Sobolev Inequalities: Isoperimetry and Symmetrization

By

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Certify:

That this dissertation presented to the Faculty of Sciences, department of Mathematics of Autonomous University of Barcelona by Walter Andrés Ortiz Vargas in fulfillment of the requirements for the degree of Doctor of Mathematics.

Bellaterra, September 17, 2019

Joaquim Martín Pedret

To my Parents

Loneliness is necessary to enjoy our own heart and to love, but To succeed in life, it is necessary to give something of our life to the greatest number of people.

Beyle Henri

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Mathematics Genealogy

Otto Menken

Jacob Bernoulli   Johann Wichmannshausen

Johann Bernoulli   Christian Hausenr

Leonhard Euler   Abraham Kaestner

Joseph Lagrange   Johann Pfaff

Jean-Baptiste Fourier   Simeon Poisson   Karl von Langsdorf   Carl Gauß

Gustav Dirichlet   Martin Omh   Christian Gerling

Rudolf Lipschitz   Julius Puckler

Eduardo Torroja   C. Felix (Christian) Klein

Julio Rey Pastor

Ricardo San Juan

Joan Augé

Rafael Aguiló

Joan Cerdá

Joaquim Martín

Walter A. Ortiz

Contents

Abstract

The first part of the thesis is devoted to obtain a Sobolev type embedding result for Besov spaces defined on a doubling metric space. This will be done by obtaining pointwise estimates between the special difference $f_\mu^{**}(t) - f_\mu^*(t)$ (called oscillation of f_μ^*) and the X−modulus of smoothness defined by

$$E_X(f,r) := \left\| \fint_{B(x,r)} |f(x) - f(y)| \, d\mu(y) \right\|_X .$$

(here f_μ^* is the decreasing rearrangement of f, $f_\mu^{**}(t) = \frac{1}{t}\int_0^t f_\mu^*(s)ds$, for all $t > 0$ and X a rearrangement invariant space on Ω.

In the second part of the thesis, to obtain symmetrization inequalities on probability metric spaces that admit a convex isoperimetric estimator which incorporate in their formulation the isoperimetric estimator and that can be applied to provide a unified treatment of sharp Sobolev-Poincaré and Nash type inequalities.

Chapter 1

Introduction

This monograph is devoted to the study of Sobolev type embedding results in the setting of:

- Besov spaces defined on doubling metric spaces (Chapter 3),
- Probability metric spaces with convex isoperimetric profile (Chapter 4).

The history of Sobolev embeddings started in the thirties of the last century with Sobolev's famous embedding theorem: [91]

$$W_p^1(\Omega) \subset L^r(\Omega) \tag{1.0.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary, L^r, $1 \leq r \leq \infty$ stands for the Lebesgue space, and $W_p^1(\Omega)$, $1 < p < \infty$, are the classical Sobolev spaces. The latter have been widely accepted as one of the crucial instruments in functional analysis – in particular in connection with PDES – and have played a significant role in numerous parts of mathematics for many years. Sobolev's famous result (1.0.1) holds for $p < n$ and r such that $\frac{1}{n} - \frac{1}{p} \geq -\frac{1}{r}$ (strictly speaking [91] covers the case $\frac{1}{n} - \frac{1}{p} > -\frac{1}{r}$ whereas the extension to $\frac{1}{n} - \frac{1}{p} = -\frac{1}{r}$ was obtained later). In the limiting case, when $p = n$, the inclusion (1.0.1) does not hold for $r = \infty$, whereas for all $1 \leq r < \infty$

$$W_n^1(\Omega) \subset L^r(\Omega). \tag{1.0.2}$$

Roughly speaking, the theory of Sobolev inequalities originated in classical inequalities from which properties of real functions can be deduced from those of its derivatives. In fact, (1.0.2) can be understood as the impossibility of specifying integrability conditions of functions in $W_n^1(\Omega)$ by means of $L^r(\Omega)$ conditions. Inequalities (1.0.1) and (1.0.2) are not optimal. In order to get further refinements, it is necessary to deal with a wider class of spaces. In the sixties of the last century, Peetre [84], Trudinguer [97] and Pohozarev [85] independently found refinements of (1.0.1) expressed in terms of Orlicz spaces. In 1979, Hansson [43] and Brezis and Wainger [11] showed independently that $W_n^1(\Omega)$ is embedded in a Lorentz–Zygmund type space. Limiting Sobolev embeddings, in more general settings, have been investigated by several authors (see [70] and the references quoted therein).

If instead of working on bounded domains with a nice boundary, we work in the full space, Sobolev's embedding theorem in $\mathbb{R}^n$ states that (see [93][1] and the references quoted therein):

$$W_p^1(\mathbb{R}^n) \subset \begin{cases} L^{\frac{np}{n-p},p}(\mathbb{R}^n) & p < n \text{ (subcritical case)}, \\ L^{\infty,p}(\log\ L)^{-1}(\mathbb{R}^n) & p = n \text{ (critical case)}, \\ L^{\infty}(\mathbb{R}^n) & p > n \text{ (supercritical case)}. \end{cases} \tag{1.0.3}$$

The Lorentz spaces $L^{p,q}(\mathbb{R}^n)$ are defined as the collection of functions of finite function quasi-norm

$$\|f\|_{L^{p,q}} = \left(\int_0^\infty \left(s^{1/p} f^*(s)\right)^q \frac{ds}{s}\right)^{1/q},$$

when $0 < p, q < \infty$, and

$$\|f\|_{p,\infty} = \sup_{0<t<1} s^{1/p} f^*(s).$$

when $q = \infty$ (f^* denotes the decreasing rearrangement of f). The Lorentz–Zygmund spaces $L^{\infty,q}(\log L)^{-1}$, $1 \le q < \infty$, are defined as the set of functions for which the quasi-norm

$$\|f\|_{L^{\infty,q}(\log L)^{-1}} = \left(\int_0^\infty \left(\frac{f^{**}(t)}{1 + \log^+\left(\frac{1}{t}\right)}\right)^q \frac{dt}{t}\right)^{1/q},$$

is finite (where $f_\mu^{**}(t) = \frac{1}{t}\int_0^t f_\mu^*(s)ds$).

Generalizations of (1.0.3) have been considered by replacing $W_p^1(\mathbb{R}^n)$ by a Besov space.

Given $0 < s < 1, 1 \le p < \infty$ and $1 \le q \le \infty$, the Besov space $\dot{B}^s_{p,q}(\mathbb{R}^n)$ is the linear set of functions $f \in L^p_{loc}(\mathbb{R}^n)$ of finite quasi-norm

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} := \left(\int_0^\infty (t^{-s}\omega_p(f,t))^q \frac{dt}{t}\right)^{1/q},$$

where

$$\omega_p(f,t) := \sup_{|h|\le t} \|f(x+h) - f(x)\|_{L^p(\mathbb{R}^n)}$$

is the L^p-modulus of continuity. Here the parameters s and q give a finer gradation of smoothness. The scales of Besov spaces $\dot{B}^s_{p,q}$, on $\mathbb{R}^n$, or in domains of $\mathbb{R}^n$,

[1] In the introduction of that paper there is an excellent history of the evolution of this problem.

were introduced between 1959 and 1975. A comprehensive treatment of these function spaces and their history can be found in Triebel's monographs [94], [95].

The Sobolev embedding in this context[2] states that

$$\dot{B}^s_{p,q}(\mathbb{R}^n) \subset \begin{cases} L^{\frac{np}{n-sp},q}(\mathbb{R}^n) & p < \frac{n}{s} \text{ (subcritical case)}, \\ L^{\infty,q}(\log L)^{-1}(\mathbb{R}^n) & p = \frac{n}{s} \text{ (critical case)}, \\ L^{\infty}(\mathbb{R}^n) & p > \frac{n}{s} \text{ (supercritical case)}. \end{cases} \tag{1.0.4}$$

One proof of the subcritical case is based on real interpolation. We recall briefly the construction of real interpolation spaces (see [6] for a complete treatment). Let (A_0, A_1) be a pair of quasi-Banach spaces that are compatible in the sense that both A_0 and A_1 are continuously embedded in some common Hausdorff topological vector space $\mathcal{H}$. The K-functional is defined, for $t > 0$ and $f \in A_0 + A_1$, by

$$K(t, f; A_0, A_1) = \inf_{f = f_0 + f_1} \{\|f_0\|_{A_0} + t\,\|f_1\|_{A_1}\}$$

For $0 < s < 1$ and $0 < q \leq \infty$, the real interpolation space $\vec{A}_{s,q} = (A_0, A_1)_{s,q}$ is the set of all $f \in A_0 + A1$ such that

$$\|f\|_{(A_0,A_1)_{s,q}} := \begin{cases} \left(\int_0^\infty \left(\frac{K(t,f;\vec{A})}{t^s}\right)^q \frac{dt}{t}\right)^{1/q}, & 0 \leq q < \infty, \\ \sup_{t>0} t^{-s} K(t, f; \vec{A}), & q = \infty, \end{cases}$$

is finite.

Since (cf. [6])

$$K(t, f; L^p(\mathbb{R}^n), \dot{W}^1_p(\mathbb{R}^n)) = \inf_{f=f_0+f_1} \{\|f_0\|_{L^p} + t\,\|f_1\|_{\dot{W}^1_p}\} \simeq \omega_p(f, t),$$

we get

$$\|f\|_{(L^p(\mathbb{R}^n), \dot{W}^1_p(\mathbb{R}^n))_{s,q}} = \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}.$$

Using the fact that $L^p(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ and that $\dot{W}^1_p(\mathbb{R}^n) \subset L^{\frac{np}{n-p},p}(\mathbb{R}^n)$, we obtain by interpolation

$$\dot{B}^s_{p,q}(\mathbb{R}^n) = (L^p, \dot{W}^1_p)_{s,q}(\mathbb{R}^n) \subset (L^p, L^{\frac{np}{n-p},p})_{s,q}(\mathbb{R}^n) = L^{\frac{np}{n-sp},q}(\mathbb{R}^n).$$

Our main objective in Chapter 3 will be to give an extension of (1.0.4) in the context of doubling metric spaces[3]. A theory of Besov spaces on metric measure

[2](See for example DeVore, Riemenschneider and Sharpley [18], Netrusov [82], Goldman and Kerman [29], Caetano and Moura [12],[13], Martín [60], or Haroske and Schneider [44]).

[3]Metric spaces play a prominent role in many fields of mathematics. In particular, they constitute natural generalizations of manifolds, admitting all kinds of singularities and still providing rich geometric structure.

spaces was developed in [38], which is a generalization of the corresponding theory of function spaces on $\mathbb{R}^n$ (see [94],[95],[96]), respectively, Ahlfors n-regular metric measure spaces (see [39],[41]).

There are several equivalent ways to define Besov spaces in the setting of a doubling metric space (see for example [27],[28],[38],[75],[74],[105],[45] and the references therein). In Chapter 3, we shall use the approach based on a generalization of the classical L^p-modulus of smoothness introduced in [27].

Let (Ω, d, μ) be a metric measure space equipped with a metric d and a Borel regular outer measure μ, for which the measure of every ball is positive and finite. Given $t > 0$, $0 < p < \infty$ and $f \in L^p_{loc}(\Omega)$, the L^p-modulus of smoothness is defined by

$$E_p(f,t) = \left(\int_\Omega \left(\fint_{B(x,t)} |f(x) - f(y)|^p \, d\mu(y) \right) d\mu(x) \right)^{1/p},$$

where $\fint_B f(x) d\mu(x) := \frac{1}{\mu(B)} \int_B f(x) d\mu(x)$ is the integral average of a locally integrable function f over B.

Definition 1.0.1. *For $0 < s < \infty$, the homogeneous Besov space $\dot{\mathcal{B}}^s_{p,q}(\Omega)$ consists of those functions $f \in L^p_{loc}(\Omega)$ for which the seminorm*

$$\|f\|_{\dot{\mathcal{B}}^s_{p,q}(\Omega)} := \begin{cases} \left(\int_0^\infty \left(\frac{E_p(f,t)}{t^s} \right)^q \frac{dt}{t} \right)^{1/q}, & 0 < q < \infty, \\ \sup\limits_{t>0} t^{-s} E_p(f,t), & q = \infty, \end{cases}$$

is finite.

This definition is rather concrete and gives the usual Besov space in the Euclidean setting since $E_p(f,t)$ is equivalent to the classical $L^p(\mathbb{R}^n)$-modulus (see (3.1.2) in Section 3.1 below). Moreover, it has been shown by Müller and Yang [74] that it coincides with the definition based on test functions used earlier by Han [40], Han and Yang [42], and Yang [103], provided that Ω, besides being doubling, also satisfies a reverse doubling condition.

The abstract variant of (1.0.4) for metric spaces is only known in the following particular case (see [27] and [45]):

Theorem 1. *Let Ω be a Q-regular metric space, i.e. there exists a $Q \geq 1$ and a constant $c_Q \geq 1$ such that*

$$c_Q^{-1} r^Q \leq \mu(B(x,r)) \leq c_Q r^Q$$

for each $x \in X$, and for all $0 < r < diam\ \Omega$ (here $diam\ \Omega$ is the diameter of Ω). Suppose that $0 < s < 1$ and $1 \leq q \leq \infty$. Then:

1. *(See ([27, Thm. 5.1])) Suppose* Ω *satisfies a* $(1,p)$*-Poincaré inequality, i.e. if there exist constants* $C_p \geq 0$ *and* $\lambda \geq 1$ *such that*

$$⨍_B |f - f_B| d\mu \leq \left(⨍_{\lambda B} g^p d\mu \right)^{1/p}$$

for any locally integrable functions f *for all upper gradients*[4] g *of* f*. Then*

$$\dot{\mathcal{B}}^s_{p,q}(\Omega) \subset L^{p(Q),q}_\mu(\Omega) \tag{1.0.5}$$

for $1 < p < Q/s$*, where* $p(Q) = Qp/(Q - sp)$*.*

2. *(See ([45, Thm. 4.4])) If* Ω *is geodesic, i.e. every pair of points can be jointed by a curve whose length is equal to the distance between the points, then (3.1.3) holds for* $1 \leq p < Q/s$*.*

The proof of this theorem is based on the real interpolation method, for example in ([27, Thm. 5.1]) under a $(1,p)$-Poincaré inequality assumption, the Besov space $\dot{\mathcal{B}}^s_{p,q}(\Omega)$ is realized as the real interpolation space $(L^p(\Omega), KS_{1,p}(\Omega))_{\alpha,q}$ between the corresponding $L^p(\Omega)$ and the Sobolev space of Korevaar and Schoen $KS_{1,p}(\Omega)$, consist of measurable functions f of finite norm[5]

$$\|f\|_{KS_{1,p}(\Omega)} := \limsup_{t\to 0} \frac{E_p(f,t)}{t}. \tag{1.0.6}$$

They proved that $E_p(f,t)$ is equivalent to the K-functional between $L_p(\Omega)$ and $KS_{1,p}(\Omega)$. Moreover if Ω is Q-regular, then

$$\|f\|_{L_\mu^{\frac{Qp}{Q-p}}(\Omega)} \leq \|f\|_{KS_{1,p}(\Omega)}, \tag{1.0.7}$$

and, consequently, interpolation allows one to obtain embedding theorems.

The key point in the previous argument is the embedding (1.0.7), which is only known for Q-regular spaces.

The purpose of Chapter 3 will be to obtain a Sobolev type embedding result for Besov spaces defined on a doubling metric space. In our investigation we will avoid the use of interpolation techniques that require the presence of a Sobolev type space. The main idea will be to extend to the metric side the Euclidean oscillation inequality

$$f^{**}(t) - f^*(t) \leq 2^{1/p} \frac{\omega_p(f, t^{1/n})}{t^{1/p}}, \quad t > 0 \; (1 \leq p < \infty).$$

[4] A non-negative Borel function g is an upper gradient of a function $f : \Omega \to \mathbb{R}$ if $|f(y) - f(x)| \leq \int_\gamma g d,s$ for every x and $y \in \Omega$ and every rectifiable path γ in Ω with endpoints x and y (see [46],[27]).

[5] When Ω is a Riemannian manifold this definition yields the usual Sobolev space and the quantity in (3.1.4) is equivalent to the usual semi-quasinorm (see [56]).

(Here $f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds$ (see [55],[57],[59],[63] and the references quoted therein)).

Chapter 3 is organized as follows: Section 3.2 contains basic definitions and technical results on doubling metric measure spaces. In Section 3.3 we obtain pointwise estimates of the oscillation $O_\mu(f,t) = f_\mu^{**}(t) - f_\mu^*(t)$ in terms of the X-modulus of smoothness defined by

$$E_X(f,r) := \left\| \fint_{B(x,r)} |f(x) - f(y)|\, d\mu(y) \right\|_X .$$

Here, X is a rearrangement invariant space[6] on Ω. In Section 4.3 we define generalized Besov type spaces and use the oscillation inequalities obtained in the previous sections to derive embedding Sobolev theorems. In Section 3.5 we deal with generalized uncertainty Sobolev inequalities in the context of Besov spaces. In Section 3.6 a criterion for essential continuity and for the embedding into $BMO\,(\Omega)$ will be obtained. Finally in Section 3.7 we will study in detail the case $\dot{\mathcal{B}}^s_{p,q}(\Omega)$ for $0 < s < 1$, $0 < p < \infty$ and $0 < p \le \infty$.

The results contained in Chapter 3 has been published in Journal of Mathematical Analysis and Applications (see [67]).

In the second part of this memoir (Chapter 4) we will study Sobolev inequalities in metric spaces with convex isoperimetric profile. In order to explain what our main objective will be, we now recall some definitions.

Let (Ω, d, μ) be a connected metric space equipped with a separable Borel probability measure μ. The perimeter or Minkowski content of a Borel set $A \subset \Omega$ is defined by

$$\mu^+(A) = \liminf_{h \to 0} \frac{\mu(A_h) - \mu(A)}{h},$$

where $A_h = \{x \in \Omega : d(x, A) < h\}$ is the open h-neighbourhood of A. The **isoperimetric profile** I_μ is defined as the pointwise maximal function $I_\mu : [0,1] \to [0,\infty)$ such that

$$\mu^+(A) \ge I_\mu(\mu(A))$$

for all Borel sets A. An isoperimetric inequality measures the relation between the boundary measure and the measure of a set, by providing a lower bound on I_μ by some function $I : [0,1] \to [0,\infty)$ which is not identically zero.

The modulus of the gradient of a Lipschitz function f on Ω (briefly $f \in Lip(\Omega)$) is defined by[7]

$$|\nabla f(x)| = \limsup_{d(x,y) \to 0} \frac{|f(x) - f(y)|}{d(x,y)}.$$

[6] I.e. such that if f and g have the same distribution function, then $\|f\|_X = \|g\|_X$ (see Section 2.2 below).

[7] In fact one can define $|\nabla f|$ for functions f that are Lipschitz on every ball in (Ω, d) (cf. [7] for more details).

The equivalence between isoperimetric inequalities and Poincaré inequalities was obtained by Maz'ya. Maz'ya's method (see [16], [62] and [70]) shows that given $X = X(\Omega)$ a rearrangement invariant space[8], the inequality

$$\left\| f - \int_\Omega f d\mu \right\|_X \leq c \left\| |\nabla f| \right\|_{L^1}, \ f \in Lip(\Omega), \tag{1.0.8}$$

holds if, and only if, there exists a constant $c = c(\Omega) > 0$ such that for all Borel sets $A \subset \Omega$

$$\min\left(\phi_X(\mu(A)), \phi_X(1 - \mu(A))\right) \leq c\mu^+(A), \tag{1.0.9}$$

where $\phi_X(t)$ is the fundamental function[9] of X :

$$\phi_X(t) = \|\chi_A\|_X, \text{ with } \mu(A) = t.$$

Motivated by this fact, we will say (Ω, d, μ) admits a concave isoperimetric estimator if there exists a function $I : [0,1] \to [0,\infty)$ which is continuous, concave, increasing on $(0, 1/2)$, symmetric about the point $1/2$, such that $I(0) = 0$ and $I(t) > 0$ on $(0,1)$, and satisfies

$$I_\mu(t) \geq I(t), \ 0 \leq t \leq 1.$$

In recent work, Milman and Martín (see [61], [63]) proved that (Ω, d, μ) admits a concave isoperimetric estimator I if, and only if, the following symmetrization inequality

$$f_\mu^{**}(t) - f_\mu^*(t) \leq \frac{t}{I(t)} |\nabla f|_\mu^{**}(t), \ (f \in Lip(\Omega)) \tag{1.0.10}$$

where $f_\mu^{**}(t) = \frac{1}{t}\int_0^t f_\mu^*(s)ds$, and f_μ^* is the non-increasing rearrangement of f with respect to the measure μ. If we apply a rearrangement invariant function norm X on Ω (see Sections 2.1 and 2.2 below) to (1.0.10), we obtain Sobolev–Poincaré type estimates of the form[10]

$$\left\| \left(f_\mu^{**}(t) - f_\mu^*(t)\right) \frac{I(t)}{t} \right\|_{\bar{X}} \leq \left\| |\nabla f|_\mu^{**} \right\|_{\bar{X}}. \tag{1.0.11}$$

To see how the isoperimetric profile helps to determine the correct spaces, consider the basic model cases (see [64], [65]).

Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain of measure 1, $X = L^p(\Omega)$, $1 \leq p \leq n$, and p^* be the usual Sobolev exponent defined by $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$. Then

$$\left\| \left(f^{**}(t) - f^*(t)\right) \frac{I(t)}{t} \right\|_{L^p} \simeq \left\| \left(f^{**}(t) - f^*(t)\right) \right\|_{L^{p^*,p}}, \tag{1.0.12}$$

[8] i.e. such that if f and g have the same distribution function then $\|f\|_X = \|g\|_X$ (see Section 2.2 below).

[9] We can assume with no loss of generality that ϕ_X is concave (see Section 2.2.1 below).

[10] $\bar{X}$ denotes the representation space of X (see Section 2.2 below).

which follows from the fact that the isoperimetric profile is equivalent to $I(t) = c_n \min(t, 1-t)^{1-1/n}$, and Hardy's inequality (here $L^{p^*,p}$ is a Lorentz space (see Section 2.2 below)). In the case of $\mathbb{R}^n$ with a Gaussian measure γ_n, and let $X = L^p$, $1 \le p < \infty$, then (compare with [23], [34]), since $I_{(\mathbb{R}^n,d,\gamma_n)}(t) \simeq t(\log 1/t)^{1/2}$ for t near zero, we have

$$\left\| \left(f^{**}_{\gamma_n}(t) - f^{*}_{\gamma_n}(t)\right) \frac{I(t)}{t} \right\|_{L^p} \simeq \left\| \left(f^{**}_{\gamma_n}(t) - f^{*}_{\gamma_n}(t)\right) \right\|_{L^p(Log)^{p/2}}, \tag{1.0.13}$$

where $L^p(logL)^{p/2}$ is a Lorentz–Zygmund space (see Section 2.2).

