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Clasificación de 4-simplices vacíos y otros politopos reticulares

Classification of empty 4-simplices and other lattice polytopes

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Contents

Resumen de la tesis doctoral

Un d-politopo es la envolvente convexa de un conjunto finito de puntos en $\mathbb{R}^d$. En particular, si un d-politopo está generado por exactamente $d+1$ puntos se dice que es un símplice o un d-símplice. Además si tomamos los puntos con coordenadas enteras, es decir, un politopo $P = \text{conv}(p_1, \cdots, p_n)$ con $p_n \in \mathbb{Z}^d$ se dice que el politopo es reticular.

A lo largo de esta tesis doctoral se estudian los politopos reticulares y, más concretamente, se estudian dos tipos de estos que son los politopos reticulares vacíos (cuyos únicos puntos reticulares son los vértices) y los politopos reticulares huecos, politopos reticulares que no poseen puntos reticulares en su interior relativo, es decir, todos sus puntos reticulares se encuentran en la frontera.

Los politopos huecos, también vacíos, aparecen como el ejemplo más sencillo de politopos reticulares al no tener puntos enteros en el interior de su envolvente convexa.

El principal resultado de la tesis doctoral es la clasificación de símplices vacíos en dimensión 4. Mientras los casos en dimensión 1 y 2 son triviales y el caso de dimensión 3 estaba concluido desde 1964 con el trabajo de White [Whi64], con este trabajo se completa esta clasificación en dimensión 4.

Artículos como el de Mori, Morrison y Morrison [MMM88] en 1988 consiguen describir algunas familias de 4-símplices vacíos de volumen primo en términos de quíntuplas. Otros trabajos como el de Haase y Ziegler [HZ00] en el 2000, obtienen resultados parciales de esta clasificación. En particular, en ese trabajo se conjeturó una lista completa de 4-símplices vacíos con anchura mayor que dos, la cual se prueba completa en esta tesis.

Empleando técnicas de geometría convexa, geometría de números y resultados previos sobre la relación entre la anchura de un politopo y su volumen, somos capaces de establecer unas cotas superiores para los 4-símplices vacíos que deseamos clasificar. Con estas cotas para el volumen de los símplices y una gran cantidad de computación de estos politopos reticulares en dimensión 4 somos capaces de completar la clasificación, explicando el método general utilizado para describir las familias de símplices vacíos que aparecen en la clasificación.

Una vez clasificados los símplices vacíos en dimensión 4, y empleando los resultados de las computaciones que han sido realizadas, se determinan todos los h^*-vectores para estos politopos reticulares y se demuestra que todo 4-símplice vacío posee al menos 2 facetas unimodulares, un resultado que había sido anunciado como cierto, pero del cual no se disponía una prueba completa.

Para finalizar la tesis doctoral se establecen una serie de generalizaciones de los resultados empleados para lograr la clasificación de los 4-símplices vacíos que dan lugar a un procedimiento de clasificación para otros tipos de politopos reticulares, como son los 5-símplices vacíos o los d-símplices huecos, con $d \geq 4$, aunque más computaciones y cálculos serían necesarios para completar dichas clasificaciones.

Chapter 1
Introduction

In this introductory chapter we define the main concepts that will appear throughout this work and we give some motivation to the study of empty and hollow polytopes, in particular, simplices.

Most of the content of this thesis appears in the following research papers: [IVS19, IVS19*] written by the author and Francisco Santos, his Phd. advisor. In the same manner, the main results coming from this thesis have been presented in several spanish and international conferences, either as a talk or a poster; this fact has resulted in some online contributions in proceedings for the following conferences:

- The abstract of my talk, Classification of empty lattice 4-simplices of width larger than 2, at European Conference on Combinatorics, Graph Theory and Applications (EUROCOMB, Wien 2017) has been published in the special issue of Electronics Notes in Discrete Mathematics with the same name of the conference [IVS17].

- The abstract of my talk, The complete classification of empty 4-simplices, at Discrete Mathematics Days 2018 hold in Sevilla has been published in the special issue of Electronics Notes in Discrete Mathematics with the same name of the conference. [IVS18]

- A survey on width and volume bounds of hollow polytopes has been published as a chapter of the book: Algebraic and Geometric Combinatorics on Lattice Polytopes [IV19]. This book has been published as result of the proceedings of the Summer Workshop on Lattice Polytopes (Osaka 2018).

This introductory chapter has a first section 1.1 where we introduce the reader to the basic and main concepts that will be needed to understand the content of this work, mainly the definitions that are related to the concept of lattice polytope and convex body and their main properties and basic characteristics and ways to measure them.

In the second section 1.2, we give a motivation for the study of lattice polytopes, and more particularly, the classification of hollow polytopes.

In this introduction we remark which are the main results obtained throughout this research. These conclusions are summarized in Chapter 1.3. We also describe there the organization of the results within this work.

At the end of that Chapter we also describe the content of the appendix where we include some data, experimental results and information that are part of this thesis but would interrupt the flow of the text, if put in the main body.

1.1 Main definitions

A *polytope* $P \in \mathbb{R}^n$ is the convex hull of a finite set of points, i. e., it is the smallest convex set that contains a certain finite set of points $p_1, \cdots, p_s$. When we refer to the convex hull of a set of points we will denote it as $\operatorname{conv}(p_1, \cdots, p_n)$. The dimension of a polytope is the dimension of its affine span. A polytope of dimension d is usually called a d-polytope.

We also need to consider a lattice $\mathcal{L}$; with this word, we mean a discrete subgroup of $\mathbb{R}^n$. Unless stated otherwise our lattice will be $\mathbb{Z}^n$. There is no loss of generality as the problems we look at are invariant under affine transformations.

Once we have introduced a lattice, if a finite set of points lies in $\mathbb{Z}^n$, then we say that its convex hull is a *lattice polytope*, i.e., $P = \operatorname{conv}(p_1, \dots, p_n)$ with $\{p_1, \dots, p_n\} \subset \mathbb{Z}^n$.

Classification will always be meant modulo unimodular equivalence: two lattice d-polytopes $P_1, P_2 \subset \mathbb{R}^d$ are said *unimodularly equivalent* if there is an affine integer isomorphism between them; that is, a map $f : \mathbb{R}^d \to \mathbb{R}^d$ with integer coefficients and determinant 1 such that $P_2 = f(P_1)$

A face of a polytope P is the intersection of P with a hyperplane H that does not cut the relative interior of P. We call *vertices* the 0-dimensional faces of P, *edges* the 1-dimensional faces of P and *facets* the $(d-1)$-dimensional faces of P.

Throughout the whole text it is convenient to measure the polytopes we classify. Two measures that are invariant under unimodular equivalence and we use to determine polytopes are their volume and their width.

We define the *normalized volume*, $\operatorname{Vol}(P) := d! \operatorname{vol}(P)$ for any $P \subset \mathbb{R}^d$, where vol denotes the Euclidean volume and Vol the normalized one. This definition makes all lattice polytopes to have integer volume, and the standard simplex, $\Delta_d = \operatorname{conv}(0, e_1, \cdots, e_d)$ has volume 1, as the volume of a simplex it is equal to its determinant:

$$\operatorname{Vol}(P) = \det \begin{pmatrix} v_0 & \dots & v_d \\ 1 & \dots & 1 \end{pmatrix}.$$

where P is a d-simplex with $d+1$ vertices $v_0, \dots, v_d$.

We say that $K \in \mathbb{R}^n$ is a *convex body* if it is a convex compact set.

The *width* of a convex body $K \in \mathbb{R}^d$ with respect to an affine functional $f : \mathbb{R}^d \to \mathbb{R}^d$ is defined as

$$\mathrm{w}_f(K) := \max_{x,y \in K} |(f(x) - f(y)|.$$

The *(lattice) width* of a lattice polytope P is the minimum value of the widths among all non-constant (integer) functionals and we will denote it as $\mathrm{w}(P)$. Lattice width can also be seen as the minimum lattice distance of two parallel hyperplanes that can enclose the given lattice polytope.

We will say that p is a lattice point of P if $p \in P \cap \mathbb{Z}^n$. If $p \in int(P)$, then p is called an interior point of P. In most of this work we will focus in lattice polytopes without interior points; these are called *hollow* polytopes.

If the only lattice points of a lattice polytope P are its vertices, then we say that P is *empty*, or more frequently, P is an empty d-polytope.

1.2 Motivation

Lattice polytopes appear in different areas of research in which their study and properties can lead to find new interesting results. Our knowledge of lattice polytopes helps to translate questions made in other mathematical fields to discrete geometry in which these problems sometimes can be more approachable. Some of these fields in which several questions can be written in their lattice polytope interpretation are discrete optimization, geometry of numbers, toric geometry, etc.

Lately, lattice polytopes, and particularly, properties of hollow-empty polytopes have been a field of research by itself [Tre08, NZ11, AW12, AWW11].

When looking at lattice polytopes, the simplest example that can be found is a simplex (simplices have the lowest number of vertices possible) and, in particular, empty simplices (their only lattice points are their vertices). Even more, every polytope can be decomposed into several empty simplices, i.e., they work as building blocks for lattice polytopes in the same way as triangulations work in the theory of subdivisions [KS03, Knu73, SR, HS, SZ13].

In this manner, it is interesting to know how to construct empty simplices as, if their classification is understood, it will be easier to understand how to build lattice polytopes as the union of empty simplices. Even more, some nice properties that occur for empty simplices can be translatable to general lattice polytopes, so having a complete classification of empty simplices may give simpler proofs for some properties for the full class of lattice polytopes or, even more, bring ideas of some facts

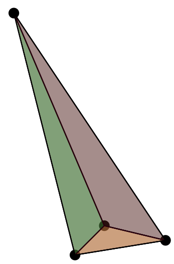


Figure 1.1: Reeve's tetrahedron with volume $V = 3$.

happening in more general lattice polytopes that we do not know yet. This is not the only factor that make empty-hollow polytopes particular. In 1983, Hensley [Hen83] realised that having interior lattice points makes a difference when referring to the volume of lattice polytopes:

Theorem (Hensley [Hen83])**.** *For fixed* d *and* $k > 0$*, there is an upper bound on the volume of lattice* d*-polytopes with* k *interior lattice points.*

This theorem ensures that every lattice polytope with interior lattice points (i.e., non-hollow) has an upper bound for its volume depending on the number of interior lattice points. Some examples for this upper bound can be found in [LZ91,Pik01]. In contrast, if we take a look at what happens in the case of hollow polytopes, there is no such bound. One example to verify this are *Reeve tetrahedra*, the following family of lattice tetrahedra:

$$P := \operatorname{conv}(0,0,0), (1,0,0), (0,1,0), (1,1,V),$$

here V is the volume of P. V can be any positive integer number and P is empty (so, hollow).

Although the volume is not bounded for hollow polytopes there are measures that can capture some type of size of hollow polytopes. One interesting example is what happens with the width. It has been known for a long time that the width of a hollow polytope is bounded in terms of its dimension:

Theorem (Flatness theorem)**.** *There exists a function* $f : \mathbb{N} \to \mathbb{R}$ *such that for any hollow convex body* $K \in \mathbb{R}^d$ *and with respect to the lattice* $\mathbb{Z}^d$*,* $w(K) \leq f(d)$*.*

Some progress has been done in last years trying to obtain better upper bounds for the width of hollow polytopes depending on the dimension. Some of the best improvements were made by Khintchine [Khi48], Kannan and Lovász [KL88] and lately Banazczyk et al. and an improvement by Rudelson got that the flatness constant is of the order $\mathrm{O}(d^{\frac{4}{3}}(1+\log d))$ [BLPS99, Rud00], even though there is still room to get a better upper bound and fully understand this flatness of hollow bodies.

It is also noticeable, that the concept of width is also an important part of the known classifications of hollow polytopes in lower dimensions:

Theorem (Folklore)**.** *A hollow polygon (2-polytope) is either of width one, or equivalent to $2\Delta_2$, the second dilation of the standard unimodular simplex.*

Theorem (Treutlin [Tre08], Theorem 1.3)**.** *Every hollow 3-polytope is in one of the following cases:*

- *It has width 1*
- *It has width 2 and projects to the polygon $2\Delta_2$.*
- *It has width≥ 2, and does not admit a projection to $2\Delta_2$. There are only finitely many of these and they are contained in* hollow-maximal *3-polytopes.*

With hollow-maximal, we refer to hollow polytopes that are not properly contained in another hollow d-polytope.

These classification results in lower dimension, were generalized by Nill and Ziegler:

Theorem (Nill and Ziegler [NZ11])**.** *For all fixed d, all hollow d-polytopes project to a hollow polytope of dimension less than d except for finitely many cases.*

This theorem ensures the finiteness of hollow polytopes that do not project to lower dimensional hollow polytopes. Even more, these polytopes are always contained in a hollow-maximal d-polytope.

Knowing these classification results, in [HZ00], Haase and Ziegler in 2000 tried to evaluate the width for the particular case of empty 4-simplices as it is the next natural step where this bound in the volume was not known at that time. In the case of dimension 3, White had found a way of completely characterizing all empty 3-simplices, i.e. empty tetrahedra:

Theorem (White [Whi64])**.** *Every empty tetrahedron of volume q is unimodularly equivalent to*

$$T(p,q) := \operatorname{conv}\{(0,0,0),(1,0,0),(0,0,1),(p,q,1)\},$$

for some $p \in \mathbb{Z}$ with $\gcd(p,q)=1$. Moreover, $T(p,q)$ is $\mathbb{Z}$-equivalent to $T(p',q)$ if and only if $p' = \pm p^{\pm 1} \pmod{q}$.

It is easy to realise that this classification has a follow-up implication for the width of all these simplices. Taking a look at $T(p,q)$, we can check that with respect to the functional $f(x,y,z) = z$, all empty tetrahedra have width one.

Haase and Ziegler found an infinite family of empty simplices of width 2 in dimension 4, showing that in dimension 4 the same phenomenon is not happening. They try to find the maximum width that empty 4-simplices can attain by enumerating them up to volume 1000. The results of the enumeration are summarized in the following theorem:

Theorem (Haase and Ziegler [HZ00]). *Among empty* 4*-simplices with volume bounded by* 1000 *there are the following cases:*

- *There are no empty simplices with width greater than four.*
- *There is a list of* 178 *empty* 4*-simplices of width three with volume between* 41 *and* 179.
- *There is a unique example of an empty* 4*-simplex with width* 4 *and volume* 101.

As the list of simplices of width greater than two that they found has maximum volume 179, they conjecture that their list is finite and contains only those simplices. Even more, Perriello in his Masters Thesis [Per08] starts a new enumeration that goes up to volume 1600, in which he finds no other empty 4-simplex of width greater than two that was not in Haase and Ziegler list.

The assumption of the finiteness of this list of empty 4-simplices makes natural the following definition that appears in [BHHS]:

We define *finiteness threshold width* to be the minimum width $w^\infty(d,n) \in \mathbb{N}$, such that there exist only finitely many lattice d-polytopes with n lattice points of width $> w^\infty(d,n)$ modulo unimodular equivalence.

Once this concept is introduced, we can rephrase the Haase Ziegler conjecture as saying that $w^\infty(4) = 2$ and that there is no other simplex of width greater than two that is not in their list.

In fact, this attempt to a classification of empty 4-simplices from Haase and Ziegler was not the first work on this topic. In 1988, Mori, Morrison and Morrison [MMM88] made some advances in this classification attacking the problem from the concept of *terminal quotient singularities* , that comes from the point of view of algebraic and toric geometry [MMM88, MS84].

In their work, the authors suggest that a terminal quotient singularity is attached to a quintuple that can be of one of the following three different types.

- 1 quintuple depending on 3 parameters

- 1 quintuple that depends on 2 parameters
- 29 particular cases.

They conjecture that this list is complete and all terminal quotient singularities are associated to one of these quintuples if the value of a number p, associated to that quintuple, is prime (In terms of lattice polytopes the value p is the volume of the empty simplex related to this singularity). This claim was proved to be true with the computations of Sankaran [San90] and Bober [Bob09].

In Chapter 2, we can see how this information about the terminal quotient singularities and the quintuples correspond to empty simplices. Although, Barile et al. claimed that the result of Sankaran and Bober can be extended to all empty 4-simplices, we prove that this is not the case, since if the volume of the simplex is a non-prime number there exist other families of empty 4-simplices that can appear. Examples of these empty simplices are showed in Chapter 2.

There are some other facts regarding the width that raise interesting questions about empty 4-simplices. It is also a peculiar fact that dimension 4 is a limit case for empty simplex regarding the volume of their facets. In [Wes89], Wessels writes in his Masters thesis the following theorem:

Theorem. *Every empty 4-simplex has at least two unimodular facets*

This may not be an impressive fact, but looking at what happens when dimension $d > 4$, Haase and Ziegler and Barile et al. give nice examples of empty 5-simplices in which all facets are non-unimodular.

As it seems, the dimension of the polytopes has a crucial part in the possible volume of the facets, and so we decide to compute the volume facet vector for every family of empty 4-simplices and the sporadic cases because we cannot verify the proof of Wessels.

Even more, classifying empty 4-simplices is another way to approach one problem of interest for lattice polytopes. For this special case, knowing the volume vector of the facets and the volume of the polytope it is equivalent to knowing the complete Ehrhart polynomial and h^*-vector.

The Ehrhart polynomial is a function that measures how many lattice points the dilations of a lattice polytope P have. The *Ehrhart polynomial* of a lattice d-polytope P is a degree d-polynomial $E(P,t) = E_t t^d + \cdots + E_0 \in \mathbb{Q}[t]$ with the property that

$$E(P,t) = |tP \cap \Lambda|, \quad \forall t \in \mathbb{N}.$$

The h^*-vector (or δ-vector) of P, a vector $h^*(P) = (h_0^*, \ldots, h_d^*) \in \mathbb{N}^{d+1}$ with the property that

$$\sum_{n=0}^{\infty} E(P,n)x^n = \frac{h_d^* x^d + \cdots + h_0^*}{(1+x)^{d+1}}.$$

That is, the h^*-vector gives (the vector of coefficients of the numerator of the rational function of) the generating function of the sequence $(E(P,n))_{n\in\mathbb{N}}$.

In recent years, the interest in knowing more facts about the Ehrhart polynomial of a lattice polytope, the meaning of its coefficients and their values for several kinds of polytopes has been raising [BJMc19, Sco76, Sta09, HKN18, BH18, LS19, HNO18, BdLPS05]. The binomial h^*-polynomials are fully characterised by Batyrev and Hofscheier [BH13], in particular empty 3-simplices are part of that classification [BH10]. The specific case of h^*-vectors of the form $(1, 0, h_2^*, h_3^*, 0)$ is the next step.

1.3 Main results and organization of the thesis

The main result of this thesis is the complete classification of empty 4-simplices. Chapter 4 contains the proof of the following theorem that states the different cases that appear for an empty 4-simplex:

Theorem (Classification of empty 4-simplices, Theorem 2.2.1)**.** *Let P be an empty 4-simplex and let $k \in \{1,2,3,4\}$ be the minimum dimension of a hollow polytope that P projects to. Then P is as follows, depending on k:*

$\boxed{k=1:}$ *P lies in the three-parameter family parametrized by the volume V of P and another two integer parameters α, β with $\gcd(\alpha,\beta,V)=1$; the 5-tuple of P is $(\alpha+\beta, -\alpha, -\beta, -1, 1)$.*

$\boxed{k=2:}$ *P lies in one of the following two two-parameter families parametrized by the volume V of P and another integer parameter α with $\gcd(\alpha, V)=1$:*

$$(1,-2,\alpha,-2\alpha,1+\alpha) \quad \textit{with odd } V, \quad \textit{and}$$
$$\frac{V}{2}(0,1,0,1,0)+(-1,-1,\alpha,-\alpha,2) \quad \textit{with } V\in 4\mathbb{Z}.$$

We call the first family primitive *and the second* nonprimitive.

$\boxed{k=3:}$ *Except for finitely many simplices, P belongs to one of the 29* primitive + *17* nonprimitive *families with quintuples shown in Tables 2.1 and 2.2 with some conditions for the volume expressed in Chapter 2*

$\boxed{k=4:}$ *There are finitely many possibilities for P, by Theorem 2.1.1. Their volumes are bounded by 419. See more details in Theorem 2.2.5.*

While trying to finish this classification during last years we have encountered several difficulties. Solving the problems found in the way to the complete classification has evolved in obtaining other partial results that form part of this work, and so, they are present in this thesis:

- In Chapter 3 we explain the techniques we use to obtain upper bounds for the volume of hollow polytopes in particular cases. One of these upper bounds can be generalized to general convex bodies with width greater than a fixed constant. In this sense, we obtain a better bound that the previous known:

  Theorem (Theorem 3.2.1)**.** *Let K be a hollow convex 3-body of lattice width w, with $w > 1 + 2/\sqrt{3} = 2.155$ and let $\mu = w^{-1}$. Then,*

$$\mathrm{vol}(K) \leq \frac{8w^3}{(w-1)^3}, \qquad \textit{if } w \geq \frac{2}{\sqrt{3}}(\sqrt{5}-1)+1 = 2.427, \textit{ and}$$
$$\mathrm{vol}(K) \leq \frac{3w^3}{4(w-(1+2/\sqrt{3}))}, \qquad \textit{if } w \leq 2.427.$$

- For the case $k = 4$, the sporadic simplices, we needed an upper bound for the volume of hollow simplices that do not project to hollow 3-polytopes. We prove this bound in Chapter 3.4 by doing a case by case proof.

- In chapter 5 we complete the study of the facet volume vector of empty 4-simplices for all the cases in the main theorem 2.2.1. Via obtaining these facet volume vectors, we get the following two results:

  1. We determine every possible h^*-vector of the form:
  $$h^* = (h_0^*, h_1^*, h_2^*, h_3^*, h_4^*) = \left(1, 0, \frac{V+S}{2} - 3, \frac{V-S}{2} + 2, 0\right).$$
  2. We verify the statement of Wessels [Wes89] that says that every empty 4-simplex has at least 2 unimodular facets and give an explicit list of the cases in which the simplex has only 2 unimodular facets.

1.3.1 Organization of the thesis

The proof of the main theorem is developed in chapter 2. There, we introduce the different families that appear in the classification and give explicit description of the simplices by referring to their hollow lattice projections. Most of the results that we use through this section apply also for hollow 4-simplices.

In Chapter 3 we calculate the upper bounds for the volume of hollow simplices in dimension 3 and 4. These upper bounds are needed for the classification of empty 4-simplices as there exist some sporadic simplices that do not belong to any family and we need to make sure that we get all of them in the enumeration process.

In Chapter 4 we give details of how the computations were made and list the main algorithms used during the exhaustive enumeration. We also compare the computation time between algorithms.

In Chapter 5 we verify the fact that every empty 4-simplex has 2 unimodular facets and explain how the enumeration of sporadic empty 4-simplices determines the h^*-vectors of the form $(1, 0, h_2^*, h_3^*, 0)$

Finally, Chapter 6 mentions some research lines related with the classification of empty simplices and we state some open questions that remain unsolved. In particular:

- We mention in which direction the classification of empty 5-simplices could be approached, even though we enumerate some of the problems that will appear trying to complete the task.

- We can generalize some of the methods used for the classification of empty simplices if the polytopes are not empty but, more generally, hollow. We start this work by giving a brief idea of how the classification of hollow 4-simplices would look like, but we do not forget about the problems that arise when trying to classify hollow polytopes instead of empty ones.

