcal{M}(\vec{f})}{v}\right\|_{L^{p,\infty}(w)}^\varepsilon .
\end{aligned}
$$

Observe that for every measurable function g, $\|M_W^c(g)\|_{L^\infty(w)} \leq \|g\|_{L^\infty(w)}$, and arguing as in the proof of Theorem 5.2.2, it is easy to show that

$$
\|M_W^c(g)\|_{L^{1,\infty}(w)} \leq 24^n [v^{-\varepsilon}]_{RH_\infty(w)} \|g\|_{L^1(w)}.
$$

In particular, and applying Marcinkiewicz's interpolation theorem (see [4, Theorem 4.13]), we conclude that

$$
\|M_W^c\|_{L^{(\frac{p}{\varepsilon})',1}(w)} \leq c_{n,p,\varepsilon} [v^{-\varepsilon}]_{RH_\infty(w)}^{1-\frac{\varepsilon}{p}} < \infty.
$$

Combining the previous estimates, we obtain that

$$\left\|\frac{\mathcal{A}_{\mathcal{S}}(\vec{f})}{v}\chi_F\right\|_{L^{\varepsilon}(w)} w(F)^{\frac{1}{p}-\frac{1}{\varepsilon}}$$
$$\leq \left(\frac{p}{p-\varepsilon}\left(\frac{3^n}{\eta}\|W\|_{A_r^{\mathcal{R}}}\right)^r c_{n,p,\varepsilon}[v^{-\varepsilon}]_{RH_\infty(w)}^{1-\frac{\varepsilon}{p}}\right)^{1/\varepsilon}\left\|\frac{\mathcal{M}(\vec{f})}{v}\right\|_{L^{p,\infty}(w)},$$

and the desired result follows, with

$$C = \inf_{r\geq 1\,:\,W\in A_r^{\mathcal{R}}}\left(\frac{p}{p-\varepsilon}\left(\frac{3^n}{\eta}\|W\|_{A_r^{\mathcal{R}}}\right)^r c_{n,p,\varepsilon}[v^{-\varepsilon}]_{RH_\infty(w)}^{1-\frac{\varepsilon}{p}}\right)^{1/\varepsilon}.$$

□

Remark 2.4.9. For $0<p\leq 1$, if we take v such that $v^\delta\in A_\infty$ for some $\delta>0$, and $w=uv^p$, with $u\in A_1$, the previous result can be established via an extrapolation argument (see [90, Theorem 1.1]).

Under the conditions that $0<p\leq 1$, and $w=uv^p$, we can find weights u and v that satisfy the hypotheses of Theorem 1.1 in [90] but not the ones of Theorem 2.4.8, and vice versa. If we take a non-constant weight $u\in A_1$, and $v=u^{-1/p}$, then $v\in RH_\infty\subseteq A_\infty$, and $uv^p=1$, but for every $0<\varepsilon\leq 1$ such that $\varepsilon<p$, we have that $v^{-\varepsilon}=u^{\varepsilon/p}\in A_1$, and since u is non-constant, $v^{-\varepsilon}\notin RH_\infty$. Similarly, if we take a non-constant weight $v\in A_1$, and $u=v^{-p}$, then $uv^p=1$, and for every $\varepsilon>0$, $uv^{p-\varepsilon}=v^{-\varepsilon}\in RH_\infty\subseteq A_\infty$, but $u\in RH_\infty$ and is non-constant, so $u\notin A_1$ (see Lemma 2.3.1).

The previous examples show that, sometimes, some of the hypotheses of Theorem 2.4.8 may be redundant. Let us be more precise on this fact. If $w\in A_\infty$, and $wv^{-\varepsilon}$ is a weight, then $[v^{-\varepsilon}]_{RH_\infty(w)}<\infty$ implies that $wv^{-\varepsilon}\in A_\infty$. Indeed, given a cube $Q\subseteq\mathbb{R}^n$, and a measurable set $E\subseteq Q$, we have that

$$\frac{wv^{-\varepsilon}(E)}{wv^{-\varepsilon}(Q)} = \frac{1}{wv^{-\varepsilon}(Q)}\int_Q \chi_E wv^{-\varepsilon}$$
$$\leq \frac{w(E)}{wv^{-\varepsilon}(Q)}\|\chi_Q v^{-\varepsilon}\|_{L^\infty(w)} \leq [v^{-\varepsilon}]_{RH_\infty(w)}\frac{w(E)}{w(Q)},$$

and since $w\in A_\infty$, there exist $\delta, C>0$ such that

$$\frac{w(E)}{w(Q)}\leq C\left(\frac{|E|}{|Q|}\right)^\delta,$$

so

$$\frac{wv^{-\varepsilon}(E)}{wv^{-\varepsilon}(Q)}\leq C[v^{-\varepsilon}]_{RH_\infty(w)}\left(\frac{|E|}{|Q|}\right)^\delta, \tag{2.4.3}$$

and hence, $wv^{-\varepsilon}\in A_\infty$ (see [39]).

The next application of Theorem 2.3.8 follows from the combination of Theorems 2.4.6 and 2.4.8, and gives us mixed restricted weak type bounds for multi-variable sparse operators that can also be deduced for other operators, such as multi-linear Calderón-Zygmund operators, using sparse domination techniques (see [71]).

Theorem 2.4.10. *Let $1 \leq p_1, \dots, p_m < \infty$, and let $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. Let $w_1, \dots, w_m$ be weights, with $w_i \in A^{\mathcal{R}}_{p_i}$, $i = 1, \dots, m$, and $\nu_{\vec{w}} = w_1^{p/p_1} \dots w_m^{p/p_m}$. Let v be a weight such that $\nu_{\vec{w}} v^p$ is a weight, and $w_i v^{p_i} \in A_\infty$, $i = 1, \dots, m$. Moreover, suppose that there exists $0 < \varepsilon \leq 1$ such that $\varepsilon < p$, $\nu_{\vec{w}} v^{p-\varepsilon} \in A_\infty$, and $[v^{-\varepsilon}]_{RH_\infty(\nu_{\vec{w}} v^p)} < \infty$. Then, there exists a constant $C > 0$ such that the inequality*

$$\left\| \frac{T(\vec{f})}{v} \right\|_{L^{p,\infty}(\nu_{\vec{w}} v^p)} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i,1}(w_i)}$$

holds for every vector of measurable functions $\vec{f} = (f_1, \dots, f_m)$, where T is either a sparse operator of the form

$$\mathcal{A}_{\mathcal{S}}(\vec{f}) := \sum_{Q \in \mathcal{S}} \left(\prod_{i=1}^m \fint_Q f_i \right) \chi_Q,$$

where $\mathcal{S}$ is an η-sparse collection of dyadic cubes, or any operator that can be conveniently dominated by such sparse operators, like m-linear ω-Calderón-Zygmund operators with ω satisfying the Dini condition.

Remark 2.4.11. In the case when $p_1 = \cdots = p_m = 1$, and T is a multi-linear Calderón-Zygmund operator, the previous result follows from Theorem 1.9 in [75].

In general, there are examples of weights that satisfy the hypotheses of Theorem 2.4.10 apart from the constant weights. For instance, if $1 \leq p_1, \dots, p_m \leq m'$, we can take $w_i = (Mh_i)^{\frac{1-p_i}{m}}$, with $h_i \in L^1_{loc}(\mathbb{R}^n)$, $i = 1, \dots, m$, and $v = \nu_{\vec{w}}^{-\frac{1}{p}}$. Indeed, in virtue of Theorem 2.7 in [9], we have that $w_i \in A^{\mathcal{R}}_{p_i}$, $i = 1, \dots, m$, and $w_i v^{p_i} = \left(\prod_{j \neq i} (Mh_j)^{1/p_j'} \right)^{p_i/m} \in A_1$. Observe that $\nu_{\vec{w}} v^p = 1$, and $v = \left(\prod_{i=1}^m (Mh_i)^{1/p_i'} \right)^{1/m} \in A_1$, so for every $\varepsilon > 0$, $\nu_{\vec{w}} v^{p-\varepsilon} = v^{-\varepsilon} \in RH_\infty \subseteq A_\infty$.

2.4.4 A Dual Sawyer-Type Inequality for M

The last application that we provide of Theorem 2.3.8 can be interpreted as a dual version of it, and generalizes [14, Proposition 2.10].

Theorem 2.4.12. *Fix $p > 1$, and let u and v be weights such that $u \in A^{\mathcal{R}}_p$, $uv^p \in A_\infty$, and for some $0 < \varepsilon \leq 1$, $uv^{p-\varepsilon}$ is a weight, and $[v^{-\varepsilon}]_{RH_\infty(uv^p)} < \infty$. Then, there exists a constant $C > 0$ such that for every measurable function f,*

$$\left\| \frac{M(fuv^{p-1})}{u} \right\|_{L^{p',\infty}(u)} \leq C \|f\|_{L^{p',1}(uv^p)}. \tag{2.4.4}$$

Proof. It is known (see [64]) that there exist a collection $\{\mathscr{D}_\alpha\}_\alpha$ of 2^n general dyadic grids in $\mathbb{R}^n$, and a collection $\{\mathcal{S}_\alpha\}_\alpha$ of $\frac{1}{2}$-sparse families of cubes, with $\mathcal{S}_\alpha \subseteq \mathscr{D}_\alpha$, such that for every measurable function F,

$$MF \leq 2 \cdot 12^n \sum_{\alpha=1}^{2^n} \mathcal{A}_{\mathcal{S}_\alpha}(|F|).$$

Hence,

$$\left\| \frac{M(fuv^{p-1})}{u} \right\|_{L^{p',\infty}(u)} \leq 2 \cdot 24^n \sum_{\alpha=1}^{2^n} \left\| \frac{\mathcal{A}_{\mathcal{S}_\alpha}(|f|uv^{p-1})}{u} \right\|_{L^{p',\infty}(u)}. \tag{2.4.5}$$

By duality, and self-adjointness of $\mathcal{A}_{\mathcal{S}_\alpha}$, and in virtue of Hölder's inequality, we have that

$$\begin{aligned} \left\| \frac{\mathcal{A}_{\mathcal{S}_\alpha}(|f|uv^{p-1})}{u} \right\|_{L^{p',\infty}(u)} &\leq p \sup_{\|g\|_{L^{p,1}(u)} \leq 1} \left\{ \int_{\mathbb{R}^n} \mathcal{A}_{\mathcal{S}_\alpha}(|f|uv^{p-1})|g| \right\} \\ &= p \sup_{\|g\|_{L^{p,1}(u)} \leq 1} \left\{ \int_{\mathbb{R}^n} |f|uv^{p-1}\mathcal{A}_{\mathcal{S}_\alpha}(|g|) \right\} \\ &\leq p \sup_{\|g\|_{L^{p,1}(u)} \leq 1} \left\{ \left\| \frac{\mathcal{A}_{\mathcal{S}_\alpha}(|g|)}{v} \right\|_{L^{p,\infty}(uv^p)} \right\} \|f\|_{L^{p',1}(uv^p)}. \end{aligned} \tag{2.4.6}$$

Given a measurable function g such that $\|g\|_{L^{p,1}(u)} \leq 1$, and in virtue of Theorem 2.4.8, we get that

$$\left\| \frac{\mathcal{A}_{\mathcal{S}_\alpha}(|g|)}{v} \right\|_{L^{p,\infty}(uv^p)} \leq \psi_{\varepsilon,u,v}([v^{-\varepsilon}]_{RH_\infty(uv^p)}) \left\| \frac{Mg}{v} \right\|_{L^{p,\infty}(uv^p)}, \tag{2.4.7}$$

with

$$\psi_{\varepsilon,u,v}([v^{-\varepsilon}]_{RH_\infty(uv^p)}) := \left(\frac{p}{p-\varepsilon} \left(2 \cdot 3^n r [uv^{p-\varepsilon}]_{A_r^{\mathcal{R}}} \right)^r c_{n,p,\varepsilon} [v^{-\varepsilon}]_{RH_\infty(uv^p)}^{1-\frac{\varepsilon}{p}} \right)^{1/\varepsilon}, \tag{2.4.8}$$

where $r \geq 1$ is such that $uv^{p-\varepsilon} \in A_r^{\mathcal{R}}$ (see (2.4.3)). Applying Theorem 2.3.8 and Lemma 2.3.10, we obtain that

$$\left\| \frac{Mg}{v} \right\|_{L^{p,\infty}(uv^p)} \leq \phi_{s,p}^n([u]_{A_p^{\mathcal{R}}}, [uv^p]_{A_s^{\mathcal{R}}}) \|g\|_{L^{p,1}(u)} \leq \phi_{s,p}^n([u]_{A_p^{\mathcal{R}}}, [uv^p]_{A_s^{\mathcal{R}}}), \tag{2.4.9}$$

where $s \geq 1$ is such that $uv^p \in A_s^{\mathcal{R}}$.

Combining the estimates (2.4.5), (2.4.6), (2.4.7), and (2.4.9), we conclude that (2.4.4) holds, with

$$C = 2 \cdot 48^n p \psi_{\varepsilon,u,v}([v^{-\varepsilon}]_{RH_\infty(uv^p)}) \phi^n_{s,p}([u]_{A_p^{\mathcal{R}}}, [uv^p]_{A_s^{\mathcal{R}}}).$$

□

Remark 2.4.13. Observe that if $v = 1$, then in Theorem 2.4.12 we can take $\varepsilon = 1$, and $C = C_{n,p}[u]_{A_p^{\mathcal{R}}}^{p+1}$, and the dependence on u of the constant C is explicit, yielding a refined version of [14, Proposition 2.10].

Note that for $p > 1$, if $u \in A_p$, and v is a weight, then for every measurable function f,

$$\begin{aligned} \left\| \frac{M(fuv^{p-1})}{u} \right\|_{L^{p',\infty}(u)} &\leq \left\| \frac{M(fuv^{p-1})}{u} \right\|_{L^{p'}(u)} = \|M(fuv^{p-1})\|_{L^{p'}(u^{1-p'})} \\ &\leq c_1 [u^{1-p'}]_{A_{p'}}^{\frac{1}{p'-1}} \|fuv^{p-1}\|_{L^{p'}(u^{1-p'})} = c_1 [u]_{A_p} \|f\|_{L^{p'}(uv^p)} \\ &\leq c_2 [u]_{A_p} \|f\|_{L^{p',1}(uv^p)}, \end{aligned} \tag{2.4.10}$$

where in the second line we have used [8, Theorem 2.5] and [44, Proposition 7.1.5]. Hence, we obtain (2.4.4) without assuming that for some $0 < \varepsilon \leq 1$, $[v^{-\varepsilon}]_{RH_\infty(uv^p)} < \infty$. We would like to prove Theorem 2.4.12 without this technical hypothesis, but unfortunately, at the time of writing, we don't know how to do it.

Chapter 3

Two-Variable Mixed Type Extrapolation

> *No, no, no, it's all right. It's supposed to be a little asymmetrical. Apparently, a small flaw somehow improves it.*
>
> Sheldon Cooper, *The Big Bang Theory*, 2018

We devote this chapter to the study of mixed type Rubio de Francia's extrapolation and its applications. For simplicity, we write the two-variable versions of our results; the extension of these to the multi-variable setting is just a matter of notation. In Section 3.1, we expose technical results that we will apply in our work. In Section 3.2, we present our first two-variable restricted weak type extrapolation theorems, precursors of our work on mixed type extrapolation. In Section 3.3, we discuss our main results on mixed type extrapolation, including downwards, upwards, and combined schemes. In Section 3.4, we apply our extrapolation results to produce bounds for product-type operators, averaging operators, bi-linear Fourier multiplier operators, and two-variable commutators.

3.1 Technical Results

In this section, we gather some technical results that we will use throughout this chapter.

The next result will be handy for future computations.

Lemma 3.1.1. *Given real numbers $A, B \geq 0$, and $0 < \vartheta < \varrho$,*

$$\inf_{\gamma>0} \left\{ A\gamma^{-\vartheta} + B\gamma^{\varrho-\vartheta} \right\} = \frac{\varrho}{\varrho-\vartheta} \left(\frac{\varrho-\vartheta}{\vartheta} \right)^{\vartheta/\varrho} A^{1-\frac{\vartheta}{\varrho}} B^{\vartheta/\varrho}.$$

Proof. If $AB = 0$, then the result is clear. Otherwise, observe that the function $h(\gamma) := A\gamma^{-\vartheta} + B\gamma^{\varrho-\vartheta}$, defined for $\gamma > 0$, achieves its minimum at

$$\gamma_0 := \left(\frac{\vartheta}{\varrho-\vartheta} \right)^{1/\varrho} \left(\frac{A}{B} \right)^{1/\varrho},$$

and hence,

$$\inf_{\gamma>0}\left\{A\gamma^{-\vartheta}+B\gamma^{\varrho-\vartheta}\right\}=h(\gamma_0)=\left(1+\frac{\vartheta}{\varrho-\vartheta}\right)\left(\frac{\varrho-\vartheta}{\vartheta}\right)^{\vartheta/\varrho}A^{1-\frac{\vartheta}{\varrho}}B^{\vartheta/\varrho}.$$

□

The next result ensures that certain functions are locally integrable.

Lemma 3.1.2. *Let $p\geq 1$, and let w be a weight.*

(a) If $w\in A_p$, and $f\in L^p(w)$, then $f\in L^1_{loc}(\mathbb{R}^n)$.

(b) If $w\in A_p^{\mathcal{R}}$, and $f\in L^{p,1}(w)$, then $f\in L^1_{loc}(\mathbb{R}^n)$.

Proof. To prove (a), in virtue of Hölder's inequality we have that for every cube $Q\subseteq\mathbb{R}^n$,

$$\begin{aligned}\int_Q|f|=\int_{\mathbb{R}^n}|f|\chi_Q w^{-1}w&\leq\|f\|_{L^p(w)}\|\chi_Q w^{-1}\|_{L^{p'}(w)}\\&\leq\|f\|_{L^p(w)}[w]_{A_p}^{1/p}\frac{|Q|}{w(Q)^{1/p}}<\infty.\end{aligned}$$

To prove (b), in virtue of Hölder's inequality we have that for every cube $Q\subseteq\mathbb{R}^n$,

$$\begin{aligned}\int_Q|f|=\int_{\mathbb{R}^n}|f|\chi_Q w^{-1}w&\leq\|f\|_{L^{p,1}(w)}\|\chi_Q w^{-1}\|_{L^{p',\infty}(w)}\\&\leq\|f\|_{L^{p,1}(w)}[w]_{A_p^{\mathcal{R}}}\frac{|Q|}{w(Q)^{1/p}}<\infty.\end{aligned}$$

□

The next result allows us to construct nice weights.

Lemma 3.1.3. *Let $1\leq q\leq p$, and let w be a weight. For a measurable function $h\in L^1_{loc}(\mathbb{R}^n)$, let $v=(Mh)^{q-p}w$.*

(a) If $1<q<p$, and $w\in A_q$, then $v\in A_p$, and

$$[v]_{A_p}\leq C[w]_{A_q}^{1+\frac{p-1}{q-1}},$$

with C independent of h.

(b) If $w\in\widehat{A}_q$, then $v\in A_p^{\mathcal{R}}$, and

$$\|v\|_{A_p^{\mathcal{R}}}\leq C\|w\|_{\widehat{A}_q}^{q/p},$$

with C independent of h.

Proof. To prove (a), since $w \in A_q$, we can find weights $u_1, u_2 \in A_1$ such that $w = u_1^{1-q} u_2$, with $[u_1]_{A_1} \le c_1 [w]_{A_q}^{\frac{1}{q-1}}$ and $[u_2]_{A_1} \le c_2 [w]_{A_q}$. The details on the construction of such A_1 weights are available in [22, Theorem 4.2]. Moreover,

$$v = \left((Mh)^{\frac{q-p}{1-p}} u_1^{\frac{1-q}{1-p}} \right)^{1-p} u_2 =: \widetilde{u}_1^{1-p} u_2,$$

and in virtue of [14, Lemma 2.12], $\widetilde{u}_1 \in A_1$, with $[\widetilde{u}_1]_{A_1} \le c_3 [u_1]_{A_1}$, and c_3 independent of h. Hence, $v \in A_p$, with

$$[v]_{A_p} \le [\widetilde{u}_1]_{A_1}^{p-1} [u_2]_{A_1} \le c_3^{p-1} [u_1]_{A_1}^{p-1} [u_2]_{A_1} \le c_1^{p-1} c_2 c_3^{p-1} [w]_{A_q}^{1+\frac{p-1}{q-1}},$$

and the desired result follows, with $C = c_1^{p-1} c_2 c_3^{p-1}$.

To prove (b), since $w \in \widehat{A}_q$, we can find a measurable function $h_1 \in L^1_{loc}(\mathbb{R}^n)$ and a weight $u \in A_1$ such that $w = (Mh_1)^{1-q} u$, with $[u]_{A_1}^{1/q} \le 2\|w\|_{\widehat{A}_q}$. Note that if $p = 1$, then $q = 1$, and $v = w = u$. If $p > 1$, then

$$v = ((Mh)^{1-p} u)^{\frac{q-p}{1-p}} ((Mh_1)^{1-p} u)^{\frac{1-q}{1-p}},$$

and it follows from [9, Lemma 3.8] that $v \in A_p^{\mathcal{R}}$, with

$$\|v\|_{A_p^{\mathcal{R}}} \le c[u]_{A_1}^{1/p} \le 2^{q/p} c \, \|w\|_{\widehat{A}_q}^{q/p},$$

and the desired result follows, with $C = 2^{q/p} c$, independent of h. □

The next result also allows us to construct nice weights.

Lemma 3.1.4. *Let $1 < p < q$, and let w be a weight. For a measurable function $h \in L^1_{loc}(\mathbb{R}^n)$, let $v = w^{\frac{p-1}{q-1}} (Mh)^{\frac{q-p}{q-1}}$.*

(a) If $w \in A_q$, then $v \in A_p$, and

$$[v]_{A_p} \le C[w]_{A_q}^{1+\frac{p-1}{q-1}},$$

with C independent of h.

(b) If $w \in \widehat{A}_q$, then $v \in \widehat{A}_p$, and

$$\|v\|_{\widehat{A}_p} \le C\|w\|_{\widehat{A}_q}^{q/p},$$

with C independent of h.

Proof. To prove (a), since $w \in A_q$, we can find weights $u_1, u_2 \in A_1$ such that $w = u_1^{1-q} u_2$, with $[u_1]_{A_1} \le c_1 [w]_{A_q}^{\frac{1}{q-1}}$ and $[u_2]_{A_1} \le c_2 [w]_{A_q}$. Moreover,

$$v = u_1^{1-p} u_2^{\frac{p-1}{q-1}} (Mh)^{\frac{q-p}{q-1}} =: u_1^{1-p} \widetilde{u}_2,$$

and in virtue of [14, Lemma 2.12], $\widetilde{u}_2 \in A_1$, with $[\widetilde{u}_2]_{A_1} \le c_3 [u_2]_{A_1}$, and c_3 independent of h. Hence, $v \in A_p$, with

$$[v]_{A_p} \le [u_1]_{A_1}^{p-1} [\widetilde{u}_2]_{A_1} \le c_3 [u_1]_{A_1}^{p-1} [u_2]_{A_1} \le c_1^{p-1} c_2 c_3 [w]_{A_q}^{1+\frac{p-1}{q-1}},$$

and the desired result follows, with $C = c_1^{p-1} c_2 c_3$.

To prove (b), since $w \in \widehat{A}_q$, we can find a measurable function $h_1 \in L^1_{loc}(\mathbb{R}^n)$ and a weight $u \in A_1$ such that $w = (Mh_1)^{1-q} u$, with $[u]_{A_1}^{1/q} \le 2\|w\|_{\widehat{A}_q}$. Note that

$$v = (Mh_1)^{1-p} u^{\frac{p-1}{q-1}} (Mh)^{\frac{q-p}{q-1}} =: (Mh_1)^{1-p} \widetilde{u}.$$

Applying [14, Lemma 2.12], we see that $\widetilde{u} \in A_1$, with $[\widetilde{u}]_{A_1} \le c[u]_{A_1}$, and c independent of h. Hence, $v \in \widehat{A}_p$, with

$$\|v\|_{\widehat{A}_p} \le [\widetilde{u}]_{A_1}^{1/p} \le c^{1/p} [u]_{A_1}^{1/p} \le 2^{q/p} c^{1/p} \|w\|_{\widehat{A}_q}^{q/p},$$

and the desired result follows, with $C = 2^{q/p} c^{1/p}$. □

The following result also lets us construct nice weights.

Lemma 3.1.5. *Let $1 \le p, q < \infty$. Let $u \in A_q$, $v \in A_p$, and take $W = \left(\frac{u}{v}\right)^{1/p}$. Then, $W \in A_{1+\frac{q}{p}}$, and*

$$[W]_{A_{1+\frac{q}{p}}} \le [u]_{A_q}^{1/p} [v]_{A_p}^{1/p}.$$

Proof. Note that $\left(1 + \frac{q}{p}\right)' = 1 + \frac{p}{q}$, and $\frac{1}{p}\left(1 - \left(1 + \frac{q}{p}\right)'\right) = -\frac{1}{q}$, so

$$[W]_{A_{1+\frac{q}{p}}} = \sup_Q \left(\fint_Q \left(\frac{u}{v}\right)^{1/p} \right) \left(\fint_Q \left(\frac{u}{v}\right)^{-\frac{1}{q}} \right)^{q/p}. \tag{3.1.1}$$

Fix a cube $Q \subseteq \mathbb{R}^n$. To estimate the first factor in (3.1.1), in virtue of Hölder's inequality with exponent $p \ge 1$, we get that

$$\fint_Q \left(\frac{u}{v}\right)^{1/p} \le \left(\fint_Q u \right)^{1/p} \left(\fint_Q v^{1-p'} \right)^{\frac{p-1}{p}}, \tag{3.1.2}$$

where the last term is interpreted as $\operatorname{ess\,sup}_{x \in Q} v(x)^{-1}$ if $p = 1$.

Similarly, to estimate the second factor in (3.1.1), in virtue of Hölder's inequality with exponent $q \geq 1$, we have that

$$\fint_Q \left(\frac{u}{v}\right)^{-\frac{1}{q}} \leq \left(\fint_Q v\right)^{1/q} \left(\fint_Q u^{1-q'}\right)^{\frac{q-1}{q}}, \tag{3.1.3}$$

where the last term is interpreted as $\operatorname{ess\,sup}_{x\in Q} u(x)^{-1}$ if $q = 1$.

Combining (3.1.1), (3.1.2) and (3.1.3), we obtain that

$$\begin{aligned}
[W]_{A_{1+\frac{q}{p}}} &= \sup_Q \left(\fint_Q \left(\frac{u}{v}\right)^{1/p}\right) \left(\fint_Q \left(\frac{u}{v}\right)^{-\frac{1}{q}}\right)^{q/p} \\
&\leq \sup_Q \left(\fint_Q u\right)^{1/p} \left(\fint_Q u^{1-q'}\right)^{\frac{q-1}{p}} \left(\fint_Q v\right)^{1/p} \left(\fint_Q v^{1-p'}\right)^{\frac{p-1}{p}} \\
&\leq [u]_{A_q}^{1/p} [v]_{A_p}^{1/p}.
\end{aligned}$$

□

The next result gives us information about a certain weight.

Lemma 3.1.6. *Let $1 \leq q_1, q_2 < \infty$, and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Let $w_1 \in A_{q_1}$, $w_2 \in \widehat{A}_{q_2}$, and take $w = w_1^{q/q_1} w_2^{q/q_2}$. Then, $w \in A_{2q}^{\mathcal{R}}$, and*

$$[w]_{A_{2q}^{\mathcal{R}}} \leq \psi([w_1]_{A_{q_1}}) \|w_2\|_{\widehat{A}_{q_2}}^{1/2},$$

where $\psi : [1,\infty) \longrightarrow [0,\infty)$ is an increasing function.

Proof. If $q_2 = 1$, then $w_2 \in A_1$, and in virtue of [44, Exercise 7.1.5], we get that $w \in A_{2q}$, with $[w]_{A_{2q}^{\mathcal{R}}} \leq [w]_{A_{2q}}^{\frac{1}{2q}}$, and

$$[w]_{A_{2q}} \leq [w_1]_{A_{q_1}}^{q/q_1} [w_2]_{A_1}^{q}. \tag{3.1.4}$$

If $q_2 > 1$, we can find a measurable function $h \in L^1_{loc}(\mathbb{R}^n)$ and a weight $u \in A_1$ such that $w_2 = (Mh)^{1-q_2} u$, with $[u]_{A_1}^{1/q_2} \leq 2\|w_2\|_{\widehat{A}_{q_2}}$. If $q_1 > 1$, we can also find weights $u_1, u_2 \in A_1$ such that $w_1 = u_1^{1-q_1} u_2$, with $[u_1]_{A_1} \leq c_1 [w_1]_{A_{q_1}}^{\frac{1}{q_1-1}}$ and $[u_2]_{A_1} \leq c_2 [w_1]_{A_{q_1}}$. Note that $2q > 1$, and $\theta := \frac{q}{q_1}\frac{1-q_1}{1-2q} \in (0,1)$, and by [14, Lemma 2.12], $\widetilde{u}_1 := u_1^{\theta}(Mh_2)^{1-\theta} \in A_1$, with $[\widetilde{u}_1]_{A_1} \leq c_3 [u_1]_{A_1}$, and c_3 independent of h. Now,

$$\begin{aligned}
\widetilde{u}_1^{1-2q} u_2^{q/q_1} u^{q/q_2} &= u_1^{\frac{q}{q_1}(1-q_1)} (Mh)^{1-q-\frac{q}{q_1}} u_2^{q/q_1} u^{q/q_2} \\
&= \left(u_1^{1-q_1} u_2\right)^{q/q_1} \left((Mh)^{1-q_2} u\right)^{q/q_2} = w,
\end{aligned}$$

and $w \in A_{2q}$, with

$$[w]_{A_{2q}} \leq [\widetilde{u}_1]_{A_1}^{2q-1}[u_2^{q/q_1}u^{q/q_2}]_{A_1}$$
$$\leq c_3^{2q-1}[u_1]_{A_1}^{2q-1}[u_2]_{A_1}^{q/q_1}[u]_{A_1}^{q/q_2} \leq 2^q c_1^{2q-1}c_2^{q/q_1}c_3^{2q-1}[w_1]_{A_{q_1}}^{\frac{q}{q_1}+\frac{2q-1}{q_1-1}}\|w_2\|_{\widehat{A}_{q_2}}^q.$$

If $q_2 > 1$ and $q_1 = 1$, then $w_1 \in A_1$, and $\frac{q}{q_2}\frac{1-q_2}{1-2q} = 1$, so

$$w = (Mh)^{1-2q}w_1^q u^{q/q_2} \in \widehat{A}_{2q},$$

with $[w]_{A_{2q}^{\mathcal{R}}} \leq c_4\|w\|_{\widehat{A}_{2q}}$, and

$$\|w\|_{\widehat{A}_{2q}} \leq [w_1^q u^{q/q_2}]_{A_1}^{\frac{1}{2q}} \leq [w_1]_{A_1}^{1/2}[u]_{A_1}^{\frac{1}{2q_2}} \leq 2^{1/2}[w_1]_{A_1}^{1/2}\|w_2\|_{\widehat{A}_{q_2}}^{1/2}.$$

In any case, $w \in A_{2q}^{\mathcal{R}}$, and $[w]_{A_{2q}^{\mathcal{R}}} \leq \psi([w_1]_{A_{q_1}})\|w_2\|_{\widehat{A}_{q_2}}^{1/2}$, with

$$\psi([w_1]_{A_{q_1}}) = \begin{cases} [w_1]_{A_{q_1}}^{\frac{1}{2q_1}}, & q_1 \geq 1,\ q_2 = 1, \\ 2^{1/2}c_1^{1-\frac{1}{2q}}c_2^{\frac{1}{2q_1}}c_3^{1-\frac{1}{2q}}[w_1]_{A_{q_1}}^{\frac{1}{2q_1}+\frac{1}{2q}\frac{2q-1}{q_1-1}}, & q_1 > 1,\ q_2 > 1, \\ 2^{1/2}c_4[w_1]_{A_1}^{1/2}, & q_1 = 1,\ q_2 > 1. \end{cases}$$

□

The following result is a precise open property for A_p weights, and it is a particular case of [55, Theorem 1.2].

Lemma 3.1.7. *Let $1 < p < \infty$, and let $w \in A_p$ and $\sigma = w^{1-p'}$. For*

$$\varepsilon = \frac{p-1}{1 + 6 \cdot 800^n[\sigma]_{A_\infty}},$$

we have that $w \in A_{p-\varepsilon}$, and

$$[w]_{A_{p-\varepsilon}} \leq 2^{2np+p-1}[w]_{A_p}.$$

The next result is the classical weak type extrapolation theorem for one-variable operators (see [42, Observation] and [101, Theorem 4]), but with explicit constants. It follows immediately from the classical Rubio de Francia's extrapolation theorem for one-variable operators with explicit constants presented in [36, Theorem 1], the bounds for the Hardy-Littlewood maximal operator proved in [54, Corollary 1.10], and the argument in the proof of [28, Corollary 3.10].

Theorem 3.1.8. *Let T be a one-variable operator defined for measurable functions. Suppose that for some $1 \leq r < \infty$, and every weight $u \in A_r$,*

$$T : L^r(u) \longrightarrow L^{r,\infty}(u),$$

with constant bounded by $\phi([u]_{A_r})$, where $\phi : [1,\infty) \longrightarrow [0,\infty)$ is an increasing function. Then, for every $1 < p < \infty$, and every weight $w \in A_p$,

$$T : L^p(w) \longrightarrow L^{p,\infty}(w),$$

with constant bounded by

$$\psi([w]_{A_p}) = \begin{cases} 2^{1/r}\phi(2(c_n p)^{\frac{p-r}{p-1}}[w]_{A_p}), & p > r, \\ \phi([w]_{A_p}), & p = r, \\ 2^{\frac{r-1}{r}}\phi(2^{r-1}((c_n p')^{p-r}[w]_{A_p})^{\frac{r-1}{p-1}}), & p < r. \end{cases}$$

The next result shows the strength of a restricted weak type operator.

Theorem 3.1.9. *Let T be a sub-linear operator defined for measurable functions. Suppose that for some $1 < r < \infty$, and every weight $v \in A_r^{\mathcal{R}}$,*

$$T : L^{r,1}(v) \longrightarrow L^{r,\infty}(v), \tag{3.1.5}$$

with constant bounded by $\varphi([v]_{A_r^{\mathcal{R}}})$, where $\varphi : [1,\infty) \longrightarrow [0,\infty)$ is an increasing function. Then, for every weight $w \in A_r$,

$$T : L^{r,1}(w) \longrightarrow L^{r,1}(w), \tag{3.1.6}$$

with constant bounded by $\Phi([w]_{A_r})$, where $\Phi : [1,\infty) \longrightarrow [0,\infty)$ is an increasing function.

Proof. To establish this result, we will perform two different extrapolation procedures and an interpolation argument, keeping track of the constants at each step. In virtue of (3.1.5), and Theorem 3.11 and Remark 3.12 in [9], there exists an increasing function $\phi : [1,\infty) \longrightarrow [0,\infty)$, depending only on φ, r, and the dimension n, such that for every weight $u \in A_r$,

$$T : L^r(u) \longrightarrow L^{r,\infty}(u), \tag{3.1.7}$$

with constant bounded by $\phi([u]_{A_r})$.

Take a weight $w \in A_r$, and let $0 < \varepsilon < r-1$ be as in Lemma 3.1.7. Combining (3.1.7) and Theorem 3.1.8, we obtain that for every weight $V \in A_{r-\varepsilon}$,

$$T : L^{r-\varepsilon}(V) \longrightarrow L^{r-\varepsilon,\infty}(V), \tag{3.1.8}$$

with constant bounded by

$$\begin{aligned} \psi_-([V]_{A_{r-\varepsilon}}) &:= 2^{\frac{r-1}{r}}\phi(2^{r-1}((c_n(r-\varepsilon)')^{-\varepsilon}[V]_{A_{r-\varepsilon}})^{\frac{r-1}{r-\varepsilon-1}}) \\ &\leq 2^{\frac{r-1}{r}}\phi(2^{r-1}[V]^2_{A_{r-\varepsilon}}), \end{aligned} \tag{3.1.9}$$

since ϕ is increasing, $0 < (c_n(r-\varepsilon)')^{-\varepsilon} \leq 1$, and

$$1 < \frac{r-1}{r-\varepsilon-1} = \frac{1}{1 - \frac{1}{1+6\cdot 800^n[w^{1-r'}]_{A_\infty}}} = 1 + \frac{1}{6\cdot 800^n[w^{1-r'}]_{A_\infty}} < 2.$$

Similarly, for every weight $W \in A_{r+\varepsilon}$,

$$T : L^{r+\varepsilon}(W) \longrightarrow L^{r+\varepsilon,\infty}(W), \tag{3.1.10}$$

with constant bounded by

$$\begin{aligned}\psi_+([W]_{A_{r+\varepsilon}}) &:= 2^{1/r}\phi(2(c_n(r+\varepsilon))^{\frac{\varepsilon}{r+\varepsilon-1}}[W]_{A_{r+\varepsilon}}) \\ &\leq 2^{1/r}\phi(2c_n(2r-1)[W]_{A_{r+\varepsilon}}),\end{aligned} \tag{3.1.11}$$

since ϕ is increasing, $1 < c_n(r+\varepsilon) < c_n(2r-1)$, and $0 < \frac{\varepsilon}{r+\varepsilon-1} < 1$.

By our choice of ε, we know that $w \in A_{r-\varepsilon}$, and $[w]_{A_{r-\varepsilon}} \leq 2^{2nr+r-1}[w]_{A_r}$, so from (3.1.8) and (3.1.9), we get that

$$T : L^{r-\varepsilon}(w) \longrightarrow L^{r-\varepsilon,\infty}(w), \tag{3.1.12}$$

with constant bounded by

$$\Psi_-([w]_{A_r}) := 2^{\frac{r-1}{r}}\phi(2^{4nr+3r-3}[w]_{A_r}^2).$$

Similarly, $w \in A_{r+\varepsilon}$, and $[w]_{A_{r+\varepsilon}} \leq [w]_{A_r}$, so from (3.1.10) and (3.1.11), we have that

$$T : L^{r+\varepsilon}(w) \longrightarrow L^{r+\varepsilon,\infty}(w), \tag{3.1.13}$$

with constant bounded by

$$\Psi_+([w]_{A_r}) := 2^{1/r}\phi(2c_n(2r-1)[w]_{A_r}).$$

Applying Marcinkiewicz's interpolation theorem with explicit constants (see [44, Theorem 1.4.19]), we can interpolate between (3.1.12) and (3.1.13), and deduce that (3.1.6) holds, with constant bounded by

$$C\left(\Psi_-([w]_{A_r})\right)^{1-\theta}\left(\Psi_+([w]_{A_r})\right)^{\theta}, \tag{3.1.14}$$

where $\theta := \frac{r+\varepsilon}{2r} \in (0,1)$, and

$$C := 8\frac{2^{1/r}\left(\frac{r+\varepsilon}{r}\right)^{2\theta}C_-^{1-\theta}C_+^{\theta}}{\left(\frac{1}{r-\varepsilon}-\frac{1}{r}\right)^{2(1-\theta)}\left(\frac{1}{r}-\frac{1}{r+\varepsilon}\right)^{2\theta}}, \tag{3.1.15}$$

with

$$C_\pm := 2^{8+\frac{4}{r\pm\varepsilon}}\left(\frac{r\pm\varepsilon}{r\pm\varepsilon-\frac{1}{2}}\right)^4 c,$$

and $c := \left(1-\frac{1}{\sqrt{2}}\right)^{-2}(\log 2)^{-2}$.

Note that

$$\begin{aligned}(\Psi_-([w]_{A_r}))^{1-\theta}(\Psi_+([w]_{A_r}))^{\theta} &\leq \max\{\Psi_-([w]_{A_r}), \Psi_+([w]_{A_r})\}\\ &\leq 2\phi(\max\{2^{4nr+3r-3}, 2c_n(2r-1)\}[w]_{A_r}^2)\\ &=: \Psi([w]_{A_r}).\end{aligned} \tag{3.1.16}$$

It remains to control (3.1.15) in terms of $[w]_{A_r}$. Since $1 < r \pm \varepsilon$, we have that $1 < \frac{r\pm\varepsilon}{r\pm\varepsilon-\frac{1}{2}} < 2$, so $C_\pm \leq 2^{16}c$, and $C_-^{1-\theta}C_+^{\theta} \leq \max\{C_-, C_+\} \leq 2^{16}c$. Hence,

$$C \leq 8\cdot 2^{16+\frac{1}{r}}c\frac{(r-\varepsilon)^{2(1-\theta)}r^{2(1-\theta)}(r+\varepsilon)^{4\theta}}{\varepsilon^2} \leq 8\cdot 2^{16+\frac{1}{r}}c\frac{r^2r^2(2r-1)^4}{\varepsilon^2} =: \frac{c_r}{\varepsilon^2}.$$

In virtue of [54, Proposition 2.2], we know that

$$1 \leq [w^{1-r'}]_{A_\infty} \leq C_n[w^{1-r'}]_{A_{r'}} = C_n[w]_{A_r}^{\frac{1}{r-1}},$$

so

$$\frac{1}{\varepsilon} = \frac{1+6\cdot 800^n[w^{1-r'}]_{A_\infty}}{r-1} \leq \frac{1+6\cdot 800^n}{r-1}C_n[w]_{A_r}^{\frac{1}{r-1}} =: C_{n,r}[w]_{A_r}^{\frac{1}{r-1}},$$

and we get that $C \leq c_rC_{n,r}^2[w]_{A_r}^{\frac{2}{r-1}}$. Finally, with this estimate, (3.1.14), and (3.1.16), we conclude that the desired result holds, with

$$\Phi([w]_{A_r}) = c_rC_{n,r}^2[w]_{A_r}^{\frac{2}{r-1}}\Psi([w]_{A_r}).$$

□

3.2 First Steps Towards Restricted Weak Type Extrapolation

In this section, we present the first results on two-variable restricted weak type extrapolation that we have obtained. The ideas and techniques behind their proofs will be crucial for our work on mixed type extrapolation, and we will refine and extend most of them in the next chapter to produce general multi-variable restricted weak type extrapolation schemes.

The following theorem is a two-variable extension of [9, Theorem 2.11]. The proofs of both results share various common points, although the two-variable setting gives rise to non-trivial differences. In Figure 3.1 you can find a pictorial representation of this scheme.

Theorem 3.2.1. *Given measurable functions f_1, f_2, and g, suppose that for some $1 < r < \infty$, and all weights $v_1, v_2 \in \widehat{A}_r$,*

$$\|g\|_{L^{\frac{r}{2},\infty}(v_1^{1/2}v_2^{1/2})} \leq \varphi(\|v_1\|_{\widehat{A}_r}, \|v_2\|_{\widehat{A}_r})\,\|f_1\|_{L^{r,1}(v_1)}\,\|f_2\|_{L^{r,1}(v_2)}, \tag{3.2.1}$$

where $\varphi : [1,\infty)^2 \longrightarrow [0,\infty)$ is a function increasing in each variable. Then, for all weights $w_1, w_2 \in A_1$,

$$\|g\|_{L^{\frac{1}{2},\infty}(w_1^{1/2}w_2^{1/2})} \le \Phi([w_1]_{A_1},[w_2]_{A_1}) \, \|f_1\|_{L^{1,\frac{1}{r}}(w_1)} \, \|f_2\|_{L^{1,\frac{1}{r}}(w_2)}, \tag{3.2.2}$$

where $\Phi : [1,\infty)^2 \longrightarrow [0,\infty)$ is a function increasing in each variable.

Proof. Pick weights $w_1, w_2 \in A_1$, and write $w = w_1^{1/2}w_2^{1/2}$. If for some $i = 1,2$, $\|f_i\|_{L^{1,\frac{1}{r}}(w_i)} = \infty$, then there is nothing to prove, so we may assume that $\|f_i\|_{L^{1,\frac{1}{r}}(w_i)} < \infty$, for $i = 1,2$. In particular, f_1 and f_2 are locally integrable (see Lemma 3.1.2). Fix $y > 0$ and $\gamma > 0$. We have that

$$\begin{aligned}
\lambda_g^w(y) &= \int_{\{|g|>y\}} w = \int_{\{|g|>y,\, Mf_1Mf_2>\gamma y\}} w + \int_{\{|g|>y,\, Mf_1Mf_2\le\gamma y\}} w \\
&\le \lambda^w_{Mf_1Mf_2}(\gamma y) + \int_{\{|g|>y,\, Mf_1Mf_2\le\gamma y\}} w \\
&\le \lambda^w_{Mf_1Mf_2}(\gamma y) + \int_{\{|g|>y\}} \left(\frac{\gamma y}{Mf_1Mf_2}\right)^{\frac{r-1}{2}} w =: I + II.
\end{aligned} \tag{3.2.3}$$

To estimate the term I, we apply Theorem 2.4.1 to obtain that

$$\begin{aligned}
I &= \frac{(\gamma y)^{1/2}}{(\gamma y)^{1/2}}\lambda^w_{Mf_1Mf_2}(\gamma y) \le \frac{1}{(\gamma y)^{1/2}}\|Mf_1Mf_2\|^{1/2}_{L^{\frac{1}{2},\infty}(w)} \\
&\le \frac{\phi^{1/2}_{w_1,w_2}}{(\gamma y)^{1/2}}\|f_1\|^{1/2}_{L^1(w_1)}\|f_2\|^{1/2}_{L^1(w_2)} \le r^{1-r}\frac{\phi^{1/2}_{w_1,w_2}}{(\gamma y)^{1/2}}\|f_1\|^{1/2}_{L^{1,\frac{1}{r}}(w_1)}\|f_2\|^{1/2}_{L^{1,\frac{1}{r}}(w_2)},
\end{aligned} \tag{3.2.4}$$

with $\phi_{w_1,w_2} := \phi([w_1]_{A_1},[w_2]_{A_1})$. The last inequality follows from the fact that for $i = 1,2$,

$$\|f_i\|_{L^1(w_i)} \le r^{1-r}\|f_i\|_{L^{1,\frac{1}{r}}(w_i)}, \tag{3.2.5}$$

as shown in (2.1.1) (see [44, Proposition 1.4.10]).

To estimate the term II, observe that for $i = 1,2$, $v_i := (Mf_i)^{1-r}w_i \in \widehat{A}_r$, so by (3.2.1) we get that

$$\begin{aligned}
II &= \frac{(\gamma y)^{r/2}}{(\gamma y)^{1/2}} \int_{\{|g|>y\}} ((Mf_1)^{1-r}w_1)^{1/2}((Mf_2)^{1-r}w_2)^{1/2} \\
&\le \frac{\gamma^{r/2}}{(\gamma y)^{1/2}}\|g\|^{r/2}_{L^{\frac{r}{2},\infty}(v_1^{1/2}v_2^{1/2})} \\
&\le \frac{\gamma^{r/2}}{(\gamma y)^{1/2}}\varphi(\|v_1\|_{\widehat{A}_r},\|v_2\|_{\widehat{A}_r})^{r/2}\,\|f_1\|^{r/2}_{L^{r,1}(v_1)}\,\|f_2\|^{r/2}_{L^{r,1}(v_2)}.
\end{aligned} \tag{3.2.6}$$

Note that for $i = 1, 2$,

$$\begin{aligned} \|f_i\|_{L^{r,1}(v_i)} &= r\int_0^\infty \left(\int_{\{|f_i|>z\}} v_i\right)^{1/r} dz \\ &\leq r\int_0^\infty z^{1/r}\left(\int_{\{|f_i|>z\}} w_i\right)^{1/r} \frac{dz}{z} = r\,\|f_i\|^{1/r}_{L^{1,\frac{1}{r}}(w_i)}, \end{aligned} \tag{3.2.7}$$

because if $x \in \{|f_i| > z\}$, then $Mf_i(x)^{1-r} \leq |f_i(x)|^{1-r} \leq z^{1-r}$. Therefore, from (3.2.6) we deduce that

$$II \leq r^r \frac{\gamma^{r/2}}{(\gamma y)^{1/2}} \varphi(\|v_1\|_{\widehat{A}_r}, \|v_2\|_{\widehat{A}_r})^{r/2} \|f_1\|^{1/2}_{L^{1,\frac{1}{r}}(w_1)} \|f_2\|^{1/2}_{L^{1,\frac{1}{r}}(w_2)}. \tag{3.2.8}$$

For $i = 1, 2$, $\|v_i\|_{\widehat{A}_r} \leq [w_i]^{1/r}_{A_1}$, so by the monotonicity of φ and combining the estimates (3.2.3), (3.2.4), and (3.2.8), we conclude that

$$\begin{aligned} \lambda^w_g(y) \leq \frac{1}{y^{1/2}} &\left(r^{1-r}\gamma^{-\frac{1}{2}}\phi^{1/2}_{w_1,w_2} + r^r\gamma^{\frac{r-1}{2}}\varphi([w_1]^{1/r}_{A_1}, [w_2]^{1/r}_{A_1})^{r/2}\right) \\ &\times \|f_1\|^{1/2}_{L^{1,\frac{1}{r}}(w_1)} \|f_2\|^{1/2}_{L^{1,\frac{1}{r}}(w_2)}, \end{aligned}$$

and taking the infimum over all $\gamma > 0$, it follows from Lemma 3.1.1 with $A := r^{1-r}\phi^{1/2}_{w_1,w_2}$, $B := r^r\varphi([w_1]^{1/r}_{A_1}, [w_2]^{1/r}_{A_1})^{r/2}$, $\varrho := \frac{r}{2}$, and $\vartheta := \frac{1}{2}$, that

$$\lambda^w_g(y) \leq r^{3-r}(r')^{1/r'} \frac{\phi^{\frac{1}{2r'}}_{w_1,w_2}}{y^{1/2}} \varphi([w_1]^{1/r}_{A_1}, [w_2]^{1/r}_{A_1})^{1/2} \|f_1\|^{1/2}_{L^{1,\frac{1}{r}}(w_1)} \|f_2\|^{1/2}_{L^{1,\frac{1}{r}}(w_2)}.$$

Finally, multiplying this last expression by $y^{1/2}$, raising everything to the power two, and taking the supremum over all $y > 0$, we see that (3.2.2) holds, with

$$\Phi([w_1]_{A_1}, [w_2]_{A_1}) = r^{6-2r}(r')^{2/r'}\phi^{1/r'}_{w_1,w_2}\varphi([w_1]^{1/r}_{A_1}, [w_2]^{1/r}_{A_1}),$$

which depends on the constants $[w_1]_{A_1}$ and $[w_2]_{A_1}$ in an increasing way. □

We have presented Theorem 3.2.1 in its general form, for triples of functions (f_1, f_2, g). In the next corollary, we deduce the corresponding extrapolation scheme for two-variable operators. For convenience, we provide a pictorial representation of it in Figure 3.1.

Corollary 3.2.2. *Let T be a two-variable operator defined for measurable functions f_1 and f_2. Suppose that for some $1 < r < \infty$, and all weights $v_1, v_2 \in \widehat{A}_r$,*

$$T : L^{r,1}(v_1) \times L^{r,1}(v_2) \longrightarrow L^{\frac{r}{2},\infty}(v_1^{1/2}v_2^{1/2}), \tag{3.2.9}$$

with constant bounded by $\varphi(\|v_1\|_{\widehat{A}_r}, \|v_2\|_{\widehat{A}_r})$ as in (3.2.1). Then, for all weights $w_1, w_2 \in A_1$,

$$T : L^{1,\frac{1}{r}}(w_1) \times L^{1,\frac{1}{r}}(w_2) \longrightarrow L^{\frac{1}{2},\infty}(w_1^{1/2}w_2^{1/2}), \tag{3.2.10}$$

with constant bounded by $\Phi([w_1]_{A_1}, [w_2]_{A_1})$ *as in* (3.2.2).

Proof. Let $g := |T(f_1, f_2)|$, and for every natural number $N \geq 1$, let $g_N := \min\{g, N\}\chi_{B(0,N)}$, where $B(0, N) := \{x \in \mathbb{R}^n : |x| < N\}$. Note that for every $N \geq 1$, $g_N \leq g$, so from (3.2.9) we have that

$$\|g_N\|_{L^{\frac{r}{2},\infty}(v_1^{1/2}v_2^{1/2})} \leq \varphi(\|v_1\|_{\widehat{A}_r}, \|v_2\|_{\widehat{A}_r}) \|f_1\|_{L^{r,1}(v_1)} \|f_2\|_{L^{r,1}(v_2)},$$

for all weights $v_1, v_2 \in \widehat{A}_r$, and by Theorem 3.2.1, we obtain that

$$\|g_N\|_{L^{\frac{1}{2},\infty}(w_1^{1/2}w_2^{1/2})} \leq \Phi([w_1]_{A_1}, [w_2]_{A_1}) \|f_1\|_{L^{1,\frac{1}{r}}(w_1)} \|f_2\|_{L^{1,\frac{1}{r}}(w_2)},$$

for all weights $w_1, w_2 \in A_1$, and we get (3.2.10) taking the supremum over all $N \geq 1$ in the previous expression, because

$$\|g\|_{L^{\frac{1}{2},\infty}(w_1^{1/2}w_2^{1/2})} = \sup_{N\geq 1} \|g_N\|_{L^{\frac{1}{2},\infty}(w_1^{1/2}w_2^{1/2})},$$

since $g_N \uparrow g$. □

Remark 3.2.3. Observe that if the operator T is defined for characteristic functions of measurable sets $E_1, E_2 \subseteq \mathbb{R}^n$, then under the hypotheses of Corollary 3.2.2, we deduce that

$$\left\|T(\chi_{E_1}, \chi_{E_2})\right\|_{L^{\frac{1}{2},\infty}(w_1^{1/2}w_2^{1/2})} \leq r^{2r}\Phi([w_1]_{A_1}, [w_2]_{A_1})w_1(E_1)w_2(E_2),$$

and hence, T is of weak type $(1, 1, \frac{1}{2})$ at least for characteristic functions.

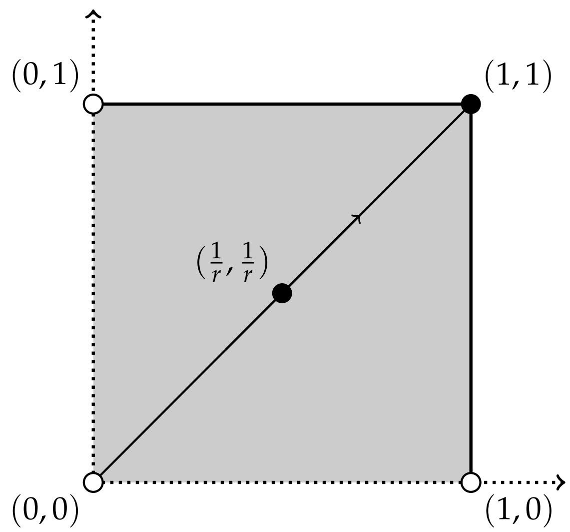

FIGURE 3.1: Pictorial representation of Theorem 3.2.1 and Corollary 3.2.2.

It is clear that the crucial point of the proof of Theorem 3.2.1 is the endpoint bound of the 2-fold product of Hardy-Littlewood maximal operators, which we manage to transfer to the original triple of functions (f_1, f_2, g). This fact suggests that a general extrapolation scheme should allow us to transfer bounds of a product-type maximal operator to a triple of functions (f_1, f_2, g) under suitable hypotheses. The following theorem quantifies this idea and allows us to obtain extrapolation schemes based on the bounds of an explicit operator.