In this fashion, in [61], [63], [64] and [65], Milman and Martín were able to provide a unified framework for studying the classical Sobolev inequalities and logarithmic Sobolev inequalities. Moreover, the embeddings (1.0.11) turn out to be the best possible in all the classical cases. However, the method used in the proof of these results cannot be applied to probability measures with heavy tails, since the isoperimetric estimators of such measures are convex, which means there exists a function $I : [0,1] \to [0,\infty)$ which is continuous, convex, increasing on $(0, 1/2)$, symmetric about the point $1/2$, such that $I(0) = 0$ and $I(t) > 0$ on $(0,1)$, and satisfying

$$I_\mu(t) \ge I(t), \;\; 0 \le t \le 1.$$

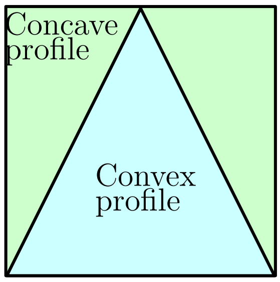

Figure 1.1: Isoperimetric profile

Therefore (unless $I(t) \simeq \min(t, 1-t)$), the Poincaré inequality

$$\left\| f - \int_\Omega f d\mu \right\|_{L^1} \le c \, \| |\nabla f| \|_{L^1}, f \in Lip(\Omega),$$

never holds, which means that we cannot deduce from $|\nabla f| \in L^1$ that $f \in L^1$. Hence, a symmetrization inequality like (1.0.10) will not be possible, since f^{**}_μ is defined if, and only if, $f \in L^1$.

Chapter 4 is organized as follows. In Section 4.2 we obtain symmetrization inequalities which incorporate in their formulation the isoperimetric convex estimator. In Section 4.3 we use the symmetrization inequalities to derive Sobolev–Poincaré and Nash type inequalities. Finally in Section 4.4 we study in detail

several examples, such as, an α-Cauchy type law (the example 4.1.2), extended p-sub-exponential laws (the example 4.1.3), and n-dimensional weighted Riemannian manifolds that satisfy the $CD(0, N)$ curvature condition with $N < 0$ (the example 4.1.2).
The results contained in that chapter 4 have been submitted for publication (see [68]).

Chapter 2

Preliminaries

In this chapter, we present the basic notation we shall use in the following chapters and briefly review some basic facts from the theory of rearrangement invariant spaces. We refer the reader to [6],[29],[53] or [86] for a complete treatment.

Throughout what follows we will work on a measure space (Ω, μ) with a separable, non-atomic, Borel measure μ. Let $\mathcal{M}(\Omega)$ be the set of all extended real-valued measurable functions on Ω. By $\mathcal{M}_0(\Omega)$ we denote the class of functions in $\mathcal{M}(\Omega)$ that are finite μ-a.e.

As usual, if $E \subset \Omega$ is μ-measurable, then, for $1 \leq p < \infty$, $L^p(E)$ is the space of μ-measurable functions f such that the norm $\|f\|_{L^p(A)} = \left(\int_A |f|^p d\mu\right)^{1/p}$ is finite. We define $L^\infty(E)$ similarly, but using $\|f\|_{L^\infty(A)} = ess\sup_A |f|$. $L^p_{loc}(\Omega)$ will denote functions that are p-integrable on balls.

The symbol $f \simeq g$ will indicate the existence of a universal constant $c > 0$ (independent of all parameters involved), thus $c^{-1} f \leq g \leq cf$, while $f \preceq g$ means that $f \leq cg$.

2.1 Decreasing rearrangement

The **distribution function** μ_f of a function f in $\mathcal{M}_0(\Omega)$ is defined by

$$\mu_f(t) = \mu\{x \in \Omega : f(x) > t\} \quad (t \in \mathbb{R}).$$

In the literature it is common to denote the distribution function of $|f|$ by μ_f, while here it is denoted by $\mu_{|f|}$ since we need to distinguish between the distribution function of f and that of $|f|$.

Two functions f and $g \in \mathcal{M}_0(\Omega)$ are said to be **equimeasurable** if $\mu_{|g|}(t) = \mu_{|f|}(t)$ for $t \geq 0$.

The **signed decreasing rearrangement** of a function $f \in \mathcal{M}_0(\Omega)$ $f^{\star}_{\mu} : [0, \mu(\Omega)) \to \mathbb{R}$ is defined by

$$f^{\star}_{\mu}(t) = \inf\left\{s \in \mathbb{R} : \mu\{x \in \Omega : \mu_f(x) > s\} \le t\right\}, \quad t \in [0, \mu(\Omega)).$$

It follows readily from the definition that f^{*}_{μ} is decreasing and that

$$(f+g)^{\star}_{\mu}(t) \le f^{\star}_{\mu}\left(\frac{t}{2}\right) + g^{\star}_{\mu}\left(\frac{t}{2}\right), \quad (t > 0).$$

Moreover,

$$f^{\star}_{\mu}(0^+) = ess\sup f \quad \text{and} \quad f^{\star}_{\mu}(\infty) = ess\inf f. \tag{2.1.1}$$

The **decreasing rearrangement** f^{*}_{μ} of f is given by

$$f^{*}_{\mu}(t) = |f|^{\star}_{\mu}(t).$$

In the next proposition we establish some basic properties of the decreasing rearrangement.

Proposition 2.1.1. *Let f, g, f_i $(i = 1, 2, \dots,)$ belong to $\mathcal{M}_0(\Omega)$ and $\alpha \in \mathbb{R}$. Then*

(i.) $f^{\star}_{\mu}(\mu_f(t)) \le t$ *for all* $t \ge 0$ *with* $\mu_f(t) < \infty$*;*

(ii.) $\mu_f(f^{\star}_{\mu}(t)) \le t$ *for all* $t \ge 0$ *with* $f^{\star}_{\mu}(t) < \infty$*;*

(iii.) If $f \le g$*, then* $f^{\star}_{\mu} \le g^{\star}_{\mu}$*;*

(iv.) $(\alpha f)^{\star}_{\mu} = \alpha f^{\star}_{\mu}$ *and* $(f+\alpha)^{\star}_{\mu} = f^{\star}_{\mu}(t) + \alpha$*;*

(v.) If $|f_i| \uparrow |f|$*, then* $(f_i)^{*}_{\mu} \uparrow f^{*}_{\mu}$*;*

(vi.) f^{*}_{μ} *is right continuous;*

(vii.) $f^{*}_{\mu}(s) = m_{\mu_{|f|}}(s)$*,* $t \ge 0$ *(where* m *denotes Lebesgue measure on* $(0, \mu(\Omega))$*;*

(viii.) $f^{\star}_{\mu}$ *and* f^{*}_{μ} *are equimeasurable with respect to Lebesgue measure on* $(0, \mu(\Omega))$*.*

Example 2.1.1. *Let f be a positive simple function, i.e.*

$$f(x) = \sum_{j=1}^{n} b_j \chi_{F_j}(x),$$

where the coefficients b_j are positive and $F_j = \{x \in \Omega : f(x) = b_j\}$.

The distribution function is given by

$$\mu_f(\lambda) = \sum_{j=1}^{n} m_j \chi_{[b_{j+1}, b_j]}(\lambda),$$

where $m_j = \sum_{i=1}^{j} \mu(F_i)$, $(j = 1, 2, \dots, n)$ and $b_{n+1} = 0$ (see Figure 2.1). The decreasing rearrangement is given by

$$f^{*}_{\mu}(t) = \sum_{j=1}^{n} b_j \chi_{[0, m_j)}(t).$$

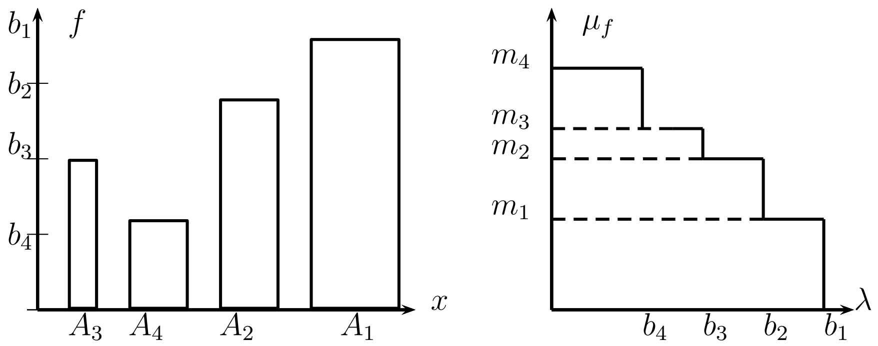

Figure 2.1: Graphs of f and μ_f

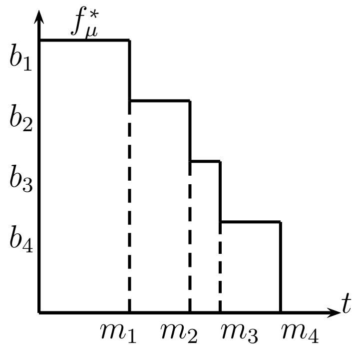

Figure 2.2: Graph of f_μ^*

Example 2.1.2. *This example shows how signed rearrangement works:*

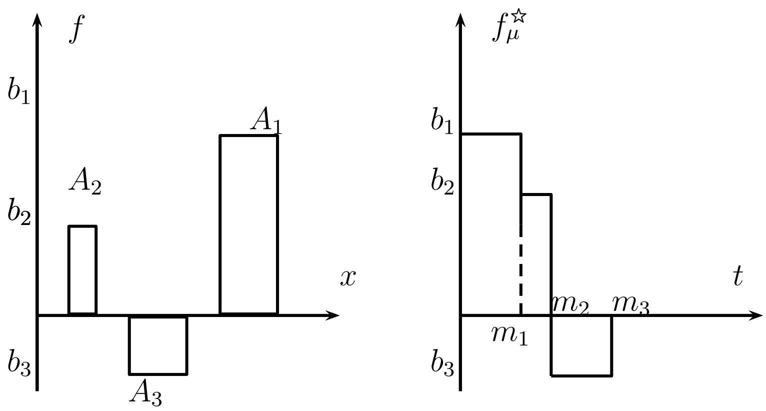

Figure 2.3: Graph of $f_\mu^{\star}$

For any measurable set $E \subset \Omega$ and $f \in \mathcal{M}_0(\Omega)$,

$$\int_E |f(x)| d\mu \leq \int_0^{\mu(E)} f_\mu^*(s) ds. \tag{2.1.2}$$

In fact,

$$\sup_{\mu(E)=t} \int_E |f(x)| d\mu = \int_0^t f_\mu^*(s) ds \tag{2.1.3}$$

and

$$\int_0^t f_\mu^{\star}(s) ds = \sup \left\{ \int_E f(s) d\mu : \mu(E) = t \right\}, \quad (t > 0). \tag{2.1.4}$$

The **signed maximal function** $f_\mu^{\star\star}$ is defined by

$$f_\mu^{\star\star} = \frac{1}{t} \int_0^t f_\mu^{\star}(s) ds, \quad (t > 0).$$

Similarly, f_μ^{**} will denote the **maximal function** of f_μ^* defined by

$$f_\mu^{**}(t) = \frac{1}{t} \int_0^t f_\mu^*(s) ds, \quad (t > 0).$$

Some elementary properties of the maximal signed function are listed below.

Proposition 2.1.2. *Let f, g and f_i $(i = 1, 2, \dots,)$ belong to $\mathcal{M}_0(\Omega)$ and $\alpha \in \mathbb{R}$. Then*

(i.) $f_\mu^{\star} \leq f_\mu^{\star\star}$;

(ii.) *If $f \leq g$, then* $f_\mu^{\star\star} \leq g_\mu^{\star\star}$;

(iii.) $(\alpha f)_\mu^{\star\star} = \alpha f_\mu^{\star\star}$;

(iv.) $(f+g)_\mu^{\star\star}(t) \leq f_\mu^{\star\star}(t) + g_\mu^{\star\star}(t), \ (t > 0)$.

Example 2.1.3. *Let $\Omega = [0, \infty)$ and μ be Lebesgue measure on Ω. Define $f : [0, \infty) \to [0, \infty)$ by*

$$f(x) = \begin{cases} 1 - (x-1)^2 & if \ \ 0 \leq x \leq 2 \\ 0 & if \ \ x > 2. \end{cases}$$

The distribution function can be easily computed:

$$\mu_f(\lambda) = \begin{cases} 2\sqrt{1-\lambda} & if \ \ 0 \leq \lambda \leq 1 \\ 0 & if \ \ \lambda > 1, \end{cases}$$

and the decreasing rearrangement becomes

$$f_\mu^*(t) = \begin{cases} 1 - \frac{t^2}{4} & if \;\; 0 \le t \le 2. \\ 0 & if \;\; t > 2, \end{cases}$$

Moreover,

$$\int_0^\infty f(x)dx = \int_0^2 1 - (1 - x^2)dx = \int_0^1 2\sqrt{1-\lambda}d\lambda = \int_0^2 1 - \frac{t^2}{4}dt = \frac{4}{3}.$$

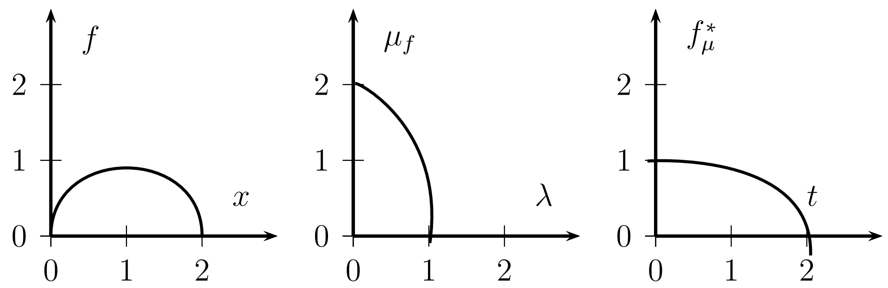

Figure 2.4: Graph f, μ_f, f_μ^*

The maximal function is given by

$$f_\mu^{**}(t) = \begin{cases} 1 - \frac{t^2}{12} & if \;\; 0 < t \le 2 \\ \frac{4}{3t} & if \;\; t > 2 \end{cases}$$

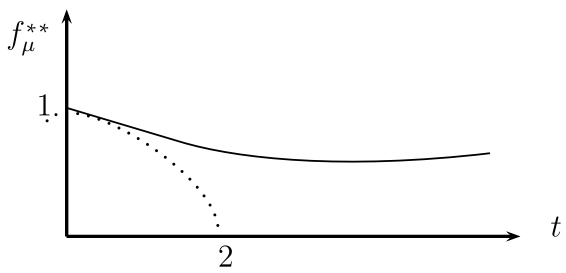

Figure 2.5: Graph f_μ^{**}

Definition 2.1.1. *Let f belong to $\mathcal{M}_0(\Omega)$. The oscillation of f_μ^* is defined by the special difference*

$$O_\mu(f,t) = f_\mu^{**}(t) - f_\mu^*(t).$$

The functional $O_\mu(f,t)$ has certain technical disadvantages. It vanishes on constant functions and the operation $f \to O_\mu(f,t)$ is not subadditive.

Lemma 2.1.1. *Let f belong to $\mathcal{M}_0(\Omega)$. Then*

$$\frac{\partial}{\partial t} f_\mu^{**}(t) = -\frac{O_\mu(f,t)}{t}, \quad t > 0, \tag{2.1.5}$$

and the function $tO_\mu(f,t)$ is increasing in t.

Proof. By the definition of f_μ^{**}, and a simple computation, we get

$$\begin{aligned}
\frac{\partial}{\partial t} f_\mu^{**}(t) &= \frac{\partial}{\partial t}\left(\frac{1}{t}\int_0^t f_\mu^*(s)\right) ds \\
&= -\frac{1}{t^2}\int_0^t f_\mu^*(s)ds + \frac{1}{t} f_\mu^*(t) \\
&= -\frac{1}{t}\left(\frac{1}{t}\int_0^t f_\mu^*(s)ds - f_\mu^*(t)\right) \\
&= -\frac{1}{t}\left(f_\mu^{**}(t) - f_\mu^*(t)\right).
\end{aligned}$$

Using the fact that (see [14])

$$O_\mu(f,t) = \frac{1}{t}\int_{f_\mu^*(t)}^{\|f\|_\infty} \mu_{|f|}(s)ds, \tag{2.1.6}$$

it follows that $tO_\mu(f,t)$ is increasing. Indeed, to see 2.1.6, let $[x]^+ = max(x,0)$. Then, for all $y > 0$, we have that

$$\int_0^\infty [f_\mu^*(x) - y]^+ dx = \int_0^\infty \mu_{[f_\mu^* - y]^+}(s)ds = \int_y^\infty \mu_{f_\mu^*}(s)ds = \int_y^{\|f\|_\infty} \mu_{|f|}(s)ds. \tag{2.1.7}$$

Inserting $y = f_\mu^*(t)$ in 2.1.7 and taking into account that f_μ^* is decreasing, we get

$$\begin{aligned}
tO_\mu(f,t) &= t(f_\mu^{**}(t) - f_\mu^*(t)) \\
&= \int_0^t (f_\mu^*(x) - f_\mu^*(t))dx \\
&= \int_0^\infty [f_\mu^*(x) - f_\mu^*(t)]^+ dx \\
&= \int_{f_\mu^*(x)}^{\|f\|_\infty} \mu_{|f|}(s)ds.
\end{aligned}$$

□

Conditions like $f_\mu^*(\infty) = 0$ will appear often. The following proposition clarifies the significance of such conditions.

Proposition 2.1.3 (See [57])**.** *If $\mu(\Omega) = \infty$, then $f_\mu^*(\infty) = 0$ if, and only if, $\mu_f(t)$ is finite for any $t > 0$*

Proof. Suppose that $\mu_f(t_0) = \infty$ for some $t_0 > 0$. From the definition of rearrangement, we have that $f_\mu^*(t) \geq t_0$ for all $t > 0$.
Therefore the condition $f_\mu^*(\infty) = 0$ implies $\mu_f(t) < \infty$, for all $t > 0$.
Conversely, assume $f_\mu^*(t) \geq \varepsilon > 0$. This means that $\mu_f(\varepsilon) = \infty$. Thus the condition $\mu_f(t) < \infty$, $t > 0$ implies $f_\mu^*(\infty) = 0$. □

Note that an application of L'Hôpital's rule to f_μ^{**} shows that the condition $f_\mu^*(\infty) = 0$ is equivalent to $f_\mu^{**}(\infty) = 0$.

Remark 2.1.1. *By (2.1.5) and the Fundamental Theorem of Calculus, and using $f_\mu^{**}(\infty) = 0$, we have*

$$f_\mu^{**}(t) = \int_t^\infty \frac{O_\mu(f,s)}{s} ds, \quad t > 0.$$

2.2 Rearrangement invariant spaces

Rearrangement invariant spaces play an important role in contemporary mathematics. They have many applications in various branches of analysis, including the theory of function spaces, interpolation theory, mathematical physics, and probability theory.

Definition 2.2.1. *A function space $X(\Omega)$ is the linear space of all $f \in \mathcal{M}_0(\Omega)$ for which $\|f\|_{X(\Omega)} < \infty$, where $\|\cdot\|_{X(\Omega)}$ is a functional norm, i.e.*

(i.) $\|\cdot\|_{X(\Omega)}$ *is a norm;*

(ii.) if $0 < g \leq f$ *a.e., then* $\|g\|_{X(\Omega)} \leq \|f\|_{X(\Omega)}$;

(iii.) if $0 < f_j \uparrow f$ *a.e., then* $\|f_j\|_{X(\Omega)} \uparrow \|f\|_{X(\Omega)}$;

(iv.) for any measurable set $E \subset \Omega$, $\|\chi_E\|_{X(\Omega)} < \infty$; *and*

(v.) $\int_E |f(x)| dx \leq \|f\|_{X(\Omega)}$.

If, in the definition of a norm, the triangle inequality is weakened to the requirement that for some constant C_X where

$$||x + y||_X \leq C_X(||x||_X + ||y||_X)$$

holds for all x and y, then we have a quasi-norm. A complete quasi-normed space is called a quasi-Banach space.

Definition 2.2.2. *A Banach function space (resp. quasi-Banach function space) $X = X(\Omega)$ is called a rearrangement invariant (r.i.) space (resp. quasi-Banach rearrangement invariant (q.r.i.) space) if $g \in X$ implies that all μ-measurable functions f with the same rearrangement function with respect to the measure μ, i.e. such that $f^*_\mu = g^*_\mu$, also belong to X, and moreover $\|f\|_X = \|g\|_X$.*

For any r.i. space $X(\Omega)$ we have

$$L^1_\mu(\Omega) \cap L^\infty_\mu(\Omega) \subset X(\Omega) \subset L^1_\mu(\Omega) + L^\infty_\mu(\Omega), \tag{2.2.1}$$

with continuous embedding, where the space $L^1_\mu(\Omega) + L^\infty_\mu(\Omega)$ consist of all functions $f \in \mathcal{M}_0(\Omega)$ that are representable as a sum $f = g + h$ of functions $g \in L^1_\mu(\Omega)$ and $h \in L^\infty_\mu(\Omega)$. The norm in $L^1_\mu(\Omega) + L^\infty_\mu(\Omega)$ is given by

$$||f||_{L^1_\mu(\Omega)+L^\infty_\mu(\Omega)} = \inf\{||g||_{L^1_\mu(\Omega)} + ||h||_{L^\infty_\mu(\Omega)}\},$$

where the infimum is taken over all representations $f = g + h$ of the kind described above.

Consider the following norm on $L^1_\mu(\Omega) \cap L^\infty_\mu(\Omega)$:

$$||f||_{L^1_\mu(\Omega)\cap L^\infty_\mu(\Omega)} = \max\{||g||_{L^1_\mu(\Omega)}, ||h||_{L^\infty_\mu(\Omega)}\}.$$

If $\mu(\Omega) < \infty$, we obviously have

$$L^\infty_\mu(\Omega) \subset X(\Omega) \subset L^1_\mu(\Omega).$$

The **associate space** $X'(\Omega)$ of $X(\Omega)$ is the r.i. space of all $h \in \mathcal{M}_0(\Omega)$ for which the r.i. norm given by

$$\|h\|_{X'(\Omega)} = \sup_{||g||\le 1} \int_\Omega |g(x)h(x)|\, d\mu \tag{2.2.2}$$

is finite.

Therefore the following generalized Hölder inequality holds

$$\int_\Omega |g(x)h(x)|\, d\mu \le \|g\|_{X(\Omega)} \|h\|_{X'(\Omega)}.$$

A useful property states that if

$$\int_0^r f^*_\mu(s) ds \le \int_0^r g^*_\mu(s) ds,$$

for all $r > 0$, then for any r.i. space $X = X(\Omega)$ we have

$$\|f\|_X \le \|g\|_X.$$

Let $X(\Omega)$ be an r.i. space. Then there exists a unique r.i. space (see [6, Theorem 4.10 and subsequent remarks]) $\bar{X} = \bar{X}(0,\mu(\Omega))$ on $((0,\mu(\Omega)),m)$, (m denotes the Lebesgue measure on the interval $(0,\mu(\Omega))$), such that

$$f \in X(\Omega) \Leftrightarrow f_\mu^* \in \bar{X}(0,\mu(\Omega)),$$

and furthermore,

$$\|f\|_{X(\Omega)} = \|f_\mu^*\|_{\bar{X}(0,\mu(\Omega))} = \|f_\mu{}^{\star\star}\|_{\bar{X}(0,\mu(\Omega))}.$$

$\bar{X}$ is called the **representation space** of $X(\Omega)$. The norm of $\bar{X}(0,\mu(\Omega))$ is given explicitly by

$$\|h\|_{\bar{X}(0,\mu(\Omega))} = \sup\left\{\int_0^{\mu(\Omega)} h^*(s)g_\mu^*(s)ds;\quad \|g\|_{X'(\Omega)} \leq 1\right\},$$

where the first rearrangement is taken with respect to the Lebesgue measure on $[0,\mu(\Omega))$.