- While the finiteness threshold width in the case of empty 4-simplices was determined with previous results [BHHS, IVS19], the value when $d > 4$ is unknown. We propose a method via generating random simplices that could give some intuition in order to determine this values.

1.4 Appendix

After the bibliography some appendices appear that include useful information which added to the main sections, helps the reader understand the thesis and explain some technical details that complete the full content of the thesis.

The appendix includes:

- Some data obtained when enumerated empty 4-simplices.

- An algorithm for creating random simplices of the type $\sigma(v)$, see more in section 6.2.1.

- Example of files with all empty 4-simplices for a given volume V separated in the different cases of Theorem 1.3.

Chapter 2
The classification of empty 4-simplices

In this Chapter we show the proof of the classification of empty 4-simplices. We determine the quintuples that are associated to an empty 4-simplex in terms of their fine family and coarse family of points that they project to, according to the definitions in the next section.

In order to finish this proof we also need the volume bounds and computations described in Chapters 3 and 4.

2.1 Classifying hollow polytopes

If there is a lattice projection $\pi : \mathbb{R}^n \to \mathbb{R}^k$ sending a polytope P to a polytope Q and Q is hollow with respect to the projected lattice $\pi(\Lambda)$, then P is automatically hollow; (the same is not true for empty). In this situation we say that π, or Q, is a *hollow projection* of P, and that P is a *lift* of Q. The starting point to a general classification of hollow lattice polytopes is the following result of Nill and Ziegler:

Theorem 2.1.1 (Nill-Ziegler [NZ11, Thm. 1.2])**.** *For each dimension d there is only a finite number of hollow d-polytopes that do not project onto a hollow $(d-1)$-polytope.*

To rephrase this statement we introduce the following definition:

Definition 2.1.2. Let $d \in \mathbb{N}$ be fixed and let Q be a k-dimensional lattice hollow polytope that does not project to any $(k-1)$-hollow polytope, with $k \leq d$. We call *coarse family* of Q the collection of all hollow d-polytopes that have Q as a hollow projection.

Corollary 2.1.3. *The hollow d-polytopes of any fixed dimension d belong to a finite number of coarse families.*

Proof. There is one family for each of the finitely many polytopes of Theorem 2.1.1, for each $k = 1, \dots, d$. □

Example 2.1.4. A lattice polytope P projects to a hollow 1-polytope if and only if P has *width one*. That is, if P is contained between two consecutive parallel lattice hyperplanes. It is easy to check that the only hollow 2-polytope without that property is the second dilation $2\Delta_2$ of a unimodular triangle. Thus, the coarse classification of hollow lattice 2-polytopes is as follows:

- The dilated unimodular triangle $2\Delta_2$ is a coarse family with a single element.
- The lattice polygons of width one form a second family. Each of them is isomorphic to a trapezoid $\{0\} \times [0, a] \cup \{1\} \times [0, b]$ with $a, b \in \mathbb{Z}_{\geq 0}$ and $a + b > 0$. (The trapezoid degenerates to a triangle if a or b equal zero).

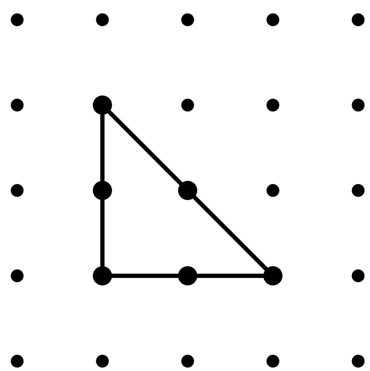


Figure 2.1: The second dilation of a unimodular triangle Δ_2, which is the only hollow 2-polytope not projecting to a unit segment.

Example 2.1.5. The coarse classification of hollow 3-polytopes is:

- The coarse family of width one; each of which can be expressed as a pair of lattice polygons.
- The coarse family projecting to $2\Delta_2$. As before, these can be written as the convex hull of six hollow lattice segments $\{p_i\} \times [a_i, b_i]$ where p_i, $i = 1, \ldots, 6$, are the six lattice points in $2\Delta_2$ and $[a_i, b_i]$ is an integer interval.
- Each of the finitely many (by Theorem 2.1.1) hollow 3-polytopes that do not project to dimension two is a coarse family in itself. These were enumerated by Averkov et al. [AWW11, AKW17], who showed that there are 12 maximal ones. See Theorem 2.4.1.

Observe that the families just defined may not be disjoint. For example, the Cartesian product of $2\Delta_2$ with a unit segment belongs to the first two families of Example 2.1.5, since it projects both to $2\Delta_2$ and to a unit segment.

We are interested in a finer classification, which takes into account the number of lattice points. A *hollow configuration* is a finite set S of lattice points such that $\mathrm{conv}(S)$ is a hollow polytope.

Definition 2.1.6. Let $d \in \mathbb{N}$ be fixed and let S be a configuration of n lattice points (perhaps with repetition) in $\mathbb{R}^k$, with $n > d \geq k$. Assume that $\mathrm{conv}(S)$ is hollow but it does not project to a hollow $(k-1)$-polytope. We call *fine family of S* the collection of all hollow d-polytopes with n vertices that admit a lattice projection sending $\mathrm{vert}(P)$ to S.

Corollary 2.1.7. *All hollow d-simplices belong to a finite number of fine families. More generally, for each fixed n, all hollow d-polytopes with n vertices belong to a finite number of fine families.*

Proof. There is one for each multisubset of size n of the lattice points in each of the finitely many polytopes of Theorem 2.1.1, for $k = 1, \dots, d$. $\square$

Example 2.1.8. There are three fine families of hollow lattice 2-polytopes:

- The dilated unimodular triangle $2\Delta_2$ is still a fine family with a single element. The corresponding S has size three (the three vertices of $2\Delta_2$).

- The lattice polygons of width one fall into two fine families, one projecting to the set $S_1 = \{0, 1, 1\}$ ($n = 3, k = 1$) and one projecting to the set $\{0, 0, 1, 1\}$ ($n = 4, k = 1$). Members of the first family are isomorphic to a triangle $\{(0,0)\} \cup (\{1\} \times [0, b])$, with $b \in \mathbb{Z}_{\geq 1}$. Members of the second family are trapezoids $(\{0\} \times [0, a]) \cup (\{1\} \times [0, b])$ with $a, b \in \mathbb{Z}_{\geq 1}$.

Example 2.1.9. There are infinitely many fine families of hollow 3-polytopes of width one, since they can have arbitrarily many vertices and all polytopes in the same fine family have the same number of vertices, by definition.

One key difference between coarse and fine families is that in the latter we fix the number n of vertices. In particular, if we take $n = d + 1$ we are looking at hollow simplices. Observe that in Example 2.1.8 each fine family is parametrized by $n - k - 1$ parameters. In the next section we analyze this phenomenon in more detail in the case of interest to us.

Let us finish this section by pointing out that these finiteness results are very similar in spirit to Theorem 2.1 in [Bor99]), which Borisov derives from the following more general statement of Lawrence [Law91]: for any open subset U of the torus

$\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ the family of subgroups of $\mathbb{T}^d$ not intersecting U has finitely many maximal elements. The relation is as follows: let U be the interior of the standard simplex in $\mathbb{T}^d$. Then, discrete subgroups $G \in \mathbb{T}^d$ not meeting U correspond to hollow d-simplices $P \subset \mathbb{R}^d$ via the correspondence $P \leftrightarrow G_P := \Lambda/\Lambda_P$. If G is not discrete (e.g., G corresponds to positive dimensional linear subspace $V \leq \mathbb{R}^d$) then the discrete subgroups of G form a fine family of hollow simplices, in the sense of Definition 2.1.6.

2.1.1 The case of cyclic simplices

In this section we relate the $(d{+}1)$-tuple of a cyclic simplex P to a hollow projection. Let us fix the following notation:

Let $P = \operatorname{conv}(v_0, \ldots, v_d) \subset \mathbb{R}^d$ be a cyclic lattice d-simplex of volume V, and let Λ_P be the affine lattice generated by its vertices (we assume without loss of generality that $0 \in \Lambda_P$). By definition of cyclic simplex, the quotient group $G(P) := \Lambda/\Lambda_P$ is cyclic of order V. Let $\pi : \mathbb{R}^d \to \mathbb{R}^k$ be a linear projection and denote

$$S := \pi(\operatorname{vert}(P)) = \{\pi(v_0), \ldots, \pi(v_d)\}.$$

Observe that both $\operatorname{vert}(P)$ and S are considered as ordered sets, and their ordering corresponds to the order of coordinates in a $(d+1)$-tuple representing P.

Let Λ_S be the affine lattice generated by S, which is a sublattice of $\pi(\Lambda)$. Then $\pi(\Lambda)/\Lambda_S$ is a cyclic group too, since π induces a surjective homomorphism

$$\tilde{\pi} : \Lambda/\Lambda_P \twoheadrightarrow \pi(\Lambda)/\Lambda_S.$$

Let I be the index of Λ_S in $\pi(\Lambda)$ which, by the above remark, divides V. We say that S, and the fine family defined by it, are *primitive* if $I = 1$; that is, if $\Lambda_S = \pi(\Lambda)$.

We need the following elementary fact about cyclic groups:

Lemma 2.1.10. *Let $\pi : \mathbb{Z}_V \to \mathbb{Z}_I$ be a surjective homomorphism between the cyclic groups of orders V and I. Then, for every generator q of $\mathbb{Z}_I$ threre is a generator p of $\mathbb{Z}_V$ with $\pi(p) = q$.*

Proof. Take as p any prime not dividing V from the arithmetic progression $\{q + nI : n \in \mathbb{Z}\}$. Such a prime exists since, by Dirichlet's prime number theorem, the arithmetic progression contains infinitely many primes. □

Proposition 2.1.11. *With the above notation, let $q \in \pi(\Lambda)$ be a generator of the quotient group $\pi(\Lambda)/\Lambda_S$. Then:*

1. *There is a vector* $a \in \frac{1}{I}\mathbb{Z}^{d+1}$ *such that*

$$q = \sum_{i=0}^{d} a_i \pi(v_i), \quad \textit{and} \quad 1 = \sum_{i=0}^{d} a_i.$$

2. *There is a generator* $p \in \Lambda$ *of the quotient group* Λ/Λ_P *such that the barycentric coordinates of* p *with respect to* $\{v_0, \dots, v_d\}$ *have the form*

$$a + \frac{1}{V} b,$$

with $b \in \mathbb{Z}^{d+1}$ *the coefficient vector of an integer affine dependence on* S.

Proof. For part (1), observe that since Λ_S has index I in $\pi(\Lambda)$, we have $\pi(\Lambda) \leq \frac{1}{I}\Lambda_S$. In particular, the point $q \in \pi(\Lambda)$ can be written as an affine combination, with coefficients in $\frac{1}{I}\mathbb{Z}$, of the points in S. The vector a is the vector of coefficients in this dependence.

For part (2), let $p \in \Lambda$ be a generator of Λ/Λ_P with $\pi(p) = q$, which exists by Lemma 2.1.10. Let $c = (c_0, \dots, c_d) \in \frac{1}{V}\mathbb{Z}^{d+1}$ be the barycentric coordinates of p with respect to $\{v_0, \dots, v_d\}$. That is, $\sum c_i = 1$ and $\sum c_i v_i = p$. By construction, $c - a \in \frac{1}{V}\mathbb{Z}^{d+1}$. The only thing that remains to be shown is that $b := V(c-a) \in \mathbb{Z}^{d+1}$ is the coefficient vector of an affine dependence among the $\pi(v_i)$s. This is easy:

$$\sum_{i=0}^{d} (c-a)_i \, \pi(v_i) = \pi\left(\sum_{i=0}^{d} c_i v_i\right) - \sum_{i=0}^{d} a_i \pi(v_i) = \pi(p) - q = 0.$$

and

$$\sum_{i=0}^{d} (c-a)_i = \sum_{i=0}^{d} c_i - \sum_{i=0}^{d} a_i = 1 - 1 = 0.$$

□

The above statement implicitly gives a parametrization of the fine family of cyclic simplices projecting to S. Let us make it more explicit.

Corollary 2.1.12. *Let* Λ_0 *be a lattice in* $\mathbb{R}^k$ *and let* S *be a multiset of* $d+1$ *lattice points affinely spanning* $\mathbb{R}^k$. *Assume that* Λ/Λ_S *is cyclic, of index* I, *and let* a *be as in part (1) of Proposition 2.1.11. Then, the cyclic* d*-simplices of a given volume* $V \in I \cdot \mathbb{N}$ *and projecting to* S *are parametrized as having* $(d+1)$*-tuples*

$$Va + b,$$

where $b \in \mathbb{Z}^{d+1}$ *runs over all the integer affine dependences of* S. *Moreover,* b *is only important modulo* V, *and satisfies* $\gcd(V, b_0, \dots, b_d) = 1$. □

The proof of Theorem 2.2.1 is by applying Corollary 2.1.12 to the case of cyclic empty 4-simplices. That is, by looking at hollow configurations of five points in $\mathbb{R}^k$, $k < 4$. Observe that for primitive families (that is, for $I = 1$), the only generator q of Λ/Λ_S is the zero class, represented (for example) by the first element of S. This gives us $a = (1, 0, \dots, 0)$ but, since we are interested in the tuples modulo the integers, we can as well take $a = 0$. This is our convention in all the primitive families of Theorem 2.2.1.

Observe that if $\mathrm{conv}(S)$ is hollow then all the cyclic simplices of Corollary 2.1.12 are automatically *hollow*, but not necessarily *empty*. Let us now address the issue of the restrictions needed for them to be empty. They are related to the volumes of facets, and the following observation.

Lemma 2.1.13. *Let P be a cyclic d-simplex of volume V with $(d+1)$-tuple $(b_0, \dots, b_d)$. Then, the volume of the ith facet of P ($i = 0, \dots, d$) equals*

$$V_i := \gcd(V, b_i). \qquad \square$$

Proposition 2.1.14. *Let P be a cyclic d-simplex of volume V with tuple $(b_0, \dots, b_d)$. A necessary condition for P to be empty is that no $d-2$ of the b_is (equivalently, no $d-2$ of the facet volumes V_i) have a factor in common with V.*

Proof. Recall that the V tuples jb, $j = 0, \dots, V-1$, represent the V classes of lattice points in Λ/Λ_P. If $d-2$ of the b_is have a factor in common with V then there is a $j \neq 0$ such that jb has three (or less) nonzero entries. That implies one of the non-zero classes in Λ/Λ_P to have representatives in a 2-plane spanned by a 2-face of P, which implies P has a 2-face that is not unimodular, hence not empty. That is a contradiction since every face of an empty simplex is empty. $\square$

Proposition 2.1.15. *Let P be a cyclic hollow 4-simplex of volume V with quintuple $(b_0, \dots, b_4)$ and, as above, let $V_i := \gcd(V, b_i)$ (the volume of the i-facet of P). The following are equivalent:*

1. *P is empty.*

2. *For each i, if $V_i \neq 1$ then the multiset $\{b_0, \dots, b_4\}$ coincides, modulo V_i, with the multiset $\{0, \alpha, -\alpha, \beta, -\beta\}$ for some α and β coprime with V_i.*

Proof. Once we know that P is hollow, it will be empty if and only if its facets are empty tetrahedra. The (classes of) lattice points in Λ/Λ_P lying in the hyperplane of the i-th facet are those that have a zero in the i-th position of their barycentric coordinates; these, as multiples of the generator $(b_0, \dots, b_4)$ for the quotient group, are precisely the multiples of $\frac{1}{V_i}(b_0, \dots, b_4)$. The necessary and sufficient condition for the facet to be empty is, by Example 2.1.16, that the four non-zero entries in

$\frac{1}{V_i}(b_0, \ldots, b_4)$ come in two pairs of opposite entries modulo V_i, and that the entries are prime with V_i. $\square$

Example 2.1.16 (Empty 3-simplices)**.** The 3-simplex of Theorem 1.2, $T(p,q) = \operatorname{conv}\{(0,0,0),(1,0,0),(0,0,1),(p,q,1)\}$, has the associated 4-tuple $(p,-p,-1,1)$, since $(0,1,0)$ is a generator for $G_P \cong \mathbb{Z}_q$ and

$$(0,1,0) = \left(1+\frac{p}{q}\right)(0,0,0) - \frac{p}{q}(1,0,0) - \frac{1}{q}(0,0,1) + \frac{1}{q}(p,q,1).$$

2.2 The complete clasification

Theorem 2.2.1 (Classification of empty 4-simplices)**.** *Let P be an empty 4-simplex and let $k \in \{1,2,3,4\}$ be the minimum dimension of a hollow polytope that P projects to. Then P is as follows, depending on k:*

$\boxed{k=1:}$ *P lies in the three-parameter family parametrized by the volume V of P and another two integer parameters α, β with $\gcd(\alpha,\beta,V)=1$; the 5-tuple of P is $(\alpha+\beta,-\alpha,-\beta,-1,1)$.*

$\boxed{k=2:}$ *P lies in one of the following two two-parameter families parametrized by the volume V of P and another integer parameter α with $\gcd(\alpha,V)=1$:*

$$(1,-2,\alpha,-2\alpha,1+\alpha) \quad \textit{with odd } V, \quad \textit{and}$$
$$\frac{V}{2}(0,1,0,1,0) + (-1,-1,\alpha,-\alpha,2) \quad \textit{with } V \in 4\mathbb{Z}.$$

We call the first family primitive *and the second* nonprimitive.

$\boxed{k=3:}$ *Except for finitely many simplices (of volumes bounded by 72, see Prop. 2.4.3) P belongs to one of the 29* primitive + *17* nonprimitive *families with quintuples shown in Tables 2.1 and 2.2, parametrized by volume alone (plus a choice of sign in some of the nonprimitive families).*

The volume needs to satisfy the modular conditions states in the caption of Table 2.1 and in Table 2.3 (from Section 2.4), respectively.

$\boxed{k=4:}$ *There are finitely many possibilities for P, by Theorem 2.1.1. Their volumes are bounded by 419. See more details in Theorem 2.2.5, below.*

Remark 2.2.2 (From a quintuple to coordinates)**.** *For the convenience of the reader, here comes an explicit recipe to translate a volume V and a tuple $b=(b_0,\ldots,b_d) \in {\mathbb{Z}_V}^{d+1}$ to actual coordinates for a cyclic d-simplex P that they represent. Suppose* *one of the entries in b, say the first one b_0, is a unit modulo V; this property is equivalent to the corresponding facet of P being unimodular (see Lemma 2.1.13) and it is a fact that all empty 4-simplices have at least two such unimodular facets (Corollary 5.1.1). Then, since we can multiply b by a unit modulo V there is no loss of generality in assuming $b_0 = -1$. Also, since the entries of b are important only modulo V and add up to zero, without loss of generality we assume that $\sum_{i=0}^{d} b_i = V$. In these conditions, the simplex P can be taken to be*

$$\operatorname{conv}(e_1, \ldots, e_d, v),$$

where $v = (b_1, \ldots, b_d)$. Indeed, this simplex is clearly of volume V (the last vertex lies at lattice distance $\sum_{i=1}^{d} b_i - 1 = V$ from the facet spanned by the standard basis, which is unimodular) and it is represented by our tuple since the origin has barycentric coordinates $\frac{1}{V}b$ for it:

$$(0,0,0,0) = \frac{b_1}{V}e_1 + \cdots + \frac{b_d}{V}e_d - \frac{1}{V}v.$$

Example 2.2.3. For a concrete example, consider the first quintuple $(9, 1, -2, -3, -5)$ of Table 2.1 and let $V = 100$. We first modify b to have sum of entries equal to V and one entry -1 (we do this with the first entry, but it could be done with the second or fourth):

$$\begin{aligned} 11 \cdot (9, 1, -2, -3, -5) &= (-1, 11, -22, -33, -55) \\ &= (-1, 11, -22, 67, 45) \pmod{100}. \end{aligned}$$

Then, the simplex P can be taken to be

$$\operatorname{conv}(e_1, e_2, e_3, e_4, (11, -22, 67, 45)).$$

As another example, the simplex of quintuple $(\alpha+\beta, -\alpha, -\beta, -1, 1)$ and volume V (case $k = 1$ of Theorem 2.2.1) is isomorphic to

$$\operatorname{conv}\{e_1, e_2, e_3, e_4, (V - \alpha - \beta, \alpha, \beta, 1)\} \cong \operatorname{conv}\{0, e_2, e_3, e_4, (V, \alpha, \beta, 1)\}.$$

Some comments about the statement of Theorem 2.2.1:

- The classification is not irredundant. The same empty simplex may belong to several families, since it may project to different lower dimensional configurations.
- The parameters α and β are only important modulo V; also, multiplying a 5-tuple by a unit modulo V does not change the simplex.

$\frac{1}{V}(9,1,-2,-3,-5)$	$\frac{1}{V}(15,1,-2,-5,-9)$	$\frac{1}{V}(9,6,4,-1,-18)$
$\frac{1}{V}(9,2,-1,-4,-6)$	$\frac{1}{V}(12,5,-3,-4,-10)$	$\frac{1}{V}(9,6,5,-2,-18)$
$\frac{1}{V}(12,3,-4,-5,-6)$	$\frac{1}{V}(15,2,-3,-4,-10)$	$\frac{1}{V}(12,9,1,-4,-18)$
$\frac{1}{V}(12,2,-3,-4,-7)$	$\frac{1}{V}(6,4,3,-1,-12)$	$\frac{1}{V}(10,7,4,-1,-20)$
$\frac{1}{V}(9,4,-2,-3,-8)$	$\frac{1}{V}(7,5,3,-1,-14)$	$\frac{1}{V}(10,8,3,-1,-20)$
$\frac{1}{V}(12,1,-2,-3,-8)$	$\frac{1}{V}(9,7,1,-3,-14)$	$\frac{1}{V}(10,9,4,-3,-20)$
$\frac{1}{V}(12,3,-1,-6,-8)$	$\frac{1}{V}(15,7,-3,-5,-14)$	$\frac{1}{V}(12,10,1,-3,-20)$
$\frac{1}{V}(15,4,-5,-6,-8)$	$\frac{1}{V}(8,5,3,-1,-15)$	$\frac{1}{V}(12,8,5,-1,-24)$
$\frac{1}{V}(12,2,-1,-4,-9)$	$\frac{1}{V}(10,6,1,-2,-15)$	$\frac{1}{V}(15,10,6,-1,-30)$
$\frac{1}{V}(10,6,-2,-5,-9)$	$\frac{1}{V}(12,5,2,-4,-15)$	

Table 2.1: The 29 stable quintuples of Mori-Morrison-Morrison.