Theorem 3.2.4. *Given measurable functions f_1, f_2, and g, suppose that for some exponents $1 \leq p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and all weights $v_1 \in A_{p_1}^{\mathcal{R}}$ and $v_2 \in A_{p_2}^{\mathcal{R}}$,*

$$\|g\|_{L^{p,\infty}(v_1^{p/p_1} v_2^{p/p_2})} \leq \varphi([v_1]_{A_{p_1}^{\mathcal{R}}}, [v_2]_{A_{p_2}^{\mathcal{R}}}) \|f_1\|_{L^{p_1,1}(v_1)} \|f_2\|_{L^{p_2,1}(v_2)}, \tag{3.2.11}$$

where $\varphi : [1,\infty)^2 \longrightarrow [0,\infty)$ is a function increasing in each variable. Then, for all exponents $1 \leq q_1 \leq p_1$, $1 \leq q_2 \leq p_2$, and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$ such that $p \neq q$, and all weights $w_1 \in \widehat{A}_{q_1}$ and $w_2 \in \widehat{A}_{q_2}$,

$$\|g\|_{L^{q,\infty}(w)} \leq \Psi(\|w_1\|_{\widehat{A}_{q_1}}, \|w_2\|_{\widehat{A}_{q_2}}) \|\mathscr{Z}\|_{L^{q,\infty}(w)}^{1-\frac{q}{p}} \|f_1\|_{L^{q_1,\frac{q_1}{p_1}}(w_1)}^{q_1/p_1} \|f_2\|_{L^{q_2,\frac{q_2}{p_2}}(w_2)}^{q_2/p_2}, \tag{3.2.12}$$

where $w = w_1^{q/q_1} w_2^{q/q_2}$, and $\mathscr{Z} := (Mf_1)^{\delta_1} w_1^{\beta_1} (Mf_2)^{\delta_2} w_2^{\beta_2}$, with

$$\delta_i := \frac{p(p_i - q_i)}{p_i(p-q)} \quad \text{and} \quad \beta_i := \frac{\frac{p}{p_i} - \frac{q}{q_i}}{q-p}, \quad \text{for } i = 1,2.$$

Moreover,

$$\Psi(\|w_1\|_{\widehat{A}_{q_1}}, \|w_2\|_{\widehat{A}_{q_2}}) = \frac{p_1 p_2}{q_1 q_2} C_{p,q} \varphi(C_1 \|w_1\|_{\widehat{A}_{q_1}}^{q_1/p_1}, C_2 \|w_2\|_{\widehat{A}_{q_2}}^{q_2/p_2}), \tag{3.2.13}$$

which depends on the constants $\|w_1\|_{\widehat{A}_{q_1}}$ and $\|w_2\|_{\widehat{A}_{q_2}}$ in an increasing way.

Proof. Pick weights $w_1 \in \widehat{A}_{q_1}$ and $w_2 \in \widehat{A}_{q_2}$, and let $w = w_1^{q/q_1} w_2^{q/q_2}$. We may assume that the quantity $\|f_i\|_{L^{q_i,\frac{q_i}{p_i}}(w_i)} < \infty$, for $i = 1,2$. In particular, f_1 and f_2 are locally integrable (see Lemma 3.1.2). Fix $y > 0$ and $\gamma > 0$. We have that

$$\begin{aligned}
\lambda_g^w(y) &= \int_{\{|g|>y\}} w = \int_{\{|g|>y,\, \mathscr{Z}>\gamma y\}} w + \int_{\{|g|>y,\, \mathscr{Z}\leq\gamma y\}} w \\
&\leq \lambda_{\mathscr{Z}}^w(\gamma y) + \int_{\{|g|>y,\, \mathscr{Z}\leq\gamma y\}} w \\
&\leq \lambda_{\mathscr{Z}}^w(\gamma y) + \int_{\{|g|>y\}} \left(\frac{\gamma y}{\mathscr{Z}}\right)^{p-q} w =: I + II.
\end{aligned} \tag{3.2.14}$$

To estimate the term I in (3.2.14), we have that

$$I = \frac{(\gamma y)^q}{(\gamma y)^q} \lambda_{\mathscr{Z}}^w(\gamma y) \leq \frac{1}{(\gamma y)^q} \|\mathscr{Z}\|_{L^{q,\infty}(w)}^q. \tag{3.2.15}$$

We proceed to estimate the term II in (3.2.14). For $i=1,2$, take

$$v_i := (Mf_i)^{\delta_i(q-p)\frac{p_i}{p}} w_i^{(\beta_i(q-p)+\frac{q}{q_i})\frac{p_i}{p}}$$
$$= (Mf_i)^{\frac{p(p_i-q_i)}{p_i(p-q)}(q-p)\frac{p_i}{p}} w_i^{(\frac{\frac{p}{p_i}-\frac{q}{q_i}}{q-p}(q-p)+\frac{q}{q_i})\frac{p_i}{p}} = (Mf_i)^{q_i-p_i} w_i.$$

Since $w_i \in \widehat{A}_{q_i}$, it follows from Lemma 3.1.3 that $v_i \in A_{p_i}^{\mathcal{R}}$, with

$$[v_i]_{A_{p_i}^{\mathcal{R}}} \le \|v_i\|_{A_{p_i}^{\mathcal{R}}} \le C_i \, \|w_i\|_{\widehat{A}_{q_i}}^{q_i/p_i}. \tag{3.2.16}$$

Observe that

$$\mathscr{Z}^{q-p} w = (Mf_1)^{\delta_1(q-p)} w_1^{\beta_1(q-p)} (Mf_2)^{\delta_2(q-p)} w_2^{\beta_2(q-p)} w_1^{q/q_1} w_2^{q/q_2}$$
$$= (Mf_1)^{\frac{p(q_1-p_1)}{p_1}} w_1^{\frac{p}{p_1}-\frac{q}{q_1}} (Mf_2)^{\frac{p(q_2-p_2)}{p_2}} w_2^{\frac{p}{p_2}-\frac{q}{q_2}} w_1^{q/q_1} w_2^{q/q_2}$$
$$= ((Mf_1)^{q_1-p_1} w_1)^{p/p_1} ((Mf_2)^{q_2-p_2} w_2)^{p/p_2} = v_1^{p/p_1} v_2^{p/p_2},$$

so by (3.2.11), (3.2.16), and the monotonicity of φ, we get that

$$\begin{aligned} II &= \frac{(\gamma y)^p}{(\gamma y)^q} \int_{\{|g|>y\}} v_1^{p/p_1} v_2^{p/p_2} \le \frac{\gamma^p}{(\gamma y)^q} \|g\|_{L^{p,\infty}(v_1^{p/p_1} v_2^{p/p_2})}^p \\ &\le \frac{\gamma^p}{(\gamma y)^q} \varphi(C_1 \, \|w_1\|_{\widehat{A}_{q_1}}^{q_1/p_1}, C_2 \, \|w_2\|_{\widehat{A}_{q_2}}^{q_2/p_2})^p \, \|f_1\|_{L^{p_1,1}(v_1)}^p \, \|f_2\|_{L^{p_2,1}(v_2)}^p, \end{aligned} \tag{3.2.17}$$

and arguing as we did in (3.2.7), we have that for $i=1,2$,

$$\|f_i\|_{L^{p_i,1}(v_i)} \le \frac{p_i}{q_i} \, \|f_i\|_{L^{q_i,\frac{q_i}{p_i}}(w_i)}^{q_i/p_i}. \tag{3.2.18}$$

If $\|\mathscr{Z}\|_{L^{q,\infty}(w)} = \infty$, then (3.2.12) holds, so we may assume that the quantity $\|\mathscr{Z}\|_{L^{q,\infty}(w)} < \infty$. Combining the estimates (3.2.14), (3.2.15), (3.2.17), and (3.2.18), we conclude that

$$y^q \lambda_g^w(y) \le \gamma^{-q} \|\mathscr{Z}\|_{L^{q,\infty}(w)}^q$$
$$+ \left(\frac{p_1 p_2}{q_1 q_2}\right)^p \gamma^{p-q} \varphi(C_1 \, \|w_1\|_{\widehat{A}_{q_1}}^{q_1/p_1}, C_2 \, \|w_2\|_{\widehat{A}_{q_2}}^{q_2/p_2})^p \, \|f_1\|_{L^{q_1,\frac{q_1}{p_1}}(w_1)}^{\frac{pq_1}{p_1}} \, \|f_2\|_{L^{q_2,\frac{q_2}{p_2}}(w_2)}^{\frac{pq_2}{p_2}},$$

and taking the infimum over all $\gamma > 0$, it follows from Lemma 3.1.1 that

$$y^q \lambda_g^w(y) \le \frac{p}{p-q} \left(\frac{p-q}{q}\right)^{q/p} \left(\frac{p_1 p_2}{q_1 q_2}\right)^q \varphi(C_1 \, \|w_1\|_{\widehat{A}_{q_1}}^{q_1/p_1}, C_2 \, \|w_2\|_{\widehat{A}_{q_2}}^{q_2/p_2})^q$$
$$\times \|\mathscr{Z}\|_{L^{q,\infty}(w)}^{q(1-\frac{q}{p})} \, \|f_1\|_{L^{q_1,\frac{q_1}{p_1}}(w_1)}^{\frac{qq_1}{p_1}} \, \|f_2\|_{L^{q_2,\frac{q_2}{p_2}}(w_2)}^{\frac{qq_2}{p_2}}.$$

Finally, raising everything to the power $\frac{1}{q}$ in this last expression, and taking the supremum over all $y > 0$, we see that (3.2.12) holds, with

$$C_{p,q} := \left(\frac{p}{p-q}\right)^{1/q} \left(\frac{p-q}{q}\right)^{1/p} \tag{3.2.19}$$

in (3.2.13). □

Remark 3.2.5. Observe that the expression (3.2.12) is homogeneous in the weights w_1 and w_2. Indeed, pick weights $w_1 \in \widehat{A}_{q_1}$ and $w_2 \in \widehat{A}_{q_2}$. For parameters $\alpha_1, \alpha_2 > 0$, we have that $\alpha_i w_i \in \widehat{A}_{q_i}$, with $\|\alpha_i w_i\|_{\widehat{A}_{q_i}} = \|w_i\|_{\widehat{A}_{q_i}}$, for $i = 1, 2$, and under the hypotheses of Theorem 3.2.4, (3.2.12) gives us that

$$\begin{aligned} \|g\|_{L^{q,\infty}(\alpha_1^{q/q_1}\alpha_2^{q/q_2}w)} \leq \Psi_{w_1,w_2} &\|\alpha_1^{\beta_1}\alpha_2^{\beta_2}\mathscr{Z}\|^{1-\frac{q}{p}}_{L^{q,\infty}(\alpha_1^{q/q_1}\alpha_2^{q/q_2}w)} \\ &\times \|f_1\|^{q_1/p_1}_{L^{q_1,\frac{q_1}{p_1}}(\alpha_1 w_1)} \|f_2\|^{q_2/p_2}_{L^{q_2,\frac{q_2}{p_2}}(\alpha_2 w_2)}, \end{aligned}$$

with $\Psi_{w_1,w_2} := \Psi(\|w_1\|_{\widehat{A}_{q_1}}, \|w_2\|_{\widehat{A}_{q_2}})$. Since

$$\|g\|_{L^{q,\infty}(\alpha_1^{q/q_1}\alpha_2^{q/q_2}w)} = \alpha_1^{1/q_1}\alpha_2^{1/q_2}\|g\|_{L^{q,\infty}(w)},$$

and

$$\|\alpha_1^{\beta_1}\alpha_2^{\beta_2}\mathscr{Z}\|_{L^{q,\infty}(\alpha_1^{q/q_1}\alpha_2^{q/q_2}w)} = \alpha_1^{\beta_1+\frac{1}{q_1}}\alpha_2^{\beta_2+\frac{1}{q_2}}\|\mathscr{Z}\|_{L^{q,\infty}(w)},$$

and for $i = 1, 2$,

$$\|f_i\|_{L^{q_i,\frac{q_i}{p_i}}(\alpha_i w_i)} = \alpha_i^{1/q_i}\|f_i\|_{L^{q_i,\frac{q_i}{p_i}}(w_i)},$$

we conclude that

$$\|g\|_{L^{q,\infty}(w)} \leq c_{\alpha_1,\alpha_2}\Psi_{w_1,w_2}\|\mathscr{Z}\|^{1-\frac{q}{p}}_{L^{q,\infty}(w)}\|f_1\|^{q_1/p_1}_{L^{q_1,\frac{q_1}{p_1}}(w_1)}\|f_2\|^{q_2/p_2}_{L^{q_2,\frac{q_2}{p_2}}(w_2)},$$

with

$$\begin{aligned} c_{\alpha_1,\alpha_2} &:= \alpha_1^{(\beta_1+\frac{1}{q_1})(1-\frac{q}{p})+\frac{1}{p_1}-\frac{1}{q_1}}\alpha_2^{(\beta_2+\frac{1}{q_2})(1-\frac{q}{p})+\frac{1}{p_2}-\frac{1}{q_2}} \\ &= \alpha_1^{\beta_1(1-\frac{q}{p})+\frac{1}{p_1}-\frac{q}{pq_1}}\alpha_2^{\beta_2(1-\frac{q}{p})+\frac{1}{p_2}-\frac{q}{pq_2}} \\ &= \alpha_1^{\frac{1}{p}(\frac{q}{q_1}-\frac{p}{p_1})+\frac{1}{p_1}-\frac{q}{pq_1}}\alpha_2^{\frac{1}{p}(\frac{q}{q_2}-\frac{p}{p_2})+\frac{1}{p_2}-\frac{q}{pq_2}} = 1, \end{aligned}$$

and we recover (3.2.12).

Note that we can easily recover the conclusion of Theorem 3.2.1 from Theorem 3.2.4. Indeed, for $r > 1$, choose $p_1 = p_2 = r$, with $p = \frac{r}{2}$, and $q_1 = q_2 = 1$, with $q = \frac{1}{2}$. For this choice of parameters, $\delta_1 = \delta_2 = 1$, and

$\beta_1 = \beta_2 = 0$, so $\mathscr{L} = Mf_1 Mf_2$, and Theorem 3.2.4 gives us that

$$\|g\|_{L^{\frac{1}{2},\infty}(w)} \leq \Psi([w_1]_{A_1}, [w_2]_{A_1}) \|Mf_1 Mf_2\|^{1/r'}_{L^{\frac{1}{2},\infty}(w)} \|f_1\|^{1/r}_{L^{1,\frac{1}{r}}(w_1)} \|f_2\|^{1/r}_{L^{1,\frac{1}{r}}(w_2)},$$

with $\Psi([w_1]_{A_1}, [w_2]_{A_1}) = r^{2+\frac{2}{r}} (r')^{2/r'} \varphi(C_1 [w_1]^{1/r}_{A_1}, C_2 [w_2]^{1/r}_{A_1})$. Applying Theorem 2.4.1, and (3.2.5), (3.2.2) follows with a slightly different Φ due to the constants C_1 and C_2.

It may seem that the term $\mathscr{L}$ in Theorem 3.2.4 is a weird-looking object in general, and it's not at all clear what kind of bounds, if any, does it satisfy. However, we have seen that under an appropriate choice of parameters, the object $\mathscr{L}$ becomes the classical operator $M^{\otimes}$, for which we have a plethora of estimates available. For this reason, we are now interested in finding all the possible combinations of parameters for which $\mathscr{L} = Mf_1 Mf_2$. The following lemma will help us in this matter. We will also use it later to find other useful candidates for $\mathscr{L}$.

Lemma 3.2.6. *In Theorem 3.2.4, the parameters involved satisfy the following relations:*

(a) $\frac{q_1}{p_1} = \frac{q}{p}$ *if, and only if* $\frac{q_2}{p_2} = \frac{q}{p}$ *if, and only if* $\frac{q_1}{p_1} = \frac{q_2}{p_2}$.

(b) $\delta_1 = 0$ *if, and only if* $\delta_2 = \frac{q_2}{q}$. *Similarly,* $\delta_1 = \frac{q_1}{q}$ *if, and only if* $\delta_2 = 0$.

(c) For $i = 1, 2$, $\delta_i = 0$ *if, and only if* $\beta_i = -\frac{1}{p_i}$ *if, and only if* $\beta_i = -\frac{1}{q_i}$.

(d) $\beta_1 + \beta_2 = 0$. *In particular,* $\beta_1 = 0$ *if, and only if* $\beta_2 = 0$.

(e) For $i = 1, 2$, $\delta_i = 1$ *if, and only if* $\beta_i = 0$ *if, and only if* $\delta_1 = \delta_2$ *if, and only if* $\frac{q_1}{p_1} = \frac{q_2}{p_2}$.

Proof. To prove (a),

$$\frac{q_1}{p_1} = \frac{q}{p} \Leftrightarrow \frac{p}{p_1} = \frac{q}{q_1} \Leftrightarrow 1 - \frac{p}{p_2} = 1 - \frac{q}{q_2} \Leftrightarrow \frac{q_2}{p_2} = \frac{q}{p}$$
$$\Leftrightarrow \frac{q_2}{q} = \frac{p_2}{p} \Leftrightarrow 1 + \frac{q_2}{q_1} = 1 + \frac{p_2}{p_1} \Leftrightarrow \frac{q_1}{p_1} = \frac{q_2}{p_2}.$$

To prove the first part of (b),

$$\delta_1 = 0 \Leftrightarrow p_1 = q_1 \Leftrightarrow \frac{p}{q_1} + \frac{p}{p_2} = 1 \Leftrightarrow \frac{p}{q_2} - \frac{p}{p_2} = \frac{p}{q_1} + \frac{p}{q_2} - 1$$
$$\Leftrightarrow \frac{p}{p_2}\left(\frac{p_2}{q_2} - 1\right) = \frac{p}{q} - 1 \Leftrightarrow \frac{p(p_2 - q_2)}{p_2 q_2} = \frac{p-q}{q} \Leftrightarrow \delta_2 = \frac{q_2}{q}.$$

The proof of the second part of (b) is entirely analogous.

To prove (c), we have that for $i = 1, 2$,

$$\delta_i = 0 \Leftrightarrow p_i = q_i \Leftrightarrow \frac{p}{p_i} - \frac{q}{q_i} = \frac{p}{p_i} - \frac{q}{p_i} \Leftrightarrow \beta_i = -\frac{1}{p_i}.$$

Similarly,

$$\beta_i = -\frac{1}{q_i} \Leftrightarrow p_i = q_i \Leftrightarrow \delta_i = 0.$$

To prove (d),

$$\beta_1 + \beta_2 = \frac{\frac{p}{p_1} - \frac{q}{q_1} + \frac{p}{p_2} - \frac{q}{q_2}}{q - p} = 0.$$

To prove (e), we have that for $i = 1, 2$,

$$\delta_i = 1 \Leftrightarrow \frac{p_i - q_i}{p_i} = \frac{p - q}{p} \overset{\text{(a)}}{\Longleftrightarrow} \frac{q_1}{p_1} = \frac{q_2}{p_2} \Leftrightarrow \frac{p_1 - q_1}{p_1} = \frac{p_2 - q_2}{p_2} \Leftrightarrow \delta_1 = \delta_2,$$

and

$$\beta_i = 0 \Leftrightarrow \frac{p}{p_i} = \frac{q}{q_i} \Leftrightarrow \frac{p_i - q_i}{p_i} = \frac{p - q}{p} \Leftrightarrow \delta_i = 1.$$

$\square$

We can see that the term $\mathscr{X}$ in Theorem 3.2.4 becomes Mf_1Mf_2 if, and only if $\delta_1 = \delta_2 = 1$ and $\beta_1 = \beta_2 = 0$, and Lemma 3.2.6 tells us that

$$\delta_1 = 1 \overset{\text{(e)}}{\Longleftrightarrow} \beta_1 = 0 \overset{\text{(d)}}{\Longleftrightarrow} \beta_2 = 0 \overset{\text{(e)}}{\Longleftrightarrow} \delta_2 = 1 \overset{\text{(e)}}{\Longleftrightarrow} \frac{q_1}{p_1} = \frac{q_2}{p_2}.$$

In conclusion, $\mathscr{X} = Mf_1Mf_2$ if, and only if the points $P = (\frac{1}{p_1}, \frac{1}{p_2})$ and $Q = (\frac{1}{q_1}, \frac{1}{q_2})$ in $\mathbb{R}^2$ lay on a straight line passing through the origin point $(0, 0)$.

We are now in the position to provide an extension of Theorem 3.2.1 that works not only in the diagonal case but also in the more general case when the exponents of the Lorentz spaces involved satisfy the alignment condition $\frac{q_1}{p_1} = \frac{q_2}{p_2}$. In Figure 3.2 you can find a pictorial representation of it.

Theorem 3.2.7. *Given measurable functions f_1, f_2, and g, suppose that for some exponents $1 \leq p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and all weights $v_1 \in A^{\mathcal{R}}_{p_1}$ and $v_2 \in A^{\mathcal{R}}_{p_2}$, (3.2.11) holds for a function $\varphi : [1, \infty)^2 \longrightarrow [0, \infty)$ that increases in each variable. Then, for all exponents $1 \leq q_1 \leq p_1$, $1 \leq q_2 \leq p_2$, and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$ such that $p \neq q$ and $\frac{q_1}{p_1} = \frac{q_2}{p_2}$, and all weights $w_1 \in \widehat{A}_{q_1}$ and $w_2 \in \widehat{A}_{q_2}$,*

$$\|g\|_{L^{q,\infty}(w_1^{q/q_1} w_2^{q/q_2})} \leq \Phi(\|w_1\|_{\widehat{A}_{q_1}}, \|w_2\|_{\widehat{A}_{q_2}}) \, \|f_1\|_{L^{q_1, \frac{q}{p}}(w_1)} \, \|f_2\|_{L^{q_2, \frac{q}{p}}(w_2)}, \tag{3.2.20}$$

where $\Phi : [1, \infty)^2 \longrightarrow [0, \infty)$ is a function increasing in each variable.

Proof. Pick weights $w_1 \in \widehat{A}_{q_1}$ and $w_2 \in \widehat{A}_{q_2}$, and write $w = w_1^{q/q_1} w_2^{q/q_2}$. From Lemma 3.2.6 we see that $\frac{q_1}{p_1} = \frac{q_2}{p_2} = \frac{q}{p}$, and in virtue Theorem 3.2.4, we have that

$$\|g\|_{L^{q,\infty}(w)} \leq \Psi_{w_1, w_2} \|Mf_1Mf_2\|^{1-\frac{q}{p}}_{L^{q,\infty}(w)} \, \|f_1\|^{q/p}_{L^{q_1, \frac{q}{p}}(w_1)} \, \|f_2\|^{q/p}_{L^{q_2, \frac{q}{p}}(w_2)}, \tag{3.2.21}$$

with

$$\Psi_{w_1,w_2} := \Psi(\|w_1\|_{\widehat{A}_{q_1}}, \|w_2\|_{\widehat{A}_{q_2}}) = \left(\frac{p}{q}\right)^2 C_{p,q}\varphi(C_1\|w_1\|_{\widehat{A}_{q_1}}^{q/p}, C_2\|w_2\|_{\widehat{A}_{q_2}}^{q/p}).$$

We apply Theorem 2.4.1 to obtain that

$$\begin{aligned}\|Mf_1Mf_2\|_{L^{q,\infty}(w)} &\leq \phi([w_1]_{A_{q_1}^{\mathcal{R}}}, [w_2]_{A_{q_2}^{\mathcal{R}}})\|f_1\|_{L^{q_1,1}(w_1)}\|f_2\|_{L^{q_2,1}(w_2)} \\ &\leq \phi(c_1\|w_1\|_{\widehat{A}_{q_1}}, c_2\|w_2\|_{\widehat{A}_{q_2}})\|f_1\|_{L^{q_1,1}(w_1)}\|f_2\|_{L^{q_2,1}(w_2)}.\end{aligned} \tag{3.2.22}$$

We know from (2.1.1) that for $i = 1, 2$,

$$\|f_i\|_{L^{q_i,1}(w_i)} \leq \left(\frac{q}{pq_i}\right)^{\frac{p}{q}-1}\|f_i\|_{L^{q_i,\frac{q}{p}}(w_i)} = p_i^{1-\frac{p}{q}}\|f_i\|_{L^{q_i,\frac{q}{p}}(w_i)}. \tag{3.2.23}$$

Combining the estimates (3.2.21), (3.2.22), and (3.2.23), we get (3.2.20) with

$$\Phi(\|w_1\|_{\widehat{A}_{q_1}}, \|w_2\|_{\widehat{A}_{q_2}}) = (p_1p_2)^{2-\frac{q}{p}-\frac{p}{q}}\phi(c_1\|w_1\|_{\widehat{A}_{q_1}}, c_2\|w_2\|_{\widehat{A}_{q_2}})^{1-\frac{q}{p}}\Psi_{w_1,w_2}.$$

□

Once again, we have presented Theorem 3.2.7 in its general form, for triples of functions (f_1, f_2, g). We can deduce the corresponding extrapolation scheme for two-variable operators arguing as in the proof of Corollary 3.2.2. For convenience, we also provide a pictorial representation of it in Figure 3.2.

Corollary 3.2.8. *Let T be a two-variable operator defined for measurable functions. Suppose that for some exponents $1 \leq p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and all weights $v_1 \in A_{p_1}^{\mathcal{R}}$ and $v_2 \in A_{p_2}^{\mathcal{R}}$,*

$$T : L^{p_1,1}(v_1) \times L^{p_2,1}(v_2) \longrightarrow L^{p,\infty}(v_1^{p/p_1}v_2^{p/p_2}),$$

with constant bounded by $\varphi([v_1]_{A_{p_1}^{\mathcal{R}}}, [v_2]_{A_{p_2}^{\mathcal{R}}})$ as in (3.2.11). Then, for all exponents $1 \leq q_1 \leq p_1$, $1 \leq q_2 \leq p_2$, and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$ such that $p \neq q$ and $\frac{q_1}{p_1} = \frac{q_2}{p_2}$, and all weights $w_1 \in \widehat{A}_{q_1}$ and $w_2 \in \widehat{A}_{q_2}$,

$$T : L^{q_1,\frac{q}{p}}(w_1) \times L^{q_2,\frac{q}{p}}(w_2) \longrightarrow L^{q,\infty}(w_1^{q/q_1}w_2^{q/q_2}),$$

with constant bounded by $\Phi(\|w_1\|_{\widehat{A}_{q_1}}, \|w_2\|_{\widehat{A}_{q_2}})$ as in (3.2.20).

Remark 3.2.9. Observe that if the operator T is defined for characteristic functions of measurable sets $E_1, E_2 \subseteq \mathbb{R}^n$, then under the hypotheses of Corollary 3.2.8, we deduce that

$$\|T(\chi_{E_1}, \chi_{E_2})\|_{L^{q,\infty}(w_1^{q/q_1}w_2^{q/q_2})} \leq C\Phi(\|w_1\|_{\widehat{A}_{q_1}}, \|w_2\|_{\widehat{A}_{q_2}})w_1(E_1)^{1/q_1}w_2(E_2)^{1/q_2},$$

with $C = \left(q_1 q_2 \frac{p^2}{q^2}\right)^{p/q}$, and hence, T is of weak type (q_1, q_2, q) at least for characteristic functions.

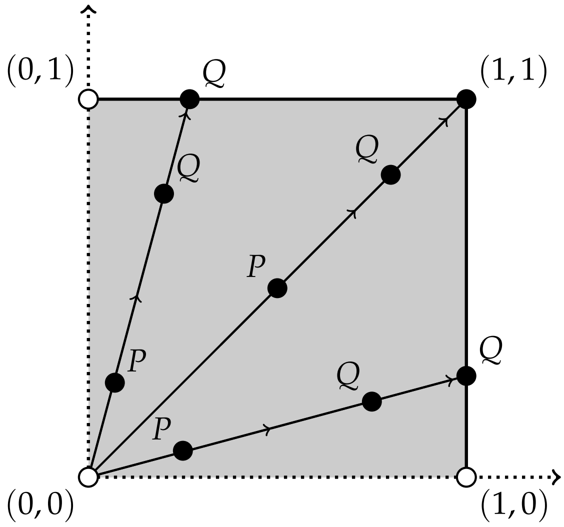

FIGURE 3.2: Pictorial representation of Theorem 3.2.7 and Corollary 3.2.8.

All the extrapolation results presented in this section produce bounds involving Lorentz spaces with lower exponents than the ones in the hypotheses. In the one-variable extrapolation setting, the distinction between increasing and decreasing the exponents was necessary, because the techniques involved were different (see [9, 14]). In the two-variable case, we observe the same phenomenon.

Now we present a result in which we manage to increase the exponents of the Lorentz spaces involved, provided that we equip them with the same weight. The key ingredient is a bound of a perturbed version of the Hardy-Littlewood maximal operator, namely $\frac{M(fw)}{w}$. We provide a pictorial representation of this scheme in Figure 3.3.

Theorem 3.2.10. *Given measurable functions f_1, f_2, and g, suppose that for some exponents $1 \leq p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and all weights $v_1 \in \widehat{A}_{p_1}$ and $v_2 \in \widehat{A}_{p_2}$,*

$$\|g\|_{L^{p,\infty}(v_1^{p/p_1} v_2^{p/p_2})} \leq \varphi(\|v_1\|_{\widehat{A}_{p_1}}, \|v_2\|_{\widehat{A}_{p_2}}) \, \|f_1\|_{L^{p_1,1}(v_1)} \, \|f_2\|_{L^{p_2,1}(v_2)}, \tag{3.2.24}$$

where $\varphi : [1,\infty)^2 \longrightarrow [0,\infty)$ is a function increasing in each variable. Then, for all exponents $q_1 = p_1 \geq 1$ or $q_1 > p_1 > 1$, $q_2 = p_2 \geq 1$ or $q_2 > p_2 > 1$, and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, and every weight $w \in \widehat{A}_{q_1} \cap \widehat{A}_{q_2}$,

$$\|g\|_{L^{q,\infty}(w)} \leq \Phi(\|w\|_{\widehat{A}_{q_1}}, \|w\|_{\widehat{A}_{q_2}}) \, \|f_1\|_{L^{q_1,1}(w)} \, \|f_2\|_{L^{q_2,1}(w)}, \tag{3.2.25}$$

where $\Phi : [1,\infty)^2 \longrightarrow [0,\infty)$ is a function increasing in each variable.

Proof. Note that if $q_1 = p_1$ and $q_2 = p_2$, then there is nothing to prove. We first discuss the case when $1 < p_1 < q_1$ and $1 < p_2 < q_2$. Pick a weight $w \in \widehat{A}_{q_1} \cap \widehat{A}_{q_2}$. As usual, we may assume that $\|f_1\|_{L^{q_1,1}(w)} < \infty$ and $\|f_2\|_{L^{q_2,1}(w)} < \infty$. For every natural number $N \geq 1$, let $g_N := |g|\chi_{B(0,N)}$. Fix $N \geq 1$. We will prove (3.2.25) for the triple (f_1, f_2, g_N). Since $g_N \leq |g|$, we already know that (3.2.24) holds for (f_1, f_2, g_N). Fix $y > 0$ such that $\lambda^w_{g_N}(y) \neq 0$. If no such y exists, then $\|g_N\|_{L^{q,\infty}(w)} = 0$ and we are done.

In order to apply (3.2.24), we want to find weights $v_1 \in \widehat{A}_{p_1}$ and $v_2 \in \widehat{A}_{p_2}$ such that for $v := v_1^{p/p_1} v_2^{p/p_2}$, $\lambda^w_{g_N}(y) \leq \lambda^v_{g_N}(y)$. For $i = 1, 2$, take

$$v_i := w^{\frac{p_i-1}{q_i-1}} \left(M(w\chi_{\{|g_N|>y\}}) \right)^{\frac{q_i-p_i}{q_i-1}}. \tag{3.2.26}$$

Note that if $q_i = p_i$, then $v_i = w$. Applying Lemma 3.1.4, we see that $v_i \in \widehat{A}_{p_i}$, with $\|v_i\|_{\widehat{A}_{p_i}} \leq C_i \|w\|^{q_i/p_i}_{\widehat{A}_{q_i}}$, and C_i independent of w, N, and y. Observe that $v_i \geq w\chi_{\{|g_N|>y\}}$, so (3.2.24) implies that

$$\begin{aligned} \lambda^w_{g_N}(y) &= \int_{\{|g_N|>y\}} w^{p/p_1} w^{p/p_2} \leq \int_{\{|g_N|>y\}} v_1^{p/p_1} v_2^{p/p_2} = \lambda^v_{g_N}(y) \\ &\leq \frac{1}{y^p} \varphi(C_1 \|w\|^{q_1/p_1}_{\widehat{A}_{q_1}}, C_2 \|w\|^{q_2/p_2}_{\widehat{A}_{q_2}})^p \, \|f_1\|^p_{L^{p_1,1}(v_1)} \, \|f_2\|^p_{L^{p_2,1}(v_2)}. \end{aligned} \tag{3.2.27}$$

For $i = 1, 2$, we want to replace $\|f_i\|_{L^{p_i,1}(v_i)}$ by $\|f_i\|_{L^{q_i,1}(w)}$ in (3.2.27). If $q_i = p_i$, then $\|f_i\|_{L^{p_i,1}(v_i)} = \|f_i\|_{L^{q_i,1}(w)}$. If $q_i > p_i$, then applying Hölder's inequality with exponent $\frac{q_i}{p_i} > 1$, we obtain that for every $t > 0$,

$$\begin{aligned} v_i(\{|f_i| > t\}) &= \int_{\{|f_i|>t\}} \left(\frac{M(w\chi_{\{|g_N|>y\}})}{w} \right)^{\frac{q_i-p_i}{q_i-1}} w \\ &\leq \|\chi_{\{|f_i|>t\}}\|_{L^{\frac{q_i}{p_i},1}(w)} \left\| \left(\frac{M(w\chi_{\{|g_N|>y\}})}{w} \right)^{\frac{q_i-p_i}{q_i-1}} \right\|_{L^{\frac{q_i}{q_i-p_i},\infty}(w)} \\ &= \frac{q_i}{p_i} w(\{|f_i| > t\})^{p_i/q_i} \left\| \frac{M(w\chi_{\{|g_N|>y\}})}{w} \right\|^{\frac{q_i-p_i}{q_i-1}}_{L^{q_i',\infty}(w)}. \end{aligned} \tag{3.2.28}$$

Now, Theorem 2.4.12, Remark 2.4.13, and [9, Corollary 2.8] give us that

$$\left\| \frac{M(w\chi_{\{|g_N|>y\}})}{w} \right\|_{L^{q_i',\infty}(w)} \leq c_i \|w\|^{q_i+1}_{\widehat{A}_{q_i}} w(\{|g_N| > y\})^{1/q_i'}, \tag{3.2.29}$$

so

$$v_i(\{|f_i| > t\}) \leq \frac{q_i}{p_i} \left(c_i \|w\|_{\widehat{A}_{q_i}}^{q_i+1} \right)^{\frac{q_i-p_i}{q_i-1}} w(\{|g_N| > y\})^{1-\frac{p_i}{q_i}} w(\{|f_i| > t\})^{p_i/q_i}, \tag{3.2.30}$$

and hence,

$$\begin{aligned} \|f_i\|_{L^{p_i,1}(v_i)} &= p_i \int_0^\infty v_i(\{|f_i| > t\})^{1/p_i} dt \\ &\leq p_i \left(\frac{q_i}{p_i}\right)^{1/p_i} \left(c_i \|w\|_{\widehat{A}_{q_i}}^{q_i+1} \right)^{\frac{1}{p_i}\frac{q_i-p_i}{q_i-1}} \\ &\times w(\{|g_N| > y\})^{\frac{1}{p_i}-\frac{1}{q_i}} \int_0^\infty w(\{|f_i| > t\})^{1/q_i} dt \\ &= \left(\frac{p_i}{q_i}\right)^{1/p_i'} \left(c_i \|w\|_{\widehat{A}_{q_i}}^{q_i+1} \right)^{\frac{1}{p_i}\frac{q_i-p_i}{q_i-1}} w(\{|g_N| > y\})^{\frac{1}{p_i}-\frac{1}{q_i}} \|f_i\|_{L^{q_i,1}(w)}. \end{aligned} \tag{3.2.31}$$

Combining the estimates (3.2.27) and (3.2.31), we have that

$$\lambda_{g_N}^w(y) \leq \frac{1}{y^p} \Phi(\|w\|_{\widehat{A}_{q_1}}, \|w\|_{\widehat{A}_{q_2}})^p \|f_1\|_{L^{q_1,1}(w)}^p \|f_2\|_{L^{q_2,1}(w)}^p \lambda_{g_N}^w(y)^{1-\frac{p}{q}}, \tag{3.2.32}$$

with

$$\begin{aligned} \Phi(\|w\|_{\widehat{A}_{q_1}}, \|w\|_{\widehat{A}_{q_2}}) = &\left(\prod_{i=1}^2 \left(\frac{p_i}{q_i}\right)^{1/p_i'} \left(c_i \|w\|_{\widehat{A}_{q_i}}^{q_i+1} \right)^{\frac{1}{p_i}\frac{q_i-p_i}{q_i-1}} \right) \\ &\times \varphi(C_1 \|w\|_{\widehat{A}_{q_1}}^{q_1/p_1}, C_2 \|w\|_{\widehat{A}_{q_2}}^{q_2/p_2}). \end{aligned}$$

By our choice of y and g_N, $0 < \lambda_{g_N}^w(y) \leq w(B(0,N)) < \infty$, so we can divide by $\lambda_{g_N}^w(y)^{1-\frac{p}{q}}$ in (3.2.32) and raise everything to the power $\frac{1}{p}$, obtaining that

$$y \lambda_{g_N}^w(y)^{1/q} \leq \Phi(\|w\|_{\widehat{A}_{q_1}}, \|w\|_{\widehat{A}_{q_2}}) \|f_1\|_{L^{q_1,1}(w)} \|f_2\|_{L^{q_2,1}(w)},$$

and taking the supremum over all $y > 0$, we deduce (3.2.25) for the triple (f_1, f_2, g_N), and the result for the triple (f_1, f_2, g) follows taking the supremum over all $N \geq 1$.

Finally, the case when for some $i = 1, 2$, but not both, $1 \leq p_i = q_i$, follows from the previous argument, since $v_i = w$ in (3.2.26), and estimate (3.2.31) is no longer necessary. □

We have presented Theorem 3.2.10 in its general form, for triples of functions (f_1, f_2, g). We can deduce the corresponding extrapolation scheme for two-variable operators arguing as in the proof of Corollary 3.2.2. For convenience, we also provide a pictorial representation of it in Figure 3.3.

Corollary 3.2.11. *Let T be a two-variable operator defined for measurable functions. Suppose that for some exponents $1 \leq p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and all weights $v_1 \in \widehat{A}_{p_1}$ and $v_2 \in \widehat{A}_{p_2}$,*

$$T : L^{p_1,1}(v_1) \times L^{p_2,1}(v_2) \longrightarrow L^{p,\infty}(v_1^{p/p_1} v_2^{p/p_2}),$$

with constant bounded by $\varphi(\|v_1\|_{\widehat{A}_{p_1}}, \|v_2\|_{\widehat{A}_{p_2}})$ as in (3.2.24). Then, for all exponents $q_1 = p_1 \geq 1$ or $q_1 > p_1 > 1$, $q_2 = p_2 \geq 1$ or $q_2 > p_2 > 1$, and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, and every weight $w \in \widehat{A}_{q_1} \cap \widehat{A}_{q_2}$,

$$T : L^{q_1,1}(w) \times L^{q_2,1}(w) \longrightarrow L^{q,\infty}(w),$$

with constant bounded by $\Phi(\|w\|_{\widehat{A}_{q_1}}, \|w\|_{\widehat{A}_{q_2}})$ as in (3.2.25).

Remark 3.2.12. Observe that if the operator T is defined for characteristic functions of measurable sets $E_1, E_2 \subseteq \mathbb{R}^n$, then under the hypotheses of Corollary 3.2.11, we deduce that

$$\|T(\chi_{E_1}, \chi_{E_2})\|_{L^{q,\infty}(w)} \leq q_1 q_2 \Phi(\|w\|_{\widehat{A}_{q_1}}, \|w\|_{\widehat{A}_{q_2}}) w(E_1)^{1/q_1} w(E_2)^{1/q_2},$$

and hence, T is of weak type (q_1, q_2, q) at least for characteristic functions.

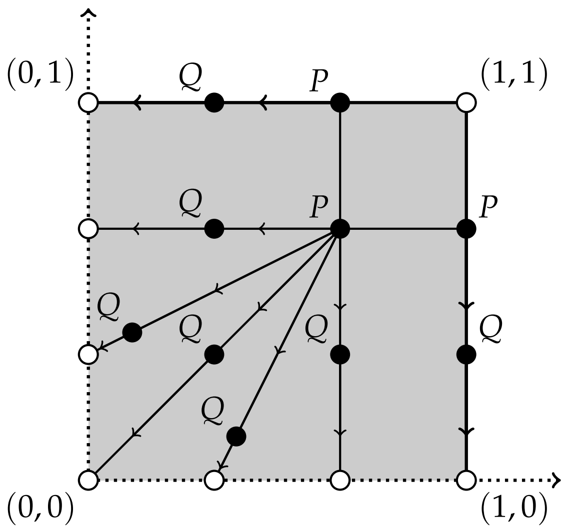

FIGURE 3.3: Pictorial representation of Theorem 3.2.10 and Corollary 3.2.11, and the results in Subsection 4.2.2.

Although Theorem 3.2.10 gives us a one-weight conclusion, we need to assume a two-weight hypothesis to get it, because, in general, the weights v_1 and v_2 that we chose in (3.2.26) are different. However, for $q_1 > p_1 > 1$ and $q_2 > p_2 > 1$ such that $\frac{p_1-1}{q_1-1} = \frac{p_2-1}{q_2-1}$, we have that $v_1 = v_2$. That is, these weights coincide if the points (p_1, p_2) and (q_1, q_2) in $\mathbb{R}^2$ lay on a straight line passing through the point $(1,1)$. Equivalently, for some $\tau > 0$, the points

$P = (\frac{1}{p_1}, \frac{1}{p_2})$ and $Q = (\frac{1}{q_1}, \frac{1}{q_2})$ in $\mathbb{R}^2$ belong to the graph of the function $F_\tau(x) := \frac{x}{(1-\tau)x+\tau}$, defined for $0 < x \leq 1$. Adding this assumption in Theorem 3.2.10 allows us to replace (3.2.24) by its one-weight analog (3.2.33), obtaining the following corollary, and the corresponding extrapolation scheme for two-variable operators. See Figure 3.4 for a pictorial representation of these results.

Corollary 3.2.13. *Given measurable functions f_1, f_2, and g, suppose that for some exponents $1 < p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and every weight $v \in \widehat{A}_{p_1} \cap \widehat{A}_{p_2}$,*

$$\|g\|_{L^{p,\infty}(v)} \leq \varphi(\|v\|_{\widehat{A}_{p_1}}, \|v\|_{\widehat{A}_{p_2}}) \, \|f_1\|_{L^{p_1,1}(v)} \, \|f_2\|_{L^{p_2,1}(v)}, \tag{3.2.33}$$

where $\varphi : [1,\infty)^2 \longrightarrow [0,\infty)$ is a function increasing in each variable. Then, for all exponents $q_1 \geq p_1$, $q_2 \geq p_2$, and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$ such that $\frac{p_1-1}{q_1-1} = \frac{p_2-1}{q_2-1}$, and every weight $w \in \widehat{A}_{q_1} \cap \widehat{A}_{q_2}$,

$$\|g\|_{L^{q,\infty}(w)} \leq \Phi(\|w\|_{\widehat{A}_{q_1}}, \|w\|_{\widehat{A}_{q_2}}) \, \|f_1\|_{L^{q_1,1}(w)} \, \|f_2\|_{L^{q_2,1}(w)}, \tag{3.2.34}$$

where $\Phi : [1,\infty)^2 \longrightarrow [0,\infty)$ is a function increasing in each variable.

Corollary 3.2.14. *Let T be a two-variable operator defined for measurable functions. Suppose that for some exponents $1 < p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and every weight $v \in \widehat{A}_{p_1} \cap \widehat{A}_{p_2}$,*

$$T : L^{p_1,1}(v) \times L^{p_2,1}(v) \longrightarrow L^{p,\infty}(v),$$

with constant bounded by $\varphi(\|v\|_{\widehat{A}_{p_1}}, \|v\|_{\widehat{A}_{p_2}})$ as in (3.2.33). Then, for all exponents $q_1 \geq p_1$, $q_2 \geq p_2$, and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$ such that $\frac{p_1-1}{q_1-1} = \frac{p_2-1}{q_2-1}$, and every weight $w \in \widehat{A}_{q_1} \cap \widehat{A}_{q_2}$,

$$T : L^{q_1,1}(w) \times L^{q_2,1}(w) \longrightarrow L^{q,\infty}(w),$$

with constant bounded by $\Phi(\|w\|_{\widehat{A}_{q_1}}, \|w\|_{\widehat{A}_{q_2}})$ as in (3.2.34).

3.3 Main Results on Mixed Type Extrapolation

In this section, we present our theorems on two-variable mixed type extrapolation. To prove them, we build upon ideas introduced in the previous section.

3.3.1 Downwards Extrapolation Results

The first result that we prove is a mixed type version of Theorem 3.2.4. The proof is more or less the same, except for some technical modifications involving the A_p condition on the weights.

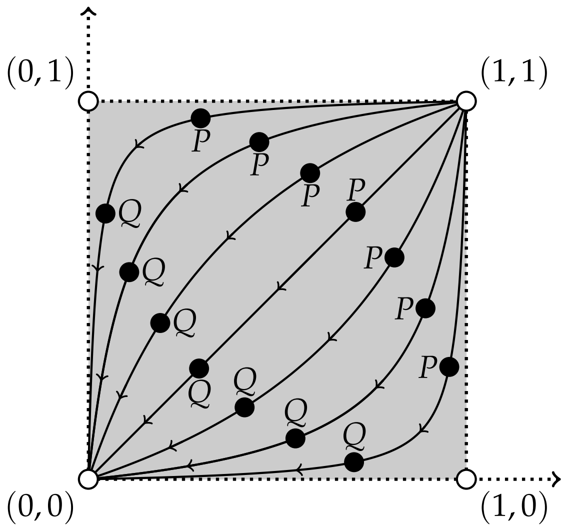

FIGURE 3.4: Pictorial representation of Corollaries 3.2.13 and 3.2.14.

Theorem 3.3.1. *Given measurable functions f_1, f_2, and g, suppose that for some exponents $1 \le p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and all weights $v_1 \in A_{p_1}$ and $v_2 \in A^{\mathcal{R}}_{p_2}$,*

$$\|g\|_{L^{p,\infty}(v_1^{p/p_1} v_2^{p/p_2})} \le \varphi([v_1]_{A_{p_1}}, [v_2]_{A^{\mathcal{R}}_{p_2}}) \, \|f_1\|_{L^{p_1}(v_1)} \, \|f_2\|_{L^{p_2,1}(v_2)}, \tag{3.3.1}$$

where $\varphi : [1,\infty)^2 \longrightarrow [0,\infty)$ is a function increasing in each variable. Then, for all exponents $1 \le q_1 = p_1$ or $1 < q_1 < p_1$, $1 \le q_2 \le p_2$, and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$ such that $p \neq q$, and all weights $w_1 \in A_{q_1}$ and $w_2 \in \widehat{A}_{q_2}$,

$$\|g\|_{L^{q,\infty}(w)} \le \Psi([w_1]_{A_{q_1}}, \|w_2\|_{\widehat{A}_{q_2}}) \|\mathscr{L}\|_{L^{q,\infty}(w)}^{1-\frac{q}{p}} \, \|f_1\|_{L^{q_1}(w_1)}^{q_1/p_1} \, \|f_2\|_{L^{q_2,\frac{q_2}{p_2}}(w_2)}^{q_2/p_2}, \tag{3.3.2}$$

where $\Psi : [1,\infty)^2 \longrightarrow [0,\infty)$ is a function increasing in each variable, $w = w_1^{q/q_1} w_2^{q/q_2}$, and $\mathscr{L}$ is as in Theorem 3.2.4.

Proof. We want to prove this result adapting the proof of Theorem 3.2.4. To do so, we have to show that the weight $v_1 := (Mf_1)^{q_1 - p_1} w_1 \in A_{p_1}$, apply (3.3.1), find an appropriate replacement for estimate (3.2.18), and keep track of the changes in the constants involved.

Let us see that $v_1 \in A_{p_1}$. If $q_1 = p_1$, then $v_1 = w_1$ and we are done. If $p_1 > q_1 > 1$, in virtue of Lemma 3.1.3, $v_1 \in A_{p_1}$, with

$$[v_1]_{A_{p_1}} \le C_1 [w_1]_{A_{q_1}}^{1 + \frac{p_1 - 1}{q_1 - 1}}.$$

Now, estimate (3.2.18) for $i = 1$ should be replaced by

$$\begin{aligned} \|f_1\|_{L^{p_1}(v_1)} &= \left(\int_{\mathbb{R}^n} |f_1|^{p_1} (Mf_1)^{q_1-p_1} w_1 \right)^{1/p_1} \\ &\leq \left(\int_{\mathbb{R}^n} |f_1|^{p_1} |f_1|^{q_1-p_1} w_1 \right)^{1/p_1} = \|f_1\|_{L^{q_1}(w_1)}^{q_1/p_1}. \end{aligned} \tag{3.3.3}$$

Finally, if we follow the proof of Theorem 3.2.4 performing the previous changes and keeping track of the constants, we conclude that (3.3.2) holds, with

$$\Psi([w_1]_{A_{q_1}}, \|w_2\|_{\widehat{A}_{q_2}}) = \frac{p_2}{q_2} C_{p,q} \varphi(\psi_1([w_1]_{A_{q_1}}), C_2 \|w_2\|_{\widehat{A}_{q_2}}^{q_2/p_2}), \tag{3.3.4}$$

where

$$\psi_1([w_1]_{A_{q_1}}) := \begin{cases} [w_1]_{A_{q_1}}, & 1 \leq q_1 = p_1, \\ C_1 [w_1]_{A_{q_1}}^{1+\frac{p_1-1}{q_1-1}}, & 1 < q_1 < p_1, \end{cases}$$

and $\Psi([w_1]_{A_{q_1}}, \|w_2\|_{\widehat{A}_{q_2}})$ depends on the constants $[w_1]_{A_{q_1}}$ and $\|w_2\|_{\widehat{A}_{q_2}}$ in an increasing way. □

If we combine Theorem 3.3.1 with Remark 2.4.2, we can deduce the following mixed type version of Theorem 3.2.7. In Figure 3.5 you can find a pictorial representation of it.

Theorem 3.3.2. *Given measurable functions f_1, f_2, and g, suppose that for some exponents $1 \leq p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and all weights $v_1 \in A_{p_1}$ and $v_2 \in A_{p_2}^{\mathcal{R}}$, (3.3.1) holds for a function $\varphi : [1,\infty)^2 \longrightarrow [0,\infty)$ that increases in each variable. Then, for all exponents $1 \leq q_1 = p_1$ or $1 < q_1 < p_1$, $1 \leq q_2 \leq p_2$, and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$ such that $p \neq q$ and $\frac{q_1}{p_1} = \frac{q_2}{p_2}$, and all weights $w_1 \in A_{q_1}$ and $w_2 \in \widehat{A}_{q_2}$,*

$$\|g\|_{L^{q,\infty}(w_1^{q/q_1} w_2^{q/q_2})} \leq \Phi([w_1]_{A_{q_1}}, \|w_2\|_{\widehat{A}_{q_2}}) \|f_1\|_{L^{q_1}(w_1)} \|f_2\|_{L^{q_2, \frac{q}{p}}(w_2)}, \tag{3.3.5}$$

where $\Phi : [1,\infty)^2 \longrightarrow [0,\infty)$ is a function increasing in each variable.

Proof. Pick weights $w_1 \in A_{q_1}$ and $w_2 \in \widehat{A}_{q_2}$, and write $w = w_1^{q/q_1} w_2^{q/q_2}$. From Lemma 3.2.6 we see that $\frac{q_1}{p_1} = \frac{q_2}{p_2} = \frac{q}{p}$, and in virtue of Theorem 3.3.1, we have that

$$\|g\|_{L^{q,\infty}(w)} \leq \Psi_{w_1,w_2} \|Mf_1 Mf_2\|_{L^{q,\infty}(w)}^{1-\frac{q}{p}} \|f_1\|_{L^{q_1}(w_1)}^{q/p} \|f_2\|_{L^{q_2,\frac{q}{p}}(w_2)}^{q/p}, \tag{3.3.6}$$

with $\Psi_{w_1,w_2} := \Psi([w_1]_{A_{q_1}}, \|w_2\|_{\widehat{A}_{q_2}})$ as in (3.3.4).

We apply Remark 2.4.2 to obtain that

$$\|Mf_1 Mf_2\|_{L^{q,\infty}(w)} \leq \phi([w_1]_{A_{q_1}}, c\|w_2\|_{\widehat{A}_{q_2}}) \|f_1\|_{L^{q_1}(w_1)} \|f_2\|_{L^{q_2,1}(w_2)}, \tag{3.3.7}$$

and as in (3.2.23),

$$\|f_2\|_{L^{q_2,1}(w_2)} \leq p_2^{1-\frac{p}{q}} \|f_2\|_{L^{q_2,\frac{q}{p}}(w_2)}. \tag{3.3.8}$$

Combining the estimates (3.3.6), (3.3.7), and (3.3.8), we get (3.3.5) with

$$\Phi([w_1]_{A_{q_1}}, \|w_2\|_{\widehat{A}_{q_2}}) = p_2^{2-\frac{q}{p}-\frac{p}{q}} \phi([w_1]_{A_{q_1}}, c\,\|w_2\|_{\widehat{A}_{q_2}})^{1-\frac{q}{p}} \Psi_{w_1,w_2}.$$

□

Remark 3.3.3. Observe that if in Theorem 3.3.2 we assume that $1 \leq q_2 = p_2$ or $1 < q_2 < p_2$, change $A_{p_2}^{\mathcal{R}}$ by A_{p_2} and $\widehat{A}_{q_2}$ by A_{q_2}, and replace (3.3.1) by

$$\|g\|_{L^{p,\infty}(v_1^{p/p_1} v_2^{p/p_2})} \leq \varphi([v_1]_{A_{p_1}}, [v_2]_{A_{p_2}}) \|f_1\|_{L^{p_1}(v_1)} \|f_2\|_{L^{p_2}(v_2)}, \tag{3.3.9}$$

then we can modify the proof of Theorem 3.2.4 as we did in the proof of Theorem 3.3.1 to obtain that

$$\|g\|_{L^{q,\infty}(w)} \leq \Psi_{w_1,w_2} \|Mf_1 Mf_2\|_{L^{q,\infty}(w)}^{1-\frac{q}{p}} \|f_1\|_{L^{q_1}(w_1)}^{q/p} \|f_2\|_{L^{q_2}(w_2)}^{q/p},$$

with

$$\Psi_{w_1,w_2} := \Psi([w_1]_{A_{q_1}}, [w_2]_{A_{q_2}}) = C_{p,q} \varphi(\psi_1([w_1]_{A_{q_1}}), \psi_2([w_2]_{A_{q_2}})),$$

where for $i = 1, 2$,

$$\psi_i([w_i]_{A_{q_i}}) := \begin{cases} [w_i]_{A_{q_i}}, & 1 \leq q_i = p_i, \\ C_i [w_i]_{A_{q_i}}^{1+\frac{p_i-1}{q_i-1}}, & 1 < q_i < p_i, \end{cases}$$

and using Remark 2.4.2, we conclude that

$$\|g\|_{L^{q,\infty}(w_1^{q/q_1} w_2^{q/q_2})} \leq \phi([w_1]_{A_{q_1}}, [w_2]_{A_{q_2}}) \Psi_{w_1,w_2} \|f_1\|_{L^{q_1}(w_1)} \|f_2\|_{L^{q_2}(w_2)}.$$

This argument gives an alternative proof of [15, Theorem 3.12] in the particular case when $1 \leq q_1 = p_1$ or $1 < q_1 < p_1$, $1 \leq q_2 = p_2$ or $1 < q_2 < p_2$, and $\frac{q_1}{p_1} = \frac{q_2}{p_2}$.