2.2.1 Indices

We will present some definitions and properties related to indices (see [10] and [107]).

The dilation operator is defined by

$$D_{\frac{1}{s}}f(t) = \begin{cases} f_\mu^*\left(\frac{t}{s}\right) & 0 < t < s, \\ 0 & s < t < \mu(\Omega). \end{cases}$$

For each $t > 0$, we denote by $h_X(s)$ the norm of $D_{\frac{1}{s}}$, i.e.

$$h_X(s) = \sup_{f \in X} \frac{\left\|D_{\frac{1}{s}}f\right\|_X}{\|f\|_X},\ s > 0.$$

The upper and lower Boyd indices associated with an r.i. space X are defined by

$$\overline{\alpha}_X = \inf_{s>1} \frac{\ln h_X(s)}{\ln s} \quad \text{and} \quad \underline{\alpha}_X = \sup_{s<1} \frac{\ln h_X(s)}{\ln s}. \tag{2.2.3}$$

The **fundamental function** of X an r.i. space is defined by

$$\phi_X(s) = \|\chi_E\|_X,$$

where E is any measurable subset of Ω with $\mu(E) = t$.

Note that the particular choice of E is immaterial, due to the rearrangement invariance of X. We can assume without loss of generality that ϕ_X is concave (see [6]). Moreover, by Hölder's inequality,

$$\phi_{X'}(t)\phi_X(t) = t. \tag{2.2.4}$$

It is also useful sometimes to consider a slightly different set of indices obtained by means of replacing $h_X(s)$ in (2.2.3) by

$$M_X(s) = \sup_{t>0} \frac{\phi_X(ts)}{\phi_X(t)}, \; s > 0.$$

The corresponding indices are denoted by $\overline{\beta}_X$, $\underline{\beta}_X$, and will be referred to as the upper and lower fundamental indices of X. Actually, the relationship between $M_X(s)$ and $h_X(s)$ is that the computation of the former is exactly the computation of the latter but done only over functions of the form $f = \chi_{(0,a)}$. Therefore we have

$$0 \le \underline{\alpha}_X \le \underline{\beta}_X \le \overline{\beta}_X \le \overline{\alpha}_X \le 1.$$

Lemma 2.2.1. *(See [60] and [107]) Let Y be an r.i. space on $(0,1)$. Let ϕ_Y be its fundamental function. Assume that $\phi_Y(0) = 0$. Then*

1. *If $\overline{\alpha}_Y < 1$, then for every $\overline{\alpha}_Y < \gamma < 1$, the function $\phi_Y(s)/s^\gamma$ is almost decreasing (i.e. $\exists c > 0$ s.t. $\phi_Y(s)/s^\gamma \le c\phi_Y(t)/t^\gamma$ whenever $t \le s$).*

2. *If $\underline{\alpha}_Y > 0$, then for every $0 < \gamma < \underline{\alpha}_Y$, the function $\phi_Y(s)/s^\gamma$ is almost increasing (i.e. $\exists c > 0$ s.t. $\phi_Y(s)/s^\gamma \le c\phi_Y(t)/t^\gamma$ whenever $t \ge s$).*

3. *If $\underline{\alpha}_Y > 0$, there exists a concave function $\hat{\phi}_Y$ and constant $c > 0$ such that*

$$\hat{\phi}_Y(t) \simeq \phi_Y(t) \quad \text{and} \quad c^{-1}\phi_Y(t)/t \le \frac{\partial}{\partial t}\hat{\phi}_Y(t) \le c\phi_Y(t)/t.$$

We shall usually formulate conditions on r.i. spaces in terms of the Hardy operators defined by

$$(P_a f)(t) = t^{-a}\int_0^t x^a f(x)\frac{dx}{x}; \qquad (Q_a f)(t) = t^{-a}\int_t^{\mu(\Omega)} x^a f(x)\frac{dx}{x}$$

for $t \in (0, \mu(\Omega))$ and $a \in (0,1]$, $f \in \mathcal{M}_0((0,\mu(\Omega)))$.

It is known that X is an r.i. (resp. q.r.i) space, that P_a is bounded if, and only if, $\overline{\alpha}_X < a$, and that Q_a is bounded if, and only if, $a < \underline{\alpha}_X$ (see [6]).

2.2.2 Examples

In this section we present some examples of rearrangement invariant spaces.

- The L^p_μ**-spaces** consist of all $f \in \mathcal{M}_0(\Omega)$ for which

$$\|f\|_{L^p_\mu} := \begin{cases} \left(\int_0^{\mu(\Omega)} \left(f^*_\mu(t)\right)^p dt\right)^{1/p}, & 0 < p < \infty, \\ \sup\limits_{t>0} f^*_\mu(t), & p = \infty, \end{cases}$$

  is finite. Its fundamental function is given by

$$\phi_{L^p_\mu}(t) = \begin{cases} t^{1/p}, & 1 \le p \le \infty, \quad t > 0 \\ 0, & p = \infty, \quad t = 0 \\ 1, & p = \infty, \quad t > 0. \end{cases}$$

  L^p_μ is q.r.i. if $0 < p < 1$ and r.i. if $1 \le p < \infty$.
  The Boyd indices are given by $\underline{\alpha}_{L^p_\mu} = \overline{\alpha}_{L^p_\mu} = \frac{1}{p}$.

- Assume that $0 < p, q \le \infty$. The **Lorentz spaces** $L^{p,q}_\mu$ consist of all $f \in \mathcal{M}_0(\Omega)$ for which

$$\|f\|_{L^{p,q}_\mu} := \begin{cases} \left(\int_0^{\mu(\Omega)} \left(t^{1/p} f^*_\mu(t)\right)^q dt\right)^{1/q}, & 0 < q < \infty, \\ \sup\limits_{t>0} \left\{t^{1/p} f^*_\mu(t)\right\}, & q = \infty, \end{cases}$$

  is finite. Its fundamental function is given by

$$\phi_{L^{p,q}_\mu}(t) = t^{1/p}, \quad t > 0,$$

  where $1 \le p < \infty$. The Boyd indices are given by $\underline{\alpha}_{L^{p,q}_\mu} = \overline{\alpha}_{L^{p,q}_\mu} = \frac{1}{p}$.

- **Lorentz Λ spaces** are defined by the functional

$$\|f\|_{\Lambda^q(v)} = \left(\int_0^{\mu(\Omega)} f^*_\mu(s)^q v(s) ds\right)^{1/q},$$

  where $0 < q < \infty$ and v is a weight decreasing ($v \ge 0$ measurable function) on $(0, \mu(\Omega))$.
  By choosing $v(s) = s^{q/p-1}$ one obtains $\Lambda^q(v) = L^{p,q}_\mu$.
  If we take $v(s) = s^{q/p-1}(1 + \log\frac{1}{s})^\alpha$, then the $\Lambda^q(v) = L^{p,q}_\mu(\log L)^\alpha$ are the **Lorentz–Zygmund spaces**; or if we take $v(s) = s^{q/p1}(1+\log\frac{1}{s})^\alpha(1+\log\frac{1}{s})^\beta$, then the $\Lambda^q(v) = L^{p,q}_\mu(\log L)^\alpha(\log\log L)^\beta$ are the generalized Lorentz–Zygmund spaces.

Definition 2.2.3. *A function* $A\colon [0,\infty) \to [0,\infty]$ *is a Young function if*

$$A(s) = \int_0^s a(t)\ dt, \tag{2.2.5}$$

where $a : [0,\infty] \to [0,\infty]$ *is an increasing, left continuous function which is neither identically zero nor identically infinite on* $(0,\infty)$ *and which satisfies* $a(0) = 0$. *A Young function is convex on the interval where it is finite.*

- For a Young function A, the **Orlicz space** L^A_μ is the collection of all functions $f \in \mathcal{M}_0(\Omega)$ for which there exists a λ such that

  $$\int_\Omega A\left(\frac{|f(x)|}{\lambda}\right) d\mu < \infty.$$

  The Orlicz space L^A_μ is endowed with the Luxemburg norm

  $$\|f\|_{L^A_\mu} = \inf\left\{\lambda > 0; \int_\Omega A\left(\frac{|f(x)|}{\lambda}\right) d\mu \le 1\right\}.$$

  The fundamental function for L^A_μ is given by

  $$\phi_{L^A_\mu}(t) = 1/A^{-1}(1/t), \tag{2.2.6}$$

  where A is a Young function on $(0,\infty)$.
  On the other hand (see [6]), we have

  $$\int_\Omega A\left(|f(x)|\right) d\mu = \int_0^{\mu(\Omega)} A(f^*_\mu)(t)dt.$$

Given an r.i. space X over $(0,\mu(\Omega))$, suppose X has been renormed so as to have a concave fundamental function ϕ_X. There are some useful **Lorentz** and **Marcinkiewicz** spaces, defined by the quasi-norms

$$\|f\|_{M(X)} = \sup_t f^{**}_\mu(t)\phi_X(t), \quad \|f\|_{\Lambda(X)} = \int_0^{\mu(\Omega)} f^*_\mu(t) d\phi_X(t). \tag{2.2.7}$$

It follows readily that

$$\phi_{M(X)}(t) = \phi_{\Lambda(X)}(t) = \phi_X(t).$$

Furthermore

$$\Lambda(X) \subset X \subset M(X), \tag{2.2.8}$$

and each of the embeddings has norm 1.

Chapter 3

A Sobolev type embedding theorem for Besov spaces defined on doubling metric spaces

3.1 Introduction

Analysis on metric measure spaces has been studied quite intensively in recent years; see, for example, Semmesś survey [87] for a more detailed discussion and references. A field of particular interest is the study of functional inequalities, like Sobolev and Poincaré inequalities on metric measure spaces; see, for example, [56],[36],[46], [32],[33],[54]. Since Hajłasz in [35] introduced Sobolev spaces on any metric measure space, a series of papers has been devoted to the construction and investigation of Sobolev spaces of various types on metric measure spaces; see, for example, [36],[46],[37],[24]. Recently, a theory of Besov spaces was developed in [38] which is a generalization of the corresponding theory of function spaces on $\mathbb{R}^n$ (see [94],[95],[96]) respectively Ahlfors n-regular metric measure spaces (see [39],[41]).

There are several equivalent ways to define Besov spaces in the setting of a doubling metric space (see for example [27], [28],[38],[75],[74],[105],[45] and the references therein), in this chapter, we shall use the approach based on a generalization of the classical L^p-modulus of smoothness introduced in [27].

Recall that the L^p-modulus of smoothness of a function $f \in L^p_{loc}(\mathbb{R}^n)$ is defined by

$$\omega_p(f,t) := \sup_{|h|\le t} \|f(x+h) - f(x)\|_{L^p(\mathbb{R}^n)},$$

where $t > 0$ and $|h|$ is the Euclidean length of the vector h. As a general metric space possesses no group structure, a modification to this definition is needed.

Let (Ω, d, μ) be a metric measure space equipped with a metric d and a Borel regular outer measure μ for which the measure of every ball is positive and finite. Given $t > 0$, $0 < p < \infty$ and $f \in L^p_{loc}(\Omega)$, the L^p-modulus of smoothness is defined by

$$E_p(f,t) = \left(\int_\Omega \left(\fint_{B(x,t)} |f(x) - f(y)|^p \, d\mu(y) \right) d\mu(x) \right)^{1/p}, \tag{3.1.1}$$

where $\fint_B f(x) d\mu(x) := \frac{1}{\mu(B)} \int_B f(x) d\mu(x)$ is the integral average of a locally integrable function f over B.

$E_p(f,t)$ is equivalent to the classical $L^p(\mathbb{R}^n)$-modulus of smoothness of a function $f \in L^p_{loc}(\mathbb{R}^n)$. Indeed (see [27]),

$$\begin{aligned} E_p(f,t) &= \left(\int_{\mathbb{R}^n} \left(\fint_{B(x,t)} |f(x) - f(y)|^p \, dy \right) dx \right)^{1/p} \qquad (3.1.2)\\ &= \left(\int_{\mathbb{R}^n} \left(\fint_{B(0,t)} |f(x+h) - f(x)|^p \, dh \right) dx \right)^{1/p} \\ &\simeq \sup_{|h| \le t} \|f(x+h) - f(x)\|_{L^p(\mathbb{R}^n)} := \omega_p(f,t), \quad \text{(see [55])}. \end{aligned}$$

Definition 3.1.1. *For $0 < s < \infty$, the homogeneous Besov space $\dot{\mathcal{B}}^s_{p,q}(\Omega)$ consists of those functions $f \in L^p_{loc}(\Omega)$ for which the seminorm*

$$\|f\|_{\dot{\mathcal{B}}^s_{p,q}(\Omega)} := \begin{cases} \left(\int_0^\infty \left(\frac{E_p(f,t)}{t^s} \right)^q \frac{dt}{t} \right)^{1/q}, & 0 < q < \infty, \\ \sup\limits_{t>0} t^{-s} E_p(f,t), & q = \infty, \end{cases}$$

is finite.

This definition is rather concrete and by (3.1.2) gives the usual Besov space in the Euclidean setting. Moreover, it has been shown by Müller and Yang [74] that it coincides with the definition based on test functions and used earlier by Han [40], Han and Yang [42] and Yang [103], provided that Ω, besides being doubling, also satisfies a reverse doubling condition

In the Euclidean setting, the Sobolev embedding theorem states (see, for example, [74, Theorem. 1.15]) that there is a constant $C > 0$ such that

$$\|f\|_{L^{p^*,q}_\mu(\mathbb{R}^n)} \le C \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)},$$

where $p^* = np/(n - sp)$, and the Lorentz space $L^{p,q}(\mathbb{R}^n)$, consist of measurable functions f of finite norm

$$\|f\|_{L^{p,q}_\mu(\mathbb{R}^n)} = \left\| t^{\frac{1}{p} - \frac{1}{q}} f^*_\mu(t) \right\|_{L^q([0,\infty))},$$

(f_μ^* denotes the decreasing rearrangement of f, see Section 2.1).

The abstract variant for metric spaces is only known in the following particular case (see [27] and [45]):

Theorem 2. *Let Ω be a Q-regular metric space, i.e. there exists a $Q \geq 1$ and a constant $c_Q \geq 1$ such that*

$$c_Q^{-1} r^Q \leq \mu(B(x,r)) \leq c_Q r^Q$$

for each $x \in X$, and for all $0 < r < diam\ \Omega$ (here $diam\ \Omega$ is the diameter of Ω). Suppose that $0 < s < 1$ and $1 \leq q \leq \infty$. Then:

1. *(See ([27, Thm. 5.1])) Suppose Ω satisfies a $(1,p)$-Poincare inequality, i.e. there exist constants $C_p \geq 0$ and $\lambda \geq 1$ such that*

$$\fint_B |f - f_B| d\mu \leq \left(\fint_{\lambda B} g^p d\mu \right)^{1/p}$$

for any locally integrable function f for all upper gradients[1] g of f. Then, if $1 < p < Q/s$,

$$\dot{\mathcal{B}}^s_{p,q}(\Omega) \subset L_\mu^{p(Q),q}(\Omega), \tag{3.1.3}$$

where $p(Q) = Qp/(Q - sp)$.

2. *(See ([45, Thm. 4.4])) Suppose that Ω is geodesic, i.e. every pair of points can be joined by a curve whose length is equal to the distance between the points. Then (3.1.3) holds for $1 \leq p < Q/s$.*

The proof of this theorem is based on the real interpolation method; for example, in ([27, Thm. 5.1]) under a $(1,p)$-Poincaré inequality assumption, the Besov space $\dot{\mathcal{B}}^s_{p,q}(\Omega)$ is realized as the real interpolation space $(L^p(\Omega), KS_{1,p}(\Omega))_{\alpha,q}$ between the corresponding $L^p(\Omega)$ and the Sobolev space of Korevaar and Schoen $KS_{1,p}(\Omega)$, consisting of measurable functions f of finite norm[2]

$$\|f\|_{KS_{1,p}(\Omega)} := \limsup_{t \to 0} \frac{E_p(f,t)}{t}. \tag{3.1.4}$$

In [27] it is proved that $E_p(f,t)$ is equivalent to the K-functional between $L^p(\Omega)$ and $KS_{1,p}(\Omega)$. Moreover if Ω is Q-regular, then

$$\|f\|_{L_\mu^{\frac{Qp}{Q-p}}(\Omega)} \leq \|f\|_{KS_{1,p}(\Omega)}, \tag{3.1.5}$$

and, consequently, interpolation allows one to obtain embedding theorems.

[1] For the definition of upper gradient, see, for example [46],[27].

[2] When Ω is a Riemannian manifold, this definition yields the usual Sobolev space and the quantity in (3.1.4) is equivalent to the usual semi-quasinorm (see [56]).

The key point in the previous argument is the embedding (3.1.5), which is only known for Q-regular spaces.

The purpose of this chapter is to obtain a Sobolev type embedding result for Besov spaces defined on a doubling metric space. In Theorem 2 we will see that if the embedding

$$\dot{\mathcal{B}}^{s}_{p,q}(\Omega) \subset L^{p^*,q}_{\mu}(\Omega)$$

holds for some $p^* > p$, then Ω does not have the collapsing volume property, i.e.

$$\inf_{x\in\Omega} \mu(B(x,1)) > 0. \tag{3.1.6}$$

In view of this result, we will need to limit the class of doubling spaces in which we going to work. Our framework will be (k,m)-**spaces** (see Section 3.2 below), i.e. there exist positive constants c_0, C_0, k, and m ($k \le m$) such that

$$c_0 \min(r^k, r^m) V_\mu(x,1) \le V_\mu(x,r) \le C_0 \max(r^k, r^m) V_\mu(x,1), \tag{3.1.7}$$

for all $x \in \Omega$ and $0 < r < \infty$, where $V_\mu(x,r) = \mu(B(x,r))$. To incorporate condition (3.1.6), we define

(i.) A (k,m)-space will be called **uniform** if there are constants $c, C > 0$ such that
$$c \min(r^k, r^m) \le V_\mu(x,r) \le C \max(r^k, r^m). \tag{3.1.8}$$

(ii.) A (k,m)-space will be called **bounded from below** if there are constants $d, D > 0$ such that
$$d \min(r^k, r^m) \le V_\mu(x,r) \le D \max(r^k, r^m) V_\mu(x,1). \tag{3.1.9}$$

In order to avoid using an embedding result like (3.1.5), which, recall, is only known for Q-regular metric spaces, we will obtain, for the above class of spaces, pointwise estimates between the oscillation of f^*_μ and the modulus of smoothness that will allow us to derive Sobolev type embedding results.

The results that we will obtain can be applied in a wide range of settings, for instance, to Ahlfors regular metric measure spaces (see [46]), Lie groups of polynomial volume growth (see [99],[100],[81],[76],[2]), Carnot–Carathéodory manifolds (see [81],[78],[79]), and to the boundaries of certain unbounded model domains of polynomial type in $\mathbb{C}^N$ appearing in the work of Nagel and Stein (see [80],[81],[78],[79]) (see Section 3.2.1 below).

This chapter is organized as follows: Section 3.2 contain basic definitions and technical results on doubling metric measure spaces. In Section 3.3 we obtain pointwise estimates of the oscillation $O_\mu(f,t) = f^{**}_\mu(t) - f^*_\mu(t)$ in terms of the X-modulus of smoothness defined by

$$E_X(f,r) := \left\| \fint_{B(x,r)} |f(x) - f(y)| \, d\mu(y) \right\|_X,$$

where X is a rearrangement invariant space on Ω. In Section 4.3 we define generalized Besov type spaces and use the oscillation inequalities obtained in the previous sections to derive embedding Sobolev theorems. In Section 3.5 we deal with generalized uncertainty Sobolev inequalities in the context of Besov spaces. In Section 3.6 a criterion for essential continuity and for the embedding into $BMO(\Omega)$ will be obtained. Finally, in Section 3.7 we will study in detail the case $\dot{\mathcal{B}}^s_{p,q}(\Omega)$ for $0 < s < 1$, $0 < p < \infty$ and $0 < p \leq \infty$.

The results contained in this chapter have been published in Journal of Mathematical Analysis and Applications (see [67]).

3.2 Doubling measures

In this section we briefly review basic facts about doubling metric spaces and some of their properties. The general framework will be a metric measure space (Ω, d, μ) with a metric d and a regular Borel measure μ for which the measure of every ball is positive and finite.

The ball with centre $x \in \Omega$ and radius $r > 0$ is defined by $B(x,r) = \{y \in \Omega : d(x,y) < r\}$. We shall denote by $V_\mu(x,r)$ the measure of the ball, i.e.

$$V_\mu(x,r) := \mu(B(x,r)).$$

Definition 3.2.1. *A metric measure space (Ω, d, μ) is called doubling if there exists a constant $C_D > 1$ such that for all $x \in \Omega$ and $r > 0$,*

$$0 < V_\mu(x,2r) \leq C_D V_\mu(x,r) < \infty. \tag{3.2.1}$$

We will now present some examples.

Example 3.2.1.

- *Let $\Omega = \mathbb{R}^n$, let $d(x,y) = |x - y|$ be the Euclidean metric, and let $\mu = \mathcal{L}^n$ be Lebesgue measure on Ω. Then $(\mathbb{R}^n, |.|, \mathcal{L}^n)$ is a doubling metric measure space with $C_D = 2^n$ (see [3],[46]).*
- *Let $\Omega = [-1,0] \times [-1,-1] \cup [0,1] \times \{0\}$, let d be the Euclidean metric, and let $\mu = \mathcal{L}^2|_\Omega + \mathcal{H}^1|_{[0,1]\times\{0\}}$, where $\mathcal{H}^1$ is the 1-dimensional Hausdorff measure. Then μ is doubling with $C_D = 4$ (see [3],[46]).*
- *Cantor sets (see [3]): Let H be a finite set having k points, $k \geq 2$, and*

  $$H^\infty = \{x = (x_i)_{i\in\mathbb{N}} : x_i \in H\}.$$

  Let $a \in (0,1)$. Then, $d_a : H^\infty \times H^\infty \to [0,\infty]$

  $$d_a(x,y) = \begin{cases} 0, & if\ x = y \\ a^j, & if\ x_i = y_i\ for\ \ i < j\ and\ \ x_j \neq y_i \end{cases}$$

is a metric in H^∞. Let ν be a uniformly distributed probability measure on H. Define the measure μ on H^∞ as the product measure of ν infinitely many times. In this case one can show that

$$\mu(B(x,a^j)) = k^{-j}$$

and that (H^∞, d_a, μ) is a doubling metric measure space with dimension s given by $a^s = k^{-1}$. If $k = 2$ and $a = \frac{1}{3}$, then H^∞ is bi-Lipschitz equivalent to the standard $\frac{1}{3}$-Cantor set.