In all the families we have stated some restrictions on the volume $V \in \mathbb{N}$ or the parameters $\alpha, \beta \in \mathbb{Z}_V$ (e.g. the condition $\gcd(V, \alpha, \beta) = 1$ when $k = 1$). Without these restrictions the 5-tuples represent *hollow* cyclic 4-simplices. That these restrictions are necessary for emptyness is part of Theorem 2.2.1, and their sufficiency is shown in propositions 2.3.2, 2.4.6 and 2.4.7. That is, we have the following converse of Theorem 2.2.1:

Theorem 2.2.4. *All the cyclic* 4*-simplices described in Theorem 2.2.1 are empty.*

Proof. The proof of cases $k = 1, \cdots, 4$ of the theorem is explained in sections 2.3,2.4 and 2.5. □

In order to have a complete classification we need to enumerate the finitely many exceptions of the cases $k = 3, 4$. For this, in Section 2.5 we first prove an upper bound for their volume (Theorem 3.4.1) and then enumerate all empty simplices up to that volume. This yields:

Theorem 2.2.5. *Apart of the* $1 + 2 + 29 + 17$ *infinite families described in Theorem 2.2.1, there are exactly 2461 sporadic empty* 4*-simplices. Their volumes range from 24 to 419 and the number of them for each volume is as listed in Table 2.6.*

Index 2:	**Index 3:**
$\frac{1}{2}(0,0,1,1,0) + \frac{1}{V}(3,-1,-6,2,2)$	$\frac{1}{3}(0,0,2,1,0) \pm \frac{1}{V}(-3,2,1,1,-1)$
$\frac{1}{2}(1,0,0,0,1) + \frac{1}{V}(4,-3,1,-4,2)$	$\frac{1}{3}(1,0,2,0,0) \pm \frac{1}{V}(3,-3,1,-2,1)$
$\frac{1}{2}(1,0,0,0,1) + \frac{1}{V}(2,3,-1,-8,4)$	$\frac{1}{3}(0,0,1,2,0) \pm \frac{1}{V}(-3,1,2,2,-2)$
$\frac{1}{2}(0,1,1,0,0) + \frac{1}{V}(1,-6,2,6,-3)$	$\frac{1}{3}(0,0,1,2,0) \pm \frac{1}{V}(4,-2,-4,1,1)$
$\frac{1}{2}(1,0,1,0,0) + \frac{1}{V}(6,-8,4,-3,1)$	$\frac{1}{3}(1,0,2,0,0) \pm \frac{1}{V}(3,-6,2,2,-1)$
$\frac{1}{2}(1,0,0,0,1) + \frac{1}{V}(4,3,-1,-12,6)$	$\frac{1}{3}(1,0,2,0,0) \pm \frac{1}{V}(4,-6,1,2,-1)$
	$\frac{1}{3}(1,0,2,0,0) \pm \frac{1}{V}(4,-3,1,-4,2)$
Index 4:	$\frac{1}{3}(0,0,1,1,1) \pm \frac{1}{V}(1,-6,2,6,-3)$
$\frac{1}{4}(2,1,1,0,0) \pm \frac{1}{V}(3,-3,1,-2,1)$	**Index 6:**
$\frac{1}{4}(0,1,1,0,2) \pm \frac{1}{V}(1,3,-1,-6,3)$	$\frac{1}{6}(1,0,0,4,1) \pm \frac{1}{V}(1,-3,1,2,-1)$

Table 2.2: The 23 non-primitive quintuples.

2.3 Proof of the main theorem, cases $k = 1, 2$

Proof of Theorem 2.2.1, case $k = 1$. There are two possibilities for a hollow configuration S of five points in dimension one. Either $S = \{0,0,0,1,1\}$ or $S = \{0,0,0,0,1\}$. We first show that every cyclic simplex projecting to the latter projects also to the former. Indeed, let $P = \operatorname{conv}(v_0,\dots,v_4)$ be a cyclic simplex and $\pi : \mathbb{R}^4 \to \mathbb{R}$ a lattice projection sending $v_0,\dots,v_3$ to 0 and v_4 to 1. Since the facet $F = \operatorname{conv}(v_0,\dots,v_3)$ is an empty tetrahedron, by Theorem 1.2 there is a lattice functional $f : \mathbb{R}^4 \to \mathbb{R}$ sending two of its vertices (say v_0 and v_1) to 0 and the other two to 1. Let $c = f(v_4) \in \mathbb{Z}$. Then the functional $f - c\cdot\pi$ sends v_0, v_1 and v_4 to 0 and v_2, v_3 to 1.

That is to say; we have a single fine family, projecting to $S = \{0,0,0,1,1\}$. It is a primitive configuration and the linear space of affine dependences among its five points equals

$$\{(\alpha+\beta,-\alpha,-\beta,-\gamma,\gamma) : \alpha,\beta,\gamma \in \mathbb{R}\}.$$

Thus, by Corollary 2.1.12, every cyclic simplex of volume V projecting to S has a 5-tuple of the form

$$(\alpha+\beta,-\alpha,-\beta,-\gamma,\gamma),$$

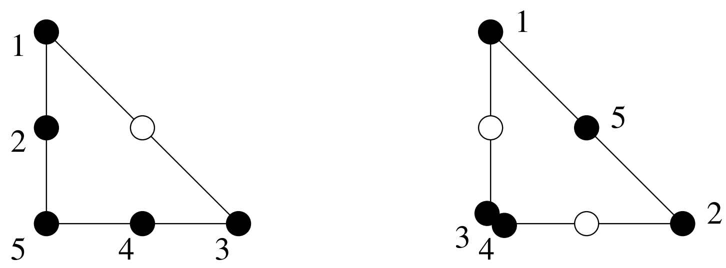


Figure 2.2: There are six possibilities for a size 5 subconfiguration of $2\Delta_2$ containing the three vertices. Only the first two arise as the projection of empty 4-simplices with $k = 2$.

with $\alpha, \beta, \gamma \in \mathbb{Z}$. Fix such a quintuple. By Proposition 2.1.14, for the simplex to be empty we need $\gcd(\gamma, V) = \gcd(\alpha, \beta, V) = 1$. Multiplying the $(d+1)$-tuple by the inverse of γ modulo V, there is no loss of generality in taking $\gamma = 1$. $\square$

Proof of Theorem 2.2.1, case $k = 2$. Our set S consists now of five of the six lattice points in $2\Delta_2$, perhaps with repetition. In order for S not to project to dimension 1, we need to use the three vertices of Δ, which leaves six possibilities for the two additional elements of S, modulo affine symmetry. But we claim that:

Claim: *no four of the five elements of S can be on the same edge of $2\Delta_2$:* Suppose that $\pi : P \to 2\Delta$ projects an empty 4-simplex P to 2Δ, with four of the vertices going to the same edge of 2Δ. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the lattice functional taking the value 0 on that edge and the value 2 at the opposite vertex. Let $\tilde{f} := f \circ \pi : \mathbb{R}^4 \to \mathbb{R}$. Since the facet of P where $\tilde{f}$ vanishes is an empty tetrahedron, by Theorem 1.2 there is a lattice functional $g : \mathbb{R}^4 \to \mathbb{R}$ sending two of its vertices to 0 and the other two to 1. Let c be the value of g at the fifth vertex of P. Then $g - \lfloor \frac{c}{2} \rfloor \tilde{f}$ takes values 0 or 1 at all vertices of P, contradicting the fact that P does not project to a hollow segment.

The claim implies that S is, modulo symmetries of $2\Delta_2$, one of the two point configurations in Figure 2.3. Their respective spaces of linear dependences are as follows, where coordinates are labeled as shown in the figure.

$$\{(\beta, -2\beta, \alpha, -2\alpha, \alpha + \beta) : \alpha, \beta \in \mathbb{R}^2\} \text{ and } \{(-\beta, -\beta, \alpha, -\alpha, 2\beta) : \alpha, \beta \in \mathbb{R}^2\};$$

the integer dependences are the same, with $\alpha, \beta \in \mathbb{Z}$. The first configuration is primitive ($I = 1$), but in the second one we have $I = 2$ and we can choose as barycentric coordinates for the unique generator of the quotient group the vector $\left(0, \frac{1}{2}, 0, \frac{1}{2}, 0\right)$. Thus, by Corollary 2.1.12, the cyclic simplices of volume V projecting to these configurations are parametrized by

$$(\beta, -2\beta, \alpha, -2\alpha, \alpha+\beta) \quad \text{and} \quad \frac{V}{2}(0,1,0,1,0) + (-\beta, -\beta, \alpha, -\alpha, 2\beta),$$

respectively. In the first case V must be odd, by Proposition 2.1.14. In the second case V must be even, since V is a multiple of the index $I = 2$. Proposition 2.1.14 also implies that $\gcd(\alpha, V) = \gcd(\beta, V) = 1$ for empty simplices. This allows us to multiply the quintuple by the inverse of β modulo V, producing quintuples in the form of Theorem 2.2.1. (In the second one, observe that both β and its inverse are odd, so that multiplying by β^{-1} leaves the part $\left(0, \frac{1}{2}, 0, \frac{1}{2}, 0\right)$ intact). Finally, in the second quintuple V must be a multiple of four since for $V = 2 \pmod 4$ the quintuple

$$\left(0, \frac{V}{2}, 0, \frac{V}{2}, 0\right) + (-1, -1, \alpha, -\alpha, 2) = \left(-1, \frac{V}{2} - 1, \alpha, \frac{V}{2} - \alpha, 2\right)$$

has two even b_is, contradicting Proposition 2.1.14 (observe that α is odd, since $\gcd(\alpha, V) = 1$ and V is even). $\square$

Let us finally check that the conditions stated in Theorem 2.2.1 for α, β and V are not only necessary but also sufficient for the corresponding simplices to be empty. To show this via Proposition 2.1.15 we need to look at the facets of volumes in each case:

Lemma 2.3.1. *Let P be an empty simplex as in Theorem 2.2.1 with $k \in \{1, 2\}$. Then, the volumes $(V_0, V_1, \dots, V_4)$ of its facets are:*

1. *If $k = 1$, we have $V_0 = \gcd(V, \alpha+\beta)$, $V_1 = \gcd(V, \alpha)$, $V_2 = \gcd(V, \beta)$ and $V_3 = V_4 = 1$. In particular, there can be up to three nonunimodular facets.*

2. *If $k = 2$ then $V_0 = V_1 = V_2 = V_3 = 1$. In the primitive case $V_4 = \gcd(V, \alpha + 1)$ and in the nonprimitive case $V_4 = 2$. In particular, there is at most one nonunimodular facet.*

Proof. This follows directly from Lemma 2.1.13 and the expression of the quintuples, taking into account that if $k = 2$ and P is primitive then V is required to be odd, while if $k = 2$ and P is nonprimitive then V is required to be a multiple of four. $\square$

Proposition 2.3.2. *All the cyclic simplices in the conditions stated in parts $k = 1$ and $k = 2$ of Theorem 2.2.1 are empty.*

Proof. By Corollary 2.1.12, all integer values of V, α and β produce hollow simplices, since they produce simplices projecting to hollow configurations in dimensions 1 and 2. Hence, we can apply Proposition 2.1.15, taking into account the facet volumes computed in Proposition 2.3.1:

- For the quintuple $(\alpha+\beta, -\alpha, -\beta, -1, 1)$ we have:

$$(\alpha+\beta, -\alpha, -\beta, -1, 1) = (0, -\alpha, \alpha, -1, 1) \pmod{\gcd(V, \alpha+\beta)},$$
$$(\alpha+\beta, -\alpha, -\beta, -1, 1) = (\beta, 0, -\beta, -1, 1) \pmod{\gcd(V, \alpha)},$$
$$(\alpha+\beta, -\alpha, -\beta, -1, 1) = (\alpha, -\alpha, 0, -1, 1) \pmod{\gcd(V, \beta)},$$

  as required.

- For the case $k = 2$, primitive, (quintuple $(1, -2, \alpha, -2\alpha, 1+\alpha)$):

$$(1, -2, \alpha, -2\alpha, 1+\alpha) = (1, -2, -1, 2, 0) \pmod{\gcd(V, \alpha+1)}.$$

- For the case $k = 2$, nonprimitive, with quintuple

$$\left(0, \frac{V}{2}, 0, \frac{V}{2}, 0\right) + (-1, -1, \alpha, -\alpha, 2) = \left(-1, \frac{V}{2} - 1, \alpha, \frac{V}{2} - \alpha, 2\right)$$

  we have

$$\left(-1, \frac{V}{2} - 1, \alpha, \frac{V}{2} - \alpha, 2\right) = (-1, 1, -1, 1, 0) \pmod{2}. \qquad \square$$

2.4 Proof of the main theorem, case $k = 3$

We now look at the case $k = 3$. That is, let $P = \operatorname{conv}(v_0, \dots, v_4)$ be a hollow cyclic 4-simplex (later we will add the constraint that P is empty) and $\pi : \mathbb{R}^4 \to \mathbb{R}^3$ be a projection map sending the vertices of P to a hollow 3-dimensional configuration $S = \{s_0, \dots, s_4\}$ with the property that S does not project to dimension two. There are finitely many possibilities for S, by Theorem 2.1.1. Their exhaustive computation was done in [AKW17] and can be summarized as follows:

Theorem 2.4.1 (Averkov et al. [AWW11, AKW17]). *There are twelve maximal* 3*-dimensional hollow polytopes that do not project to dimension two. Their volumes are bounded by* 36 *(attained by the tetrahedron* $\operatorname{conv}(0, 6e_1, 3e_2, 2e_3)$*).*

Observe that $\operatorname{conv}(S)$ is a 3-polytope with four or five vertices, for which there are combinatorially three cases: it is either a tetrahedron, a pyramid over a quadrilateral, or a triangular bipyramid (a convex union of two tetrahedra with a common facet). Our proof mixes a (computationally straightforward) enumeration of the subconfigurations of the twelve maximal 3-polytopes from Theorem 2.4.1 with some theoretical observations. What we need from the enumeration is summarized in the following statement. The computations giving it were done by Mónica Blanco:

Lemma 2.4.2. *The twelve polytopes of Theorem 2.4.1 contain exactly the following subconfigurations of size five and which do not project to dimension two, according to the combinatorics of their convex hull:*

1. *A certain number of tetrahedra (with an additional boundary point).*
2. *24 quadrilateral pyramids, all of them containing some lattice point in the interior of the quadrilateral facet.*
3. *29 primitive bipyramids, whose affine dependences are generated by the quintuples in Table 2.1.*
4. *23 nonprimitive bipyramids, whose data are specified in Table 2.3.* □

The following statement shows that we do not need to care much about tetrahedra and quadrilateral pyramids:

Proposition 2.4.3. *Let S be a hollow configuration of five points in $\mathbb{R}^3$ such that* $\mathrm{conv}(S)$ *is one of the tetrahedra or quadrilateral pyramids of Lemma 2.4.2. Then, any empty 4-simplex projecting to S has volume bounded by* 72.

Remark 2.4.4. *The existence of a global bound in Proposition 2.4.3 follows from results in [BHHS]. For tetrahedra this is Corollary 4.2 in that paper, and for pyramids over non-hollow polytopes it is the combination of Corollary 4.4 and Lemma 4.1. We include a proof of Proposition 2.4.3 in order to give the explicit bound of 72.*

Proof. Let $P = \mathrm{conv}(v_0, \dots, v_4)$ be empty and projecting to $S = \{s_0, \dots, s_4)$ and let $\pi : \mathbb{R}^4 \to \mathbb{R}^3$ be the projection map. (We assume $\pi(v_i) = s_i$).

Suppose first that $\mathrm{conv}(S)$ is a tetrahedon, with vertices s_1, s_2, s_3, s_4. Let s_0 be the fifth element of S (which may or may not coincide with one of the vertices). Since P is not empty, $\pi^{-1}(s_0) \cap P$ is a segment having v_0 as one end-point and of length at most one. It is easy to show (see [IVS19, Lemma 3.1]; in our case s_0 is the "Radon point of S") that

$$\mathrm{Vol}(P) = \mathrm{Vol}(\mathrm{conv}(S)) \times \mathrm{length}(\pi^{-1}(s_0) \cap P) \leq \mathrm{Vol}(\mathrm{conv}(S)). \tag{2.1}$$

For the tetrahedra of Lemma 2.4.2 this gives us a bound of 36, via Theorem 2.4.1.

Suppose now that $\mathrm{conv}(S)$ is a pyramid over a quadrilateral Q, with apex at s_0. Let ℓ be the lattice distance between the plane spanned by Q and s_0, and let

$a \in \frac{1}{I}\mathbb{Z}^5$	$b \in \mathbb{Z}^5$	restrictions for no two entries of $Va \pm b$ to have a common divisor with I
$\frac{1}{2}(0,0,1,1,0)$	$(3,-1,-6,2,2)$	$V = 2 \pmod 4$
$\frac{1}{2}(1,0,0,0,1)$	$(4,-3,1,-4,2)$	$V = 2 \pmod 4$
$\frac{1}{2}(0,0,1,0,1)$	$(4,-2,-6,3,1)$	$V \in \emptyset$
$\frac{1}{2}(1,0,0,0,1)$	$(2,3,-1,-8,4)$	$V = 2 \pmod 4$
$\frac{1}{2}(0,1,1,0,0)$	$(1,-6,2,6,-3)$	$V = 2 \pmod 4$
$\frac{1}{2}(1,0,1,0,0)$	$(6,-8,4,-3,1)$	$V = 2 \pmod 4$
$\frac{1}{2}(0,1,0,0,1)$	$(1,6,-4,-6,3)$	$V \in \emptyset$
$\frac{1}{2}(1,0,0,0,1)$	$(4,3,-1,-12,6)$	$V = 2 \pmod 4$
$\frac{1}{2}(0,1,0,0,1)$	$(3,-1,4,-12,6)$	$V \in \emptyset$
$\frac{1}{4}(2,1,1,0,0)$	$(3,-3,1,-2,1)$	$V = 0 \pmod 8$
$\frac{1}{4}(0,1,1,0,2)$	$(1,2,-1,-4,2)$	$V \in \emptyset$
$\frac{1}{4}(0,0,1,2,1)$	$(1,-4,1,4,-2)$	$V \in \emptyset$
$\frac{1}{4}(0,1,1,0,2)$	$(1,3,-1,-6,3)$	$V = 0 \pmod 8$
$\frac{1}{3}(0,0,2,1,0)$	$(-3,2,1,1,-1)$	$V = 0 \pmod 9$
$\frac{1}{3}(1,0,2,0,0)$	$(3,-3,1,-2,1)$	$V = \pm 6 \pmod 9$
$\frac{1}{3}(0,0,1,2,0)$	$(-3,1,2,2,-2)$	$V = 0 \pmod 9$
$\frac{1}{3}(0,0,1,2,0)$	$(4,-2,-4,1,1)$	$V \in \{0,\pm 6\} \pmod 9$
$\frac{1}{3}(1,0,2,0,0)$	$(3,-6,2,2,-1)$	$V = \pm 3 \pmod 9$
$\frac{1}{3}(1,0,2,0,0)$	$(4,-6,1,2,-1)$	$V = 0 \pmod 9$
$\frac{1}{3}(1,0,2,0,0)$	$(4,-3,1,-4,2)$	$V = 0 \pmod 9$
$\frac{1}{3}(1,0,2,0,0)$	$(2,-1,2,-6,3)$	$V \in \emptyset$
$\frac{1}{3}(0,0,1,1,1)$	$(1,-6,2,6,-3)$	$V = \pm 6 \pmod 9$
$\frac{1}{6}(1,0,0,4,1)$	$(1,-3,1,2,-1)$	$V = 0 \pmod{36}$

Table 2.3: The 23 non-primitive hollow triangular bipyramids of Lemma 2.4.2. In each of them we give a generator $a \in \frac{1}{I}\mathbb{Z}^5$ for the quotient group $\mathbb{Z}^3/\Lambda_S \cong \mathbb{Z}_I$ (written in barycentric coordinates with respect to the vertex set S of the bipyramid) and the primitive affine dependence $b \in \mathbb{Z}^5$ among S.

$F = \text{conv}(v_1, v_2, v_3, v_4)$ be the facet of P that projects to Q. Observe that the lattice distance between the hyperplane spanned by F and v_0 divides ℓ. In particular,

$$\text{Vol}(\text{conv}(S)) = \ell \times \text{Vol}(Q), \qquad \textit{and} \qquad \text{Vol}(P) \leq \ell \times \text{Vol}(F).$$

Let x be the intersection of the two diagonals of Q, which is in this case the Radon point of S as used in [IVS19, Lemma 3.1]. As before, that lemma says

$$\text{Vol}(F) = \text{Vol}(Q) \times \text{length}(\pi^{-1}(x) \cap F),$$

so that

$$\text{Vol}(P) \leq \text{Vol}(\text{conv}(S)) \times \text{length}(\pi^{-1}(x) \cap F).$$

It is no longer true that $\text{length}(\pi^{-1}(x) \cap F) \leq 1$, but we can bound this length as follows. Let y be an interior lattice point in Q and let z be the last point in Q along the ray from x through y (if $x = y$, let z be an arbitrary boundary point of Q). Then, it follows from the proof of [IVS19, Lemma 3.1] (see also the related result [IVS19, Lemma 3.5]) that

$$\text{length}(\pi^{-1}(x) \cap F) = \frac{|xz|}{|yz|} \, \text{length}(\pi^{-1}(y) \cap P) \leq \frac{|xz|}{|yz|}.$$

This gives us the desired upper bound on the volume of P:

$$\text{Vol}(P) \leq \text{Vol}(\text{conv}(S)) \times \frac{|xz|}{|yz|}. \tag{2.2}$$

For the 24 pyramids of Lemma 2.4.2 this formula (taking the best possibility for the interior point y whenever there is a choice) gives the claimed bound of 72. $\square$

With this we can now prove the case $k = 3$ in our main theorem:

Proof of Theorem 2.2.1, case $k = 3$. Let $P = \text{conv}(v_0, \dots, v_4)$ be an empty 4-simplex projecting to a hollow configuration $S = \{s_0, \dots, s_4\} \subset \mathbb{R}^3$ that does not project to dimension two. By Proposition 2.4.3, if $\text{conv}(S)$ is not a triangular bipyramid then $\text{Vol}(P) \leq 72$. For each of the 29 primitive plus 23 nonprimitive triangular bipyramids of Lemma 2.4.2, Corollary 2.1.12 tells us how to parametrize the $(d+1)$-tuples of empty simplices projecting to them. More precisely:

- When the bipyramid is primitive, the quintuple is an integer affine dependence b among S. A priori there are different possibilities for b, since the affine dependence of S is only unique modulo multiplication by a scalar. But by Proposition 2.1.14 we can assume b to not have a common divisor with V. This implies that b equals, modulo V, the primitive affine dependence times a factor coprime with V. Since multiplying a quintuple by a unit modulo V does not change the cyclic simplex it represents, there is no loss of generality in taking b to be the primitive dependence as we do in Table 2.1.

- When the bipyramid is not primitive, the quintuple is of the form $Va + b$ where a are the barycentric coordinates of a generator of $\mathbb{Z}^3/\Lambda_S$ and b is an integer affine dependence among S. Observe that Corollary 2.1.12 allows us to choose our preferred a (even if $\mathbb{Z}^3/\Lambda_S$ may have several generators) but it does not, a priori, allow us to choose b. But, as before, every two valid choices of b are related via a unit modulo V. That is, every empty simplex of volume V for one of these bipyramids can be represented as a quintuple of the form

  $$Va + rb,$$

  where a and b are the choices in Table 2.3, and $r \in \mathbb{Z}$ is coprime with V. Multiplying such a quintuple by $r^{-1} \pmod V$ we find that the same simplex is represented by

  $$Vr^{-1}a + b.$$

  Now, since I divides V, r is also a unit modulo I, which implies that $r^{-1}a$ is also a generator of $\mathbb{Z}^3/\Lambda_S$. In all our cases $I = \{1, 2, 3, 6\}$, so $\mathbb{Z}^3/\Lambda_S \cong \mathbb{Z}_I$ has only two generators, $\pm a$. Thus, our simplex is represented by the quintuple $\pm Va + b$.