As always, we have presented Theorem 3.3.2 in its general form. We can obtain the corresponding extrapolation scheme for two-variable operators arguing as in the proof of Corollary 3.2.2. See Figure 3.5 for a pictorial representation of such scheme.

Corollary 3.3.4. *Let T be a two-variable operator defined for measurable functions. Suppose that for some exponents $1 \leq p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and all weights $v_1 \in A_{p_1}$ and $v_2 \in A_{p_2}^{\mathcal{R}}$,*

$$T : L^{p_1}(v_1) \times L^{p_2,1}(v_2) \longrightarrow L^{p,\infty}(v_1^{p/p_1} v_2^{p/p_2}),$$

with constant bounded by $\varphi([v_1]_{A_{p_1}}, [v_2]_{A^{\mathcal{R}}_{p_2}})$ as in (3.3.1). Then, for all exponents $1 \leq q_1 = p_1$ or $1 < q_1 < p_1$, $1 \leq q_2 \leq p_2$, and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$ such that $p \neq q$ and $\frac{q_1}{p_1} = \frac{q_2}{p_2}$, and all weights $w_1 \in A_{q_1}$ and $w_2 \in \widehat{A}_{q_2}$,

$$T : L^{q_1}(w_1) \times L^{q_2,\frac{q}{p}}(w_2) \longrightarrow L^{q,\infty}(w_1^{q/q_1} w_2^{q/q_2}),$$

with constant bounded by $\Phi([w_1]_{A_{q_1}}, \|w_2\|_{\widehat{A}_{q_2}})$ as in (3.3.5).

Remark 3.3.5. Observe that if the operator T is defined for characteristic functions of measurable sets $E_1, E_2 \subseteq \mathbb{R}^n$, then under the hypotheses of Corollary 3.3.4, we deduce that

$$\|T(\chi_{E_1}, \chi_{E_2})\|_{L^{q,\infty}(w_1^{q/q_1} w_2^{q/q_2})} \leq C\Phi([w_1]_{A_{q_1}}, \|w_2\|_{\widehat{A}_{q_2}}) w_1(E_1)^{1/q_1} w_2(E_2)^{1/q_2},$$

with $C = \left(q_2 \frac{p}{q}\right)^{p/q}$, and hence, T is of weak type (q_1, q_2, q) at least for characteristic functions.

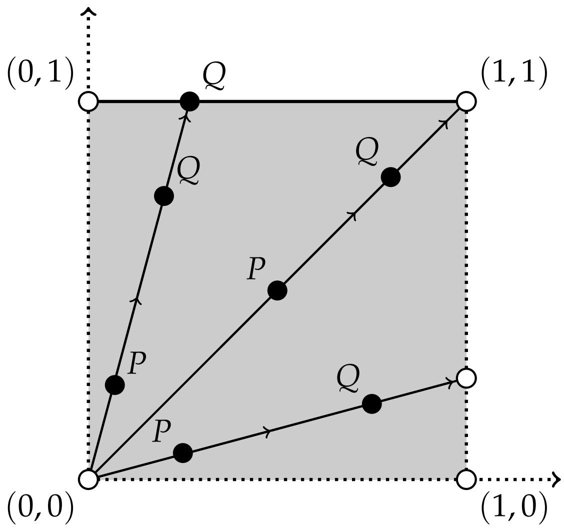

FIGURE 3.5: Pictorial representation of Theorem 3.3.2 and Corollary 3.3.4.

So far we have produced extrapolation schemes for the case when the exponents involved satisfy the alignment condition $\frac{q_1}{p_1} = \frac{q_2}{p_2}$. Now, we are interested in studying the case when we fix the value of the second exponent, that is, when $p_2 = q_2$. To do so, we can use Theorem 3.3.1, but this time the term $\mathscr{L}$ differs from Mf_1Mf_2. Indeed, since $p_2 = q_2$, we have that $\delta_2 = 0$, and in virtue of Lemma 3.2.6 (b), (c), and (d), we deduce that $\delta_1 = \frac{q_1}{q}$, $\beta_1 = \frac{1}{p_2}$, and $\beta_2 = -\frac{1}{p_2}$. Hence,

$$\mathscr{L} = (Mf_1)^{q_1/q} \left(\frac{w_1}{w_2}\right)^{1/p_2}, \tag{3.3.10}$$

and a suitable bound of this operator allows us to produce the following extrapolation scheme. In Figure 3.6 you can find a pictorial representation of it.

Theorem 3.3.6. *Given measurable functions f_1, f_2, and g, suppose that for some exponents $1 \le p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and all weights $v_1 \in A_{p_1}$ and $v_2 \in A^{\mathcal{R}}_{p_2}$, (3.3.1) holds for a function $\varphi : [1,\infty)^2 \longrightarrow [0,\infty)$ that increases in each variable. Then, for every exponent $1 \le q_1 = p_1$ or $1 < q_1 < p_1$, and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{p_2}$, and all weights $w_1 \in A_{q_1}$ and $w_2 \in A^{\mathcal{R}}_{p_2}$,*

$$\|g\|_{L^{q,\infty}(w_1^{q/q_1} w_2^{q/p_2})} \le \Phi([w_1]_{A_{q_1}}, [w_2]_{A^{\mathcal{R}}_{p_2}}) \, \|f_1\|_{L^{q_1}(w_1)} \, \|f_2\|_{L^{p_2,1}(w_2)}, \tag{3.3.11}$$

where $\Phi : [1,\infty)^2 \longrightarrow [0,\infty)$ is a function increasing in each variable.

Proof. Note that if $q_1 = p_1$, then there is nothing to prove, so we may assume that $1 < q_1 < p_1$. Pick weights $w_1 \in A_{q_1}$ and $w_2 \in A^{\mathcal{R}}_{p_2}$, and write $w = w_1^{q/q_1} w_2^{q/p_2}$. Observe that in the proof of Theorem 3.2.4, the weight v_2 that we chose becomes $w_2 \in A^{\mathcal{R}}_{p_2}$, and estimate (3.2.16) is no longer necessary. This fact allows us to work with $A^{\mathcal{R}}_{p_2}$ instead of $\widehat{A}_{p_2}$ in Theorem 3.2.4 when $q_2 = p_2$, and also in Theorem 3.3.1, and hence, in virtue of (3.3.10), we have that

$$\|g\|_{L^{q,\infty}(w)} \le \Psi_{w_1,w_2} \left\| (Mf_1)^{q_1/q} \left(\frac{w_1}{w_2}\right)^{1/p_2} \right\|_{L^{q,\infty}(w)}^{1-\frac{q}{p}} \|f_1\|_{L^{q_1}(w_1)}^{q_1/p_1} \|f_2\|_{L^{p_2,1}(w_2)}, \tag{3.3.12}$$

with

$$\Psi_{w_1,w_2} := \Psi([w_1]_{A_{q_1}}, [w_2]_{A^{\mathcal{R}}_{p_2}}) = C_{p,q}\varphi(C_1[w_1]_{A_{q_1}}^{1+\frac{p_1-1}{q_1-1}}, [w_2]_{A^{\mathcal{R}}_{p_2}}).$$

Note that

$$\begin{aligned} \left\| (Mf_1)^{q_1/q} \left(\frac{w_1}{w_2}\right)^{1/p_2} \right\|_{L^{q,\infty}(w)} &\le \left\| (Mf_1)^{q_1/q} \left(\frac{w_1}{w_2}\right)^{1/p_2} \right\|_{L^{q}(w)} \\ &= \left(\int_{\mathbb{R}^n} (Mf_1)^{q_1} w_1^{\frac{q}{q_1}+\frac{q}{p_2}} \right)^{1/q} \\ &= \|Mf_1\|_{L^{q_1}(w_1)}^{q_1/q} \le c[w_1]_{A_{q_1}}^{q_1'/q} \|f_1\|_{L^{q_1}(w_1)}^{q_1/q}, \end{aligned} \tag{3.3.13}$$

where the first inequality follows from [44, Proposition 1.1.6], and the second inequality follows from [8, Theorem 2.5].

Combining the estimates (3.3.12) and (3.3.13), we get that

$$\begin{aligned}\|g\|_{L^{q,\infty}(w)} &\leq C[w_1]_{A_{q_1}}^{\frac{q_1'}{q}(1-\frac{q}{p})}\Psi_{w_1,w_2}\|f_1\|_{L^{q_1}(w_1)}^{\frac{q_1}{p_1}+\frac{q_1}{q}(1-\frac{q}{p})}\|f_2\|_{L^{p_2,1}(w_2)}\\ &= C[w_1]_{A_{q_1}}^{\frac{1}{p_1}\frac{p_1-q_1}{q_1-1}}\Psi_{w_1,w_2}\|f_1\|_{L^{q_1}(w_1)}\|f_2\|_{L^{p_2,1}(w_2)},\end{aligned}$$

and (3.3.11) holds, with

$$\Phi([w_1]_{A_{q_1}},[w_2]_{A_{p_2}^{\mathcal{R}}}) = C[w_1]_{A_{q_1}}^{\frac{1}{p_1}\frac{p_1-q_1}{q_1-1}}\Psi_{w_1,w_2}.$$

□

Remark 3.3.7. Note that given $0<\alpha\leq p_1$, if in Theorem 3.3.6 we replace (3.3.1) by

$$\|g\|_{L^{p,\infty}(v_1^{p/p_1}v_2^{p/p_2})} \leq \varphi([v_1]_{A_{p_1}},[v_2]_{A_{p_2}^{\mathcal{R}}})\|f_1\|_{L^{p_1,\alpha}(v_1)}\|f_2\|_{L^{p_2,1}(v_2)},$$

and also in Theorem 3.3.1, then we can replace estimate (3.3.3) by

$$\|f_1\|_{L^{p_1,\alpha}(v_1)} \leq p_1^{1/\alpha}\left(\int_0^\infty t^{\frac{\alpha q_1}{p_1}}\lambda_{f_1}^{w_1}(t)^{\alpha/p_1}\frac{dt}{t}\right)^{1/\alpha} = \left(\frac{p_1}{q_1}\right)^{1/\alpha}\|f_1\|_{L^{q_1,\frac{\alpha q_1}{p_1}}(w_1)}^{q_1/p_1},$$

and follow the proof of Theorem 3.3.6, using that

$$\|f_1\|_{L^{q_1}(w_1)} \leq \left(\frac{\alpha}{p_1}\right)^{\frac{p_1-\alpha}{\alpha q_1}}\|f_1\|_{L^{q_1,\frac{\alpha q_1}{p_1}}(w_1)}$$

in (3.3.13), to conclude that

$$\|g\|_{L^{q,\infty}(w_1^{q/q_1}w_2^{q/p_2})} \leq \Phi_\alpha([w_1]_{A_{q_1}},[w_2]_{A_{p_2}^{\mathcal{R}}})\|f_1\|_{L^{q_1,\frac{\alpha q_1}{p_1}}(w_1)}\|f_2\|_{L^{p_2,1}(w_2)},$$

with $\Phi_\alpha := \left(\frac{p_1}{q_1}\right)^{1/\alpha}\left(\frac{\alpha}{p_1}\right)^{(\frac{p_1}{\alpha}-1)(\frac{1}{q_1}-\frac{1}{p_1})}\Phi$.

Remark 3.3.8. Observe that if in Theorem 3.3.6 we change $A_{p_2}^{\mathcal{R}}$ by A_{p_2}, and replace (3.3.1) by (3.3.9), then we can modify the proof of Theorem 3.3.1 arguing as in the proof of Theorem 3.3.6 to obtain that for $1<q_1<p_1$,

$$\|g\|_{L^{q,\infty}(w)} \leq \Psi_{w_1,w_2}\left\|(Mf_1)^{q_1/q}\left(\frac{w_1}{w_2}\right)^{1/p_2}\right\|_{L^{q,\infty}(w)}^{1-\frac{q}{p}}\|f_1\|_{L^{q_1}(w_1)}^{q_1/p_1}\|f_2\|_{L^{p_2}(w_2)},$$

with

$$\Psi_{w_1,w_2} := \Psi([w_1]_{A_{q_1}},[w_2]_{A_{p_2}}) = C_{p,q}\varphi(C_1[w_1]_{A_{q_1}}^{1+\frac{p_1-1}{q_1-1}},[w_2]_{A_{p_2}}),$$

and in virtue of (3.3.13), we conclude that

$$\|g\|_{L^{q,\infty}(w_1^{q/q_1} w_2^{q/p_2})} \leq \Phi([w_1]_{A_{q_1}}, [w_2]_{A_{p_2}}) \, \|f_1\|_{L^{q_1}(w_1)} \, \|f_2\|_{L^{p_2}(w_2)}, \tag{3.3.14}$$

with

$$\Phi([w_1]_{A_{q_1}}, [w_2]_{A_{p_2}}) = c[w_1]_{A_{q_1}}^{\frac{1}{p_1}\frac{p_1-q_1}{q_1-1}} \Psi_{w_1,w_2}.$$

In the case when $1 \leq q_1 = p_1$, (3.3.14) is just (3.3.9). We can now use (3.3.14) as a starting condition to extrapolate again, playing the role of (3.3.9) in the previous argument, but this time we fix the value of q_1, obtaining that for $1 < q_2 < p_2$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, and all weights $w_1 \in A_{q_1}$ and $w_2 \in A_{q_2}$,

$$\|g\|_{L^{q,\infty}(w_1^{q/q_1} w_2^{q/q_2})} \leq \Phi([w_1]_{A_{q_1}}, [w_2]_{A_{q_2}}) \, \|f_1\|_{L^{q_1}(w_1)} \, \|f_2\|_{L^{q_2}(w_2)}, \tag{3.3.15}$$

with

$$\Phi([w_1]_{A_{q_1}}, [w_2]_{A_{q_2}}) = C[w_1]_{A_{q_1}}^{\frac{1}{p_1}\frac{p_1-q_1}{q_1-1}} [w_2]_{A_{q_2}}^{\frac{1}{p_2}\frac{p_2-q_2}{q_2-1}} \varphi(C_1[w_1]_{A_{q_1}}^{1+\frac{p_1-1}{q_1-1}}, C_2[w_2]_{A_{q_2}}^{1+\frac{p_2-1}{q_2-1}}).$$

In the case when $1 \leq q_2 = p_2$, (3.3.15) is just (3.3.14). This argument gives an alternative proof of [15, Theorem 3.12] in the case when $1 \leq q_1 = p_1$ or $1 < q_1 < p_1$, and $1 \leq q_2 = p_2$ or $1 < q_2 < p_2$.

Remark 3.3.9. We can use Rubio de Francia's algorithm (see [101]) to improve Theorem 3.3.6, producing a better function Φ. Indeed, for $q_1 > 1$, in Theorem 3.3.1 we can take

$$\mathscr{Z} := (Rf_1)^{q_1/q} \left(\frac{w_1}{w_2}\right)^{1/p_2},$$

and in its proof, we can take

$$v_1 := (Rf_1)^{q_1-p_1} w_1,$$

where for a measurable function $h \in L^{q_1}(w_1)$,

$$Rh := \sum_{k=0}^{m} \frac{M^k h}{\left(2\|M\|_{L^{q_1}(w_1)}\right)^k}$$

is the Rubio de Francia's algorithm (see [37, 101]).

In virtue of [37, Lemma 2.2], we have that $\|Rh\|_{L^{q_1}(w_1)} \leq 2\|h\|_{L^{q_1}(w_1)}$, $h \leq Rh$, and $Rh \in A_1$, with $[Rh]_{A_1} \leq 2\|M\|_{L^{q_1}(w_1)} \leq \widetilde{c}_1[w_1]_{A_{q_1}}^{\frac{1}{q_1-1}}$ (see [8, Theorem 2.5]). Moreover, applying [37, Lemma 2.1], we obtain that $v_1 \in A_{p_1}$, with $[v_1]_{A_{p_1}} \leq \widetilde{C}_1[w_1]_{A_{q_1}}^{\frac{p_1-1}{q_1-1}}$. Hence, we can rewrite the proof of Theorem 3.3.6 to conclude that (3.3.11) holds, with

$$\Phi([w_1]_{A_{q_1}},[w_2]_{A_{p_2}^{\mathcal{R}}}) = 2^{1-\frac{q_1}{p_1}} C_{p,q}\varphi(\widetilde{C}_1[w_1]_{A_{q_1}}^{\frac{p_1-1}{q_1-1}},[w_2]_{A_{p_2}^{\mathcal{R}}}). \tag{3.3.16}$$

As usual, we have presented Theorem 3.3.6 in its general form. We can obtain the corresponding extrapolation scheme for two-variable operators arguing as in the proof of Corollary 3.2.2. See Figure 3.6 for a pictorial representation of such scheme.

Corollary 3.3.10. *Let T be a two-variable operator defined for measurable functions. Suppose that for some exponents $1 \le p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and all weights $v_1 \in A_{p_1}$ and $v_2 \in A_{p_2}^{\mathcal{R}}$,*

$$T : L^{p_1}(v_1) \times L^{p_2,1}(v_2) \longrightarrow L^{p,\infty}(v_1^{p/p_1} v_2^{p/p_2}),$$

with constant bounded by $\varphi([v_1]_{A_{p_1}},[v_2]_{A_{p_2}^{\mathcal{R}}})$ as in (3.3.1). Then, for every exponent $1 \le q_1 = p_1$ or $1 < q_1 < p_1$, and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{p_2}$, and all weights $w_1 \in A_{q_1}$ and $w_2 \in A_{p_2}^{\mathcal{R}}$,

$$T : L^{q_1}(w_1) \times L^{p_2,1}(w_2) \longrightarrow L^{q,\infty}(w_1^{q/q_1} w_2^{q/p_2}),$$

with constant bounded by $\Phi([w_1]_{A_{q_1}},[w_2]_{A_{p_2}^{\mathcal{R}}})$ as in (3.3.11).

Remark 3.3.11. Observe that if the operator T is defined for characteristic functions of measurable sets $E_1, E_2 \subseteq \mathbb{R}^n$, then under the hypotheses of Corollary 3.3.10, we deduce that

$$\left\|T(\chi_{E_1},\chi_{E_2})\right\|_{L^{q,\infty}(w_1^{q/q_1} w_2^{q/p_2})} \le p_2 \Phi([w_1]_{A_{q_1}},[w_2]_{A_{p_2}^{\mathcal{R}}}) w_1(E_1)^{1/q_1} w_2(E_2)^{1/p_2},$$

and hence, T is of weak type (q_1, p_2, q) at least for characteristic functions.

In Theorem 3.3.6, we manage to fix the second exponent p_2 and decrease the first exponent p_1 down to q_1 exploiting the A_{q_1} condition on the weight w_1. We can also fix the value of p_1 and decrease the second exponent p_2 down to q_2 exploiting the $\widehat{A}_{q_2}$ condition on the weight w_2, as we show in the next result. The proof is similar to the one of Theorem 3.3.6, but we need to use Theorem 2.3.8 to control the quantity $\|\mathscr{Z}\|_{L^{q,\infty}}(w)$. We include a pictorial representation of this scheme in Figure 3.7.

Theorem 3.3.12. *Given measurable functions f_1, f_2, and g, suppose that for some exponents $1 \le p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and all weights $v_1 \in A_{p_1}$ and $v_2 \in A_{p_2}^{\mathcal{R}}$, (3.3.1) holds for a function $\varphi : [1,\infty)^2 \longrightarrow [0,\infty)$ that increases in each variable. Then, for every exponent $1 \le q_2 \le p_2$, and $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{q_2}$, and all weights $w_1 \in A_{p_1}$ and $w_2 \in \widehat{A}_{q_2}$,*

$$\|g\|_{L^{q,\infty}(w_1^{q/p_1} w_2^{q/q_2})} \le \Phi([w_1]_{A_{p_1}}, \|w_2\|_{\widehat{A}_{q_2}}) \|f_1\|_{L^{p_1}(w_1)} \|f_2\|_{L^{q_2,\frac{q_2}{p_2}}(w_2)}, \tag{3.3.17}$$

where $\Phi : [1,\infty)^2 \longrightarrow [0,\infty)$ is a function increasing in each variable.

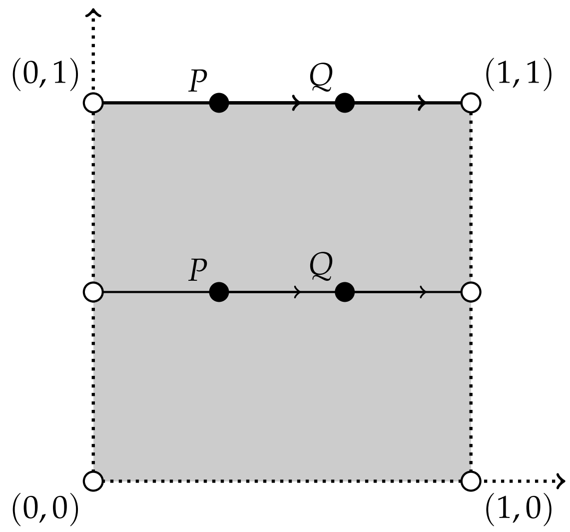

FIGURE 3.6: Pictorial representation of Theorem 3.3.6 and Corollary 3.3.10.

Proof. Note that if $q_2 = p_2$, then there is nothing to prove, so we may assume that $q_2 < p_2$. Pick weights $w_1 \in A_{p_1}$ and $w_2 \in \widehat{A}_{q_2}$, and write $w = w_1^{q/p_1} w_2^{q/q_2}$. In virtue of Theorem 3.3.1, we have that

$$\|g\|_{L^{q,\infty}(w)} \leq \Psi_{w_1,w_2} \|\mathscr{Z}\|_{L^{q,\infty}(w)}^{1-\frac{q}{p}} \|f_1\|_{L^{p_1}(w_1)} \|f_2\|_{L^{q_2,\frac{q_2}{p_2}}(w_2)}^{q_2/p_2}, \tag{3.3.18}$$

with

$$\Psi_{w_1,w_2} := \Psi([w_1]_{A_{p_1}}, \|w_2\|_{\widehat{A}_{q_2}}) = \frac{p_2}{q_2} C_{p,q} \varphi([w_1]_{A_{p_1}}, C_2 \|w_2\|_{\widehat{A}_{q_2}}^{q_2/p_2}).$$

We want to control the term $\|\mathscr{Z}\|_{L^{q,\infty}(w)}$ in (3.3.18). Observe that

$$\mathscr{Z} = (Mf_2)^{q_2/q} \left(\frac{w_2}{w_1} \right)^{1/p_1},$$

since $\delta_1 = 0$, and from Lemma 3.2.6 (b), (c), and (d), we deduce that $\delta_2 = \frac{q_2}{q}$, $\beta_1 = -\frac{1}{p_1}$, and $\beta_2 = \frac{1}{p_1}$. If we take $W := \left(\frac{w_1}{w_2}\right)^{\frac{q}{p_1 q_2}}$, in virtue of Lemma 3.1.5, $W \in A_\infty$, and $w_2 W^{q_2} = w_1^{q/p_1} w_2^{1-\frac{q}{p_1}} = w$. Applying Lemma 3.1.6, we see that $w \in A_{2q}^{\mathcal{R}}$, with $[w]_{A_{2q}^{\mathcal{R}}} \leq \psi([w_1]_{A_{p_1}}) \|w_2\|_{\widehat{A}_{q_2}}^{1/2}$, so by Theorem 2.3.8, Lemma 2.3.10, and (2.1.1), we obtain that

$$
\begin{aligned}
\|\mathscr{Z}\|_{L^{q,\infty}(w)} &= \left\|\frac{Mf_2}{W}\right\|^{q_2/q}_{L^{q_2,\infty}(w_2W^{q_2})} \leq \phi([w_2]_{A^{\mathcal{R}}_{q_2}},[w]_{A^{\mathcal{R}}_{2q}})^{q_2/q}\|f_2\|^{q_2/q}_{L^{q_2,1}(w_2)} \\
&\leq p_2^{\frac{1}{q}(q_2-p_2)}\phi(c\|w_2\|_{\widehat{A}_{q_2}},\psi([w_1]_{A_{p_1}})\|w_2\|^{1/2}_{\widehat{A}_{q_2}})^{q_2/q}\|f_2\|^{q_2/q}_{L^{q_2,\frac{q_2}{p_2}}(w_2)} \\
&= p_2^{\frac{1}{q}(q_2-p_2)}\phi_{w_1,w_2}^{q_2/q}\|f_2\|^{q_2/q}_{L^{q_2,\frac{q_2}{p_2}}(w_2)},
\end{aligned}
\tag{3.3.19}
$$

with

$$
\phi_{w_1,w_2} := \phi(c\|w_2\|_{\widehat{A}_{q_2}},\psi([w_1]_{A_{p_1}})\|w_2\|^{1/2}_{\widehat{A}_{q_2}}). \tag{3.3.20}
$$

Combining estimates (3.3.18) and (3.3.19), we get that

$$
\begin{aligned}
\|g\|_{L^{q,\infty}(w)} &\leq p_2^{\frac{1}{q}(q_2-p_2)(1-\frac{q}{p})}\phi_{w_1,w_2}^{\frac{q_2}{q}(1-\frac{q}{p})}\Psi_{w_1,w_2}\|f_1\|_{L^{p_1}(w_1)}\|f_2\|^{\frac{q_2}{p_2}+\frac{q_2}{q}(1-\frac{q}{p})}_{L^{q_2,\frac{q_2}{p_2}}(w_2)} \\
&= p_2^{2-\frac{q_2}{p_2}-\frac{p_2}{q_2}}\phi_{w_1,w_2}^{1-\frac{q_2}{p_2}}\Psi_{w_1,w_2}\|f_1\|_{L^{p_1}(w_1)}\|f_2\|_{L^{q_2,\frac{q_2}{p_2}}(w_2)},
\end{aligned}
$$

and (3.3.17) holds, with

$$
\Phi([w_1]_{A_{p_1}},\|w_2\|_{\widehat{A}_{q_2}}) = p_2^{2-\frac{q_2}{p_2}-\frac{p_2}{q_2}}\phi_{w_1,w_2}^{1-\frac{q_2}{p_2}}\Psi_{w_1,w_2}.
$$

□

Remark 3.3.13. Note that given $\alpha > 0$, if in Theorem 3.3.12 we replace (3.3.1) by

$$
\|g\|_{L^{p,\infty}(v_1^{p/p_1}v_2^{p/p_2})} \leq \varphi([v_1]_{A_{p_1}},[v_2]_{A^{\mathcal{R}}_{p_2}})\|f_1\|_{L^{p_1,\alpha}(v_1)}\|f_2\|_{L^{p_2,1}(v_2)},
$$

then we can replace estimate (3.3.18) by

$$
\|g\|_{L^{q,\infty}(w)} \leq \Psi_{w_1,w_2}\|\mathscr{Z}\|^{1-\frac{q}{p}}_{L^{q,\infty}(w)}\|f_1\|_{L^{p_1,\alpha}(w_1)}\|f_2\|^{q_2/p_2}_{L^{q_2,\frac{q_2}{p_2}}(w_2)},
$$

and follow the proof of Theorem 3.3.12 to conclude that

$$
\|g\|_{L^{q,\infty}(w_1^{q/p_1}w_2^{q/q_2})} \leq \Phi([w_1]_{A_{p_1}},\|w_2\|_{\widehat{A}_{q_2}})\|f_1\|_{L^{p_1,\alpha}(w_1)}\|f_2\|_{L^{q_2,\frac{q_2}{p_2}}(w_2)}.
$$

As before, from Theorem 3.3.12 we can obtain the corresponding extrapolation scheme for two-variable operators arguing as in the proof of Corollary 3.2.2. See Figure 3.7 for a pictorial representation of such scheme.

Corollary 3.3.14. *Let T be a two-variable operator defined for measurable functions. Suppose that for some exponents $1 \leq p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and all weights*

$v_1 \in A_{p_1}$ and $v_2 \in A_{p_2}^{\mathcal{R}}$,

$$T : L^{p_1}(v_1) \times L^{p_2,1}(v_2) \longrightarrow L^{p,\infty}(v_1^{p/p_1} v_2^{p/p_2}),$$

with constant bounded by $\varphi([v_1]_{A_{p_1}}, [v_2]_{A_{p_2}^{\mathcal{R}}})$ as in (3.3.1). Then, for every exponent $1 \leq q_2 \leq p_2$, and $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{q_2}$, and all weights $w_1 \in A_{p_1}$ and $w_2 \in \widehat{A}_{q_2}$,

$$T : L^{p_1}(w_1) \times L^{q_2,\frac{q_2}{p_2}}(w_2) \longrightarrow L^{q,\infty}(w_1^{q/p_1} w_2^{q/q_2}),$$

with constant bounded by $\Phi([w_1]_{A_{p_1}}, [w_2]_{A_{q_2}^{\mathcal{R}}})$ as in (3.3.17).

Remark 3.3.15. Observe that if the operator T is defined for characteristic functions of measurable sets $E_1, E_2 \subseteq \mathbb{R}^n$, then under the hypotheses of Corollary 3.3.14, we deduce that

$$\left\| T(\chi_{E_1}, \chi_{E_2}) \right\|_{L^{q,\infty}(w_1^{q/p_1} w_2^{q/q_2})} \leq C\Phi([w_1]_{A_{p_1}}, [w_2]_{A_{q_2}^{\mathcal{R}}}) w_1(E_1)^{1/p_1} w_2(E_2)^{1/q_2},$$

with $C = p_2^{p_2/q_2}$, and hence, T is of weak type (p_1, q_2, q) at least for characteristic functions.

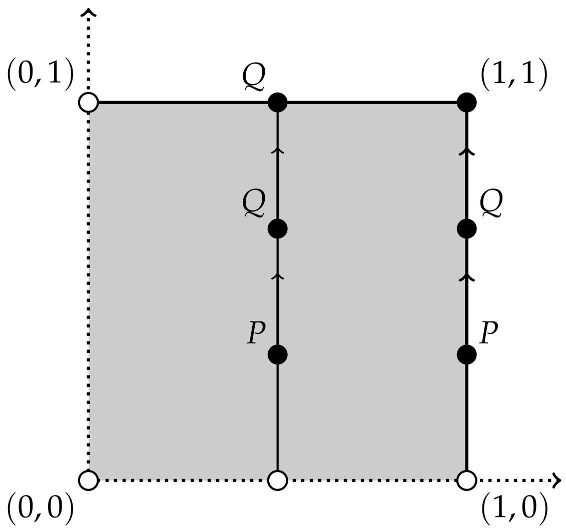

FIGURE 3.7: Pictorial representation of Theorem 3.3.12 and Corollary 3.3.14.

We can combine Theorem 3.3.6 and Theorem 3.3.12 to produce a more general extrapolation scheme that decreases both exponents p_1 and p_2 down to q_1 and q_2, respectively. The monotonicity of the functions φ and Φ in both theorems is crucial for the iteration process. For convenience, we include a pictorial representation of this scheme in Figure 3.8.

Theorem 3.3.16. *Given measurable functions f_1, f_2, and g, suppose that for some exponents $1 \leq p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and all weights $v_1 \in A_{p_1}$ and $v_2 \in A_{p_2}^{\mathcal{R}}$,* *(3.3.1) holds for a function $\varphi : [1,\infty)^2 \longrightarrow [0,\infty)$ that increases in each variable. Then, for all exponents $1 \leq q_1 = p_1$ or $1 < q_1 < p_1$, $1 \leq q_2 \leq p_2$, and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, and all weights $w_1 \in A_{q_1}$ and $w_2 \in \widehat{A}_{q_2}$,*

$$\|g\|_{L^{q,\infty}(w_1^{q/q_1} w_2^{q/q_2})} \leq \Phi([w_1]_{A_{q_1}}, \|w_2\|_{\widehat{A}_{q_2}}) \|f_1\|_{L^{q_1}(w_1)} \|f_2\|_{L^{q_2,\frac{q_2}{p_2}}(w_2)}, \tag{3.3.21}$$

where $\Phi : [1,\infty)^2 \longrightarrow [0,\infty)$ is a function increasing in each variable.

Proof. Note that if $q_1 = p_1$ and $q_2 = p_2$, then there is nothing to prove. If $q_1 < p_1$ and $q_2 = p_2$, then the result follows immediately from Theorem 3.3.6, and if $q_1 = p_1$ and $q_2 < p_2$, then the result follows immediately from Theorem 3.3.12. Let us assume that $1 < q_1 < p_1$ and $1 \leq q_2 < p_2$. In virtue of Theorem 3.3.6 and (3.3.16), we have that for $\frac{1}{r} := \frac{1}{q_1} + \frac{1}{p_2}$, and all weights $W_1 \in A_{q_1}$ and $W_2 \in A_{p_2}^{\mathcal{R}}$,

$$\|g\|_{L^{r,\infty}(W_1^{r/q_1} W_2^{r/p_2})} \leq \widetilde{\Phi}([W_1]_{A_{q_1}}, [W_2]_{A_{p_2}^{\mathcal{R}}}) \|f_1\|_{L^{q_1}(W_1)} \|f_2\|_{L^{p_2,1}(W_2)}, \tag{3.3.22}$$

with

$$\widetilde{\Phi}([W_1]_{A_{q_1}}, [W_2]_{A_{p_2}^{\mathcal{R}}}) := 2^{1-\frac{q_1}{p_1}} C_{p,r} \varphi(\widetilde{C}_1 [W_1]_{A_{q_1}}^{\frac{p_1-1}{q_1-1}}, [W_2]_{A_{p_2}^{\mathcal{R}}}),$$

and applying Theorem 3.3.12 replacing (3.3.1) by (3.3.22), we conclude that (3.3.21) holds, with

$$\Phi([w_1]_{A_{q_1}}, \|w_2\|_{\widehat{A}_{q_2}}) = p_2^{3-\frac{q_2}{p_2}-\frac{p_2}{q_2}} \frac{C_{r,q}}{q_2} \phi_{w_1,w_2}^{1-\frac{q_2}{p_2}} \widetilde{\Phi}([w_1]_{A_{q_1}}, C_2 \|w_2\|_{\widehat{A}_{q_2}}^{q_2/p_2}),$$

where ϕ_{w_1,w_2} is as in (3.3.20) with the obvious modifications in the parameters. $\square$

Remark 3.3.17. Observe that in the case when $\frac{q_1}{p_1} = \frac{q_2}{p_2}$, we can extrapolate with either Theorem 3.3.2 or Theorem 3.3.16, but the function Φ that we obtain with Theorem 3.3.16 is slightly better due to Remark 3.3.9.

As usual, from Theorem 3.3.16 we can obtain the corresponding extrapolation scheme for two-variable operators arguing as in the proof of Corollary 3.2.2. See Figure 3.8 for a pictorial representation of such scheme.

Corollary 3.3.18. *Let T be a two-variable operator defined for measurable functions. Suppose that for some exponents $1 \leq p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and all weights $v_1 \in A_{p_1}$ and $v_2 \in A_{p_2}^{\mathcal{R}}$,*

$$T : L^{p_1}(v_1) \times L^{p_2,1}(v_2) \longrightarrow L^{p,\infty}(v_1^{p/p_1} v_2^{p/p_2}),$$

with constant bounded by $\varphi([v_1]_{A_{p_1}}, [v_2]_{A_{p_2}^{\mathcal{R}}})$ as in (3.3.1). Then, for all exponents $1 \leq q_1 = p_1$ or $1 < q_1 < p_1$, $1 \leq q_2 \leq p_2$, and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, and all weights $w_1 \in A_{q_1}$ and $w_2 \in \widehat{A}_{q_2}$,

$$T : L^{q_1}(w_1) \times L^{q_2,\frac{q_2}{p_2}}(w_2) \longrightarrow L^{q,\infty}(w_1^{q/q_1} w_2^{q/q_2}),$$

with constant bounded by $\Phi([w_1]_{A_{q_1}}, \|w_2\|_{\widehat{A}_{q_2}})$ *as in* (3.3.21).

Remark 3.3.19. Observe that if the operator T is defined for characteristic functions of measurable sets $E_1, E_2 \subseteq \mathbb{R}^n$, then under the hypotheses of Corollary 3.3.18, we deduce that

$$\|T(\chi_{E_1}, \chi_{E_2})\|_{L^{q,\infty}(w_1^{q/q_1} w_2^{q/q_2})} \le C\Phi([w_1]_{A_{q_1}}, \|w_2\|_{\widehat{A}_{q_2}}) w_1(E_1)^{1/q_1} w_2(E_2)^{1/q_2},$$

with $C = p_2^{p_2/q_2}$, and hence, T is of weak type (q_1, q_2, q) at least for characteristic functions.

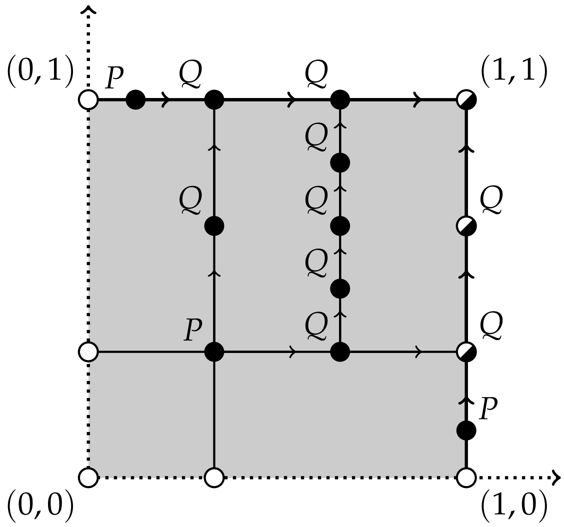

FIGURE 3.8: Pictorial representation of Theorem 3.3.16 and Corollary 3.3.18.

3.3.2 Upwards and Combined Extrapolation Results

Now we want to produce extrapolation schemes that allow us to increase the value of the exponents of the Lorentz spaces involved, like we did in Theorem 3.2.10. The next result shows that we can increase the value of p_1 while fixing p_2. The proof is different from the one of Theorem 3.3.6, and borrows some ideas from the proof of Theorem 3.2.10. See Figure 3.9 for a pictorial representation of this new scheme.

Theorem 3.3.20. *Given measurable functions f_1, f_2, and g, suppose that for some exponents $1 \le p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and all weights $v_1 \in A_{p_1}$ and $v_2 \in A_{p_2}^{\mathcal{R}}$, (3.3.1) holds for a function $\varphi : [1,\infty)^2 \longrightarrow [0,\infty)$ that increases in each variable. Then, for every exponent $q_1 \ge p_1$, and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{p_2}$, and all weights $w_1 \in A_{q_1}$ and $w_2 \in A_{p_2}^{\mathcal{R}}$,*

$$\|g\|_{L^{q,\infty}(w_1^{q/q_1} w_2^{q/p_2})} \le \Phi([w_1]_{A_{q_1}}, [w_2]_{A_{p_2}^{\mathcal{R}}}) \|f_1\|_{L^{q_1,p_1}(w_1)} \|f_2\|_{L^{p_2,1}(w_2)}, \quad (3.3.23)$$

where $\Phi : [1,\infty)^2 \longrightarrow [0,\infty)$ *is a function increasing in each variable.*

Proof. Note that if $q_1 = p_1$, then there is nothing to prove, so we may assume that $q_1 > p_1$. Pick weights $w_1 \in A_{q_1}$ and $w_2 \in A_{p_2}^{\mathcal{R}}$, and write $w = w_1^{q/q_1} w_2^{q/p_2}$. We may also assume that $\|f_1\|_{L^{q_1,p_1}(w_1)} < \infty$ and $\|f_2\|_{L^{p_2,1}(w_2)} < \infty$. For every natural number $N \geq 1$, let $g_N := |g|\chi_{B(0,N)}$. Fix $N \geq 1$. We will prove (3.3.23) for the triple (f_1, f_2, g_N). Since $g_N \leq |g|$, we already know that (3.3.1) holds for (f_1, f_2, g_N). Fix $y > 0$ such that $\lambda_{g_N}^w(y) \neq 0$. If no such y exists, then $\|g_N\|_{L^{q,\infty}(w)} = 0$ and we are done.

In order to apply (3.3.1), we want to find weights $v_1 \in A_{p_1}$ and $v_2 \in A_{p_2}^{\mathcal{R}}$ such that for $v := v_1^{p/p_1} v_2^{p/p_2}$, $\lambda_{g_N}^w(y) \leq \lambda_{g_N}^v(y)$. Since $q_1 > 1$ and $w_1 \in A_{q_1}$, we can find weights $u_1, u_2 \in A_1$ such that $w_1 = u_1^{1-q_1} u_2$, with $[u_1]_{A_1} \leq c_1 [w_1]_{A_{q_1}}^{\frac{1}{q_1-1}}$ and $[u_2]_{A_1} \leq c_2 [w_1]_{A_{q_1}}$. Consider the function

$$W := u_1^{-\frac{q}{p_2}(1+p_2)} u_2^{q/q_1} w_2^{q/p_2}.$$

We have that $W \in A_\infty$, and in particular, it is locally integrable. Indeed, since $-\frac{q}{p_2}(1+p_2) < 0$, $u_1^{-\frac{q}{p_2}(1+p_2)} \in RH_\infty$, and since $\frac{q}{q_1} + \frac{q}{p_2} = 1$, $u_2^{q/q_1} w_2^{q/p_2} \in A_\infty$. Hence, $W \in A_\infty$. Now take $v_2 := w_2$, and

$$v_1 := u_1^{1-p_1} u_2^{p_1/q_1} \left(M(W\chi_{\{|g_N|>y\}}) \right)^{1-\frac{p_1}{q_1}} =: u_1^{1-p_1} \widetilde{u}_2.$$

Applying [14, Lemma 2.12], we see that $\widetilde{u}_2 \in A_1$, with $[\widetilde{u}_2]_{A_1} \leq c_3 [u_2]_{A_1}$, and c_3 independent of W, N, and y. Hence, $v_1 \in A_{p_1}$, with

$$[v_1]_{A_{p_1}} \leq [u_1]_{A_1}^{p_1-1} [\widetilde{u}_2]_{A_1} \leq c_3 [u_1]_{A_1}^{p_1-1} [u_2]_{A_1} \leq C_1 [w_1]_{A_{q_1}}^{1+\frac{p_1-1}{q_1-1}}. \tag{3.3.24}$$

Observe that

$$v_1^{p/p_1} v_2^{p/p_2} \geq u_1^{\frac{p}{p_1}(1-p_1)} u_2^{p/q_1} W^{\frac{p}{p_1}(1-\frac{p_1}{q_1})} w_2^{p/p_2} \chi_{\{|g_N|>y\}} = u_1^{\alpha_1} u_2^{\alpha_2} w_2^{\alpha_3} \chi_{\{|g_N|>y\}},$$

with

$$\begin{aligned}
\alpha_1 &:= \frac{p}{p_1}(1-p_1) - \frac{pq}{p_1 p_2}\left(1 - \frac{p_1}{q_1}\right)(1+p_2) \\
&= \frac{p_2}{(p_1+p_2)(q_1+p_2)}\left((1-p_1)(q_1+p_2) + (p_1-q_1)(1+p_2)\right) \\
&= \frac{p_2}{(p_1+p_2)(q_1+p_2)}(p_1+p_2)(1-q_1) = \frac{p_2(1-q_1)}{q_1+p_2} = \frac{q}{q_1}(1-q_1), \\
\alpha_2 &:= \frac{p}{q_1} + \frac{pq}{p_1 q_1}\left(1 - \frac{p_1}{q_1}\right) = \frac{p}{q_1}\left(1 + \frac{q}{p_1} - \frac{q}{q_1}\right) = \frac{p}{q_1}\left(\frac{q}{p_1} + \frac{q}{p_2}\right) = \frac{q}{q_1}, \\
\alpha_3 &:= \frac{p}{p_2} + \frac{pq}{p_1 p_2}\left(1 - \frac{p_1}{q_1}\right) = \frac{p}{p_2}\left(1 + \frac{q}{p_1} - \frac{q}{q_1}\right) = \frac{p}{p_2}\left(\frac{q}{p_1} + \frac{q}{p_2}\right) = \frac{q}{p_2},
\end{aligned}$$

so $v \geq w\chi_{\{|g_N|>y\}}$, and hence, (3.3.1) and (3.3.24) imply that

$$\lambda_{g_N}^{w}(y) \leq \lambda_{g_N}^{v}(y) \leq \frac{1}{y^p}\varphi(C_1[w_1]_{A_{q_1}}^{1+\frac{p_1-1}{q_1-1}}, [w_2]_{A_{p_2}^{\mathcal{R}}})^p \, \|f_1\|_{L^{p_1}(v_1)}^p \, \|f_2\|_{L^{p_2,1}(w_2)}^p . \tag{3.3.25}$$

Now we want to replace the term $\|f_1\|_{L^{p_1}(v_1)}$ in (3.3.25) by $\|f_1\|_{L^{q_1,p_1}(w_1)}$. Applying Hölder's inequality with exponent $\frac{q_1}{p_1} > 1$, we obtain that for every $t > 0$,

$$\begin{aligned} v_1(\{|f_1| > t\}) &= \int_{\{|f_1|>t\}} \left(\frac{M(W\chi_{\{|g_N|>y\}})}{u_1^{-q_1}u_2}\right)^{1-\frac{p_1}{q_1}} w_1 \\ &\leq \|\chi_{\{|f_1|>t\}}\|_{L^{\frac{q_1}{p_1},1}(w_1)} \left\| \left(\frac{M(W\chi_{\{|g_N|>y\}})}{u_1^{-q_1}u_2}\right)^{1-\frac{p_1}{q_1}} \right\|_{L^{\frac{q_1}{q_1-p_1},\infty}(w_1)} \\ &= \frac{q_1}{p_1} w_1(\{|f_1| > t\})^{p_1/q_1} \left\| \frac{M(W\chi_{\{|g_N|>y\}})}{u_1^{-q_1}u_2} \right\|_{L^{1,\infty}(w_1)}^{1-\frac{p_1}{q_1}} . \end{aligned}$$

Let $F_N := W\chi_{\{|g_N|>y\}}$, $U := u_1 \in A_1$, and $V := u_1^{-q_1}u_2 \in A_{q_1+1}$. Note that

$$\begin{aligned} UF_N &= u_1^{1-\frac{q}{p_2}(1+p_2)} u_2^{q/q_1} w_2^{q/p_2} \chi_{\{|g_N|>y\}} \\ &= u_1^{\frac{q}{q_1}(1-q_1)} u_2^{q/q_1} w_2^{q/p_2} \chi_{\{|g_N|>y\}} = w\chi_{\{|g_N|>y\}}, \end{aligned}$$

and since $UV = w_1 \in A_{q_1} \subseteq A_{q_1}^{\mathcal{R}}$, and $[w_1]_{A_{q_1}^{\mathcal{R}}} \leq [w_1]_{A_{q_1}}^{1/q_1}$, it follows from Theorem 2.3.8 and Lemma 2.3.10 that

$$\begin{aligned} \left\| \frac{M(W\chi_{\{|g_N|>y\}})}{u_1^{-q_1}u_2} \right\|_{L^{1,\infty}(w_1)} &= \left\| \frac{M(F_N)}{V} \right\|_{L^{1,\infty}(UV)} \\ &\leq \phi([U]_{A_1}, [UV]_{A_{q_1}^{\mathcal{R}}}) \int_{\mathbb{R}^n} UF_N \\ &\leq \phi(c_1[w_1]_{A_{q_1}}^{\frac{1}{q_1-1}}, [w_1]_{A_{q_1}}^{1/q_1}) w(\{|g_N| > y\}) \\ &=: \psi([w_1]_{A_{q_1}}) w(\{|g_N| > y\}). \end{aligned} \tag{3.3.26}$$

Hence,

$$v_1(\{|f_1| > t\}) \leq \frac{q_1}{p_1}\psi([w_1]_{A_{q_1}})^{1-\frac{p_1}{q_1}} w(\{|g_N| > y\})^{1-\frac{p_1}{q_1}} w_1(\{|f_1| > t\})^{p_1/q_1},$$

so

$$\begin{aligned} \|f_1\|_{L^{p_1}(v_1)} &= p_1^{1/p_1}\left(\int_0^\infty t^{p_1} v_1(\{|f_1|>t\})\frac{dt}{t}\right)^{1/p_1} \\ &\le \psi([w_1]_{A_{q_1}})^{\frac{1}{p_1}-\frac{1}{q_1}} w(\{|g_N|>y\})^{\frac{1}{p_1}-\frac{1}{q_1}} \\ &\times q_1^{1/p_1}\left(\int_0^\infty t^{p_1} w_1(\{|f_1|>t\})^{p_1/q_1}\frac{dt}{t}\right)^{1/p_1} \\ &= \psi([w_1]_{A_{q_1}})^{\frac{1}{p_1}-\frac{1}{q_1}} w(\{|g_N|>y\})^{\frac{1}{p_1}-\frac{1}{q_1}} \|f_1\|_{L^{q_1,p_1}(w_1)}. \end{aligned} \tag{3.3.27}$$

Combining the estimates (3.3.25) and (3.3.27), we have that

$$\lambda_{g_N}^w(y) \le \frac{1}{y^p}\Phi([w_1]_{A_{q_1}},[w_2]_{A_{p_2}^{\mathcal{R}}})^p \lambda_{g_N}^w(y)^{\frac{p}{p_1}-\frac{p}{q_1}} \|f_1\|_{L^{q_1,p_1}(w_1)}^p \|f_2\|_{L^{p_2,1}(w_2)}^p, \tag{3.3.28}$$

with

$$\Phi([w_1]_{A_{q_1}},[w_2]_{A_{p_2}^{\mathcal{R}}}) = \varphi(C_1[w_1]_{A_{q_1}}^{1+\frac{p_1-1}{q_1-1}},[w_2]_{A_{p_2}^{\mathcal{R}}})\psi([w_1]_{A_{q_1}})^{\frac{1}{p_1}-\frac{1}{q_1}}.$$

By our choice of y and g_N, $0<\lambda_{g_N}^w(y)\le w(B(0,N))<\infty$, so we can divide by $\lambda_{g_N}^w(y)^{\frac{p}{p_1}-\frac{p}{q_1}}$ in (3.3.28) and raise everything to the power $\frac{1}{p}$, obtaining that

$$y\lambda_{g_N}^w(y)^{1/q} \le \Phi([w_1]_{A_{q_1}},[w_2]_{A_{p_2}^{\mathcal{R}}}) \|f_1\|_{L^{q_1,p_1}(w_1)} \|f_2\|_{L^{p_2,1}(w_2)},$$

and taking the supremum over all $y>0$, we deduce (3.3.23) for the triple (f_1,f_2,g_N). Finally, (3.3.23) for the triple (f_1,f_2,g) follows taking the supremum over all $N\ge 1$, because

$$\|g\|_{L^{q,\infty}(w)} = \sup_{N\ge 1}\|g_N\|_{L^{q,\infty}(w)},$$

since $g_N \uparrow |g|$. □

Remark 3.3.21. Note that given $\alpha>0$, if in Theorem 3.3.20 we replace (3.3.1) by

$$\|g\|_{L^{p,\infty}(v_1^{p/p_1}v_2^{p/p_2})} \le \varphi([v_1]_{A_{p_1}},[v_2]_{A_{p_2}^{\mathcal{R}}}) \|f_1\|_{L^{p_1,\alpha}(v_1)} \|f_2\|_{L^{p_2,1}(v_2)},$$

then we can replace estimate (3.3.27) by

$$\|f_1\|_{L^{p_1,\alpha}(v_1)} \le \left(\frac{q_1}{p_1}\right)^{\frac{1}{p_1}-\frac{1}{\alpha}} \psi([w_1]_{A_{q_1}})^{\frac{1}{p_1}-\frac{1}{q_1}} w(\{|g_N|>y\})^{\frac{1}{p_1}-\frac{1}{q_1}} \|f_1\|_{L^{q_1,\alpha}(w_1)},$$

and follow the proof of Theorem 3.3.20 to conclude that

$$\|g\|_{L^{q,\infty}(w_1^{q/q_1}w_2^{q/p_2})} \le \left(\frac{q_1}{p_1}\right)^{\frac{1}{p_1}-\frac{1}{\alpha}} \Phi([w_1]_{A_{q_1}},[w_2]_{A_{p_2}^{\mathcal{R}}}) \|f_1\|_{L^{q_1,\alpha}(w_1)} \|f_2\|_{L^{p_2,1}(w_2)}.$$

Once again, we have presented Theorem 3.3.20 in its general form. We can obtain the corresponding extrapolation scheme for two-variable operators arguing as in the proof of Corollary 3.2.2. See Figure 3.9 for a pictorial representation of such scheme.

Corollary 3.3.22. *Let T be a two-variable operator defined for measurable functions. Suppose that for some exponents $1 \leq p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and all weights $v_1 \in A_{p_1}$ and $v_2 \in A^{\mathcal{R}}_{p_2}$,*

$$T : L^{p_1}(v_1) \times L^{p_2,1}(v_2) \longrightarrow L^{p,\infty}(v_1^{p/p_1} v_2^{p/p_2}),$$

with constant bounded by $\varphi([v_1]_{A_{p_1}}, [v_2]_{A^{\mathcal{R}}_{p_2}})$ as in (3.3.1). Then, for every exponent $q_1 \geq p_1$, and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{p_2}$, and all weights $w_1 \in A_{q_1}$ and $w_2 \in A^{\mathcal{R}}_{p_2}$,

$$T : L^{q_1,p_1}(w_1) \times L^{p_2,1}(w_2) \longrightarrow L^{q,\infty}(w_1^{q/q_1} w_2^{q/p_2}),$$

with constant bounded by $\Phi([w_1]_{A_{q_1}}, [w_2]_{A^{\mathcal{R}}_{p_2}})$ as in (3.3.23).

Remark 3.3.23. Observe that if the operator T is defined for characteristic functions of measurable sets $E_1, E_2 \subseteq \mathbb{R}^n$, then under the hypotheses of Corollary 3.3.22, we deduce that

$$\|T(\chi_{E_1}, \chi_{E_2})\|_{L^{q,\infty}(w_1^{q/q_1} w_2^{q/p_2})} \leq C\Phi([w_1]_{A_{q_1}}, [w_2]_{A^{\mathcal{R}}_{p_2}}) w_1(E_1)^{1/q_1} w_2(E_2)^{1/p_2},$$

with $C = p_2 \left(\frac{q_1}{p_1}\right)^{1/p_1}$, and hence, T is of weak type (q_1, p_2, q) at least for characteristic functions.

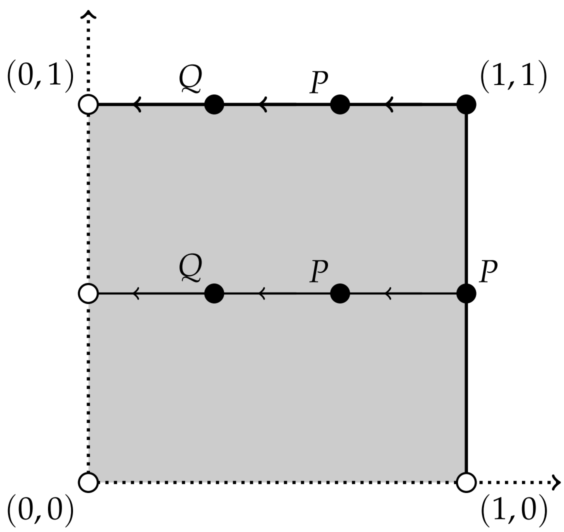

FIGURE 3.9: Pictorial representation of Theorem 3.3.20 and Corollary 3.3.22.