- *It is proved in [89], [90], and [58] that some curvature-dimension condition on metric measure spaces implies the doubling property of the considered measure.*

Remark 3.2.1. *Given $x \in \Omega$, the function $r \to \mu(B(x,r))$ is (usually) not continuous, thus given $t > 0$ there does not necessarily exist a ball $B(x)$ centred at x such that $\mu(B(x)) = t$. However there is a ball $B(x)$ centred at x such that $t/C_D \leq \mu(B(x)) \leq t$. Indeed, consider $r_0 = \sup\{r : V_\mu(x,r) < t/C_D\}$. Then*

$$V_\mu(x,r) \leq t/C_D \leq V_\mu(x,2r) \leq C_D V_\mu(x,r) \leq t.$$

Following the proofs of [102, Theorem 1] and [92, Theorem 1.4], we obtain the following result:

Lemma 3.2.1. *Let (Ω, d, μ) be a doubling measure space. Then, for all bounded subsets $A \subset \Omega$ with $\mu(A) > 0$, $x \in A$ and $0 < r < diam(A)$, we have*

$$\frac{V_\mu(x,r)}{\mu(A)} \geq 2^{-m}\left(\frac{r}{diam(A)}\right)^m \tag{3.2.2}$$

where $m = \log_2 C_D$ and $diam(A) = \sup\limits_{x,y \in A} d(x,y)$[3]*.*

Proof. (See [102] and [31]). Let $B(x,r) \subset A$, suppose given $0 < r < diam(A)$, and put $m = log_2\left(\frac{diam(A)}{r}\right)$. Then

$$\begin{aligned}\mu(A) \leq V_\mu(x,r) &= V_\mu\left(x, \frac{2^m r}{2^m}\right) \leq C_D^m V_\mu\left(x, \frac{r}{2^m}\right)\\ &\leq C_D^{m+1} V_\mu(x,r) = C_D C_D^{log_2\left(\frac{diam(A)}{r}\right)} V_\mu(x,r)\\ &\leq C_D\left(\frac{diam(A)}{r}\right)^{log_2 C_D} V_\mu(x,r),\end{aligned}$$

[3] Inequality (3.2.2) is actually equivalent to the doubling property of the measure taking $B(x,2r)$ as the set A.

Finally, we have

$$\mu(A)C_D^{-1} \leq \left(\frac{diam(A)}{r}\right)^{log_2 C_D} V_\mu(x,r).$$

Therefore

$$\frac{V_\mu(x,r)}{\mu(A)} \geq 2^{-m}\left(\frac{r}{diam(A)}\right)^m.$$

□

In order to state the opposite inequality in Lemma 3.2.1 we shall need the following definition.

Definition 3.2.2. *A metric space* (Ω, d) *is called uniformly perfect (with constant* a*) if it is not a singleton and if there exists a constant* $a > 1$ *such that*

$$\Omega \setminus B(x;r) \neq \varnothing \Rightarrow B(x;r) \setminus B(x;r/a) \neq \varnothing$$

for all $x \in \Omega$ *and* $r > 0$.

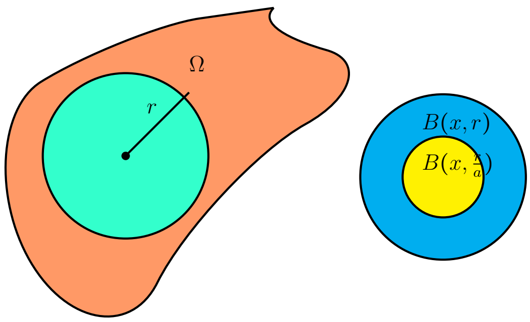

Figure 3.1: Definition 3.2.2

Lemma 3.2.2. *Let* (Ω, d, μ) *be doubling and uniformly perfect. Then there exist constants* $D \geq 1$ *and* $k > 0$*, depending only on the doubling constant* C_D *and the uniform perfectness constant* a*, such that*

$$\frac{V_\mu(x,r)}{V_\mu(x,R)} \leq D\left(\frac{r}{R}\right)^k \tag{3.2.3}$$

for all $x \in \Omega$ *and* $0 < r \leq R < diam(\Omega)$.

Proof. (See [38]).

By definition 3.2.2 we have that for any constant $a > 1$, $B(x;ar) \setminus B(x;r) \neq \emptyset$. We shall show that there exist constants $C > 1$ and $D > 1$ such that for all $x \in X$ and $0 < ar < diam(X)/2$

$$V_\mu(x, Cr) \geq DV_\mu(x, r). \tag{3.2.4}$$

To this end, fix any $0 < \sigma \leq 1$. Then, if $0 < r < diam(X)/2$, we have $a(1+\sigma)r < diam(X)$. Thus, by assumption,

$$B(x, a(1+\sigma)r) \backslash B(x, (1+\sigma)r) \neq \emptyset. \tag{3.2.5}$$

Let y be a point in the annulus. It is then easy to see that

$$B(y, \sigma r) \backslash B(x, r) = \emptyset, \; B(y, \sigma r) \subset B(x, (\sigma + a_0(1+\sigma))\, r)$$

and

$$B(x, (\sigma + a(1+\sigma))\, r) \subset B(y, (\sigma + 2a(1+\sigma))\, r), \tag{3.2.6}$$

and so

$$\begin{aligned} V_\mu(x, (\sigma + a(1+\sigma))\, r) &\geq V_\mu(x, r) + V_\mu(y, \sigma r) \quad \text{(by (3.2.5))} \\ &\geq V_\mu(x, r) + C_2^{-1} \left(\frac{\sigma}{\sigma + 2a(1+\sigma)} \right)^m V_\mu(y, (\sigma + 2a(1+\sigma))\, r) \\ &\geq V_\mu(x, r) + C_2^{-1} \left(\frac{\sigma}{\sigma + 2a(1+\sigma)} \right)^m V_\mu(x, (\sigma + a(1+\sigma))\, r) \quad \text{(by (3.2.6))}. \end{aligned}$$

This implies (3.2.3) with $C = \sigma + a(1+\sigma)$ and

$$D = \left(1 - C_2^{-1} \left(\frac{\sigma}{\sigma + 2a(1+\sigma)} \right)^m \right)^{-1} > 1.$$

Let $0 < \rho \leq r$ and $1 \leq \lambda\rho \leq diam(X)/2$. Let $n := [\log_C \lambda] \geq \log_C \lambda - 1$. Then $C^n \rho \leq diam(X)/2$ and for any $0 \leq k \leq n-1$,

$$V(x, C^{k+1}\rho) \geq DV(x, C^k \rho).$$

Iterating this inequality, we obtain

$$\begin{aligned} V(x, \lambda\rho) &\geq D^n V(x, \rho) \geq D^{\log_C \lambda - 1} V(x, \rho) \\ &= D^{-1} \lambda^{\log_C D} V(x, \rho). \end{aligned}$$

□

Note that if a measure satisfies the above inequality with some constants $D \geq 1$ and $k > 0$, then by choosing $r < D^{1/k} R$ in (3.2.3) we have that the space is uniformly perfect with any constant bigger than $D^{1/k}$.

Example 3.2.2.

- *Q-regular spaces are uniformly perfect (see [51]).*
- *Connected spaces are uniformly perfect (see [46]). Geodesic metric spaces and spaces that support a $(1,p)$-Poincare inequality are connected (see [52],[45]), and therefore are uniformly perfect.*

Combining Lemmas (3.2.1) and (3.2.2), the following is true in doubling uniformly perfect measure metric spaces: There exist positive constants c_0, C_0, k, and m ($k \le m$) depending only on the doubling constant of measure and the uniform perfectness constant of the space (Ω, d, μ) such that

$$c_0 \min(r^k, r^m) V_\mu(x,1) \le V_\mu(x,r) \le C_0 \max(r^k, r^m) V_\mu(x,1), \tag{3.2.7}$$

for all $x \in \Omega$ and $0 < r < \infty$.

Note that if $diam(\Omega) < \infty$, from (3.2.2) and (3.2.3) it follows that there exists constants c_1, C_1 such that

$$c_1 r^m \le V_\mu(x,r) \le C_1 r^k, \tag{3.2.8}$$

for all $x \in \Omega$ and $0 < r < diam(\Omega)$.

Definition 3.2.3. *Let $0 < k \le m$. Let (Ω, d, μ) be a metric measure space.*

(i.) (Ω, d, μ) *will be called a (k,m)-**space** if (3.2.7) holds*[4].

*(ii.) A (k,m)-space will be called **uniform** if there are constants $c, C > 0$, such that*

$$c \min(r^k, r^m) \le V_\mu(x,r) \le C \max(r^k, r^m). \tag{3.2.9}$$

*(iii.) A (k,m)-space will be called **bounded from below** if there are constants $d, D > 0$ such that*

$$d \min(r^k, r^m) \le V_\mu(x,r) \le D \max(r^k, r^m) V_\mu(x,1). \tag{3.2.10}$$

From (3.2.8), we have that doubling uniformly perfect measure metric spaces with $diam(\Omega) < \infty$ are uniform (k,m)-spaces.

Remark 3.2.2. *It follows from (3.2.7) that a (k,m)-space is uniform (resp. bounded from below) if, and only if, $0 < \inf_{x \in \Omega} V_\mu(x,1) \le \sup_{x \in \Omega} V_\mu(x,1) < \infty$, (resp. $0 < \inf_{x \in \Omega} V_\mu(x,1)$).*

[4]In fact (see ([105])) (Ω, d, μ) is a (k,m)-space if, and only if, it is doubling and uniformly perfect.

In the rest of this chapter, we shall use the following notation:

Notation 3. *Let (Ω, d, μ) be a (k, m)-space.*

(i.) *For $t > 0$,*

$$R(t) = \max\left(t^{m/k}, t^{k/m}\right), \ r(t) = \max\left(t^{1/k}, t^{1/m}\right).$$

(ii.) *If (Ω, d, μ) is uniform, we write*

$$\kappa_0 = 2C_D/c$$

where c is the same constant as in (3.2.9).

(iii.) *If (Ω, d, μ) is bounded from below, we write*

$$\kappa_1 = 2C_D/d$$

where d is the same constant as in (3.2.10).

Given (Ω, d, μ) a (k, m) space, we associate to the measure μ a new measure $\tilde{\mu}$ defined by

$$\tilde{\mu}(E) = \int_E \frac{d\mu(x)}{V(x,1)}$$

for any Borel set $E \subset \Omega$.

In the following lemma we obtain some properties of the measure $\tilde{\mu}$.

Lemma 3.2.3. *Let (Ω, d, μ) be a (k, m)-space. Let $f \in \mathcal{M}_0(\Omega)$. Then:*

(i.) *For all $r > 0$,*

$$\begin{aligned} \min(r^k, r^m) \fint_{B(x,r)} |f(y)|\, d\mu(y) &\leq \int_{B(x,r)} |f(y)|\, d\tilde{\mu}(y) \qquad (3.2.11) \\ &\leq \max(r^k, r^m) \fint_{B(x,r)} |f(y)|\, d\mu(y). \end{aligned}$$

Thus f is μ-locally integrable if, and only if, f is $\tilde{\mu}$-locally integrable. Moreover, $(\Omega, d, \tilde{\mu})$ is a uniform (k, m)-space.

(ii.) *If (Ω, d, μ) is uniform, then, for all measurable $E \subset \Omega$, we have that*

$$\tilde{\mu}(E) \simeq \mu(E).$$

(iii.) *If (Ω, d, μ) is bounded from below, then for all $f \in L^1_{\tilde{\mu}}(\Omega) + L^\infty_{\tilde{\mu}}(\Omega)$*

$$f^*_{\tilde{\mu}}(t) \leq f^*_{\mu}(dt), \quad (t > 0),$$

where d is the same constant that appears in (3.2.10).

Proof. (i.) Using the doubling property and the fact that $B(x,r) \subset B(y,2r)$ whenever $y \in B(x,r)$, we get

$$\begin{aligned}
\int_{B(x,r)} |f(y)|\, d\tilde{\mu}(y) &= \int_{B(x,r)} |f(y)| \frac{d\mu(y)}{V_\mu(y,1)} = \int_{B(x,r)} |f(y)| \frac{V_\mu(y,r)}{V_\mu(y,1)} \frac{d\mu(y)}{V_\mu(y,r)} \\
&\leq C_0 \max(r^k, r^m) \int_{B(x,r)} |f(y)| \frac{d\mu(y)}{V_\mu(y,r)} \text{ (by (3.2.7))} \\
&\leq C_D C_0 \max(r^k, r^m) \int_{B(x,r)} |f(y)| \frac{d\mu(y)}{V_\mu(y,2r)} \text{ (by (3.2.1))} \\
&\leq C_D C_0 \max(r^k, r^m) \frac{1}{V_\mu(x,r)} \int_{B(x,r)} |f(y)|\, d\mu(y).
\end{aligned}$$

Similarly, if $y \in B(x,r)$, then $B(y,r) \subset B(x,2r)$, thus

$$\begin{aligned}
\int_{B(x,r)} |f(y)|\, d\tilde{\mu}(y) &= \int_{B(x,r)} |f(y)| \frac{d\mu(y)}{V_\mu(y,1)} = \int_{B(x,r)} |f(y)| \frac{V_\mu(y,r)}{V_\mu(y,1)} \frac{d\mu(y)}{V_\mu(y,r)} \\
&\geq c_0 \min(r^k, r^m) \int_{B(x,r)} |f(y)| \frac{d\mu(y)}{V_\mu(y,r)} \\
&\geq c_0 \min(r^k, r^m) \int_{B(x,r)} |f(y)| \frac{d\mu(y)}{V_\mu(x,2r)} \\
&\geq \frac{c_0}{C_D} \min(r^k, r^m) \frac{1}{V_\mu(x,r)} \int_{B(x,r)} |f(y)|\, d\mu(y).
\end{aligned}$$

Taking $f = 1$ in (3.2.11), we obtain that $(\Omega, d, \tilde{\mu})$ is a uniform (k,m)-space.

(ii.) This is obvious.

(iii.) From (3.2.10), we get

$$\tilde{\mu}_f(y) = \int_{\{x \in \Omega : |f(x)| > y\}} \frac{d\mu(y)}{V(y,1)} \leq \frac{1}{d} \int_{\{x \in \Omega : |f(x)| > y\}} d\mu(y) = \frac{\mu_f(y)}{d}.$$

Therefore,

$$\mu_f(y) \leq dt \Rightarrow \tilde{\mu}_f(y) \leq t;$$

thus

$$f^*_{\tilde{\mu}}(t) = \inf\left\{y : \tilde{\mu}_f(y) \leq t\right\} \leq \inf\left\{y : \mu_f(y) \leq dt\right\} = f^*_\mu(dt).$$

□

We end this section by giving some examples of spaces that satisfy Definition 3.2.3.

3.2.1 Examples

Closed subsets of $\mathbb{R}^n$ (see [50])

We denote by m_n the n-dimensional Lebesgue measure on $\mathbb{R}^n$ and by d_n the n-Euclidean distance.

(i.) Consider $F \subset \mathbb{R}^2 = \{(x_1, x_2)\}$ defined by $F = F_1 \cup F_2$, where $F_1 = \{(x_1+1)^2 + x_2^2 \leq 1\}$ and $F_2 = \{0 \leq x_1 \leq 2,\ x_2 = 0\}$. Let m_n denote the n-dimensional Lebesgue measure, for $n = 1$ distributed over the x_1-axes, and put $d\lambda = x_1 dm_1$. Put $\mu = m_{2|F_1} + \lambda_{|F_2}$. Then (F, d_2, μ) is a $(1,2)$-uniform space.

(ii.) Let $F \subset \mathbb{R}^2$ be the set $F = \{0 \leq x_1 \leq 1,\ 0 \leq x_2 \leq x_1^\gamma\}$ where $\gamma > 1$, and $d\nu = x_1^{1-\gamma} dm_2$ and $\mu = \nu_{|F}$. Then (F, d_2, μ) is a $(1,2)$-uniform space.

(iii.) (See [50, Proposition 1]) For every closed subset $F \subset \mathbb{R}^n$ there is a measure μ with support F satisfying

$$\mu(B(x,r)) \leq c_\mu r^n \mu(B(x,1)) \text{ and } c_1 < \mu(B(x,1)) < c_2, \quad x \in F.$$

Thus F is uniformly perfect, and there is a $k > 0$, depending only on c_μ and on the uniform perfectness constant of F, such that (F, d_n, μ) is a (k,n)-uniform space.

Muckenhoupt weights

A weight is a positive, locally integrable function on $\mathbb{R}^n$. For a given subset E of $\mathbb{R}^n$, let $w(E) := \int_E w(x)dx$ and $|E| := \int_E dx$. A weight w on $\mathbb{R}^n$ is said to belong to the Muckenhoupt class A_p, $1 \leq p < \infty$, (see [75]) if

$$[w]_{A_p} := \begin{cases} \sup_B \left(\frac{1}{|B|} \int_B w(x)dx \right) \left(\frac{1}{|B|} \int_B \left(\frac{1}{w(x)} \right)^{\frac{1}{p-1}} dx \right)^{p-1} < \infty, & \text{if } 1 < p < \infty, \\ \sup_B \frac{\frac{1}{|B|} \int_B w(x)dx}{ess\ \inf_{x \in B} w(x)} < \infty, & \text{if } p = 1, \end{cases} \tag{3.2.12}$$

where the supremum is over all balls $B \subset \mathbb{R}^n$. For $p = \infty$, we define $A_\infty = \cup_{1 \leq p < \infty} A_p$. Given $w \in A_\infty$, we define

$$[w]_{A_\infty} := \sup_B \frac{1}{w(B)} \int_B M(w\chi_B)(x)dx$$

where M denotes the usual uncentred Hardy–Littlewood maxinal operator. It is known that there is a positive dimensional constant c_n such that $[w]_{A_\infty} \leq c_n [w]_{A_p}$.

Given $w \in A_p$, it follows easily from (3.2.12) that if there exists an $M > 0$ such that $ess \inf_{|x| \geq M} w(x) = 0$, then $\inf_{x \in \mathbb{R}^n} V_\mu(x,1) = 0$. Similarly, if $ess \sup_{|x| \geq M} w(x) = \infty$, then $\sup_{x \in \mathbb{R}^n} V_\mu(x,1) = \infty$.

Proposition 3.2.1. *Given $w \in A_p$ and $p \geq 1$, then $(\mathbb{R}^n, d_n, w)$ is a $\left(\frac{n}{2^{n+1}[w]_{A_\infty}}, pn\right)$-space.*

Proof. Since $w \in A_p$, by [49, Theorem 2.3], we have that

$$\frac{1}{|B|}\int_B w^r(x)dx \leq 2\left(\frac{1}{|B|}\int_B w(x)dx\right)^r$$

where $r = 1 + \frac{1}{2^{n+1}[w]_{A_\infty} - 1}$. Therefore (see [26]), there exist constants $c, C > 0$ such that

$$c\left(\frac{|E|}{|B|}\right)^p \leq \frac{w(E)}{w(B)} \leq C\left(\frac{|E|}{|B|}\right)^{(r-1)/r} \tag{3.2.13}$$

for any measurable set E of the ball B. Considering in (3.2.13) $E = B(x,r) \subset B(x,1) = B$ if $r < 1$, or $E = B(x,1) \subset B(x,r) = B$ if $r > 1$, and elementary computation shows

$$\min\left(r^{\frac{n}{2^{n+1}[w]_{A_\infty}}}, r^{pn}\right) w(B(x,1)) \leq w(B(x,r)) \leq \max\left(r^{\frac{n}{2^{n+1}[w]_{A_\infty}}}, r^{pn}\right) w(B(x,1)).$$

□

Example 3.2.3. *Suppose* $1 \leq p < \infty$, $-n < \alpha \leq \beta < n(p-1)$, *and*

$$w_{\alpha,\beta}(x) = \begin{cases} |x|^\alpha & \text{if } |x| \leq 1, \\ |x|^\beta & \text{if } |x| > 1. \end{cases}$$

Then $w_{\alpha,\beta}(x) \in A_p$, *and*

(i.) *If* $-n < \beta < 0$, *then* $\inf_{x\in\mathbb{R}^n} V_\mu(x,1) = 0$.

(ii.) *If* $\beta = 0$, *then* $\sup_{x\in\mathbb{R}^n} V_\mu(x,1) < \infty$.

(iii.) *If* $0 < \beta < n(p-1)$, *then* $\sup_{x\in\mathbb{R}^n} V_\mu(x,1) = \infty$.

Riemannian manifolds with nonnegative Ricci curvature (see [106])

.

For an n-manifold with nonnegative Ricci curvature, it is known that

$$c(n)Vol(B(x,1))r \leq Vol(B(x,r)) \leq \omega_n r^n$$

where $c(n)$ is a constant and ω_n is the volume of the unit ball in $\mathbb{R}^n$. Thus, under the assumption that the manifolds do not have the collapsing volume property[5], i.e. $\inf_x Vol(B(x,1)) > 0$, they are $(1,n)$-uniform spaces.

[5]In this context, this condition is related with the property of having finite topological type (see [88]).

Carnot–Carathéodory spaces (see [38])

Let Ω be a connected smooth manifold and suppose given k smooth real vector fields $\{X_1, ..., X_k\}$ on Ω satisfying Hörmander's condition of order m, that is, these vector fields together with their commutators of order at most m span the tangent space to Ω at each point. The control distances associated to the vector fields are defined as follows: for $x, y \in \Omega$ and $\delta > 0$, let $AC(x, y, \delta)$ denote the collection of absolutely continuous mappings $\varphi : [0, 1] \to \Omega$ with $\varphi(0) = x$ and $\varphi(1) = y$ such that for almost every $t \in [0, 1]$, $\varphi'(t) = \sum_{j=1}^{k} a_j X_j(\varphi(t))$, with $|a_j| \leq \delta$. Then the control metric $d(x, y)$ from x to y is the infimum of the set of $\delta > 0$ such that $AC(x, y, \delta) \neq \varnothing$. Hörmander's condition ensures that $d(x, y) < \infty$ for every $x, y \in \Omega$.

In this context, we have the following examples:

(i.) (Compact case). If X is a compact n-dimensional Carnot–Carathéodory space with the distance associated to the vector fields and endowed with any fixed smooth measure μ with strictly positive density, then (X, d, μ) is a uniform (n, nm)-space.

(ii.) (Noncompact case). Let $\Omega = \{(z, w) \in \mathbb{C}^2 : Im[w] > P(x)\}$, where P is a real, subharmonic, nonharmonic polynomial of degree m. Namely, Ω is an unbounded model domain of polynomial type in $\mathbb{C}^2$. Then $X = \partial\Omega$ can be identified with $\mathbb{C} \times \mathbb{R} = \{(z, t) : z \in \mathbb{C}, t \in \mathbb{R}\}$ The basic $(0, 1)$ Levi vector field is then $\overline{Z} = \partial/\partial\overline{z} - i(\partial P/\partial\overline{z})(\partial/\partial t)$, and we write $\overline{Z} = X_1 + iX_2$. The real vector fields $\{X_1, X_2\}$ and their commutators of orders $\leq m$ span the tangent space at each point. If we endow $\mathbb{C} \times \mathbb{R}$ with Lebesgue measure, then $X = \mathbb{C} \times \mathbb{R}$ is a $(4, m + 2)$-space.

(iii.) (Noncompact case). Let G be a connected Lie group and fix a left invariant Haar measure μ on G. We assume that G has polynomial volume growth, that is, if U is a compact neighbourhood of the identity element e of G, then there is a constant $C > 0$ such that $\mu(U^n) \leq n^C$ for all $n \in \mathbb{N}$. Then there is a nonnegative integer n_∞ such that $\mu(U^n) \simeq n^{n_\infty}$ as $n \to \infty$. Let $X_1, ..., X_n$ be left invariant vector fields on G that satisfy Hörmander's condition, that is, they together with their successive Lie brackets $[X_{i_1}, [X_{i_2}, [\ldots, X_{i_k}] \ldots]$ span the tangent space of G at every point of G. Let d be the associated control metric. Then this metric is left invariant and compatible with the topology on G and there is an $n_0 \in \mathbb{N}$, independent of x, such that $\mu(B(x, r)) \simeq r^{n_0}$ when $0 < r \leq 1$, and $\mu(B(x, r)) \simeq r^{n_0}$ when $r \geq 1$. From this, it follows that (G, d, μ) is a uniform $(\min\{n_0, n_\infty\}, \max\{n_0, n_\infty\})$-space.