This finishes the proof, except for the fact that in Table 2.3 we have 23 non-primitive quintuples while in Theorem 2.2.1 (Table 2.2) only seventeen appear, and except for the restrictions on V displayed in tables 2.1 and 2.3. These restriction are proved in Propositions 2.4.6 and 2.4.7 below and imply, in particular, that the six nonprimitive quintuples that have "$V \in \emptyset$" as a restriction in Table 2.3 do not produce any empty 4-simplex. □

To show that the restrictions on V shown in tables 2.1 and 2.3 are necessary and sufficient for the quintuple to produce an empty 4-simplex, we use Propositions 2.1.14 and 2.1.15, as we did in the previous section.

Lemma 2.4.5. *For the empty simplices of the case "$k = 3$, primitive", all facets are unimodular except for the 12 quintuples of Table 2.4, which can have up to three nonunimodular facets, as indicated.*

Proof. By Lemma 2.1.13, the volume of the ith facet of the primitive cyclic simplex of volume V with quintuple b equals $\gcd(V, b_i)$. On the other hand, by Proposition 2.1.14, no two facets can have volumes with a common factor. Thus, the vector of facet volumes divides (coordinate-wise) the vector obtained from b by removing

quintuple	condition on V for emptynes	max. volumes of facets
$(15,1,-2,-5,-9)$	$V \notin 3\mathbb{Z}$	$(1,1,2,1,1)$
$(9,7,1,-3,-14)$	$V \notin 3\mathbb{Z} \cup 7\mathbb{Z}$	$(1,1,1,1,2)$
$(15,7,-3,-5,-14)$	$V \notin 3\mathbb{Z} \cup 5\mathbb{Z} \cup 7\mathbb{Z}$	$(1,1,1,1,2)$
$(10,8,3,-1,-20)$	$V \notin 2\mathbb{Z} \cup 5\mathbb{Z}$	$(1,1,3,1,1)$
$(12,3,-4,-5,-6)$		$(1,1,1,5,1)$
$(9,6,5,-2,-18)$	$V \notin 2\mathbb{Z} \cup 3\mathbb{Z}$	$(1,1,5,1,1)$
$(12,8,5,-1,-24)$		$(1,1,5,1,1)$
$(12,2,-3,-4,-7)$	$V \notin 3\mathbb{Z}$	$(1,1,1,1,7)$
$(10,7,4,-1,-20)$	$V \notin 3\mathbb{Z} \cup 7\mathbb{Z}$	$(1,7,1,1,1)$
$(8,5,3,-1,-15)$	$V \notin 3\mathbb{Z} \cup 5\mathbb{Z}$	$(8,1,1,1,1)$
$(9,1,-2,-3,-5)$	$V \notin 3\mathbb{Z}$	$(1,1,2,1,5)$
$(7,5,3,-1,-14)$	$V \notin 7\mathbb{Z}$	$(1,5,3,1,2)$

Table 2.4: Possible nonunimodular facets in the case "$k = 3$, primitive". The facet volumes depend on the actual V. More precisely, an entry $(v_0, v_1, v_2, v_3, v_4)$ in the last column means that the volume of the ith facet equals $\gcd(V, v_i)$.

the prime factors that divide two or more entries of b. These vectors are precisely what we show in the last column in Table 2.4. □

Proposition 2.4.6. *The conditions on V stated in Table 2.1 are necessary and sufficient for the quintuples to represent empty simplices.*

Proof. Necessity follows from Proposition 2.1.14, since in all cases the restriction can be restated as "V has no factor in common with two of the entries in B". Sufficiency follows from Proposition 2.1.15 and the description of facet volumes in Table 2.4. Let us look at the first case in detail and leave the rest to the interested reader. Our quintuple is $(9, 1, -2, -3, -5)$, and the worst values for the $(V_0, V_1, V_2, V_3, V_4)$ of Proposition 2.1.15 are $(1, 1, 2, 1, 5)$, as expressed in Table 2.4. We say "worst" because V_2 is only 2 if V is even and V_4 is only 5 if $V \in 5\mathbb{Z}$, but if this is not the case then the corresponding V_i equals 1 and the condition in part (2) of Proposition 2.1.15 is void. So, assuming the worst case, what we need to check is that

$$(9, 1, -2, -3, -5) = (-1, 1, -2, 2, 0) \pmod 5$$

and

$$(9, 1, -2, -3, -5) = (-1, 1, 0, -1, 1) \pmod 2$$

have their non-zero entries forming two pairs of opposite residues modulo the respective $V_i \in \{5, 2\}$, which is indeed the case. $\square$

Proposition 2.4.7. *Let a and b be one of the 23 possibilities in Table 2.3. Let $I \in \{2, 3, 4, 6\}$ be its index. Let $k \in \mathbb{N}$ and $V = kI$. The following are equivalent:*

1. *k satisfies the restrictions modulo 2 and 3 stated in Table 2.5.*
2. *No factor of V divides two entries of $\pm Va + b$.*
3. *The simplex of volume V represented by the quintuple $\pm Va + b$ is empty.*

Proof. In the first column of Table 2.5 we have written the vector $\pm Va + b$, in terms of k. Observe that we have $\pm Va + b = \pm ka' + b$, where $a' := Ia$ is the integer vector from the first column of Table 2.3. From this, the reader can easily check the implication (2)⇔(1); if k does not satisfy one of the restrictions, then 2 or 3 is a common factor of $V = kI$ and at least two entries of $\pm a + \frac{V}{b}$. For the reverse implication, first observe that if a prime p divides kI and some entry of $\pm Va + b$ then $p \in \{2, 3\}$; indeed, if p divides $I \in \{2, 3, 4, 6\}$ then this is obvious and if p divides k then for it to divide an entry of $\pm Va + b = \pm ka' + b$ it must divide the corresponding entry of b, and these have only 2 and 3 as prime factors. Once this is established, it is clear that the conditions for part (2) can all be expressed as restrictions on k modulo 2 and 3, and direct inspection shows that they are the ones stated in the table. Let us show this in a couple of cases and leave the rest to the reader:

- For the forth quintuple of index 3, $a = \frac{1}{3}(0, 0, 1, 2, 0), b = (4, -2, -4, 1, 1)$, we have that the quintuple is

  $$\pm Va + b = (4, -2, \pm k - 4, \pm 2k + 1, 1).$$

  Since the first two entries are even V, hence k, must be odd. Modulo three, the third and forth entries of $Va + b$ are multiples of three if $k = 1 \pmod 3$ and the same holds for $-Va + b$ if $k = -1 \pmod 3$. In the table we abbreviate this as $\pm k \neq 1 \pmod 3$ meaning that the plus sign is taken for $Va + b$ and the minus sign for $-Va + b$.

  The interpretation of this is that for $k = 0 \pmod 3$ (that is, $V = 0 \pmod 9$) this case produces *two* empty simplices, with quintuples $Va + b$ and $-Va + b$, while for $k \neq 0 \pmod 3$ it produces only one of the two.

- Consider now the second quintuple of index four, with $a = \frac{1}{4}(0, 1, 1, 0, 2)$ and $b = (1, 2, -1, -4, 2)$. We have that

  $$\pm Va + b = (1, k + 2, k - 1, -4, 2k + 2).$$

I	$\pm Va+b$, written in terms of $k := V/I$	conditions on $\pm k$ mod 2	conditions on $\pm k$ mod 3	max. vol. of facets
	$(3,-1,\pm k-6,\pm k+2,2)$	$=1$	$\neq 0$	$(1,1,1,1,\mathbf{2})$
	$(\pm k+4,-3,1,-4,\pm k+2)$	$=1$		$(1,3,1,\mathbf{2},1)$
	$(4,-2,\pm k-6,3,\pm k+1)$	$\in\emptyset$		
	$(\pm k+2,3,-1,-8,\pm k+4)$	$=1$		$(1,3,1,\mathbf{2},1)$
2	$(1,\pm k-6,\pm k+2,6,-3)$	$=1$	$\neq 0$	$(1,1,1,\mathbf{2},1)$
	$(\pm k+6,-8,\pm k+4,-3,1)$	$=1$	$\neq 0$	$(1,\mathbf{2},1,1,1)$
	$(1,\pm k+6,-4,-6,\pm k+3)$	$\in\emptyset$		
	$(\pm k+4,3,-1,-12,\pm k+6)$	$=1$	$\neq 0$	$(1,1,1,\mathbf{2},1)$
	$(3,\pm k-1,4,-12,\pm k+6)$	$\in\emptyset$		
	$(\pm 2k+3,\pm k-3,\pm k+1,-2,1)$	$=0$	$\neq 0$	$(1,1,1,\mathbf{2},1)$
4	$(1,\pm k+2,\pm k-1,-4,\pm 2k+2)$	$\in\emptyset$		
	$(1,-4,\pm k+1,\pm 2k+4,\pm k-2)$	$\in\emptyset$		
	$(1,\pm k+3,\pm k-1,-6,\pm 2k+3)$	$=0$	$\neq 0$	$(1,1,1,\mathbf{2},1)$
	$(-3,2,\pm 2k+1,\pm k+1,-1)$		$=0$	$(\mathbf{3},2,1,1,1)$
	$(\pm k+3,-3,\pm 2k+1,-2,1)$		$=2$	$(1,\mathbf{3},1,2,1)$
	$(-3,1,\pm k+2,\pm 2k+2,-2)$	$=1$	$=0$	$(\mathbf{3},1,1,1,1)$
	$(4,-2,\pm k-4,\pm 2k+1,1)$	$=1$	$\neq 1$	$(1,1,1,1,1)$
3	$(\pm k+3,-6,\pm 2k+2,2,-1)$	$=1$	$=1$	$(1,\mathbf{3},1,1,1)$
	$(\pm k+4,-6,\pm 2k+1,2,-1)$	$=1$	$=0$	$(1,\mathbf{3},1,1,1)$
	$(\pm k+4,-3,\pm 2k+1,-4,2)$	$=1$	$=0$	$(1,\mathbf{3},1,1,1)$
	$(\pm k+2,-1,\pm 2k+2,-6,3)$	$\in\emptyset$		
	$(1,-6,\pm k+2,\pm k+6,\pm k-3)$	$=1$	$=2$	$(1,\mathbf{3},1,1,1)$
6	$(\pm k+1,-3,1,\pm 4k+2,\pm k-1)$	$=0$	$=0$	$(1,\mathbf{3},1,\mathbf{2},1)$

Table 2.5: The first column shows the 23 possibilities for the quintuple $\pm Va+b$ for cyclic simplices in the case "$k=3$, nonprimitive" (compare with Table 2.3). The second and third columns the restrictions on $k := V/I$ that make the simplex empty. The last column shows the possible facet volumes. As in Table 2.4, an entry (v_0,v_1,v_2,v_3,v_4) in the last column means that the volume of the ith facet equals $\gcd(V,v_i)$. When v_i divides the index I one automatically has $\gcd(V,v_i)=v_i$. In this case the corresponding entry v_i is marked in boldface.

It turns out that no matter what the value of k is, this quintuple contains two even entries (the 4th one is always even, the 3rd and 5th are even when k is respectively odd and even). Thus, condition (2) is never satisfied. In the table we mark this by putting $\pm k \in \emptyset$ as the restriction modulo 2.

This implies, by Proposition 2.1.14, these simplices not to be empty, no matter what the value of V is). The same happens for the other five quintuples that contain the restriction $\pm k \in \emptyset$.

The implication (3)⇒(2) is Proposition 2.1.14, so we only need to show (2)⇒(3). Part (2) implies that for each nonunimodular facet, of volume V_i, the vector $\pm Va + b$ has a single zero entry, modulo V_i. The condition in part (2) of Proposition 2.1.15 is then automatic: modulo $V_i \in \{2, 3\}$, every four non-zero integers adding up to zero form two opposite pairs. □

Corollary 2.4.8. *The facet volumes of the nonprimitie empty* 4*-simplices with* $k = 3$ *are as indicated in Table 2.5.* □

2.5 Case k=4

We call empty 4-simplices the simplices that do not projecto to hollow 3-polytopes *sporadic*. In the next Chapter (see Theorem 3.3.6) we show that their volume is bounded by 5184. Hence, in order to complete the list of sporadic empty 4-simplices we need to enumerate all empty 4-simplices up to that volume.

In Chapter 4 we describe how we implemented an exhaustive enumeration of empty 4-simplices up to volume 7600 [1]. The result of that enumeration gives us the total number of sporadic empty 4-simplices that belong to the case $k = 4$ of Theorem 2.2.1. Here we summarize the result of these computations:

Theorem 2.5.1. *List of sporadic empty* 4*-simplices:*

- *There is no empty* 4*-simplex of width greater than* 4*.*
- *There is only one empty* 4*-simplex of width* 4*, it has volume* 101 *and corresponds to the following quintuple* $\sigma()$*.*
- *Every empty* 4*-simplex of width* 3 *has volume between* 41 *and* 179*, and there are exactly q cases that do not belong to any family.*

[1] The fact of enumerating up to volume 7600 comes from early stages of the project where this number was the best upper bound known for empty 4-simplices with width greater than two [IVS19].

V	#	V	#	V	#	V	#	V	#	V	#	V	#
24	1	65	27	102	3	140	5	178	2	219	4	274	1
27	1	66	3	103	51	141	6	179	21	220	1	275	1
29	3	67	41	104	8	142	9	180	1	221	3	278	2
30	2	68	13	105	7	143	13	181	13	222	1	283	2
31	2	69	26	106	8	144	1	182	5	223	7	287	1
32	3	70	4	107	54	145	14	183	5	225	2	289	4
33	4	71	50	108	5	146	5	184	5	226	4	290	1
34	5	72	3	109	44	147	10	185	7	227	9	291	1
35	3	73	44	110	5	148	7	186	2	229	6	292	1
37	6	74	18	111	13	149	26	187	7	230	3	293	5
38	8	75	22	112	2	150	2	188	5	232	1	299	2
39	9	76	14	113	40	151	19	189	2	233	9	304	1
40	1	77	19	114	4	152	6	190	2	234	1	308	1
41	14	78	3	115	21	153	9	191	8	235	3	310	1
42	5	79	55	116	11	154	3	192	1	237	1	311	1
43	20	80	7	117	10	155	12	193	12	238	2	313	1
44	8	81	18	118	9	156	2	194	3	239	3	314	1
45	6	82	13	119	22	157	11	196	4	241	6	317	1
46	7	83	60	120	3	158	10	197	13	244	2	319	2
47	30	84	7	121	18	159	9	199	11	245	3	321	1
48	5	85	27	122	9	160	3	200	4	247	3	323	1
49	17	86	11	123	17	161	13	201	3	248	3	331	1
50	8	87	24	124	8	163	17	202	2	249	2	332	1
51	16	88	5	125	25	164	6	203	7	250	1	334	2
52	6	89	55	127	24	165	1	204	1	251	5	335	1
53	38	90	6	128	9	166	7	205	4	254	1	347	1
54	11	91	18	129	17	167	18	206	4	256	2	349	2
55	20	92	9	130	2	168	3	207	2	257	3	353	1
56	3	93	17	131	29	169	13	208	1	259	2	355	1
57	16	94	12	132	5	170	2	209	10	261	1	356	1
58	13	95	35	133	14	171	6	211	4	263	7	376	1
59	51	96	3	134	8	172	3	212	2	265	1	377	2
60	4	97	46	135	6	173	15	213	3	267	1	397	1
61	38	98	9	136	6	174	3	214	2	268	1	398	1
62	26	99	13	137	28	175	8	215	5	269	2	419	1
63	17	100	8	138	2	176	4	216	1	271	4		
64	9	101	41	139	37	177	5	218	5	272	1		

Table 2.6: Number of sporadic empty 4-simplices for each normalized volume V

- *There are exactly* 2281 *sporadic empty* 4*-simplices of width* 2*. Their volumes are between* 24 *and* 419*.*

In Table 2.6, the number of sporadic empty 4-simplices for each volume from 24 to 419 is shown. This list agrees with Conjecture 1.4 in [MMM88] in terms of terminal quotient singularities. This conjecture states that there are no empty 4-simplices with prime volume bigger than 419. This conjecture for prime volumes was first proved by Sankaran [San90]. Bober [Bob09] gave a simplified proof.

Chapter 3
Upper bounds for the volume of hollow polytopes of dimensions 3 and 4

In this Chapter we obtain upper bounds for the volume of hollow polytopes. First, we introduce some concepts of convex geometry related with the measures of a convex body, such as successive minima, covering minima and rational diameter.

After defining the geometric tools that we are going to use, we focus in obtaining an upper bound for the volume of hollow 4-simplices, in particular, empty simplices of width greater than two. This result allows us to verify the conjecture proposed by Haase and Ziegler with their enumeration of empty 4-simplices up to volume 1000.

As a tool for this upper bound we also obtain an upper bound for the volume of hollow convex bodies in dimension 3 of width greater than 2.155, improving the bounds that were known before.

In the second part of the Chapter we focus in getting an upper bound for the volume of empty 4-simplices of width two. Within this proof, we have to deal with case by case analysis in order to obtain the desired bound.

The objective of obtaining all these volume upper boundes is knowing up to which volume we have to enumerate all sporadic empty 4-simplices(those that do not belong to any infinite family, case $k = 4$ of Theorem 2.2.1) to guarantee that there will not exist other particular examples from that volume on.

3.1 Successive minima and Covering minima

In order to state the upper bound and prove it, we need to introduce the concepts of *successive minima* and *covering minima* of convex bodies with respect to a lattice $\mathcal{L}$. Remember that:

1. For a centrally symmetric convex body $C \subset \mathbb{R}^d$, the i-th successive minimum ($i \in \{1, \dots, d\}$) of C with respect to $\mathcal{L}$ is:

$$\lambda_i(C) := \inf\{\lambda > 0 : \dim(\lambda C \cap \mathcal{L}) \geq i\}.$$

That is to say, λ_i is the minimum dilation factor such that λC contains i linearly independent lattice vectors. Clearly $\lambda_1 \leq \cdots \leq \lambda_d$.

2. For a (not necessarily symmetric) convex body $K \subset \mathbb{R}^d$, the i-th covering minimum ($i \in \{1, \ldots, d\}$) of K with respect to $\mathcal{L}$ is defined as

   $\mu_i(K) := \inf\{\mu > 0 : \mu K + \mathcal{L}$ intersects every affine subspace of dimension $d-i\}$.

   Clearly $\mu_1 \leq \cdots \leq \mu_d$.

For example, $\mu_1(K)$ is nothing but the reciprocal of the lattice width of K, while $\mu_d(K)$ equals the covering radius of K (the minimum dilation factor μ such that $\mu K + \mathcal{L}$ covers $\mathbb{R}^d$). Similarly, $\lambda_1(C)$ equals twice the packing radius of C (the maximum dilation such that λC does not overlap any lattice translation of it).

Minkowski's Second Theorem [Gru93] relates successive minima and volume of a centrally symmetric convex body as follows:

$$\lambda_1(C)\lambda_2(C)\cdots\lambda_d(C)\,\mathrm{vol}(C) \leq 2^d. \tag{3.1}$$

Successive minima are not usually defined for a non-centrally symmetric body K (but see [HHH16]), but in this case the successive minima of the *difference body* $K - K := \{x - y : x, y \in K\}$ have a natural geometric meaning. For example, $\lambda_1(K - K)^{-1}$ equals the maximum (lattice) length of a rational segment contained in K. We call this the *rational diameter* of K. Observe that this is not the same as the “lattice diameter" used in [AKN15], defined as the maximum length of a *lattice* segment contained in K.

3.2 Upper bound for hollow 3-bodies

Once we have introduced the concepts of successive minima, covering minima and rational diameter of a convex body K we use well-known relations between these concepts and the volume of K, so we can obtain low upper bounds for empty simplices.

The volume of the difference body $K - K$ is bounded from below and from above by the Brunn-Minkowski and the Rogers-Shephard inequalities [Gru93], respectively:

$$2^d\,\mathrm{vol}(K) \leq \mathrm{vol}(K - K) \leq \binom{2d}{d}\mathrm{vol}(K). \tag{3.2}$$

The lower bound (resp., the upper bound) is an equality if and only if K is centrally symmetric (resp., a simplex).

From this we can derive the following inequality relating the volume of K and the first successive minimum of its difference body:

$$\mathrm{vol}(K) \leq \frac{\mathrm{vol}(K-K)}{2^d} \leq \frac{1}{\prod_{i=1}^{d} \lambda_i(K-K)} \leq \frac{1}{\lambda_1(K-K)^d}. \tag{3.3}$$

Less is known about the covering radii, but the following inequalities relating covering minima of K and successive minimum of $K-K$ are known ([KL88,Hur90] or see, e.g., [AKW17, Section 4]):

$$\mu_{i+1}(K) - \mu_i(K) \leq \lambda_{d-i}(K-K), \qquad \forall i \in \{1, \ldots, d-1\}, \tag{3.4}$$

$$\mu_2(K) \leq \left(1 + \frac{2}{\sqrt{3}}\right) \mu_1(K). \tag{3.5}$$

Once we have introduced the main tools that we will use in measuring convex bodies, we can now prove the main result in this section. Both our statement and proof are based on [AKW17, Proposition 11], which is the case $w = 3$. The theorem can also be considered the three-dimensional version of [AW12, Thm. 4.1], which gives bounds for the volume of a convex 2-body of width larger than 1.

Theorem 3.2.1. *Let K be a hollow convex 3-body of lattice width w, with $w > 1 + 2/\sqrt{3} = 2.155$ and let $\mu = w^{-1}$ be its first covering minimum. Then,* $\mathrm{vol}(K)$ *is bounded above by:*

$$\frac{8}{(1-\mu)^3} = \frac{8w^3}{(w-1)^3}, \qquad \text{if } w \geq \frac{2}{\sqrt{3}}(\sqrt{5}-1)+1 = 2.427, \text{ and}$$

$$\frac{3}{4\mu^2(1-\mu(1+2/\sqrt{3}))} = \frac{3w^3}{4(w-(1+2/\sqrt{3}))}, \qquad \text{if } w \leq 2.427.$$

Figure 3.1 plots the upper bound of Theorem 3.2.1 in the interval $w \in [2.4, 5]$ that will be of interest for us.

Proof. Throughout the proof we denote $\mu_i = \mu_i(K)$ and $\lambda_i = \lambda_i(K-K)$.

We use the following slightly modified version of Equation (3.3):

$$\mathrm{vol}(K) \leq \frac{1}{2^d} \mathrm{vol}(K-K) \leq (\lambda_1\lambda_2\lambda_3)^{-1} \leq \max\{{\lambda_1}^3, \lambda_1{\lambda_2}^2\}^{-1}.$$

Our goal is to lower bound $\max\{{\lambda_1}^3, \lambda_1{\lambda_2}^2\}$. For this we combine equations (3.4) as follows:

$$\lambda_2 \geq \mu_2 - \mu_1 \geq \mu_3 - \mu_1 - \lambda_1 \geq 1 - \mu_1 - \lambda_1,$$

where $\mu_3 \geq 1$ since K is hollow.