To extend Theorem 3.3.20 to the restricted weak type case, we have to adapt the estimate (3.3.26), and we could do it with a version of Theorem 2.3.8

in which the weight $U \in A_1$ is replaced by Mh, for $h \in L^1_{loc}(\mathbb{R}^n)$. More precisely, we would have to prove that given $q > 1$, there exists an increasing function $\phi : [1,\infty) \longrightarrow [0,\infty)$ such that for every measurable function f, every function $h \in L^1_{loc}(\mathbb{R}^n)$, and every weight $u \in A_1$, we have that

$$\left\| \frac{Mf}{(Mh)^{-q}u} \right\|_{L^{1,\infty}((Mh)^{1-q}u)} \leq \phi([u]_{A_1}) \|f\|_{L^1(Mh)}.$$

At the time of writing, we don't know if such a result is true. In Theorem 3.3.35 and Theorem 4.2.7 we present an alternative approach, using Theorem 2.4.12.

We can also use ideas from Theorem 2.4.12, and Rubio de Francia's algorithm (see [101]), to improve Theorem 3.3.20, producing a better function Φ. We will adapt the proof of Theorem 3.2.10. To do so, we have to choose appropriate weights $v_1 \in A_{p_1}$ and $v_2 \in A^{\mathcal{R}}_{p_2}$, and find suitable replacements for estimates (3.2.28), (3.2.29), (3.2.30), and (3.2.31) to control $\|f_1\|_{L^{p_1}(v_1)}$ by $\|f_1\|_{L^{q_1}(w_1)}$, and keep track of the changes in the constants involved.

First, assume that $p_1 > 1$. Instead of (3.2.26), we take

$$v_1 := w_1^{\frac{p_1-1}{q_1-1}} \left(M(w_1^{1/q_1} w^{1/q_1'} \chi_{\{|g_N|>y\}}) \right)^{\frac{q_1-p_1}{q_1-1}},$$

and $v_2 := w_2$. Let us see that $v_1 \in A_{p_1}$. If $q_1 = p_1$, then $v_1 = w_1$ and we are done. If $q_1 > p_1 > 1$, in virtue of Lemma 3.1.4, $v_1 \in A_{p_1}$, with

$$[v_1]_{A_{p_1}} \leq C_1 [w_1]_{A_{q_1}}^{1+\frac{p_1-1}{q_1-1}}.$$

Observe that $v_1^{p/p_1} v_2^{p/p_2} \geq w_1^{\frac{p}{q_1-1}(1-\frac{1}{q_1})} w_2^{p/p_2} w^{\frac{p}{p_1}(1-\frac{p_1}{q_1})} \chi_{\{|g_N|>y\}} = w\chi_{\{|g_N|>y\}}$.

Now, estimate (3.2.28) for $i = 1$ should be replaced by

$$v_1(\{|f_1| > t\}) \leq w_1(\{|f_1| > t\})^{p_1/q_1} \left\| \frac{M(w_1^{1/q_1} w^{1/q_1'} \chi_{\{|g_N|>y\}})}{w_1} \right\|_{L^{q_1'}(w_1)}^{\frac{q_1-p_1}{q_1-1}},$$

and in virtue of (2.4.10), we can replace (3.2.29) by

$$\left\| \frac{M(w_1^{1/q_1} w^{1/q_1'} \chi_{\{|g_N|>y\}})}{w_1} \right\|_{L^{q_1'}(w_1)} \leq c_1 [w_1]_{A_{q_1}} w(\{|g_N| > y\})^{1/q_1'}.$$

Hence, we can replace (3.2.30) by

$$v_1(\{|f_1| > t\}) \leq \left(c_1 [w_1]_{A_{q_1}} \right)^{\frac{q_1-p_1}{q_1-1}} w(\{|g_N| > y\})^{1-\frac{p_1}{q_1}} w_1(\{|f_1| > t\})^{p_1/q_1},$$

and (3.2.31) by

$$\|f_1\|_{L^{p_1}(v_1)} \le \left(\frac{p_1}{q_1}\right)^{1/p_1} \left(c_1[w_1]_{A_{q_1}}\right)^{\frac{1}{p_1}\frac{q_1-p_1}{q_1-1}} w(\{|g_N|>y\})^{\frac{1}{p_1}-\frac{1}{q_1}} \|f_1\|_{L^{q_1,p_1}(w_1)}.$$

Finally, if we follow the proof of Theorem 3.2.10 performing the previous changes and keeping track of the constants, we conclude that (3.3.23) holds, with

$$\Phi([w_1]_{A_{q_1}},[w_2]_{A^{\mathcal{R}}_{p_2}}) = \left(\frac{p_1}{q_1}\right)^{1/p_1} \left(c_1[w_1]_{A_{q_1}}\right)^{\frac{1}{p_1}\frac{q_1-p_1}{q_1-1}} \varphi(C_1[w_1]_{A_{q_1}}^{1+\frac{p_1-1}{q_1-1}},[w_2]_{A^{\mathcal{R}}_{p_2}}).$$

Alternatively, if $p_1 \ge 1$, and $q_1 > p_1$, then we can take

$$v_1 := w_1^{\frac{p_1-1}{q_1-1}} \left(R'(w_1^{1/q_1} w^{1/q_1'} \chi_{\{|g_N|>y\}})\right)^{\frac{q_1-p_1}{q_1-1}},$$

where for a measurable function $h \in L^{q_1'}(w_1^{1-q_1'})$,

$$R'h := \sum_{k=0}^{m} \frac{M^k h}{\left(2\|M\|_{L^{q_1'}(w_1^{1-q_1'})}\right)^k}$$

is the Rubio de Francia's algorithm (see [37, 101]).

In virtue of [37, Lemma 2.2], we have that $h \le R'h$, $\|R'h\|_{L^{q_1'}(w_1^{1-q_1'})} \le 2\|h\|_{L^{q_1'}(w_1^{1-q_1'})}$, and $R'h \in A_1$, with $[R'h]_{A_1} \le 2\|M\|_{L^{q_1'}(w_1^{1-q_1'})} \le \widetilde{c}_1[w_1]_{A_{q_1}}$ (see [8, Theorem 2.5] and [44, Proposition 7.1.5]). Moreover, applying [37, Lemma 2.1], we obtain that $v_1 \in A_{p_1}$, with $[v_1]_{A_{p_1}} \le \widetilde{C}_1[w_1]_{A_{q_1}}$. Hence, we can argue as before to conclude that (3.3.23) holds, with

$$\Phi([w_1]_{A_{q_1}},[w_2]_{A^{\mathcal{R}}_{p_2}}) = \left(\frac{p_1}{q_1}\right)^{1/p_1} 2^{\frac{1}{p_1}\frac{q_1-p_1}{q_1-1}} \varphi(\widetilde{C}_1[w_1]_{A_{q_1}},[w_2]_{A^{\mathcal{R}}_{p_2}}). \tag{3.3.29}$$

We can combine Theorem 3.3.20 and Theorem 3.3.12 to produce a more general extrapolation scheme that increases the exponent p_1 up to q_1 and decreases the exponent p_2 down to q_2. The monotonicity of the functions φ and Φ in both theorems is crucial for the iteration process. For convenience, we include a pictorial representation of this scheme in Figure 3.10.

Theorem 3.3.24. *Given measurable functions f_1, f_2, and g, suppose that for some exponents $1 \le p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and all weights $v_1 \in A_{p_1}$ and $v_2 \in A^{\mathcal{R}}_{p_2}$, (3.3.1) holds for a function $\varphi : [1,\infty)^2 \longrightarrow [0,\infty)$ that increases in each variable. Then, for all exponents $q_1 \ge p_1$, $1 \le q_2 \le p_2$, and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, and all weights*

$w_1 \in A_{q_1}$ and $w_2 \in \widehat{A}_{q_2}$,

$$\|g\|_{L^{q,\infty}(w_1^{q/q_1}w_2^{q/q_2})} \le \Phi([w_1]_{A_{q_1}}, \|w_2\|_{\widehat{A}_{q_2}}) \, \|f_1\|_{L^{q_1,p_1}(w_1)} \, \|f_2\|_{L^{q_2,\frac{q_2}{p_2}}(w_2)}, \tag{3.3.30}$$

where $\Phi : [1,\infty)^2 \longrightarrow [0,\infty)$ is a function increasing in each variable.

Proof. Note that if $q_1 = p_1$ and $q_2 = p_2$, then there is nothing to prove. If $q_1 > p_1$ and $q_2 = p_2$, then the result follows immediately from Theorem 3.3.20, and if $q_1 = p_1$ and $q_2 < p_2$, then the result follows immediately from Theorem 3.3.12. Let us assume that $q_1 > p_1$ and $1 \le q_2 < p_2$. In virtue of Theorem 3.3.20 and (3.3.29), we have that for $\frac{1}{r} := \frac{1}{q_1} + \frac{1}{p_2}$, and all weights $W_1 \in A_{q_1}$ and $W_2 \in A_{p_2}^{\mathcal{R}}$,

$$\|g\|_{L^{r,\infty}(W_1^{r/q_1}W_2^{r/p_2})} \le \widetilde{\Phi}([W_1]_{A_{q_1}}, [W_2]_{A_{p_2}^{\mathcal{R}}}) \, \|f_1\|_{L^{q_1,p_1}(W_1)} \, \|f_2\|_{L^{p_2,1}(W_2)}, \tag{3.3.31}$$

with

$$\widetilde{\Phi}([W_1]_{A_{q_1}}, [W_2]_{A_{p_2}^{\mathcal{R}}}) := \left(\frac{p_1}{q_1}\right)^{1/p_1} 2^{\frac{1}{p_1}\frac{q_1-p_1}{q_1-1}} \varphi(\widetilde{C}_1[W_1]_{A_{q_1}}, [W_2]_{A_{p_2}^{\mathcal{R}}}).$$

Applying Theorem 3.3.12 replacing (3.3.1) by (3.3.31), and in virtue of Remark 3.3.13, we conclude that (3.3.30) holds, with

$$\Phi([w_1]_{A_{q_1}}, \|w_2\|_{\widehat{A}_{q_2}}) = p_2^{3-\frac{q_2}{p_2}-\frac{p_2}{q_2}} \frac{C_{r,q}}{q_2} \phi_{w_1,w_2}^{1-\frac{q_2}{p_2}} \widetilde{\Phi}([w_1]_{A_{q_1}}, C_2\|w_2\|_{\widehat{A}_{q_2}}^{q_2/p_2}),$$

where ϕ_{w_1,w_2} is as in (3.3.20) with the obvious modifications in the parameters. □

From Theorem 3.3.24 we can obtain the corresponding extrapolation result for two-variable operators arguing as in the proof of Corollary 3.2.2. See Figure 3.10 for a pictorial representation of such scheme.

Corollary 3.3.25. *Let T be a two-variable operator defined for measurable functions. Suppose that for some exponents $1 \le p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and all weights $v_1 \in A_{p_1}$ and $v_2 \in A_{p_2}^{\mathcal{R}}$,*

$$T : L^{p_1}(v_1) \times L^{p_2,1}(v_2) \longrightarrow L^{p,\infty}(v_1^{p/p_1}v_2^{p/p_2}),$$

with constant bounded by $\varphi([v_1]_{A_{p_1}}, [v_2]_{A_{p_2}^{\mathcal{R}}})$ as in (3.3.1). Then, for all exponents $q_1 \ge p_1$, $1 \le q_2 \le p_2$, and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, and all weights $w_1 \in A_{q_1}$ and $w_2 \in \widehat{A}_{q_2}$,

$$T : L^{q_1,p_1}(w_1) \times L^{q_2,\frac{q_2}{p_2}}(w_2) \longrightarrow L^{q,\infty}(w_1^{q/q_1}w_2^{q/q_2}),$$

with constant bounded by $\Phi([w_1]_{A_{q_1}}, \|w_2\|_{\widehat{A}_{q_2}})$ as in (3.3.30).

Remark 3.3.26. Observe that if the operator T is defined for characteristic functions of measurable sets $E_1, E_2 \subseteq \mathbb{R}^n$, then under the hypotheses of

Corollary 3.3.25, we deduce that

$$\left\|T(\chi_{E_1},\chi_{E_2})\right\|_{L^{q,\infty}(w_1^{q/q_1}w_2^{q/q_2})} \leq C\Phi([w_1]_{A_{q_1}},\|w_2\|_{\widehat{A}_{q_2}})w_1(E_1)^{1/q_1}w_2(E_2)^{1/q_2},$$

with $C=\left(\frac{q_1}{p_1}\right)^{1/p_1}p_2^{p_2/q_2}$, and hence, T is of weak type (q_1,q_2,q) at least for characteristic functions.

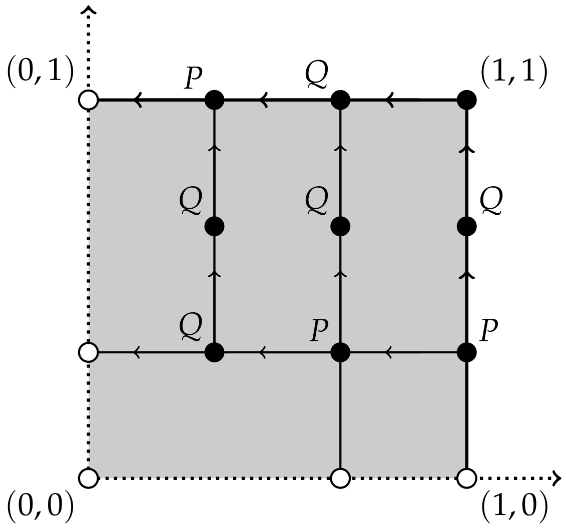

FIGURE 3.10: Pictorial representation of Theorem 3.3.24 and Corollary 3.3.25.

For convenience, in the next result, we gather the principal extrapolation schemes of this section, presented in Theorem 3.3.16 and Theorem 3.3.24. We provide a pictorial representation of this result in Figure 3.11.

Theorem 3.3.27. *Given measurable functions f_1, f_2, and g, suppose that for some exponents $1\leq p_1,p_2<\infty$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$, and all weights $v_1\in A_{p_1}$ and $v_2\in A_{p_2}^{\mathcal{R}}$, (3.3.1) holds for a function $\varphi:[1,\infty)^2\longrightarrow[0,\infty)$ that increases in each variable. Then, for all exponents $1\leq q_1=p_1$ or $1<q_1<p_1$ or $q_1>p_1$, $1\leq q_2\leq p_2$, and $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}$, and all weights $w_1\in A_{q_1}$ and $w_2\in\widehat{A}_{q_2}$,*

$$\|g\|_{L^{q,\infty}(w_1^{q/q_1}w_2^{q/q_2})} \leq \Phi([w_1]_{A_{q_1}},\|w_2\|_{\widehat{A}_{q_2}})\,\|f_1\|_{L^{q_1,\min\{p_1,q_1\}}(w_1)}\,\|f_2\|_{L^{q_2,\frac{q_2}{p_2}}(w_2)}, \tag{3.3.32}$$

where $\Phi:[1,\infty)^2\longrightarrow[0,\infty)$ is a function increasing in each variable.

Remark 3.3.28. Note that given $0<\alpha\leq p_1$, if in Theorem 3.3.27 we replace (3.3.1) by

$$\|g\|_{L^{p,\infty}(v_1^{p/p_1}v_2^{p/p_2})} \leq \varphi([v_1]_{A_{p_1}},[v_2]_{A_{p_2}^{\mathcal{R}}})\,\|f_1\|_{L^{p_1,\alpha}(v_1)}\,\|f_2\|_{L^{p_2,1}(v_2)},$$

then taking into account Remarks 3.3.7, 3.3.13, and 3.3.21, we conclude that

$$\|g\|_{L^{q,\infty}(w_1^{q/q_1}w_2^{q/q_2})} \leq \Phi_\alpha([w_1]_{A_{q_1}},\|w_2\|_{\widehat{A}_{q_2}})\,\|f_1\|_{L^{q_1,\alpha\min\{1,\frac{q_1}{p_1}\}}(w_1)}\,\|f_2\|_{L^{q_2,\frac{q_2}{p_2}}(w_2)},$$

with $\Phi_\alpha := C_{\alpha,p_1,q_1}\Phi$.

Arguing as in the proof of Corollary 3.2.2, we can obtain the corresponding extrapolation scheme for two-variable operators from Theorem 3.3.27. We provide a pictorial representation of this result in Figure 3.11.

Corollary 3.3.29. *Let T be a two-variable operator defined for measurable functions. Suppose that for some exponents $1 \leq p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and all weights $v_1 \in A_{p_1}$ and $v_2 \in A_{p_2}^{\mathcal{R}}$,*

$$T : L^{p_1}(v_1) \times L^{p_2,1}(v_2) \longrightarrow L^{p,\infty}(v_1^{p/p_1} v_2^{p/p_2}),$$

with constant bounded by $\varphi([v_1]_{A_{p_1}}, [v_2]_{A_{p_2}^{\mathcal{R}}})$ as in (3.3.1). Then, for all exponents $1 \leq q_1 = p_1$ or $1 < q_1 < p_1$ or $q_1 > p_1$, $1 \leq q_2 \leq p_2$, and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, and all weights $w_1 \in A_{q_1}$ and $w_2 \in \widehat{A}_{q_2}$,

$$T : L^{q_1,\min\{p_1,q_1\}}(w_1) \times L^{q_2,\frac{q_2}{p_2}}(w_2) \longrightarrow L^{q,\infty}(w_1^{q/q_1} w_2^{q/q_2}),$$

with constant bounded by $\Phi([w_1]_{A_{q_1}}, \|w_2\|_{\widehat{A}_{q_2}})$ as in (3.3.32).

Remark 3.3.30. Observe that if the operator T is defined for characteristic functions of measurable sets $E_1, E_2 \subseteq \mathbb{R}^n$, then under the hypotheses of Corollary 3.3.29, we deduce that

$$\left\| T(\chi_{E_1}, \chi_{E_2}) \right\|_{L^{q,\infty}(w_1^{q/q_1} w_2^{q/q_2})} \leq C\Phi([w_1]_{A_{q_1}}, \|w_2\|_{\widehat{A}_{q_2}}) w_1(E_1)^{1/q_1} w_2(E_2)^{1/q_2},$$

with $C = \left(\frac{q_1}{\min\{p_1,q_1\}}\right)^{\frac{1}{\min\{p_1,q_1\}}} p_2^{p_2/q_2}$, and hence, T is of weak type (q_1, q_2, q) at least for characteristic functions.

Observe that Theorem 3.3.27 doesn't include the case when p_2 is smaller than q_2. A way to obtain results in this direction is to rewrite Theorem 3.3.16 and Theorem 3.3.24 swapping the variables and to add the symmetric hypothesis (3.3.33) to Theorem 3.3.27, as we show in the next theorem. For a pictorial representation of it, see Figure 3.12.

Theorem 3.3.31. *Given measurable functions f_1, f_2, and g, suppose that for some exponents $1 \leq p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and all weights $v_1 \in A_{p_1}$ and $v_2 \in A_{p_2}^{\mathcal{R}}$, (3.3.1) holds for a function $\varphi : [1,\infty)^2 \longrightarrow [0,\infty)$ that increases in each variable. Suppose also that for all weights $\widetilde{v}_1 \in A_{p_1}^{\mathcal{R}}$ and $\widetilde{v}_2 \in A_{p_2}$,*

$$\|g\|_{L^{p,\infty}(\widetilde{v}_1^{p/p_1} \widetilde{v}_2^{p/p_2})} \leq \widetilde{\varphi}([\widetilde{v}_1]_{A_{p_1}^{\mathcal{R}}}, [\widetilde{v}_2]_{A_{p_2}}) \|f_1\|_{L^{p_1,1}(\widetilde{v}_1)} \|f_2\|_{L^{p_2}(\widetilde{v}_2)}, \tag{3.3.33}$$

where $\widetilde{\varphi} : [1,\infty)^2 \longrightarrow [0,\infty)$ is a function that increases in each variable. Then, for all exponents $1 \leq q_1 = p_1$ or $1 < q_1 < p_1$ or $q_1 > p_1$, $1 \leq q_2 \leq p_2$, and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, and all weights $w_1 \in A_{q_1}$ and $w_2 \in \widehat{A}_{q_2}$, (3.3.32) holds for a function $\Phi : [1,\infty)^2 \longrightarrow [0,\infty)$ that increases in each variable. Moreover, for all exponents

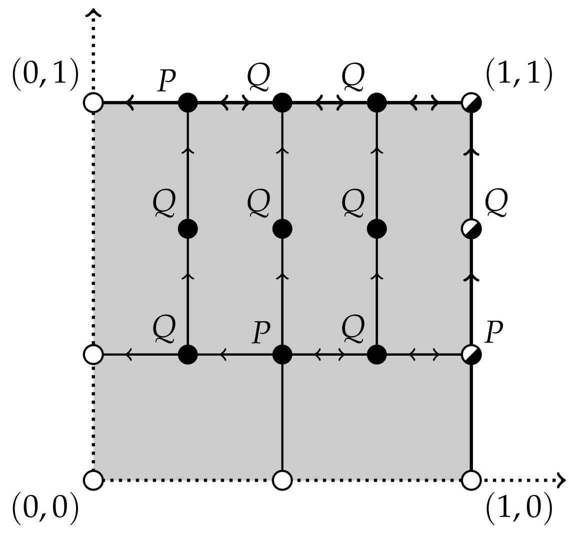

FIGURE 3.11: Pictorial representation of Theorem 3.3.27 and Corollary 3.3.29.

$1 \leq q_1 \leq p_1$, and $1 \leq q_2 = p_2$ or $1 < q_2 < p_2$ or $q_2 > p_2$, and all weights $\widetilde{w}_1 \in \widehat{A}_{q_1}$ and $\widetilde{w}_2 \in A_{q_2}$,

$$\|g\|_{L^{q,\infty}(\widetilde{w}_1^{q/q_1}\widetilde{w}_2^{q/q_2})} \leq \widetilde{\Phi}(\|\widetilde{w}_1\|_{\widehat{A}_{q_1}}, [\widetilde{w}_2]_{A_{q_2}}) \|f_1\|_{L^{q_1,\frac{q_1}{p_1}}(\widetilde{w}_1)} \|f_2\|_{L^{q_2,\min\{p_2,q_2\}}(\widetilde{w}_2)}, \tag{3.3.34}$$

where $\widetilde{\Phi} : [1,\infty)^2 \longrightarrow [0,\infty)$ is a function increasing in each variable.

Remark 3.3.32. Observe that in Theorem 3.3.31, we can not decrease both exponents p_1 and p_2 down to 1. To do so, we need to impose the restricted weak type hypothesis (3.2.11), and we will study this case in full detail in the next chapter. Note that this hypothesis implies both (3.3.1) and (3.3.33).

Arguing as in the proof of Corollary 3.2.2, we can obtain the corresponding extrapolation scheme for two-variable operators from Theorem 3.3.31. We provide a pictorial representation of this result in Figure 3.12.

Corollary 3.3.33. *Let T be a two-variable operator defined for measurable functions. Suppose that for some exponents $1 \leq p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and all weights $v_1 \in A_{p_1}$ and $v_2 \in A_{p_2}^{\mathcal{R}}$,*

$$T : L^{p_1}(v_1) \times L^{p_2,1}(v_2) \longrightarrow L^{p,\infty}(v_1^{p/p_1} v_2^{p/p_2}),$$

with constant bounded by $\varphi([v_1]_{A_{p_1}}, [v_2]_{A_{p_2}^{\mathcal{R}}})$ as in (3.3.1). Suppose also that for all weights $\widetilde{v}_1 \in A_{p_1}^{\mathcal{R}}$ and $\widetilde{v}_2 \in A_{p_2}$,

$$T : L^{p_1,1}(\widetilde{v}_1) \times L^{p_2}(\widetilde{v}_2) \longrightarrow L^{p,\infty}(\widetilde{v}_1^{p/p_1} \widetilde{v}_2^{p/p_2}),$$

with constant bounded by $\widetilde{\varphi}([\widetilde{v}_1]_{A_{p_1}^{\mathcal{R}}}, [\widetilde{v}_2]_{A_{p_2}})$ as in (3.3.33). Then, for all exponents $1 \leq q_1 = p_1$ or $1 < q_1 < p_1$ or $q_1 > p_1$, $1 \leq q_2 \leq p_2$, and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, and all *weights* $w_1 \in A_{q_1}$ *and* $w_2 \in \widehat{A}_{q_2}$,

$$T : L^{q_1,\min\{p_1,q_1\}}(w_1) \times L^{q_2,\frac{q_2}{p_2}}(w_2) \longrightarrow L^{q,\infty}(w_1^{q/q_1} w_2^{q/q_2}),$$

with constant bounded by $\Phi([w_1]_{A_{q_1}}, \|w_2\|_{\widehat{A}_{q_2}})$ *as in* (3.3.32). *Moreover, for all exponents* $1 \leq q_1 \leq p_1$, *and* $1 \leq q_2 = p_2$ *or* $1 < q_2 < p_2$ *or* $q_2 > p_2$, *and all weights* $\widetilde{w}_1 \in \widehat{A}_{q_1}$ *and* $\widetilde{w}_2 \in A_{q_2}$,

$$T : L^{q_1,\frac{q_1}{p_1}}(\widetilde{w}_1) \times L^{q_2,\min\{p_2,q_2\}}(\widetilde{w}_2) \longrightarrow L^{q,\infty}(\widetilde{w}_1^{q/q_1} \widetilde{w}_2^{q/q_2}),$$

with constant bounded by $\widetilde{\Phi}(\|\widetilde{w}_1\|_{\widehat{A}_{q_1}}, [\widetilde{w}_2]_{A_{q_2}})$ *as in* (3.3.34).

Remark 3.3.34. Observe that if the operator T is defined for characteristic functions of measurable sets $E_1, E_2 \subseteq \mathbb{R}^n$, then under the hypotheses of Corollary 3.3.33, we can extend Remark 3.3.30 and also deduce that

$$\|T(\chi_{E_1}, \chi_{E_2})\|_{L^{q,\infty}(\widetilde{w}_1^{q/q_1} \widetilde{w}_2^{q/q_2})} \leq \widetilde{C}\widetilde{\Phi}(\|\widetilde{w}_1\|_{\widehat{A}_{q_1}}, [\widetilde{w}_2]_{A_{q_2}}) \widetilde{w}_1(E_1)^{1/q_1} \widetilde{w}_2(E_2)^{1/q_2},$$

with $\widetilde{C} = p_1^{p_1/q_1} \left(\frac{q_2}{\min\{p_2,q_2\}} \right)^{\frac{1}{\min\{p_2,q_2\}}}$.

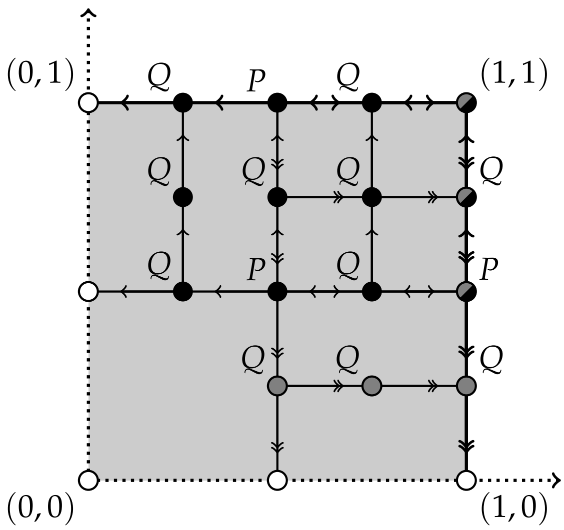

FIGURE 3.12: Pictorial representation of Theorem 3.3.31 and Corollary 3.3.33.

In Theorem 3.3.31, we managed to increase the exponent p_2 up to q_2 exploiting the A_{q_2} condition on the weight w_2. It remains to produce extrapolation schemes that increase p_2 up to q_2 using the $\widehat{A}_{q_2}$ condition on w_2. We can achieve this using Theorem 2.4.12, but we will need to add an extra technical hypothesis, as we show in the next result, depicted in Figure 3.13. Its proof is essentially the same as the one of Theorem 4.2.7, taking into account Lemma 3.1.4 in (4.2.18), and Lemma 3.1.6 in (4.2.20).

Theorem 3.3.35. *Given measurable functions f_1, f_2, and g, suppose that for some exponents $1 \le p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and all weights $v_1 \in A_{p_1}$ and $v_2 \in A^{\mathcal{R}}_{p_2}$, (3.3.1) holds for a function $\varphi : [1,\infty)^2 \longrightarrow [0,\infty)$ that increases in each variable. Take an exponent $q_2 = p_2 \ge 1$ or $q_2 > p_2 > 1$, and $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{q_2}$, and weights $w_1 \in A_{p_1}$ and $w_2 \in \widehat{A}_{q_2}$, and $w = w_1^{q/p_1} w_2^{q/q_2}$. If $q_2 > p_2$, suppose that there exists $0 < \varepsilon \le 1$ such that $wW^{-\varepsilon}$ is a weight, and $[W^{-\varepsilon}]_{RH_\infty(w)} < \infty$, with $W = \left(\frac{w}{w_2}\right)^{1/q_2}$. Then,*

$$\|g\|_{L^{q,\infty}(w)} \le \Phi_{\varepsilon,w_1,w_2}([w_1]_{A_{p_1}}, \|w_2\|_{\widehat{A}_{q_2}}) \, \|f_1\|_{L^{p_1}(w_1)} \, \|f_2\|_{L^{q_2,1}(w_2)}, \tag{3.3.35}$$

where $\Phi_{\varepsilon,w_1,w_2} : [1,\infty)^2 \longrightarrow [0,\infty)$ is a function that increases in each variable, given by

$$\Phi_{\varepsilon,w_1,w_2}([w_1]_{A_{p_1}}, \|w_2\|_{\widehat{A}_{q_2}}) = \left(\frac{p_2}{q_2}\right)^{1/p_2'} (q_2'\phi)^{\frac{1}{p_2}\frac{q_2-p_2}{q_2-1}} \varphi([w_1]_{A_{p_1}}, C\|w_2\|_{\widehat{A}_{q_2}}^{q_2/p_2}),$$

where if $q_2 = p_2$, then $\phi = 1$, and if $q_2 > p_2$, then

$$\phi = 2 \cdot 48^n q_2 \psi_{\varepsilon,w_2,W}([W^{-\varepsilon}]_{RH_\infty(w)}) \phi^n_{2q,q_2}(c\|w_2\|_{\widehat{A}_{q_2}}, \psi([w_1]_{A_{q_1}})\|w_2\|^{1/2}_{\widehat{A}_{q_2}}),$$

with $\psi_{\varepsilon,w_2,W}$ as in (2.4.8), ψ as in Lemma 3.1.6, and ϕ^n_{2q,q_2} as in Lemma 2.3.10. If $W = 1$, in virtue of Remark 2.4.13, one can take $\phi = C_{n,q_2}(c\|w_2\|_{\widehat{A}_{q_2}})^{q_2+1}$.

From Theorem 3.3.35 we can obtain the corresponding extrapolation result for two-variable operators arguing as in the proof of Corollary 3.2.2. See Figure 3.13 for a pictorial representation of such scheme.

Corollary 3.3.36. *Let T be a two-variable operator defined for measurable functions. Suppose that for some exponents $1 \le p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and all weights $v_1 \in A_{p_1}$ and $v_2 \in A^{\mathcal{R}}_{p_2}$,*

$$T : L^{p_1}(v_1) \times L^{p_2,1}(v_2) \longrightarrow L^{p,\infty}(v_1^{p/p_1} v_2^{p/p_2}),$$

with constant bounded by $\varphi([v_1]_{A_{p_1}}, [v_2]_{A^{\mathcal{R}}_{p_2}})$ as in (3.3.1). Take an exponent $q_2 = p_2 \ge 1$ or $q_2 > p_2 > 1$, and $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{q_2}$, and weights $w_1 \in A_{p_1}$ and $w_2 \in \widehat{A}_{q_2}$, and $w = w_1^{q/p_1} w_2^{q/q_2}$. If $q_2 > p_2$, suppose that there exists $0 < \varepsilon \le 1$ such that $wW^{-\varepsilon}$ is a weight, and $[W^{-\varepsilon}]_{RH_\infty(w)} < \infty$, with $W = \left(\frac{w}{w_2}\right)^{1/q_2}$. Then,

$$T : L^{p_1}(w_1) \times L^{q_2,1}(w_2) \longrightarrow L^{q,\infty}(w),$$

with constant bounded by $\Phi_{\varepsilon,w_1,w_2}([w_1]_{A_{p_1}}, \|w_2\|_{\widehat{A}_{q_2}})$ as in (3.3.35).

Remark 3.3.37. In fact, in Theorem 3.3.35 and Corollary 3.3.36 we don't need to assume that $v_2 \in A^{\mathcal{R}}_{p_2}$, since the argument in their proofs works for $v_2 \in \widehat{A}_{p_2}$.

Remark 3.3.38. Observe that if the operator T is defined for characteristic functions of measurable sets $E_1, E_2 \subseteq \mathbb{R}^n$, then under the hypotheses of Corollary 3.3.36, we deduce that

$$\|T(\chi_{E_1}, \chi_{E_2})\|_{L^{q,\infty}(w)} \leq q_2 \Phi([w_1]_{A_{p_1}}, \|w_2\|_{\widehat{A}_{q_2}}) w_1(E_1)^{1/p_1} w_2(E_2)^{1/q_2},$$

and hence, T is of weak type (p_1, q_2, q) at least for characteristic functions.

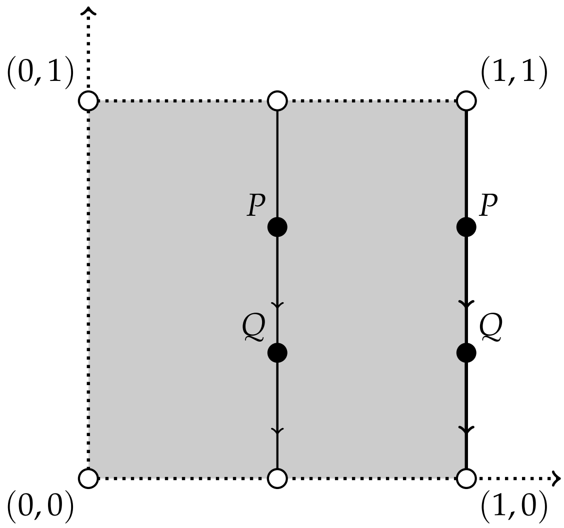

FIGURE 3.13: Pictorial representation of Theorem 3.3.35 and Corollary 3.3.36.

3.4 Applications

In this section, we present some applications for the extrapolation results previously introduced.

3.4.1 Product-Type Operators, and Averages

We start with the following result, that gives us restricted weak type bounds for products of one-variable operators.

Proposition 3.4.1. *Let S and T be one-variable operators defined for measurable functions. Suppose that S is sub-linear, and for some $p_1 > 1$, and every weight $v_1 \in A_{p_1}^{\mathcal{R}}$,*

$$S : L^{p_1,1}(v_1) \longrightarrow L^{p_1,\infty}(v_1), \tag{3.4.1}$$

with constant bounded by $\varphi_1([v_1]_{A_{p_1}^{\mathcal{R}}})$, where $\varphi_1 : [1, \infty) \longrightarrow [0, \infty)$ is an increasing function. Suppose also that for some $p_2 \geq 1$, and every weight $v_2 \in A_{p_2}^{\mathcal{R}}$,

$$T : L^{p_2,1}(v_2) \longrightarrow L^{p_2,\infty}(v_2), \tag{3.4.2}$$

with constant bounded by $\varphi_2([v_2]_{A^{\mathcal{R}}_{p_2}})$*, where* $\varphi_2 : [1,\infty) \longrightarrow [0,\infty)$ *is an increasing function. If* $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} < 1$*, then for all weights* $w_1 \in A_{p_1}$ *and* $w_2 \in A^{\mathcal{R}}_{p_2}$*, and all measurable functions* f *and* g*,*

$$\|(Sf)(Tg)\|_{L^{p,\infty}(w_1^{p/p_1}w_2^{p/p_2})} \leq \Phi([w_1]_{A_{p_1}}, [w_2]_{A^{\mathcal{R}}_{p_2}})\|f\|_{L^{p_1,1}(w_1)}\|g\|_{L^{p_2,1}(w_2)},$$

where $\Phi : [1,\infty)^2 \longrightarrow [0,\infty)$ *is a function increasing in each variable.*

Proof. We may assume that $f \in L^{p_1,1}(w_1)$ and $g \in L^{p_2,1}(w_2)$. In virtue of Lemma 2.2.1, (3.4.1) and Theorem 3.1.9, and (3.4.2), we have that

$$\begin{aligned}\|(Sf)(Tg)\|_{L^{p,\infty}(w_1^{p/p_1}w_2^{p/p_2})} &\leq c_{p_1,p_2}\|Sf\|_{L^{p_1,1}(w_1)}\|Tg\|_{L^{p_2,\infty}(w_2)}\\ &\leq \Phi([w_1]_{A_{p_1}}, [w_2]_{A^{\mathcal{R}}_{p_2}})\|f\|_{L^{p_1,1}(w_1)}\|g\|_{L^{p_2,1}(w_2)},\end{aligned}$$

with

$$\Phi([w_1]_{A_{p_1}}, [w_2]_{A^{\mathcal{R}}_{p_2}}) = c_{p_1,p_2}\phi_1([w_1]_{A_{p_1}})\varphi_2([w_2]_{A^{\mathcal{R}}_{p_2}}),$$

and $\phi_1([w_1]_{A_{p_1}})$ as in (3.1.6). □

Ideally, we should be able to extend Proposition 3.4.1 to the more general case when $p_1, p_2 \geq 1$, without restrictions on p, and $w_1 \in A^{\mathcal{R}}_{p_1}$, but since we don't have a version of Hölder's inequality for Lorentz spaces with the change of measures, this question is still open, although we managed to do the job for the particular case of the point-wise product of Hardy-Littlewood maximal operators (see Theorem 2.4.1). Fortunately, we can use our mixed type extrapolation theorems to improve the conclusion of Proposition 3.4.1.

Theorem 3.4.2. *Let* T_1 *and* T_2 *be sub-linear operators defined for measurable functions. For* $i = 1, 2$*, suppose that for some* $p_i > 1$*, and every weight* $v_i \in A^{\mathcal{R}}_{p_i}$*,*

$$T_i : L^{p_i,1}(v_i) \longrightarrow L^{p_i,\infty}(v_i), \tag{3.4.3}$$

with constant bounded by $\varphi_i([v_i]_{A^{\mathcal{R}}_{p_i}})$*, where* $\varphi_i : [1,\infty) \longrightarrow [0,\infty)$ *is an increasing function. Consider the operator*

$$T^{\otimes}(f,g) := (T_1 f)(T_2 g),$$

defined for measurable functions f *and* g*. If* $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} < 1$*, then for all exponents* $1 < q_1 < \infty$*,* $1 \leq q_2 \leq p_2$*, and* $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$*, and all weights* $w_1 \in A_{q_1}$ *and* $w_2 \in \widehat{A}_{q_2}$*,*

$$T^{\otimes} : L^{q_1,\min\{1,\frac{q_1}{p_1}\}}(w_1) \times L^{q_2,\frac{q_2}{p_2}}(w_2) \longrightarrow L^{q,\infty}(w_1^{q/q_1}w_2^{q/q_2}), \tag{3.4.4}$$

with constant bounded by $\Phi([w_1]_{A_{q_1}}, \|w_2\|_{\widehat{A}_{q_2}})$*, where* $\Phi : [1,\infty)^2 \longrightarrow [0,\infty)$ *is a function increasing in each variable. Moreover, for all exponents* $1 \leq q_1 \leq p_1$*, and*

$1 < q_2 < \infty$, and all weights $\widetilde{w}_1 \in \widehat{A}_{q_1}$ and $\widetilde{w}_2 \in A_{q_2}$,

$$T^{\otimes} : L^{q_1, \frac{q_1}{p_1}}(\widetilde{w}_1) \times L^{q_2, \min\{1, \frac{q_2}{p_2}\}}(\widetilde{w}_2) \longrightarrow L^{q,\infty}(\widetilde{w}_1^{q/q_1} \widetilde{w}_2^{q/q_2}), \tag{3.4.5}$$

with constant bounded by $\widetilde{\Phi}(\|\widetilde{w}_1\|_{\widehat{A}_{q_1}}, [\widetilde{w}_2]_{A_{q_2}})$, where $\widetilde{\Phi} : [1,\infty)^2 \longrightarrow [0,\infty)$ is a function increasing in each variable.

Proof. If we apply Proposition 3.4.1 to $(S,T) = (T_1, T_2)$ and $(S,T) = (T_2, T_1)$, we obtain that the operator $T^{\otimes}$ satisfies that for all weights $v_1 \in A_{p_1}$ and $v_2 \in A^{\mathcal{R}}_{p_2}$,

$$T^{\otimes} : L^{p_1,1}(v_1) \times L^{p_2,1}(v_2) \longrightarrow L^{p,\infty}(v_1^{p/p_1} v_2^{p/p_2}),$$

with constant bounded by $\varphi([v_1]_{A_{p_1}}, [v_2]_{A^{\mathcal{R}}_{p_2}})$, where $\varphi : [1,\infty)^2 \longrightarrow [0,\infty)$ is a function that increases in each variable. Also, for all weights $\widetilde{v}_1 \in A^{\mathcal{R}}_{p_1}$ and $\widetilde{v}_2 \in A_{p_2}$,

$$T^{\otimes} : L^{p_1,1}(\widetilde{v}_1) \times L^{p_2,1}(\widetilde{v}_2) \longrightarrow L^{p,\infty}(\widetilde{v}_1^{p/p_1} \widetilde{v}_2^{p/p_2}),$$

with constant bounded by $\widetilde{\varphi}([\widetilde{v}_1]_{A^{\mathcal{R}}_{p_1}}, [\widetilde{v}_2]_{A_{p_2}})$, where $\widetilde{\varphi} : [1,\infty)^2 \longrightarrow [0,\infty)$ is a function that increases in each variable. Taking into account Remark 3.3.28, the desired result follows from Corollary 3.3.33. □

Remark 3.4.3. Observe that if for $i = 1, 2$, T_i satisfies (3.4.3) for every $p_i > 1$, as it is the case of the Hardy-Littlewood maximal operator, then we can deduce bounds like (3.4.4) for any exponents $1 < q_1 < \infty$ and $1 \le q_2 < \infty$ by choosing $p_1 = q_1$ and $p_2 > \max\{q_2, q_1'\}$, and applying Theorem 3.4.2. Similarly, we can obtain bounds like (3.4.5) for any exponents $1 \le q_1 < \infty$ and $1 < q_2 < \infty$ by choosing $p_1 > \max\{q_1, q_2'\}$ and $p_2 = q_2$.

We have seen in Theorem 3.4.2 that, sometimes, we can use extrapolation techniques to avoid the application of some Hölder-type inequalities for Lorentz spaces. In the next result, we will see that we can also use extrapolation theorems to overcome the lack of Minkowski's integral inequality for the Lorentz quasi-norm $\|\cdot\|_{L^{q,\infty}(w)}$ when $q \le 1$.

Theorem 3.4.4. *Let $\{T_1^r\}_{r\in\mathbb{R}}$ and $\{T_2^s\}_{s\in\mathbb{R}}$ be families of sub-linear operators defined for measurable functions. For $i = 1, 2$, suppose that for some $p_i > 1$, every $t \in \mathbb{R}$, and every weight $v_i \in A^{\mathcal{R}}_{p_i}$,*

$$T_i^t : L^{p_i,1}(v_i) \longrightarrow L^{p_i,\infty}(v_i), \tag{3.4.6}$$

with constant bounded by $\varphi_i([v_i]_{A^{\mathcal{R}}_{p_i}})$, where $\varphi_i : [1,\infty) \longrightarrow [0,\infty)$ is an increasing function independent of t. For a measure μ on $\mathbb{R}^2$ such that $|\mu|(\mathbb{R}^2) < \infty$, consider the averaging operator

$$T_\mu(f,g) := \int_{\mathbb{R}^2} (T_1^r f)(T_2^s g)\, d\mu(r,s),$$

defined for measurable functions f and g. If $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} < 1$, then for all exponents $1 < q_1 < \infty$, $1 \le q_2 \le p_2$, and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, and all weights $w_1 \in A_{q_1}$ and

$w_2 \in \widehat{A}_{q_2}$,

$$T_\mu : L^{q_1,\min\{1,\frac{q_1}{p_1}\}}(w_1) \times L^{q_2,\frac{q_2}{p_2}}(w_2) \longrightarrow L^{q,\infty}(w_1^{q/q_1} w_2^{q/q_2}), \tag{3.4.7}$$

with constant bounded by $\Phi([w_1]_{A_{q_1}}, \|w_2\|_{\widehat{A}_{q_2}})$, *where* $\Phi : [1,\infty)^2 \longrightarrow [0,\infty)$ *is a function increasing in each variable. Moreover, for all exponents* $1 \leq q_1 \leq p_1$, *and* $1 < q_2 < \infty$, *and all weights* $\widetilde{w}_1 \in \widehat{A}_{q_1}$ *and* $\widetilde{w}_2 \in A_{q_2}$,

$$T_\mu : L^{q_1,\frac{q_1}{p_1}}(\widetilde{w}_1) \times L^{q_2,\min\{1,\frac{q_2}{p_2}\}}(\widetilde{w}_2) \longrightarrow L^{q,\infty}(\widetilde{w}_1^{q/q_1} \widetilde{w}_2^{q/q_2}), \tag{3.4.8}$$

with constant bounded by $\widetilde{\Phi}(\|\widetilde{w}_1\|_{\widehat{A}_{q_1}}, [\widetilde{w}_2]_{A_{q_2}})$, *where* $\widetilde{\Phi} : [1,\infty)^2 \longrightarrow [0,\infty)$ *is a function increasing in each variable.*

Proof. Since $p > 1$, in virtue of Minkowski's integral inequality (see [104, Proposition 2.1] and [3, Theorem 4.4]), we have that for all weights $v_1 \in A_{p_1}$, $v_2 \in A_{p_2}^{\mathcal{R}}$, and $v := v_1^{p/p_1} v_2^{p/p_2}$,

$$\|T_\mu(f,g)\|_{L^{p,\infty}(v)} \leq p' \int_{\mathbb{R}^2} \|(T_1^r f)(T_2^s g)\|_{L^{p,\infty}(v)} d|\mu|(r,s),$$

and applying Proposition 3.4.1 to $(S,T) = (T_1^r, T_2^s)$, we get that

$$\|T_\mu(f,g)\|_{L^{p,\infty}(v)} \leq p' |\mu|(\mathbb{R}^2) \varphi([v_1]_{A_{p_1}}, [v_2]_{A_{p_2}^{\mathcal{R}}}) \|f\|_{L^{p_1,1}(v_1)} \|g\|_{L^{p_2,1}(v_2)},$$

where $\varphi : [1,\infty)^2 \longrightarrow [0,\infty)$ is a function that increases in each variable.

Similarly, we also have that for all weights $\widetilde{v}_1 \in A_{p_1}^{\mathcal{R}}$, $\widetilde{v}_2 \in A_{p_2}$, and $\widetilde{v} = \widetilde{v}_1^{p/p_1} \widetilde{v}_2^{p/p_2}$,

$$\|T_\mu(f,g)\|_{L^{p,\infty}(\widetilde{v})} \leq p' |\mu|(\mathbb{R}^2) \widetilde{\varphi}([\widetilde{v}_1]_{A_{p_1}^{\mathcal{R}}}, [\widetilde{v}_2]_{A_{p_2}}) \|f\|_{L^{p_1,1}(\widetilde{v}_1)} \|g\|_{L^{p_2,1}(\widetilde{v}_2)},$$

where $\widetilde{\varphi} : [1,\infty)^2 \longrightarrow [0,\infty)$ is a function that increases in each variable. We can now apply Corollary 3.3.33 to deduce the desired result, taking into account Remark 3.3.28. □

Remark 3.4.5. Note that if for $i = 1,2$, and every $t \in \mathbb{R}$, T_i^t satisfies (3.4.6) for every $p_i > 1$, then we can deduce bounds like (3.4.7) for any exponents $1 < q_1 < \infty$ and $1 \leq q_2 < \infty$ by choosing $p_1 = q_1$ and $p_2 > \max\{q_2, q_1'\}$, and applying Theorem 3.4.4. Similarly, we can obtain bounds like (3.4.8) for any exponents $1 \leq q_1 < \infty$ and $1 < q_2 < \infty$ by choosing $p_1 > \max\{q_1, q_2'\}$ and $p_2 = q_2$.

3.4.2 Bi-Linear Fourier Multiplier Operators

Let us start with some classical definitions from [102, Chapter 8].

Definition 3.4.6. Given a function $f : \mathbb{R} \longrightarrow \mathbb{R}$, we say that f is of *bounded variation* if

$$V(f) := \sup \sum_{j=1}^{N} |f(x_j) - f(x_{j-1})| \in \mathbb{R},$$

where the supremum is taken over all N and over all choices of $x_0, \dots, x_N$ such that $-\infty < x_0 < x_1 < \cdots < x_N < \infty$. We call $V(f)$ the *total variation of* f. The class of all functions f of bounded variation will be denoted by $BV(\mathbb{R})$.

We say that a function $f \in BV(\mathbb{R})$ is *normalized* if f is left-continuous at every point of $\mathbb{R}$, and $\lim_{x\to-\infty} f(x) = 0$. The class of these functions will be denoted by $NBV(\mathbb{R})$.

We say that a function f is *absolutely continuous* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sum_{j=1}^{N}(b_j - a_j) < \delta \quad \Longrightarrow \quad \sum_{j=1}^{N} |f(b_j) - f(a_j)| < \varepsilon,$$

whenever $(a_1, b_1), \dots, (a_N, b_N)$ are disjoint segments. The class of all such functions will be denoted by $AC(\mathbb{R})$.

Let us focus our attention to [38, Corollary 3.8]. This result tells us that for a function $m \in NBV(\mathbb{R})$, we can write

$$m(\xi) = \int_{-\infty}^{\xi} dm(t) = \int_{\mathbb{R}} \chi_{(-\infty,\xi)}(t)dm(t) = \int_{\mathbb{R}} \chi_{(t,\infty)}(\xi)dm(t), \qquad (3.4.9)$$

where dm denotes the Lebesgue-Stieltjes measure associated with m. Therefore, the linear multiplier operator T_m given by

$$\widehat{T_m f}(\xi) := m(\xi)\widehat{f}(\xi) = \int_{\mathbb{R}} \chi_{(t,\infty)}(\xi)\widehat{f}(\xi)dm(t) =: \int_{\mathbb{R}} \widehat{S_{t,\infty} f}(\xi)dm(t), \quad \xi \in \mathbb{R},$$

initially defined for Schwartz functions f on $\mathbb{R}$, can be written as

$$T_m f(x) = \int_{\mathbb{R}} S_{t,\infty} f(x)dm(t), \quad x \in \mathbb{R},$$

where

$$S_{t,\infty} f(x) := \frac{1}{2}f(x) + \frac{i}{2}e^{2\pi i t x} H(e^{-2\pi i t \cdot} f)(x) =: \frac{1}{2}f(x) + \frac{i}{2}e^{2\pi i t x} H_t f(x).$$

As usual, H denotes the *Hilbert transform on* $\mathbb{R}$, defined as

$$Hf(x) := \frac{1}{\pi} \lim_{\varepsilon\to 0^+} \int_{\{y\in\mathbb{R}\,:\,|x-y|>\varepsilon\}} \frac{f(y)}{x-y}dy, \quad x \in \mathbb{R},$$

and for a Schwartz function $f : \mathbb{R} \longrightarrow \mathbb{R}$, we denote by $\widehat{f}$ its *Fourier transform*, given by

$$\widehat{f}(\xi) := \int_{\mathbb{R}} f(y)e^{-2\pi i y\xi} dy, \quad \xi \in \mathbb{R}.$$

Since for $1 < p < \infty$, $H : L^p(\mathbb{R}) \longrightarrow L^p(\mathbb{R})$ with constant bounded by C_p, in virtue of Minkowski's integral inequality we conclude that

$$\|T_m f\|_{L^p(\mathbb{R})} \leq \int_{\mathbb{R}} \|S_{t,\infty} f\|_{L^p(\mathbb{R})} d|m|(t) \leq \frac{1+C_p}{2} V(m)\|f\|_{L^p(\mathbb{R})},$$

and m is an L^p *Fourier multiplier* for every $1 < p < \infty$.

Inspired by this result, let us take a measure μ on $\mathbb{R}^2$ such that $|\mu|(\mathbb{R}^2) < \infty$, and define the function

$$\begin{aligned} m_\mu(\xi,\eta) &:= \int_{\{(r,t)\in\mathbb{R}^2 : r\leq\xi, t\leq\eta\}} d\mu(r,t) = \int_{\mathbb{R}^2} \chi_{(-\infty,\xi)}(r)\chi_{(-\infty,\eta)}(t) d\mu(r,t) \\ &= \int_{\mathbb{R}^2} \chi_{(r,\infty)}(\xi)\chi_{(t,\infty)}(\eta) d\mu(r,t), \end{aligned} \tag{3.4.10}$$

for $\xi, \eta \in \mathbb{R}$. It is clear that $\|m_\mu\|_{L^\infty(\mathbb{R}^2)} \leq |\mu|(\mathbb{R}^2) < \infty$, so it makes sense to consider the bi-linear multiplier operator

$$T_{m_\mu}(f,g)(x) := \int_{\mathbb{R}}\int_{\mathbb{R}} m_\mu(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x(\xi+\eta)} d\xi d\eta, \tag{3.4.11}$$

initially defined for Schwartz functions f and g, and $x \in \mathbb{R}$. Arguing as we did in the linear case, and applying Fubini's theorem, we have that

$$\begin{aligned} &T_{m_\mu}(f,g)(x) \\ &\quad = \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} \chi_{(r,\infty)}(\xi)\widehat{f}(\xi)e^{2\pi i x\xi} d\xi \right) \left(\int_{\mathbb{R}} \chi_{(t,\infty)}(\eta)\widehat{g}(\eta)e^{2\pi i x\eta} d\eta \right) d\mu(r,t) \\ &\quad = \int_{\mathbb{R}^2} S_{r,\infty} f(x) S_{t,\infty} g(x) d\mu(r,t), \end{aligned}$$

so T_{m_μ} is, in fact, a two-variable averaging operator, and we can follow the approach of Theorem 3.4.4 to prove weighted bounds for it, exploiting known restricted weak type bounds for the Hilbert transform, as we show in the next result.