3.3 Symmetrization inequalities for moduli

of continuity

Our first result in this section is to obtain embedding results for Besov spaces built on doubling measure spaces. This will be possible only if our space does not have the collapsing volume property.

Theorem 4. *Let (Ω, d, μ) be a doubling metric space. Let X be a rearrangement invariant space with $1/p > \overline{\beta}_X$. Assume that the following embedding holds*

$$\dot{\mathcal{B}}^s_{p,q}(\Omega) \subset X.$$

Then Ω does not have the collapsing volume property, i.e.

$$\inf_{x \in \Omega} V_\mu(x,1) > 0. \tag{3.3.1}$$

In particular, $\dot{\mathcal{B}}^s_{p,q}(\Omega) \subset L^{p^*,q}_\mu(\Omega)$ for some $p^* > p$ implies (3.3.1).

Proof. We claim that the conditions on the indices imply that for $1/p > \varepsilon > \overline{\beta}_X$ and t sufficiently small

$$\frac{t^{1/p}}{\varphi_X(t)} \preceq t^{\frac{1}{p} - \overline{\beta}_X - \varepsilon}. \tag{3.3.2}$$

Indeed, suppose $s, t > 0$. Then

$$\frac{t^{1/p}}{\varphi_X(t)} = \frac{t^{1/p}}{\varphi_X(st)} \frac{\varphi_X(st)}{\varphi_X(t)} \le \frac{t^{1/p}}{\varphi_X(st)} M_X(s)$$

and hence for $s = 1/t$ we get

$$\frac{t^{1/p}}{\varphi_X(t)} \le \frac{t^{1/p}}{\varphi_X(1)} M_X(\frac{1}{t}).$$

Let $1/p > \varepsilon > \overline{\beta}_X$. Then (see Lemma 2.2.1) for t sufficiently small,

$$\begin{aligned} \frac{t^{1/p}}{\varphi_X(t)} &\le \frac{t^{1/p}}{\varphi_X(1)} M_X(\frac{1}{t}) \\ &\le \frac{t^{1/p}}{\varphi_X(1)} \left(\frac{1}{t}\right)^{\overline{\beta}_X + \varepsilon} \\ &= \frac{t^{\frac{1}{p} - \overline{\beta}_X - \varepsilon}}{\varphi_X(1)}, \end{aligned}$$

as we wanted to see.

For a fixed $x_0 \in \Omega$, we define the Lipschitz function

$$u_{x_0}(y) := \begin{cases} (2 - d(x_0, y)) & \text{if } y \in B(x_0, 2) \setminus B(x_0, 1) \\ 1 & \text{if } y \in B(x_0, 1) \\ 0 & \text{if } y \in \Omega \setminus B(x_0, 2). \end{cases}$$

It is easily seen that

$$g_{x_0}(y) = \chi_{B(x_0,2)}(y)$$

is a generalized gradient, i.e.

$$|u_{x_0}(x) - u_{x_0}(y)| \le d(x,y)\,|g_{x_0}(x) + g_{x_0}(y)|\,. \tag{3.3.3}$$

By Fubini's theorem,

$$\begin{aligned} E_p(u_{x_0},t)^p &\le 2^p \int_\Omega |u_{x_0}(x)|^p\,d\mu(x) + 2^p \int_\Omega \fint_{B(x,t)} |u_{x_0}(y)|^p\,d\mu(y)d\mu(x) \qquad (3.3.4)\\ &\le 2^p \|u_{x_0}\|_p^p + 2^p \int_\Omega |u_{x_0}(y)|^p \left(\int_{B(y,t)} \frac{1}{\mu(B(x,t))} d\mu(x) \right) d\mu(y) \\ &\le \|u_{x_0}\|_p^p, \end{aligned}$$

the last estimate following from the doubling property of μ and since $B(y,t) \subset B(x,2t)$ whenever $x \in B(y,t)$.

By (3.3.3) and using a similar argument as in (3.3.4), we get

$$\begin{aligned} E_p(u_{x_0},t)^p &= \int_\Omega \left(\fint_{B(x,t)} |u_{x_0}(x) - u_{x_0}(y)|^p\,d\mu(y) \right) d\mu(x) \qquad (3.3.5)\\ &\le \int_\Omega \left(\fint_{B(x,t)} d(x,y)^p\,|g_{x_0}(x) + g_{x_0}(y)|^p\,d\mu(y) \right) d\mu(x) \\ &\le t^p \left(\int_\Omega |g_{x_0}(x)|^p\,d\mu(x) + \int_\Omega \fint_{B(x,t)} |g_{x_0}(y)|^p\,d\mu(y)d\mu(x) \right) \\ &\le t^p \|g_{x_0}\|_p^p\,. \end{aligned}$$

Thus, combining (3.3.4) and (3.3.5) with the doubling property, we get

$$\begin{aligned} E_p(u_{x_0},t) &\le \min(\|u_{x_0}\|_p\,, t\,\|g_{x_0}\|_p) \\ &\le \min(V_\mu(x_0,2)^{1/p}, tV_\mu(x_0,2)^{1/p}) \\ &\le \min(1,t)V_\mu(x_0,1)^{1/p}. \end{aligned}$$

Therefore

$$\|u_{x_0}\|_{\dot{\mathcal{B}}^s_{p,q}(\Omega)} \le V_\mu(x_0,1)^{1/p}.$$

Since

$$\|u_{x_0}\|_X \ge \varphi_X(V_\mu(x_0,1))$$

by hypothesis, we have that

$$1 \le \frac{V_\mu(x_0,1)^{1/p}}{\varphi_X(V_\mu(x_0,1))}. \tag{3.3.6}$$

If $\inf_{x\in\Omega} V_\mu(x,1) = 0$, we can select a sequence $V_\mu(x_n,1) \to 0$. Thus, for n big enough, (3.3.6) and (3.3.2) imply

$$1 \le V_\mu(x_n,1)^{\frac{1}{p} - \overline{\beta}_X - \varepsilon},$$

which is impossible since $\frac{1}{p} - \overline{\beta}_X - \varepsilon > 0$. $\square$

Recall that our aim is to obtain embedding results for Besov spaces built on doubling measure spaces. Therefore, in view of Theorem 2, it is reasonable to assume that Ω is uniformly perfect (since Q-regular spaces are uniformly perfect). Moreover, if we make an additional hypothesis (e.g. Ω supports a $(1,p)$-Poincaré inequality or Ω is geodesic), then Ω is connected and therefore uniformly perfect. Taking into account these considerations and the previous theorem, our framework in what follows will be a (k,m)-space uniformly or bounded from below.

In order to simplify the notation, throughout what follows we will assume that $\mu(\Omega)=\infty$, since all the results that we obtain can be immediately adapted to the case of finite measure.

3.3.1 Pointwise estimates for the rearrangement

For $f\in L^1_\mu(\Omega)+L^\infty_\mu(\Omega)$ and X an r.i. space on Ω, we define:

(i) The gradient at scale r

$$\left(\nabla^\mu_r f\right)(x)=\fint_{B(x,r)}|f(x)-f(y)|\,d\mu(y),\qquad (r>0).$$

(ii.) The X-modulus of continuity $E_X:(0,\infty)\times X\to[0,\infty)$,

$$E_X(f,r):=\left\|\left(\nabla^\mu_r f\right)\right\|_X.$$

Remark 3.3.1. *If $X=L^p_\mu$, by Hölder's inequality,*

$$\begin{aligned}E_{L^p_\mu}(f,r)&=\left(\int_\Omega\left(\fint_{B(x,r)}|f(x)-f(y)|\,d\mu(y)\right)^p d\mu(x)\right)^{1/p}\\&\le\left(\int_\Omega\left(\fint_{B(x,t)}|f(x)-f(y)|^p\,d\mu(y)\right)d\mu(x)\right)^{1/p}\\&=E_p(f,r)\end{aligned}$$

where $E_p(f,r)$ is the L^p_μ-modulus of smoothness defined in (3.1.1).

The aim of this section is to obtain pointwise estimates for the oscillation $O_\mu(f,t)$ in terms of the functional $E_X(f,t)$ (see [55],[61] for some related results). The next lemma will be useful in what follows.

Lemma 3.3.1. *Let $f\in L^1_\mu(\Omega)+L^\infty_\mu(\Omega)$. Let $x\in\Omega$ and $t>0$ be such that there is a ball $B_t(x)$ centred at x with $\mu(B_t(x))=t$. Then*

$$f^{**}_\mu(t/2)-f^{**}_\mu(t)\le\left(\delta^\mu_t f\right)^{**}_\mu(t/2),$$

where

$$\left(\delta^\mu_t f\right)(x)=\frac{1}{t}\int_{B_t(x)}|f(x)-f(y)|\,d\mu(y).$$

Proof. Since

$$|f(x)|\chi_{B_t(x)}(y) \leq |f(x) - f(y)|\chi_{B_t(x)}(y) + |f(y)|\chi_{B_t(x)}(y),$$

integrating with respect to $d\mu(y)$ yields that

$$\begin{aligned}|f(x)|t &\leq \int_{B_t(x)} |f(x) - f(y)|d\mu(y) + \int_{B_t(x)} |f(y)|d\mu(y)\\ &\leq \int_{B_t(x)} |f(x) - f(y)|d\mu(y) + \int_0^t f_\mu^*(s)ds \quad \text{(by (2.1.2))}.\end{aligned}$$

Now integrating with respect to $d\mu(x)$ over a subset $E \subset \Omega$ with $\mu(E) = t/2$, we get

$$\begin{aligned}\int_E |f(x)|d\mu(x) &\leq \int_E \frac{1}{t} \int_{B_t(x)} |f(x) - f(y)|d\mu(y)d\mu(x) + \int_E \frac{1}{t}\left(\int_0^t f_\mu^*(s)ds\right)d\mu(x)\\ &= \int_E \left(\delta_t^\mu f\right)(x)d\mu(x) + \frac{1}{2}\int_0^t f_\mu^*(s)ds\end{aligned}$$

By (2.1.3), taking the supremum over all such sets E, we obtain

$$\int_0^{t/2} f_\mu^*(s)ds \leq \int_0^{t/2} \left(\delta_t^\mu f\right)_\mu^*(s)ds + \frac{1}{2}\int_0^t f_\mu^*(s)ds,$$

or equivalently

$$f_\mu^{**}(t/2) - f_\mu^{**}(t) \leq \left(\delta_t^\mu f\right)_\mu^{**}(t/2).$$

□

Theorem 5. *Let $f \in L_\mu^1(\Omega) + L_\mu^\infty(\Omega)$. Let X be an r.i. space on Ω.*

(i.) If (Ω, d, μ) is uniform, then for all $t > 0$,

$$O_\mu(f,t) \leq \frac{1}{\kappa_0 t}\frac{R(\kappa_0 t)}{\phi_X(\kappa_0 t)}E_X(f, r(\kappa_0 t)). \tag{3.3.7}$$

(ii.) If (Ω, d, μ) is bounded form below, then for all $t > 0$,

$$O_{\tilde{\mu}}(f,t) \leq \frac{1}{\kappa_1 1t}\frac{R(\kappa_1 t)}{\phi_X(\kappa_1 t)}E_X(f, r(\kappa_1 t)).$$

Proof. (i.) Given $x \in \Omega$ and $t > 0$, by Remark 9 there is a ball $B(x)$ centred at x such that $t/C_D \leq \mu(B(x)) \leq t$. We denote by z the measure of this ball, i.e. $\mu(B_z(x)) = z$, with $t/C_D \leq z \leq t$. From (3.2.9) it follows that

$$\begin{cases} \mu(B_z(x)) \leq t \leq V_\mu(x, (t/c)^{1/m}) \leq C\,(t/c)^{k/m} & \text{if } t < c,\\ \mu(B_z(x)) \leq t \leq V_\mu(x, (t/c)^{1/k}) \leq C\,(t/c)^{m/k} & \text{if } t \geq c,\end{cases}$$

i.e.

$$\mu(B_z(x)) \le t \le V_\mu(x, r(t/c)) \le CR(t/c). \tag{3.3.8}$$

Obviously, $B_z(x) \subset B(x, r(t/c))$, and thus

$$\begin{aligned}
(\delta_z^\mu f)(x) &= \frac{1}{z} \int_{B_z(x)} |f(x) - f(y)| \, d\mu(y) \\
&\le \frac{C_D}{t} \int_{B(x,r(t/c))} |f(x) - f(y)| \, d\mu(y) \\
&\le CC_D \frac{R(t/c)}{t} \fint_{B(x,r(t/c))} |f(x) - f(y)| \, d\mu(y) \quad \text{(by (3.3.8))} \\
&= CC_D \frac{R(t/c)}{t} \left(\nabla_{r(t/c)}^\mu f \right)(x).
\end{aligned}$$

Taking rearrangements, we get

$$(\delta_z^\mu f)_\mu^* (s) \le CC_D \frac{R(t/c)}{t} \left(\nabla_{r(t/c)}^\mu f \right)_\mu^* (s), \ \ s > 0,$$

which implies

$$(\delta_z^\mu f)_\mu^{**} (s) \le CC_D \frac{R(t/c)}{t} \left(\nabla_{r(t/c)}^\mu f \right)_\mu^{**} (s), \ \ s > 0.$$

On the other hand,

$$\begin{aligned}
\left(\nabla_{r(t/c)}^\mu f \right)_\mu^{**} (s) &\le \frac{1}{\phi_X(s)} \sup_s \left(\phi_X(s) \left(\nabla_{r(t/c)}^\mu f \right)_\mu^{**} (s) \right) \\
&= \frac{1}{\phi_X(s)} \left\| \left(\nabla_{r(t/c)}^\mu f \right) \right\|_{M(X)} \quad \text{(by (2.2.7))} \\
&\le \frac{1}{\phi_X(s)} \left\| \left(\nabla_{r(t/c)}^\mu f \right) \right\|_{X} \quad \text{(by (2.2.8))} \\
&= \frac{1}{\phi_X(s)} E_X(f, r(t/c)).
\end{aligned}$$

Combining this inequality and Lemma 3.3.1, we obtain

$$f_\mu^{**}(z/2) - f_\mu^{**}(z) \le (\delta_z^\mu f)_\mu^{**} (z/2) \le CC_D \frac{R(t/c)}{t\phi_X(z/2)} E_X(f, r(t/c)).$$

By Remark 2.1.1, we get

$$f_\mu^{**}(z/2) - f_\mu^{**}(z) = \int_{z/2}^{z} \left(f_\mu^{**}(s) - f_\mu^*(s) \right) \frac{ds}{s} \ge \frac{f_\mu^{**}(z/2) - f_\mu^*(z/2)}{2}.$$

In summary,

$$O_\mu(f, z/2) \le 2CC_D \frac{R(t/c)}{t\phi_X(z/2)} E_X(f, r(t/c)). \tag{3.3.9}$$

Finally, using that $t/C_D \leq z \leq t$, we get

$$\begin{aligned}
\frac{t}{2C_D} O_\mu\left(f, \frac{t}{2C_D}\right) &\leq \frac{z}{2} O_\mu\left(f, \frac{z}{2}\right) \text{ (by Remark 2.1.1)} \\
&\leq 2CC_D \frac{z/2}{\phi_X(z/2)} \frac{R(t/c)}{t} E_X(f, r(t/c)) \quad \text{(by (3.3.9))} \\
&\leq 4CC_D \frac{R(t/c)}{\phi_X(t/2)} E_X(f, r(t/c)) \quad \left(\text{since } \frac{s}{\phi_X(s)} \text{ increases}\right) \\
&\leq 4CC_D \left(\sup_{t>0} \frac{\phi_X(2t/c)}{\phi_X(t/2)}\right) \frac{R(t/c)\, E_X(f, r(t/c))}{\phi_X(t/c)} \\
&= 4CC_D M_X(2/c) \frac{R(t/c)\, E_X(f, r(t/c))}{\phi_X(t/c)},
\end{aligned}$$

which implies (3.3.7).

(ii.) By Lemma 3.2.3, we get $L^1_\mu(\Omega) + L^\infty_\mu(\Omega) \subset L^1_{\tilde{\mu}}(\Omega) + L^\infty_{\tilde{\mu}}(\Omega)$ since $\tilde{\mu}$ is doubling. By Remark 9, given $x \in \Omega$ and $t > 0$, there is a ball $B_z(x)$ centred at x such that $t/C_D \leq \tilde{\mu}(B_z(x)) = z \leq t$. Then

$$\begin{aligned}
\left(\delta^{\tilde{\mu}}_z f\right)(x) &\leq \frac{C_D}{t} \int_{B(x, r(t/d))} |f(x) - f(y)| \frac{d\mu(y)}{V(y,1)} \\
&\leq C_D D \frac{R(t/d)}{t} \fint_{B(x, r(t/d))} |f(x) - f(y)|\, d\mu(y) \quad \text{(by (3.2.11))} \\
&= C_D D \frac{R(t/d)}{t} \left(\nabla^\mu_{r(t/d)} f\right)(x).
\end{aligned}$$

Taking rearrangements with respect to $\tilde{\mu}$, we have that for all $s > 0$

$$\begin{aligned}
\left(\delta^{\tilde{\mu}}_z f\right)^*_{\tilde{\mu}}(s) &\leq C_D D \frac{R(t/d)}{t} \left(\nabla^\mu_{r(t/d)} f\right)^*_{\tilde{\mu}}(s) \\
&\leq C_D D \frac{R(t/d)}{t} \left(\nabla^\mu_{r(t/d)} f\right)^*_\mu(sd) \quad \text{(by Lemma 3.2.3).}
\end{aligned}$$

Hence,

$$\begin{aligned}
\left(\delta^{\tilde{\mu}}_z f\right)^{**}_{\tilde{\mu}}(s) &= \frac{1}{s} \int_0^s \left(\delta^{\tilde{\mu}}_z f\right)^*_{\tilde{\mu}}(y)\, dy \\
&\leq C_D D \frac{R(t/d)}{t} \frac{1}{s} \int_0^s \left(\nabla^\mu_{r(\frac{t}{d})} f\right)^*_\mu(yd)\, dy \\
&= C_D D \frac{R(t/d)}{t} \frac{1}{sd} \int_0^{sd} \left(\nabla^\mu_{r(\frac{t}{d})} f\right)^*_\mu(y)\, dy \\
&= C_D D \frac{R(t/d)}{t} \left(\nabla^\mu_{r(t/d)} f\right)^{**}_\mu(sd).
\end{aligned}$$

and

$$\left(\nabla^\mu_{r(t/d)} f\right)^{**}_\mu(sd) \leq \frac{1}{\phi_X(sd)} E_X(f, r(t/d)).$$

Thus,

$$f_{\tilde{\mu}}^{**}(z/2) - f_{\tilde{\mu}}^{**}(z) \leq DC_D \frac{R(t/d)}{t\phi_X(zd/2)} E_X(f, r(t/d)).$$

Now we finish the proof as in part (i). □

3.4 A Sobolev type embedding result for Besov spaces

Definition 3.4.1. *Let (Ω, d, μ) be a (k,m)-space. Let X be a r.i. space on Ω and let Y be an r.i. space on $[0,\infty)$ over $[0,\infty)$ with respect to Lebesgue measure. Let $0 < s < 1$. The* ***Besov space*** $\mathring{B}^s_{(k,m),X,Y}(\Omega)$ *is the set of those functions in* $L^1_\mu(\Omega) + L^\infty_\mu(\Omega)$ *for which the semi-norm*

$$\|f\|_{\mathring{B}^s_{(k,m),X,Y}(\Omega)} := \left\| \frac{r(t)^{-s} E_X(f, r(t))}{\phi_Y(t)} \right\|_Y$$

is finite.

Remark 3.4.1. *We write* $\mathring{B}^s_{(k,m),p,q}(\Omega)$ *if* $X = L^p_\mu(\Omega)$ *and* $Y = L^q([0,\infty))$ $(1 \leq p < \infty,\ 1 \leq q \leq \infty)$. *In this case* $\phi_Y(t) = t^{1/q}$, *thus if* $1 \leq q < \infty$,

$$\begin{aligned}
\|f\|_{\mathring{B}^s_{(k,m),p,q}(\Omega)} &\leq \left\| \frac{E_p(f, r(t))}{(r(t))^s t^{1/q}} \right\|_{L^q} \qquad \textit{(by Remark 3.3.1)} \\
&= \left(\int_0^\infty \left(\frac{E_p(f, \max(t^{1/k}, t^{1/m}))}{\max(t^{1/k}, t^{1/m})^s} \right)^q \frac{dt}{t} \right)^{1/q} \\
&= \left(\int_0^1 \left(\frac{E_p(f, t^{1/m})}{t^{s/m}} \right)^q \frac{dt}{t} + \int_1^\infty \left(\frac{E_p(f, t^{1/k})}{t^{s/k}} \right)^q \frac{dt}{t} \right)^{1/q} \\
&\simeq \left(\int_0^1 \left(\frac{E_p(f, t)}{t^s} \right)^q \frac{dt}{t} + \int_1^\infty \left(\frac{E_p(f, t)}{t^s} \right)^q \frac{dt}{t} \right)^{1/q} \\
&= \left(\int_0^\infty \left(\frac{E_q(f, t)}{t^s} \right)^q \frac{dt}{t} \right)^{1/q}.
\end{aligned}$$

Therefore,

$$\dot{\mathcal{B}}^s_{p,q}(\Omega) \subset \mathring{B}^s_{(k,m),p,q}(\Omega).$$

Similarly,

$$\dot{\mathcal{B}}^s_{p,\infty}(\Omega) \subset \mathring{B}^s_{(k,m),p,\infty}(\Omega).$$

3.4.1 Some new function spaces

Following [59], we shall now construct the range spaces for our generalized Besov–Sobolev embedding theorem.

Definition 3.4.2. *Let* (Ω, d, μ) *be a* (k,m)*-space. Given* $s \in \mathbb{R}$*, we define*

$$v_s(t) := \frac{t}{R(t)r(t)^s} = \min\left(t^{1-\frac{m+s}{k}}, t^{1-\frac{k+s}{m}}\right)$$

and

$$S_\mu^{X,Y}(v_s) = \left\{ f : \|f\|_{S_\mu^{X,Y}(v_s)} = \left\| v_s(t)\frac{\phi_X(t)}{\phi_Y(t)} O_\mu(f,t) \right\|_Y < \infty \right\},$$

where ϕ_X *is the fundamental function of* X*, an r.i. space on* Ω*, and* Y *is an r.i. space on* $[0,\infty)$ *with respect to Lebesgue measure.*

Note that these spaces are not necessarily linear and, in particular, $\|.\|_{S_{\tilde{\mu}}^{X,Y}(v_s)}$ is not necessarily a norm.

Given an r.i. space X, we shall say that Y satisfies the $Q(s,(k,m),X)$-condition if there exists a constant $C > 0$ such that

$$\left\| v_s(t)\frac{\phi_X(t)}{\phi_Y(t)} Qf(t) \right\|_Y \le C \left\| v_s(t)\frac{\phi_X(t)}{\phi_Y(t)} f(t) \right\|_Y .$$

The following lemmas will be useful in what follows. A consequence of our first lemma is that if Y satisfies the $Q(s,(k,m),X)$-condition, then $S_\mu^{X,Y}(v_s)$ is a Banach space.