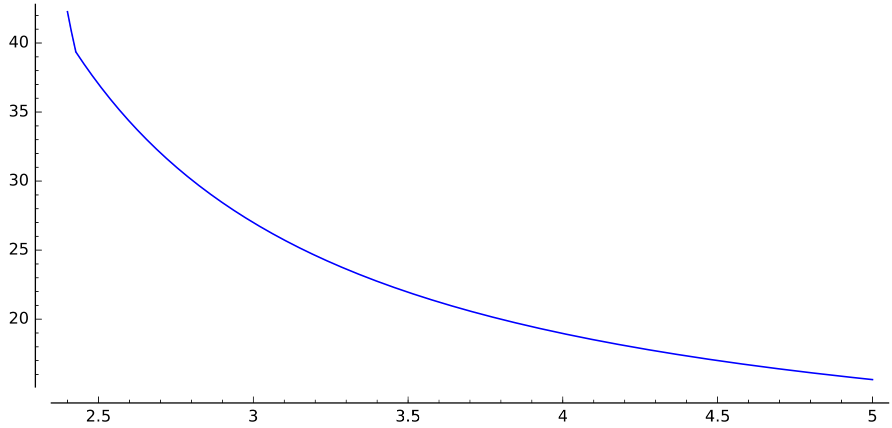


Figure 3.1: The upper bound of Theorem 3.2.1 for the Euclidean volume of a hollow 3-body (Y axis) in terms of its width (X axis).

That is:

$$\lambda_1\lambda_2\lambda_3 \geq \max\{\lambda_1^3, \lambda_1\lambda_2^2\} \geq \max\{\lambda_1^3, \lambda_1(1-\mu_1-\lambda_1)^2\}.$$

There are the following possibilities for the maximum on the right:

- If $\lambda_1 \geq (1-\mu_1)/2$ then either $1-\lambda_1-\mu_1$ is negative (and then smaller than λ_1 in absolute value, since $1-\mu_1$ is positive) or positive but smaller than λ_1. In both cases the maximum is λ_1^3, which in turn is at least $(1-\mu_1)^3/8$.

- If $\lambda_1 \leq (1-\mu_1)/2$ then $1-\lambda_1-\mu_1$ is positive and bigger than λ_1, so the maximum is $\lambda_1(1-\lambda_1-\mu_1)^2$. Now, by equations 3.4 and 3.5 we have

  $$\lambda_1 \geq \mu_3 - \mu_2 \geq 1-(1+2/\sqrt{3})\mu_1,$$

  we take as lower bound for $\lambda_1(1-\lambda_1-\mu_1)^2$ the absolute minimum of $f(\lambda) := \lambda(1-\lambda-\mu_1)^2$ in the interval

  $$1-(1+2/\sqrt{3})\mu_1 \leq \lambda \leq (1-\mu_1)/2.$$

  Since the only local minimum of f is in $\lambda = 1-\mu_1$, which is outside the interval, the minimum is achieved at one of the extremes. That is,

  $$f(\lambda) \geq \min\left\{\left(1-(1+2/\sqrt{3})\mu_1\right)\frac{4\mu_1{}^2}{3}, (1-\mu_1)^3/8\right\}.$$

  Now there are two things to take into account:

– For the interval to be non-empty we need

$$1 - \mu_1(1 + 2/\sqrt{3}) \leq (1 - \mu_1)/2,$$

which is equivalent to

$${\mu_1}^{-1} \leq 1 + 4/\sqrt{3} = 3.31.$$

– Whenever ${\mu_1}^{-1}$ is between $\frac{2}{\sqrt{3}}(\sqrt{5}-1)+1 = 2.427$ and $1+4/\sqrt{3} = 3.31$ we have

$$\min\left\{\left(1 - \mu_1(1 + 2/\sqrt{3})\right)\frac{4{\mu_1}^2}{3}, (1-\mu_1)^3/8\right\} = (1-\mu_1)^3/8,$$

while for ${\mu_1}^{-1} \leq \frac{2}{\sqrt{3}}(\sqrt{5}-1)+1 = 2.427$ the minimum is $(1 - \mu_1(1 + 2/\sqrt{3}))\frac{4{\mu_1}^2}{3}$.

□

The upper bound in this theorem is certainly not tight, but the threshold $w > 1+2/\sqrt{3} = 2.155$ is. Since there is a hollow (non-lattice) triangle of width $1+2/\sqrt{3}$ (see [Hur90, Figure 2]), taking prisms over it we can construct hollow 3-polytopes of arbitrarily large volume. In fact, inequality (3.5) is equivalent to the statement "no hollow polygon can have lattice width larger than $1 + 2/\sqrt{3}$". Hurkens was able to realise this fact before in [Hur90, Figure 2] and he showed that this is tight.

Remark 3.2.2. *Theorem 3.2.1 may be a step towards computing the exact value of the flatness constant in dimension three. By [AKW17] no hollow* 3*-polytope has width larger than three, but hollow polytopes with vertices not in the lattice can have larger width. Recently, Codenotti and Santos have constructed a hollow (non-lattice) tetrahedron of width* $2 + \sqrt{3} = 3.414$*, and conjectured that this attains the flatness constant in dimension* 3 *[CS19].*

In the particular case that concerns us, projections of empty 4-simplices, we can use that our polytopes have at most 5 vertices, and so, get better upper bounds.

The case of 5 points

Observe that in the proof of Theorem 3.2.1 what we bound is actually $\mathrm{vol}(K - K)$, and then use te Brunn-Minkowski inequality $\mathrm{vol}(K) \leq \mathrm{vol}(K - K)/8$ to transfer the bound to $\mathrm{vol}(K)$. This means that with additional information on K a sharper bound can be obtained. For example, if K is a tetrahedron then we know $\mathrm{vol}(K) =$

$\mathrm{vol}(K-K)/20$, which means that the bounds in the statement can all be multiplied by a factor of $8/20 = 0.4$. The case of interest to us in Section 3.3 is the somewhat similar case where Q is a 3-dimensional polytope expressed as the convex hull of five points (that is, either a tetrahedron, a square pyramid, or a triangular bipyramid). We now analyze this case in detail.

Most of what we want to say about this case is valid in arbitrary dimension, so let $A = \{a_1, \ldots, a_{d+2}\} \subset \mathbb{R}^d$ be $d+2$ points affinely spanning $\mathbb{R}^d$. In particular, there is a unique (modulo a scalar factor) vector $\lambda = (\lambda_1, \ldots, \lambda_{d+2})$ such that

$$\sum_{i=1}^{d+2} \lambda_i a_i = 0.$$

This naturally partitions A into three subsets (of which only A^0 can be empty):

$$A^+ := \{a_i : \lambda_i > 0\}, \quad A^0 := \{a_i : \lambda_i = 0\}, \quad A^- := \{a_i : \lambda_i < 0\}.$$

Of course, A^+ and A^- are interchanged when multiplying λ by a negative constant, but A^0 and the partition of $A \setminus A^0$ into two parts are independent of the choice of λ. In fact:

1. $a_i \in A^0$ if, and only if, $A \setminus \{a_i\}$ is affinely dependent (equivalently, A is a pyramid over $A \setminus \{a_i\}$).
2. (A^+, A^-) is the only partition of $A \setminus A^0$ into two parts such that $\mathrm{conv}(A^+) \cap \mathrm{conv}(A^-) \neq \emptyset$.
3. In fact, $\mathrm{conv}(A^+)\,\mathrm{conv}(A^-)$ is a single point. It is the unique point of $\mathbb{R}^d$ that can be expressed as a convex combination of each of two disjoint subsets of A. We call this point the *Radon point* of A, since its existence and the partition of A into three parts is basically Radon's theorem [Zie95].

Observe that both $\mathrm{conv}(A^-)$ and $\mathrm{conv}(A^0 \cup A^+)$ are simplices, by property (1) above, and that their affine spans are complementary: their dimensions add up to d and they intersect only in the Radon point. By an affine change of coordinates, we can make the Radon point to be the origin, and the affine subspace containing $\mathrm{conv}(A^-)$ and $\mathrm{conv}(A^0 \cup A^+)$ be complementary coordinate subspaces. In these conditions, $\mathrm{conv}(A)$ is the *direct sum* of $\mathrm{conv}(A^-)$ and $\mathrm{conv}(A^0 \cup A^+)$, where the direct sum of polytopes $P \subset \mathbb{R}^p$ and $Q \subset \mathbb{R}^q$ both containing the origin is defined as

$$P \oplus Q := \mathrm{conv}(P \times \{0\} \cup \{0\} \times Q) \subset \mathbb{R}^{p+q}.$$

Since the volume of a direct sum has the following simple formula

$$\mathrm{vol}(P \oplus Q) = \frac{\mathrm{vol}(P)\,\mathrm{vol}(Q)}{\binom{p+q}{p}},$$

we have the following result:

Lemma 3.2.3. *Let $A \subset \mathbb{R}^d$ be a set of $d+2$ points affinely spanning $\mathbb{R}^d$, and let $p = |A^-| - 1$ and $q = |A^0 \cup A^+| - 1$, so that $p + q = d$. Let $K = \operatorname{conv}(A)$. Then:*

$$\operatorname{vol}(K - K) \geq \binom{2p}{p}\binom{2q}{q} \operatorname{vol}(K).$$

Proof. By an affine transformation, let $A^- \subset \mathbb{R}^p \times \{0\}$ and $A^0 \cup A^+ \subset \{0\} \times \mathbb{R}^q$, and let $P \subset \mathbb{R}^p$ and $Q \subset \mathbb{R}^q$ be the corresponding convex hulls, which are simplices of dimensions p and q respectively. By the Rogers-Shephard inequality:

$$\operatorname{vol}(P - P) = \binom{2p}{p} \operatorname{vol}(P), \qquad \operatorname{vol}(Q - Q) = \binom{2q}{q} \operatorname{vol}(Q).$$

Now, $K = P \oplus Q$ implies

$$K - K = (P \oplus Q) - (P \oplus Q) \supseteq (P - P) \oplus (Q - Q).$$

In particular, $\operatorname{vol}(K - K)$ is at least

$$\operatorname{vol}((P - P) \oplus (Q - Q)) = \frac{\operatorname{vol}(P - P)\operatorname{vol}(Q - Q)}{\binom{p+q}{p}} = \frac{\binom{2p}{p}\binom{2q}{q}}{\binom{p+q}{p}} \operatorname{vol}(P)\operatorname{vol}(Q).$$

,

$$\operatorname{vol}(K) = \operatorname{vol}(P \oplus Q) = \frac{\operatorname{vol}(P)\operatorname{vol}(Q)}{\binom{p+q}{p}}.$$

□

Corollary 3.2.4. *Let K be the convex hull of five points affinely spanning $\mathbb{R}^3$. Then*

$$\operatorname{vol}(K - K) \geq 12 \operatorname{vol}(K).$$

Proof. Since the p and q in Lemma 3.2.3 are non-negative and add up to three, there are only two possibilities: $(p, q) \in \{(0, 3), (3, 0)\}$ or $(p, q) \in \{(1, 2), (2, 1)\}$. The lemma gives $\operatorname{vol}(K - K) \geq 20 \operatorname{vol}(K)$ and $\operatorname{vol}(K - K) \geq 12 \operatorname{vol}(K)$ respectively. □

We do not expect the factor 12 in the statement of Corollary 3.2.4 to be sharp, but it is not far from sharp; if K is a pyramid with square base then $\operatorname{vol}(K - K) = 14 \operatorname{vol}(K)$

Corollary 3.2.5. *Let K be the convex hull of five points affinely spanning $\mathbb{R}^3$, and assume it to be hollow. Let $w > 1 + 2/\sqrt{3} = 2.155$ be the width of K and let $\mu = w^{-1}$ be its first covering minimum. Then,* $\mathrm{vol}(K)$ *is bounded above by:*

$$\frac{16}{3(1-\mu)^3} = \frac{16w^3}{3(w-1)^3}, \qquad \text{if } w \geq \frac{2}{\sqrt{3}}(\sqrt{5}-1)+1 = 2.427, \text{ and}$$
$$\frac{1}{2\mu^2(1-\mu(1+2/\sqrt{3}))} = \frac{w^3}{2(w-(1+2/\sqrt{3}))}, \qquad \text{if } w \leq 2.427.$$

Proof. With the same notation as in the proof of Theorem 3.2.1, thanks to Lemma 3.2.3 we have

$$\mathrm{vol}(K) \leq \frac{1}{12}\,\mathrm{vol}(K-K) \leq \frac{2}{3}(\lambda_1\lambda_2\lambda_3)^{-1} \leq \frac{2}{3}\max\{{\lambda_1}^3, \lambda_1{\lambda_2}^2\}^{-1}.$$

In order to lower bound $\max\{{\lambda_1}^3, \lambda_1{\lambda_2}^2\}$ follow word by word the proof of Theorem 3.2.1. $\square$

3.3 Maximum volume of wide hollow 4-simplices

Here we give an upper bound on the determinant (equivalently, the volume) of any hollow 4-simplex of width at least three. Our main idea is to consider an integer projection $\pi : P \to Q \subset \mathbb{R}^3$ and transfer to P the bound for the volume of Q that we have in Corollary 3.2.5. Observe that Q is the convex hull of 5 points and it has width at least three (because any affine integer functional on Q can be lifted to P, with the same width) but it will not necessarily be hollow. Thus, some extra work is needed. A road map to the proof is as follows:

- If a projection π exists for which Q is hollow, then Q is a hollow polytope of width at least three. Such polytopes have been classified in [AKW17, AWW11]: there are only five, with maximum volume 27. It is easy to prove (looking at the five possibilities) an upper bound of 27 for the determinant of the simplex P. See details in Proposition 3.3.2.

- If such a π does not exist, then we show that that ${\lambda_1}^{-1}(P-P) \leq 42$ (see part (1) of Theorem 3.3.6, based on Corollary 3.3.4). We then have a dichotomy:

  - If Q is "close to hollow" (that is, if it contains a hollow polytope of about the same width) then we can still use Corollary 3.2.5 to get a good bound on its volume, hence on the volume of P.

- If Q is “far from hollow” (that is, if it has interior lattice points far from the boundary) then it is easy to get much better bounds on ${\lambda_1}^{-1}(P-P)$, which by Minkowski's Theorem directly give us a bound on the volume of P.

P has a hollow projection Q

We start with a general result about projections of a simplex to codimension one. Observe that if $P \subset \mathbb{R}^d$ is a (perhaps not-lattice) d-simplex and $\pi : P \to Q \subset \mathbb{R}^{d-1}$ is a projection of it then $Q = \pi(P) =\subset \mathbb{R}^{d-1}$ can be written as the convex hull of $d+1$ points, the images under π of the vertices of P. In this situation we call *Radon point of* Q the Radon point of $\pi(\mathrm{vert}(P))$, introduced in Section 3.2.

Lemma 3.3.1. *Let $\pi : P \to Q \subset \mathbb{R}^{d-1}$ be an integer projection of a d-simplex P. Let $x \in Q$ be the Radon point of Q and let $s = \pi^{-1}(x) \subset P$ be the fiber of x in P (a segment). Then:*

1. $\mathrm{Vol}(P) = \mathrm{Vol}(Q) \times \mathrm{length}(s)$, *where* $\mathrm{length}(s)$ *is the lattice length of* s.
2. *s maximizes the lattice length among all segments in P in the projection direction.*

Proof. Observe that every facet F of Q not containing x is a simplex (because its vertices are affinely independent), and that π is a bijection from $\pi^{-1}(F)$ to F. Let us consider Q triangulated by coning x to each of those facets. Let $S = \mathrm{conv}(F \cup \{x\})$ be one of the maximal simplices in this triangulation. Then, $\pi^{-1}(S)$ is also a simplex, with one vertex projecting to each vertex of F and the segment s projecting to x. This implies part (2) of the statement, and also the following analogue of part (1) for Euclidean (as opposed to normalized) volumes:

$$\mathrm{vol}(P) = \mathrm{vol}(Q) \times \mathrm{length}(s)/d.$$

From this, $\mathrm{Vol}(P) = d!\,\mathrm{vol}(P)$ and $\mathrm{Vol}(Q) = (d-1)!\,\mathrm{vol}(Q)$ gives part (1). □

Proposition 3.3.2. *If an empty 4-simplex P of width at least three can be projected to a hollow lattice 3-polytope Q, then the normalized volume of P is at most 27.*

Proof. Q is one of the five hollow lattice 3-polytopes of width at least three, classified in [AKW17].

Their normalized volumes are 27, 25, 27, 27, and 27, respectively. Q_5 cannot be the projection of P, since it has six vertices. For the other four, we claim that the Radon point x of $\pi(\mathrm{vert}(P))$ is always a lattice point:

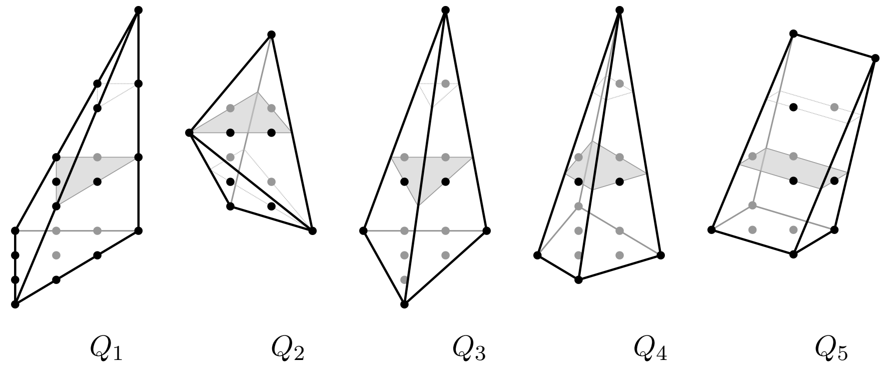


Figure 3.2: The five hollow 3-polytopes of width three. Figure taken from [AKW17]

- In the three tetrahedra Q_1, Q_2 and Q_3 this is automatic: four vertices of P project to the four vertices of the tetrahedron Q and the fifth vertex projects to a lattice point in Q which necessarily equals the Radon point.
- In Q_4, a pyramid over a quadrilateral, the Radon point is the intersection of the two diagonals of the quadrilateral, which happens to be a lattice point.

Now, by Lemma 3.3.1, $\mathrm{Vol}(P) = \mathrm{Vol}(Q) \times \mathrm{length}(s)$, where $\mathrm{length}(s)$ is the lattice length of the fiber of the Radon point. Since the Radon point is a lattice point and since P is hollow, the length of this fiber is at most 1. On the other hand, $\mathrm{Vol}(Q)$ is at most 27. $\square$

P has no hollow projection

Our first tool is Corollary 3.3.4 which guarantees that for every non-hollow lattice simplex T and facet F of T there is a lattice point in the interior of T not too close to F. The lower bound obtained is expressed in terms of the Sylvester sequence $(s_i)_{i\in\mathbb{N}}$. In particular, for dimension 4 our bound is $1/(s_4 - 1) = 1/42$.

Lemma 3.3.3. *Let T be a simplex and let $S \subset T$ be a finite set of points including the vertices of T and at least one point in the interior of T.*

Then, for each vertex a of T there is a subsimplex $T' \subset T$ with exactly one point of S in its interior and with $a \in \mathrm{vert}(T') \subset S$.

Here T and T' are not assumed to be full-dimensional, or to have the same dimension. In particular, by "interior" we mean the relative interior. They are also not assumed to be lattice simplices unless S is a set of lattice points (which is the case of interest to us).

Proof. If T has a unique point of S in its interior then there is nothing to prove, simply take $T' = T$ for every a. If T has more than one such point we argue by induction on the number of them.

Let $y \in \text{int}(T) \cap S$ be an interior point minimizing the barycentric coordinate with respect to a and let $\mathcal{T}$ be the stellar triangulation of T from y. (The maximal simplices in $\mathcal{T}$ are $\text{conv}(F \cup \{y\})$ for the facets F of T). Let $y' \in S$ be another interior point in T and let T' be the minimal simplex in $\mathcal{T}$ containing y'. Then:

- By minimality of T', y' is in the interior of T'. In particular, T' is not hollow.
- T' uses a as a vertex, since T' contains the interior point y and all simplices of $\mathcal{T}$ that do not contain a are contained in the boundary of T.
- By construction, $\text{int}(T') \subset \text{int}(T)$ (remark: T' may be not full-dimensional; by $\text{int}(T')$ we mean the relative interior). Since y is an interior point in T but not in T', T' has less interior points than T and we can apply the induction hypothesis to it.

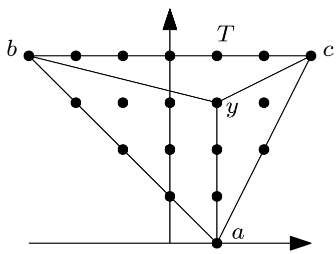


Figure 3.3: Illustration of Lema 3.3.3. Once the point y is chosen, the other six points of S (in the figure, a lattice point set) in the interior of T are valid choices for y'. Depending of the choice of y' we get a different T': $T' = \{a, b, y\}$ if y' is one of $(-1, 3)$, $(0, 3)$ or $(0, 2)$; $T' = \{a, y\}$ if y' is one of $(1, 1)$ or $(1, 2)$; and $T' = \{a, c, y\}$ if y' is $(2, 3)$. Choice of y guarantees that no interior point of S lies in $\{y, b, c\}$.

□

Corollary 3.3.4. *Let T be a non-hollow lattice d-simplex and let a be a vertex of it. Then, there is an interior lattice point in T whose barycentric coordinate with respect to a is at least $1/(s_{d+1} - 1)$.*

Proof. Let $S = T \cap \mathbb{Z}^d$ and let T' be as in Lemma 3.3.3. Observe that $\dim(T') \leq \dim(T)$ so that $s_{d'+1} \leq s_{d+1}$. Since T' has a unique interior lattice point, the statement is the case $i = d + 1$ of [AKN15, Theorem 2.1]. □

Our next result makes precise what we mean by "far from hollow" in the description at the beginning of this section, and how to use that property to upper bound the rational diameter of P (and hence its volume, via Lemma 3.3.1 and Corollary 3.2.5).

Lemma 3.3.5. *Let $P \subset \mathbb{R}^d$ be a hollow convex body. Consider an integer projection $\pi : P \to Q \subset \mathbb{R}^{d-1}$ of P. Let $x \in Q$ be an arbitrary point and let $s_x = \pi^{-1}(x) \subset P$ be its fiber.*

Then, $\operatorname{length}(s_x)^{-1} \geq 1 - r$*, where* $\operatorname{length}(s)$ *is the lattice length of s.*

Proof. Observe there if Q is hollow then $r \geq 1$, so the statement is trivial. Thus, without loss of generality we assume Q is not hollow. Also, if x is a lattice point in the interior of Q then $r = 0$ and the fact that P is hollow implies $\operatorname{length}(s_x) \geq 1$, so the statement holds. Thus, we assume that Q has at least an interior lattice point that is not x.

Let $y \in Q \setminus \{x\}$ be an interior lattice point of Q closest to x with respect to the seminorm induced by Q with center at x. That is, for each interior lattice point p in Q call $||p||_{Q,x}$ the smallest dilation factor r_p such that $p \in x + r_p(Q - x)$, and let y be a lattice point minimizing that quantity. Observe that $r = ||y||_{Q,x}$. (Remark: we do not assume x to be in the interior of Q. If Q lies in the boundary the seminorm $||p||_{Q,x}$ may be infinite for points outside Q, but it is always finite and smaller than one for points in the interior).