Theorem 3.4.7. *Given exponents $1 < q_1 < \infty$, $1 \leq q_2 < \infty$, and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, if $q > 1$, then for all weights $w_1 \in A_{q_1}$, $w_2 \in A^{\mathcal{R}}_{q_2}$, and $w = w_1^{q/q_1} w_2^{q/q_2}$,*

$$T_{m_\mu} : L^{q_1,1}(w_1) \times L^{q_2,1}(w_2) \longrightarrow L^{q,\infty}(w), \tag{3.4.12}$$

with constant bounded by $\Phi([w_1]_{A_{q_1}}, [w_2]_{A^{\mathcal{R}}_{q_2}})$, where $\Phi : [1,\infty)^2 \longrightarrow [0,\infty)$ is a function increasing in each variable. Moreover, if $q \leq 1$, then for every exponent

$p_2 > \max\{q_2, q_1'\}$, and all weights $w_1 \in A_{q_1}$ and $w_2 \in \widehat{A}_{q_2}$,

$$T_{m_\mu} : L^{q_1,1}(w_1) \times L^{q_2,\frac{q_2}{p_2}}(w_2) \longrightarrow L^{q,\infty}(w), \tag{3.4.13}$$

and if also $q_2 > 1$, then for every exponent $p_1 > \max\{q_1, q_2'\}$, and all weights $w_1 \in A_{q_1}$ and $w_2 \in A_{q_2}^{\mathcal{R}}$,

$$T_{m_\mu} : L^{q_1,\frac{q_1}{p_1}}(w_1) \times L^{q_2,1}(w_2) \longrightarrow L^{q,\infty}(w), \tag{3.4.14}$$

with constants bounded by $\Phi_1([w_1]_{A_{q_1}}, \|w_2\|_{\widehat{A}_{q_2}})$ and $\Phi_2([w_1]_{A_{q_1}}, [w_2]_{A_{q_2}^{\mathcal{R}}})$ respectively, where for $i = 1,2$, $\Phi_i : [1,\infty)^2 \longrightarrow [0,\infty)$ is a function increasing in each variable. Analogously, we also have the symmetric bounds for $w_1 \in A_{q_1}^{\mathcal{R}}$ or $w_1 \in \widehat{A}_{q_1}$, and $w_2 \in A_{q_2}$.

Proof. It follows from Theorem 5.2.7, and [67, Theorem 1.2], that for every $p \geq 1$, and every weight $v \in A_p^{\mathcal{R}}$, $H : L^{p,1}(v) \longrightarrow L^{p,\infty}(v)$, with constant bounded by

$$\phi([v]_{A_p^{\mathcal{R}}}) := \begin{cases} C_{n,p}[v]_{A_p^{\mathcal{R}}}^{p+1}, & p > 1, \\ C_n[v]_{A_1}(1+\log^+[v]_{A_1})(1+\log^+\log^+[v]_{A_1}), & p = 1, \end{cases}$$

so for every $\sigma \in \mathbb{R}$, and every $h \in L^{p,1}(v)$,

$$\|S_{\sigma,\infty}h\|_{L^{p,\infty}(v)} \leq \|h\|_{L^{p,\infty}(v)} + \|H_\sigma h\|_{L^{p,\infty}(v)} \leq \left(\frac{1}{p} + \phi([v]_{A_p^{\mathcal{R}}})\right)\|h\|_{L^{p,1}(v)}.$$

Hence, applying Proposition 3.4.1 to $(S,T) = (S_{r,\infty}, S_{t,\infty})$, we get that if $q > 1$, then for all weights $w_1 \in A_{q_1}$ and $w_2 \in A_{q_2}^{\mathcal{R}}$, and all measurable functions $f \in L^{q_1,1}(w_1)$ and $g \in L^{q_2,1}(w_2)$,

$$\|(S_{r,\infty}f)(S_{t,\infty}g)\|_{L^{q,\infty}(w_1^{q/q_1}w_2^{q/q_2})} \leq \varphi([w_1]_{A_{q_1}}, [w_2]_{A_{q_2}^{\mathcal{R}}})\|f\|_{L^{q_1,1}(w_1)}\|g\|_{L^{q_2,1}(w_2)},$$

where $\varphi : [1,\infty)^2 \longrightarrow [0,\infty)$ is a function increasing in each variable. Therefore, in virtue of Minkowski's integral inequality, we deduce that (3.4.12) holds, with $\Phi = q'|\mu|(\mathbb{R}^2)\varphi$.

To discuss the case when $q \leq 1$, we will use our extrapolation results. Note that for every $p_2 > \max\{q_2, q_1'\}$, we have that $\frac{1}{q_1} + \frac{1}{p_2} =: \frac{1}{\ell} < 1$, and by (3.4.12), we get that for all weights $v_1 \in A_{q_1}$, $v_2 \in A_{p_2}^{\mathcal{R}}$, and $v := v_1^{\ell/q_1}v_2^{\ell/p_2}$,

$$T_{m_\mu} : L^{q_1,1}(v_1) \times L^{p_2,1}(v_2) \longrightarrow L^{\ell,\infty}(v),$$

with constant bounded by $\varphi_1([v_1]_{A_{q_1}}, [v_2]_{A_{p_2}^{\mathcal{R}}})$, where $\varphi_1 : [1,\infty)^2 \longrightarrow [0,\infty)$ is a function increasing in each variable. Hence, applying Corollary 3.3.14 and Remark 3.3.13, we can extrapolate downwards and obtain (3.4.13). Alternatively, if $q \leq 1$ and $q_2 > 1$, then for every $p_1 > \max\{q_1, q_2'\}$, we have that $\frac{1}{p_1} + \frac{1}{q_2} =: \frac{1}{\ell} < 1$, and by (3.4.12), we get that for all weights $v_1 \in A_{p_1}$,

$v_2 \in A_{q_2}^{\mathcal{R}}$, and $v := v_1^{\ell/p_1} v_2^{\ell/q_2}$,

$$T_{m_\mu} : L^{p_1,1}(v_1) \times L^{q_2,1}(v_2) \longrightarrow L^{\ell,\infty}(v),$$

with constant bounded by $\varphi_2([v_1]_{A_{p_1}}, [v_2]_{A_{q_2}^{\mathcal{R}}})$, where $\varphi_2 : [1,\infty)^2 \longrightarrow [0,\infty)$ is a function increasing in each variable. Hence, applying Corollary 3.3.10 and Remark 3.3.7, we can extrapolate downwards and obtain (3.4.14). □

Remark 3.4.8. Observe that for $1 \leq p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $v_1 \in A_{p_1}^{\mathcal{R}}$, $v_2 \in A_{p_2}^{\mathcal{R}}$, and $v = v_1^{p/p_1} v_2^{p/p_2}$, and virtue of Lemma 2.2.1, we get that

$$\begin{aligned}
&\|(S_{r,\infty} f)(S_{t,\infty} g)\|_{L^{p,\infty}(v)} \\
&\leq \|fg\|_{L^{p,\infty}(v)} + \|f H_t g\|_{L^{p,\infty}(v)} + \|g H_r f\|_{L^{p,\infty}(v)} + \|(H_r f)(H_t g)\|_{L^{p,\infty}(v)} \\
&\leq C_{n,p_1,p_2} \varphi([v_1]_{A_{p_1}^{\mathcal{R}}}, [v_2]_{A_{p_2}^{\mathcal{R}}}) \|f\|_{L^{p_1,1}(v_1)} \|g\|_{L^{p_2,1}(v_2)} + \|(H_r f)(H_t g)\|_{L^{p,\infty}(v)},
\end{aligned}$$

where $\varphi : [1,\infty)^2 \longrightarrow [0,\infty)$ is a function increasing in each variable. Hence, if we could prove restricted weak type bounds for the point-wise product of Hilbert transforms, we would be able to transfer them to the operator T_{m_μ} using our extrapolation results from Chapter 4, arguing as in the proof of Theorem 3.4.7.

If we take functions $m_1, m_2 \in NBV(\mathbb{R})$, we can easily construct a function like (3.4.10) by merely considering their product, since by (3.4.9),

$$(m_1 \otimes m_2)(\xi, \eta) := m_1(\xi) m_2(\eta) = \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_{(r,\infty)}(\xi) \chi_{(t,\infty)}(\eta) dm_1(r) dm_2(t),$$

and

$$\|m_1 \otimes m_2\|_{L^\infty(\mathbb{R}^2)} \leq \int_{\mathbb{R}} \int_{\mathbb{R}} d|m_1|(r) d|m_2|(t) = V(m_1) V(m_2) < \infty.$$

The following result, which is a combination of Theorems 8.17 and 8.18 in [102], will allow us to construct another simple yet more elaborate example of a function like (3.4.10), along with many examples of functions in $NBV(\mathbb{R})$.

Theorem 3.4.9. *If $\psi \in L^1(\mathbb{R})$, and for every $x \in \mathbb{R}$,*

$$f(x) := \int_{-\infty}^{x} \psi(r) dr,$$

then $f \in NBV(\mathbb{R})$, f is absolutely continuous, and $f' = \psi$ almost everywhere.

Conversely, if $f \in NBV(\mathbb{R}) \cap AC(\mathbb{R})$, then f is differentiable almost everywhere, $f' \in L^1(\mathbb{R})$, and for every $x \in \mathbb{R}$,

$$f(x) = \int_{-\infty}^{x} f'(r) dr.$$

An immediate consequence of Theorem 3.4.9 is the next lemma.

Lemma 3.4.10. *Given a function $m \in NBV(\mathbb{R}) \cap AC(\mathbb{R})$, for all $\xi, \eta \in \mathbb{R}$,*

$$\widetilde{m}(\xi,\eta) := m(\min\{\xi,\eta\}) = \int_{\mathbb{R}} \chi_{(r,\infty)}(\xi)\chi_{(r,\infty)}(\eta)m'(r)dr,$$

and $\|\widetilde{m}\|_{L^\infty(\mathbb{R}^2)} \leq \|m'\|_{L^1(\mathbb{R})} < \infty$.

Proof. Observe that for $r \in \mathbb{R}$, $-\infty < r < \min\{\xi,\eta\}$ if, and only if $-\infty < r < \xi$ and $-\infty < r < \eta$, so by Theorem 3.4.9, we have that

$$\begin{aligned} m(\min\{\xi,\eta\}) &= \int_{-\infty}^{\min\{\xi,\eta\}} m'(r)dr = \int_{\mathbb{R}} \chi_{(-\infty,\xi)}(r)\chi_{(-\infty,\eta)}(r)m'(r)dr \\ &= \int_{\mathbb{R}} \chi_{(r,\infty)}(\xi)\chi_{(r,\infty)}(\eta)m'(r)dr. \end{aligned}$$

□

The function $\widetilde{m}$ is an example of function like (3.4.10) where the measure μ is restricted to $\mathbb{R}$. More generally, we can take a subset $E \subseteq \mathbb{R}^2$, and a measure ν on E such that $|\nu|(E) < \infty$, and consider the function

$$m_{\nu,E}(\xi,\eta) := \nu(E \cap R_{\xi,\eta}),$$

with $R_{\xi,\eta} := \{(r,t) \in \mathbb{R}^2 : r \leq \xi,\ t \leq \eta\}$, and $\|m_{\nu,E}\|_{L^\infty(\mathbb{R}^2)} \leq |\nu|(E) < \infty$.

In the particular case when $E = (0,1)^2$, the unit square, and ν is the Lebesgue measure on E, we obtain that

$$m_{\nu,E}(\xi,\eta) = \min\{1,\xi\}\min\{1,\eta\}\chi_{\{r\in\mathbb{R}:r>0\}}(\xi)\chi_{\{r\in\mathbb{R}:r>0\}}(\eta). \tag{3.4.15}$$

For a pictorial representation of such function, see Figure 3.14.

We obtain a more elaborate example if we consider the unit half-disk

$$E = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1, y \geq 0\},$$

and ν the Lebesgue measure on E.

In this case, we get that $m_{\nu,E}(\xi,\eta)$ is equal to

$$\begin{cases} 0, & -\infty < \eta \leq 0, \\ & -\infty < \xi < \infty, \\ 0, & 0 < \eta < 1, \\ & -\infty < \xi \leq -1, \\ \frac{\pi}{4} + \frac{1}{2}\left(\xi\sqrt{1-\xi^2} + \arcsin(\xi)\right), & 0 < \eta < 1, \\ & -1 < \xi \leq -\sqrt{1-\eta^2}, \\ \frac{1}{2}\left(\eta\sqrt{1-\eta^2} + \arcsin(\eta)\right) + \eta\xi, & 0 < \eta < 1, \\ & -\sqrt{1-\eta^2} < \xi \leq \sqrt{1-\eta^2}, \\ \eta\sqrt{1-\eta^2} + \arcsin(\eta) + \frac{1}{2}\left(\xi\sqrt{1-\xi^2} - \arccos(\xi)\right), & 0 < \eta < 1, \\ & \sqrt{1-\eta^2} < \xi \leq 1, \\ \eta\sqrt{1-\eta^2} + \arcsin(\eta), & 0 < \eta < 1, \\ & 1 < \xi < \infty, \\ 0, & 1 \leq \eta < \infty, \\ & -\infty < \xi \leq -1, \\ \frac{\pi}{4} + \frac{1}{2}\left(\xi\sqrt{1-\xi^2} + \arcsin(\xi)\right), & 1 \leq \eta < \infty, \\ & -1 < \xi \leq 1, \\ \frac{\pi}{2}, & 1 \leq \eta < \infty, \\ & 1 < \xi < \infty. \end{cases} \tag{3.4.16}$$

For a three-dimensional plot of such function, see Figure 3.15.

3.4.3 Two-Variable Commutators

Given one-variable operators T_1 and T_2, defined for measurable functions on $\mathbb{R}^n$, and measurable functions b_1 and b_2, with $\vec{b} = (b_1, b_2)$, let us consider the two-variable commutators $[\vec{b}, T^{\otimes}]_1$ and $[\vec{b}, T^{\otimes}]_2$. Observe that for measurable functions f_1 and f_2,

$$[\vec{b}, T^{\otimes}]_1(f_1, f_2) := b_1(T_1 f_1)(T_2 f_2) - T_1(b_1 f_1)(T_2 f_2) = ([b_1, T_1]f_1)(T_2 f_2),$$

and similarly,

$$[\vec{b}, T^{\otimes}]_2(f_1, f_2) = (T_1 f_1)([b_2, T_2]f_2).$$

Hence, these operators are, in fact, product-type operators, and we can follow the approach of Theorem 3.4.2 to prove weighted bounds for them, using known estimates for commutators of one-variable operators, as we show in the next result.

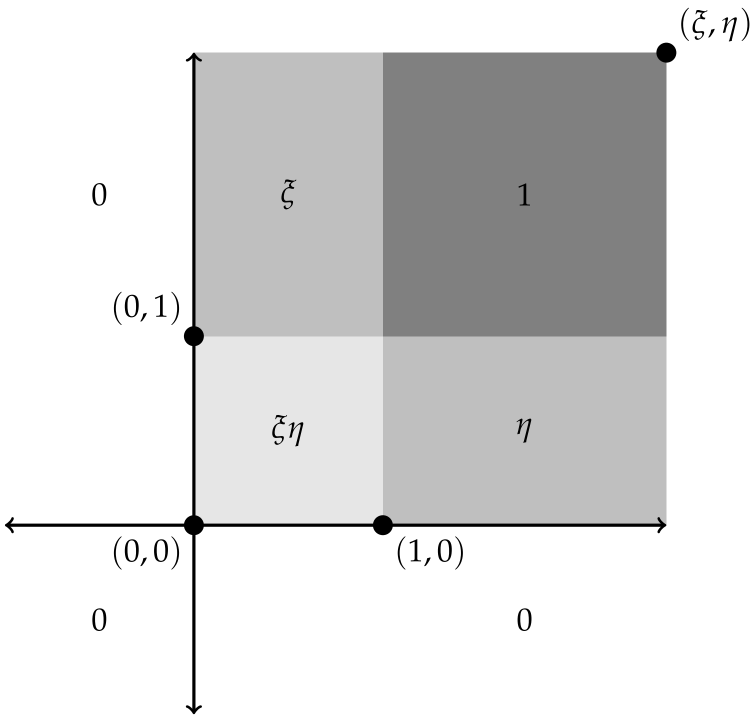

FIGURE 3.14: Pictorial representation of the function in (3.4.15).

Theorem 3.4.11. *Let T_1 be a linear operator such that for every weight $u \in A_2$,*

$$T_1 : L^2(u) \longrightarrow L^2(u),$$

with constant bounded by $\varphi_1([u]_{A_2})$, and let T_2 be a one-variable operator such that for some $p_2 > 1$, and every weight $v_2 \in A^{\mathcal{R}}_{p_2}$,

$$T_2 : L^{p_2,1}(v_2) \longrightarrow L^{p_2,\infty}(v_2), \tag{3.4.17}$$

with constant bounded by $\varphi_2([v_2]_{A^{\mathcal{R}}_{p_2}})$, where for $i = 1,2$, $\varphi_i : [1,\infty) \longrightarrow [0,\infty)$ is an increasing function. Let $b_1, b_2 \in BMO$. Given exponents $1 < q_1 < \infty$, $1 \leq q_2 \leq p_2$, $p_1 \geq q_1$ such that $p_1 > p_2'$, and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, and weights $w_1 \in A_{q_1}$, $w_2 \in \widehat{A}_{q_2}$, and $w = w_1^{q/q_1} w_2^{q/q_2}$,

$$[\vec{b}, T^{\otimes}]_1 : L^{q_1,\frac{q_1}{p_1}}(w_1) \times L^{q_2,\frac{q_2}{p_2}}(w_2) \longrightarrow L^{q,\infty}(w), \tag{3.4.18}$$

with constant bounded by $\Phi_{\vec{b}}([w_1]_{A_{q_1}}, \|w_2\|_{\widehat{A}_{q_2}})$, where $\Phi_{\vec{b}} : [1,\infty)^2 \longrightarrow [0,\infty)$ is a function increasing in each variable. An analogous result can be produced for $[\vec{b}, T^{\otimes}]_2$.

Proof. In virtue of [16, Corollary 3.3], we have that for $1 < r < \infty$, and every weight $v_1 \in A_r$,

$$[b_1, T_1] : L^r(v_1) \longrightarrow L^r(v_1),$$

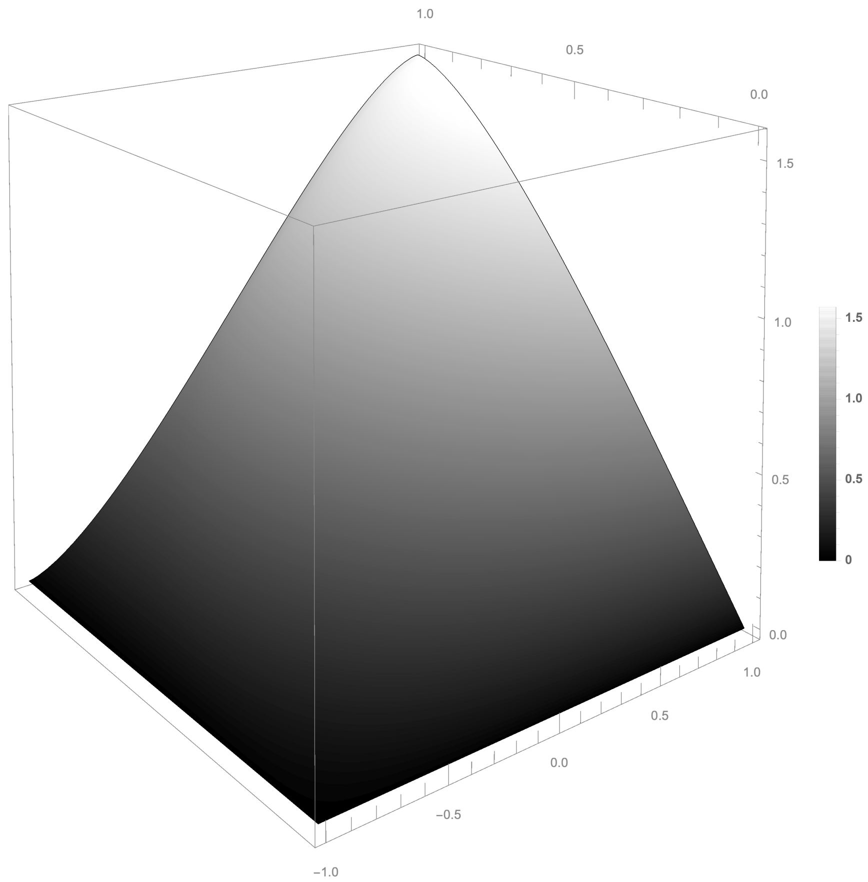

FIGURE 3.15: *Mathematica*'s 3D plot of the function in (3.4.16) on $[-1,1] \times [0,1]$.

with constant bounded by

$$\phi([v_1]_{A_r}) := c_{n,r}\varphi_1(C_{n,r}[v_1]_{A_r}^{\max\{1,\frac{1}{r-1}\}})[v_1]_{A_r}^{\max\{1,\frac{1}{r-1}\}}\|b_1\|_{BMO}.$$

In particular,

$$[b_1, T_1] : L^r(v_1) \longrightarrow L^{r,\infty}(v_1),$$

with constant also bounded by $\phi([v_1]_{A_r})$, and arguing as in the proof of Theorem 3.1.9, we deduce that for every weight $v_1 \in A_r$,

$$[b_1, T_1] : L^{r,1}(v_1) \longrightarrow L^{r,1}(v_1),$$

with constant bounded by

$$\psi([v_1]_{A_r}) := \widetilde{c}_{n,r}[v_1]_{A_r}^{\frac{2}{r-1}}\phi(\widetilde{C}_{n,r}[v_1]_{A_r}^2).$$

For $r = p_1$, and applying Lemma 2.2.1, since $\frac{1}{p} := \frac{1}{p_1} + \frac{1}{p_2} < 1$, we get that for all weights $v_1 \in A_{p_1}$, $v_2 \in A_{p_2}^{\mathcal{R}}$, and $v := v_1^{p/p_1}v_2^{p/p_2}$, and all measurable functions $f_1 \in L^{p_1,1}(v_1)$ and $f_2 \in L^{p_2,1}(v_2)$,

$$\begin{aligned}\|[\vec{b}, T^{\otimes}]_1(f_1, f_2)\|_{L^{p,\infty}(v)} &= \|([b_1, T_1]f_1)(T_2 f_2)\|_{L^{p,\infty}(v)} \\ &\leq c_{p_1,p_2} \|[b_1, T_1]f_1\|_{L^{p_1,1}(v_1)} \|T_2 f_2\|_{L^{p_2,\infty}(v_2)} \\ &\leq c_{p_1,p_2} \psi([v_1]_{A_{p_1}}) \varphi_2([v_2]_{A^{\mathcal{R}}_{p_2}}) \prod_{i=1}^{2} \|f_i\|_{L^{p_i,1}(v_i)},\end{aligned}$$

and (3.4.18) follows extrapolating downwards with Corollary 3.3.18, taking into account Remark 3.3.7 and Remark 3.3.13.

It is worth mentioning that the function $\Phi_{\vec{b}}$ that we obtain is of the form $\Phi_{\vec{b}} = \|b_1\|_{BMO} \widetilde{\Phi}$, where $\widetilde{\Phi} : [1, \infty)^2 \longrightarrow [0, \infty)$ is a function increasing in each variable and independent of $\vec{b}$. $\square$

Remark 3.4.12. Observe that if T_2 satisfies (3.4.17) for every $p_2 > 1$, as it is the case of linear Calderón-Zygmund operators, then we can deduce bounds like (3.4.18) for any exponents $1 < q_1 < \infty$ and $1 \leq q_2 < \infty$ by choosing $p_1 = q_1$ and $p_2 > \max\{q_2, q_1'\}$, and applying Theorem 3.4.11.

Chapter 4

Multi-Variable Restricted Weak Type Extrapolation

> *We choose to go to the Moon in this decade and do the other things, not because they are easy, but because they are hard.*
>
> John Fitzgerald Kennedy, *Address at Rice University*, 1962

We devote this chapter to the study of multi-variable restricted weak type Rubio de Francia's extrapolation and its applications. In Section 4.1, we expose more technical results that we will use in our work. In Section 4.2, we present our main results on restricted weak type extrapolation, including downwards, upwards, and one-variable off-diagonal schemes. In Section 4.3, we apply our extrapolation results to produce bounds for sums of product-type operators, and the corresponding averaging operators.

4.1 More Technical Results

Let us start defining the following class of weights, which was introduced in an unpublished version of [9].

Definition 4.1.1. Given $1 \leq p < \infty$, and $1 \leq N \in \mathbb{N}$, we say that a weight w belongs to the class $\widehat{A}_{p,N}$ if there exist functions $f_1, \dots, f_N \in L^1_{loc}(\mathbb{R}^n)$, parameters $\theta_1, \dots, \theta_N \in (0,1]$, with $\theta_1 + \cdots + \theta_N = 1$, and a weight $u \in A_1$ such that

$$w = \left(\prod_{i=1}^{N} (Mf_i)^{\theta_i} \right)^{1-p} u. \tag{4.1.1}$$

We can associate a constant to this class of weights, given by

$$\|w\|_{\widehat{A}_{p,N}} := \inf\, [u]_{A_1}^{1/p},$$

where the infimum is taken over all weights $u \in A_1$ such that w can be written as (4.1.1). We also define

$$\widehat{A}_{p,\infty} := \bigcup_{N=1}^{\infty} \widehat{A}_{p,N},$$

with the corresponding associated constant, given by

$$\|w\|_{\widehat{A}_{p,\infty}} := \inf_{N\geq 1} \|w\|_{\widehat{A}_{p,N}}.$$

It is clear that $\widehat{A}_{1,\infty} = A_1$, and $\widehat{A}_{p,1} = \widehat{A}_p$. Also, observe that for every $N \geq 1$, $\widehat{A}_{p,N} \subseteq \widehat{A}_{p,N+1}$, and $\|w\|_{\widehat{A}_{p,N+1}} \leq \|w\|_{\widehat{A}_{p,N}}$, but we don't know if these inclusion relations are strict.

The following lemma will be helpful for future computations.

Lemma 4.1.2. *Given real numbers $A, B \geq 0$, and $0 < \theta < 1$,*

$$\int_0^{\infty} \frac{\min\{A, tB\}}{t^{\theta}} \frac{dt}{t} = \frac{A^{1-\theta}B^{\theta}}{\theta(1-\theta)}.$$

Proof. If $AB = 0$, then the result is clear. Otherwise, the result follows from the fact that

$$\min\{A, tB\} = \begin{cases} A, & t \geq \frac{A}{B}, \\ tB, & t < \frac{A}{B}. \end{cases}$$

□

The next lemma gives us a restricted weak type interpolation result for weights.

Lemma 4.1.3. *Fix $0 < p < \infty$, and $0 < \theta < 1$. Let u_1, u_2, v_1, v_2 be weights, and write $u = u_1^{1-\theta}u_2^{\theta}$, and $v = v_1^{1-\theta}v_2^{\theta}$. Let T be a sub-linear operator defined for characteristic functions. Suppose that for $i = 1, 2$, there exists a constant $C_i > 0$ such that for every measurable set E,*

$$\|T(\chi_E)\|_{L^{p,\infty}(u_i)} \leq C_i \|\chi_E\|_{L^{p,1}(v_i)}. \tag{4.1.2}$$

Then, for $C = C_1 + C_2$, and every measurable set E,

$$\|T(\chi_E)\|_{L^{p,\infty}(u)} \leq C \|\chi_E\|_{L^{p,1}(v)}.$$

Proof. Fix $t, y > 0$, and $0 < \gamma < 1$. Since T is sub-linear, we have that

$$\begin{aligned} \int_{\{|T(\chi_E)|>y\}} \min\{u_1(x), tu_2(x)\}dx \leq & \int_{\{|T(\chi_{E_1})|>\gamma y\}} u_1(x)dx \\ & + t\int_{\{|T(\chi_{E_2})|>(1-\gamma)y\}} u_2(x)dx, \end{aligned} \tag{4.1.3}$$

where $E_1 := \{x \in E : v_1(x) \leq tv_2(x)\}$, and $E_2 := E \setminus E_1$. Applying (4.1.2), and choosing $\gamma = \frac{C_1}{C_1+C_2}$, we can bound (4.1.3) by

$$\left(\frac{pC_1}{\gamma y}\right)^p v_1(E_1) + t\left(\frac{pC_2}{(1-\gamma)y}\right)^p v_2(E_2) = \left(\frac{pC}{y}\right)^p (v_1(E_1) + tv_2(E_2))$$
$$= \left(\frac{pC}{y}\right)^p \int_E \min\{v_1(x), tv_2(x)\}dx,$$

and in virtue of Lemma 4.1.2, we conclude that

$$\int_{\{|T(\chi_E)|>y\}} u(x)dx = \theta(1-\theta)\int_0^\infty \int_{\{|T(\chi_E)|>y\}} \frac{\min\{u_1(x), tu_2(x)\}}{t^\theta}dx\frac{dt}{t}$$
$$\leq \theta(1-\theta)\left(\frac{pC}{y}\right)^p \int_0^\infty \int_E \frac{\min\{v_1(x), tv_2(x)\}}{t^\theta}dx\frac{dt}{t}$$
$$= \left(\frac{pC}{y}\right)^p v(E),$$

and the desired result follows. □

Remark 4.1.4. At the time of writing, we don't know if it is possible to prove Lemma 4.1.3 with $C = \max\{C_1, C_2\}$.

We will use Lemma 4.1.3 to show that for $p \geq 1$, $\widehat{A}_{p,\infty} \subseteq A_p^{\mathcal{R}}$, but due to Remark 4.1.4, we can't work with $\|\cdot\|_{\widehat{A}_{p,\infty}}$, and we need to introduce a new constant for weights in $\widehat{A}_{p,\infty}$.

Definition 4.1.5. Given $1 \leq p < \infty$, and $w \in \widehat{A}_{p,\infty}$, we define the constant

$$[\![w]\!]_{\widehat{A}_{p,\infty}} := \inf_{N\geq 1} N\|w\|_{\widehat{A}_{p,N}}.$$

We can see that $[\![w]\!]_{\widehat{A}_{1,\infty}} = \|w\|_{\widehat{A}_{1,\infty}} = [w]_{A_1}$, and in general, $\|w\|_{\widehat{A}_{p,\infty}} \leq [\![w]\!]_{\widehat{A}_{p,\infty}}$. Moreover, $[\![w]\!]_{\widehat{A}_{p,\infty}} < \infty$ if, and only if $\|w\|_{\widehat{A}_{p,\infty}} < \infty$, but we don't know if there exists an increasing function $\psi : [1,\infty) \longrightarrow [0,\infty)$ such that $[\![w]\!]_{\widehat{A}_{p,\infty}} \leq \psi(\|w\|_{\widehat{A}_{p,\infty}})$.

We can now prove that for $p \geq 1$, $\widehat{A}_{p,\infty} \subseteq A_p^{\mathcal{R}}$.

Theorem 4.1.6. *Given $1 \leq p < \infty$, there exists a constant $C > 0$, depending only on p and the dimension n, such that for every $N \geq 1$, and every weight $w \in \widehat{A}_{p,N}$,*

$$[w]_{A_p^{\mathcal{R}}} \leq CN\|w\|_{\widehat{A}_{p,N}}. \tag{4.1.4}$$

In particular, if $w \in \widehat{A}_{p,\infty}$, then $w \in A_p^{\mathcal{R}}$, and

$$[w]_{A_p^{\mathcal{R}}} \leq C[\![w]\!]_{\widehat{A}_{p,\infty}}.$$

Proof. Observe that if $p = 1$, then the result is true for any $C \geq 1$, so we will assume that $p > 1$.

For $N = 1$, if $w \in \widehat{A}_{p,1}$, then we can find a locally integrable function f, and a weight $u \in A_1$ such that $w = (Mf)^{1-p}u$. It was proved in [9, Corollary 2.8] that $[(Mf)^{1-p}u]_{A_p^{\mathcal{R}}} \leq c_{n,p}[u]_{A_1}^{1/p}$, and taking the infimum over all such weights $u \in A_1$, we get that $[w]_{A_p^{\mathcal{R}}} \leq c_{n,p}\|w\|_{\widehat{A}_{p,1}}$.

For $N \geq 2$, if $w \in \widehat{A}_{p,N}$, then we can find locally integrable functions $f_1, \dots, f_N$, a weight $u \in A_1$, and real values $0 < \theta_1, \dots, \theta_N \leq 1$, with $\sum_{i=1}^N \theta_i = 1$, such that $w = \left(\prod_{i=1}^N (Mf_i)^{\theta_i}\right)^{1-p} u$. We will proceed by applying Lemma 4.1.3 iteratively $N-1$ times. Let us assume that we are performing the kth iteration, with $1 \leq k \leq N-1$, and that all the previous iterations are already done. We choose the weights

$$w_1^{(k)} := \left(\prod_{i=1}^k (Mf_i)^{\frac{\theta_i}{1-\sum_{\ell=k+1}^N \theta_\ell}}\right)^{1-p} u, \quad w_2^{(k)} := (Mf_{k+1})^{1-p}u,$$

and the exponent

$$\theta^{(k)} := \frac{\theta_{k+1}}{1 - \sum_{\ell=k+2}^N \theta_\ell}.$$

Observe that $w_2^{(k)} \in \widehat{A}_{p,1}$, and we already know that $\widehat{A}_{p,1} \subseteq A_p^{\mathcal{R}}$, so in virtue of Remark 5.2.3 and the case $N = 1$, we have that for every measurable set E,

$$\begin{aligned}\|M(\chi_E)\|_{L^{p,\infty}(w_2^{(k)})} &\leq 2^n 24^{n/p} [w_2^{(k)}]_{A_p^{\mathcal{R}}} \|\chi_E\|_{L^{p,1}(w_2^{(k)})} \\ &\leq 2^n 24^{n/p} c_{n,p} \|w_2^{(k)}\|_{\widehat{A}_{p,1}} \|\chi_E\|_{L^{p,1}(w_2^{(k)})} \\ &\leq 2^n 24^{n/p} c_{n,p} [u]_{A_1}^{1/p} \|\chi_E\|_{L^{p,1}(w_2^{(k)})}.\end{aligned} \tag{4.1.5}$$

Similarly, if $k = 1$, then $w_1^{(k)} = (Mf_1)^{1-p}u \in \widehat{A}_{p,1}$, and we also have that for every measurable set E,

$$\|M(\chi_E)\|_{L^{p,\infty}(w_1^{(k)})} \leq 2^n 24^{n/p} c_{n,p} [u]_{A_1}^{1/p} \|\chi_E\|_{L^{p,1}(w_1^{(k)})}. \tag{4.1.6}$$

If $k > 1$, we know from the previous iterations that for every measurable set E,

$$\|M(\chi_E)\|_{L^{p,\infty}(w_1^{(k)})} \leq 2^n 24^{n/p} c_{n,p} k [u]_{A_1}^{1/p} \|\chi_E\|_{L^{p,1}(w_1^{(k)})}. \tag{4.1.7}$$

In virtue of (4.1.5), (4.1.6), and (4.1.7), if we apply Lemma 4.1.3 for the kth time, with exponent $\theta^{(k)}$ and weights $w_1^{(k)}$ and $w_2^{(k)}$, then we get that for every measurable set E,

$$\|M(\chi_E)\|_{L^{p,\infty}(w_1^{(k+1)})} \leq 2^n 24^{n/p} c_{n,p} (k+1) [u]_{A_1}^{1/p} \|\chi_E\|_{L^{p,1}(w_1^{(k+1)})},$$

because $\left(w_1^{(k)}\right)^{1-\theta^{(k)}}\left(w_2^{(k)}\right)^{\theta^{(k)}} = w_1^{(k+1)}$. At the end of the iteration process, $k = N-1$, and we have that for every measurable set E,

$$\|M(\chi_E)\|_{L^{p,\infty}(w)} \leq 2^n 24^{n/p} c_{n,p} N [u]_{A_1}^{1/p} \|\chi_E\|_{L^{p,1}(w)}, \tag{4.1.8}$$

since $w_1^{(N)} = w$.

Now, we apply Theorem 5.2.6 to deduce from (4.1.8) that

$$[w]_{A_p^{\mathcal{R}}} \leq \|w\|_{A_p^{\mathcal{R}}} \leq 2^n 24^{n/p} p c_{n,p} N [u]_{A_1}^{1/p},$$

and taking the infimum over all suitable representations of w, we get that

$$[w]_{A_p^{\mathcal{R}}} \leq 2^n 24^{n/p} p c_{n,p} N \|w\|_{\widehat{A}_{p,N}},$$

and hence, (4.1.4) holds taking $C = 2^n 24^{n/p} p c_{n,p}$.

Finally, given $w \in \widehat{A}_{p,\infty}$, we have that

$$[w]_{A_p^{\mathcal{R}}} \leq C \inf_{N \geq 1 : w \in \widehat{A}_{p,N}} N \|w\|_{\widehat{A}_{p,N}} = C [\![w]\!]_{\widehat{A}_{p,\infty}},$$

because if $N \geq 1$ is such that $w \notin \widehat{A}_{p,N}$, then $\|w\|_{\widehat{A}_{p,N}} = \inf \emptyset = \infty$. □

The following result allows us to construct $A_p^{\mathcal{R}}$ weights.

Lemma 4.1.7. *Given $1 \leq p < \infty$, $0 < \theta_1, \dots, \theta_m \leq 1$, with $\theta_1 + \dots + \theta_m = 1$, and weights $w_1, \dots, w_m \in A_p^{\mathcal{R}}$, the weight $w = w_1^{\theta_1} \dots w_m^{\theta_m}$ is in $A_p^{\mathcal{R}}$, and*

$$[w]_{A_p^{\mathcal{R}}} \leq C \sum_{i=1}^m [w_i]_{A_p^{\mathcal{R}}}.$$

Proof. If $m = 1$, there is nothing to prove, so we may assume that $m > 1$.

If $p = 1$, it follows from Hölder's inequality that

$$[w]_{A_1} \leq \prod_{i=1}^m [w_i]_{A_1}^{\theta_i} \leq \max_{i=1,\dots,m} \{[w_i]_{A_1}\} \leq \sum_{i=1}^m [w_i]_{A_1}.$$

If $p > 1$, it follows from Remark 5.2.3 that for $i = 1, \dots, m$, and every measurable set E,

$$\|M(\chi_E)\|_{L^{p,\infty}(w_i)} \leq 2^n 24^{n/p} [w_i]_{A_p^{\mathcal{R}}} \|\chi_E\|_{L^{p,1}(w_i)}. \tag{4.1.9}$$

We now proceed by applying Lemma 4.1.3 iteratively $m-1$ times. Let us assume that we are performing the kth iteration, with $1 \leq k \leq m-1$, and that all the previous iterations are already done. We choose the weights

$$w_1^{(k)} := \prod_{i=1}^k w_i^{\frac{\theta_i}{1-\sum_{\ell=k+1}^m \theta_\ell}}, \quad w_2^{(k)} := w_{k+1},$$

and the exponent

$$\theta^{(k)} := \frac{\theta_{k+1}}{1 - \sum_{\ell=k+2}^{m} \theta_\ell}.$$

In virtue of (4.1.9), if we apply Lemma 4.1.3 for the kth time, with exponent $\theta^{(k)}$ and weights $w_1^{(k)}$ and $w_2^{(k)}$, then we get that for every measurable set E,

$$\|M(\chi_E)\|_{L^{p,\infty}(w_1^{(k+1)})} \leq 2^n 24^{n/p} \left(\sum_{i=1}^{k+1} [w_i]_{A_p^{\mathcal{R}}} \right) \|\chi_E\|_{L^{p,1}(w_1^{(k+1)})},$$

since $w_1^{(1)} = w_1$, $w_2^{(1)} = w_2$, and $\left(w_1^{(k)}\right)^{1-\theta^{(k)}} \left(w_2^{(k)}\right)^{\theta^{(k)}} = w_1^{(k+1)}$. In particular, for $k = m-1$, $w_1^{(m)} = w$, and we conclude that

$$\|M(\chi_E)\|_{L^{p,\infty}(w)} \leq 2^n 24^{n/p} \left(\sum_{i=1}^{m} [w_i]_{A_p^{\mathcal{R}}} \right) \|\chi_E\|_{L^{p,1}(w)}.$$

Finally, applying Theorem 5.2.6, we obtain that

$$[w]_{A_p^{\mathcal{R}}} \leq 2^n 24^{n/p} p \sum_{i=1}^{m} [w_i]_{A_p^{\mathcal{R}}},$$

and the desired result follows, with $C = 2^n 24^{n/p} p$. □

Remark 4.1.8. For $p > 1$, we don't know if $[w]_{A_p^{\mathcal{R}}} \leq C_{m,n,p,\theta_1,\dots,\theta_m} \prod_{i=1}^{m} [w_i]_{A_p^{\mathcal{R}}}^{\theta_i}$.

The next lemma allows us to construct nice weights.

Lemma 4.1.9. *Let $1 \leq q \leq p$, and let w be a weight. For a measurable function $h \in L^1_{loc}(\mathbb{R}^n)$, let $v = (Mh)^{q-p} w$. If $w \in \widehat{A}_{q,N}$, then $v \in \widehat{A}_{p,N+1}$, and*

$$\|v\|_{\widehat{A}_{p,N+1}} \leq C \|w\|_{\widehat{A}_{q,N}}^{q/p}, \tag{4.1.10}$$

with C independent of h. In particular, if $w \in \widehat{A}_{q,\infty}$, then $v \in \widehat{A}_{p,\infty}$, and

$$[\![v]\!]_{\widehat{A}_{p,\infty}} \leq 2C [\![w]\!]_{\widehat{A}_{q,\infty}}. \tag{4.1.11}$$

Proof. For a weight $w \in \widehat{A}_{q,N}$, we can find measurable functions $h_1, \dots, h_N \in L^1_{loc}(\mathbb{R}^n)$, parameters $\theta_1, \dots, \theta_N \in (0,1]$, with $\theta_1 + \dots + \theta_N = 1$, and a weight $u \in A_1$ such that $w = \left(\prod_{i=1}^{N} (Mh_i)^{\theta_i} \right)^{1-q} u$, with $[u]_{A_1}^{1/q} \leq 2 \|w\|_{\widehat{A}_{q,N}}$. Note that if $p = 1$, then $q = 1$, and $v = w = u$, so (4.1.10) holds for every $C \geq 1$. If $p > 1$, then

$$v = \left((Mh)^{\frac{q-p}{1-p}} (Mh_1)^{\theta_1 \frac{1-q}{1-p}} \dots (Mh_N)^{\theta_N \frac{1-q}{1-p}} \right)^{1-p} u,$$

and since $\frac{q-p}{1-p} + (\theta_1 + \cdots + \theta_N)\frac{1-q}{1-p} = 1$, we have that $v \in \widehat{A}_{p,N+1}$, with

$$\|v\|_{\widehat{A}_{p,N+1}} \leq [u]_{A_1}^{1/p} \leq 2^{q/p} \|w\|_{\widehat{A}_{q,N}}^{q/p},$$

and (4.1.10) follows, with $C = 2^{q/p}$.

Finally, if $w \in \widehat{A}_{q,\infty}$, we can find a natural number $N \geq 1$ such that $w \in \widehat{A}_{q,N}$, and in virtue of (4.1.10), we get that

$$[\![v]\!]_{\widehat{A}_{p,\infty}} \leq (N+1) \|v\|_{\widehat{A}_{p,N+1}} \leq 2CN \|w\|_{\widehat{A}_{q,N}}^{q/p} \leq 2CN \|w\|_{\widehat{A}_{q,N}},$$

and taking the infimum over all such $N \geq 1$, we obtain (4.1.11). □

The next result also allows us to construct nice weights. It is an extension of Lemma 3.1.4, and the proof is similar.

Lemma 4.1.10. *Let $1 < p < q$, and $1 \leq N \in \mathbb{N}$, and let $w \in \widehat{A}_{q,N}$. For a measurable function $h \in L^1_{loc}(\mathbb{R}^n)$, let $v = w^{\frac{p-1}{q-1}}(Mh)^{\frac{q-p}{q-1}}$. Then, $v \in \widehat{A}_{p,N}$, and*

$$\|v\|_{\widehat{A}_{p,N}} \leq C\|w\|_{\widehat{A}_{q,N}}^{q/p}, \tag{4.1.12}$$

with C independent of h. In particular, if $w \in \widehat{A}_{q,\infty}$, then $v \in \widehat{A}_{p,\infty}$, and

$$[\![v]\!]_{\widehat{A}_{p,\infty}} \leq C[\![w]\!]_{\widehat{A}_{q,\infty}}^{q/p}. \tag{4.1.13}$$

Proof. For a weight $w \in \widehat{A}_{q,N}$, we can find measurable functions $h_1, \ldots, h_N \in L^1_{loc}(\mathbb{R}^n)$, parameters $\theta_1, \ldots, \theta_N \in (0,1]$, with $\theta_1 + \cdots + \theta_N = 1$, and a weight $u \in A_1$ such that $w = \left(\prod_{i=1}^N (Mh_i)^{\theta_i}\right)^{1-q} u$, with $[u]_{A_1}^{1/q} \leq 2\|w\|_{\widehat{A}_{q,N}}$. Note that

$$v = \left(\prod_{i=1}^N (Mh_i)^{\theta_i}\right)^{1-p} u^{\frac{p-1}{q-1}}(Mh)^{\frac{q-p}{q-1}} =: \left(\prod_{i=1}^N (Mh_i)^{\theta_i}\right)^{1-p} \widetilde{u}.$$

Applying [14, Lemma 2.12], we see that $\widetilde{u} \in A_1$, with $[\widetilde{u}]_{A_1} \leq c[u]_{A_1}$, and c independent of h. Hence, $v \in \widehat{A}_{p,N}$, with

$$\|v\|_{\widehat{A}_{p,N}} \leq [\widetilde{u}]_{A_1}^{1/p} \leq c^{1/p}[u]_{A_1}^{1/p} \leq 2^{q/p}c^{1/p}\|w\|_{\widehat{A}_{q,N}}^{q/p},$$

and (4.1.12) holds, with $C = 2^{q/p}c^{1/p}$.

Finally, if $w \in \widehat{A}_{q,\infty}$, we can find a natural number $N \geq 1$ such that $w \in \widehat{A}_{q,N}$, and in virtue of (4.1.12), we get that

$$[\![v]\!]_{\widehat{A}_{p,\infty}} \leq N \|v\|_{\widehat{A}_{p,N}} \leq CN \|w\|_{\widehat{A}_{q,N}}^{q/p} \leq C(N \|w\|_{\widehat{A}_{q,N}})^{q/p},$$

and taking the infimum over all such $N \geq 1$, we obtain (4.1.13). □

The following lemma gives us information about certain weights.

Lemma 4.1.11. *Let $1 \leq q_1, \dots, q_m < \infty$, and $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$. Let $1 \leq N_1, \dots, N_m \in \mathbb{N}$. Choose weights $w_1 \in \widehat{A}_{q_1,N_1}, \dots, w_m \in \widehat{A}_{q_m,N_m}$, and take $w = w_1^{q/q_1} \dots w_m^{q/q_m}$. Then, $w \in A^{\mathcal{R}}_{mq}$, and*

$$[w]_{A^{\mathcal{R}}_{mq}} \leq C(N_1 + \cdots + N_m) \prod_{i=1}^{m} \|w_i\|^{1/m}_{\widehat{A}_{q_i,N_i}}. \tag{4.1.14}$$

Alternatively, if $w_1 \in \widehat{A}_{q_1,\infty}, \dots, w_m \in \widehat{A}_{q_m,\infty}$, then $w \in A^{\mathcal{R}}_{\max\{q_1,\dots,q_m\}}$, and

$$[w]_{A^{\mathcal{R}}_{\max\{q_1,\dots,q_m\}}} \leq \widetilde{C} \sum_{i=1}^{m} [\![w_i]\!]_{\widehat{A}_{q_i,\infty}}.$$

Proof. For $i = 1, \dots, m$, $w_i \in \widehat{A}_{q_i,N_i}$, and we can find functions $h^i_1, \dots, h^i_{N_i} \in L^1_{loc}(\mathbb{R}^n)$, parameters $\theta^i_1, \dots, \theta^i_{N_i} \in (0,1]$, with $\theta^i_1 + \cdots + \theta^i_{N_i} = 1$, and a weight $u_i \in A_1$ such that $w_i = \left(\prod_{j=1}^{N_i} (Mh^i_j)^{\theta^i_j}\right)^{1-q_i} u_i$, with $[u_i]^{1/q_i}_{A_1} \leq 2\|w_i\|_{\widehat{A}_{q_i,N_i}}$.

If $mq = 1$, then $q_1 = \cdots = q_m = 1$, so $w = u_1^{1/m} \dots u_m^{1/m} \in A_1$, and (4.1.14) holds. If $mq > 1$, then

$$w = \left(\prod_{i=1}^{m} \left(\prod_{j=1}^{N_i} (Mh^i_j)^{\theta^i_j} \right)^{\frac{q}{q_i} \frac{1-q_i}{1-mq}} \right)^{1-mq} u,$$

with $u = u_1^{q/q_1} \dots u_m^{q/q_m} \in A_1$, and since

$$\sum_{i=1}^{m} \sum_{j=1}^{N_i} \theta^i_j \frac{q}{q_i} \frac{1-q_i}{1-mq} = \sum_{i=1}^{m} \frac{q}{q_i} \frac{1-q_i}{1-mq} = 1,$$

we have that $w \in \widehat{A}_{mq,N_1+\cdots+N_m}$, with

$$\|w\|_{\widehat{A}_{mq,N_1+\cdots+N_m}} \leq [u]^{\frac{1}{mq}}_{A_1} \leq \prod_{i=1}^{m} [u_i]^{\frac{1}{mq_i}}_{A_1} \leq 2 \prod_{i=1}^{m} \|w_i\|^{1/m}_{\widehat{A}_{q_i,N_i}},$$

and (4.1.14) follows from Theorem 4.1.6.

Finally, for $i = 1, \dots, m$, $\widehat{A}_{q_i,\infty} \subseteq A^{\mathcal{R}}_{q_i} \subseteq A^{\mathcal{R}}_{\max\{q_1,\dots,q_m\}}$, and in virtue of Lemma 4.1.7, (2.1.2), and Theorem 4.1.6, we get that

$$[w]_{A^{\mathcal{R}}_{\max\{q_1,\dots,q_m\}}} \leq C_1 \sum_{i=1}^{m} [w_i]_{A^{\mathcal{R}}_{\max\{q_1,\dots,q_m\}}} \leq C_2 \sum_{i=1}^{m} [w_i]_{A^{\mathcal{R}}_{q_i}} \leq \widetilde{C} \sum_{i=1}^{m} [\![w_i]\!]_{\widehat{A}_{q_i,\infty}}.$$

□

4.2 Main Results on Restricted Weak Type Extrapolation

In this section, we present our theorems on multi-variable restricted weak type extrapolation. To prove them, we build upon ideas introduced in the previous chapter.

4.2.1 Downwards Extrapolation Theorems

The first result that we prove allows us to fix the exponents $p_2, \dots, p_m$ and decrease the first exponent p_1 down to q_1 exploiting the $\widehat{A}_{q_1,\infty}$ condition on the weight w_1. We include a pictorial representation of this scheme in Figure 4.1. Such scheme is a multi-variable restricted weak type version of both Theorem 3.3.6 and Theorem 3.3.12.

Theorem 4.2.1. *Given measurable functions $f_1, \dots, f_m$, and g, suppose that for some exponents $1 \leq p_1, \dots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, and all weights $v_i \in \widehat{A}_{p_i,\infty}$, $i = 1, \dots, m$,*

$$\|g\|_{L^{p,\infty}(v_1^{p/p_1} \dots v_m^{p/p_m})} \leq \varphi([\![v_1]\!]_{\widehat{A}_{p_1,\infty}}, \dots, [\![v_m]\!]_{\widehat{A}_{p_m,\infty}}) \prod_{i=1}^{m} \|f_i\|_{L^{p_i,1}(v_i)}, \tag{4.2.1}$$

where $\varphi : [1,\infty)^m \longrightarrow [0,\infty)$ is a function increasing in each variable. Then, for every exponent $1 \leq q_1 \leq p_1$, and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m}$, and all weights $w_1 \in \widehat{A}_{q_1,\infty}$ and $w_i \in \widehat{A}_{p_i,\infty}$, $i = 2, \dots, m$,

$$\begin{aligned}\|g\|_{L^{q,\infty}(w_1^{q/q_1} w_2^{q/p_2} \dots w_m^{q/p_m})} &\leq \Phi([\![w_1]\!]_{\widehat{A}_{q_1,\infty}}, [\![w_2]\!]_{\widehat{A}_{p_2,\infty}}, \dots, [\![w_m]\!]_{\widehat{A}_{p_m,\infty}}) \\ &\quad \times \|f_1\|_{L^{q_1,\frac{q_1}{p_1}}(w_1)} \prod_{i=2}^{m} \|f_i\|_{L^{p_i,1}(w_i)},\end{aligned} \tag{4.2.2}$$

where $\Phi : [1,\infty)^m \longrightarrow [0,\infty)$ is a function increasing in each variable.

Proof. We will follow the steps of the proof of Theorem 3.2.4. Note that if $q_1 = p_1$, then there is nothing to prove, so we may assume that $q_1 < p_1$.

Pick weights $w_1 \in \widehat{A}_{q_1,\infty}$ and $w_i \in \widehat{A}_{p_i,\infty}$, $i = 2, \dots, m$, and let $w = w_1^{q/q_1} w_2^{q/p_2} \dots w_m^{q/p_m}$. We may assume that the quantity $\|f_1\|_{L^{q_1,\frac{q_1}{p_1}}(w_1)} < \infty$. In particular, f_1 is locally integrable (see Lemma 3.1.2). Fix $y, \gamma > 0$, and take

$$\mathscr{X} := (Mf_1)^{q_1/q} w_1^{\frac{1}{p_2} + \cdots + \frac{1}{p_m}} w_2^{-\frac{1}{p_2}} \dots w_m^{-\frac{1}{p_m}}.$$

We have that

$$
\begin{aligned}
\lambda_g^w(y) &= \int_{\{|g|>y,\, \mathscr{Z}>\gamma y\}} w + \int_{\{|g|>y,\, \mathscr{Z}\leq \gamma y\}} w \\
&\leq \lambda_{\mathscr{Z}}^w(\gamma y) + \int_{\{|g|>y\}} \left(\frac{\gamma y}{\mathscr{Z}}\right)^{p-q} w =: I + II.
\end{aligned}
\tag{4.2.3}
$$

To estimate the term I in (4.2.3), we have that

$$
I = \frac{(\gamma y)^q}{(\gamma y)^q}\lambda_{\mathscr{Z}}^w(\gamma y) \leq \frac{1}{(\gamma y)^q}\|\mathscr{Z}\|_{L^{q,\infty}(w)}^q = \frac{1}{(\gamma y)^q}\left\|\frac{Mf_1}{W}\right\|_{L^{q_1,\infty}(w_1 W^{q_1})}^{q_1}, \tag{4.2.4}
$$

with

$$
W := \left(w_1^{-\frac{1}{p_2}-\dots-\frac{1}{p_m}} w_2^{1/p_2}\dots w_m^{1/p_m}\right)^{q/q_1} = \left(\frac{w}{w_1}\right)^{1/q_1}. \tag{4.2.5}
$$

Note that $W \in A_\infty$. Indeed, if $m = 1$, then $W = 1$, and if $m > 1$, for $\varepsilon := \frac{q_1 q}{q_1 - q} > 0$, $\widehat{A}_{q_1,\infty} \subseteq A_{q_1+\varepsilon}$, so in virtue of Lemma 3.1.5, for $i = 2, \dots, m$, $W_i := \left(\frac{w_i}{w_1}\right)^{\frac{1}{q_1+\varepsilon}} \in A_\infty$, and $W = W_2^{\theta_2}\dots W_m^{\theta_m}$, where $\theta_i := \frac{q}{q_1 p_i}(q_1+\varepsilon) \in (0,1]$, and $\theta_2 + \dots + \theta_m = \frac{q}{q_1}(q_1+\varepsilon)\left(\frac{1}{q} - \frac{1}{q_1}\right) = 1$.