Lemma 3.4.1. *Let* X, Y *be two r.i. spaces. If* Y *satisfies the* $Q(s,(k,m),X)$*-condition, then for all* $f_\mu^*(\infty) = 0$,

$$\|f\|_{S_\mu^{X,Y}(v_s)} \simeq \left\| v_s(t)\frac{\phi_X(t)}{\phi_Y(t)} f_\mu^{**}(t) \right\|_Y , \tag{3.4.1}$$

with constants of equivalence independent of f.

Proof. Obviously,

$$\left\| v_s(t)\frac{\phi_X(t)}{\phi_Y(t)} O_\mu(f,t) \right\|_Y \le \left\| v_s(t)\frac{\phi_X(t)}{\phi_Y(t)} f_\mu^{**}(t) \right\|_Y .$$

Conversely, from $\frac{d}{dt} f_\mu^{**}(t) = -\frac{f_\mu^{**}(t) - f_\mu^*(t)}{t}$ and the Fundamental Theorem of Calculus, we have

$$f_\mu^{**}(t) = \int_t^\infty \left(f_\mu^{**}(s) - f_\mu^*(s)\right)\frac{ds}{s} = Q\left(f_\mu^{**} - f_\mu^*\right)(t),$$

and the result follows by the $Q(s,(k,m),X)$-condition. □

The next result gives a useful criterion for checking the validity of a $Q(s,(k,m),X)$-condition.

Lemma 3.4.2. *Let X,Y be two r.i. spaces. Suppose that*

$$\int_1^\infty t^{\frac{m+s}{k}-1} h_Y(1/t) M_X(1/t) M_Y(t) \frac{dt}{t} < \infty. \tag{3.4.2}$$

Then Y satisfies the $Q(s,(k,m),X)$-condition.

Proof. Let us write $v_s \coloneqq v$. We have

$$\begin{aligned} v(t)\frac{\phi_X(t)}{\phi_Y(t)} Qf(t) &= \int_t^\infty v(t)\frac{\phi_X(t)}{\phi_Y(t)} f(x)\frac{dx}{x} = \int_1^\infty v(t)\frac{\phi_X(t)}{\phi_Y(t)} f(tx)\frac{dx}{x} \\ &\le \int_1^\infty v(tx) f(tx) \frac{\phi_X(xt)}{\phi_Y(xt)} \sup_{t>0}\frac{v(t)}{v(tx)} \sup_{t>0}\frac{\phi_X(t)}{\phi_X(xt)} \sup_{t>0}\frac{\phi_Y(xt)}{\phi_Y(t)}\frac{dx}{x} \\ &= \int_1^\infty v(tx) f(tx) \frac{\phi_X(xt)}{\phi_Y(xt)} \sup_{t>0}\frac{v(t)}{v(tx)} M_X(1/x) M_Y(x)\frac{dx}{x}. \end{aligned}$$

Applying Minkowski's inequality, we obtain

$$\begin{aligned} \left\| v(t)\frac{\phi_X(t)}{\phi_Y(t)} Qf(t)\right\|_Y &\le \int_1^\infty \left\| v(tx) f(tx)\frac{\phi_X(xt)}{\phi_Y(xt)}\right\|_Y \sup_{t>0}\frac{v(t)}{v(tx)} M_X(1/x) M_Y(x)\frac{dx}{x} \\ &\le \int_1^\infty \left(\sup_{t>0}\frac{v(t)}{v(tx)}\right) h_Y(1/x) M_X(1/x) M_Y(x)\frac{dx}{x} \left\| v(t) f(t)\frac{\phi_X(t)}{\phi_Y(t)}\right\|_Y. \end{aligned}$$

Finally, an elementary computation shows that if $x > 1$, then

$$\sup_{t>0}\frac{v(t)}{v(tx)} = x^{\frac{m+s}{k}-1}.$$

□

Remark 3.4.2. *In terms of indices, it is easy to see that (3.4.2) is equivalent to the inequality*

$$\frac{m+s}{k} - 1 < \underline{\alpha}_Y - \overline{\beta}_Y + \underline{\beta}_X.$$

Moreover, if $\phi_X(t)/\phi_Y(t)$ is equivalent to an increasing function, then starting from $\varphi_{s,(k,m)}(t)\frac{\phi_X(t)}{\phi_Y(t)} Qf(t) \le c\int_t^\infty \varphi_{s,(k,m)}(t)\frac{\phi_X(x)}{\phi_Y(x)} f(x)\frac{dx}{x}$ and following the same steps as in the proof of the previous lemma, we see that

$$\int_1^\infty t^{\frac{m+s}{k}-1} h_Y\left(\frac{1}{t}\right)\frac{dt}{t} < \infty$$

implies that Y satisfies the $Q(s,(k,m),X)$-condition.

In the particular case that $Y = L^q$ we can obtain a more specific result.

Lemma 3.4.3. *Let X be an r.i. space. Let $s \in (0,1)$ and suppose $q \geq 1$. Let $v(t) = \varphi_s(t) t^{-1/q} \phi_X(t)$. Then the following statements are equivalent.*

i) $\frac{m+s}{k} - 1 < \underline{\beta}_X$.

ii) L^q *satisfies the* $Q(s,(k,m),X)$*-condition.*

iii) If $f^*(\infty) = 0$,

$$\left\| v(t) \left(f_\mu^{**}(t) - f_\mu^*(t) \right) \right\|_{L_\mu^q} \simeq \left\| v(t) f_\mu^{**}(t) \right\|_{L_\mu^q}.$$

Proof. $(i) \to (ii)$ Since $\underline{\alpha}_Y = \overline{\beta}_Y = \frac{1}{q}$, Remark 3.4.2 applies. $ii) \to iii)$ follows from Lemma 3.4.1. To conclude the proof, we show $iii) \to i)$: By Fubini we readily see that (we need $f_\mu^*(\infty) = 0$, otherwise Qf_μ^* does not exist)

$$Qf_\mu^* = Q \circ Pf_\mu^* = Qf_\mu^* + Pf_\mu^* = P \circ Qf_\mu^* = \left(Qf_\mu^* \right)^{**}.$$

Thus

$$\left(Qf_\mu^* \right)^{**}(t) - Qf_\mu^*(t) = Pf_\mu^*(t) = f_\mu^{**}(t). \tag{3.4.3}$$

Consider now the r.i. space H defined by the norm

$$\|h\|_H = \left\| v(t) f_\mu^{**}(t) \right\|_{L_\mu^q}.$$

Then, by condition iii),

$$\begin{aligned} \phi_H(r) = \left\| \chi_{[0,r]} \right\|_H &\simeq \left\| v(t) \left(\chi_{[0,r]}^{**}(t) - \chi_{[0,r]}^*(t) \right) \right\|_{L_\mu^q} \\ &= r \left(\int_r^\infty v(t)^q \frac{dt}{t^q} \right)^{1/q}. \end{aligned}$$

On the other hand, since ϕ_X is increasing and $\varphi_s(t)/t$ is decreasing,

$$\begin{aligned} \int_r^\infty v(t)^q \frac{dt}{t^q} &= \int_r^\infty \left(\varphi_s(t) t^{-1/q} \phi_X(t) \right)^q \frac{dt}{t^q} \\ &\geq \int_r^{2r} \phi_X(t)^q \left(\frac{\varphi_s(t)}{t} \right)^q \frac{dt}{t} \\ &\geq \phi_X(r)^q \left(\frac{\varphi_s(2r)}{2r} \right)^q. \end{aligned}$$

Similarly, since $\phi_X(t)/t$ is decreasing,

$$\begin{aligned} \int_r^\infty \left(\varphi_s(t) t^{-1/q} \phi_X(t) \right)^q \frac{dt}{t^q} &\leq \left(\frac{\phi_X(r)}{r} \right)^q \int_r^\infty \varphi_s(t)^q \frac{dt}{t} \\ &= \left(\frac{\phi_X(r)}{r} \right)^q \int_r^\infty \left(\min\left(t^{1-\frac{m+s}{k}}, t^{1-\frac{k+s}{m}} \right) \right)^q \frac{dt}{t} \\ &\leq \left(\frac{\phi_X(r)}{r} \right)^q \int_r^\infty \left(t^{1-\frac{m+s}{k}} \right)^q \frac{dt}{t} \\ &\simeq \left(\frac{\phi_X(r)}{r} \right)^q \left(r^{1-\frac{m+s}{k}} \right)^q \quad \text{(since } k \leq m\text{)}. \end{aligned}$$

Thus, if $r > 1$,

$$\phi_H(r) \simeq \phi_X(r) r^{1-\frac{m+s}{k}}$$

and

$$\underline{\beta}_H = \left(1 - \frac{m+s}{k}\right) + \underline{\beta}_X.$$

Finally, since

$$\begin{aligned}
\|f\|_H &= \left\| v(t)\left(\left(Qf_\mu^*\right)^{**}(t) - Qf_\mu^*(t)\right)\right\|_{L^q_\mu} \quad \text{(by (3.4.3))} \\
&\simeq \left\| v(t)\left(Qf_\mu^*\right)^{**}(t)\right\|_{L^q_\mu} \quad \text{(by condition } iii\text{))} \\
&= \left\|Qf_\mu^*\right\|_H,
\end{aligned}$$

it follows that $Q : H \to H$ is bounded, which implies that $\underline{\beta}_H > 0$, thus

$$\underline{\beta}_X > \frac{m+s}{k} - 1.$$

□

From Theorem 5 we get immediately the following generalization of the Sobolev embedding theorem for Besov spaces.

Theorem 6. *Let (Ω, d, μ) be a (k,m)-space, X, Y r.i. spaces, and $0 < s < 1$. Then*

(i.) If (Ω, d, μ) is uniform,

$$\mathring{B}^s_{(k,m),X,Y}(\Omega) \subset S^{X,Y}_\mu(v_s).$$

Moreover if Y satisfies the $Q(s,(k,m),X)$-condition, then for all $f^*_\mu(\infty) = 0$,

$$\left\| v_s(t)\frac{\phi_X(t)}{\phi_Y(t)} f^{**}_\mu(t)\right\|_Y \preceq \|f\|_{\mathring{B}^s_{(k,m),X,Y}(\Omega)}.$$

(ii.) If (Ω, d, μ) is bounded from below,

$$\mathring{B}^s_{(k,m),X,Y}(\Omega) \subset S^{X,Y}_{\tilde{\mu}}(v_s).$$

Moreover if Y satisfies the $Q(s,(k,m),X)$-condition, then for all $f^*_{\tilde{\mu}}(\infty) = 0$,

$$\left\| v_s(t)\frac{\phi_X(t)}{\phi_Y(t)} f^{**}_{\tilde{\mu}}(t)\right\|_Y \preceq \|f\|_{\mathring{B}^s_{(k,m),X,Y}(\Omega)}.$$

Proof. (i.) Let $f \in L^1_\mu(\Omega) + L^\infty_\mu(\Omega)$. Then from (3.3.7) we know that there is a constant $\kappa > 0$ such that

$$O_\mu(f,t) \leq \frac{1}{\kappa_0 t} \frac{R(\kappa_0 t)}{\phi_X(\kappa_0 t)} E_X(f, r(\kappa_0 t)), \quad t > 0.$$

Thus,

$$\begin{aligned}
\|f\|_{S^{X,Y}_\mu(v_s)} &\leq \left\| \frac{t}{R(t) r(t)^s} \frac{\phi_X(t)}{\phi_Y(t)} \frac{1}{\kappa_0 t} \frac{R(\kappa_0 t)}{\phi_X(\kappa_0 t)} E_X(f, r(\kappa_0 t)) \right\|_Y \\
&= \left\| \frac{R(\kappa_0 t)\, r(\kappa_0 t)}{R(t) r(t)^s} \frac{\phi_X(t)\phi_Y(\kappa_0 t)}{\phi_X(\kappa_0 t)\phi_Y(t)} \frac{1}{\kappa_0} \frac{r(\kappa_0 t)^{-s}}{\phi_Y(\kappa_0 t)} E_X(f, r(\kappa_0 t)) \right\|_Y \\
&\leq \sup_{t>0} \left(\frac{R(\kappa_0 t)\, r(\kappa_0 t)}{R(t) r(t)^s} \frac{\phi_X(t)\phi_Y(\kappa_0 t)}{\phi_X(\kappa_0 t)\phi_Y(t)} \frac{1}{\kappa_0} \right) \left\| \frac{r(\kappa_0 t)^{-s}}{\phi_Y(\kappa_0 t)} E_X(f, r(\kappa_0 t)) \right\|_Y \\
&\leq h_Y(1/\kappa_0) \left\| \frac{r(t)^{-s}}{\phi_Y(t)} E_X(f, r(t)) \right\|_Y \\
&\leq \|f\|_{\mathring{B}^s_{(k,m),X,Y}(\Omega)}.
\end{aligned}$$

Part (ii) is analogous. □

3.5 Uncertainty type inequalities

The purpose of this section is to extend the generalized uncertainty Sobolev inequalities obtained in [66] to the context of Besov spaces.

Definition 3.5.1. *Let (Ω, d, μ) be a (k,m)-space, X, Y r.i. spaces, and $0 < s < 1$. We will say that a μ-measurable function $w : \Omega \to (0, \infty)$ is an $(s, (k,m), X, Y)$-admissible weight if*

$$[w] := \sup_{t>0} \left(\left(\left(\frac{1}{w} \right)^*_\mu (t) \right)^s \frac{1}{v_s(t) \frac{\phi_X(t)}{\phi_Y(t)}} \right) < \infty.$$

Theorem 7. *Let (Ω, d, μ) be a uniform (k,m)-space, let X, Y be r.i. spaces, suppose $0 < s < 1$, and let w be an $(s, (k,m), X, Y)$-admissible weight. Assume that Y satisfies the $Q(s, (k,m), X)$-condition. Let $\alpha > 0$. Then for all $f \in L^1_\mu(\Omega) + L^\infty_\mu(\Omega)$ such that $f^*_\mu(\infty) = 0$, we have that*

$$\|f\|_Y \leq [w]^{\frac{\alpha}{\alpha+1}} \|f\|^{\frac{\alpha}{\alpha+1}}_{\mathring{B}^s_{(k,m),X,Y}(\Omega)} \|w^{\alpha s} f\|^{\frac{1}{\alpha+1}}_Y. \tag{3.5.1}$$

Proof. Since $f_\mu^*(\infty) = 0$, by the Fundamental Theorem of Calculus and (3.3.7), we get

$$\begin{aligned} f_\mu^{**}(t) &= \int_t^\infty \left(f_\mu^{**}(s) - f_\mu^*(s)\right)\frac{ds}{s} \\ &\le \int_t^\infty \frac{1}{\kappa_0 s}\frac{R(\kappa_0 s)}{\phi_X(\kappa_0 s)} E_X(f, r(\kappa_0 s))\frac{ds}{s}. \end{aligned} \tag{3.5.2}$$

Then

$$\begin{aligned} \|f\|_Y &= \left\|f\left(\frac{w}{w}\right)^s\right\|_Y \le \left\|f\left(\frac{w}{w}\right)^s \chi_{\{w\le r^{1/s}\}}\right\|_Y + \left\|f\left(\frac{w}{w}\right)^s \chi_{\{w>r^{1/s}\}}\right\|_Y \\ &\le r\left\|f\left(\frac{1}{w}\right)^s\right\|_Y + \left\|f\left(\frac{w}{w}\right)^{s\alpha} \chi_{\{w>r^{1/s}\}}\right\|_Y \\ &\le r\left\|f\left(\frac{1}{w}\right)^s\right\|_Y + r^{-\alpha}\left\|w^{\alpha s} f\right\|_Y. \end{aligned}$$

Now we estimate the first term:

$$\begin{aligned} \left\|f\left(\frac{1}{w}\right)^s\right\|_Y &\le \left\|f_\mu^*(t)\left(\left(\frac{1}{w}\right)_\mu^*(t)\right)^s \frac{v_s(t)\frac{\phi_X(t)}{\phi_Y(t)}}{v_s(t)\frac{\phi_X(t)}{\phi_Y(t)}}\right\|_Y \\ &\le [w]\left\|f_\mu^*(t) v_s(t)\frac{\phi_X(t)}{\phi_Y(t)}\right\|_Y \\ &\le [w]\left\|f_\mu^{**}(t/2) v_s(t)\frac{\phi_X(t)}{\phi_Y(t)}\right\|_Y \\ &\le [w]\left\|v_s(t)(t)\frac{\phi_X(t)}{\phi_Y(t)}\int_t^\infty \frac{1}{\kappa_0 s}\frac{R(\kappa_0 s)}{\phi_X(\kappa_0 s)} E_X(f, r(\kappa_0 s))\frac{ds}{s}\right\|_Y \quad \text{(by (3.5.2))} \\ &\le [w]\left\|\frac{r(t)^{-s}E_X(f, r(t))}{\phi_Y(t)}\right\|_Y \quad \text{(by the } Q(s,(k,m),X)-\text{condition)} \\ &\le [w]\,\|f\|_{\mathring{B}^s_{(k,m),X,Y}(\Omega)}. \end{aligned}$$

In summary, we have proved that there is an absolute constant $A > 0$ such that

$$\|f\|_Y \le A\,[w]\,r\,\|f\|_{\mathring{B}^s_{(k,m),X,Y}(\Omega)} + r^{-\alpha}\left\|w^{\alpha s} f\right\|_Y. \tag{3.5.3}$$

Selecting the value $r = \left(\frac{\|w^{s\alpha} f\|_Y}{2A[w]\|f\|_{\mathring{B}^s_{(k,m),X,Y}(\Omega)}}\right)^{\frac{1}{1+\alpha}}$ to compute (3.5.3) balances the two terms and we obtain the multiplicative inequality (3.5.1). □

Remark 3.5.1. *The connection with the notion of isoperimetric weight introduced in [66] is the following: consider the case $(\mathbb{R}^n, |\cdot|, \mathcal{L}^n)$. Obviously, it is a uniform (k,m)-space with $k = m = \frac{1}{n}$. Let $X = Y = L^q$. Then*

$$[w] := \sup_{t>0}\left(\left(\left(\frac{1}{w}\right)_\mu^*(t)\right)^s t^{\frac{s}{n}}\right) = \sup_{t>0}\left(\left(\left(\frac{1}{w}\right)_\mu^*(t)\right) t^{\frac{1}{n}}\right)^s.$$

Thus w is admissible if, and only if, $\frac{1}{w} \in L^{n,\infty}$ (i.e. w is an isoperimetric weight).

Let $\alpha > 0$, $1 \le q < \infty$, and $0 < s < 1$ with $s < n/q$. By Remark 3.4.2, L^q satisfies the $Q(s, (\frac{1}{n}, \frac{1}{n}), L^q)$-condition. Then by Theorem 7, if $\frac{1}{w} \in L^{n,\infty}$,

$$\|f\|_{l^q} \le (2\kappa[w])^{\frac{\alpha}{\alpha+1}} \|f\|_{\mathring{B}^s_{p,q}(\mathbb{R}^n)}^{\frac{\alpha}{\alpha+1}} \|w^{\alpha s} f\|_{L^q}^{\frac{1}{\alpha+1}},$$

where $\mathring{B}^s_{p,q}(\mathbb{R}^n)$ is the classical Euclidean Besov space.

3.6 Embedding into BMO and essential continuity

Definition 3.6.1. *Let $f : \Omega \to \mathbb{R}$ be a locally integrable function on Ω. Then f is said to have bounded mean oscillation (written $f \in BMO$) if the seminorm is given by*

$$\|f\|_{BMO_\mu(\Omega)} = \sup_B \left\{ \fint_B |f - f_B| \, d\mu \right\} < \infty.$$

Here B denotes any ball of Ω.

Theorem 8. *Let (Ω, d, μ) be a uniform (k, m)-space. Then*

$$\|f\|_{BMO_\mu(\Omega)} \le \sup_{t>0} \frac{R(t)}{t\phi_X(t)} E_X(f, r(t))$$

Proof. Let $B := B(x)$ be a ball centred at x. Since (Ω, d, μ) is a uniform (k, m)-space, we have that

$$\mu(B) \le \mu(B(x, r(\mu(B)/c)) \le CR(\mu(B)/c).$$

Then

$$\begin{aligned}
I &:= \int_B \left| f(y) - \fint_{B(x)} f(s) d\mu(s) \right| d\mu(y) \\
&\le \fint_B \left(\int_{B(x)} |f(y) - f(s)| \, d\mu(s) \right) d\mu(y) \\
&\le \fint_B \int_{B(x, r(\mu(B)/c))} |f(y) - f(s)| \, d\mu(s) d\mu(y) \\
&\le C \frac{R(\mu(B)/c)}{\mu(B)} \int_B \left(\fint_{B(x, r(\mu(B)/c))} |f(s) - f(y)| \, d\mu(s) \right) d\mu(y) \\
&\le C \frac{R(\mu(B)/c)}{\mu(B)} E_X(f, r(\mu(B)/c)) \left\| \chi_{B(x)} \right\|_{X'} \quad \text{(by (2.2.4))} \\
&= C \frac{R(\mu(B)/c)}{\phi_X(\mu(B))} E_X(f, r(\mu(B)/c)).
\end{aligned}$$

Using this estimate and Remark 3.2.1, we get

$$
\begin{aligned}
\|f\|_{BMO_\mu(\Omega)} &= \sup_B \fint_B \left| f(y) - \fint_B f(s)d\mu(s) \right| d\mu(y) \\
&\leq \sup_{\mu(B)} C \frac{R(\mu(B)/c)}{\mu(B)\phi_X(\mu(B))} E_X(f, r(\mu(B)/c)) \\
&\leq \sup_{t>0} \sup_{t/C_D \leq \mu(B) \leq t} C \frac{R(\mu(B)/c)}{\mu(B)\phi_X(t/C_D)} E_X(f, r(\mu(B)/c)) \\
&\leq \sup_{t>0} \frac{R(t/c)}{t\phi_X(t)} E_X(f, r(t/c)) \leq \sup_{t>0} \frac{R(t)}{t\phi_X(t)} E_X(f, r(t)).
\end{aligned}
$$

□

3.6.1 Essential continuity

Theorem 9. *Let (Ω, d, μ) be a uniform (k, m)-space. Let X be an r.i. space on Ω. Now suppose $f \in L^1_\mu(\Omega) + L^\infty_\mu(\Omega)$ satisfies*

$$
\int_0^\infty \frac{R(t)}{t\phi_X(t)} E_X(f, r(t)) \frac{dt}{t} < \infty.
$$

Then f is μ-locally essentially continuous.