Let $s_y = \pi^{-1}(y) \subset P$ be the fiber of y. The length of s_y must be at most 1, because P is hollow. Consider the ray from x through y and let z be the point where it hits the boundary of Q. We have:

$$\operatorname{length}(s_x) \leq \frac{\operatorname{length}(s_x)}{\operatorname{length}(s_y)} \leq \frac{|xz|}{|yz|},$$

where the second inequality follows from convexity of P. Then:

$$\operatorname{length}(s_x)^{-1} \geq \frac{|yz|}{|xz|} = \frac{|xz| - |xy|}{|xz|} = 1 - \frac{|xy|}{|xz|} = 1 - ||y||_{Q,x} = 1 - r.$$

□

With this we can prove the main result in this section. In it, we consider the projection $\pi : P \to Q$ along the direction where $\lambda_1(P - P)$ is achieved. Equivalently, this is the direction of the longest (with respect to the lattice) rational segment contained in P. (Recall that this length is called the *rational diameter* of P, and it equals $\lambda_1(P - P)^{-1}$). In particular, if we let the point x of Lemma 3.3.5 be the point whose fiber achieves the rational diameter, we have $\lambda_1(P - P) \geq 1 - r$. Moreover, Lemma 3.3.1 tells us that x is the Radon point of Q and that $\operatorname{Vol}(P) = \operatorname{Vol}(Q)/\lambda_1(P - P)$.

Theorem 3.3.6. *Let P be a hollow 4-simplex of width at least three and that does not project to a hollow 3-polytope. Then:*

1. $\lambda_1(P-P) \geq 1/42$.
2. $\operatorname{Vol}(P) \leq 5058$.

Proof. Let Q be the projection of P along the direction where $\lambda_1(P-P)$ is attained. We know Q is not hollow, and has width at least three. For the rest of the proof we denote $\lambda := \lambda_1(P-P)$.

Suppose first that $\lambda \geq 0.19$, in which case part (1) obviously holds. For part (2) we can simply bound the volume of P in the manner of Equation (3.3), except that for a d-simplex P we can use the Rogers-Shephard equality, see Eq. (3.2):

$$\operatorname{vol}(P-P) = \binom{2d}{d} \operatorname{vol}(P).$$

Thus:

$$\operatorname{Vol}(P) = 24 \operatorname{vol}(P) = \frac{24}{70} \operatorname{vol}(P-P) \leq \frac{24 \cdot 16}{70\lambda^4} \leq \frac{24 \cdot 16}{70 \cdot 0.19^4} \leq 4209.38.$$

So, *for the rest of the proof we assume* $\lambda < 0.19$.

Let x be the Radon point of Q, that is, the image of the segment where λ is achieved. Let r be as in Lemma 3.3.5, so that $r \geq 1-\lambda \geq 0.81$ and $Q_r := x + r(Q-x)$ is hollow. Together with Lemma 3.3.1 this gives:

$$\operatorname{Vol}(Q) = \frac{\operatorname{Vol}(Q_r)}{r^3} \leq \frac{6 \operatorname{vol}(Q_r)}{(1-\lambda)^3}. \tag{3.6}$$

Observe that the width of Q_r is r times the width of Q and, in particular, $w(Q_r) \geq 3 \cdot 0.81 = 2.43$. Since Q_r is the convex hull of five points (the projection of the five vertices of P), Corollary 3.2.5 gives

$$\operatorname{vol}(Q_r) \leq \frac{16}{3(1-\mu_1(Q_r))^3} \leq \frac{16}{3\left(1-\frac{1}{3(1-\lambda)}\right)^3} = \frac{2^4 3^2 (1-\lambda)^3}{(2-3\lambda)^3}, \tag{3.7}$$

where the inequality in the middle follows from $\mu_1(Q) \leq 1/3$ (Q has width at least three) and $\mu_1(Q_r) = \mu_1(Q)/r \leq \mu_1(Q)/(1-\lambda)$.

Consider Q triangulated centrally from the Radon point x. That is, for each facet F of Q not containing x we consider the tetrahedron $\operatorname{conv}(F \cup \{x\})$. (Observe that all facets of Q not containing x are triangles, since the only affine dependence among the vertices of Q is precisely the one that gives the Radon point). We call such tetrahedra the *Radon tetrahedra* in Q.

Let $v_1, \ldots, v_5$ be the five vertices of Q. (Remark: if Q has only four vertices then the Radon point is the projection of the fifth vertex of P. But the Radon point being integer implies $\lambda \geq 1$). For each $i \in \{1, \ldots, 5\}$ denote $Q_i := \mathrm{conv}(\{v_1, \ldots, v_5\} \setminus \{v_i\})$ the lattice tetrahedron contained in Q and not using vertex v_i.

There are two possibilities:

- Suppose first that one of the Q_i's has the following property: only one of the Radon tetrahedra contained in that Q_i contains interior lattice points of Q. Let $T := \mathrm{conv}(F \cup \{x\})$ be that Radon tetrahedron. Let $T' := \mathrm{conv}(F' \cup \{x\})$ be a minimal face of T that contains some interior lattice point y of Q. Then: Minimality of F' implies that y is in the interior of T', by minimality.

  Let $v_j \notin T$ be the vertex of Q_i not in F and let $Q'_i = \mathrm{conv}(F' \cup \{v_j\})$. Observe that $\mathrm{conv}(F' \cup \{x\}) \subset \mathrm{conv}(F' \cup \{v_j\})$ (because the Radon point of Q is contained in every simplex spanned by vertices of Q and intersecting the interior of Q).

  By Corollary 3.3.4 the non-hollow lattice simplex $\mathrm{conv}(F' \cup \{v_j\})$ contains an interior point z whose barycentric coordinate with respect to the facet F' is at least $1/42$. This is the same as the barycentric coordinate of z in T with respect to the facet F. Now, by uniqueness of $\mathrm{conv}(F \cup \{x\})$ as a Radon tetrahedron in T, y is also contained in $\mathrm{conv}(F \cup \{x\})$. Moreover, the barycentric coordinate of y in $\mathrm{conv}(F \cup \{x\})$, which is a lower bound for λ by the same arguments as in Lemma 3.3.5, is greater than in $\mathrm{conv}(F \cup \{a\})$. Thus, $\lambda \geq 1/42$.

- Suppose now that every Q_i contains either zero or at least two Radon tetrahedra with interior lattice points of Q. An easy case study shows that then at least four Radon tetrahedra contain interior lattice points of Q. (The cases are that Q is a triangular bipyramid with six Radon tetrahedra or a quadrangular pyramid with four Radon tetrahedra). Let $\mathrm{conv}(F_i \cup \{x\})$ be such Radon tetrahedra, and let $y_i \in \mathrm{conv}(F_i \cup \{x\})$ be an interior lattice point of Q, for $i \in \{1, 2, 3, 4\}$. (Some of the y_i's may coincide, since we do not assume them to be interior in $\mathrm{conv}(F_i \cup \{x\})$, only in Q). Then, $1/\lambda$ is smaller than $\frac{\mathrm{Vol}(\mathrm{conv}(F_i \cup \{x\}))}{\mathrm{Vol}(\mathrm{conv}(F_i \cup \{y_i\}))}$, for each i, and this is smaller than $\mathrm{Vol}(\mathrm{conv}(F_i \cup \{x\}))$ since $\mathrm{conv}(F_i \cup \{y_i\})$ is a lattice tetrahedron. That is:

  $$\mathrm{Vol}(Q) \geq \sum_{i=1}^{4} \mathrm{Vol}(\mathrm{conv}(F_i \cup \{x\})) \geq \frac{4}{\lambda}.$$

  This together with equations (3.6) and (3.7) gives

  $$\frac{4}{\lambda} \leq \mathrm{Vol}(Q) \leq \frac{2^5 3^3}{(2 - 3\lambda)^3},$$

which implies $(\frac{2}{3} - \lambda)^3 \leq 8\lambda$ or, equivalently, $\lambda \geq 0.03196 > 1/42$.

So, in both cases we have $\lambda \geq 1/42$, which finishes the proof of part (1).
For part (2), Equations (3.6) and (3.7) give

$$\mathrm{Vol}(P) = \frac{\mathrm{Vol}(Q)}{\lambda} \leq \frac{2^5 3^3}{(2-3\lambda)^3\lambda}. \tag{3.8}$$

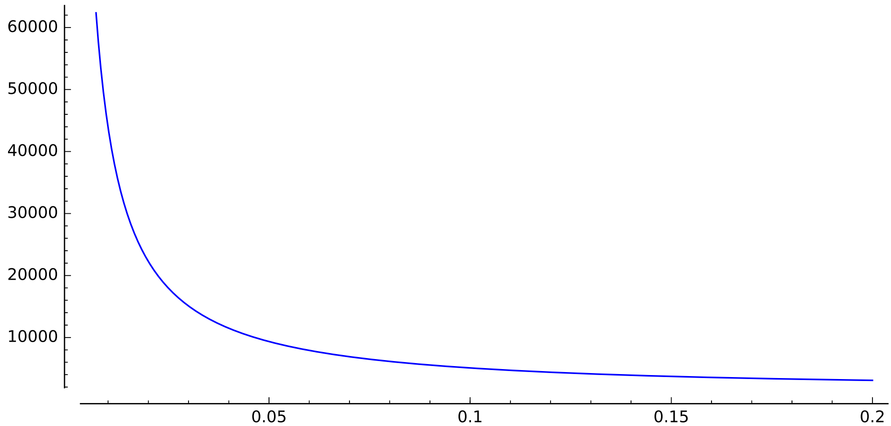


Figure 3.4: Plot of the upper bound in Eq. (3.8) for $\lambda \in [0.02, 0.20]$.

Figure 3.4 plots this function in the relevant range $\lambda \in [1/42, 0.19]$. Although the function is slightly increasing after its local minimum at $\lambda = 1/6$, its maximum in the interval is clearly at $\lambda = 1/42$, where it takes the value

$$\mathrm{Vol}(P) \leq \frac{2^5\, 3^3\, 42}{(2-1/14)^3} = \frac{14^3\, 2^5\, 3^3\, 42}{27^3} = \frac{2^9\, 7^4}{3^5} = 5058.897.$$

We can round this down to 5058 since $\mathrm{Vol}(P) \in \mathbb{Z}$. $\square$

3.4 Upper bound for the volume of hollow 4-simplices of width 2

Theorem 3.4.1. *Let P be a hollow 4-simplex which does not project to a hollow 3-polytope. Then,* $\mathrm{Vol}(P) \leq 5184$.

For simplices of width at least three this statement was stated in Theorem 3.3.6. Since the width of P cannot be one (for that would imply P to project to a hollow 1-polytope) in the rest of the section we assume that P is an empty 4-simplex of width two. Thus, without loss of generality, we take $P \subset \mathbb{R}^3 \times [-1, 1]$.

Let $Q := P \cap \{x_4 = 0\}$ be the middle 3-dimensional slice with respect to the last coordinate. If we get a good bound for the volume of Q then we can transfer it to P via the following lemma:

Lemma 3.4.2. *Let $K \subset \mathbb{R}^d$ be a convex body with supporting hyperplanes $\mathbb{R}^{d-1} \times \{-a\}$ and $\mathbb{R}^{d-1} \times \{b\}$, with $0 < a \leq b$. Let $K_0 := P \cap \{x_d = 0\}$. Then,*

$$\mathrm{Vol}(K) \leq a \left(\frac{a+b}{a} \right)^d \mathrm{Vol}(K_0).$$

A more general version of Lemma 3.4.2 has been published in [G20].

Proof. The proof is based on applying Schwarz symmetrization (see, e.g., [Gru93, Sect. 9.3]) to our convex body K.

For each $t \in [-a, b]$ let $K_t := K \cap \{x_d = t\}$, and let $B_t \subset \mathbb{R}^{d-1}$ be the Euclidean ball centered at the origin $O \in \mathbb{R}^{d-1}$ and with the same volume as K_t. The *Schwarz symmetrization* of K is defined to be

$$K^S := \cup_{t \in [-a,b]} B_t \times \{t\}.$$

Then, $K^S \subset \mathbb{R}^d \times [-a, b]$ is a convex body (as proved by Schwarz), it has the same volume as K, and it is symmetric around the line $\{O\} \times \mathbb{R}$. In particular, K^S is contained in a truncated cone C of the form

$$C = \mathrm{conv}(C_{-a} \times \{-a\} \cup C_b \times \{b\}),$$

where C_{-a} and C_b are two Euclidean balls with the property that the slice at $t = 0$ of C coincides with that of K^S. (To prove this, consider a supporting hyperplane of K^S at a boundary point with $t = 0$ and rotate it around the line $O \times \mathbb{R}$).

Let r be the radius of K_0, and let $r + \lambda t$ be the radius of $C \cap \{x_d = t\}$. Then,

$$\mathrm{Vol}(K) \leq \mathrm{Vol}(C) = d \int_{-a}^{b} \kappa_{d-1} (r + \lambda t)^{d-1} \mathrm{d}t = \frac{1}{\lambda} \left[(r + \lambda b)^d - (r - \lambda a)^d \right] \kappa_{d-1},$$

where κ_{d-1} denotes the normalized volume of the $(d-1)$-dimensional unit ball.

The slope λ must lie between $-r/b$ and r/a (the values for which the truncated cone is actually a cone). Within this range the maximum of the right-hand-side is achieved for $\lambda = r/a$, where we have

$$\mathrm{Vol}(K) = \frac{a}{r} (r + rb/a)^d \kappa_{d-1} = a r^{d-1} \kappa_{d-1} \left(\frac{a+b}{a} \right)^d = a \, \mathrm{Vol}(K_0) \left(\frac{a+b}{a} \right)^d.$$

$\square$

Corollary 3.4.3.

$$\mathrm{Vol}(P) \leq 16\,\mathrm{Vol}(Q).$$

To bound the volume of Q we now observe that Q is a hollow 3-polytope with half-integer vertices; in particular, its width is half-integer. We also know that Q does not project to a hollow 2-polytope (otherwise P would project to a hollow 3-polytope). We distinguish three cases:

(I) $\mathrm{width}(Q) \geq 5/2$, then by Theorem 3.3.6 we have the following bound.

$$\mathrm{Vol}(Q) \leq \frac{3! \cdot 8 \cdot w^3}{(w-1)^3} \leq \frac{6 \cdot 8 \cdot (5/2)^3}{(3/2)^3} = \frac{2000}{9} = 222.22.$$

(Remark: in fact the bound could be multiplied by $2/3$ using Corollary 3.2.5, since Q has at most five vertices). Via Lemma 3.4.2 we get that

$$\mathrm{Vol}(P) \leq 16\,\mathrm{Vol}(Q) \leq \frac{16000}{9} = 3555.55.$$

(II) If $\mathrm{width}(Q) \leq 3/2$, or $\mathrm{width}(Q) = 2$ with respect to a functional whose minimum and maximum are integer, then we assume without loss of generality that $Q \subset \mathbb{R}^2 \times [-1,1] \times \{0\}$. In this case we can apply to the slice $R := Q \cap \{x_3 = 0\}$ the same ideas that we applied to $P \cap \{x_4 = 0\}$, since R is hollow and does not project to a hollow segment.

(III) If Q has width two, but with respect only to functionals whose minimum and maximum are half-integer, then we can assume $Q \subset \mathbb{R}^2 \times [-1/2, 3/2] \times \{0\}$. There are two integer slices $R := Q \cap \{x_3 = 0\}$ and $R' := Q \cap \{x_3 = 1\}$. We have two subcases:

  (III.a) If one of R or R' (say R) does not project to a hollow segment, we do the same as in case (II). See details below, in particular Corollary 3.4.7.

  (III.b) If both R and R' project to hollow segments, then they are contained in respective lattice bands of width one. These lattice bands have to be of different direction, since otherwise the projection of Q along that direction is hollow.

In what follows we give details on Cases (II) and (III), obtaining bounds of 324 and 192 for the volume of Q (see Corollary 3.4.7. and Lemma 3.4.4). This finishes the proof of Theroem 3.3.6, via Corollary 3.4.3. In all cases we will resort to 3- or 2-dimensional cases of Lemma 3.4.2.

The easiest case is Case (III.b):

Lemma 3.4.4. *If Q is in case (III.b) then* $\mathrm{Vol}(Q) \leq 192$*, hence* $\mathrm{Vol}(P) \leq 3072$.

Proof. Let us recall our hypotheses: $Q \subset \mathbb{R}^3$ is a 3-dimensional hollow polytope with supporting hyperplanes $\{x_3 = -1/2\}$ and $\{x_3 = 3/2\}$, and the slices $R := Q \cap \{x_3 = 0\}$ and $R' = Q \cap \{x_3 = 1\}$ both have width one and project to hollow segments, but with respect to different projection directions.

Applying Lemma 3.4.2 to $R \subset Q$ with $a = 1/2$ and $b = 3/2$ we get that

$$\mathrm{Vol}(Q) \leq \frac{1}{2}\left(\frac{2}{1/2}\right)^3 \mathrm{Vol}(R) = 32\,\mathrm{Vol}(R).$$

Now, R has width one with respect to a certain direction, and width at most three with respect to a second one. (For the latter, observe that R is contained in a band of width three along the direction of the band of width one containing R'). This implies $\mathrm{Vol}(R) \leq 6$, from which we deduce $\mathrm{Vol}(Q) \leq 192$ and $\mathrm{Vol}(P) \leq 192 \cdot 16 = 3072$. □

For cases (II) and (III.a) we need to use that the coordinates of vertices of Q are rational with small denominators:

Lemma 3.4.5. *In the conditions of cases (II) or (III), all vertices of R and R' have coordinates in* $\frac{1}{6}\mathbb{Z}^2 \cup \frac{1}{8}\mathbb{Z}^2$.

Proof. In case (III) the situations in R and R' are symmetric to one another, so for the rest of the proof we only look at $R = Q \cap \{x_3 = 0\}$. Since Q is the middle slice of a lattice polytope P of width two, Q is a half-integer 3-polytope. That is, the vertices of Q have integer or half-integer coordinates. Let now p be a vertex of R. Either p is also a vertex of Q (in which case it has half-integer coordinates) or p is the intersection of an edge uv of Q with the plane $x_3 = 0$. Let $\lambda \in (0, 1)$ be the coefficient such that

$$p = \lambda u + (1 - \lambda)v.$$

Assume without loss of generality that u is in $\{x_3 < 0\}$ and v in $\{x_3 > 0\}$. In case (II) we have that u has its third coordinate in $\{-1, -\frac{1}{2}\}$ and v in $\{\frac{1}{2}, 1\}$. This implies that $\lambda \in \{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\}$. In case (III) we have that u has its third coordinate equal to $-\frac{1}{2}$ and v in $\{\frac{1}{2}, 1, \frac{3}{2}\}$, which implies $\lambda \in \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}\}$. Since $u, v \in \frac{1}{2}\mathbb{Z}^2$, in all cases we get $p \in \frac{1}{6}\mathbb{Z}^2 \cup \frac{1}{8}\mathbb{Z}^2$.

□

Lemma 3.4.6. *Let R be a hollow polygon with vertices in $\bigcup_{i \leq k} \frac{1}{i}\mathbb{Z}^2$ for an integer $k \geq 1$ and such that R does not project to a hollow segment. Then,*

$$\mathrm{Vol}(R) \leq \frac{(k+1)^2}{k}.$$

Proof. Averkow and Wagner [AW12, Theorem 2.2] have given upper bounds for the maximum area of a hollow polygon in terms of its width w, depending on whether w lies in $[0, 1]$, $[1, 2]$ or $[2, 1 + 2/\sqrt{3}]$. (That $1 + 2/\sqrt{3} \sim 2.15$ is the maximum possible width was previously shown by Hurkens [Hur90]). We prove the statement separately in the three cases:

- If $w \in (0, 1]$ then R is contained in a strip of width one, say $R \subset \mathbb{R} \times [\alpha - 1, \alpha]$, with $\alpha \in (0, 1)$ (α cannot be an integer, because R does not project to a hollow segment). We can assume without loss of generality that $\alpha \leq 1/2$ and then, since $R \cap \{x_2 = 0\}$ has length at most 1, Lemma 3.4.2 implies

$$\mathrm{Vol}(R) \leq \alpha \left(\frac{1}{\alpha}\right)^2 = \frac{1}{\alpha} \leq k,$$

  where the last inequality comes from the fact that $\alpha \in \bigcup_{i \leq k} \frac{1}{i}\mathbb{Z}^2$.

- If $w \in (1, 2]$ then the bound from [AW12, Theorem 2.2] is

$$\mathrm{Vol}(R) \leq \frac{w^2}{w - 1}. \tag{3.9}$$

  (Observe we have multiplied the formula in [AW12] by two, since our volume is normalized to the unimodular triangle and theirs is not). Since $w > 1$ must be in $\bigcup_{i \leq k} \frac{1}{i}\mathbb{Z}^2$, we have $w \geq (k+1)/k$. Since the function $\frac{w^2}{w-1}$ is decreasing for $w \leq 2$, we get

$$\mathrm{Vol}(R) \leq \frac{w^2}{w - 1} \leq \frac{(k + 1)^2/k^2}{1/k} = \frac{(k + 1)^2}{k}.$$

- If $w \in [2, 1 + 2/\sqrt{3}]$ then the bound in [AW12] implies $\mathrm{Vol}(R) \leq 4$. On the other hand $k \geq 2$ (no hollow lattice polygon has width larger than two), so indeed

$$\mathrm{Vol}(R) \leq 4 \leq \frac{(k + 1)^2}{k}.$$

□

We can now address cases (II) and (III.a) together:

Corollary 3.4.7. *If Q is in one of cases (II) or (III.a) then* $\mathrm{Vol}(Q) \leq 324$*, hence* $\mathrm{Vol}(P) \leq 5184$.

Proof. By Lemma 3.4.6 we have $\mathrm{Vol}(R) \leq 81/8$. Moreover, we can apply Lemma 3.4.2 to Q and its slice R, with $(a, b) \in \{(1/2, 1/2), (1/2, 1), (1/2, 3/2), (1, 1)\}$. These four cases give, respectively,

$$a\left(\frac{a+b}{a}\right)^3 \in \left\{4, \frac{27}{2}, 32, 8\right\}.$$

Hence,

$$\mathrm{Vol}(Q) \leq 32\frac{81}{8} = 324, \quad \mathrm{Vol}(P) \leq 16\,\mathrm{Vol}(Q) \leq 5184.$$

$\square$

Chapter 4
Enumeration of empty 4-simplices

In this section we describe the algorithmic methods used for the exhaustive enumeration of all empty 4-simplices up to volume 7600. Although eventually we got upper bounds of $V = 5500$, the computations were done up to volume 7600 as our previous bounds were close to that number.

Having all empty 4-simplices up to that volume bound allows us to finish the classification of the sporadic simplices (case $k = 4$) thanks to Theorem 3.4.1.

A pseudo-code of the main algorithms that enumerate the simplices and verify if they are empty is described in this section. We also shown how we obtain empty simplices of a non-prime volume $V = ab$ by using 2 empty simplices of volume a and b. This method increase the speed of the computations (in most of the cases) by using the simplices calculated before with lower volumes. The speed of this second algorithm depends heavily on the prime factorization for the volume V (Figure 4.1).

4.1 Strategy of the enumeration/quintuples

Let's remember that in section 2.1.1, in particular, in propositions 2.1.11 and 2.2.2 we define what is a quintuple for an empty 4-simplex and how to obtain the coordinates of our polytopes vertices from the quintuple. The algorithms stated in this section express the simplices in terms of their quintuples.

4.2 Algorithms

Let P be a lattice simplex $P \in \mathbb{R}^d$, and let $\Lambda(P)$ be the lattice generated by vertices of P. We assume with no loss of generality that the origin is a vertex of P, so that $\Lambda(P)$ is a linear lattice and $G(P) := \mathbb{Z}^d/\Lambda(P)$ is a finite group of order equal to the determinant of P. One way to store P is via generators of $G(P)$ as a subgroup of $\mathbb{R}^d/\Lambda(P)$, with barycentric coordinates used in $\mathbb{R}^d$. Let us be more precise:

- The barycentric coordinates of a point $x \in \mathbb{R}^d$ with respect to the simplex P are the vector $(x_0, \ldots, x_d)$ of coefficients of the unique expression of x as an affine combination of the vertices of P (the vertices of P are assumed given in a particular order). They add up to one; conversely, any $(d+1)$-vector with real coefficients and sum of coordinates equal to one represents a unique point in $\mathbb{R}^d$ in barycentric coordinates. If x is a lattice point and P a lattice simplex of determinant D, then all the x_i's lie in $\frac{1}{D}\mathbb{Z}$.