Applying Lemma 4.1.11, we get that $w \in A^{\mathcal{R}}_{\max\{q_1,p_2,\dots,p_m\}}$, with

$$
[w]_{A^{\mathcal{R}}_{\max\{q_1,p_2,\dots,p_m\}}} \leq \widetilde{C}\left([\![w_1]\!]_{\widehat{A}_{q_1,\infty}} + \sum_{i=2}^m [\![w_i]\!]_{\widehat{A}_{p_i,\infty}}\right) =: \psi_{w_1,\dots,w_m}, \tag{4.2.6}
$$

so in virtue of Theorem 2.3.8, Lemma 2.3.10, Theorem 4.1.6, and (2.1.1), we deduce that

$$
\begin{aligned}
\left\|\frac{Mf_1}{W}\right\|_{L^{q_1,\infty}(w_1 W^{q_1})} &\leq \phi([w_1]_{A^{\mathcal{R}}_{q_1}}, [w]_{A^{\mathcal{R}}_{\max\{q_1,p_2,\dots,p_m\}}})\|f_1\|_{L^{q_1,1}(w_1)} \\
&\leq \phi(c[\![w_1]\!]_{\widehat{A}_{q_1,\infty}}, \psi_{w_1,\dots,w_m})\|f_1\|_{L^{q_1,1}(w_1)} \\
&\leq p_1^{1-\frac{p_1}{q_1}}\phi(c[\![w_1]\!]_{\widehat{A}_{q_1,\infty}}, \psi_{w_1,\dots,w_m})\|f_1\|_{L^{q_1,\frac{q_1}{p_1}}(w_1)} \\
&=: p_1^{1-\frac{p_1}{q_1}}\phi_{w_1,\dots,w_m}\|f_1\|_{L^{q_1,\frac{q_1}{p_1}}(w_1)},
\end{aligned}
\tag{4.2.7}
$$

and combining the estimates (4.2.4) and (4.2.7), we obtain that

$$
I \leq \frac{1}{(\gamma y)^q} p_1^{q_1-p_1}\phi_{w_1,\dots,w_m}^{q_1}\|f_1\|_{L^{q_1,\frac{q_1}{p_1}}(w_1)}^{q_1}. \tag{4.2.8}
$$

We proceed to estimate the term II in (4.2.3). Take $v_1 := (Mf_1)^{q_1-p_1}w_1$, and for $i = 2, \dots, m$, take $v_i := w_i$. Since $w_1 \in \widehat{A}_{q_1,\infty}$, it follows from

Lemma 4.1.9 that $v_1 \in \widehat{A}_{p_1,\infty}$, with

$$[\![v_1]\!]_{\widehat{A}_{p_1,\infty}} \leq C[\![w_1]\!]_{\widehat{A}_{q_1,\infty}}. \tag{4.2.9}$$

Observe that

$$\begin{aligned}\mathscr{Z}^{q-p}w &= (Mf_1)^{\frac{q_1}{q}(q-p)} w_1^{(\frac{1}{p_2}+\cdots+\frac{1}{p_m})(q-p)+\frac{q}{q_1}} w_2^{\frac{p-q}{p_2}+\frac{q}{p_2}} \dots w_m^{\frac{p-q}{p_m}+\frac{q}{p_m}} \\ &= (Mf_1)^{\frac{p}{p_1}(q_1-p_1)} w_1^{p/p_1} w_2^{p/p_2} \dots w_m^{p/p_m} = v_1^{p/p_1} \dots v_m^{p/p_m},\end{aligned}$$

so by (4.2.1), (4.2.9), and the monotonicity of φ, we get that

$$\begin{aligned}II &= \frac{(\gamma y)^p}{(\gamma y)^q} \int_{\{|g|>y\}} v_1^{p/p_1} \dots v_m^{p/p_m} \leq \frac{\gamma^p}{(\gamma y)^q} \|g\|^p_{L^{p,\infty}(v_1^{p/p_1} \dots v_m^{p/p_m})} \\ &\leq \frac{\gamma^p}{(\gamma y)^q} \varphi(C[\![w_1]\!]_{\widehat{A}_{q_1,\infty}}, [\![w_2]\!]_{\widehat{A}_{p_2,\infty}}, \dots, [\![w_m]\!]_{\widehat{A}_{p_m,\infty}})^p \\ &\times \|f_1\|^p_{L^{p_1,1}(v_1)} \prod_{i=2}^m \|f_i\|^p_{L^{p_i,1}(w_i)} =: \frac{\gamma^p}{(\gamma y)^q} \varphi^p_{w_1,\dots,w_m} \|f_1\|^p_{L^{p_1,1}(v_1)} \prod_{i=2}^m \|f_i\|^p_{L^{p_i,1}(w_i)},\end{aligned} \tag{4.2.10}$$

and arguing as we did in (3.2.7), we have that

$$\|f_1\|_{L^{p_1,1}(v_1)} \leq \frac{p_1}{q_1} \|f_1\|^{q_1/p_1}_{L^{q_1,\frac{q_1}{p_1}}(w_1)}. \tag{4.2.11}$$

Combining the estimates (4.2.3), (4.2.8), (4.2.10), and (4.2.11), we conclude that

$$\begin{aligned}\lambda^w_g(y) &\leq \frac{1}{(\gamma y)^q} p_1^{q_1-p_1} \phi^{q_1}_{w_1,\dots,w_m} \|f_1\|^{q_1}_{L^{q_1,\frac{q_1}{p_1}}(w_1)} \\ &+ \frac{\gamma^p}{(\gamma y)^q} \left(\frac{p_1}{q_1}\right)^p \varphi^p_{w_1,\dots,w_m} \|f_1\|^{\frac{pq_1}{p_1}}_{L^{q_1,\frac{q_1}{p_1}}(w_1)} \prod_{i=2}^m \|f_i\|^p_{L^{p_i,1}(w_i)},\end{aligned}$$

and taking the infimum over all $\gamma > 0$, it follows from Lemma 3.1.1 that

$$\begin{aligned}y^q \lambda^w_g(y) &\leq \frac{p}{p-q} \left(\frac{p-q}{q}\right)^{q/p} \left(\frac{p_1}{q_1}\right)^q p_1^{q(2-\frac{q_1}{p_1}-\frac{p_1}{q_1})} \phi^{q(1-\frac{q_1}{p_1})}_{w_1,\dots,w_m} \varphi^q_{w_1,\dots,w_m} \\ &\times \|f_1\|^q_{L^{q_1,\frac{q_1}{p_1}}(w_1)} \prod_{i=2}^m \|f_i\|^q_{L^{p_i,1}(w_i)}.\end{aligned}$$

Finally, raising everything to the power $\frac{1}{q}$ in this last expression and taking the supremum over all $y > 0$, we see that (4.2.2) holds, with

$$\Phi([\![w_1]\!]_{\widehat{A}_{q_1,\infty}}, [\![w_2]\!]_{\widehat{A}_{p_2,\infty}}, \dots, [\![w_m]\!]_{\widehat{A}_{p_m,\infty}}) = p_1^{3-\frac{q_1}{p_1}-\frac{p_1}{q_1}} \frac{C_{p,q}}{q_1} \phi^{1-\frac{q_1}{p_1}}_{w_1,\dots,w_m} \varphi_{w_1,\dots,w_m}, \tag{4.2.12}$$

and $C_{p,q}$ as in (3.2.19). □

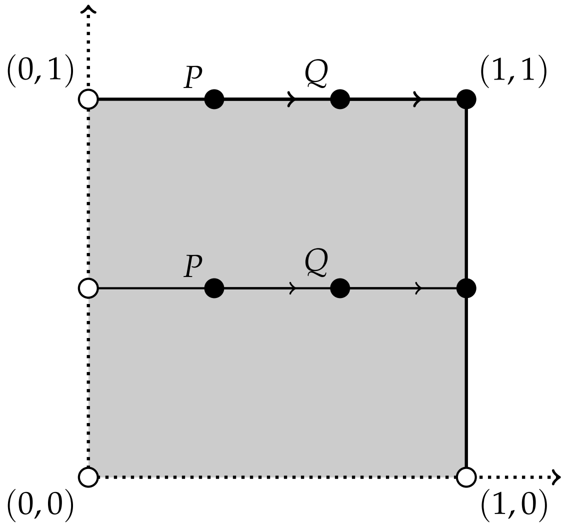

FIGURE 4.1: Pictorial representation of Theorem 4.2.1 and Theorem 4.2.5 for $m = 2$.

Observe that in Theorem 4.2.1 and its proof, for $i = 2, \dots, m$, the quantity $\|f_i\|_{L^{p_i,1}(w_i)}$ plays no role and can be replaced by $\|f_i\|_{L^{p_i,\alpha_i}(w_i)}$, with $\alpha_i > 0$. This fact allows us to iterate m times the argument in the proof of Theorem 4.2.1, one for each variable, to produce the following general downwards extrapolation scheme, depicted in Figure 4.2. Note that the monotonicity of the function Φ in (4.2.12) is of utmost importance for the iteration process. Such scheme generalizes and extends to the multi-variable case both Theorem 3.2.1 and Theorem 3.2.7, and produces multi-variable restricted weak type versions of Theorem 3.3.2, Theorem 3.3.6, Theorem 3.3.12, and Theorem 3.3.16.

Theorem 4.2.2. *Given measurable functions $f_1, \dots, f_m$, and g, suppose that for some exponents $1 \leq p_1, \dots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, and all weights $v_i \in \widehat{A}_{p_i,\infty}$, $i = 1, \dots, m$,*

$$\|g\|_{L^{p,\infty}(v_1^{p/p_1} \dots v_m^{p/p_m})} \leq \varphi([\![v_1]\!]_{\widehat{A}_{p_1,\infty}}, \dots, [\![v_m]\!]_{\widehat{A}_{p_m,\infty}}) \prod_{i=1}^m \|f_i\|_{L^{p_i,1}(v_i)}, \tag{4.2.13}$$

where $\varphi : [1,\infty)^m \longrightarrow [0,\infty)$ is a function increasing in each variable. Then, for all exponents $1 \leq q_1 \leq p_1, \dots, 1 \leq q_m \leq p_m$, $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, and all weights $w_i \in \widehat{A}_{q_i,\infty}$, $i = 1, \dots, m$,

$$\|g\|_{L^{q,\infty}(w_1^{q/q_1} \dots w_m^{q/q_m})} \leq \Phi([\![w_1]\!]_{\widehat{A}_{q_1,\infty}}, \dots, [\![w_m]\!]_{\widehat{A}_{q_m,\infty}}) \prod_{i=1}^m \|f_i\|_{L^{q_i,\frac{q_i}{p_i}}(w_i)}, \tag{4.2.14}$$

where $\Phi : [1,\infty)^m \longrightarrow [0,\infty)$ *is a function increasing in each variable, given by*

$$\Phi(t_1,\dots,t_m) = \left(\prod_{i=1}^m p_i^{3-\frac{q_i}{p_i}-\frac{p_i}{q_i}} \frac{C_{r_{i-1},r_i}}{q_i} \phi_i(c_i t_i, \widetilde{C}_i(t_1+\cdots+t_m))^{1-\frac{q_i}{p_i}}\right) \\ \times \varphi(C_1 t_1,\dots,C_m t_m),$$

with $r_0 = p$, *and for* $i = 1,\dots,m$, $\frac{1}{r_i} = \frac{1}{q_1}+\cdots+\frac{1}{q_i}+\frac{1}{p_{i+1}}+\cdots+\frac{1}{p_m}$, C_{r_{i-1},r_i} *as in* (3.2.19), *and* $\phi_i = \phi^n_{\max\{q_1,\dots,q_i,p_{i+1},\dots,p_m\},q_i}$ *as in Lemma* 2.3.10. *If* $q_i = p_i$, *then we can take* $\phi_i = 1$.

We have presented Theorem 4.2.2 in its general form, for $(m+1)$-tuples of functions $(f_1,\dots,f_m,g)$. We can deduce the corresponding extrapolation scheme for m-variable operators arguing as in the proof of Corollary 3.2.2. For convenience, we also provide a pictorial representation of it in Figure 4.2.

Corollary 4.2.3. *Let* T *be an* m*-variable operator defined for measurable functions. Suppose that for some exponents* $1 \le p_1,\dots,p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1}+\cdots+\frac{1}{p_m}$, *and all weights* $v_i \in \widehat{A}_{p_i,\infty}$, $i = 1,\dots,m$,

$$T : L^{p_1,1}(v_1)\times\cdots\times L^{p_m,1}(v_m) \longrightarrow L^{p,\infty}(v_1^{p/p_1}\dots v_m^{p/p_m}),$$

with constant bounded by $\varphi([\![v_1]\!]_{\widehat{A}_{p_1,\infty}},\dots,[\![v_m]\!]_{\widehat{A}_{p_m,\infty}})$ *as in* (4.2.13). *Then, for all exponents* $1 \le q_1 \le p_1,\dots,1 \le q_m \le p_m$, $\frac{1}{q} = \frac{1}{q_1}+\cdots+\frac{1}{q_m}$, *and all weights* $w_i \in \widehat{A}_{q_i,\infty}$, $i = 1,\dots,m$,

$$T : L^{q_1,\frac{q_1}{p_1}}(w_1)\times\cdots\times L^{q_m,\frac{q_m}{p_m}}(w_m) \longrightarrow L^{q,\infty}(w_1^{q/q_1}\dots w_m^{q/q_m}),$$

with constant bounded by $\Phi([\![w_1]\!]_{\widehat{A}_{q_1,\infty}},\dots,[\![w_m]\!]_{\widehat{A}_{q_m,\infty}})$ *as in* (4.2.14).

Remark 4.2.4. Observe that if the operator T is defined for characteristic functions of measurable sets $E_1,\dots,E_m \subseteq \mathbb{R}^n$, then under the hypotheses of Corollary 4.2.3, we deduce that

$$\left\|T(\chi_{E_1},\dots,\chi_{E_m})\right\|_{L^{q,\infty}(w_1^{q/q_1}\dots w_m^{q/q_m})} \le C\Phi([\![w_1]\!]_{\widehat{A}_{q_1,\infty}},\dots,[\![w_m]\!]_{\widehat{A}_{q_m,\infty}}) \\ \times w_1(E_1)^{1/q_1}\dots w_m(E_m)^{1/q_m},$$

with $C = p_1^{p_1/q_1}\dots p_m^{p_m/q_m}$, and hence, T is of weak type $(q_1,\dots,q_m,q)$ at least for characteristic functions.

Let us point out that if in Theorem 4.2.1 we replace $\widehat{A}_{p_1,\infty}$ by $\widehat{A}_{p_1,N_1+1}$, and $\widehat{A}_{q_1,\infty}$ by $\widehat{A}_{q_1,N_1}$, and for $i = 2,\dots,m$, we replace $\widehat{A}_{p_i,\infty}$ by $\widehat{A}_{p_i,N_i}$, with $1 \le N_1,\dots,N_m \in \mathbb{N}$, then we can replace (4.2.6) by

$$[w]_{A^{\mathcal{R}}_{\vec{mq}}} \le \widetilde{C}(N_1+\cdots+N_m)\|w_1\|^{1/m}_{\widehat{A}_{q_1,N_1}} \prod_{i=2}^m \|w_i\|^{1/m}_{\widehat{A}_{p_i,N_i}},$$

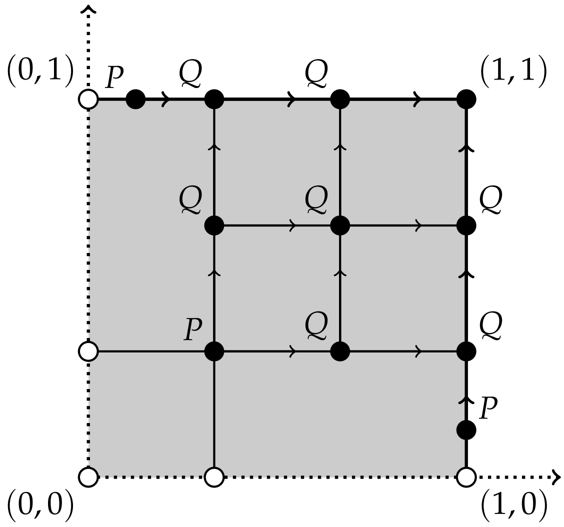

FIGURE 4.2: Pictorial representation of Theorem 4.2.2, Corollary 4.2.3, and Theorem 4.2.6 for $m = 2$.

and (4.2.9) by

$$\|v_1\|_{\widehat{A}_{p_1,N_1+1}} \leq C\|w_1\|_{\widehat{A}_{q_1,N_1}}^{q_1/p_1},$$

and use that

$$[w_1]_{A_{q_1}^{\mathcal{R}}} \leq cN_1\|w_1\|_{\widehat{A}_{q_1,N_1}}$$

in (4.2.7), and obtain the following variant of Theorem 4.2.1, depicted in Figure 4.1.

Theorem 4.2.5. *Given measurable functions $f_1, \dots, f_m$, and g, suppose that for some exponents $1 \leq p_1, \dots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, and $1 \leq N_1, \dots, N_m \in \mathbb{N}$, and all weights $v_1 \in \widehat{A}_{p_1,N_1+1}$, and $v_i \in \widehat{A}_{p_i,N_i}$, $i = 2, \dots, m$,*

$$\|g\|_{L^{p,\infty}(v_1^{p/p_1} \dots v_m^{p/p_m})} \leq \varphi(\|v_1\|_{\widehat{A}_{p_1,N_1+1}}, \|v_2\|_{\widehat{A}_{p_2,N_2}}, \dots, \|v_m\|_{\widehat{A}_{p_m,N_m}}) \times \prod_{i=1}^{m} \|f_i\|_{L^{p_i,1}(v_i)},$$

where $\varphi : [1,\infty)^m \longrightarrow [0,\infty)$ is a function increasing in each variable. Then, for every exponent $1 \leq q_1 \leq p_1$, and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m}$, and all weights $w_1 \in \widehat{A}_{q_1,N_1}$, and $w_i \in \widehat{A}_{p_i,N_i}$, $i = 2, \dots, m$,

$$\|g\|_{L^{q,\infty}(w_1^{q/q_1} w_2^{q/p_2} \dots w_m^{q/p_m})} \leq \Phi(\|w_1\|_{\widehat{A}_{q_1,N_1}}, \|w_2\|_{\widehat{A}_{p_2,N_2}}, \dots, \|w_m\|_{\widehat{A}_{p_m,N_m}}) \times \|f_1\|_{L^{q_1,\frac{q_1}{p_1}}(w_1)} \prod_{i=2}^{m} \|f_i\|_{L^{p_i,1}(w_i)},$$

where $\Phi : [1,\infty)^m \longrightarrow [0,\infty)$ *is a function increasing in each variable, given by*

$$\Phi(t_1,\dots,t_m) = p_1^{3-\frac{q_1}{p_1}-\frac{p_1}{q_1}}\frac{C_{p,q}}{q_1}\phi(cN_1t_1,\widetilde{C}(N_1+\cdots+N_m)t_1^{1/m}\dots t_m^{1/m})^{1-\frac{q_1}{p_1}}$$
$$\times\,\varphi(Ct_1^{q_1/p_1},t_2,\dots,t_m),$$

with $C_{p,q}$ *as in (3.2.19), and* $\phi=\phi^n_{mq,q_1}$ *as in Lemma 2.3.10. If* $q_1=p_1$*, then we can take* $\phi=1$.

Once again, in Theorem 4.2.5, for $i=2,\dots,m$, the quantity $\|f_i\|_{L^{p_i,1}(w_i)}$ plays no role and can be replaced by $\|f_i\|_{L^{p_i,\alpha_i}(w_i)}$, with $\alpha_i>0$. Hence, we can iterate m times Theorem 4.2.5, one for each variable, to produce the following alternative version of Theorem 4.2.2, depicted in Figure 4.2.

Theorem 4.2.6. *Given measurable functions* $f_1,\dots,f_m$*, and* g*, suppose that for some exponents* $1\le p_1,\dots,p_m<\infty$, $\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}$*, and* $1\le N_1,\dots,N_m\in\mathbb{N}$*, and all weights* $v_i\in\widehat{A}_{p_i,N_i+1}$, $i=1,\dots,m$,

$$\|g\|_{L^{p,\infty}(v_1^{p/p_1}\dots v_m^{p/p_m})}\le\varphi(\|v_1\|_{\widehat{A}_{p_1,N_1+1}},\dots,\|v_m\|_{\widehat{A}_{p_m,N_m+1}})\prod_{i=1}^m\|f_i\|_{L^{p_i,1}(v_i)},$$

where $\varphi:[1,\infty)^m\longrightarrow[0,\infty)$ *is a function increasing in each variable. Then, for all exponents* $1\le q_1\le p_1,\dots,1\le q_m\le p_m$, $\frac{1}{q}=\frac{1}{q_1}+\cdots+\frac{1}{q_m}$*, and all weights* $w_i\in\widehat{A}_{q_i,N_i}$, $i=1,\dots,m$,

$$\|g\|_{L^{q,\infty}(w_1^{q/q_1}\dots w_m^{q/q_m})}\le\Phi(\|w_1\|_{\widehat{A}_{q_1,N_1}},\dots,\|w_m\|_{\widehat{A}_{q_m,N_m}})\prod_{i=1}^m\|f_i\|_{L^{q_i,\frac{q_i}{p_i}}(w_i)},$$

where $\Phi : [1,\infty)^m \longrightarrow [0,\infty)$ *is a function increasing in each variable, given by*

$$\Phi(t_1,\dots,t_m)=\left(\prod_{i=1}^m p_i^{3-\frac{q_i}{p_i}-\frac{p_i}{q_i}}\frac{C_{r_{i-1},r_i}}{q_i}\psi_i(t_1,\dots,t_m)^{1-\frac{q_i}{p_i}}\right)$$
$$\times\,\varphi(C_1t_1^{q_1/p_1},\dots,C_mt_m^{q_m/p_m}),$$

and

$$\psi_i(t_1,\dots,t_m)=\phi_i(c_iN_it_i,\widetilde{C}_i(N_1+\cdots+N_m+m-i)t_1^{\frac{1}{m}}\dots t_i^{\frac{1}{m}}t_{i+1}^{\frac{q_{i+1}}{mp_{i+1}}}\dots t_m^{\frac{q_m}{mp_m}}),$$

with $r_0=p$*, and for* $i=1,\dots,m$, $\frac{1}{r_i}=\frac{1}{q_1}+\cdots+\frac{1}{q_i}+\frac{1}{p_{i+1}}+\cdots+\frac{1}{p_m}$, C_{r_{i-1},r_i} *as in (3.2.19), and* $\phi_i=\phi^n_{mr_i,q_i}$ *as in Lemma 2.3.10. If* $q_i=p_i$*, then we can take* $\psi_i=1$.

4.2.2 Upwards Extrapolation Theorems

The next result that we present allows us to increase all the exponents exploiting the $\widehat{A}_{p,\infty}$ condition on the weights involved, but we need to assume some technical hypotheses coming from Theorem 2.4.12. See Figure 3.3 for a pictorial representation of this extrapolation result when $m = 2$. Such result generalizes Theorem 3.2.10 and extends it to the multi-variable case, and produces a multi-variable restricted weak type version of Theorem 3.3.20 and Theorem 3.3.35.

Theorem 4.2.7. *Given measurable functions $f_1, \dots, f_m$, and g, suppose that for some exponents $1 \leq p_1, \dots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$, and all weights $v_i \in \widehat{A}_{p_i,\infty}$, $i = 1, \dots, m$,*

$$\|g\|_{L^{p,\infty}(v_1^{p/p_1} \dots v_m^{p/p_m})} \leq \varphi([\![v_1]\!]_{\widehat{A}_{p_1,\infty}}, \dots, [\![v_m]\!]_{\widehat{A}_{p_m,\infty}}) \prod_{i=1}^{m} \|f_i\|_{L^{p_i,1}(v_i)}, \tag{4.2.15}$$

where $\varphi : [1,\infty)^m \longrightarrow [0,\infty)$ is a function increasing in each variable. Given exponents $q_1 = p_1 \geq 1$ or $q_1 > p_1 > 1, \dots, q_m = p_m \geq 1$ or $q_m > p_m > 1$, and $\frac{1}{q} = \frac{1}{q_1} + \dots + \frac{1}{q_m}$, and weights $w_i \in \widehat{A}_{q_i,\infty}$, $i = 1, \dots, m$, and $w = w_1^{q/q_1} \dots w_m^{q/q_m}$, if for every $i = 1, \dots, m$ for which $q_i > p_i$, there exists $0 < \varepsilon_i \leq 1$ such that $wW_i^{-\varepsilon_i}$ is a weight, and $[W_i^{-\varepsilon_i}]_{RH_\infty(w)} < \infty$, with $W_i = \left(\frac{w}{w_i}\right)^{1/q_i}$, then

$$\|g\|_{L^{q,\infty}(w)} \leq \Phi_{\vec{\varepsilon},\vec{w}}([\![w_1]\!]_{\widehat{A}_{q_1,\infty}}, \dots, [\![w_m]\!]_{\widehat{A}_{q_m,\infty}}) \prod_{i=1}^{m} \|f_i\|_{L^{q_i,1}(w_i)}, \tag{4.2.16}$$

where $\Phi_{\vec{\varepsilon},\vec{w}} : [1,\infty)^m \longrightarrow [0,\infty)$ is a function that increases in each variable and depends on $\vec{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_m)$, and $\vec{w} = (w_1, \dots, w_m)$.

Proof. We will follow the steps of the proof of Theorem 3.2.10. Note that if $q_1 = p_1, \dots, q_m = p_m$, then there is nothing to prove. We first discuss the case when $1 < p_i < q_i$, $i = 1, \dots, m$. For $i = 1, \dots, m$, pick a weight $w_i \in \widehat{A}_{q_i,\infty}$, and write $w = w_1^{q/q_1} \dots w_m^{q/q_m}$. As usual, we may assume that $\|f_i\|_{L^{q_i,1}(w_i)} < \infty$. For every natural number $N \geq 1$, let $g_N := |g|\chi_{B(0,N)}$. Fix $N \geq 1$. We will prove (4.2.16) for the tuple $(f_1, \dots, f_m, g_N)$. Since $g_N \leq |g|$, we already know that (4.2.15) holds for $(f_1, \dots, f_m, g_N)$. Fix $y > 0$ such that $\lambda^w_{g_N}(y) \neq 0$. If no such y exists, then $\|g_N\|_{L^{q,\infty}(w)} = 0$ and we are done.

In order to apply (4.2.15), we want to find weights $v_1 \in \widehat{A}_{p_1,\infty}, \dots, v_m \in \widehat{A}_{p_m,\infty}$ such that for $v := v_1^{p/p_1} \dots v_m^{p/p_m}$, $\lambda^w_{g_N}(y) \leq \lambda^v_{g_N}(y)$. For $i = 1, \dots, m$, take

$$v_i := w_i^{\frac{p_i-1}{q_i-1}} \left(M(w_i^{1/q_i} w^{1/q_i'} \chi_{\{|g_N|>y\}}) \right)^{\frac{q_i-p_i}{q_i-1}}. \tag{4.2.17}$$

Note that if $q_i = p_i$, then $v_i = w_i$. Applying Lemma 4.1.10, we see that $v_i \in \widehat{A}_{p_i,\infty}$, with

$$[\![v_i]\!]_{\widehat{A}_{p_i,\infty}} \leq C_i [\![w_i]\!]^{q_i/p_i}_{\widehat{A}_{q_i,\infty}}, \tag{4.2.18}$$

and C_i independent of w_i, w, N, and y. Observe that

$$\begin{aligned}
v_1^{p/p_1} \dots v_m^{p/p_m} &\geq \left(\prod_{i=1}^m w_i^{\frac{p}{p_i}\frac{p_i-1}{q_i-1}+\frac{p}{p_iq_i}\frac{q_i-p_i}{q_i-1}} w^{\frac{p}{p_iq_i'}\frac{q_i-p_i}{q_i-1}} \right) \chi_{\{|g_N|>y\}} \\
&= \left(\prod_{i=1}^m w_i^{\frac{p}{p_i(q_i-1)}(p_i-\frac{p_i}{q_i})} \right) w^{\sum_{i=1}^m \frac{p}{p_iq_i}(q_i-p_i)} \chi_{\{|g_N|>y\}} \\
&= \left(\prod_{i=1}^m w_i^{\frac{p}{q_i-1}(1-\frac{1}{q_i})} \right) w^{\sum_{i=1}^m \frac{p}{p_i}-\frac{p}{q_i}} \chi_{\{|g_N|>y\}} \\
&= \left(\prod_{i=1}^m w_i^{p/q_i} \right) w^{1-\frac{p}{q}} \chi_{\{|g_N|>y\}} \\
&= \left(\prod_{i=1}^m w_i^{\frac{p}{q_i}+\frac{q}{q_i}(1-\frac{p}{q})} \right) \chi_{\{|g_N|>y\}} = w_1^{q/q_1} \dots w_m^{q/q_m} \chi_{\{|g_N|>y\}},
\end{aligned}$$

so (4.2.15) and (4.2.18) imply that

$$\begin{aligned}
\lambda_{g_N}^w(y) &= \int_{\{|g_N|>y\}} w_1^{q/q_1} \dots w_m^{q/q_m} \leq \int_{\{|g_N|>y\}} v_1^{p/p_1} \dots v_m^{p/p_m} = \lambda_{g_N}^v(y) \\
&\leq \frac{1}{y^p} \varphi([\![v_1]\!]_{\widehat{A}_{p_1,\infty}}, \dots, [\![v_m]\!]_{\widehat{A}_{p_m,\infty}})^p \prod_{i=1}^m \|f_i\|_{L^{p_i,1}(v_i)}^p \\
&\leq \frac{1}{y^p} \varphi(C_1[\![w_1]\!]_{\widehat{A}_{q_1,\infty}}^{q_1/p_1}, \dots, C_m[\![w_m]\!]_{\widehat{A}_{q_m,\infty}}^{q_m/p_m})^p \prod_{i=1}^m \|f_i\|_{L^{p_i,1}(v_i)}^p .
\end{aligned} \tag{4.2.19}$$

For $i = 1, \dots, m$, we want to replace $\|f_i\|_{L^{p_i,1}(v_i)}$ by $\|f_i\|_{L^{q_i,1}(w_i)}$ in (4.2.19). If $q_i = p_i$, then $\|f_i\|_{L^{p_i,1}(v_i)} = \|f_i\|_{L^{q_i,1}(w_i)}$. If $q_i > p_i$, then applying Hölder's inequality with exponent $\frac{q_i}{p_i} > 1$, we obtain that for every $t > 0$,

$$\begin{aligned}
\lambda_{f_i}^{v_i}(t) &= \int_{\{|f_i|>t\}} \left(\frac{M(w_i^{1/q_i} w^{1/q_i'} \chi_{\{|g_N|>y\}})}{w_i} \right)^{\frac{q_i-p_i}{q_i-1}} w_i \\
&\leq \|\chi_{\{|f_i|>t\}}\|_{L^{\frac{q_i}{p_i},1}(w_i)} \left\| \left(\frac{M(w_i^{1/q_i} w^{1/q_i'} \chi_{\{|g_N|>y\}})}{w_i} \right)^{\frac{q_i-p_i}{q_i-1}} \right\|_{L^{\frac{q_i}{q_i-p_i},\infty}(w_i)} \\
&= \frac{q_i}{p_i} w_i(\{|f_i| > t\})^{p_i/q_i} \left\| \frac{M(w_i^{1/q_i} w^{1/q_i'} \chi_{\{|g_N|>y\}})}{w_i} \right\|_{L^{q_i',\infty}(w_i)}^{\frac{q_i-p_i}{q_i-1}} .
\end{aligned}$$

Now, for $F := \chi_{\{|g_N|>y\}}$, $U_i := w_i \in A_{q_i}^{\mathcal{R}}$, and $W_i = \left(\frac{w}{w_i}\right)^{1/q_i} \in A_\infty$ (see (4.2.5)), we have that $U_i W_i^{q_i-1} = w_i^{1/q_i} w^{1/q_i'}$, and $U_i W_i^{q_i} = w \in A_\infty$, so

Theorem 2.4.12, Theorem 4.1.6, and Lemma 4.1.11 give us that

$$\begin{aligned}\left\|\frac{M(FU_iW_i^{q_i-1})}{U_i}\right\|_{L^{q_i',\infty}(U_i)} &= \left\|\frac{M(w_i^{1/q_i}w^{1/q_i'}\chi_{\{|g_N|>y\}})}{w_i}\right\|_{L^{q_i',\infty}(w_i)}\\ &\leq \phi_i([w_i]_{A_{q_i}^{\mathcal{R}}},[w]_{A^{\mathcal{R}}_{\max\{q_1,\dots,q_m\}}})\|F\|_{L^{q_i',1}(U_iW_i^{q_i})}\\ &\leq q_i'\phi_i(c_i[\![w_i]\!]_{\widehat{A}_{q_i,\infty}},\widetilde{C}\sum_{j=1}^m[\![w_j]\!]_{\widehat{A}_{q_j,\infty}})w(\{|g_N|>y\})^{1/q_i'}\\ &=: q_i'\phi_i w(\{|g_N|>y\})^{1/q_i'},\end{aligned} \tag{4.2.20}$$

so

$$\lambda_{f_i}^{v_i}(t)\leq\frac{q_i}{p_i}\left(q_i'\phi_i\right)^{\frac{q_i-p_i}{q_i-1}}w(\{|g_N|>y\})^{1-\frac{p_i}{q_i}}w_i(\{|f_i|>t\})^{p_i/q_i},$$

and hence,

$$\begin{aligned}\|f_i\|_{L^{p_i,1}(v_i)} &= p_i\int_0^\infty\lambda_{f_i}^{v_i}(t)^{1/p_i}dt\leq p_i\left(\frac{q_i}{p_i}\right)^{1/p_i}\left(q_i'\phi_i\right)^{\frac{1}{p_i}\frac{q_i-p_i}{q_i-1}}\\ &\quad\times w(\{|g_N|>y\})^{\frac{1}{p_i}-\frac{1}{q_i}}\int_0^\infty w_i(\{|f_i|>t\})^{1/q_i}dt\\ &=\left(\frac{p_i}{q_i}\right)^{1/p_i'}\left(q_i'\phi_i\right)^{\frac{1}{p_i}\frac{q_i-p_i}{q_i-1}}w(\{|g_N|>y\})^{\frac{1}{p_i}-\frac{1}{q_i}}\|f_i\|_{L^{q_i,1}(w_i)}.\end{aligned} \tag{4.2.21}$$

Combining the estimates (4.2.19) and (4.2.21), we have that

$$\lambda_{g_N}^w(y)\leq\frac{1}{y^p}\Phi_{\vec{\varepsilon},\vec{w}}([\![w_1]\!]_{\widehat{A}_{q_1,\infty}},\dots,[\![w_m]\!]_{\widehat{A}_{q_m,\infty}})^p\left(\prod_{i=1}^m\|f_i\|^p_{L^{q_i,1}(w_i)}\right)\lambda_{g_N}^w(y)^{1-\frac{p}{q}}, \tag{4.2.22}$$

with

$$\begin{aligned}\Phi_{\vec{\varepsilon},\vec{w}}([\![w_1]\!]_{\widehat{A}_{q_1,\infty}},\dots,[\![w_m]\!]_{\widehat{A}_{q_m,\infty}}) &= \left(\prod_{i=1}^m\left(\frac{p_i}{q_i}\right)^{1/p_i'}\left(q_i'\phi_i\right)^{\frac{1}{p_i}\frac{q_i-p_i}{q_i-1}}\right)\\ &\quad\times\varphi(C_1[\![w_1]\!]_{\widehat{A}_{q_1,\infty}}^{q_1/p_1},\dots,C_m[\![w_m]\!]_{\widehat{A}_{q_m,\infty}}^{q_m/p_m}).\end{aligned}$$

By our choice of y and g_N, $0<\lambda_{g_N}^w(y)\leq w(B(0,N))<\infty$, so we can divide by $\lambda_{g_N}^w(y)^{1-\frac{p}{q}}$ in (4.2.22) and raise everything to the power $\frac{1}{p}$, obtaining that

$$y\lambda_{g_N}^w(y)^{1/q}\leq\Phi_{\vec{\varepsilon},\vec{w}}([\![w_1]\!]_{\widehat{A}_{q_1,\infty}},\dots,[\![w_m]\!]_{\widehat{A}_{q_m,\infty}})\prod_{i=1}^m\|f_i\|_{L^{q_i,1}(w_i)},$$

and taking the supremum over all $y>0$, we deduce (4.2.16) for the tuple $(f_1,\dots,f_m,g_N)$, and the result for the tuple $(f_1,\dots,f_m,g)$ follows taking the supremum over all $N\geq 1$.

Finally, the case when for some $i = 1, \dots, m$, but not all, $1 \leq p_i = q_i$, follows from the previous argument, since $v_i = w_i$ in (4.2.17), and estimate (4.2.21) is no longer necessary. □

Remark 4.2.8. Note that, in general, there are weights that satisfy the hypotheses of Theorem 4.2.7. For instance, take $w_1 = \dots = w_m \in \bigcap_{i=1}^m \widehat{A}_{q_i,\infty}$.

Remark 4.2.9. In practice, we may desire an extrapolation result that allows us to increase some exponents and decrease others. We can achieve this by merely applying first Theorem 4.2.2 to lower the corresponding exponents, and then using Theorem 4.2.7 to raise the other ones, taking into account that in Theorem 4.2.7, if $q_i = p_i$, then we can replace the space $L^{p_i,1}(v_i)$ by $L^{p_i,\alpha_i}(v_i)$, with $\alpha_i > 0$.

From Theorem 4.2.7 we can obtain the corresponding extrapolation result for m-variable operators arguing as in the proof of Corollary 3.2.2. See Figure 3.3 for a pictorial representation of such scheme when $m = 2$.

Corollary 4.2.10. *Let T be an m-variable operator defined for measurable functions. Suppose that for some exponents $1 \leq p_1, \dots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$, and all weights $v_i \in \widehat{A}_{p_i,\infty}$, $i = 1, \dots, m$,*

$$T : L^{p_1,1}(v_1) \times \dots \times L^{p_m,1}(v_m) \longrightarrow L^{p,\infty}(v_1^{p/p_1} \dots v_m^{p/p_m}),$$

with constant bounded by $\varphi([\![v_1]\!]_{\widehat{A}_{p_1,\infty}}, \dots, [\![v_m]\!]_{\widehat{A}_{p_m,\infty}})$ as in (4.2.15). Given exponents $q_1 = p_1 \geq 1$ or $q_1 > p_1 > 1, \dots, q_m = p_m \geq 1$ or $q_m > p_m > 1$, and $\frac{1}{q} = \frac{1}{q_1} + \dots + \frac{1}{q_m}$, and weights $w_i \in \widehat{A}_{q_i,\infty}$, $i = 1, \dots, m$, and $w = w_1^{q/q_1} \dots w_m^{q/q_m}$, if for every $i = 1, \dots, m$ for which $q_i > p_i$, there exists $0 < \varepsilon_i \leq 1$ such that $wW_i^{-\varepsilon_i}$ is a weight, and $[W_i^{-\varepsilon_i}]_{RH_\infty(w)} < \infty$, with $W_i = \left(\frac{w}{w_i}\right)^{1/q_i}$, then

$$T : L^{q_1,1}(w_1) \times \dots \times L^{q_m,1}(w_m) \longrightarrow L^{q,\infty}(w),$$

with constant bounded by $\Phi_{\vec{\varepsilon},\vec{w}}([\![w_1]\!]_{\widehat{A}_{q_1,\infty}}, \dots, [\![w_m]\!]_{\widehat{A}_{q_m,\infty}})$ as in (4.2.16).

Remark 4.2.11. Observe that if the operator T is defined for characteristic functions of measurable sets $E_1, \dots, E_m \subseteq \mathbb{R}^n$, then under the hypotheses of Corollary 4.2.10, we deduce that

$$\begin{aligned} \left\| T(\chi_{E_1}, \dots, \chi_{E_m}) \right\|_{L^{q,\infty}(w)} \leq C \Phi_{\vec{\varepsilon},\vec{w}}([\![w_1]\!]_{\widehat{A}_{q_1,\infty}}, \dots, [\![w_m]\!]_{\widehat{A}_{q_m,\infty}}) \\ \times w_1(E_1)^{1/q_1} \dots w_m(E_m)^{1/q_m}, \end{aligned}$$

with $C = q_1 \dots q_m$, and hence, T is of weak type $(q_1, \dots, q_m, q)$ at least for characteristic functions.

Note that if in Theorem 4.2.7 we replace $\widehat{A}_{p_i,\infty}$ by $\widehat{A}_{p_i,N_i}$, and $\widehat{A}_{q_i,\infty}$ by $\widehat{A}_{q_i,N_i}$, $i = 1, \dots, m$, with $1 \leq N_1, \dots, N_m \in \mathbb{N}$, then in (4.2.20) we can use that

$$[w]_{A_{mq}^{\mathcal{R}}} \leq \widetilde{C}(N_1 + \dots + N_m) \prod_{i=1}^m \|w_i\|_{\widehat{A}_{q_i,N_i}}^{1/m},$$

and

$$[w_i]_{A^{\mathcal{R}}_{q_i}} \leq c_i N_i \|w_i\|_{\widehat{A}_{q_i,N_i}},$$

and replace (4.2.18) by

$$\|v_i\|_{\widehat{A}_{p_i,N_i}} \leq C_i \|w_i\|^{q_i/p_i}_{\widehat{A}_{q_i,N_i}},$$

and obtain the following variant of Theorem 4.2.7, depicted in Figure 3.3 for $m = 2$.

Theorem 4.2.12. *Given measurable functions $f_1, \dots, f_m$, and g, suppose that for some exponents $1 \leq p_1, \dots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, and $1 \leq N_1, \dots, N_m \in \mathbb{N}$, and all weights $v_i \in \widehat{A}_{p_i,N_i}$, $i = 1, \dots, m$,*

$$\|g\|_{L^{p,\infty}(v_1^{p/p_1} \dots v_m^{p/p_m})} \leq \varphi(\|v_1\|_{\widehat{A}_{p_1,N_1}}, \dots, \|v_m\|_{\widehat{A}_{p_m,N_m}}) \prod_{i=1}^m \|f_i\|_{L^{p_i,1}(v_i)},$$

where $\varphi : [1,\infty)^m \longrightarrow [0,\infty)$ is a function increasing in each variable. Given exponents $q_1 = p_1 \geq 1$ or $q_1 > p_1 > 1, \dots, q_m = p_m \geq 1$ or $q_m > p_m > 1$, and $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, and weights $w_i \in \widehat{A}_{q_i,N_i}$, $i = 1, \dots, m$, and $w = w_1^{q/q_1} \dots w_m^{q/q_m}$, if for every $i = 1, \dots, m$ for which $q_i > p_i$, there exists $0 < \varepsilon_i \leq 1$ such that $wW_i^{-\varepsilon_i}$ is a weight, and $[W_i^{-\varepsilon_i}]_{RH_\infty(w)} < \infty$, with $W_i = \left(\frac{w}{w_i}\right)^{1/q_i}$, then

$$\|g\|_{L^{q,\infty}(w)} \leq \Phi_{\vec{\varepsilon},\vec{w}}(\|w_1\|_{\widehat{A}_{q_1,N_1}}, \dots, \|w_m\|_{\widehat{A}_{q_m,N_m}}) \prod_{i=1}^m \|f_i\|_{L^{q_i,1}(w_i)},$$

where $\Phi_{\vec{\varepsilon},\vec{w}} : [1,\infty)^m \longrightarrow [0,\infty)$ is a function that increases in each variable, given by

$$\Phi_{\vec{\varepsilon},\vec{w}}(t_1, \dots, t_m) = \left(\prod_{i=1}^m \left(\frac{p_i}{q_i}\right)^{1/p_i'} (q_i' \phi_i(t_1, \dots, t_m))^{\frac{1}{p_i}\frac{q_i - p_i}{q_i - 1}} \right) \times \varphi(C_1 t_1^{q_1/p_1}, \dots, C_m t_m^{q_m/p_m}),$$

where for $i = 1, \dots, m$, if $q_i = p_i$, then $\phi_i(t_1, \dots, t_m) = 1$, and if $q_i > p_i$, then

$$\phi_i(t_1, \dots, t_m) = 2 \cdot 48^n q_i \psi_{\varepsilon_i,w_i,W_i}([W_i^{-\varepsilon_i}]_{RH_\infty(w)}) \times \phi^n_{mq,q_i}(c_i N_i t_i, \widetilde{C}(N_1 + \cdots + N_m) t_1^{1/m} \dots t_m^{1/m}),$$

with $\psi_{\varepsilon_i,w_i,W_i}$ as in (2.4.8), and ϕ^n_{mq,q_i} as in Lemma 2.3.10. If $W_i = 1$, in virtue of Remark 2.4.13, one can take $\phi_i(t_1, \dots, t_m) = C_{n,q_i}(c_i N_i t_i)^{q_i+1}$.

Remark 4.2.13. Observe that when we extrapolate the variable i downwards with Theorem 4.2.6, the space $L^{p_i,1}(v_i)$ becomes $L^{q_i,\frac{q_i}{p_i}}(w_i)$, and the class of weights $\widehat{A}_{p_i,N_i+1}$ becomes $\widehat{A}_{q_i,N_i}$. Something different happens when we extrapolate such variable upwards with Theorem 4.2.12. In this case, the space $L^{p_i,1}(v_i)$ becomes $L^{q_i,1}(w_i)$, and the class of weights $\widehat{A}_{p_i,N_i}$ becomes $\widehat{A}_{q_i,N_i}$.

4.2.3 One-Variable Off-Diagonal Extrapolation Theorems

In [37], multi-variable strong type extrapolation theorems were obtained as corollaries of one-variable off-diagonal strong type extrapolation theorems; that is, results in which the target space is different from the domain, both in terms of exponents and weights. In the case of multi-variable restricted weak type extrapolation, we observe a similar phenomenon, and we can also deduce our results from one-variable off-diagonal restricted weak type extrapolation theorems.

Let us start with the downwards extrapolation. The following theorem will allow us to obtain alternative proofs of Theorem 4.2.1 and Theorem 4.2.2. It is no surprise that its proof is similar to the one of Theorem 4.2.1.

Theorem 4.2.14. *Let $0 \leq \alpha < \infty$, and let $u \in A_\infty$. Given measurable functions f and g, suppose that for some exponent $1 \leq p < \infty$, and every weight $v \in \widehat{A}_{p,\infty}$,*

$$\|g\|_{L^{p_\alpha,\infty}(v_\alpha)} \leq \psi([\![v]\!]_{\widehat{A}_{p,\infty}})\|f\|_{L^{p,1}(v)}, \tag{4.2.23}$$

where $\frac{1}{p_\alpha} = \frac{1}{p} + \alpha$, $v_\alpha = v^{p_\alpha/p}u^{\alpha p_\alpha}$, and $\psi : [1,\infty) \longrightarrow [0,\infty)$ is an increasing function. Then, for every exponent $1 \leq q \leq p$, and every weight $w \in \widehat{A}_{q,\infty}$,

$$\|g\|_{L^{q_\alpha,\infty}(w_\alpha)} \leq \Psi([\![w]\!]_{\widehat{A}_{q,\infty}})\|f\|_{L^{q,\frac{q}{p}}(w)}, \tag{4.2.24}$$

where $\frac{1}{q_\alpha} = \frac{1}{q} + \alpha$, $w_\alpha = w^{q_\alpha/q}u^{\alpha q_\alpha}$, and $\Psi : [1,\infty) \longrightarrow [0,\infty)$ is an increasing function.

Proof. Observe that if $q = p$, then there is nothing to prove, so we may assume that $q < p$. Pick a weight $w \in \widehat{A}_{q,\infty}$. We may also assume that $\|f\|_{L^{q,\frac{q}{p}}(w)} < \infty$. In particular, f is locally integrable. Fix $y > 0$ and $\gamma > 0$. We have that

$$\lambda_g^{w_\alpha}(y) = \int_{\{|g|>y\}} w_\alpha \leq \lambda_{\mathscr{Z}}^{w_\alpha}(\gamma y) + \int_{\{|g|>y\}} \left(\frac{\gamma y}{\mathscr{Z}}\right)^{p_\alpha - q_\alpha} w_\alpha =: I + II, \tag{4.2.25}$$

where $\mathscr{Z} := (Mf)^{q/q_\alpha}\left(\frac{w}{u}\right)^\alpha$.

To estimate the term I in (4.2.25), we have that

$$I = \frac{(\gamma y)^{q_\alpha}}{(\gamma y)^{q_\alpha}}\lambda_{\mathscr{Z}}^{w_\alpha}(\gamma y) \leq \frac{1}{(\gamma y)^{q_\alpha}}\|\mathscr{Z}\|_{L^{q_\alpha,\infty}(w_\alpha)}^{q_\alpha} = \frac{1}{(\gamma y)^{q_\alpha}}\left\|\frac{Mf}{W}\right\|_{L^{q,\infty}(wW^q)}^q, \tag{4.2.26}$$

with $W := \left(\frac{u}{w}\right)^{\frac{\alpha q_\alpha}{q}}$. Note that $W \in A_\infty$. Indeed, if $\alpha = 0$, then $W = 1$, and if $\alpha > 0$, then $\widehat{A}_{q,\infty} \subseteq A_{q+\frac{1}{\alpha}}$, so in virtue of Lemma 3.1.5, $W = \left(\frac{u}{w}\right)^{\frac{1}{q+\frac{1}{\alpha}}} \in A_\infty$.

Moreover, since $u \in A_\infty$, and $\frac{q_\alpha}{q} + \alpha q_\alpha = 1$, $w_\alpha \in A_\infty$, and there exists $r \geq 1$ such that $w_\alpha \in A_r^{\mathcal{R}}$. If $s \geq 1$ is such that $u \in A_s^{\mathcal{R}}$, then we can choose $r := \max\{q, s\}$, and applying Lemma 4.1.7 and (2.1.2), we get that $[w_\alpha]_{A_r^{\mathcal{R}}} \leq \widetilde{C}([w]_{A_q^{\mathcal{R}}} + [u]_{A_s^{\mathcal{R}}})$.

In virtue of Theorem 2.3.8, Lemma 2.3.10, Theorem 4.1.6, and (2.1.1), we deduce that

$$\begin{aligned} \left\| \frac{Mf}{W} \right\|_{L^{q,\infty}(wW^q)} &\leq \phi([w]_{A_q^{\mathcal{R}}}, [w_\alpha]_{A_r^{\mathcal{R}}}) \|f\|_{L^{q,1}(w)} \\ &\leq \phi(c[\![w]\!]_{\widehat{A}_{q,\infty}}, \widetilde{C}(c[\![w]\!]_{\widehat{A}_{q,\infty}} + [u]_{A_s^{\mathcal{R}}})) \|f\|_{L^{q,1}(w)} \\ &\leq p^{1-\frac{p}{q}} \phi(c[\![w]\!]_{\widehat{A}_{q,\infty}}, \widetilde{C}(c[\![w]\!]_{\widehat{A}_{q,\infty}} + [u]_{A_s^{\mathcal{R}}})) \|f\|_{L^{q,\frac{q}{p}}(w)} \\ &=: p^{1-\frac{p}{q}} \phi_{u,w} \|f\|_{L^{q,\frac{q}{p}}(w)}, \end{aligned} \tag{4.2.27}$$

and combining the estimates (4.2.26) and (4.2.27), we obtain that

$$I \leq \frac{1}{(\gamma y)^{q_\alpha}} p^{q-p} \phi_{u,w}^q \|f\|_{L^{q,\frac{q}{p}}(w)}^q. \tag{4.2.28}$$

We proceed to estimate the term II in (4.2.25). Take $v := (Mf)^{q-p} w$. Since $w \in \widehat{A}_{q,\infty}$, it follows from Lemma 4.1.9 that $v \in \widehat{A}_{p,\infty}$, with $[\![v]\!]_{\widehat{A}_{p,\infty}} \leq C[\![w]\!]_{\widehat{A}_{q,\infty}}$. Observe that

$$\begin{aligned} \mathscr{Z}^{q_\alpha - p_\alpha} w_\alpha &= (Mf)^{q(1-\frac{p_\alpha}{q_\alpha})} w^{\alpha(q_\alpha - p_\alpha) + \frac{q_\alpha}{q}} u^{\alpha(p_\alpha - q_\alpha) + \alpha q_\alpha} \\ &= (Mf)^{\frac{p_\alpha}{p}(q-p)} w^{p_\alpha/p} u^{\alpha p_\alpha} = v^{p_\alpha/p} u^{\alpha p_\alpha}, \end{aligned}$$

so by (4.2.23) and the monotonicity of ψ, we get that

$$\begin{aligned} II = \frac{(\gamma y)^{p_\alpha}}{(\gamma y)^{q_\alpha}} \int_{\{|g|>y\}} v^{p_\alpha/p} u^{\alpha p_\alpha} &\leq \frac{\gamma^{p_\alpha}}{(\gamma y)^{q_\alpha}} \|g\|_{L^{p_\alpha,\infty}(v^{p_\alpha/p} u^{\alpha p_\alpha})}^{p_\alpha} \\ &\leq \frac{\gamma^{p_\alpha}}{(\gamma y)^{q_\alpha}} \psi(C[\![w]\!]_{\widehat{A}_{q,\infty}})^{p_\alpha} \|f\|_{L^{p,1}(v)}^{p_\alpha}, \end{aligned} \tag{4.2.29}$$

and arguing as we did in (3.2.7), we have that

$$\|f\|_{L^{p,1}(v)} \leq \frac{p}{q} \|f\|_{L^{q,\frac{q}{p}}(w)}^{q/p}. \tag{4.2.30}$$

Combining the estimates (4.2.25), (4.2.28), (4.2.29), and (4.2.30), we conclude that

$$\begin{aligned} \lambda_g^{w_\alpha}(y) &\leq \frac{1}{(\gamma y)^{q_\alpha}} p^{q-p} \phi_{u,w}^q \|f\|_{L^{q,\frac{q}{p}}(w)}^q \\ &\quad + \frac{\gamma^{p_\alpha}}{(\gamma y)^{q_\alpha}} \left(\frac{p}{q}\right)^{p_\alpha} \psi(C[\![w]\!]_{\widehat{A}_{q,\infty}})^{p_\alpha} \|f\|_{L^{q,\frac{q}{p}}(w)}^{\frac{p_\alpha q}{p}}, \end{aligned}$$

and taking the infimum over all $\gamma > 0$, it follows from Lemma 3.1.1 that

$$y^{q_\alpha} \lambda_g^{w_\alpha}(y) \leq \frac{p_\alpha}{p_\alpha - q_\alpha} \left(\frac{p_\alpha - q_\alpha}{q_\alpha} \right)^{q_\alpha / p_\alpha} \left(\frac{p}{q} \right)^{q_\alpha} p^{q_\alpha (2 - \frac{q}{p} - \frac{p}{q})} \\ \times \phi_{u,w}^{q_\alpha (1 - \frac{q}{p})} \psi(C[\![w]\!]_{\widehat{A}_{q,\infty}})^{q_\alpha} \|f\|^{q_\alpha}_{L^{q,\frac{q}{p}}(w)}.$$

Finally, raising everything to the power $\frac{1}{q_\alpha}$ in this last expression, and taking the supremum over all $y > 0$, we see that (4.2.24) holds, with

$$\Psi([\![w]\!]_{\widehat{A}_{q,\infty}}) = p^{3 - \frac{q}{p} - \frac{p}{q}} \frac{C_{p_\alpha, q_\alpha}}{q} \phi_{u,w}^{1 - \frac{q}{p}} \psi(C[\![w]\!]_{\widehat{A}_{q,\infty}}), \tag{4.2.31}$$

and C_{p_α, q_α} as in (3.2.19). □

From Theorem 4.2.14 we can obtain the corresponding extrapolation result for one-variable operators arguing as in the proof of Corollary 3.2.2.