Proof. If $f \geq 0$, Theorem 5 implies

$$
f_\mu^{\star\star}(t) - f_\mu^{\star}(t) = f_\mu^{**}(t) - f_\mu^{*}(t) \leq \frac{1}{\kappa_0 t} \frac{R(\kappa_0 t)}{\phi_X(\kappa_0 t)} E_X(f, r(\kappa_0 t)).
$$

If f is bounded from below and $c = \inf(f)$, then $f - c \geq 0$, and therefore

$$
\begin{aligned}
O_\mu(f-c)(t) &\leq \frac{1}{\kappa_0 t} \frac{R(\kappa_0 t)}{\phi_X(\kappa_0 t)} E_X(f - c, r(\kappa_0 t)) \\
&\leq \frac{1}{\kappa_0 t} \frac{R(\kappa_0 t)}{\phi_X(\kappa_0 t)} E_X(f, r(\kappa_0 t)).
\end{aligned}
$$

By Proposition 2.1.1 (ix),

$$
f_\mu^{\star\star}(t) - f_\mu^{\star}(t) = (f-c)_\mu^{**}(t) - (f-c)_\mu^{*}(t),
$$

and thus

$$
f_\mu^{\star\star}(t) - f_\mu^{\star}(t) \leq \frac{1}{\kappa_0 t} \frac{R(\kappa_0 t)}{\phi_X(\kappa_0 t)} E_X(f, r(\kappa_0 t)).
$$

Let $f \in L^1_\mu(\Omega) + L^\infty_\mu(\Omega)$, and let B be a ball. Given $n \in \mathbb{N}$, we consider $f_n = \max(f\chi_B, -n)$. Since f_n is bounded from below, we get

$$
\begin{aligned}
(f_n)_\mu^{\star\star}(t) - (f_n)_\mu^{\star}(t) &\leq \frac{1}{\kappa_0 t} \frac{R(\kappa_0 t)}{\phi_X(\kappa_0 t)} E_X(f_n, r(\kappa_0 t)) \\
&\leq \frac{1}{\kappa_0 t} \frac{R(\kappa_0 t)}{\phi_X(\kappa_0 t)} E_X(f, r(\kappa_0 t)).
\end{aligned}
$$

Let $0 < a < \mu(B)$. By the Fundamental Theorem of Calculus,

$$\int_a^{\mu(B)} \left((f_n)_\mu^{\star\star}(t) - (f_n)_\mu^{\star}(t)\right)\frac{dt}{t} = \frac{1}{a}\int_0^a (f_n)_\mu^{\star}(t)dt - \frac{1}{\mu(B)}\int_0^{\mu(B)} (f_n)_\mu^{\star}(t)dt.$$

Since $f_n(z) \to f\chi_B(z)$ μ-a.e. and $|f_n| \le |f\chi_B|$, we have

$$\frac{1}{a}\int_0^a (f_n)_\mu^{\star}(t)dt - \frac{1}{\mu(B)}\int_0^{\mu(B)} (f_n)_\mu^{\star}(t)dt := (F)$$
$$(F) \underset{n\to\infty}{\to} \frac{1}{a}\int_0^a (f\chi_B)_\mu^{\star}(t)dt - \frac{1}{\mu(B)}\int_0^{\mu(B)} (f\chi_B)_\mu^{\star}(t)dt.$$

Letting $a \to 0$, we get

$$\begin{aligned}(f\chi_B)_\mu^{\star\star}(0) - (f\chi_B)_\mu^{\star\star}(\mu(B)) &\le \int_0^{\mu(B)} \frac{1}{\kappa_0 t}\frac{R(\kappa_0 t)}{\phi_X(\kappa_0 t)}E_X(f, r(\kappa_0 t))\frac{dt}{t}\\ &\le \int_0^{k\mu(B)} \frac{R(t)}{t\phi_X(t)}E_X(f, r(t))\frac{dt}{t}.\end{aligned}$$

By (2.1.1),

$$\begin{aligned}&ess\sup f\chi_B - \frac{1}{\mu(B)}\int_0^{\mu(B)} (f\chi_B)_\mu^{\star}(t)dt\\ &\le \int_0^{\kappa_0\mu(B)} \frac{R(t)}{t\phi_X(t)}E_X(f, r(t))\frac{dt}{t}.\end{aligned}$$

Similarly, considering $-f\chi_B$ instead of $f\chi_B$, we obtain

$$\begin{aligned}&\frac{1}{\mu(B)}\int_0^{\mu(B)} (-f\chi_B)_\mu^{\star}(s)ds - ess\inf(f\chi_B)\\ &\le \int_0^{\kappa_0\mu(B)} \frac{R(t)}{t\phi_X(t)}E_X(f, r(t))\frac{dt}{t}.\end{aligned}$$

Since $f\chi_B$ and $-f\chi_B$ are both supported on B, we have that

$$\int_0^{\mu(B)} (f\chi_G)_\mu^{\star}(s)ds = \int_B f d\mu \quad \text{and} \quad \int_0^{\mu(B)} (-f\chi_G)_\mu^{\star}(s)ds = -\int_B f d\mu.$$

Adding these results, we have that for μ-almost every $x, y \in B$

$$\begin{aligned}|f(x) - f(y)| &\le ess\sup(f\chi_B) - ess\inf(f\chi_B)\\ &\le 2\int_0^{\kappa_0\mu(B)} \frac{R(t)}{t\phi_X(t)}E_X(f, r(t))\frac{dt}{t}.\end{aligned}$$

and μ-locally essentially continuity follows. $\square$

3.7 Sobolev type embeddings for homogeneous Besov spaces

In this section we going to consider in detail Sobolev type embeddings for the homogeneous Besov spaces $\dot{\mathcal{B}}^s_{p,q}(\Omega)$ where $0 < p < \infty$, $0 < q \leq \infty$.

First of all, note that an elementary computation shows (see Remark 3.4.1) that

$$\|f\|_{\dot{\mathcal{B}}^s_{p,q}(\Omega)} \simeq \begin{cases} \left(\int_0^\infty \left(\frac{E_p(f,r(t))}{r(t)^s}\right)^q \frac{dt}{t}\right)^{1/q}, & 0 < q < \infty, \\ \sup_{t>0} r(t)^{-s} E_p(f, r(t)), & q = \infty. \end{cases}$$

In case $1 \leq p < \infty$ and $1 \leq q \leq \infty$, our results will be a consequence of the theory developed in the previous sections. However, for $0 < p, q < 1$, $L^p_\mu(\Omega)$ and $L^q([0,\infty))$ are not Banach spaces, thus the previous theory cannot be applied.

Lemma 3.7.1. *Let $0 < p < 1$. Let $f \in L^p_\mu(\Omega) + L^\infty_\mu(\Omega)$. Then:*

1. *If (Ω, d, μ) is uniform, then for all $t > 0$ we have that*

$$O_\mu(|f|^p, t) \leq \frac{R(\kappa_0 t)}{(\kappa_0 t)^2} E_p(f, \kappa_0 t)^p.$$

2. *If (Ω, d, μ) is bounded from below, then for all $t > 0$ we have that*

$$O_{\tilde{\mu}}(|f|^p, t) \leq \frac{R(\kappa_1 t)}{(\kappa_1 t)^2} E_p(f, r(\kappa_1 t))^p.$$

Proof. Let $B = B(x)$ be a ball centred at x. Since $0 < p < 1$, we have that

$$|f(x)|^p \chi_{B(x)}(y) \leq |f(x) - f(y)|^p \chi_{B(x)}(y) + |f(y)|^p \chi_{B(x)}(y).$$

Integrating with respect to $d\mu(y)$, we have that

$$\begin{aligned} |f(x)|^p \mu(B) &\leq \int_{B(x)} |f(x) - f(y)|^p d\mu(y) + \int_{B(x)} |f(y)|^p d\mu(y) \\ &\leq \int_{B(x)} |f(x) - f(y)|^p d\mu(y) + \int_0^{\mu(B)} (|f|^p)^*_\mu(s) ds \quad \text{(by (2.1.3))}. \end{aligned}$$

Now integrating with respect to $d\mu(x)$ over a subset $E \subset \Omega$ with $\mu(E) = \mu(B)/2$, we get

$$\begin{aligned} \int_E |f(x)|^p d\mu(x) &\leq \int_E \fint_{B(x)} |f(x) - f(y)|^p d\mu(y) d\mu(x) + \int_E \frac{1}{\mu(B)} \left(\int_0^{\mu(B)} f^*_\mu(s) ds\right) d\mu(x) \\ &\leq \int_\Omega \fint_{B(x)} |f(x) - f(y)|^p d\mu(y) d\mu(x) + \frac{1}{2} \int_0^{\mu(B)} f^*_\mu(s) ds \end{aligned}$$

By (2.2.2), taking the supremum over all such sets E, we obtain

$$\int_0^{\mu(B)/2} (|f|^p)_\mu^*(s)ds \leq \int_\Omega \fint_{B(x)} |f(x)-f(y)|^p d\mu(y)d\mu(x) + \frac{1}{2}\int_0^{\mu(B)} (|f|^p)_\mu^*(s)ds.$$

Equivalently,

$$(|f|^p)_\mu^{**}(\mu(B)/2) - (|f|^p)_\mu^{**}(\mu(B)) \leq \frac{1}{\mu(B)}\int_\Omega \fint_{B(x)} |f(x)-f(y)|^p d\mu(y)d\mu(x)$$

Now (i) and (ii) follow in the same way as Theorem 5. □

Definition 3.7.1. *Suppose $0 < p < \infty$ and $0 < q \leq \infty$ and let v be a weight on $(0,\infty)$. The space $S_\mu^{p,q}(v)$ is the collection of all μ-measurable functions such that $\|f\|_{S_\mu^{p,q}(v)} < \infty$, where*

$$\|f\|_{S_\mu^{p,q}(v)} = \left(\int_0^\infty O_\mu(|f|^p, t)^{\frac{q}{p}} v(t)dt\right)^{1/q}.$$

Remark 3.7.1. *For $p = 1$ the spaces $S^{1,q}(v)$ were introduced in [14]. Note that if $1 \leq p < \infty$ and $1 \leq q \leq \infty$, then*

$$S_\mu^{L^p,L^q}(v_s) = S_\mu^{1,q}(v_s).$$

Corollary 10. *Let (Ω, d, μ) be a (k,m)-space. Let $0 < s < 1$ and $0 < p < \infty$, $0 < q \leq \infty$. Let*

$$v(t) = \min\left(t^{1+\frac{1}{\max(1,p)}-\frac{m+s\min(1,p)}{k}}, t^{1+\frac{1}{\max(1,p)}-\frac{k+s\min(1,p)}{m}}\right)^{\frac{q}{\min(1,p)}} \frac{1}{t}.$$

Then:

(i.) If (Ω, d, μ) is uniform, then

$$\dot{\mathcal{B}}_{p,q}^s(\Omega) \subset S_\mu^{\min(1,p),q}(v).$$

(ii.) If (Ω, d, μ) is bounded from below, then

$$\dot{\mathcal{B}}_{p,q}^s(\Omega) \subset S_{\tilde{\mu}}^{\min(1,p),q}(v).$$

Proof. Part (i.) In the case $1 \leq p < \infty$ the proof given in Theorem 6 works. In case $0 < p < 1$, then from Lemma 3.7.1 it follows that

$$\left(\frac{t^2}{t^{p/q}R(t)\,r(t)^{sp}} O_\mu(|f|^p, t)\right)^{1/p} \preceq \frac{r(\kappa_0 t)^{-s} E_p(f, r(\kappa_0 t))}{(\kappa_0 t)^{1/q}},$$

and the result is obtained by taking $L^q([0,\infty))$-(quasi) norms on both sides.

Part (ii) can be proved in the same way. □

The following lemma will be useful in what follows.

Lemma 3.7.2. *(see [5, Lemma 5.4]). Let* $1 \leq q < \infty$*, and suppose that* (w, v) *is a pair of weights satisfying the following condition: there exists* $C > 0$ *such that for all* $0 < t < 1$,

$$\left(\int_0^t w(s)ds\right)^{1/q}\left(\int_t^1 \frac{v(s)^{\frac{-1}{q-1}}}{s^{\frac{q}{q-1}}}ds\right)^{(q-1)/q} \leq C.$$

Then

$$\left(\int_0^1 f_\mu^{**}(s)^q w(s)ds\right)^{1/q} \leq \left(\int_0^1 \left(f_\mu^{**}(s) - f_\mu^*(s)\right)^q v(s)ds\right)^{1/q} + \left(\int_0^1 w(s)ds\right)^{1/q}\int_0^1 f_\mu^*(t)ds.$$

Lemma 3.7.3. *Let* $0 < q < 1$ *and* $b > 0$*, then*

$$\left(\int_0^1 t^b f_\mu^{**}(t)^q \frac{dt}{t}\right)^{1/q} \leq \left(\int_0^1 t^b \left(f_\mu^{**}(t) - f_\mu^*(t)\right)^q \frac{dt}{t}\right)^{1/q} + f_\mu^{**}(1).$$

Proof. We integrate by parts and obtain

$$\begin{aligned}\int_0^1 t^b f_\mu^{**}(t)^q \frac{dt}{t} &= \frac{1}{b}\left[t^b f_\mu^{**}(t)^q\right]_0^1 + \frac{q}{b}\int_0^1 t^b f_\mu^{**}(t)^{q-1}\left(f_\mu^{**}(t) - f_\mu^*(t)\right)\frac{dt}{t}\\ &\leq \frac{1}{b}\left[t^b f^{**}(t)^q\right]_0^1 + \frac{q}{b}\int_0^1 t^b\left(f_\mu^{**}(t) - f_\mu^*(t)\right)^q \frac{dt}{t} \text{ (since } q < 1).\end{aligned}$$

Now

$$\left[t^b f_\mu^{**}(t)^q\right]_0^b = f_\mu^{**}(1)^q - \lim_{t\to 0} t^b f_\mu^{**}(t)^q.$$

To finish the proof we need to show that the previous limit is finite. This is obvious if $f_\mu^{**}(0) < \infty$. If $f_\mu^{**}(0) = \infty$, taking into account that $t\left(f_\mu^{**}(t) - f_\mu^*(t)\right)$ is increasing, we get

$$\begin{aligned}t\left(f_\mu^{**}(t) - f_\mu^*(t)\right)\left(\int_t^1 s^{b-1-q}\right)^{1/q} &\leq \left(\int_t^1 s^q\left(f_\mu^{**}(s) - f_\mu^*(s)\right)^q s^{b-1-q}\right)^{1/q}\\ &= \left(\int_t^1 s^b\left(f_\mu^{**}(s) - f_\mu^*(s)\right)^q \frac{ds}{s}\right)^{1/q}.\end{aligned}$$

If $t < 1/2$, then

$$\left(\int_t^1 s^{b-1-q}\right)^{1/q} \geq \left(\int_t^{2t} s^{b-1-q}\right)^{1/q} \simeq t^{b/q-1}, \text{ if } b \neq q,$$

and

$$\left(\int_t^1 s^{b-1-q}\right)^{1/q} \geq \left(\int_t^{2t} \frac{1}{s}\right)^{1/q} \simeq 1, \text{ if } b = q.$$

Thus

$$t^{b/q}\left(f_\mu^{**}(t)-f_\mu^*(t)\right) \leq \left(\int_t^1 s^b\left(f_\mu^{**}(s)-f_\mu^*(s)\right)^q \frac{ds}{s}\right)^{1/q}.$$

Finally, by L'Hopital's rule,

$$\begin{aligned}\lim_{t\to 0} t^{b/q} f_\mu^{**}(t) &= \lim_{t\to 0}\frac{f_\mu^{**}(t)}{t^{-b/q}} = \lim_{t\to 0}\frac{-(f_\mu^{**}(t)-f_\mu^*(t))/t}{-\frac{b}{q}t^{-b/q-1}} \\ &= \lim_{t\to 0}\frac{t^{b/q}(f_\mu^{**}(t)-f_\mu^*(t)}{b/q} \leq \left(\int_0^1 s^b\left(f_\mu^{**}(s)-f_\mu^*(s)\right)^q \frac{ds}{s}\right)^{1/q}.\end{aligned}$$

□

Lemma 3.7.4. *Given $a < b < \infty$, we define*

$$v(t) = \frac{\min(t^a, t^b)}{t}.$$

Let $0 < q \leq \infty$ and $f \in L_\mu^1(\Omega) + L_\mu^\infty(\Omega)$, with $f_\mu^*(\infty) = 0$. Then

(i.) If $0 < a < b < \infty$, then

$$\|f\|_{S_\mu^{1,q}(v)} \simeq \left(\int_0^\infty f_\mu^{**}(t)^q v(t)dt\right)^{1/q}.$$

(ii.) If $a \leq 0$, then

$$\left(\int_0^1 f_\mu^{**}(t)^q v(t)dt\right)^{1/q} \leq \|f\|_{S_\mu^{1,q}(v)} + f_\mu^{**}(1).$$

(iii.) If $b = 0$ and $q > 1$, then

$$\left(\int_0^\infty \left(\frac{f_\mu^{**}(t)^q}{1+\ln\frac{1}{t}}\right)^q \frac{dt}{t}\right)^{1/q} \leq \|f\|_{S_\mu^{1,q}(v)} + f_\mu^{**}(1).$$

(iv.) If $b = 0$ and $q \leq 1$ or $b < 0$ and $0 < q \leq \infty$, then

$$\|f\|_\infty \leq \|f\|_{S_\mu^{1,q}(v)} + f_\mu^{**}(1).$$

Proof. (i.) By [14, Corollary 4.3.] we only need to check that

$$\int_0^r v(t)dt \leq r^q \int_r^\infty \frac{v(t)}{t^q}dt, \quad r > 0.$$

Pick $0 < \varepsilon < a$. Then

$$\begin{aligned}\int_0^r v(t)dt &= \int_0^r \min(t^{a-\varepsilon}, t^{b-\varepsilon})\frac{dt}{t^{1-\varepsilon}} \leq \min(r^{a-\varepsilon}, r^{b-\varepsilon})\int_0^r \frac{dt}{t^{1-\varepsilon}}\\ &\leq \min(r^a, r^b) \leq \min(r^a, r^b)\frac{r^q}{r^q}\\ &\leq \min(r^a, r^b) r^q \int_r^{2r} \frac{dt}{s^{q+1}} \leq r^q \int_r^{2r} v(t)\frac{dt}{s^q}\\ &\leq r^q \int_r^\infty \frac{v(t)}{t^q} dt.\end{aligned}$$

(ii.) By Lemma 3.7.3,

$$\left(\int_0^1 t^b f_\mu^{**}(t)^q \frac{dt}{t}\right)^{1/q} \leq \left(\int_0^1 t^b O_\mu(f,t)^q \frac{dt}{t}\right)^{1/q} + f_\mu^{**}(1).$$

(iii.) By Lemma 3.7.2 with $w(t) = \left(\frac{1}{1+\ln\left(\frac{1}{t}\right)}\right)^q \frac{1}{s}$ and $v(t) = \frac{1}{t}$, we get

$$\begin{aligned}\left(\int_0^1 \left(\frac{f_\mu^{**}(t)}{1+\ln\left(\frac{1}{t}\right)}\right)^q \frac{dt}{t}\right)^{1/q} &\leq \left(\int_0^1 O_\mu(f,t)^q \frac{dt}{t}\right)^{1/q} + f_\mu^{**}(1)\\ &\leq \|f\|_{S_\mu^{1,q}(v)} + f_\mu^{**}(1)\end{aligned}$$

and

$$\left(\int_1^\infty \left(\frac{f_\mu^{**}(t)}{1+\ln\left(\frac{1}{t}\right)}\right)^q \frac{dt}{t}\right)^{1/q} \leq f_\mu^{**}(1)\left(\int_1^\infty \left(\frac{1}{1+\ln\left(\frac{1}{t}\right)}\right)^q \frac{dt}{t}\right)^{1/q} \leq f_\mu^{**}(1).$$

(iv.) If $b = 0$ and $q = 1$, then

$$\|f\|_\infty = f_\mu^{**}(0) = \int_0^1 O_\mu(f,t)\frac{dt}{t} + f_\mu^{**}(1) \leq \|f\|_{S_\mu^{1,q}(v)} + f_\mu^{**}(1).$$

If $0 < q < 1$, let $0 < r < 1$. Then

$$\begin{aligned}f_\mu^{**}(r) - f_\mu^{**}(1) &= \int_r^1 O_\mu(f,t)\frac{dt}{t} = f_\mu^{**}(r)^{1-q}\int_r^1 \frac{O_\mu(f,t)}{f_\mu^{**}(r)^{1-q}}\frac{dt}{t} \qquad (3.7.1)\\ &\leq \int_r^1 \frac{O_\mu(f,t)}{f_\mu^{**}(t)^{1-q}}\frac{dt}{t} \leq \int_r^1 \frac{O_\mu(f,t)}{O_\mu(f,t)^{1-q}}\frac{dt}{t}\\ &= f_\mu^{**}(r)^{1-q}\int_r^1 O_\mu(f,t)^q\frac{dt}{t}\\ &\leq (1-q)f_\mu^{**}(r) + q\left(\int_0^1 O_\mu(f,t)^q\frac{dt}{t}\right)^{1/q} \qquad (3.7.2)\end{aligned}$$

thus

$$q f_\mu^{**}(r) \leq q\left(\int_0^1 O_\mu(f,t)^q\frac{dt}{t}\right)^{1/q} + f_\mu^{**}(1)$$

which implies

$$\|f\|_\infty = f_\mu^{**}(0) \le \left(\int_0^1 O_\mu(f,t)^q \frac{dt}{t}\right)^{1/q} + \frac{f_\mu^{**}(1)}{q}.$$

If $b < 0$ and $1 \le q \le \infty$, then

$$\begin{aligned} f_\mu^{**}(0) - f_\mu^{**}(1) &= \int_0^1 O_\mu(f,t)\frac{dt}{t} \\ &\le \left(\int_0^1 \left(t^b O_\mu(f,t)\right)^q \frac{dt}{t}\right)^{1/q} \left(\int_0^1 \left(t^{-b}\right)^{\frac{q}{q-1}} \frac{dt}{t}\right)^{\frac{q-1}{q}} \\ &\lesssim \left(\int_0^1 \left(t^b O_\mu(f,t)\right)^q \frac{dt}{t}\right)^{1/q}. \end{aligned}$$

If $b < 0$ and $0 < q < 1$, then

$$f_\mu^{**}(0) - f_\mu^{**}(1) = \int_0^1 O_\mu(f,t)\frac{dt}{t} \le \int_0^1 t^b O_\mu(f,t)\frac{dt}{t}$$

and we finish the proof in the same way as in (3.7.1). □

Now we are ready to establish our Sobolev embedding theorem for homogeneous Besov spaces $\dot{\mathcal{B}}^s_{p,q}(\Omega)$. Motivated by the classical theory, we will distinguish three cases: The subcritical case, when there is an embedding into a Lorentz type space; the critical case, when $\dot{\mathcal{B}}^s_{p,q}(\Omega)$ is embedded into a logarithmic Lorentz space; and the supercritical case, when the Besov space is embedded into L^∞.