- Looking at x in the quotient $\mathbb{R}^d/\Lambda(P)$ is equivalent to looking at the x_i's modulo $\mathbb{Z}$; that is, looking only at the fractional part of them. In particular, every lattice point $u \in \mathbb{Z}^d$, considered as an element of the quotient $\mathbb{Z}^d/\Lambda \subset \mathbb{R}^d/\Lambda$, can be represented as a vector $(u_0, \ldots, u_d) \in (\mathbb{Z}_D)^{d+1}$ with sum of coefficients equal to 0 modulo D.

- In this manner, to every lattice simplex P of determinant D we associate a subgroup $G(P)$ of order D of the group

$$\mathbb{T}^d_D := \{(u_0, \ldots, u_d) \in \mathbb{Z}_D^{d+1} : \sum u_i = 0 \pmod{D}\}.$$

We call $\mathbb{T}^d_D$ the discrete d-torus of order D, since it is isomorphic to $(\mathbb{Z}/D\mathbb{Z})^d$. In this setting we have:

Lemma 4.2.1. *Let G_1 and G_2 be two subgroups of order D of $\mathbb{T}^d_D$. Then, G_1 and G_2 represent equivalent simplices of determinant D if, and only if, they are the same subgroup modulo permutation of coordinates.*

Proof. The "if" part is obvious. For the "only if" observe that a unimodular equivalence $f : P_1 \to P_2$ between two lattice simplices preserves barycentric coordinates, modulo the permutation of vertices induced by f. That is, if $(x_0, \ldots, x_d)$ are the barycentric coordinates of a point x with respect to P_1, then the barycentric coordinates of $f(x)$ with respect to P_2 are a permutation of them: the i-th barycentric coordinate of $f(x)$ with respect to P_2 equals x_j, where j and i are such that the f maps the j-th vertex of P_1 to the i-th vertex of P_2. □

This formalism is specially useful if P is a cyclic simplex, that is, if $G(P)$ is a cyclic group. In this case we represent $G(P)$ by giving a generator of it, as we did with quintuples in chapter 2.Observe that this includes all empty 4-simplices, since they are all cyclic (Theorem 1 in Barile et al. [BBBK11]). Hence, we introduce the following definition:

Definition 4.2.2. Let P be a cyclic lattice 4-simplex of determinant D and let $u \in \mathbb{T}^4_D$. We say that the quintuple u generates P if the barycentric coordinates of every element in $\mathbb{Z}^4/\Lambda(P)$ with respect to P are multiples of u (modulo D). That is, if the point of $\mathbb{R}^4/\Lambda(P)$ with barycentric coordinates $\frac{1}{D}u$ is a generator for the cyclic group $\mathbb{Z}^4/\Lambda(P)$.

Observe the the entries of $\frac{1}{D}u$ may not add up to 1, but they add up to an integer. This is what they need to satisfy in order to represent a point of $\mathbb{R}^4/\Lambda(P)$, since the quotient by $\Lambda(P)$ makes barycentric coordinates be defined only modulo the integers.

Summing up: every primitive element $u \in \mathbb{T}_D^4$ is the quintuple of a cyclic lattice simplex of determinant D. Moreover:

- Two quintuples $u, v \in \mathbb{T}_D^4$ generate equivalent simplices if, and only if, one is obtained from the other one by permutation of entries and/or multiplication by a scalar coprime with D.

- The width of P equals the minimum $k \in \mathbb{N}$ such that there are $\lambda_0, \dots \lambda_4 \in \{0, 1, \dots, k\}$ (not all zero) with $\sum_i \lambda_i u_i = 0 \pmod{D}$.

- The element ku of $\mathbb{T}_D$ represents a lattice point in P if, and only if, when writing it with all entries in $\{0, \dots, D-1\}$ the sum of entries equals D. (Indeed, this means that the lattice point of $\mathbb{R}^d$ whose barycentric coordinates are $\frac{1}{D}ku$ is a convex combination of the vertices of P. In order to check whether P is empty one can check that this does not happen for any $k = 1, 2, \dots, D-1$.

Remark 4.2.3. *As a remark we would like to remember the result by Sebö [Seb99] that says: It is NP-complete to decide whether the width of* $\operatorname{conv}(e_1, \dots, e_n, v)$ *is at most 1.*

4.2.1 Algorithm 1

This algorithm enumerates all empty 4-simplices of given volume V. In order to obtain all empty 4-simplices in this way, V must not have more than 5 different prime factors. This condition guarantees that our simplices have at least one unimodular facet, since two different facets must have coprime volumes. We are going to fix that unimodular facet to be the one generated by $\{v_1, v_2, v_3, v_4\}$.

Observe that, a posteriori, the classification we obtain tell us that all empty 4-simplices have this unimodular facet but we did not know this fact a priori.

The first part is starting a loop that create all 4-tuples (v_0, v_1, v_2, v_3) with $2 \leq v_0 \leq v_1 \leq v_2 \leq v_3 < D$ and satisfying the equation $\operatorname{Vol}(\sigma(v)) = V$, which can be restated as $v_0 + v_1 + v_2 + v_3 \pmod{V} \equiv 1$.

In the second part of Algorithm 1, we check emptiness for each quintuple, calculate the width and compute the orbits of this simplex.

Algorithm 1: Enumeration of the 4-tuples (v_0, v_1, v_2, v_3) with first pruning.

```
for v0 in [i for i in range(2,D) if (inverses[i])%D>=i]:
u0=inverses[v0]
g0=gcd(D,v0) # Compute upper limits for v1 and v2 for given v0.
v1max=int((2*D+1-v0)/3);       v2max=D-v0
for v1 in range(v0,v1max+1):
g1=gcd(D,v1) u1=inverses[v1]
mymin1=min((u1)%D,(-u1*v0)%D,(-v1*u0)%D)
if mymin1 < v0:
continue
for v2 in [i for i in range(v1,v2max+1) if (1-v0-v1-i)%D>=i]:
v3=(1-v0-v1-v2)%D # Fixing (v0+v1+v2+v3)%D=1
u2=inverses[v2]       u3=inverses[v3]
flag=0
if v3 < v2:
flag=1       # The simplex is not the smallest representative in its class.
continue
```

Check that the closest point to each facet are not in the simplex

```
closest_inner_points=[D-1,u2,u1,u0]
if flag==0:
for i in closest_inner_points:
if (v0*i)%D +(v1*i)%D +(v2*i)%D +(v3*i)%D <D:
flag=2 # The simplex is not empty break
```

Do a partial check that the representative is lexicographically smallest.

```
if flag==0:
mymin2=min((u2)%D, (u3)%D, (-u2*v0)%D, (-v2*u0)%D, (-u2*v1)%D,
(-v2*u1)%D, (-u3*v0)%D, (-v3*u0)%D, (-u3*v1)%D, (-v3*u1)%D, (-
u3*v2)%D, (-u2*v3)%D) if mymin2 < v0:
flag=1       # The simplex is not the smallest representative in its class
continue
```

Discarding non-reduced simplices

```
if flag==0:
if mymin1==v0 or mymin2==v0:
for j in closest_inner_points:
repr=sorted([(-v0*j)%D,(-v1*j)%D,(-v2*j)%D,(-v3*j)%D,j%D])
orbit=sorted([[v0,v1,v2,v3,D-1],repr])
if orbit[0]!=[v0,v1,v2,v3,D-1]:
flag=1  The simplex is not the smallest possible in its class
break
```

Algorithm 1: Once the enumeration is done, check of emptiness and calculate width for the list of 4-simplices.

Checking emptiness [Sca85, HZ00]

```
if flag==0:
# Traditional method
# Taking i's in reverse order (minus signs in next line) seems to be more efficient
for i in range(3,D):
if (-v0*i)%D +(-v1*i)%D +(-v2*i)%D +(-v3*i)%D +(-v4*i)%D<D:
flag=2 # the simplex is not empty
break
```

Width check

```
producto=[]
flag=1
for i in range(0,len(L4)-1):
c=[a*b for a,b in zip(L4[i],[v0,v1,v2,v3,v4,D-1])]
c=sum(c)
producto.append(c%D)
resultado=[ai<anchura+1 for ai in producto]
if sum(resultado)==0:
flag=0
```

Computing orbits

```
if flag==0:
orbit=[[v0,v1,v2,v3,v4,D-1]]
sorted_orbit=[[v0,v1,v2,v3,v4,D-1]]
uu=D-1
for u in [u4,u3,u2,u1,u0]:
if gcd(u,D)==1 and u!=uu:
repr=[(-v0*u)%D,(-v1*u)%D,(-v2*u)%D,(-v3*u)%D,(-v4*u)%D,%D]
uu=u
if sorted(repr) in sorted_orbit:
continue
orbit.append(repr)
sorted_orbit.append(sorted(repr))
if flag==0:
empty.append(orbit)
```

4.2.2 Algorithm 2

This algorithm enumerates all empty 4-simplices of a volume V that is not a prime number, by factoring it as $V = ab$ of it and performing a method after loading all empty 4-simplices of volume the factors, a and b.

"Glueing" the empty simplices of volumes a and b together

Previous to apply Algorithm 2, we get a "nice" factorization of V with the subroutine that is explained below:

Obtaining the factors p and q of V for Algorithm 2

The idea of this subroutine is obtaining 2 factors p and q of the non-prime volume of a simplex $V = pq$. As it is shown in Figure 4.1, the lower is $|p - q|$, the faster the algorithm 2 runs, so ideally we would like to get the factorization $V = pq$, with $p \sim q$.

Algorithm:
Fix a initial $p = 1$:
for i in sorted(divisors(V)):
if i in primefactors(V):
d=1
while i^d in divisors(V):
d=d+1
if $i^{d-1} > p$:
$p = i^{d-1}$
$q = \frac{V}{p}$

Once we have the factorization and we have the both list of empty 4-simplices of volume a and b we can continue with the algorithm 2:

Algorithm 2: Algorithm obtaining empty simplices in volume V from the complete list of empty simplices of prime volumes a and b, with $V = ab$

- Read quintuples of empty 4-simplices of volume a
- Read quintuples of empty 4-simplices of volume b
- $p\Delta_a + q\Delta_b (\mod V)$.
- Check if the new simplex is empty.

Once this simplices are created we need to check the emptiness and width as in Algorithm 1.

4.3 Computation time

We have implemented the above algorithms in `python` and run them in the Altamira Supercomputer at the Institute of Physics of Cantabria (IFCA-CSIC) for every $D \in \{1, 2, \ldots, 7600\}$.

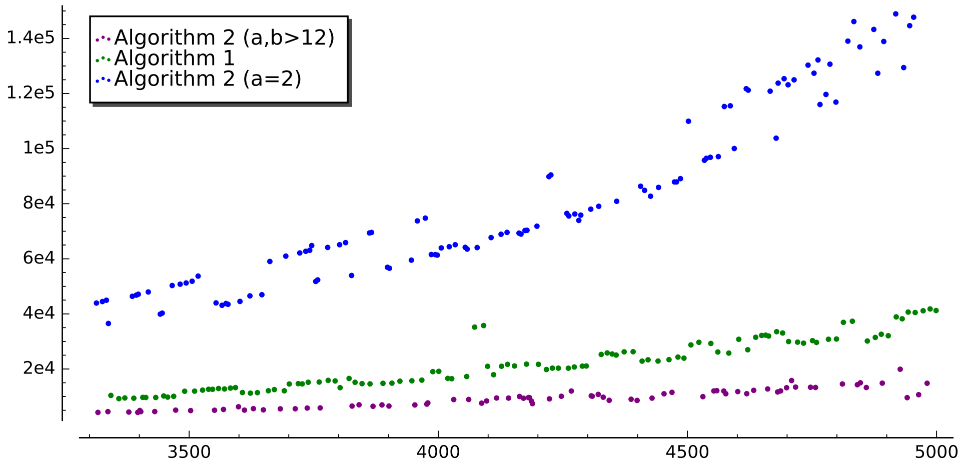


Figure 4.1: Computation times (seconds) for the enumeration of empty 4-simplices of a given determinant D between 3000 and 5000

For many values of V (those with two, three, or four prime factors) both algorithms work and we have chosen one of them. Also, for Algorithm 2 there is often several choices of how to split V as a product of two coprime numbers a and b. Experimentally we have found that Algorithm 2 runs much faster if a and b are chosen of about the same size, and in this case it outperforms Algorithm 1. This is seen in Figure 4.1 where some computation times are plotted for the two algorithms. Blue points in the figure show the time taken for Algorithm 2 to compute all empty 4-simplices for a given determinant of the form $V = 2b$ with b a prime number. Purple points correspond to the same computation for $V = ab$ with both a and b primes bigger than 12. Green points are prime determinants, where Algorithm 1 needs to be used for the enumeration. In Algorithm 2 the time to precompute empty simplices of determinants a and b is not taken into account, since we obviously had that already

2	0	73	220	179	105	283	10
3	0	79	275	181	65	293	25
5	0	83	300	191	40	307	0
7	0	89	275	193	60	311	5
11	0	97	230	197	65	313	5
13	0	101	201	199	55	317	5
17	9	103	255	211	20	331	5
19	13	107	270	223	35	337	0
23	28	109	220	227	45	347	5
29	39	113	200	229	30	349	10
31	30	127	120	233	45	353	5
37	50	131	145	239	15	359	0
41	76	137	140	241	30	367	0
43	110	139	185	251	25	373	0
47	100	149	130	257	15	379	0
53	195	151	95	263	35	383	0
59	260	157	55	269	10	389	0
61	186	163	85	271	20	397	5
67	205	167	90	277	0	409	0
71	250	173	75	281	0	419	5

17	0	107	270	227	45
19	0	109	220	229	30
23	0	113	200	233	45
29	15	127	120	239	15
31	10	131	145	241	30
37	30	137	140	251	25
41	66	139	185	257	15
43	100	149	130	263	35
47	150	151	95	269	10
53	190	157	55	271	20
59	255	163	85	283	10
61	186	167	90	293	25
67	205	173	75	311	5
71	250	179	105	313	5
73	220	181	65	317	5
79	275	191	40	331	5
83	300	193	60	347	5
89	275	197	65	349	10
97	230	199	55	353	5
101	201	211	20	397	5
103	255	223	35	419	5

Figure 4.2: Comparative in the numbers of empty 4-simplices obtained by Mori, Morrison and Morrison and our calculations

stored from the previous values of V. As seen in the figure, about 100 000 seconds (that is, about 1 day) computing time was needed in some cases with $V \sim 5000$. The total computation time for the whole set of values of V was about 10 000 hours ($\sim$1 year).

4.4 Differences with Mori et al. results

We have compared our computation of sporadic examples with the one by Mori et al., who listed the number of empty 4-simplices that they obtain for each prime volume up to 419 in their computations; see the left part of Figure 4.4, which is Table 1.14 in [MMM88]. The right part of the same table is our count of them. This is not exactly the same count as in Table 2.6 since we are here counting terminal quotient singularities rather than simplices; that is, each simplex is counted as many times as orbits of vertices are there in its affine-unimodular symmetry group. As seen in the table, there are some discrepancies between our results and those from [MMM88]. We approached the authors of [MMM88] about this issue and I. Morrison (personal communication) told us that they no longer have their full output, so it is not possible to verify their numbers, or to look at what particular simplices produce the discrepancies. Observe that, when there is a discrepancy, the value in [MMM88] is higher than ours (with a single exception for $V = 47$ that might be a typographic error).

Our guess is that their mistake was not in the enumeration part but in the search for redundancies, where quintuples defining isomorphic simplices may look different, specially when V is not big with respect to the other entries in the quintuple. This guess is consistent with the facts that all discrepancies have $V < 60$ and discrepancies are bigger for smaller values of V. Most entries, and most discrepancies between the two tables, are multiples of five since most simplices have no symmetries.

Chapter 5

Facets of empty 4-simplices

In this section, we describe all the possible combinations of volume for the facets of empty simplices that happen in the sporadic simplices and the different families described in chapter 2. Knowing all these possible combinations allow us to prove that all empty 4-simplices have at least 2 unimodular facets. This result was announced in [Wes89] but we could not verify the proof as that Master's Thesis is written in German and it seems that has not been verified.

In the particular case of empty 4-simplices, knowing the volume and the volume of its facets is equivalent to having the Ehrhart polynomial of the simplex, so we include some results and figures representing how the h^*-vector looks like for the different families.

As a result of the complete enumeration of the facets volume vector, we get all the possible h^*-vectors of the form $(1, 0, h_2^*, h_3^*, 0)$.

5.1 Unimodular facets of empty simplices

Through chapter 2 we have verified the vector of facet volumes for the empty 4-simplices that project to hollow 3-polytopes.

From the output of the computations described in Chapter 5 we can also compute the facet-volume vector of all sporadic 4-simplices. The number of non-unimodular facets for that cases are given in Table 5.1.

With the content of this table and the results of Chapter 3 we can finally verify the following theorem announced by Wessels [Wes89].

Corollary 5.1.1. *Every empty 4-simplex has* at least *two unimodular facets. The ones that have* only *two unimodular facets are:*

- *The simplices with $k = 1$ (equivalently, of width 1) when their 5-tuple $(\alpha + \beta, -\alpha, -\beta, -1, 1)$ has the property that V has prime factors in common with the three of α, β and $\alpha + \beta$ (such factors are automatically distinct, since $\gcd(\alpha, \beta, V) = 1$).*

- *The simplices with $k = 3$ (hence, of width two) in the primitive family with quintuple $(7, 5, 3, -1, -14)$, whenever V is a multiple of* 30.
- *The following* 3 *sporadic empty* 4*-simplices of width two:*

5-tuple	volume	facet volumes	h^*-vector
$(4, 7, 15, 17, 41)$	42	$(2, 7, 3, 1, 1)$	$(1, 0, 25, 16, 0)$
$(2, 13, 21, 25, 59)$	60	$(2, 1, 3, 5, 1)$	$(1, 0, 33, 26, 0)$
$(2, 13, 25, 81, 119)$	120	$(2, 1, 5, 3, 1)$	$(1, 0, 63, 56, 0)$

In higher dimension it is no longer true that all empty simplices have some unimodular facet: there is an empty 5-simplex of volume 54 whose facet volumes are $(6, 6, 9, 54, 54, 54)$ [Wes89, p. 21]; see also [BBBK11, Remark 1] for a 3-parameter infinite family of noncyclic empty 5-simplices projecting to $2\Delta_2$ and with all facets of the same, arbitrarily large, volume.

Dimension of the projection	Empty 4-simplices	$h_2^* - h_3^*$	# of non unimodular facets
$d' = 2$	Projecting to hollow 2-triangle à la Mori	Unbounded	0 or 1
	Projecting to hollow 2- triangle	1	1
$d' = 3$	Bipyramid of index 2	1 or 3	1 or 2
	Bipyramid of index 3	0, 2 or 3	$0, 1$ or 2
	Bipyramid of index 4	1	1
	Bipyramid of index 6	3	2
	Primitive bipyramids	[0,. . . ,7]	$0, 1, 2$ or 3
$d' = 4$	Sporadic empty 4-simplices	[0,. . . ,12] except 10	$0, 1, 2$ or 3

Table 5.1: Table with different numbers of non-unimodular facets depending on the classification of Theorem 2.2.1

5.2 h*-vector of empty 4-simplices

As a tool to study the restrictions on the parameters α, β and V in Theorem 2.2.1, in Sections 2.3 and 2.4 we have studied the possible facet volumes of empty 4-simplices in the infinite families. We here complete that information including a summary of the data for sporadic families, and relate it to h^*-vectors and Ehrhart polynomials.

Recall that the *Ehrhart polynomial* of a lattice d-polytope P is a degree d-polynomial $E(P,t) = E_t t^d + \cdots + E_0 \in \mathbb{Q}[t]$ with the property that

$$E(P,t) = |tP \cap \Lambda|, \quad \forall t \in \mathbb{N}.$$

Some well-known facts about it are that $E_d = \mathrm{Vol}(P)/d!$, $E_{d-1} = \mathrm{Surf}(P)/2(d-1)!$, where Vol and Surf denote the normalized volume and surface area (the sum of normalized volumes of facets). Also, Ehrhart reciprocity states that

$$E(P,-t) = |\text{interior}(tP) \cap \Lambda|, \quad \forall t \in \mathbb{N}.$$

An alternative way of giving the same information is via the h^*-vector (or δ-vector) of P, a vector $h^*(P) = (h_0^*, \ldots, h_d^*) \in \mathbb{N}^{d+1}$ with the property that

$$\sum_{n=0}^{\infty} E(P,n)x^n = \frac{h_d^* x^d + \cdots + h_0^*}{(1+x)^{d+1}}.$$

That is, the h^*-vector gives (the vector of coefficients of the numerator of the rational function of) the generating function of the sequence $(E(P,n))_{n\in\mathbb{N}}$. See [BR07, Ehr62,Sta80] for more information on Ehrhart polynomials and h^*-vectors, and [Sco76, Sta09, HKN18, BH18, LS19, HNO18] for results on their classification.

For empty 4-simplices, the h^*-vector admits the following simple expression in terms of volume and surface area:

Proposition 5.2.1. *Let P be an empty 4-simplex of volume V and facet volumes $(V_0, V_1, V_2, V_3, V_4)$. Let $S = V_0 + \cdots + V_4$ be the surface area of P. Then, the h^*-vector of P is*

$$h_0^* = 1, \quad h_1^* = 0, \quad h_2^* = \frac{V+S}{2} - 3, \quad h_3^* = \frac{V-S}{2} + 2, \quad h_4^* = 0.$$

Proof. From the two coefficients $E_4 = V/24$ and $E_3 = S/12$ and the three values $E(P,-1) = 0$, $E(P,0) = 1$, $E(P,1) = 5$ we can recover the whole Ehrhart polynomial, which turns out to be

$$E_P(n) = \frac{V}{24}n^4 + \frac{S}{12}n^3 + \left(\frac{3}{2} - \frac{V}{24}\right)n^2 + \left(\frac{5}{2} - \frac{S}{12}\right)n + 1.$$

From this, routine computations give the h^*-vector. □

Remark 5.2.2. *The values $h_0^* = 1$, $h_1^* = 0$, and $h_d^* = 0$ hold for empty simplices in arbitrary dimensions, by the following general formulas for arbitrary lattice polytopes [BR07, Section 3.4]:*

$$h_0^* = 1, \qquad h_1^* = |P \cap \mathbb{Z}^d| - (d+1), \qquad h_d^* = |\mathit{interior}(P) \cap \mathbb{Z}^d|.$$

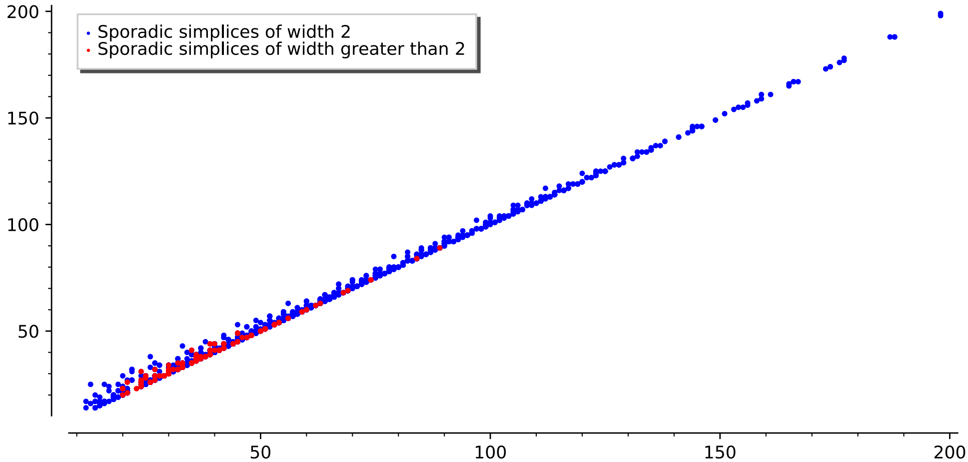


Figure 5.1: "Diference of volume in the facets" for sporadic empty 4-simplices in terms of h_3^* (x axis) and h_2^* (y axis).