Corollary 4.2.15. *Let $0 \leq \alpha < \infty$, and let $u \in A_\infty$. Let T be a one-variable operator defined for measurable functions. Suppose that for some exponent $1 \leq p < \infty$, and every weight $v \in \widehat{A}_{p,\infty}$,*

$$T : L^{p,1}(v) \longrightarrow L^{p_\alpha, \infty}(v_\alpha),$$

with constant bounded by $\psi([\![v]\!]_{\widehat{A}_{p,\infty}})$ as in (4.2.23), where $\frac{1}{p_\alpha} = \frac{1}{p} + \alpha$, and $v_\alpha = v^{p_\alpha / p} u^{\alpha p_\alpha}$. Then, for every exponent $1 \leq q \leq p$, and every weight $w \in \widehat{A}_{q,\infty}$,

$$T : L^{q,\frac{q}{p}}(w) \longrightarrow L^{q_\alpha, \infty}(w_\alpha),$$

with constant bounded by $\Psi([\![w]\!]_{\widehat{A}_{q,\infty}})$ as in (4.2.24), where $\frac{1}{q_\alpha} = \frac{1}{q} + \alpha$, and $w_\alpha = w^{q_\alpha / q} u^{\alpha q_\alpha}$.

Remark 4.2.16. Observe that if the operator T is defined for characteristic functions of measurable sets $E \subseteq \mathbb{R}^n$, then under the hypotheses of Corollary 4.2.15, we deduce that

$$\|T(\chi_E)\|_{L^{q_\alpha, \infty}(w_\alpha)} \leq p^{p/q} \Psi([\![w]\!]_{\widehat{A}_{q,\infty}}) w(E)^{1/q},$$

and hence, T is of weak type (q, q_α) at least for characteristic functions.

Remark 4.2.17. Note that in Theorem 4.2.14 and Corollary 4.2.15, we can replace the class of weights $\widehat{A}_{p,\infty}$ by $\widehat{A}_{p,N+1}$, and the class of weights $\widehat{A}_{q,\infty}$ by $\widehat{A}_{q,N}$, with $1 \leq N \in \mathbb{N}$, as we did in Theorem 4.2.5 and Theorem 4.2.6. If $N = 1$, $\alpha = 0$, and T is a sub-linear operator, one can work with the class of weights $\widehat{A}_{p,1}$ instead of $\widehat{A}_{p,2}$ in Corollary 4.2.15, via an interpolation argument based on Lemma 4.1.3, as shown in [9, Theorem 2.13].

As we mentioned before, we can deduce Theorem 4.2.1 from Theorem 4.2.14. Indeed, under the hypotheses of Theorem 4.2.1, we apply Theorem

4.2.14, where if $m = 1$, we take $\alpha = 0$, and $u = 1$, and if $m > 1$, we take $\alpha = \frac{1}{p_2} + \cdots + \frac{1}{p_m}$, and

$$u = \left(v_2^{p/p_2} \dots v_m^{p/p_m}\right)^{\frac{p_1}{p_1 - p}} = \left(v_2^{q/p_2} \dots v_m^{q/p_m}\right)^{\frac{q_1}{q_1 - q}}, \tag{4.2.32}$$

which is an A_∞ weight since $\sum_{i=2}^m \frac{p}{p_i} \frac{p_1}{p_1 - p} = \sum_{i=2}^m \frac{q}{p_i} \frac{q_1}{q_1 - q} = 1$. We also take

$$\psi([\![v_1]\!]_{\widehat{A}_{p_1,\infty}}) = \varphi([\![v_1]\!]_{\widehat{A}_{p_1,\infty}}, \dots, [\![v_m]\!]_{\widehat{A}_{p_m,\infty}}) \prod_{i=2}^m \|f_i\|_{L^{p_i,1}(v_i)}. \tag{4.2.33}$$

The estimate (4.2.2) follows immediately from (4.2.24). Moreover, we can iterate the same argument m times, one for each variable, to deduce Theorem 4.2.2.

Similarly, Theorem 3.3.12 follows at once from Theorem 4.2.14 and Theorem 4.1.6, with a slightly worse function Φ. Indeed, under the hypotheses of Theorem 3.3.12, we apply Theorem 4.2.14 with $\alpha = \frac{1}{p_1}$, $u = v_1$, and

$$\psi([\![v_2]\!]_{\widehat{A}_{p_2,\infty}}) = \varphi([v_1]_{A_{p_1}}, C[\![v_2]\!]_{\widehat{A}_{p_2,\infty}}) \|f_1\|_{L^{p_1}(v_1)}, \tag{4.2.34}$$

with C given by Theorem 4.1.6. The estimate (3.3.17) follows immediately from (4.2.24) and the fact that $[\![w_2]\!]_{\widehat{A}_{q_2,\infty}} \leq \|w_2\|_{\widehat{A}_{q_2}}$, but we have lost a power of $\frac{q_2}{p_2} \leq 1$ in the dependence on $\|w_2\|_{\widehat{A}_{q_2}}$ of Φ. Such loss can be avoided using Remark 4.2.17 and Lemma 4.1.9.

We now discuss the upwards extrapolation. The next result will give us an alternative proof of Theorem 4.2.7. Once again, it is no surprise that both proofs are similar.

Theorem 4.2.18. *Let $0 \leq \alpha < \infty$, and let $u \in A_\infty$. Given measurable functions f and g, suppose that for some exponent $1 \leq p < \infty$, and every weight $v \in \widehat{A}_{p,\infty}$,*

$$\|g\|_{L^{p_\alpha,\infty}(v_\alpha)} \leq \psi([\![v]\!]_{\widehat{A}_{p,\infty}}) \|f\|_{L^{p,1}(v)}, \tag{4.2.35}$$

where $\frac{1}{p_\alpha} = \frac{1}{p} + \alpha$, $v_\alpha = v^{p_\alpha/p} u^{\alpha p_\alpha}$, and $\psi : [1, \infty) \longrightarrow [0, \infty)$ is an increasing function. Take an exponent $q = p \geq 1$ or $q > p > 1$, and a weight $w \in \widehat{A}_{q,\infty}$. If $q > p$, suppose that there exists $0 < \varepsilon \leq 1$ such that $w_\alpha W^{-\varepsilon}$ is a weight, and $[W^{-\varepsilon}]_{RH_\infty(w_\alpha)} < \infty$, with $W = \left(\frac{u}{w}\right)^{\frac{\alpha q_\alpha}{q}}$, and $w_\alpha = w^{q_\alpha/q} u^{\alpha q_\alpha}$, where $\frac{1}{q_\alpha} = \frac{1}{q} + \alpha$. Then,

$$\|g\|_{L^{q_\alpha,\infty}(w_\alpha)} \leq \Psi_{\varepsilon,u,w}([\![w]\!]_{\widehat{A}_{q,\infty}}) \|f\|_{L^{q,1}(w)}, \tag{4.2.36}$$

where $\Psi_{\varepsilon,u,w} : [1, \infty) \longrightarrow [0, \infty)$ is an increasing function that depends on ε, u, and w.

Proof. Note that if $q = p$, then there is nothing to prove, so we may assume that $1 < p < q$. Pick a weight $w \in \widehat{A}_{q,\infty}$. As usual, we may assume that $\|f\|_{L^{q,1}(w)} < \infty$. For every natural number $N \geq 1$, let $g_N := |g| \chi_{B(0,N)}$. Fix $N \geq 1$. We will prove (4.2.36) for the pair (f, g_N). Since $g_N \leq |g|$, we already know that (4.2.35) holds for (f, g_N). Fix $y > 0$ such that $\lambda_{g_N}^{w_\alpha}(y) \neq 0$. If no such y exists, then $\|g_N\|_{L^{q_\alpha,\infty}(w_\alpha)} = 0$ and we are done.

In order to apply (4.2.35), we want to find a weight $v \in \widehat{A}_{p,\infty}$ such that $\lambda_{g_N}^{w_\alpha}(y) \leq \lambda_{g_N}^{v_\alpha}(y)$. We take

$$v := w^{\frac{p-1}{q-1}} \left(M(w^{1/q} w_\alpha^{1/q'} \chi_{\{|g_N|>y\}}) \right)^{\frac{q-p}{q-1}}. \tag{4.2.37}$$

Applying Lemma 4.1.10, we see that $v \in \widehat{A}_{p,\infty}$, with $[\![v]\!]_{\widehat{A}_{p,\infty}} \leq C[\![w]\!]_{\widehat{A}_{q,\infty}}^{q/p}$, and C independent of w, w_α, N, and y. Observe that

$$\begin{aligned} v^{p_\alpha/p} u^{\alpha p_\alpha} &\geq w^{\frac{p_\alpha}{p}\frac{p-1}{q-1} + \frac{p_\alpha}{pq}\frac{q-p}{q-1}} w_\alpha^{\frac{p_\alpha}{pq'}\frac{q-p}{q-1}} u^{\alpha p_\alpha} \chi_{\{|g_N|>y\}} \\ &= w^{\frac{p_\alpha}{q}} w_\alpha^{1-\frac{p_\alpha}{q_\alpha}} u^{\alpha p_\alpha} \chi_{\{|g_N|>y\}} = w_\alpha \chi_{\{|g_N|>y\}}, \end{aligned}$$

so (4.2.35) implies that

$$\lambda_{g_N}^{w_\alpha}(y) \leq \int_{\{|g_N|>y\}} v^{p_\alpha/p} u^{\alpha p_\alpha} = \lambda_{g_N}^{v_\alpha}(y) \leq \frac{1}{y^{p_\alpha}} \psi(C[\![w]\!]_{\widehat{A}_{q,\infty}}^{q/p})^{p_\alpha} \|f\|_{L^{p,1}(v)}^{p_\alpha}. \tag{4.2.38}$$

We want to replace $\|f\|_{L^{p,1}(v)}$ by $\|f\|_{L^{q,1}(w)}$ in (4.2.38). Applying Hölder's inequality with exponent $\frac{q}{p} > 1$, we obtain that for every $t > 0$,

$$\begin{aligned} \lambda_f^v(t) &= \int_{\{|f|>t\}} \left(\frac{M(w^{1/q} w_\alpha^{1/q'} \chi_{\{|g_N|>y\}})}{w} \right)^{\frac{q-p}{q-1}} w \\ &= \frac{q}{p} w(\{|f| > t\})^{p/q} \left\| \frac{M(w^{1/q} w_\alpha^{1/q'} \chi_{\{|g_N|>y\}})}{w} \right\|_{L^{q',\infty}(w)}^{\frac{q-p}{q-1}}. \end{aligned}$$

Arguing as we did in the proof of Theorem 4.2.14, we know that the weights $w_\alpha, W \in A_\infty$. Moreover, if $s \geq 1$ is such that $u \in A_s^{\mathcal{R}}$, then $w_\alpha \in A_r^{\mathcal{R}}$, with $r := \max\{q, s\}$, and applying Lemma 4.1.7 and (2.1.2), we get that $[w_\alpha]_{A_r^{\mathcal{R}}} \leq \widetilde{C}([w]_{A_q^{\mathcal{R}}} + [u]_{A_s^{\mathcal{R}}})$. Also, note that $W = \left(\frac{w_\alpha}{w}\right)^{1/q}$, so $wW^{q-1} = w^{1/q} w_\alpha^{1/q'}$, and $wW^q = w_\alpha$. Hence, Theorem 2.4.12 and Theorem 4.1.6 give us that

$$\begin{aligned} \left\| \frac{M(wW^{q-1}\chi_{\{|g_N|>y\}})}{w} \right\|_{L^{q',\infty}(w)} &\leq q' \phi([w]_{A_q^{\mathcal{R}}}, [w_\alpha]_{A_r^{\mathcal{R}}}) w_\alpha(\{|g_N| > y\})^{1/q'} \\ &\leq q' \phi_{\varepsilon,u,w} w_\alpha(\{|g_N| > y\})^{1/q'}, \end{aligned} \tag{4.2.39}$$

with
$$\phi_{\varepsilon,u,w} := \phi(c[\![w]\!]_{\widehat{A}_{q,\infty}}, \widetilde{C}(c[\![w]\!]_{\widehat{A}_{q,\infty}} + [u]_{A_s^{\mathcal{R}}})),$$
so
$$\lambda_f^v(t) \leq \frac{q}{p}(q'\phi_{\varepsilon,u,w})^{\frac{q-p}{q-1}} w_\alpha(\{|g_N| > y\})^{1-\frac{p}{q}} w(\{|f| > t\})^{p/q},$$
and hence,
$$\begin{aligned} \|f\|_{L^{p,1}(v)} &= p\int_0^\infty \lambda_f^v(t)^{1/p}dt \leq p\left(\frac{q}{p}\right)^{1/p} (q'\phi_{\varepsilon,u,w})^{\frac{1}{p}\frac{q-p}{q-1}} \\ &\times w_\alpha(\{|g_N| > y\})^{\frac{1}{p}-\frac{1}{q}} \int_0^\infty w(\{|f| > t\})^{1/q}dt \qquad (4.2.40) \\ &= \left(\frac{p}{q}\right)^{1/p'} (q'\phi_{\varepsilon,u,w})^{\frac{1}{p}\frac{q-p}{q-1}} w_\alpha(\{|g_N| > y\})^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^{q,1}(w)}. \end{aligned}$$

Combining the estimates (4.2.38) and (4.2.40), we have that
$$\lambda_{g_N}^{w_\alpha}(y) \leq \frac{1}{y^{p_\alpha}} \Psi_{\varepsilon,u,w}([\![w]\!]_{\widehat{A}_{q,\infty}})^{p_\alpha} \|f\|_{L^{q,1}(w)}^{p_\alpha} \lambda_{g_N}^{w_\alpha}(y)^{1-\frac{p_\alpha}{q_\alpha}}, \qquad (4.2.41)$$
with
$$\Psi_{\varepsilon,u,w}([\![w]\!]_{\widehat{A}_{q,\infty}}) = \left(\frac{p}{q}\right)^{1/p'} (q'\phi_{\varepsilon,u,w})^{\frac{1}{p}\frac{q-p}{q-1}} \psi(C[\![w]\!]_{\widehat{A}_{q,\infty}}^{q/p}).$$

By our choice of y and g_N, $0 < \lambda_{g_N}^{w_\alpha}(y) \leq w_\alpha(B(0,N)) < \infty$, so we can divide by $\lambda_{g_N}^{w_\alpha}(y)^{1-\frac{p_\alpha}{q_\alpha}}$ in (4.2.41) and raise everything to the power $\frac{1}{p_\alpha}$, obtaining that
$$y\lambda_{g_N}^{w_\alpha}(y)^{1/q_\alpha} \leq \Psi_{\varepsilon,u,w}([\![w]\!]_{\widehat{A}_{q,\infty}}) \|f\|_{L^{q,1}(w)},$$
and taking the supremum over all $y > 0$, we deduce (4.2.36) for the pair (f, g_N), and the result for the pair (f, g) follows taking the supremum over all $N \geq 1$. □

As usual, from Theorem 4.2.18 we can obtain the corresponding extrapolation result for one-variable operators arguing as in the proof of Corollary 3.2.2.

Corollary 4.2.19. *Let $0 \leq \alpha < \infty$, and let $u \in A_\infty$. Let T be a one-variable operator defined for measurable functions. Suppose that for some exponent $1 \leq p < \infty$, and every weight $v \in \widehat{A}_{p,\infty}$,*
$$T : L^{p,1}(v) \longrightarrow L^{p_\alpha,\infty}(v_\alpha),$$
with constant bounded by $\psi([\![v]\!]_{\widehat{A}_{p,\infty}})$ as in (4.2.35), where $\frac{1}{p_\alpha} = \frac{1}{p} + \alpha$, and $v_\alpha = v^{p_\alpha/p}u^{\alpha p_\alpha}$. Take an exponent $q = p \geq 1$ or $q > p > 1$, and a weight $w \in \widehat{A}_{q,\infty}$. If $q > p$, suppose that there exists $0 < \varepsilon \leq 1$ such that $w_\alpha W^{-\varepsilon}$ is a weight, and

$[W^{-\varepsilon}]_{RH_\infty(w_\alpha)} < \infty$, with $W = \left(\frac{u}{w}\right)^{\frac{\alpha q_\alpha}{q}}$, and $w_\alpha = w^{q_\alpha/q}u^{\alpha q_\alpha}$, where $\frac{1}{q_\alpha} = \frac{1}{q} + \alpha$. Then,

$$T : L^{q,1}(w) \longrightarrow L^{q_\alpha,\infty}(w_\alpha),$$

with constant bounded by $\Psi_{\varepsilon,u,w}([\![w]\!]_{\widehat{A}_{q,\infty}})$ as in (4.2.36).

Remark 4.2.20. Observe that if the operator T is defined for characteristic functions of measurable sets $E \subseteq \mathbb{R}^n$, then under the hypotheses of Corollary 4.2.19, we deduce that

$$\|T(\chi_E)\|_{L^{q_\alpha,\infty}(w_\alpha)} \leq q\Psi_{\varepsilon,u,w}([\![w]\!]_{\widehat{A}_{q,\infty}})w(E)^{1/q},$$

and hence, T is of weak type (q, q_α) at least for characteristic functions.

Remark 4.2.21. Note that in Theorem 4.2.18 and Corollary 4.2.19, we can replace the class of weights $\widehat{A}_{p,\infty}$ by $\widehat{A}_{p,N}$, and the class of weights $\widehat{A}_{q,\infty}$ by $\widehat{A}_{q,N}$, with $1 \leq N \in \mathbb{N}$, as we did in Theorem 4.2.12. If $N = 1$, and $\alpha = 0$, Theorem 4.2.18 gives us an alternative proof of [14, Theorem 3.1].

Observe that we can obtain Theorem 4.2.7 from Theorem 4.2.18 in the case when $q_2 = p_2, \dots, q_m = p_m$ by taking $\alpha = \frac{1}{p_2} + \cdots + \frac{1}{p_m}$, u as in (4.2.32), and $\psi([\![v_1]\!]_{\widehat{A}_{p_1,\infty}})$ as in (4.2.33). Similarly, we can recover Theorem 3.3.35 from Theorem 4.2.18 and Theorem 4.1.6 by choosing $\alpha = \frac{1}{p_1}$, $u = v_1$, and $\psi([\![v_2]\!]_{\widehat{A}_{p_2,\infty}})$ as in (4.2.34).

4.3 Applications to Sums of Products, and Averages

In this section, we present some applications for our multi-variable extrapolation theorems.

The following result gives us bounds for sums of products of functions. Once again, thanks to our extrapolation techniques, we manage to produce estimates in the case when $q \leq 1$ that we don't know how to obtain directly due to the lack of the corresponding Hölder-type inequality for Lorentz spaces.

Theorem 4.3.1. *Let $T_1, \dots, T_m$ be one-variable operators defined for measurable functions. For $i = 1, \dots, m$, suppose that for some $p_i > 1$, and every weight $v_i \in \widehat{A}_{p_i,\infty}$,*

$$T_i : L^{p_i,1}(v_i) \longrightarrow L^{p_i,\infty}(v_i), \tag{4.3.1}$$

with constant bounded by $\varphi_i([\![v_i]\!]_{\widehat{A}_{p_i,\infty}})$, where $\varphi_i : [1,\infty) \longrightarrow [0,\infty)$ is an increasing function. Suppose also that $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p} < 1$. Consider the operator $T_\boxslash$, defined for measurable functions $f_1, \dots, f_m$ by

$$T_\boxslash(f_1, \dots, f_m) := \sum_{i=1}^{m} f_1 \dots f_{i-1}(T_i f_i) f_{i+1} \dots f_m.$$

Take exponents $1 \le q_1, \dots, q_m$, and $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, and weights $w_i \in \widehat{A}_{q_i,\infty}$, $i = 1, \dots, m$, and $w = w_1^{q/q_1} \dots w_m^{q/q_m}$. If $q_i > p_i$, assume that $p_i > 1$, and that there exists $0 < \varepsilon_i \le 1$ such that $wW_i^{-\varepsilon_i}$ is a weight, and $[W_i^{-\varepsilon_i}]_{RH_\infty(w)} < \infty$, with $W_i = \left(\frac{w}{w_i}\right)^{1/q_i}$. Then,

$$T_{\boxslash} : L^{q_1,\min\{1,\frac{q_1}{p_1}\}}(w_1) \times \cdots \times L^{q_m,\min\{1,\frac{q_m}{p_m}\}}(w_m) \longrightarrow L^{q,\infty}(w), \tag{4.3.2}$$

with constant bounded by $\Phi([\![w_1]\!]_{\widehat{A}_{q_1,\infty}}, \dots, [\![w_m]\!]_{\widehat{A}_{q_m,\infty}})$, where $\Phi : [1,\infty)^m \longrightarrow [0,\infty)$ is a function increasing in each variable.

Proof. Given measurable functions $f_1, \dots, f_m$, and weights $v_i \in \widehat{A}_{p_i,\infty}$, $i = 1, \dots, m$, and $v := v_1^{p/p_1} \dots v_m^{p/p_m}$, in virtue of Lemma 2.2.1, and (4.3.1), we have that

$$\begin{aligned}
\|T_{\boxslash}(f_1, \dots, f_m)\|_{L^{p,\infty}(v)} &\le m \sum_{i=1}^m \|f_1 \dots f_{i-1}(T_i f_i) f_{i+1} \dots f_m\|_{L^{p,\infty}(v)} \\
&\le mC \sum_{i=1}^m \left(\prod_{j \ne i} \|f_j\|_{L^{p_j,1}(v_j)} \right) \|T_i f_i\|_{L^{p_i,\infty}(v_i)} \\
&\le mC \left(\sum_{i=1}^m \varphi_i([\![v_i]\!]_{\widehat{A}_{p_i,\infty}}) \right) \prod_{i=1}^m \|f_i\|_{L^{p_i,1}(v_i)}.
\end{aligned}$$

Hence, (4.3.2) follows from Corollary 4.2.3, Corollary 4.2.10, and Remark 4.2.9. □

In Theorem 4.3.1, we used our multi-variable extrapolation results to overcome the lack of some Hölder-type inequalities for Lorentz spaces. In the next theorem, an extension of Theorem 3.4.4, we will use our multi-variable extrapolation techniques to avoid the application of Minkowski's integral inequality for $\|\cdot\|_{L^{q,\infty}(w)}$ when $q \le 1$, which is not available, and we will produce bounds for averages of operators like $T_{\boxslash}$.

Theorem 4.3.2. *Let $\{T_1^{t_1}\}_{t_1 \in \mathbb{R}}, \dots, \{T_m^{t_m}\}_{t_m \in \mathbb{R}}$ be families of sub-linear operators defined for measurable functions. For $i = 1, \dots, m$, suppose that for some $p_i > 1$, every $t_i \in \mathbb{R}$, and every weight $v_i \in \widehat{A}_{p_i,\infty}$,*

$$T_i^{t_i} : L^{p_i,1}(v_i) \longrightarrow L^{p_i,\infty}(v_i), \tag{4.3.3}$$

with constant bounded by $\varphi_i([\![v_i]\!]_{\widehat{A}_{p_i,\infty}})$, where $\varphi_i : [1,\infty) \longrightarrow [0,\infty)$ is an increasing function independent of t_i. Suppose also that $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p} < 1$. For a measure μ on $\mathbb{R}^m$ such that $|\mu|(\mathbb{R}^m) < \infty$, consider the averaging operator

$$T_{\boxslash,\mu}(f_1, \dots, f_m) := \int_{\mathbb{R}^m} \left(\sum_{i=1}^m f_1 \dots f_{i-1}(T_i^{t_i} f_i) f_{i+1} \dots f_m \right) d\mu(t_1, \dots, t_m),$$

defined for measurable functions $f_1, \dots, f_m$. *Take exponents* $1 \le q_1, \dots, q_m$, *and* $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, *and weights* $w_i \in \widehat{A}_{q_i,\infty}$, $i = 1, \dots, m$, *and write* $w = w_1^{q/q_1} \dots w_m^{q/q_m}$. *If* $q_i > p_i$, *assume that* $p_i > 1$, *and that there exists* $0 < \varepsilon_i \le 1$ *such that* $wW_i^{-\varepsilon_i}$ *is a weight, and* $[W_i^{-\varepsilon_i}]_{RH_\infty(w)} < \infty$, *with* $W_i = \left(\frac{w}{w_i}\right)^{1/q_i}$. *Then,*

$$T_{\boxslash,\mu} : L^{q_1,\min\{1,\frac{q_1}{p_1}\}}(w_1) \times \cdots \times L^{q_m,\min\{1,\frac{q_m}{p_m}\}}(w_m) \longrightarrow L^{q,\infty}(w), \tag{4.3.4}$$

with constant bounded by $\Phi([\![w_1]\!]_{\widehat{A}_{q_1,\infty}}, \dots, [\![w_m]\!]_{\widehat{A}_{q_m,\infty}})$, *where* $\Phi : [1,\infty)^m \longrightarrow [0,\infty)$ *is a function increasing in each variable.*

Proof. Since $p > 1$, given measurable functions $f_1, \dots, f_m$, and weights $v_i \in \widehat{A}_{p_i,\infty}$, $i = 1, \dots, m$, and $v := v_1^{p/p_1} \dots v_m^{p/p_m}$, in virtue of Minkowski's integral inequality (see [104, Proposition 2.1] and [3, Theorem 4.4]), we have that

$$\|T_{\boxslash,\mu}(f_1, \dots, f_m)\|_{L^{p,\infty}(v)} \le p' \int_{\mathbb{R}^m} \left\| \sum_{i=1}^m (T_i^{t_i} f_i) \prod_{j \ne i} f_j \right\|_{L^{p,\infty}(v)} d|\mu|(t_1, \dots, t_m),$$

and arguing as we did in the proof of Theorem 4.3.1, we get that

$$\|T_{\boxslash,\mu}(f_1, \dots, f_m)\|_{L^{p,\infty}(v)} \le mCp'|\mu|(\mathbb{R}^m) \left(\sum_{i=1}^m \varphi_i([\![v_i]\!]_{\widehat{A}_{p_i,\infty}}) \right) \prod_{i=1}^m \|f_i\|_{L^{p_i,1}(v_i)},$$

and hence, (4.3.4) follows from Corollary 4.2.3, Corollary 4.2.10, and Remark 4.2.9. □

Chapter 5

Fractional and Singular Integrals, and Commutators

> *'I have thought of a nice ending for my book: and he lived happily ever after to the end of his days.' Gandalf laughed. 'I hope he will. But nobody will read the book, however it ends.'*

John Ronald Reuel Tolkien, *The Lord of the Rings*, 1954

We devote this chapter to the study of restricted weak type inequalities for fractional operators, Calderón-Zygmund operators, and their commutators. In Section 5.1, we provide general information about fractional operators, and Orlicz and weak Orlicz spaces. In Section 5.2, we present our restricted weak type bounds for multi-variable fractional operators and Calderón-Zygmund operators. In Section 5.3, we discuss our restricted weak type bounds for commutators of Calderón-Zygmund operators and fractional integrals. In Section 5.4, we apply our bounds for the multi-linear fractional integral $\mathcal{I}_\alpha$ to produce Poincaré and Sobolev-type inequalities for products of functions. The contents of this chapter are part of a joint work with David V. Cruz-Uribe (see [33]).

5.1 Special Preliminaries

In this section, we present some technical results that we will use throughout this chapter.

5.1.1 Fractional Operators

We define some of the operators that we will study, and prove some useful relations between them.

Definition 5.1.1. Let $0 \leq \alpha < nm$. Given $\vec{f} = (f_1, \dots, f_m)$, with $f_i \in L^1_{loc}(\mathbb{R}^n)$, $i = 1, \dots, m$, we define the *centered fractional maximal operator* $\mathcal{M}^c_\alpha$ by

$$\mathcal{M}^c_\alpha(\vec{f})(x) := \sup_{r>0} \prod_{i=1}^m \frac{1}{|Q(x,r)|^{1-\frac{\alpha}{nm}}} \int_{Q(x,r)} |f_i|,$$

where $Q(x,r) := (x_1 - r, x_1 + r) \times \cdots \times (x_n - r, x_n + r)$ denotes an open cube with side length $2r$ centered at $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Similarly, we define the *fractional maximal operator* $\mathcal{M}_\alpha$ by

$$\mathcal{M}_\alpha(\vec{f})(x) := \sup_{Q \ni x} \prod_{i=1}^{m} \frac{1}{|Q|^{1-\frac{\alpha}{nm}}} \int_Q |f_i|,$$

where the supremum is taken over all cubes Q containing x and with sides parallel to the coordinate axes. If $\alpha = 0$, then $\mathcal{M}_\alpha$ is just $\mathcal{M}$.

The operators $\mathcal{M}_\alpha^c$ and $\mathcal{M}_\alpha$ are comparable, as the next lemma shows.

Lemma 5.1.2. $\mathcal{M}_\alpha^c(\vec{f}) \leq \mathcal{M}_\alpha(\vec{f}) \leq 2^{nm-\alpha}\mathcal{M}_\alpha^c(\vec{f})$.

Proof. Clearly, $\mathcal{M}_\alpha^c(\vec{f}) \leq \mathcal{M}_\alpha(\vec{f})$. Now, fix $x \in \mathbb{R}^n$, and Q containing x. If we denote by c_Q and ℓ_Q the center and the side length of Q, respectively, we have that

$$Q \subseteq \overline{Q(c_Q, \ell_Q/2)} \subseteq \overline{Q(x, \ell_Q)}.$$

The first inclusion is clear. For the second one, we use that

$$\overline{Q(c_Q, \ell_Q/2)} = \{y \in \mathbb{R}^n : \|c_Q - y\|_\infty \leq \ell_Q/2\},$$

where for $z = (z_1, \ldots, z_n) \in \mathbb{R}^n$, $\|z\|_\infty := \max_{1\leq j\leq n}\{|z_j|\}$. Since $x \in Q$, we have that $\|c_Q - x\|_\infty \leq \ell_Q/2$, so for any $y \in \overline{Q(c_Q, \ell_Q/2)}$, by the triangular inequality we obtain that

$$\|x - y\|_\infty \leq \|c_Q - y\|_\infty + \|c_Q - x\|_\infty \leq \ell_Q,$$

and $y \in \overline{Q(x, \ell_Q)}$. The sides of an n-dimensional cube have Lebesgue measure 0 in $\mathbb{R}^n$ and $1 - \frac{\alpha}{nm} > 0$, so we get that

$$|Q|^{1-\frac{\alpha}{nm}} \leq |Q(x, \ell_Q)|^{1-\frac{\alpha}{nm}} = 2^{n-\frac{\alpha}{m}}|Q|^{1-\frac{\alpha}{nm}}.$$

Hence,

$$\prod_{i=1}^{m} \frac{1}{|Q|^{1-\frac{\alpha}{nm}}} \int_Q |f_i| \leq \prod_{i=1}^{m} \frac{2^{n-\frac{\alpha}{m}}}{|Q(x,\ell_Q)|^{1-\frac{\alpha}{nm}}} \int_{Q(x,\ell_Q)} |f_i| \leq 2^{nm-\alpha}\mathcal{M}_\alpha^c(\vec{f})(x).$$

□

Definition 5.1.3. Let $0 < \alpha < nm$, and $\vec{f} = (f_1, \ldots, f_m)$, where each f_i is a measurable function on $\mathbb{R}^n$. We define the *multi-linear fractional integral* as

$$\mathcal{I}_\alpha(\vec{f})(x) := \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \frac{f_1(y_1) \ldots f_m(y_m)}{(|x - y_1| + \cdots + |x - y_m|)^{nm-\alpha}} dy_1 \ldots dy_m, \quad x \in \mathbb{R}^n,$$

where the integrals converge if $\vec{f} \in \mathscr{S}(\mathbb{R}^n) \times \cdots \times \mathscr{S}(\mathbb{R}^n)$. If $m = 1$, then we will use the notation $I_\alpha(f)$.

The next result shows that the fractional integral $\mathcal{I}_\alpha$ is controlled by the fractional maximal operator $\mathcal{M}_\alpha$.

Theorem 5.1.4. *Let $0 < \alpha < nm$, $0 < q < \infty$, and $v \in A_\infty$. Then,*

$$\|\mathcal{I}_\alpha(\vec{f})\|_{L^{q,\infty}(v)} \approx_v \|\mathcal{M}_\alpha(\vec{f})\|_{L^{q,\infty}(v)},$$

for every $\vec{f} = (f_1, \dots, f_m)$ with $0 \le f_i \in L^\infty_c(\mathbb{R}^n)$, $i = 1, \dots, m$.

Proof. One inequality follows from the fact that $\mathcal{M}_\alpha(\vec{f}) \lesssim \mathcal{I}_\alpha(\vec{f})$ (see [82, Section 3]), while the other one follows from Theorem 3.1 in [82] and Theorem 2.1 in [35]. □

Remark 5.1.5. The equivalence in Theorem 5.1.4 is true for many other quasi-norms rather than $\|\cdot\|_{L^{q,\infty}(v)}$ since Theorem 2.1 in [35] works for a large class of rearrangement invariant quasi-Banach function spaces.

5.1.2 Orlicz and Weak Orlicz Spaces

We recall some facts about Orlicz and weak Orlicz spaces. For more information, see [76, 96, 97].

A function $\phi : [0, \infty) \longrightarrow [0, \infty)$ is a *Young function* if it is continuous, convex, strictly increasing, $\phi(0) = 0$, and $\frac{\phi(t)}{t} \to \infty$ as $t \to \infty$. Note that $\phi(t) = \mathrm{Id}(t) := t$ is not properly a Young function, but in many cases, what we say also applies to it. A particular case of interest in this chapter is the Young function

$$\phi(t) = B(t) := t \log(e + t).$$

Given a Young function ϕ, a weight ν, and a cube Q, $L^\phi(Q, \frac{\nu}{\nu(Q)})$ is the *Orlicz space* of measurable functions f on $\mathbb{R}^n$ such that for some $\lambda > 0$,

$$\frac{1}{\nu(Q)} \int_Q \phi(\lambda|f|)\nu < \infty.$$

For $f \in L^\phi(Q, \frac{\nu}{\nu(Q)})$, define the *Luxemburg norm with respect to ϕ and ν* by

$$\|f\|_{L^\phi(Q,\nu/\nu(Q))} := \|f\|_{\phi,Q,\nu} := \inf\left\{\lambda > 0 : \frac{1}{\nu(Q)} \int_Q \phi(\lambda^{-1}|f|)\nu \le 1\right\}.$$

Given a measurable set $E \subseteq \mathbb{R}^n$ with $0 < \nu(E) < \infty$,

$$\|\chi_E\|_{\phi,Q,\nu} = \frac{1}{\phi^{-1}\left(\frac{\nu(Q)}{\nu(E\cap Q)}\right)}.$$

If we consider the function $\phi(t) = t^p$, with $0 < p < \infty$, which is a Young function for $p > 1$, then

$$\|f\|_{\phi,Q,\nu} = \left(\frac{1}{\nu(Q)} \int_Q |f|^p \nu\right)^{1/p} = \|f\chi_Q\|_{L^p(\nu/\nu(Q))} =: \|f\|_{L^p(Q,\nu/\nu(Q))}.$$

For a Young function ϕ, one can define a *complementary function*

$$\bar{\phi}(s) := \sup_{t>0}\{st - \phi(t)\}.$$

Such $\bar{\phi}$ is also a Young function, and satisfies that for all $s, t \geq 0$,

$$st \leq \phi(t) + \bar{\phi}(s).$$

This is known as *Young's inequality* (see [105]). The classical examples of complementary Young functions are $\phi(t) = \frac{t^p}{p}$ and $\bar{\phi}(t) = \frac{t^{p'}}{p'}$, with $1 < p < \infty$. One can also show that $\bar{B}(t) \leq e^t - 1$.

As an application of Young's inequality, we can prove the following generalization of Hölder's inequality.

Theorem 5.1.6. *Let ϕ and $\bar{\phi}$ be complementary Young functions, and let ν be a weight. Then, for every pair of measurable functions f and g, and every cube Q,*

$$\frac{1}{\nu(Q)}\int_Q |fg|\nu \leq 2\|f\|_{\phi,Q,\nu}\|g\|_{\bar{\phi},Q,\nu}. \tag{5.1.1}$$

Proof. Without loss of generality, we may assume that $\|f\|_{\phi,Q,\nu}$ and $\|g\|_{\bar{\phi},Q,\nu}$ are non-zero and finite, since otherwise the inequality (5.1.1) is immediate. By homogeneity, we may further assume that $\|f\|_{\phi,Q,\nu} = \|g\|_{\bar{\phi},Q,\nu} = 1$. We now have that

$$\frac{1}{\nu(Q)}\int_Q \phi(|f|)\nu \leq 1.$$

Indeed, it follows from the definition of the Orlicz norm that for every $\lambda > 1$,

$$\frac{1}{\nu(Q)}\int_Q \phi(\lambda^{-1}|f|)\nu \leq 1.$$

Let $\{\lambda_k\}_{k\geq 1} \subseteq [1,\infty)$ be a sequence decreasing to 1. Then, by Fatou's lemma,

$$\frac{1}{\nu(Q)}\int_Q \phi(|f|)\nu \leq \liminf_{k\to\infty}\frac{1}{\nu(Q)}\int_Q \phi(\lambda_k^{-1}|f|)\nu \leq 1.$$

Similarly, we also have that

$$\frac{1}{\nu(Q)}\int_Q \bar{\phi}(|g|)\nu \leq 1.$$

Finally, applying Young's inequality, we conclude that

$$\begin{aligned}\frac{1}{\nu(Q)}\int_Q |fg|\nu &\leq \frac{1}{\nu(Q)}\int_Q \phi(|f|)\nu + \frac{1}{\nu(Q)}\int_Q \bar{\phi}(|g|)\nu \\ &\leq 2 = 2\|f\|_{\phi,Q,\nu}\|g\|_{\bar{\phi},Q,\nu}.\end{aligned}$$

□

Given a Young function ϕ, a weight ν, and a cube Q, $L^{\phi,\infty}(Q, \frac{\nu}{\nu(Q)})$ is the *weak Orlicz space* of measurable functions f on $\mathbb{R}^n$ such that for some $\lambda > 0$,

$$\sup_{t>0} \phi(\lambda t)\nu(|f|\chi_Q > t) < \infty.$$

For $f \in L^{\phi,\infty}(Q, \frac{\nu}{\nu(Q)})$, define the quasi-norm

$$\|f\|_{L^{\phi,\infty}(Q,\nu/\nu(Q))} := \inf\left\{\lambda > 0 : \sup_{t>0} \phi(\lambda^{-1}t)\nu(|f|\chi_Q > t) \leq \nu(Q)\right\}.$$

Given a measurable set $E \subseteq \mathbb{R}^n$ with $0 < \nu(E) < \infty$,

$$\|\chi_E\|_{L^{\phi,\infty}(Q,\nu/\nu(Q))} = \|\chi_E\|_{L^{\phi}(Q,\nu/\nu(Q))} = \frac{1}{\phi^{-1}(\frac{\nu(Q)}{\nu(E\cap Q)})},$$

and in general, $\|f\|_{L^{\phi,\infty}(Q,\nu/\nu(Q))} \leq \|f\|_{L^{\phi}(Q,\nu/\nu(Q))}$. If we take $\phi(t) = t^p$, with $0 < p < \infty$, then

$$\begin{aligned}\|f\|_{L^{\phi,\infty}(Q,\nu/\nu(Q))} &= \sup_{t>0} t\left(\frac{\nu(|f|\chi_Q > t)}{\nu(Q)}\right)^{1/p} \\ &= \|f\chi_Q\|_{L^{p,\infty}(\nu/\nu(Q))} =: \|f\|_{L^{p,\infty}(Q,\nu/\nu(Q))}.\end{aligned}$$

5.2 Bounds for Fractional Operators

In this section we will prove weighted restricted weak type bounds for the fractional maximal operators $\mathcal{M}_\alpha^c$ and $\mathcal{M}_\alpha$ and, as a consequence of Theorem 5.1.4, we will be able to obtain the same type of bounds for the fractional integral $\mathcal{I}_\alpha$.

Let us start by proving the following summation lemma. It is an extension of [17, Lemma 2.5], and the proof is similar.

Lemma 5.2.1. *Let $0 < p, q < \infty$ and $\gamma \geq \max\{p, q\}$. Given a measurable function f and a weight w, if $\{E_j\}_{j\geq 1}$ is a collection of measurable sets such that $\sum_{j\geq 1} \chi_{E_j} \leq C$, then*

$$\sum_{j\geq 1} \|\chi_{E_j} f\|^{\gamma}_{L^{p,q}(w)} \leq C^{\gamma/p} \|f\|^{\gamma}_{L^{p,q}(w)}.$$

Proof. Since $\frac{\gamma}{q} \geq 1$, applying Minkowski's integral inequality we have that

$$
\begin{aligned}
I &:= \left(\sum_{j\geq 1} \|\chi_{E_j} f\|^{\gamma}_{L^{p,q}(w)} \right)^{q/\gamma} = \left\| \left\{ \|\chi_{E_j} f\|^{q}_{L^{p,q}(w)} \right\}_{j\geq 1} \right\|_{\ell^{\gamma/q}} \\
&= \left\| \left\{ p \int_0^\infty w(\{x \in E_j : |f(x)| > y\})^{q/p} y^{q-1} dy \right\}_{j\geq 1} \right\|_{\ell^{\gamma/q}} \\
&\leq p \int_0^\infty \left\| \left\{ w(\{x \in E_j : |f(x)| > y\})^{q/p} \right\}_{j\geq 1} \right\|_{\ell^{\gamma/q}} y^{q-1} dy \\
&= p \int_0^\infty \left(\sum_{j\geq 1} w(\{x \in E_j : |f(x)| > y\})^{\gamma/p} \right)^{q/\gamma} y^{q-1} dy =: II.
\end{aligned}
$$

Since $\frac{\gamma}{p} \geq 1$, by [44, Exercise 1.1.4.(b)] and the hypotheses, we get that

$$
\begin{aligned}
&\sum_{j\geq 1} w(\{x \in E_j : |f(x)| > y\})^{\gamma/p} \leq \left(\sum_{j\geq 1} w(\{x \in E_j : |f(x)| > y\}) \right)^{\gamma/p} \\
&= \left(\int_{\mathbb{R}^n} \sum_{j\geq 1} \chi_{E_j} \chi_{\{|f|>y\}} w \right)^{\gamma/p} \leq C^{\gamma/p} w(\{|f| > y\})^{\gamma/p}.
\end{aligned}
$$

Hence, $I \leq II \leq C^{q/p} \|f\|^q_{L^{p,q}(w)}$, and the result follows. $\square$

We can now give the characterization of the weights for which the operators $\mathcal{M}_\alpha$ and $\mathcal{M}^c_\alpha$ are bounded in the restricted weak type setting. We use ideas from [17, Section 3] and [44, Theorem 7.1.9].

Theorem 5.2.2. *Let $0 \leq \alpha < nm$, $1 \leq p_1, \ldots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, and $p \leq q$. Let $w_1, \ldots, w_m$, and v be weights. The inequality*

$$
\|\mathcal{M}_\alpha(\vec{f})\|_{L^{q,\infty}(v)} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i,1}(w_i)} \tag{5.2.1}
$$

holds for every vector of measurable functions $\vec{f}$ if, and only if

$$
[\vec{w}, v]_{A^{\mathcal{R}}_{\vec{P},q,\alpha}} := \sup_Q v(Q)^{1/q} \prod_{i=1}^m \frac{\|\chi_Q w_i^{-1}\|_{L^{p_i',\infty}(w_i)}}{|Q|^{1-\frac{\alpha}{nm}}} < \infty. \tag{5.2.2}
$$

Proof. First, recall that by [44, Theorem 1.4.16.(v)] (see also [3, Theorem 4.4]), we have that

$$
\frac{1}{p_i} \|g\|_{L^{p_i',\infty}(w_i)} \leq \sup \left\{ \int_{\mathbb{R}^n} |fg| w_i : \|f\|_{L^{p_i,1}(w_i)} \leq 1 \right\} \leq \|g\|_{L^{p_i',\infty}(w_i)}.
$$

Now, fix a cube Q, and $\varepsilon > 1$, and for $i = 1, \dots, m$, choose a non-negative function f_i such that $\|f_i\|_{L^{p_i,1}(w_i)} \leq 1$ and

$$\int_Q f_i = \int_{\mathbb{R}^n} f_i(\chi_Q w_i^{-1}) w_i \geq \frac{1}{\varepsilon p_i} \|\chi_Q w_i^{-1}\|_{L^{p_i',\infty}(w_i)}. \tag{5.2.3}$$

Since

$$\left(\prod_{i=1}^m \frac{1}{|Q|^{1-\frac{\alpha}{nm}}} \int_Q |f_i| \right) \chi_Q \leq \mathcal{M}_\alpha(\vec{f}),$$

the hypothesis (5.2.1) and (5.2.3) imply that

$$\nu(Q)^{1/q} \prod_{i=1}^m \frac{\|\chi_Q w_i^{-1}\|_{L^{p_i',\infty}(w_i)}}{|Q|^{1-\frac{\alpha}{nm}}} \leq \varepsilon^m p_1 \dots p_m C,$$

and hence, $[\vec{w}, \nu]_{A^{\mathcal{R}}_{\vec{P},q,\alpha}} \leq p_1 \dots p_m C < \infty$.

For the converse, suppose that the quantity $[\vec{w}, \nu]_{A^{\mathcal{R}}_{\vec{P},q,\alpha}} < \infty$. Observe that by Lemma 5.1.2, it suffices to establish the result for the operator $\mathcal{M}^c_\alpha$.

If for some $i = 1, \dots, m$, $\|f_i\|_{L^{p_i,1}(w_i)} = \infty$, then there is nothing to prove, so we may assume that $\|f_i\|_{L^{p_i,1}(w_i)} < \infty$ for every $i = 1, \dots, m$. Fix $\lambda > 0$, and let $E_\lambda := \{x \in \mathbb{R}^n : \mathcal{M}^c_\alpha(\vec{f})(x) > \lambda\}$. We first show that this set is open. If for some $i = 1, \dots, m$, $f_i \notin L^1_{loc}(\mathbb{R}^n)$, then $E_\lambda = \mathbb{R}^n$. Otherwise, observe that for any $r > 0$, and $x \in \mathbb{R}^n$, the function

$$x \longmapsto \prod_{i=1}^m \frac{1}{|Q(x,r)|^{1-\frac{\alpha}{nm}}} \int_{Q(x,r)} |f_i|$$

is continuous. Indeed, if $x_n \to x_0$, then $|Q(x_n,r)|^{1-\frac{\alpha}{nm}} \to |Q(x_0,r)|^{1-\frac{\alpha}{nm}}$, and also $\int_{Q(x_n,r)} |f_i| \to \int_{Q(x_0,r)} |f_i|$ by Lebesgue's dominated convergence theorem. Since $|Q(x_0,r)|^{1-\frac{\alpha}{nm}} \neq 0$, the result follows. This implies that $\mathcal{M}^c_\alpha(\vec{f})$ is the supremum of continuous functions and hence, it is lower semi-continuous, and the set E_λ is open.

Given K a compact subset of E_λ, for any $x \in K$, select an open cube Q_x centered at x such that

$$\prod_{i=1}^m \frac{1}{|Q_x|^{1-\frac{\alpha}{nm}}} \int_{Q_x} |f_i| > \lambda.$$

In virtue of [44, Lemma 7.1.10], we find a subset $\{Q_j\}_{j=1}^N$ of $\{Q_x : x \in K\}$ such that $K \subseteq \bigcup_{j=1}^N Q_j$, and $\sum_{j=1}^N \chi_{Q_j} \leq 24^n$. Then, by Hölder's inequality, (5.2.2), discrete Hölder's inequality with exponents $\frac{q_i}{q} := \frac{p_i}{p}$, and Lemma 5.2.1,

$$\begin{aligned}
\nu(K) &\leq \sum_{j=1}^{N} \nu(Q_j) \leq \frac{1}{\lambda^q} \sum_{j=1}^{N} \nu(Q_j) \left(\prod_{i=1}^{m} \frac{1}{|Q_j|^{1-\frac{\alpha}{nm}}} \int_{Q_j} |f_i| \right)^q \\
&\leq \frac{1}{\lambda^q} \sum_{j=1}^{N} \nu(Q_j) \prod_{i=1}^{m} |Q_j|^{\frac{q\alpha}{nm}-q} \|f_i \chi_{Q_j}\|^q_{L^{p_i,1}(w_i)} \|\chi_{Q_j} w_i^{-1}\|^q_{L^{p_i',\infty}(w_i)} \\
&\leq \frac{[\vec{w},\nu]^q_{A^{\mathcal{R}}_{\vec{P},q,\alpha}}}{\lambda^q} \sum_{j=1}^{N} \prod_{i=1}^{m} \|f_i \chi_{Q_j}\|^q_{L^{p_i,1}(w_i)} \\
&\leq \frac{[\vec{w},\nu]^q_{A^{\mathcal{R}}_{\vec{P},q,\alpha}}}{\lambda^q} \prod_{i=1}^{m} \left(\sum_{j=1}^{N} \|f_i \chi_{Q_j}\|^{q_i}_{L^{p_i,1}(w_i)} \right)^{q/q_i} \\
&\leq 24^{\frac{qn}{p}} \frac{[\vec{w},\nu]^q_{A^{\mathcal{R}}_{\vec{P},q,\alpha}}}{\lambda^q} \prod_{i=1}^{m} \|f_i\|^q_{L^{p_i,1}(w_i)}.
\end{aligned}$$

Taking the supremum over all compact subsets K of E_λ, using the inner regularity of $\nu(x)dx$, and applying Lemma 5.1.2, we obtain (5.2.1) with constant

$$C = 2^{nm-\alpha} 24^{n/p} [\vec{w},\nu]_{A^{\mathcal{R}}_{\vec{P},q,\alpha}}.$$

□

Remark 5.2.3. In fact, we have proved that

$$\frac{2^{\alpha-nm}}{\prod_{i=1}^m p_i} [\vec{w},\nu]_{A^{\mathcal{R}}_{\vec{P},q,\alpha}} \leq \|\mathcal{M}^c_\alpha\|_{\prod_{i=1}^m L^{p_i,1}(w_i) \to L^{q,\infty}(\nu)} \leq 24^{n/p} [\vec{w},\nu]_{A^{\mathcal{R}}_{\vec{P},q,\alpha}},$$

and that

$$\frac{1}{\prod_{i=1}^m p_i} [\vec{w},\nu]_{A^{\mathcal{R}}_{\vec{P},q,\alpha}} \leq \|\mathcal{M}_\alpha\|_{\prod_{i=1}^m L^{p_i,1}(w_i) \to L^{q,\infty}(\nu)} \leq 2^{nm-\alpha} 24^{n/p} [\vec{w},\nu]_{A^{\mathcal{R}}_{\vec{P},q,\alpha}}.$$

Remark 5.2.4. Observe that for $w_1 = \cdots = w_m = \nu = 1$, $[\vec{w},\nu]_{A^{\mathcal{R}}_{\vec{P},q,\alpha}} < \infty$ if, and only if $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$. In general, if $\mathcal{M}_\alpha$ is bounded as in (5.2.1), then for every cube Q, if we choose $f_1 = \cdots = f_m = \chi_Q$, we get that

$$|Q|^{\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}} \left(\fint_Q \nu \right)^{1/q} \leq p_1 \dots p_m C \prod_{i=1}^{m} \left(\fint_Q w_i \right)^{1/p_i},$$

and this condition implies that $\frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{n}$. Indeed, if this is not the case, then Lebesgue's differentiation theorem implies that $\nu = 0$ a.e., which is not true.

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In virtue of Theorem 5.2.2, we define the following class of weights.

Definition 5.2.5. Let $0 \leq \alpha < nm$, $1 \leq p_1, \ldots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, and $p \leq q$. Let $w_1, \ldots, w_m$, and ν be weights. We say that $(w_1, \ldots, w_m, \nu)$ belongs to the class $A^{\mathcal{R}}_{\vec{P},q,\alpha}$ if $[\vec{w}, \nu]_{A^{\mathcal{R}}_{\vec{P},q,\alpha}} < \infty$. For $\alpha = 0$ and $q = p$, we write $A^{\mathcal{R}}_{\vec{P}} := A^{\mathcal{R}}_{\vec{P},p,0}$.

The condition that defines the class of $A^{\mathcal{R}}_{\vec{P},q,\alpha}$ weights depends on their behavior on cubes, and has been obtained following the ideas of Chung, Hunt and Kurtz (see [17]). One can ask if it is possible to obtain a different condition, resembling the one obtained by Kerman and Torchinsky (see [58]). Our next theorem gives a positive answer to this question, recovering their results in the case when $m = 1$, $\alpha = 0$, $p_1 = p = q$, and $w_1 = \nu$.

Theorem 5.2.6. *Let $0 \leq \alpha < nm$, $1 \leq p_1, \ldots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, and $p \leq q$. Let $w_1, \ldots, w_m$, and ν be weights. The following statements are equivalent:*

(a) $\|\mathcal{M}_\alpha(\vec{f})\|_{L^{q,\infty}(\nu)} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i,1}(w_i)}$, *for every* $\vec{f}$.

(b) $\|\mathcal{M}_\alpha(\vec{\chi})\|_{L^{q,\infty}(\nu)} \leq c \prod_{i=1}^m w_i(E_i)^{1/p_i}$, *for every* $\vec{\chi} = (\chi_{E_1}, \ldots, \chi_{E_m})$.

(c)

$$\|\vec{w}, \nu\|_{A^{\mathcal{R}}_{\vec{P},q,\alpha}} := \sup_Q \nu(Q)^{1/q} \prod_{i=1}^m \sup_{0<w_i(E_i)<\infty} \frac{|E_i \cap Q|}{|Q|^{1-\frac{\alpha}{nm}}} w_i(E_i)^{-\frac{1}{p_i}} < \infty.$$

(d) $(w_1, \ldots, w_m, \nu) \in A^{\mathcal{R}}_{\vec{P},q,\alpha}$.

Proof. It is clear that (a) implies (b), and we have already proved in Theorem 5.2.2 that (a) and (d) are equivalent. Let us show that (b) implies (c). Fix a cube Q and measurable sets E_i, for $i = 1, \ldots, m$, with $0 < w_i(E_i) < \infty$. Since

$$\left(\prod_{i=1}^m \frac{|E_i \cap Q|}{|Q|^{1-\frac{\alpha}{nm}}} \right) \chi_Q \leq \mathcal{M}_\alpha(\vec{\chi}),$$

we apply (b) to conclude that

$$\nu(Q)^{1/q} \prod_{i=1}^m \frac{|E_i \cap Q|}{|Q|^{1-\frac{\alpha}{nm}}} \leq c \prod_{i=1}^m w_i(E_i)^{1/p_i},$$

and hence, $\|\vec{w}, \nu\|_{A^{\mathcal{R}}_{\vec{P},q,\alpha}} \leq c < \infty$.

To finish the proof, we will prove that (c) is equivalent to (d). First, observe that for every $i = 1, \ldots, m$,

$$\sup_{0<w_i(E_i)<\infty} \frac{|E_i \cap Q|}{w_i(E_i)^{1/p_i}} = \sup_{E_i \subseteq Q} \frac{|E_i|}{w_i(E_i)^{1/p_i}},$$

where the first supremum is taken over all measurable sets E_i such that $0 < w_i(E_i) < \infty$, and the second one is taken over all non-empty measurable sets $E_i \subseteq Q$. Now, in virtue of [17, Lemma 2.8] and Kolmogorov's inequalities, we have that

$$\|\chi_Q w_i^{-1}\|_{L^{p_i',\infty}(w_i)} \leq \sup_{E_i \subseteq Q} \frac{|E_i|}{w_i(E_i)^{1/p_i}} \leq p_i \|\chi_Q w_i^{-1}\|_{L^{p_i',\infty}(w_i)},$$

and hence, $[\vec{w}, \nu]_{A^{\mathcal{R}}_{\vec{P},q,\alpha}} \leq \|\vec{w}, \nu\|_{A^{\mathcal{R}}_{\vec{P},q,\alpha}} \leq p_1 \dots p_m [\vec{w}, \nu]_{A^{\mathcal{R}}_{\vec{P},q,\alpha}}$. □

The following theorem gives some properties of the class of $A^{\mathcal{R}}_{\vec{P}}$ weights.