Another general formula is $\sum_{i=0}^{d} h_i^* = \mathrm{Vol}(P)$ *[BR07, Cor. 3.21], which in the case of empty simplices directly gives*

$$h_2^* + \cdots + h_{d-1}^* = V - 1.$$

Observe also that Proposition 5.2.1 agrees with the Hibi inequality $h_2^* \geq h_3^*$ *[Hib90].*

Proposition 5.2.1 implies that the Ehrhart polynomial and h^*-vector of an empty 4-simplex is determined by h_2^* and h_3^* or, equivalently, by

$$h_2^* + h_3^* = V - 1 \quad \text{and} \quad h_2^* - h_3^* = S - 5. \tag{5.1}$$

These two parameters quantities are nonnegative and measure how far is P from being unimodular or from having unimodular facets. We call them the *volume excess* and the *surface area excess* of P.

In Figure 5.2 we show the statistics of volume and surface area excess for the 2461 sporadic simplices. The reason to express some of the simplices in colour red and the others in colour blue is that the ones in blue are the ones that appear in [IVS19] classified as having width 3 and the ones in red are the rest of the sporadic empty 4-simplices.

As seen in the figure, the maximum value of the latter is 12. It is achieved exactly twice, for the simplices of volumes 39 and 65 defined by the quintuples

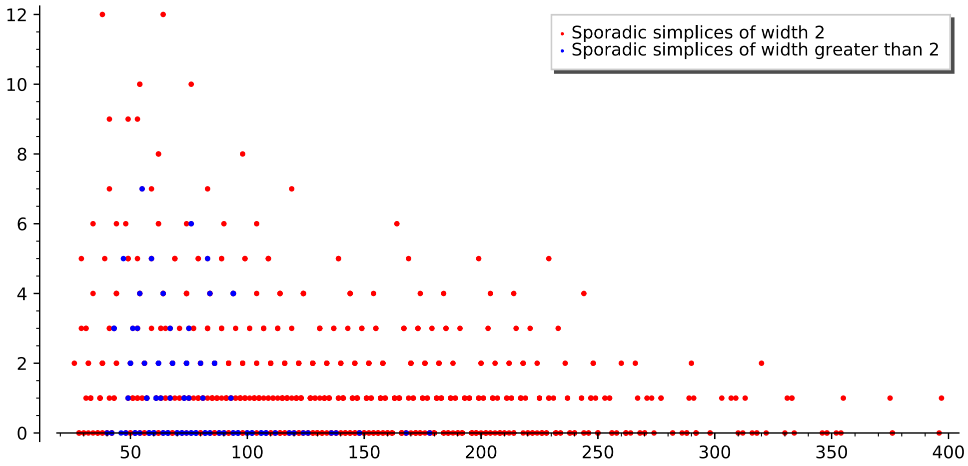


Figure 5.2: Possible values of $V-1 = h_2^*+h_3^*$ (horizontal axis) and $S-5 = h_2^*-h_3^*$ (vertical axis) for the 2461 sporadic empty 4-simplices

$(5, 8, 13, 14, 38)$ and $(3, 14, 23, 26, 64)$ respectively. They both have width two and a single nonunimodular facet, of volume 13 in both. [1]

These enumerations allow us to give a complete characterization of the h^*-vectors of the form $(1, 0, h_2^*, h_3^*, 0)$.

[1] Here we mention width of the different examples since this was a crucial invariant for the the bound in Section 2.5 and in [HZ00, IVS19]. Observe that $k = 1$ is equivalent to width one, $k \in \{2, 3\}$ implies width two, and the sporadic simplices can have width between two and four

Chapter 6

Open questions and ongoing work

In this last section we would like to have some words about research topics related or derived from our work and which are some of the missing points and problems that make these questions interesting and challenging.

Once empty 4-simplices are completely classified, it is natural to think about classifying more general polytopes or simplices in higher dimensions. Even more, one can ask new questions that may allow us to understand better properties of these polytopes and convex bodies.

6.1 Classification of hollow n-polytopes

Through this section, we discuss how to approach classification of empty 5-simplices, the limitations and the disadvantages that have the methods described in sections above for dimension 4 if we apply them to dimension 5.

On one hand, we would like to mention that as explained in section 4.3, the approach used to enumerate empty 4-simplices is computationally expensive. For example, exhaustively enumerating all empty 4-simplices of volume 5000 with Algorithm 1 takes more than 50 hours.

Using the same approach in order to enumerate empty simplices in dimension 5 seems that would require an amount of time that make this task useless.

Even more, there exist additional problems to face when trying to describe empty 5-simplices:

- The quotient group of $\mathbb{Z}^d$ by the sublattice generated by the vertices of an empty d-simplex is cyclic when $d \leq 4$ (Theorem 1, [BBBK11]), but this is not true in higher dimension.

- Every empty 4-simplex has an unimodular facet. In dimension 5, there are examples of empty simplices without unimodular facets. The simplex generated by the vertex columns of the following matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 & 1 & 4 \\ 0 & 1 & 0 & 0 & 6 & 0 \\ 0 & 1 & 0 & 0 & 0 & 9 \end{pmatrix}$$

is empty without any unimodular facet [HZ00], as expressed in section 5.1.

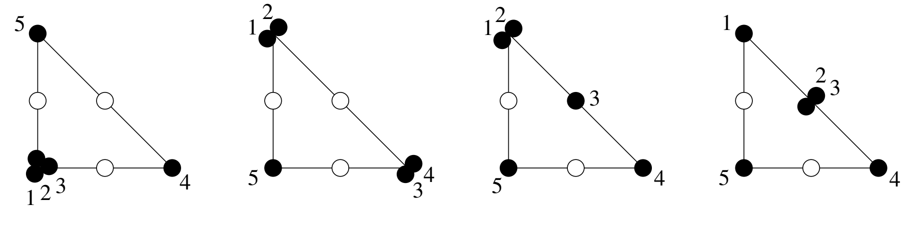
These two facts have been used in the classification work to speed up the computations in Chapter 4 and make the enumeration easier. The impossibility to do them in dimension 5 makes things even more difficult.

On the positive side, some of the ingredients in the proof of our classification work not only for empty 4-simplices, but for all hollow ones. Putting those things together we have the following not-so-explicit classification of hollow 4-simplices.

Theorem 6.1.1 (Classification of hollow 4-simplices)**.** *Let P be a hollow 4-simplex of volume $V \in \mathbb{N}$ and let $k \in \{1, 2, 3, 4\}$ be the minimum dimension of a hollow polytope that P projects to. Then P belongs to one of the following fine families:*

$\boxed{k = 1:}$ *Two fine families projecting to the multisets $\{0, 0, 0, 0, 1\}$ and $\{0, 0, 0, 1, 1\}$. The cyclic members of these families are parametrized by 5-tuples of the form $(\alpha + \beta + \gamma, -\alpha, -\beta, -\gamma, 0)$ and $(\alpha + \beta, -\alpha, -\beta, -\gamma, \gamma)$, respectively, where $\alpha, \beta, \gamma \in \mathbb{Z}_V$ are arbitrary.*

$\boxed{k = 2:}$ *Six fine families projecting to the two multisubsets of $2\Delta_2 \cap \mathbb{Z}^2$ displayed in Figure 2.3 or to the following four additional ones:*



Cyclic members of the families can be parametrized, respectively, by the fol-

lowing 5*-tuples, where* $\alpha, \beta \in \mathbb{Z}_V$ *are arbitrary:*

$$\begin{aligned}
&(\beta, -2\beta, \alpha, -2\alpha, \beta + \alpha),\\
\frac{V}{2}(0,1,0,1,0) \; &+ (\beta, \beta, \alpha, -\alpha, -2\beta),\\
\frac{V}{2}(0,0,0,1,1) \; &+ (\alpha + \beta, -\alpha, -\beta, 0, 0),\\
\frac{V}{2}(0,0,0,1,1) \; &+ (\alpha, -\alpha, \beta, -\beta, 0),\\
\frac{V}{2}(0,0,0,1,1) \; &+ (\alpha + \beta, -\alpha, -2\beta, \beta, 0),\\
\frac{V}{2}(0,0,0,1,1) \; &+ (\beta, \alpha - 2\beta, -\alpha, \beta, 0).
\end{aligned}$$

$\boxed{k = 3\text{:}}$ *P belongs to a finite set of fine families, one corresponding to each of the tetrahedra, square pyramids, or triangular bipyramids of Lemma 2.4.2.*

$\boxed{k = 4\text{:}}$ *There are finitely many possibilities for P, by Theorem 2.1.1. Their volumes are bounded by 5184.*

Proof. In all the cases $k = 1, 2$ it is easy to show that the claimed cases exhaust all possibilities for S. Let us see this for $k = 2$. Once we know that three of the elements of S are the vertices of $2\Delta_2$ the six possibilities come from the fact that the other two can either be also vertices (either the same or two different ones), they can both be midpoints of edges (either the same or two different ones), or they can be a vertex and a midpoint (either opposite or consecutive). The expression for the 5-tuples follows from Proposition 2.1.11 and the easy computation of the spaces of affine dependences of the $2 + 6$ cases of S. (In all nonprimitive cases the index is two, so there is no choice for the generator q of $\pi(\Lambda)/\Lambda_S$). For $k = 3$ our statement follows from Lemma 2.4.2 and the definition of fine family.

For $k = 4$ just observe that Theorem 3.3.6 applies to all hollow 4-simplices, not necessarily empty or cyclic ones. $\square$

The two missing ingredients to turn Theorem 6.1.1 into a more explicit description are:

- An analysis of what finite non-cyclic groups can arise as $G_P = \Lambda/\Lambda_P$. Since they are (isomorphic to) quotients of $\mathbb{Z}^4$, they can be written as $\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \mathbb{Z}_{n_3} \oplus \mathbb{Z}_{n_4}$ with n_i dividing n_{i+1}, $i = 1, 2, 3$. This implies each simplex to be representable by (at most) four 5-tuples like the ones used in the cyclic case, but we would expect a simpler description to be possible.

- An enumeration of all hollow 4-simplices of volume up to 5184, pruning those that belong to the infinite families with $k \leq 3$.

6.2 Threshold width and flatness constant

As mentioned in the introduction, when having a hollow polytope, with a width larger than a threshold given, there exists an upper bound for the volume of it.

It would be quite interesting to define the exact value for the threshold width, as defined in Chapter 1.1, for every dimension d for a convex body or, even, answering this question for polytopes.

Some small dimension values of the finiteness threshold width are known, $w^{\infty}(3) = 1$, $w^{\infty}(4) = 2$ and we do not know the exact value for any dimension greater than 5, even though it is known that it is bounded from below $w^{\infty}(5) \geq 4$. This fact is true because of the existence of an empty 4-simplex of width 4 [BHHS].

One possible question before attaining the exact value of $w^{\infty}(d)$ in every dimension d would be knowing things about the asymptotic behavior of the width. In this direction Codenotti and Santos [CS19] show that the maximum lattice widths grow with d. In particular, they show constructions of hollow lattice polytopes and simplices that have arbitrarily large dimension d with width $\simeq 1.14d$, respectively $\simeq 1.01d$ for simplices.

The question 1.12 in [BHHS] is an interesting one. Blanco et al. ask if the finiteness threshold width of an arbitrary dimension with $d > 4$ is always a value that does not depend on the number of lattice points of a polytope.

$$w^{\infty}(d) = w^{\infty}(d, d+1) \qquad \text{for all } d > 4$$

This would imply that it is just sufficient to find the value of the threshold width for lattice simplices.

In order to have some intuition about these problems, we propose an alternative method for estimating the values of $w^{\infty}(d)$ for $d > 4$ via generating random empty simplices:

6.2.1 Random empty simplices

As said before, classifying lattice polytopes in dimension bigger than 3 is a difficult task. One of the main problems is finding good upper bounds for the volume of these lattice polytopes in order to enumerate them.

In this thesis, upper bounds for hollow and empty 4-simplices are obtained with new techniques. The upper bounds obtained in chapter 3 are not tight and the exhaustive enumeration of the empty simplices that is necessary to complete the classification of empty 4-simplices requires more than 10000 hours of computation time.

In order to get new classifications of polytopes with more lattice points or in bigger dimensions, the method for obtaining good upper bounds should be improved since the enumeration part of the classification grows quite fast and the amount of computation time needed for giving an exhaustive answer to these problems would be non feasible.

We propose a method to generate random lattice simplices with a Poisson distribution and measuring statistics of them in order to gain computationally speed estimating the upper bounds needed[1].

The reasoning for selection the Poisson distribution is the existence of some results for random polytopes generated by using this distribution. Some central limit theorems have been obtained for the volume, number of i-dimensional faces, f-vector, etc [BR10, Rei05].

In the method we construct simplices of the form $\sigma(v) := conv\{e_1, e_2, e_3, e_4, v\}$, where $v \in \mathbb{R}^4$ is obtained by generating every v_i component of v as a random integer number following the Poisson distribution, that allow us to get a random $\sigma(v)$. Once we obtain the $\sigma(v)$ simplices we calculate how many of them are empty and their width. The algorithm that we use to obtain these simplices is shown at Appendix .3.

With these results we hope to deduce an approximated range for the volume of polytopes being empty and having a bounded width. We compare this range with the theoretical bounds for this class of lattice polytopes and the actual range that contains them in order to check if the method constructed gives a valid evidence to give good upper bounds.

If we take as example empty 4-simplices of width greater than 2, the complete list has lattice volume between 41 and 179, where most of this type of empty simplices are in the interval $[41, 127]$ with 6 sporadic examples with volumes from 127 to 179 [IVS19]. Trying to construct a range for the volume of these simplices, we have constructed 10000 4-simplices of type $\sigma(v)$ for different parameters that give polytopes with different volumes and enumerate all of them that had width greater than two.

As a result, we can see, in Figure 6.2.1, that we can recover most part of the range where the empty 4-simplices of width greater than 2, but 5 cases with volume bigger than 127 are not detected.

We think that using these approach we can gain some interesting information and intuitions about the problem such as the following:

[1] This approach is based in discussions with C.Borger, J-M. Brunink, A. Grosdos and some other colleagues during the Graduate Student Meeting on Applied Algebra and Combinatorics at Max-Panck-Institut Leipzig in 2019.

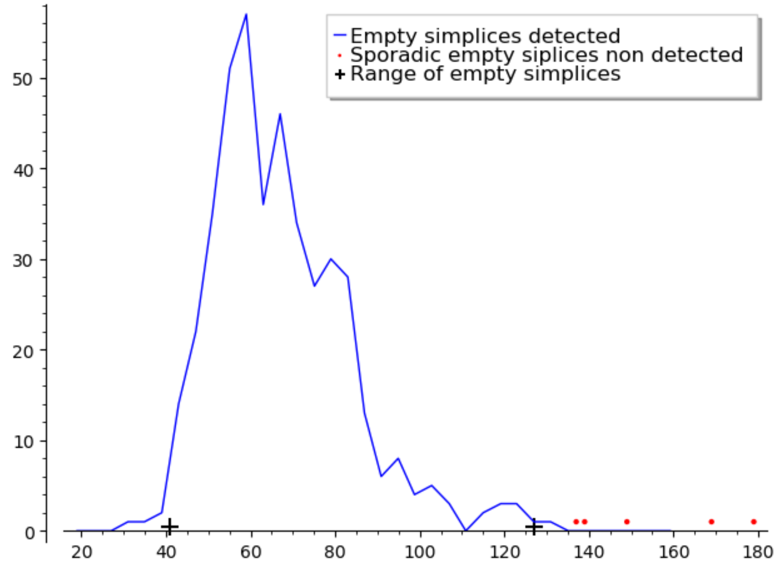


Figure 6.1: Empty 4-simplices of width greater than two detected .

1. Try to answer some open questions and conjectures obtaining concrete examples (it is easier to get examples of large widths or large volume that do not arise from the geometrical intuition).

2. It could help to give conjectural classifications of hollow d-polytopes. Maybe this could lead to some polytopes that give better bounds for the maximum width of hollow d-polyltopes or convex bodies (In terms of the flatness theorem).

3. By random search it is possible to find some data which allow us to realize some properties that follow the simplices obtain with a certain property or find some families of polytopes that we could not figure out by ourselves.

4. We obtain this information much faster than going through the exhaustive enumeration.

Even though this method can be useful, it would be needed to check the performance of this algorithm when d increases. If the frequency of simplices that are hollow (equiv. empty) goes down quickly with the dimension some adaptations might be needed.

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Appendices

.1 Number of empty 4-simplices for particular families:

From volume $V \geq 59$ onwards, the 29 families of empty simplices belonging to the empty 4-simplices that project to a primitive bipyramid appear for every volume V. The condition is (No prime that divides more than one entry in the quintuple can divide V).

Volume	Empty simplices projecting to primitive bipyramids	Volume	Empty simplices projecting to primitive bipyramids
≥ 59:	29	53	28
47	28	43	25
41	26	37	23
31	18	29	16
23	12	19	8
17	6	13	2
11	1	< 11	0

Table 1: Empty 4-simplices that project to the 29 primitive bipyramids described by Mori et al. appearing in low prime volumes

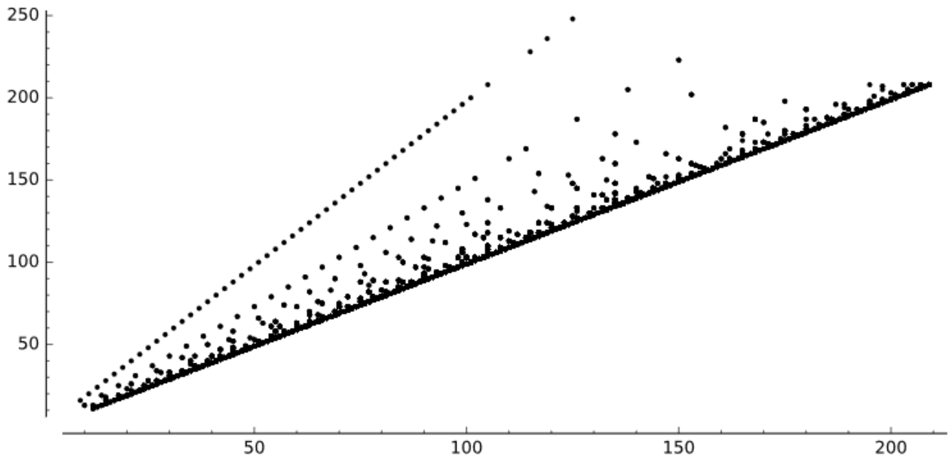

Figure 2: h_3^* (x axis) and h_2^* (y axis) for simplices that project to the hollow 2-simplex (Primitive case).

Volume	Empty simplices $k = 2$ (Primitive case)	Volume	Empty simplices $k = 2$ (Primitive case)
29	9	23	4
19	1	17	0
13	2	11	1

Table 2: Number of empty 4-simplices when $k = 2$, primitive case, for volume < 30

.2 Example of classification for some values of V?

Here we show how the files look like for a certain value of V. As mention before some example files are available at `https://personales.unican.es/iglesiasvo/`.

Empty simplices of width two:
Determinant =56
Projecting to triangle as (2,-1,D/2-1,a,D/2-a), ($k = 2$): 5
[[2, 3, 25, 27, 55]]
[[2, 5, 23, 27, 55]]
[[2, 9, 19, 27, 55]]
[[2, 11, 17, 27, 55]]
[[2, 13, 15, 27, 55]]
Projecting to triangle as $(1, -2, \alpha, -2\alpha, 1 + \alpha)$, ($k = 2$): 0
Projecting to bipyramids of index 2 ($k = 3$): 0
Projecting to bipyramids of index 3, ($k = 3$): 0
Projecting to bipyramids of index 4, ($k = 3$): 3
[[2, 9, 13, 33, 55]]
[[2, 13, 17, 25, 55]]
[[2, 25, 41, 45, 55]]
Projecting to bipyramids of index 6 ($k = 3$): 0
Projecting to primitive bipyramids ($k = 3$): 3
[[2, 3, 5, 47, 55]]
[[2, 5, 9, 41, 55]]
[[3, 5, 8, 41, 55]]
Sporadic, ($k = 4$): 3
[[2, 3, 19, 33, 55]]
[[[2, 5, 19, 31, 55]]
[[[3, 5, 17, 32, 55]]

With this example we can see that we order the values v_0, v_1, v_2, v_3 and write $v_4 = D - 1$ in order to eliminate unimodular equivalent simplices when enumerating them.

.3 Algorithm for generating random polytopes and estimating maximum width

Creating a sigma-polytope

```
def sigma_polytope(d,n,l):
points=[[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]]
points.append(r.rpois(d,l)._sage_())
return Polyhedron(points)
```

As stated in Section 6.2.1 we generate simplices of the form $\sigma(v)$. We have implemented a code to generate the sigma polytopes with the software CoCalc. In order to get the random point of $\sigma(v)$ we use the command rpois(d,l)._sage_() that generates a list of d random values following a Poisson distribution with average of the values l. By giving different values to l we can get simplices with different volumes as the volumes varies with the value of l.

Counting the number of empty d-simplices from the total of random d-simplices randomly generated

```
numero=10000
d=5
anchura=4
for l in range(v_min,v_max):
full_d = 0
spanning = 0
vol = 0
primos=0
emp=0
empty=0
while full_d < numero:
P = sigma_polytope(d,d+1,l)
pointlist = P.integral_points()
volumen = factorial(d)*P.volume()
```

During the algorithm we use the following methods from CoCalc:

- P.polytope(points)
- P.integral_points()
- Polyhedron(points)
- P.vertices_list()

We discard the configurations of points that do not form a d-dimensional simplex.

```
if dim(P)==d:
full_d += 1
vol = volumen+vol
if len(pointlist)==d+1:
emp +=1
if is_prime(int(volumen))==1:
primos+=1
producto=[]
vector=P.vertices_list()[d]
for i in range(0,len(L3)-1):
c=[a*b for a,b in zip(L3[i],vector)]
c=sum(c)
producto.append(c%volumen)
resultado=[ai < anchura for ai in producto]
if sum(resultado)==0:
print pointlist
empty +=1

print "For dimension " + str(d) + " and lambda " + str(l) + ":"
print "volume " + str(vol/numero)
print "Emptiness ", str(emp) + " / " + str(full_d)
print "Prime volume empty", str(primos)
print "Empty width greater than three", str(empty) + " / " + str(full_d)
```

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