Theorem 5.2.7. *Let $1 \leq p_1, \dots, p_m < \infty$, with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. Let $w_1, \dots, w_m$, and ν be weights. The following statements are equivalent:*

(a) $\|\mathcal{M}(\vec{f})\|_{L^{p,\infty}(\nu)} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i,1}(w_i)}$, *for every* $\vec{f}$.

(b) $\|\mathcal{M}(\vec{\chi})\|_{L^{p,\infty}(\nu)} \leq c \prod_{i=1}^m w_i(E_i)^{1/p_i}$, *for every* $\vec{\chi} = (\chi_{E_1}, \dots, \chi_{E_m})$.

(c)

$$\|\vec{w}, \nu\|_{A^{\mathcal{R}}_{\vec{P}}} := \sup_Q \nu(Q)^{1/p} \prod_{i=1}^m \sup_{0 < w_i(E_i) < \infty} \frac{|E_i \cap Q|}{|Q|} w_i(E_i)^{-\frac{1}{p_i}} < \infty.$$

(d) $(w_1, \dots, w_m, \nu) \in A^{\mathcal{R}}_{\vec{P}}$.

Moreover, if $(w_1, \dots, w_m, \nu) \in A^{\mathcal{R}}_{\vec{P}}$, and $\nu \in A_\infty$, then

$$T : L^{p_1,1}(w_1) \times \cdots \times L^{p_m,1}(w_m) \longrightarrow L^{p,\infty}(\nu), \tag{5.2.4}$$

where T is either a sparse operator of the form

$$\mathcal{A}_{\mathcal{S}}(\vec{f}) := \sum_{Q \in \mathcal{S}} \left(\prod_{i=1}^m \fint_Q f_i \right) \chi_Q,$$

where $\mathcal{S}$ is an η-sparse collection of dyadic cubes, or any operator that can be conveniently dominated by such sparse operators, like m-linear ω-Calderón-Zygmund operators with ω satisfying the Dini condition.

Proof. The equivalences follow directly from Theorem 5.2.6 with $\alpha = 0$.

A similar argument to the one in the proof of Theorem 2.4.8 shows that for $0 < \varepsilon \leq 1$ such that $\varepsilon < p$, and $r \geq 1$ such that $\nu \in A^{\mathcal{R}}_r$,

$$\|\mathcal{M}_{\mathcal{S}}(\vec{f})\|_{L^{p,\infty}(\nu)} \leq \|\mathcal{A}_{\mathcal{S}}(|\vec{f}|)\|_{L^{p,\infty}(\nu)} \leq C_{\varepsilon,\eta,n,p,r} [\nu]^{r/\varepsilon}_{A^{\mathcal{R}}_r} \|\mathcal{M}_{\mathcal{S}}(\vec{f})\|_{L^{p,\infty}(\nu)}, \tag{5.2.5}$$

where

$$\mathcal{M}_{\mathcal{S}}(\vec{f}) := \sup_{Q \in \mathcal{S}} \left(\prod_{i=1}^m \fint_Q |f_i| \right) \chi_Q,$$

and since $\mathcal{S}$ is a countable collection of dyadic cubes, the proof of Theorem 5.2.2 can be rewritten to show that

$$\mathcal{M}_{\mathcal{S}} : L^{p_1,1}(w_1) \times \cdots \times L^{p_m,1}(w_m) \longrightarrow L^{p,\infty}(\nu)$$

if, and only if

$$[w,\nu]_{A^{\mathcal{R}}_{\vec{p},\mathcal{S}}} := \sup_{Q\in\mathcal{S}} \nu(Q)^{1/p} \prod_{i=1}^{m} \frac{\|\chi_Q w_i^{-1}\|_{L^{p_i',\infty}(w_i)}}{|Q|} < \infty,$$

which is true, since $[w,\nu]_{A^{\mathcal{R}}_{\vec{p},\mathcal{S}}} \leq [w,\nu]_{A^{\mathcal{R}}_{\vec{p}}} < \infty$. Moreover,

$$\frac{1}{\prod_{i=1}^m p_i}[w,\nu]_{A^{\mathcal{R}}_{\vec{p},\mathcal{S}}} \leq \|\mathcal{M}_{\mathcal{S}}\|_{\prod_{i=1}^m L^{p_i,1}(w_i)\to L^{p,\infty}(\nu)} \leq [w,\nu]_{A^{\mathcal{R}}_{\vec{p},\mathcal{S}}},$$

so (5.2.5) implies that

$$\frac{1}{\prod_{i=1}^m p_i}[w,\nu]_{A^{\mathcal{R}}_{\vec{p},\mathcal{S}}} \leq \|\mathcal{A}_{\mathcal{S}}\|_{\prod_{i=1}^m L^{p_i,1}(w_i)\to L^{p,\infty}(\nu)} \leq C_{\varepsilon,\eta,n,p,r}[\nu]^{r/\varepsilon}_{A^{\mathcal{R}}_r}[w,\nu]_{A^{\mathcal{R}}_{\vec{p},\mathcal{S}}}. \tag{5.2.6}$$

Finally, in virtue of Theorem 1.2 and Proposition 3.1 in [71] (see also [65, Theorem 3.1]), if T is an m-linear ω-Calderón-Zygmund operator with ω satisfying the Dini condition, then there exists a dimensional constant $0 < \eta < 1$ such that given compactly supported functions $f_i \in L^1(\mathbb{R}^n)$, $i = 1, \dots, m$, there exists an η-sparse collection of dyadic cubes $\mathcal{S}$ such that

$$|T(f_1,\dots,f_m)| \leq c_n C_T \mathcal{A}_{\mathcal{S}}(|\vec{f}|).$$

Hence, (5.2.4) follows from (5.2.6) and the standard density argument in [44, Exercise 1.4.17]. Moreover,

$$\|T\|_{\prod_{i=1}^m L^{p_i,1}(w_i)\to L^{p,\infty}(\nu)} \leq c_n C_T C_{\varepsilon,\eta,n,p,r}[\nu]^{r/\varepsilon}_{A^{\mathcal{R}}_r}[w,\nu]_{A^{\mathcal{R}}_{\vec{p}}}.$$

□

Remark 5.2.8. Given weights $w_1, \dots, w_m$, and $\nu = \prod_{i=1}^m w_i^{p/p_i}$, the equivalence between (b) and (c) in Theorem 5.2.7 can be found in [5]. Moreover, if $p_1 = \cdots = p_m = 1$, then the equivalence between (a) and (d) can be found in [69]. Observe that if $\vec{w} \in A_{\vec{p}}$, then $(w_1, \dots, w_m, \nu_{\vec{w}}) \in A^{\mathcal{R}}_{\vec{p}}$. In particular, $A_{\vec{p}} \subseteq A^{\mathcal{R}}_{\vec{p}}$. In [69], strong and weak type bounds for m-linear Calderón-Zygmund operators were established for tuples of weights in $A_{\vec{p}}$. In [77], these results were extended to m-linear ω-Calderón-Zygmund operators with $\|\omega\|_{\text{Dini}} < \infty$.

Remark 5.2.9. Concerning mixed type bounds, the proof of Theorem 5.2.2 can be easily modified to show that for $1 \leq \ell < m$,

$$\mathcal{M} : L^{p_1}(w_1) \times \cdots \times L^{p_\ell}(w_\ell) \times L^{p_{\ell+1},1}(w_{\ell+1}) \times \cdots \times L^{p_m,1}(w_m) \longrightarrow L^{p,\infty}(\nu)$$

if, and only if

$$[\vec{w},\nu]_{A^{\mathfrak{M}}_{\vec{P},\ell}} := \sup_Q \nu(Q)^{1/p} \left(\prod_{i=1}^{\ell} \frac{\|\chi_Q w_i^{-1}\|_{L^{p_i'}(w_i)}}{|Q|} \right) \left(\prod_{i=\ell+1}^{m} \frac{\|\chi_Q w_i^{-1}\|_{L^{p_i',\infty}(w_i)}}{|Q|} \right) < \infty.$$

The case $\ell = m$ is the weak type bound for $\mathcal{M}$ proved in [69, Theorem 3.3]. Moreover, if $[\vec{w},\nu]_{A^{\mathfrak{M}}_{\vec{P},\ell}} < \infty$, and $\nu \in A_\infty$, then we can adapt the proof of Theorem 5.2.7 to obtain that

$$T : L^{p_1}(w_1) \times \cdots \times L^{p_\ell}(w_\ell) \times L^{p_{\ell+1},1}(w_{\ell+1}) \times \cdots \times L^{p_m,1}(w_m) \longrightarrow L^{p,\infty}(\nu),$$

where T is either $\mathcal{A}_S$ or any operator that can be conveniently dominated by such sparse operators.

To conclude this section, we give the restricted weak type bounds for the fractional integral $\mathcal{I}_\alpha$, that follow immediately from Theorem 5.1.4, Theorem 5.2.2, and the standard density argument in [44, Exercise 1.4.17].

Corollary 5.2.10. *Let* $0 < \alpha < nm$, $1 \le p_1, \dots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, *and* $p \le q$. *Let* $w_1, \dots, w_m$, *and* ν *be weights, with* $\nu \in A_\infty$. *The inequality*

$$\|\mathcal{I}_\alpha(\vec{f})\|_{L^{q,\infty}(\nu)} \le C \prod_{i=1}^{m} \|f_i\|_{L^{p_i,1}(w_i)}$$

holds for every vector of measurable functions $\vec{f}$ *if, and only if* $(w_1, \dots, w_m, \nu) \in A^{\mathcal{R}}_{\vec{P},q,\alpha}$.

Note that in virtue of the embedding relations between Lorentz spaces, the class $A^{\mathcal{R}}_{\vec{P},q,\alpha}$ contains the tuples of weights for which

$$\mathcal{I}_\alpha : \prod_{i=1}^{m} L^{p_i}(w_i) \longrightarrow L^{q,\infty}(\nu), \quad \text{or} \quad \mathcal{I}_\alpha : \prod_{i=1}^{m} L^{p_i}(w_i) \longrightarrow L^{q}(\nu).$$

This type of bounds were studied in [82].

5.3 Commutators of Linear Fractional and Singular Integrals

We will devote this section to the commutators of Calderón-Zygmund operators and fractional integrals. Our proofs will exploit sparse domination results that, to our knowledge, are only available in the literature for the one-variable case (see [21, 23]). We do believe that our techniques can be adapted and extended to cover the general case when combined with the right multi-variable sparse domination results. We are currently working on these topics, but we will not discuss our findings here.

Let us start with the following definition.

Definition 5.3.1. Given $0 \leq \alpha < n$, a Young function ϕ, a weight ν, a set of cubes $\mathcal{Q}$, and a measurable function f on $\mathbb{R}^n$, define

$$M^{\mathcal{Q}}_{\alpha,\phi,\nu}(f) := \sup_{Q\in\mathcal{Q}} \nu(Q)^{\alpha/n} \|f\|_{\phi,Q,\nu}\chi_Q.$$

If $\alpha = 0$, we omit the subindex α, and if $\nu = 1$, we omit the subindex ν. If $\mathcal{Q}$ is the set of all cubes in $\mathbb{R}^n$, we omit the symbol $\mathcal{Q}$ in the notation.

Now we state and prove some properties of the operator that we have just defined.

Lemma 5.3.2. *For $1 < p < \infty$, $B(t) = t\log(e+t)$, and a dyadic grid $\mathscr{D}$,*

$$M^{\mathscr{D}}_{B,\nu} : L^{p,1}(\nu) \longrightarrow L^{p,1}(\nu),$$

with constant independent of ν and $\mathscr{D}$.

Proof. We have that

$$\|f\|_{B,Q,\nu} \leq C_p \left(\frac{1}{\nu(Q)}\int_Q |f|^p \nu\right)^{1/p} \leq C_p \|f\|_{L^\infty(\nu)},$$

so $M^{\mathscr{D}}_{B,\nu} : L^\infty(\nu) \longrightarrow L^\infty(\nu)$, with constant independent of ν and $\mathscr{D}$. In virtue of [21, Lemma 2.6], for $1 < r < p$, $M^{\mathscr{D}}_{B,\nu} : L^r(\nu) \longrightarrow L^r(\nu)$, with constant independent of ν and $\mathscr{D}$. Finally, the desired result follows applying Marcinkiewicz's interpolation theorem (see [4, Theorem 4.13]). □

Theorem 5.3.3. *Let $0 \leq \alpha < n$, $1 < q$, and $1 \leq p \leq q$. Let w and ν be weights. Let ϕ be a Young function. The inequality*

$$\left\|M_{\alpha,\phi}(f)\right\|_{L^{q,\infty}(\nu)} \leq C \left\|f\right\|_{L^{p,1}(w)}$$

holds for every measurable function f if, and only if

$$\|w,\nu\|_{A^{\mathcal{R}}_{p,q,\alpha,\phi}} := \sup_Q \sup_{E\subseteq Q} |Q|^{\alpha/n} \left\|\chi_E\right\|_{\phi,Q} \frac{\nu(Q)^{1/q}}{w(E)^{1/p}} < \infty.$$

Proof. Fix a cube Q and a measurable set $E \subseteq Q$. We know that

$$|Q|^{\alpha/n}\|\chi_E\|_{\phi,Q}\chi_Q \leq M_{\alpha,\phi}(\chi_E),$$

and by hypothesis, we obtain that

$$|Q|^{\alpha/n}\|\chi_E\|_{\phi,Q}\nu(Q)^{1/q} \leq pCw(E)^{1/p},$$

so $\|w,\nu\|_{A^{\mathcal{R}}_{p,q,\alpha,\phi}} \leq pC < \infty$.

Conversely, assume that $\|w,\nu\|_{A^{\mathcal{R}}_{p,q,\alpha,\phi}} < \infty$. This implies that for every cube Q and every measurable set E,

$$|Q|^{\alpha/n}\|\chi_{E\cap Q}\|_{\phi,Q} \leq \|w,\nu\|_{A^{\mathcal{R}}_{p,q,\alpha,\phi}} \frac{w(E\cap Q)^{1/p}}{\nu(Q)^{1/q}}.$$

Now, fix a cube Q and a measurable set E. For every $x \in Q$, $Q \subseteq Q^x := Q(x,\ell_Q)$, and since $\ell_{Q^x} = 2\ell_Q$, we deduce that $\|\chi_{E\cap Q}\|_{\phi,Q} \leq 2^n \|\chi_{E\cap Q^x}\|_{\phi,Q^x}$. In particular,

$$\begin{aligned}|Q|^{\alpha/n}\|\chi_E\|_{\phi,Q} = |Q|^{\alpha/n}\|\chi_{E\cap Q}\|_{\phi,Q} &\leq 2^n|Q^x|^{\alpha/n}\|\chi_{E\cap Q^x}\|_{\phi,Q^x}\\ &\leq 2^n\|w,\nu\|_{A^{\mathcal{R}}_{p,q,\alpha,\phi}} \frac{w(E\cap Q^x)^{1/p}}{\nu(Q^x)^{1/q}},\end{aligned}$$

and hence, $M_{\alpha,\phi}(\chi_E)(x) \leq 2^n \|w,\nu\|_{A^{\mathcal{R}}_{p,q,\alpha,\phi}} \mathcal{N}_{p,q}(\chi_E)(x)^{1/p}$, where

$$\mathcal{N}_{p,q}(f)(x) := \sup_{r>0} \nu(Q(x,r))^{-\frac{p}{q}} \int_{Q(x,r)} |f|w.$$

Observe that the second part of the proof of Theorem 5.2.2 can be rewritten to show that

$$\|\mathcal{N}_{p,q}(f)\|_{L^{\frac{q}{p},\infty}(\nu)} \leq 24^n\|f\|_{L^1(w)},$$

and we can conclude that

$$\|M_{\alpha,\phi}(\chi_E)\|_{L^{q,\infty}(\nu)} \leq 2^n 24^{n/p}\|w,\nu\|_{A^{\mathcal{R}}_{p,q,\alpha,\phi}} w(E)^{1/p}.$$

Up to this point, everything works for $q = 1$, but the following extension argument only works if $L^{q,\infty}(\nu)$ is a Banach space, and that is why we need $q > 1$. Take $f \in L^{p,1}(w)$, and for every integer k, write $E_k := \{2^k < |f| \leq 2^{k+1}\}$. It is clear that $|f| \leq 2\sum_{k\in\mathbb{Z}} 2^k \chi_{E_k}$ a.e., so $M_{\alpha,\phi}(f) \leq 2\sum_{k\in\mathbb{Z}} 2^k M_{\alpha,\phi}(\chi_{E_k})$ and hence,

$$\begin{aligned}\|M_{\alpha,\phi}(f)\|_{L^{q,\infty}(\nu)} &\leq 2q' \sum_{k\in\mathbb{Z}} 2^k \|M_{\alpha,\phi}(\chi_{E_k})\|_{L^{q,\infty}(\nu)}\\ &\leq 2q'2^n24^{n/p}\|w,\nu\|_{A^{\mathcal{R}}_{p,q,\alpha,\phi}} \sum_{k\in\mathbb{Z}} 2^k w(E_k)^{1/p}\\ &\leq 2q'2^n24^{n/p}\|w,\nu\|_{A^{\mathcal{R}}_{p,q,\alpha,\phi}} \sum_{k\in\mathbb{Z}} 2^k w(\{|f|>2^k\})^{1/p}\\ &\leq 4q'2^n24^{n/p}\|w,\nu\|_{A^{\mathcal{R}}_{p,q,\alpha,\phi}} \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} w(\{|f|>t\})^{1/p}dt\\ &\leq 4\frac{q'}{p}2^n24^{n/p}\|w,\nu\|_{A^{\mathcal{R}}_{p,q,\alpha,\phi}} \|f\|_{L^{p,1}(w)}.\end{aligned}$$

□

Remark 5.3.4. In fact, we have proved that

$$\frac{1}{p}\|w,v\|_{A^{\mathcal{R}}_{p,q,\alpha,\phi}} \leq \|M_{\alpha,\phi}\|_{L^{p,1}(w)\to L^{q,\infty}(v)} \leq 4\frac{q'}{p}2^n 24^{n/p}\|w,v\|_{A^{\mathcal{R}}_{p,q,\alpha,\phi}}.$$

Observe that for $\phi(t)=t$, $\|w,v\|_{A^{\mathcal{R}}_{p,q,\alpha,\phi}} = \|w,v\|_{A^{\mathcal{R}}_{p,q,\alpha}}$, and we recover the condition that we obtained in Theorem 5.2.6 for $m=1$. Moreover, we can characterize the weights such that $\|w,v\|_{A^{\mathcal{R}}_{p,q,\alpha}} < \infty$ in terms of their behavior on cubes. At this point, one can ask if it is possible to do the same for a general Young function ϕ, or at least for $\phi(t)=B(t)=t\log(e+t)$, which will appear naturally in the study of commutators. The following result provides a partial answer to this question.

Theorem 5.3.5. *Let $0\leq\alpha<n$, and $1<p\leq q$. Let w and v be weights. Take $\psi(t)=B(t)^{p'}$. Then,*

$$[w,v]_{A^{\mathcal{R}}_{p,q,\alpha}} \lesssim \|w,v\|_{A^{\mathcal{R}}_{p,q,\alpha,B}} \lesssim \sup_Q |Q|^{\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}}\left(\fint_Q v\right)^{1/q}\|w^{-\frac{1}{p}}\|_{L^{\psi,\infty}(Q,1/|Q|)}.$$

Proof. To prove the first inequality, observe that for every cube Q and every measurable set $E\subseteq Q$,

$$\frac{|E|}{|Q|} = \|\chi_E\|_{\mathrm{Id},Q} \leq \|\chi_E\|_{B,Q},$$

and applying Theorem 5.2.6, we have that

$$[w,v]_{A^{\mathcal{R}}_{p,q,\alpha}} \approx \|w,v\|_{A^{\mathcal{R}}_{p,q,\alpha}} \leq \|w,v\|_{A^{\mathcal{R}}_{p,q,\alpha,B}}.$$

To prove the second inequality, it suffices to show that

$$\sup_{E\subseteq Q}\frac{\|\chi_E\|_{B,Q}}{w(E)^{1/p}} \lesssim |Q|^{-\frac{1}{p}}\|w^{-\frac{1}{p}}\|_{L^{\psi,\infty}(Q,1/|Q|)}. \tag{5.3.1}$$

For a non-empty measurable set E, and applying Hölder's inequality twice, we get that

$$1=\fint_E w^{\frac{1}{pp'}}w^{\frac{-1}{pp'}} \leq \left(\fint_E w^{1/p'}\right)^{1/p}\left(\fint_E w^{-\frac{1}{p}}\right)^{1/p'} \leq \left(\fint_E w\right)^{\frac{1}{pp'}}\left(\fint_E w^{-\frac{1}{p}}\right)^{1/p'}$$

and hence,

$$\left(\frac{|E|}{w(E)}\right)^{1/p} \leq \fint_E w^{-\frac{1}{p}}. \tag{5.3.2}$$

It is easy to see that

$$q_\psi := \inf_{t>0}\frac{t\psi'(t)}{\psi(t)} = p'\inf_{t>0}\left(1+\frac{t}{(e+t)\log(e+t)}\right) = p' > 1,$$

so in virtue of Kolmogorov's inequality for weak Orlicz spaces (see [76, Theorem 3.1]), and (5.3.2), we have that

$$\begin{aligned}\sup_{E\subseteq Q}\|\chi_E\|_{\psi,Q}\left(\frac{|E|}{w(E)}\right)^{1/p} &\le \sup_{E\subseteq Q}\|\chi_E\|_{\psi,Q}\fint_E w^{-\frac{1}{p}}\\ &= \sup_{0<|E|<\infty}\frac{\|\chi_E\|_{\psi,Q}}{|E|}\int_{E\cap Q} w^{-\frac{1}{p}}\\ &\le q'_\psi\|w^{-\frac{1}{p}}\|_{L^{\psi,\infty}(Q,1/|Q|)} = p\|w^{-\frac{1}{p}}\|_{L^{\psi,\infty}(Q,1/|Q|)}.\end{aligned} \tag{5.3.3}$$

Since $\psi(t)=B(t)^{p'}$, $\psi^{-1}(t)=B^{-1}(t^{1/p'})$. For $t\ge 0$, a straightforward computation shows that

$$B^{-1}(t)\approx\frac{t}{\log(e+t)}\approx t^{1/p}\frac{t^{1/p'}}{\log(e+t^{1/p'})}\approx t^{1/p}\psi^{-1}(t),$$

concluding that

$$\begin{aligned}\|\chi_E\|_{\psi,Q} &= \psi^{-1}\left(|Q|/|E|\right)^{-1}\\ &\approx B^{-1}\left(|Q|/|E|\right)^{-1}\left(\frac{|Q|}{|E|}\right)^{1/p} = \|\chi_E\|_{B,Q}\left(\frac{|Q|}{|E|}\right)^{1/p},\end{aligned}$$

and hence, (5.3.1) follows from (5.3.3). □

The case when $p=1$ in Theorem 5.3.5 is open. For $p>1$, our original guess was that the contribution of the weak Orlicz space should be in terms of $\|\chi_Q w^{-1}\|_{L^{\psi,\infty}(\mathbb{R}^n,w)}$, since that is the case when we work with $\phi(t)=t$ instead of $\phi(t)=B(t)$. But by the same type of argument,

$$\|\chi_Q w^{-1}\|_{L^{\psi,\infty}(\mathbb{R}^n,w)}\approx\sup_{E\subseteq Q}\frac{|E|}{w(E)^{1+\frac{1}{p}}}B^{-1}(w(E)^{-1})^{-1},$$

and this doesn't seem to work because there is no clear relation between $B^{-1}(w(E)^{-1})^{-1}$ and $\|\chi_E\|_{B,Q}$. Similarly, we have that

$$\|w^{-1}\|_{L^{\psi,\infty}(Q,w/w(Q))}\approx\sup_{E\subseteq Q}\frac{|E|}{w(E)^{1/p}}\frac{w(Q)^{1/p}}{w(E)}\|\chi_E\|_{B,Q,w},$$

and again, there is no clear relation between $\|\chi_E\|_{B,Q,w}$ and $\|\chi_E\|_{B,Q}$.

Now we define an operator that will play a significant role in the study of commutators.

Definition 5.3.6. Given $b\in BMO$, $0\le\alpha<n$, a set of cubes $\mathcal{Q}$, and a measurable function f on $\mathbb{R}^n$, we define the operator $C_b^{\mathcal{Q}}$ as

$$C_b^{\mathcal{Q}}(f)(x):=\sum_{Q\in\mathcal{Q}}|Q|^{\alpha/n}\left(\fint_Q|b(x)-b(y)|f(y)dy\right)\chi_Q(x),\quad x\in\mathbb{R}^n.$$

We have the following bound for such operator.

Theorem 5.3.7. *Let $0 \leq \alpha < n$, $1 < q$, and $1 \leq p \leq q$. Let w and v be weights, with $v \in A_\infty$ and $[w,v]_{A^{\mathcal{R}}_{p,q,\alpha,B}} < \infty$. Let $b \in BMO$. Let $\mathscr{D}$ be a dyadic grid, and let $\mathcal{S} \subseteq \mathscr{D}$ be an η-sparse collection of cubes. Then,*

$$C_b^{\mathcal{S}} : L^{p,1}(w) \longrightarrow L^{q,\infty}(v).$$

Proof. Without loss of generality, we can assume that $f \geq 0$. In virtue of Kolmogorov's inequality (see [44, Exercise 1.1.12]), and since $q > 1$, we have that

$$\|C_b^{\mathcal{S}}(f)\|_{L^{q,\infty}(v)} \leq \sup_{0<v(F)<\infty} \|C_b^{\mathcal{S}}(f)\chi_F\|_{L^1(v)} v(F)^{-\frac{1}{q'}},$$

where the supremum is taken over all measurable sets F with $0 < v(F) < \infty$. For one of such sets F, we have that

$$\begin{aligned}
&\|C_b^{\mathcal{S}}(f)\chi_F\|_{L^1(v)} \\
&= \int_{\mathbb{R}^n} \sum_{Q\in\mathcal{S}} \chi_Q(x)|Q|^{\alpha/n} \left(\fint_Q |b(x)-b(y)|f(y)dy \right) \chi_F(x)v(x)dx \\
&\leq \int_{\mathbb{R}^n} \sum_{Q\in\mathcal{S}} \chi_Q(x)|Q|^{\alpha/n} \left(\fint_Q |b(x)-b_Q|f(y)dy \right) \chi_F(x)v(x)dx \\
&+ \int_{\mathbb{R}^n} \sum_{Q\in\mathcal{S}} \chi_Q(x)|Q|^{\alpha/n} \left(\fint_Q |b(y)-b_Q|f(y)dy \right) \chi_F(x)v(x)dx =: I + II.
\end{aligned} \tag{5.3.4}$$

Since $v \in A_\infty$, there exists $s \geq 1$ such that $v \in A_s^{\mathcal{R}}$, and hence,

$$\sup_Q \sup_{E\subseteq Q} \frac{|E|}{|Q|} \left(\frac{v(Q)}{v(E)} \right)^{1/s} = \|v\|_{A_s^{\mathcal{R}}} < \infty.$$

In particular, for each $Q \in \mathcal{S}$, $v(3Q) \leq \left(\frac{3^n}{\eta} \|v\|_{A_s^{\mathcal{R}}} \right)^s v(E_Q)$. Observe that the sides of an n-dimensional cube have Lebesgue measure 0 in $\mathbb{R}^n$, so we can assume that the cubes in $\mathcal{S}$ are open. For $Q \in \mathcal{S}$ and $z \in E_Q$, we define $Q^z := Q(z, \ell_Q)$, the open cube of center z and side length twice the side length of Q. We have that $E_Q \subseteq Q \subseteq Q^z \subseteq 3Q$, so

$$|Q|^{\alpha/n} \|f\|_{B,Q} \chi_Q(z) \leq M_{\alpha,B}^{\mathscr{D}}(f)(z),$$

and

$$\frac{1}{v(3Q)} \int_Q \chi_F v \leq \frac{1}{v(Q^z)} \int_{Q^z} \chi_F v \leq M_v^c(\chi_F)(z).$$

Using these estimates, Theorem 5.1.6, and the fact that for $B(t) = t\log(e+t)$,

$$\|b - b_Q\|_{\bar{B},Q} \leq C_{n,B} \|b\|_{BMO}, \tag{5.3.5}$$

as shown in [21, Lemma 5.1], we can bound the term II in (5.3.4) as follows:

$$
\begin{aligned}
II &\le 2\sum_{Q\in\mathcal{S}} \|b-b_Q\|_{\bar{B},Q}|Q|^{\alpha/n}\|f\|_{B,Q}\left(\frac{1}{\nu(3Q)}\int_Q \chi_F \nu\right)\nu(3Q)\\
&\le 2C_{n,B}\left(\frac{3^n}{\eta}\|\nu\|_{A_s^{\mathcal{R}}}\right)^s \|b\|_{BMO}\sum_{Q\in\mathcal{S}}\int_{E_Q}|Q|^{\alpha/n}\|f\|_{B,Q}\chi_Q M_\nu^c(\chi_F)\nu\\
&\le 2C_{n,B}\left(\frac{3^n}{\eta}\|\nu\|_{A_s^{\mathcal{R}}}\right)^s \|b\|_{BMO}\int_{\mathbb{R}^n} M_{\alpha,B}^{\mathcal{D}}(f)M_\nu^c(\chi_F)\nu \qquad (5.3.6)\\
&\le 2C_{n,B}\left(\frac{3^n}{\eta}\|\nu\|_{A_s^{\mathcal{R}}}\right)^s \|b\|_{BMO}\|M_{\alpha,B}^{\mathcal{D}}(f)\|_{L^{q,\infty}(\nu)}\|M_\nu^c(\chi_F)\|_{L^{q',1}(\nu)}\\
&\le C_{n,q,B}^{II}\left(\frac{3^n}{\eta}\|\nu\|_{A_s^{\mathcal{R}}}\right)^s \|b\|_{BMO}\|M_{\alpha,B}^{\mathcal{D}}(f)\|_{L^{q,\infty}(\nu)}\nu(F)^{1/q'},
\end{aligned}
$$

where $C_{n,q,B}^{II} := 2q'C_{n,B}\|M_\nu^c\|_{L^{q',1}(\nu)}$, independent of ν (see [44, Theorem 7.1.9] and [4, Theorem 4.13]).

Similarly, we can bound the term I in (5.3.4) using Theorem 5.1.6, Lemma 5.3.2, and the fact that

$$
\|b-b_Q\|_{\bar{B},Q,\nu} \le C_{n,B}[\nu]_{A_\infty}\|b\|_{BMO}, \qquad (5.3.7)
$$

proved in [21, Lemma 5.1], obtaining that

$$
\begin{aligned}
I &\le 2\left(\frac{3^n}{\eta}\|\nu\|_{A_s^{\mathcal{R}}}\right)^s \sum_{Q\in\mathcal{S}}|Q|^{\alpha/n}\left(⨍_Q f\right)\|b-b_Q\|_{\bar{B},Q,\nu}\|\chi_F\|_{B,Q,\nu}\nu(E_Q)\\
&\le 2C_{n,B}\left(\frac{3^n}{\eta}\|\nu\|_{A_s^{\mathcal{R}}}\right)^s [\nu]_{A_\infty}\|b\|_{BMO}\int_{\mathbb{R}^n} M_\alpha^{\mathcal{D}}(f)M_{B,\nu}^{\mathcal{D}}(\chi_F)\nu\\
&\le 2C_{n,B}\left(\frac{3^n}{\eta}\|\nu\|_{A_s^{\mathcal{R}}}\right)^s [\nu]_{A_\infty}\|b\|_{BMO}\|M_\alpha^{\mathcal{D}}(f)\|_{L^{q,\infty}(\nu)}\|M_{B,\nu}^{\mathcal{D}}(\chi_F)\|_{L^{q',1}(\nu)}\\
&\le C_{n,q,B}^{I}\left(\frac{3^n}{\eta}\|\nu\|_{A_s^{\mathcal{R}}}\right)^s [\nu]_{A_\infty}\|b\|_{BMO}\|M_\alpha^{\mathcal{D}}(f)\|_{L^{q,\infty}(\nu)}\nu(F)^{1/q'},
\end{aligned}
\qquad (5.3.8)
$$

where $C_{n,q,B}^{I} := 2q'C_{n,B}\|M_{B,\nu}^{\mathcal{D}}\|_{L^{q',1}(\nu)}$, independent of ν and $\mathcal{D}$. It is worth mentioning that it follows immediately from Theorem 5.1.6 (taking $g=1$) that $M_\alpha^{\mathcal{D}}(f) \le c_B M_{\alpha,B}^{\mathcal{D}}(f)$.

Combining all the previous estimates, we get that

$$
\|C_b^{\mathcal{S}}(f)\|_{L^{q,\infty}(\nu)} \le C_{n,q,B}\left(\frac{3^n}{\eta}\|\nu\|_{A_s^{\mathcal{R}}}\right)^s [\nu]_{A_\infty}\|b\|_{BMO}\|M_{\alpha,B}^{\mathcal{D}}(f)\|_{L^{q,\infty}(\nu)},
$$

with $C_{n,q,B} := 2q'C_{n,B}\max\{c_B\|M_{B,\nu}^{\mathcal{D}}\|_{L^{q',1}(\nu)}, \|M_\nu^c\|_{L^{q',1}(\nu)}\}$.

Finally, the desired result follows from Theorem 5.3.3, and

$$\begin{aligned}&\|C_b^{\mathcal{S}}\|_{L^{p,1}(w)\to L^{q,\infty}(v)}\\ &\leq 4\frac{q'}{p}2^n 24^{n/p} C_{n,q,B}\|w,v\|_{A^{\mathcal{R}}_{p,q,\alpha,B}}\left(\frac{3^n}{\eta}\|v\|_{A_s^{\mathcal{R}}}\right)^s [v]_{A_\infty}\|b\|_{BMO}.\end{aligned}$$

□

Remark 5.3.8. If $b \in L^\infty(\mathbb{R}^n)$, in the previous argument one can apply Kolmogorov's inequality with exponent $0 < r < 1$ and control the terms I and II with the operator M_α. With these modifications, one can bound $C_b^{\mathcal{S}}$ for weights w and v such that $[w,v]_{A^{\mathcal{R}}_{p,q,\alpha}} < \infty$ (no logarithmic bump is required), and also for the case $q = 1$. Nevertheless, for $\alpha \neq 0$, the classical exponent q given by the relation $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ is strictly bigger than 1, so this case is also covered by the previous theorem.

We can now derive restricted weak type bounds for commutators of Calderón-Zygmund operators.

Theorem 5.3.9. *Let T be a ω-Calderón-Zygmund operator with ω satisfying the Dini condition, and let $b \in BMO$. Let $1 < q$, and $1 \leq p \leq q$. Let w and v be weights, with $v \in A_\infty$. Moreover, suppose that*

$$\|w,v\|_{A^{\mathcal{R}}_{p,q,B}} := \sup_{Q}\sup_{E\subseteq Q}\|\chi_E\|_{B,Q}\frac{v(Q)^{1/q}}{w(E)^{1/p}} < \infty,$$

where $B(t) = t\log(e+t)$. Then,

$$[b,T] : L^{p,1}(w) \longrightarrow L^{q,\infty}(v).$$

Proof. Without loss of generality, let f be a bounded function with compact support. In virtue of Theorem 1.1 in [70], there exist 3^n dyadic grids $\mathscr{D}_j$, and $\frac{1}{2\cdot 9^n}$-sparse families $\mathcal{S}_j \subseteq \mathscr{D}_j$ such that for a.e. $x \in \mathbb{R}^n$,

$$|[b,T]f(x)| \leq c_n C_T \sum_{j=1}^{3^n}\left(\mathcal{T}_{\mathcal{S}_j,b}(|f|)(x) + \mathcal{T}^\star_{\mathcal{S}_j,b}(|f|)(x)\right),$$

where

$$\mathcal{T}_{\mathcal{S}_j,b}(|f|)(x) := \sum_{Q\in\mathcal{S}_j}\left(\fint_Q |b(x)-b_Q||f(y)|dy\right)\chi_Q(x),$$

and

$$\mathcal{T}^\star_{\mathcal{S}_j,b}(|f|)(x) := \sum_{Q\in\mathcal{S}_j}\left(\fint_Q |b(y)-b_Q||f(y)|dy\right)\chi_Q(x).$$

In particular,

$$\|[b,T]f\|_{L^{q,\infty}(v)} \leq 2\cdot 3^n c_n C_T\sum_{j=1}^{3^n}\left(\|\mathcal{T}_{\mathcal{S}_j,b}(|f|)\|_{L^{q,\infty}(v)} + \|\mathcal{T}^\star_{\mathcal{S}_j,b}(|f|)\|_{L^{q,\infty}(v)}\right).$$

Applying Kolmogorov's inequality, and since $q > 1$, we have that

$$\|\mathcal{T}_{\mathcal{S}_j,b}(|f|)\|_{L^{q,\infty}(\nu)} \le \sup_{0<\nu(F)<\infty} \|\mathcal{T}_{\mathcal{S}_j,b}(|f|)\chi_F\|_{L^1(\nu)}\nu(F)^{-\frac{1}{q'}},$$

where the supremum is taken over all measurable sets F with $0 < \nu(F) < \infty$. For one of such sets F, we have that

$$\begin{aligned}&\|\mathcal{T}_{\mathcal{S}_j,b}(|f|)\chi_F\|_{L^1(\nu)}\\ &= \int_{\mathbb{R}^n} \sum_{Q\in\mathcal{S}_j} \chi_Q(x) \left(\fint_Q |b(x) - b_Q||f(y)|dy \right) \chi_F(x)\nu(x)dx.\end{aligned}$$

Observe that we have recovered the term I that appeared in (5.3.4), with $\alpha = 0$, and we can bound it as in (5.3.8), obtaining that

$$\|\mathcal{T}_{\mathcal{S}_j,b}(|f|)\|_{L^{q,\infty}(\nu)} \le c_B C^I_{n,q,B} \left(2\cdot 27^n \|\nu\|_{A_s^{\mathcal{R}}}\right)^s [\nu]_{A_\infty} \|b\|_{BMO} \|M_B(f)\|_{L^{q,\infty}(\nu)}.$$

Similarly, and following the computations in (5.3.6) to bound the term II in (5.3.4), we obtain that

$$\|\mathcal{T}^\star_{\mathcal{S}_j,b}(|f|)\|_{L^{q,\infty}(\nu)} \le C^{II}_{n,q,B} \left(2\cdot 27^n \|\nu\|_{A_s^{\mathcal{R}}}\right)^s \|b\|_{BMO} \|M_B(f)\|_{L^{q,\infty}(\nu)}.$$

Since the constants involved don't depend on j, combining all the previous estimates, we get that

$$\begin{aligned}&\|[b,T]f\|_{L^{q,\infty}(\nu)}\\ &\le 4\cdot 9^n c_n C_T C_{n,q,B} \left(2\cdot 27^n \|\nu\|_{A_s^{\mathcal{R}}}\right)^s [\nu]_{A_\infty} \|b\|_{BMO} \|M_B(f)\|_{L^{q,\infty}(\nu)},\end{aligned}$$

and the desired result follows from Theorem 5.3.3, with

$$\begin{aligned}&\|[b,T]\|_{L^{p,1}(w)\to L^{q,\infty}(\nu)}\\ &\le 16\frac{q'}{p} 18^n 24^{n/p} c_n C_T C_{n,q,B} \|w,\nu\|_{A^{\mathcal{R}}_{p,q,B}} \left(2\cdot 27^n \|\nu\|_{A_s^{\mathcal{R}}}\right)^s [\nu]_{A_\infty} \|b\|_{BMO}.\end{aligned}$$

□

Remark 5.3.10. Observe that for $1 < q = p$, and $w = \nu$, if $\|w,\nu\|_{A^{\mathcal{R}}_{p,q,B}} < \infty$, then it follows from Theorem 5.3.5 that $w \in A_p^{\mathcal{R}}$, and it is not difficult to check that if $w \in A_p$, then $\|w,w\|_{A^{\mathcal{R}}_{p,p,B}} < \infty$, using the open property of A_p. In particular, the classical diagonal bounds are covered by this result.

However, it is well known that, in general, $[b,T] : L^1(\mathbb{R}^n) \not\longrightarrow L^{1,\infty}(\mathbb{R}^n)$. For instance, following the ideas in [91, Section 5] and [69, Remark 7.7], one can show that for $b(x) = \log|x+1| \in BMO$ (see [98, Corollary 3]), $T = H$, the Hilbert transform on $\mathbb{R}$, and $f = \chi_{(0,1)}$,

$$\|[b,H]\chi_{(0,1)}\|_{L^{1,\infty}(\mathbb{R}^n)} \gtrsim \sup_{t>0} t \left|\left\{x > e : \frac{\log(x)}{x} > t\right\}\right| \geq \sup_{1\leq k\in\mathbb{N}} \frac{k}{e^k}(e^k - e) = \infty.$$

To finish this section, we present the restricted weak type bounds for the commutators of the linear fractional integral I_α.

Theorem 5.3.11. *Let $0 < \alpha < n$, $1 < q$, and $1 \leq p \leq q$. Let w and ν be weights, with $\nu \in A_\infty$ and $[w,\nu]_{A^{\mathcal{R}}_{p,q,\alpha,B}} < \infty$. Let $b \in BMO$. Then,*

$$[b, I_\alpha] : L^{p,1}(w) \longrightarrow L^{q,\infty}(\nu).$$

Proof. Without loss of generality, let f be a non-negative and bounded function with compact support. In virtue of Proposition 3.4 in [23], there exist 3^n dyadic grids $\mathscr{D}_j$ such that for a.e. $x \in \mathbb{R}^n$,

$$|[b, I_\alpha]f(x)| \leq C_{n,\alpha} \sum_{j=1}^{3^n} C_b^{\mathscr{D}_j}(f)(x).$$

In particular,

$$\|[b, I_\alpha]f\|_{L^{q,\infty}(\nu)} \leq 3^n C_{n,\alpha} \sum_{j=1}^{3^n} \|C_b^{\mathscr{D}_j}(f)\|_{L^{q,\infty}(\nu)}.$$

Observe that we can not apply Theorem 5.3.7 directly, because the grids $\mathscr{D}_j$ are not sparse, so we will have to work a little bit more. Applying Kolmogorov's inequality, and since $q > 1$, we have that

$$\|C_b^{\mathscr{D}_j}(f)\|_{L^{q,\infty}(\nu)} \leq \sup_{0<\nu(F)<\infty} \|C_b^{\mathscr{D}_j}(f)\chi_F\|_{L^1(\nu)}\nu(F)^{-\frac{1}{q'}},$$

where the supremum is taken over all measurable sets F with $0 < \nu(F) < \infty$. For one of such sets F, we have that

$$\begin{aligned}
&\|C_b^{\mathscr{D}_j}(f)\chi_F\|_{L^1(\nu)} \\
&= \int_{\mathbb{R}^n} \sum_{Q\in\mathscr{D}_j} \chi_Q(x)|Q|^{\alpha/n} \left(\fint_Q |b(x) - b(y)|f(y)dy\right) \chi_F(x)\nu(x)dx \\
&\leq \int_{\mathbb{R}^n} \sum_{Q\in\mathscr{D}_j} \chi_Q(x)|Q|^{\alpha/n} \left(\fint_Q |b(x) - b_Q|f(y)dy\right) \chi_F(x)\nu(x)dx \\
&+ \int_{\mathbb{R}^n} \sum_{Q\in\mathscr{D}_j} \chi_Q(x)|Q|^{\alpha/n} \left(\fint_Q |b(y) - b_Q|f(y)dy\right) \chi_F(x)\nu(x)dx =: I + II.
\end{aligned}$$

In virtue of Theorem 5.1.6 and (5.3.7), we have that

$$I \leq 2C_{n,B}[\nu]_{A_\infty}\|b\|_{BMO} \sum_{Q\in\mathscr{D}_j} |Q|^{\alpha/n}\nu(Q)\|\chi_F\|_{B,Q,\nu}\left(\fint_Q f\right).$$

We now want to replace the summation over cubes in $\mathscr{D}_j$ by a summation over a sparse subset $\mathcal{S}_j$ of $\mathscr{D}_j$. We achieve this using an argument from [21] (see also [25], [28, Appendix A] and [30]). Fix $a = 2^{n+1}$, and for each $k \in \mathbb{Z}$, consider the sets $\Omega_k^j := \{M^{\mathscr{D}_j}(f) > a^k\}$. Each of these sets is the union of a collection $\mathcal{S}_k^j$ of maximal, disjoint cubes in $\mathscr{D}_j$ such that $a^k < \fint_Q f \leq 2^n a^k$. Moreover, the set $\mathcal{S}_j := \bigcup_k \mathcal{S}_k^j$ is $\frac{1}{2}$-sparse. Let $\mathcal{C}_k^j := \{Q \in \mathscr{D}_j : a^k < \fint_Q f \leq a^{k+1}\}$. By the maximality of the cubes in $\mathcal{S}_k^j$, every $P \in \mathcal{C}_k^j$ is contained in a unique cube in $\mathcal{S}_k^j$. Therefore, applying Lemma 5.2 in [21] we get that

$$\begin{aligned}
I &\leq 2C_{n,B}[\nu]_{A_\infty}\|b\|_{BMO} \sum_{k\in\mathbb{Z}}\sum_{Q\in\mathcal{C}_k^j} |Q|^{\alpha/n}\nu(Q)\|\chi_F\|_{B,Q,\nu}\left(\fint_Q f\right)\\
&\leq 2^{n+2}C_{n,B}[\nu]_{A_\infty}\|b\|_{BMO} \sum_{k\in\mathbb{Z}} a^k \sum_{P\in\mathcal{S}_k^j}\sum_{Q\subseteq P} |Q|^{\alpha/n}\nu(Q)\|\chi_F\|_{B,Q,\nu}\\
&\leq 2^{n+2}c_\alpha C_{n,B}[\nu]_{A_\infty}\|b\|_{BMO} \sum_{k\in\mathbb{Z}}\sum_{P\in\mathcal{S}_k^j} |P|^{\alpha/n}\nu(P)\|\chi_F\|_{B,P,\nu}\left(\fint_P f\right)\\
&= 2^{n+2}c_\alpha C_{n,B}[\nu]_{A_\infty}\|b\|_{BMO} \sum_{P\in\mathcal{S}_j} |P|^{\alpha/n}\nu(P)\|\chi_F\|_{B,P,\nu}\left(\fint_P f\right).
\end{aligned}$$

Repeating the computations in (5.3.8), we conclude that

$$I \leq 2^{n+1}c_\alpha c_B C^I_{n,q,B}\left(2\cdot 3^n\|\nu\|_{A_s^{\mathcal{R}}}\right)^s [\nu]_{A_\infty}\|b\|_{BMO}\|M_{\alpha,B}(f)\|_{L^{q,\infty}(\nu)}\nu(F)^{1/q'}.$$

The argument to bound the term II is similar. In virtue of Theorem 5.1.6 and (5.3.5), we have that

$$II \leq 2C_{n,B}\|b\|_{BMO}\sum_{Q\in\mathscr{D}_j}|Q|^{\alpha/n}|Q|\|f\|_{B,Q}\left(\fint_Q \chi_F\nu\right).$$

Fix $a = 2^{n+1}$, and for each $k \in \mathbb{Z}$, consider the sets $\widetilde{\Omega}_k^j := \{M^{\mathscr{D}_j}(\chi_F\nu) > a^k\}$. Since $\fint_Q \chi_F\nu \to 0$ as $|Q| \to \infty$, each of these sets is the union of a collection $\widetilde{\mathcal{S}}_k^j$ of maximal, disjoint cubes in $\mathscr{D}_j$ such that $a^k < \fint_Q \chi_F\nu \leq 2^n a^k$. Moreover, the set $\widetilde{\mathcal{S}}_j := \bigcup_k \widetilde{\mathcal{S}}_k^j$ is $\frac{1}{2}$-sparse. Let $\widetilde{\mathcal{C}}_k^j := \{Q \in \mathscr{D}_j : a^k < \fint_Q \chi_F\nu \leq a^{k+1}\}$. By the maximality of the cubes in $\widetilde{\mathcal{S}}_k^j$, every $P \in \widetilde{\mathcal{C}}_k^j$ is contained in a unique cube in $\widetilde{\mathcal{S}}_k^j$. Therefore, and applying Lemma 5.2 in [21], we get that

$$\begin{aligned} II &\leq 2C_{n,B}\|b\|_{BMO}\sum_{k\in\mathbb{Z}}\sum_{Q\in\widetilde{\mathcal{C}}_k^j}|Q|^{\alpha/n}|Q|\|f\|_{B,Q}\left(\fint_Q \chi_F\nu\right) \\ &\leq 2^{n+2}C_{n,B}\|b\|_{BMO}\sum_{k\in\mathbb{Z}}a^k\sum_{P\in\widetilde{\mathcal{S}}_k^j}\sum_{Q\subseteq P}|Q|^{\alpha/n}|Q|\|f\|_{B,Q} \\ &\leq 2^{n+2}c_\alpha C_{n,B}\|b\|_{BMO}\sum_{k\in\mathbb{Z}}\sum_{P\in\widetilde{\mathcal{S}}_k^j}|P|^{\alpha/n}|P|\|f\|_{B,P}\left(\fint_P \chi_F\nu\right) \\ &= 2^{n+2}c_\alpha C_{n,B}\|b\|_{BMO}\sum_{P\in\widetilde{\mathcal{S}}_j}|P|^{\alpha/n}|P|\|f\|_{B,P}\left(\fint_P \chi_F\nu\right). \end{aligned}$$

Repeating the arguments in (5.3.6), we conclude that

$$II \leq 2^{n+1}c_\alpha C_{n,q,B}^{II}\left(2\cdot 3^n\|\nu\|_{A_s^{\mathcal{R}}}\right)^s[\nu]_{A_\infty}\|b\|_{BMO}\|M_{\alpha,B}(f)\|_{L^{q,\infty}(\nu)}\nu(F)^{1/q'}.$$

Since the constants involved don't depend on j, combining all the previous estimates, we get that

$$\begin{aligned} &\|[b,I_\alpha]f\|_{L^{q,\infty}(\nu)} \\ &\leq 2\cdot 18^n c_\alpha C_{n,\alpha}C_{n,q,B}\left(2\cdot 3^n\|\nu\|_{A_s^{\mathcal{R}}}\right)^s[\nu]_{A_\infty}\|b\|_{BMO}\|M_{\alpha,B}(f)\|_{L^{q,\infty}(\nu)}, \end{aligned}$$

and the desired result follows from Theorem 5.3.3, with

$$\begin{aligned} &\|[b,I_\alpha]\|_{L^{p,1}(w)\to L^{q,\infty}(\nu)} \\ &\leq 8\frac{q'}{p}36^n 24^{n/p}c_\alpha C_{n,\alpha}C_{n,q,B}\|w,\nu\|_{A_{p,q,\alpha,B}^{\mathcal{R}}}\left(2\cdot 3^n\|\nu\|_{A_s^{\mathcal{R}}}\right)^s[\nu]_{A_\infty}\|b\|_{BMO}. \end{aligned}$$

□

Remark 5.3.12. One can prove the previous result following an alternative approach. In virtue of Theorem 1.3 in [24], for every non-negative function f,

$$M^\#([b,I_\alpha]f) \lesssim \|b\|_{BMO}(I_\alpha(f)+M_{\alpha,B}(f)),$$

and applying Corollary 5.2.10 and Theorem 5.3.3,

$$\begin{aligned} \|M^\#([b,I_\alpha]f)\|_{L^{q,\infty}(\nu)} &\lesssim C_\nu\|b\|_{BMO}([w,\nu]_{A_{p,q,\alpha}^{\mathcal{R}}}+\|w,\nu\|_{A_{p,q,\alpha,B}^{\mathcal{R}}})\|f\|_{L^{p,1}(w)} \\ &\lesssim C_\nu\|b\|_{BMO}\|w,\nu\|_{A_{p,q,\alpha,B}^{\mathcal{R}}}\|f\|_{L^{p,1}(w)}. \end{aligned}$$

Finally, by Lemma 7.1 in [24] (see also [38, Page 144] and [56]) and Theorem 2.1 in [35],

$$\|[b,I_\alpha]f\|_{L^{q,\infty}(\nu)} \leq \|M^d([b,I_\alpha]f)\|_{L^{q,\infty}(\nu)} \lesssim \widetilde{C}_\nu\|M^\#([b,I_\alpha]f)\|_{L^{q,\infty}(\nu)},$$

where M^d is the *dyadic Hardy-Littlewood maximal operator*, defined for locally integrable functions h by

$$M^d(h) := \sup_{Q\in\mathscr{D}_0} \left(\fint_Q |h| \right) \chi_Q,$$

where $\mathscr{D}_0$ is the standard dyadic grid in $\mathbb{R}^n$.

5.4 Applications to Poincaré and Sobolev-Type Inequalities

Following ideas in [82], we can produce Poincaré and Sobolev-type inequalities for products of functions from the bounds for the operator $\mathcal{I}_\alpha$.

Theorem 5.4.1. *Let $1 \le p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and $p \le q$. Let w_1, w_2, and ν be weights, with $\nu \in A_\infty$ and $(w_1, w_2, \nu) \in A^{\mathcal{R}}_{\vec{P},q,1}$. Then, there exists a constant $C > 0$ such that the inequality*

$$\|fg\|_{L^{q,\infty}(\nu)} \le C \left(\|\nabla f\|_{L^{p_1,1}(w_1)} \|g\|_{L^{p_2,1}(w_2)} + \|f\|_{L^{p_1,1}(w_1)} \|\nabla g\|_{L^{p_2,1}(w_2)} \right)$$

holds for all functions $f, g \in \mathscr{C}_c^\infty(\mathbb{R}^n)$.

Proof. The proof of Theorem 7.1 in [82] establishes that

$$|fg| \lesssim \mathcal{I}_1(|\nabla f|, |g|) + \mathcal{I}_1(|f|, |\nabla g|),$$

and the desired inequality follows immediately from Corollary 5.2.10 with $\alpha = 1$. □

Theorem 5.4.2. *Let $1 < n$, $1 \le p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and $p \le q$. Let w_1, w_2, and ν be weights, with $\nu \in A_\infty$ and $(w_1, w_2, \nu) \in A^{\mathcal{R}}_{\vec{P},q,2}$. Then, there exists a constant $C > 0$ such that the inequality*

$$\|fg\|_{L^{q,\infty}(\nu)} \le C \left(\|\Delta f\|_{L^{p_1,1}(w_1)} \|g\|_{L^{p_2,1}(w_2)} + \|f\|_{L^{p_1,1}(w_1)} \|\Delta g\|_{L^{p_2,1}(w_2)} \right)$$

holds for all functions $f, g \in \mathscr{C}_c^\infty(\mathbb{R}^n)$.

Proof. The proof of Theorem 7.2 in [82] shows that

$$|fg| \lesssim \mathcal{I}_2(|\Delta f|, |g|) + \mathcal{I}_2(|f|, |\Delta g|),$$

and again, the desired result follows immediately from Corollary 5.2.10 with $\alpha = 2$. □

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