Theorem 11. *Let (Ω, d, μ) be a uniform (k,m)-space. Let $0 < s < 1$, $0 < p < \infty$, $0 < q \le \infty$ and $f \in L_\mu^{\min(1,p)}(\Omega) + L_\mu^\infty(\Omega)$, with $\left(|f|^{\min(1,p)}\right)^*_\mu(\infty) = 0$. Then:*

1. ***Subcritical case:***

   a) If $s\min(1,p) < k(1 + \frac{1}{\max(1,p)}) - m$, then

   $$\|f\|_{L_\mu^{\alpha,q}(\Omega)+L_\mu^{\beta,q}(\Omega)} \lesssim \|f\|_{\dot{\mathcal{B}}^s_{p,q}(\Omega)},$$

   where

   $$\frac{\min(1,p)}{\alpha} = 1 + \frac{1}{\max(1,p)} - \frac{k + s\min(1,p)}{m},$$

   and

   $$\frac{\min(1,p)}{\beta} = 1 + \frac{1}{\max(1,p)} - \frac{m + s\min(1,p)}{k}.$$

   b) If $k(1 + \frac{1}{\max(1,p)}) - m < s\min(1,p) < m(1 + \frac{1}{\max(1,p)}) - k$, then

   $$\left(\int_0^1 \left(t^{\frac{1}{\alpha}} f_\mu^{**}(t)\right)^q \frac{dt}{t}\right)^{1/q} \lesssim \|f\|_{\dot{\mathcal{B}}^s_{p,q}(\Omega)} + \|f\|_{L_\mu^{\min(1,p)}(\Omega)+L_\mu^\infty(\Omega)}. \quad (3.7.3)$$

   Here,

   $$\frac{\min(1,p)}{\alpha} = 1 + \frac{1}{\max(1,p)} - \frac{m + s\min(1,p)}{k}.$$

2. ***Critical case:***
If $s\min(1,p) = m(1+\frac{1}{\max(1,p)}) - k$*, then*

   a) *If* $q > 1$*, we get*

$$\left(\int_0^\infty \left(\frac{f_\mu^{**}(t)}{1+\ln\left(\frac{1}{t}\right)}\right)^q \frac{dt}{t}\right)^{1/q} \preceq \|f\|_{\dot{\mathcal{B}}^s_{p,q}(\Omega)} + \|f\|_{L_\mu^{\min(1,p)}(\Omega)+L_\mu^\infty(\Omega)}.$$

   b) *If* $0 < q \le 1$*, we get*

$$\|f\|_\infty \preceq \|f\|_{\dot{\mathcal{B}}^s_{p,q}(\Omega)} + \|f\|_{L_\mu^{\min(1,p)}(\Omega)+L_\mu^\infty(\Omega)}.$$

3. ***Supercritical case:***
If $s\min(1,p) > m(1+\frac{1}{\max(1,p)}) - k$*, then*

$$\|f\|_\infty \preceq \|f\|_{\dot{\mathcal{B}}^s_{p,q}(\Omega)} + \|f\|_{L_\mu^{\min(1,p)}(\Omega)+L_\mu^\infty(\Omega)}.$$

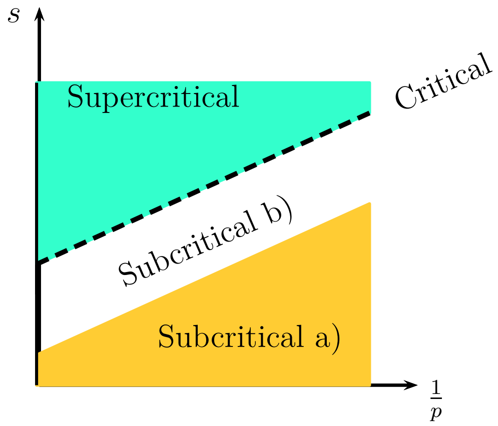

Figure 3.2: Theorem 11.

Proof. The proof follows from Lemma 3.7.4. We will employ (3.7.3) if $0 < p < 1$. Then

$$v(t) = \min\left(t^{2-\frac{m+sp}{k}}, t^{2-\frac{k+sp}{m}}\right)^{\frac{q}{p}} \frac{1}{t}$$

with $2 - \frac{m+sp}{k} \le 0$. Now by Lemma 3.7.4, applied to $|f|^p$ and q/p, we have that

$$\left(\int_0^1 \left(\frac{1}{t}\int_0^t f_\mu^*(s)^p ds\right)^{q/p} t^{\left(2-\frac{k+sp}{m}\right)\frac{q}{p}} \frac{dt}{t}\right)^{p/q} \preceq \||f|^p\|_{S_\mu^{1,q/p}(v)} + (|f|^p)_\mu^{**}(1).$$

Obviously

$$\left(\int_0^1 \left(f_\mu^*(s)t^{\left(2-\frac{k+sp}{m}\right)\frac{1}{p}}\right)^q dt\right)^{p/q} \le \left(\int_0^1 \left(\frac{1}{t}\int_0^t f_\mu^*(s)^p ds\right)^{q/p} t^{\left(2-\frac{k+sp}{m}\right)\frac{q}{p}} \frac{dt}{t}\right)^{p/q}.$$

Since

$$\begin{aligned}\left\||f|^p\right\|_{S_\mu^{1,q/p}(v)} &= \left(\int_0^\infty O_\mu(|f|^p,t)^{\frac{q}{p}} v(t)dt\right)^{p/q} = \|f\|^p_{S_\mu^{p,q}(v)} \\ &\le \|f\|^p_{\dot{\mathcal{B}}^s_{p,q}(\Omega)} \text{ (by Corollary 10),}\end{aligned}$$

we have that

$$\left(\int_0^1 \left(f_\mu^*(s)t^{\left(2-\frac{k+sp}{m}\right)\frac{1}{p}}\right)^q dt\right)^{p/q} \le \|f\|^p_{\dot{\mathcal{B}}^s_{p,q}(\Omega)} + \left(|f|^p\right)^{**}_\mu(1)$$

and thus

$$\begin{aligned}\left(\int_0^1 \left(f_\mu^*(s)t^{\left(2-\frac{k+sp}{m}\right)\frac{1}{p}}\right)^q dt\right)^{1/q} &\le \|f\|_{\dot{\mathcal{B}}^s_{p,q}(\Omega)} + \left(\left(|f|^p\right)^{**}_\mu(1)\right)^{1/p} \\ &= \|f\|_{\dot{\mathcal{B}}^s_{p,q}(\Omega)} + \|f\|_{L^p_\mu(\Omega)+L^\infty_\mu(\Omega)}.\end{aligned}$$

All the other cases can be proved in the same way. □

With the same proof as that of Theorem 11, we obtain the following theorem.

Theorem 12. *Let (Ω,d,μ) be a (k,m)-space bounded from below. Suppose $f \in L^1_\mu(\Omega)+L^\infty_\mu(\Omega)$. Then Theorem 11 holds, considering $f^{**}_{\tilde{\mu}}$ and $f^*_{\tilde{\mu}}$ instead of f^{**}_μ and f^*_μ.*

Proof. The proof is done using the arguments of Lemma 3.2.3, Theorem 5, and Corollary 6. □

By Theorems 8, 9 and 7, we obtain

Corollary 13. *Let (Ω,d,μ) be a uniform (k,m)-space. Then*

(i.)
$$\|f\|_{BMO_\mu(\Omega)} \le \sup_{0<t<1} t^{k-m(1+1/p)}E_p(f,t) + \sup_{t>1} t^{m-k(1+1/p)}E_p(f,t).$$

(ii.) If
$$\int_0^1 t^{k-m(1+1/p)}E_p(f,t)\frac{dt}{t} + \int_1^\infty t^{m-k(1+1/p)}E_p(f,t)\frac{dt}{t} < \infty,$$
then f is μ-locally essentially continuous.

(iii.) Let $s < k(1+\frac{1}{p}) - m$, and let $w > 0$ be such that

$$[w] := \sup_{t>0}\left(\left(\left(\frac{1}{w}\right)^*_\mu (t)\right)^s \frac{1}{\min\left(t^{1-\frac{m+s}{k}}, t^{1-\frac{k+s}{m}}\right) t^{\frac{1}{p}-\frac{1}{q}}}\right) < \infty.$$

Then, for all $\alpha > 0$ and $q \geq 1$, we have that

$$\|f\|_{L^q} \leq (2\kappa [w])^{\frac{\alpha}{\alpha+1}} \|f\|^{\frac{\alpha}{\alpha+1}}_{\dot{\mathcal{B}}^s_{p,q}(\Omega)} \|w^{\alpha s} f\|^{\frac{1}{\alpha+1}}_{L^q}.$$

Now, we collect the results for the case when (Ω, d, μ) is Q-regular.

Theorem 14. *Let (Ω, d, μ) be Q-regular. Let $0 < p < \infty$, $0 < q \leq \infty$, $0 < s < 1$, and $f \in L^{\min(1,p)}_\mu(\Omega) + L^\infty_\mu(\Omega)$ with $\left(|f|^{\min(1,p)}\right)^*_\mu(\infty) = 0$. Then:*

(i.) ***Subcritical case*** $s < \frac{Q}{p}$:

$$\|f\|_{L^{p(Q),q}_\mu(\Omega)} \preceq \|f\|_{\dot{\mathcal{B}}^s_{p,q}(\Omega)}$$

where $p(Q) = Qp/(Q - sp)$.

Moreover, let $w > 0$ be such that

$$[w] := \sup_{t>0}\left(\left(\left(\frac{1}{w}\right)^*_\mu (t)\right)^s \frac{1}{t^{\frac{1}{p}-\frac{1}{q}-\frac{s}{Q}}}\right) < \infty.$$

If $1 \leq p, q < \infty$, then for all $\alpha > 0$, we have that

$$\|f\|_{L^q} \preceq [w]^{\frac{\alpha}{\alpha+1}} \|f\|^{\frac{\alpha}{\alpha+1}}_{\dot{\mathcal{B}}^s_{p,q}(\Omega)} \|w^{\alpha s} f\|^{\frac{1}{\alpha+1}}_{L^q}.$$

(ii.) ***Critical case*** $s = \frac{Q}{p}$:

a) If $q > 1$, then

$$\left(\int_0^\infty \left(\frac{f^{**}_\mu(t)}{1 + \ln\left(\frac{1}{t}\right)}\right)^q \frac{dt}{t}\right)^{1/q} \preceq \|f\|_{\dot{\mathcal{B}}^{Q/p}_{p,q}(\Omega)} + \|f\|_{L^{\min(1,p)}_\mu(\Omega) + L^\infty_\mu(\Omega)}.$$

b) If $0 < q \leq 1$, then

$$\|f\|_\infty \preceq \|f\|_{\dot{\mathcal{B}}^s_{p,q}(\Omega)} + \|f\|_{L^{\min(1,p)}_\mu(\Omega) + L^\infty_\mu(\Omega)}.$$

c) If $p \geq 1$, we get:

i.

$$\|f\|_{BMO_\mu(\Omega)} \preceq \|f\|_{\dot{\mathcal{B}}^{Q/p}_{p,\infty}(\Omega)}.$$

ii. If $f \in \|f\|_{\dot{\mathcal{B}}^{Q/p}_{p,1}(\Omega)}$, *then* f *is* μ*-locally essentially continuous.*

(iii.) ***Supercritical case*** $s > \frac{Q}{p}$

$$\|f\|_\infty \lesssim \|f\|_{\dot{\mathcal{B}}^s_{p,q}(\Omega)} + \|f\|_{L^{\min(1,p)}_\mu(\Omega)+L^\infty_\mu(\Omega)}.$$

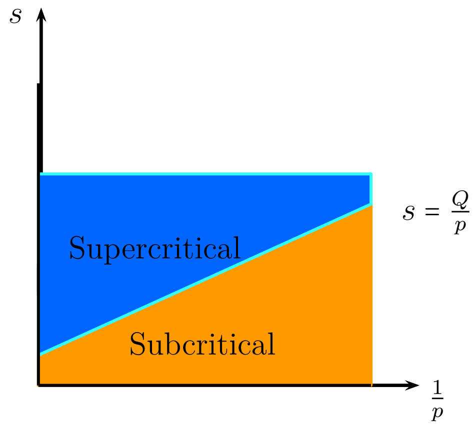

Figure 3.3: Theorem 14

Chapter 4

Symmetrization inequalities for probability metric spaces with convex isoperimetric profile

4.1 Introduction

Let (Ω, d, μ) be a connected metric space equipped with a separable Borel probability measure μ. The perimeter or Minkowski content of a Borel set $A \subset \Omega$ is defined by

$$\mu^{+}(A) = \liminf_{h \to 0} \frac{\mu(A_h) - \mu(A)}{h},$$

where $A_h = \{x \in \Omega : d(x, A) < h\}$ is an open h-neighbourhood of A. The **isoperimetric profile** I_μ is defined as the pointwise maximal function $I_\mu : [0,1] \to [0,\infty)$ such that

$$\mu^{+}(A) \geq I_\mu(\mu(A)),$$

holds for all Borel sets A. An isoperimetric inequality measures the relation between the boundary measure and the measure of a set, by providing a lower bound on I_μ by some function $I : [0,1] \to [0,\infty)$ which is not identically zero.

The modulus of the gradient of a Lipschitz function f on Ω (briefly $f \in Lip(\Omega)$) is defined by[1]

$$|\nabla f(x)| = \limsup_{d(x,y) \to 0} \frac{|f(x) - f(y)|}{d(x,y)}.$$

The equivalence between isoperimetric inequalities and Poincaré inequalities was obtained by Maz'ya, whose method (see [70], [62] and [16]) shows that given

[1] In fact one can define $|\nabla f|$ for functions f that are Lipschitz on every ball in (Ω, d) (cf. [7] for more details).

$X = X(\Omega)$ a rearrangement invariant space[2], the inequality

$$\left\| f - \int_\Omega f d\mu \right\|_X \leq c \, \||\nabla f|\|_{L^1}, \ f \in Lip(\Omega), \tag{4.1.1}$$

holds if, and only if, there exists a constant $c = c(\Omega) > 0$ such that for all Borel sets $A \subset \Omega$,

$$\min\left(\phi_X(\mu(A)), \phi_X(1-\mu(A))\right) \leq c\mu^+(A), \tag{4.1.2}$$

where $\phi_X(t)$ is the fundamental function[3] of X :

$$\phi_X(t) = \|\chi_A\|_X, \text{ with } \mu(A) = t.$$

Motivated by this fact, we will say (Ω, d, μ) admits a concave isoperimetric estimator if there exists a function $I : [0,1] \to [0,\infty)$ that is continuous, concave, increasing on $(0, 1/2)$, symmetric about the point $1/2$, satisfies $I(0) = 0$, and $I(t) > 0$ on $(0,1)$, such that

$$I_\mu(t) \geq I(t), \ 0 \leq t \leq 1.$$

In the recent work of Milman and Martín (see [61], [63]) it was proved that (Ω, d, μ) admits a concave isoperimetric estimator I if, and only if, the following symmetrization inequality holds,

$$f_\mu^{**}(t) - f_\mu^*(t) \leq \frac{t}{I(t)} |\nabla f|_\mu^{**}(t), \ (f \in Lip(\Omega)) \tag{4.1.3}$$

where $f_\mu^{**}(t) = \frac{1}{t}\int_0^t f_\mu^*(s)ds$, and f_μ^* is the non-increasing rearrangement of f with respect to the measure μ. If we apply a rearrangement invariant function norm X on Ω (see Section 2.2) to (4.1.3) we obtain Sobolev–Poincaré type estimates of the form[4]

$$\left\| \left(f_\mu^{**}(t) - f_\mu^*(t)\right) \frac{I(t)}{t} \right\|_{\bar{X}} \leq \left\| |\nabla f|_\mu^{**} \right\|_{\bar{X}}. \tag{4.1.4}$$

Example 4.1.1. *(See [64], [65].) Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain of measure 1, $X = L^p(\Omega)$, $1 \leq p \leq n$, and p^* be the usual Sobolev exponent defined by $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$. Then*

$$\left\| \left(f^{**}(t) - f^*(t)\right) \frac{I(t)}{t} \right\|_{L^p} \simeq \left\| \left(f^{**}(t) - f^*(t)\right) \right\|_{L^{p^*,p}}, \tag{4.1.5}$$

which follows from the fact that the isoperimetric profile is equivalent to $I(t) = c_n \min(t, 1-t)^{1-1/n}$ and from Hardy's inequality (here $L^{p^*,p}$ is a Lorentz space (see Section 2.2.2)). In the case where we consider $\mathbb{R}^n$ with Gaussian measure

[2] i.e. such that if f and g have the same distribution function then $\|f\|_X = \|g\|_X$ (see Section 2.2).

[3] We can assume with no loss of generality that ϕ_X is concave.

[4] The spaces $\bar{X}$ were defined in Section 2.2.

γ_n, and let $X = L^p$, $1 \le p < \infty$, then (compare with [34], [23]) since $I_{(\mathbb{R}^n,d,\gamma_n)}(t) \simeq t(\log 1/t)^{1/2}$ for t near zero, we have

$$\left\| \left(f^{**}_{\gamma_n}(t) - f^{*}_{\gamma_n}(t) \right) \frac{I(t)}{t} \right\|_{L^p} \simeq \left\| \left(f^{**}_{\gamma_n}(t) - f^{*}_{\gamma_n}(t) \right) \right\|_{L^p(Log)^{p/2}}, \tag{4.1.6}$$

where $L^p(logL)^{p/2}$ is a Lorentz–Zygmund space (see Section 2.2.2).

In this fashion, in [61], [63], [64] and [65], Martín and Milman were able to provide a unified framework to study the classical Sobolev inequalities and logarithmic Sobolev inequalities. Moreover, the embeddings (4.1.4) turn out to be the best possible in all the classical cases. However the method used in the proof of the previous results cannot be applied with probability measures with heavy tails, since isoperimetric estimators of such measures are non concave.

Let us illustrate this phenomenon with some examples (see [15, Propositions 4.3 and 4.4] for examples 4.1.2 and 4.1.3 and [72] for example 4.1.4).

Example 4.1.2. *(α-Cauchy type law). Let $\alpha > 0$. Consider the probability measure space $(\mathbb{R}^n, d, \mu)$ where d is the Euclidean distance and μ is defined by $d\mu(x) = V^{-(n+\alpha)}dx$ with $V : \mathbb{R}^n \to (0,\infty)$ convex. Then there exists a $C > 0$ such that for any measurable set $A \subset \mathbb{R}^n$*

$$\mu^+(A) \ge C \min\left(\mu(A), 1 - \mu(A)\right)^{1+1/\alpha}.$$

Example 4.1.3. *(Extended p-sub-exponential law). Let $p \in (0,1)$. Consider the probability measure on $\mathbb{R}^n$ defined by $d\mu(x) = (1/Z_p)\, e^{-V^p(x)}dx$ for some positive convex function $V : \mathbb{R}^n \to (0,\infty)$. Then there exists a $C > 0$ such that for any measurable set $A \subset \mathbb{R}^n$*

$$\mu^+(A) \ge C \min\left(\mu(A), 1 - \mu(A)\right) \left(\log \frac{1}{\min\left(\mu(A), 1 - \mu(A)\right)} \right)^{1-1/p}.$$

Example 4.1.4. *Let (M^n, g, μ) be an n-dimensional weighted Riemannian manifold ($n \ge 2$) that satisfies the $CD(0,N)$ curvature condition with $N < 0$. Then for every Borel set $A \subset (M^n, g)$,*

$$\mu^+(A) \ge C \min\left(\mu(A), 1 - \mu(A)\right)^{-1/N}.$$

Motivated by these examples, we will say (Ω, d, μ) admits a convex isoperimetric estimator if there exists a function $I : [0,1] \to [0,\infty)$ that is continuous, convex, increasing on $(0, 1/2)$, symmetric about the point $1/2$, such that $I(0) = 0$ and $I(t) > 0$ on $(0,1)$, and satisfies

$$I_\mu(t) \ge I(t), \; 0 \le t \le 1.$$

The purpose of this chapter is to obtain symmetrization inequalities on probability metric spaces that admit a convex isoperimetric estimator which incorporate in their formulation the isoperimetric estimator and that can be applied to provide a unified treatment of sharp Sobolev–Poincaré and Nash type inequalities. Note that if I is a convex isoperimetric estimator, then

$$I(t) \leq \min(t, 1-t).$$

Therefore (unless $I(t) \simeq \min(t, 1-t)$), the Poincaré inequality

$$\left\| f - \int_\Omega f d\mu \right\|_{L^1} \leq c \left\| |\nabla f| \right\|_{L^1}, f \in Lip(\Omega),$$

never holds, which means that we cannot use $|\nabla f| \in L^1$ to deduce that $f \in L^1$. Hence a symmetrization inequality like (4.1.3) will not be possible since f_μ^{**} is defined if, and only if, $f \in L^1$.

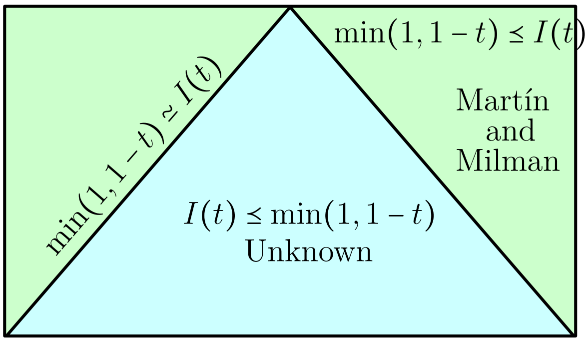

Figure 4.1: Isoperimetric profile

This chapter is organized as follows. In Section 4.2 we obtain symmetrization inequalities which incorporate in their formulation the isoperimetric convex estimator. In Section 4.3 we use the symmetrization inequalities to derive Sobolev–Poincaré and Nash type inequalities. Finally, in Section 4.4, we study in detail Examples 4.1.2, 4.1.3 and 4.1.4.
The results contained in this chapter have been submitted for publication (see [68]).

Definition 4.1.1. *Let $f \in \mathcal{M}_0(\Omega)$. We say that $m(f)$ is a median value of f if*

$$\mu\{f \geq m(f)\} \geq \frac{1}{2} \text{ and } \mu\{f \leq m(f)\} \geq \frac{1}{2}.$$

Lemma 4.1.1. *$f_\mu^{\star}(1/2)$ is a median value of f.*

Proof. Definition 4.1.1 is equivalent to

$$\mu\{f > m(f)\} \le 1/2; \quad and \quad \mu\{f < m(f)\} \le 1/2.$$

Now,

$$\mu\left\{f < f_\mu^*(1/2)\right\} = \mu\left\{-f > -f_\mu^*(1/2)\right\},$$

but from

$$(-f)_\mu^*(t) = -f_\mu^*(1-t)$$

it follows that

$$(-f)_\mu^*(1/2) = -f_\mu^*(1/2).$$

Consequently

$$\begin{aligned}\mu\left\{f < f_\mu^*(1/2)\right\} &= \mu\left\{-f > -f_\mu^*(1/2)\right\}\\ &= \mu\left\{-f > (-f)_\mu^*(1/2)\right\}\\ &\le 1/2.\end{aligned}$$

Therefore $f_\mu^{\star}\left(\frac{1}{2}\right)$ is a median value as was to be shown. $\square$

Remark 4.1.1. *If f has zero median and $f_\mu^{\star}$ is continuous, then* $f_\mu^{\star}\left(\frac{1}{2}\right) = 0$.

4.2 Symmetrization and Isoperimetry

We will assume in what follows that (Ω, d, μ) is a connected measure metric space equipped with with a separable, non-atomic, Borel probability measure μ which admits a convex isoperimetric estimator.

In order to balance generality with power and simplicity, we will assume throughout the paper that our spaces satisfy the following condition.

Condition 4.2.1. *We assume that Ω is such that for every $f \in Lip(\Omega)$ and every $c \in \mathbb{R}$ we have that $|\nabla f(x)| = 0$, a.e. on the set $\{x : f(x) = c\}$.*

Theorem 15. *Let $I : [0,1] \to [0,\infty)$ be a convex isoperimetric estimator. The following statements are equivalent:*

(i.) Isoperimetric inequality: for all Borel sets $A \subset \Omega$,

$$\mu^+(A) \ge I(\mu(A)). \tag{4.2.1}$$

(ii.) *Ledoux's inequality (cf [57]): for all* $f \in Lip(\Omega)$,

$$\int_{-\infty}^{\infty} I(\mu_f(s)) \leq \int_{\Omega} |\nabla f(x)| \, d\mu. \tag{4.2.2}$$

(iii.) *For all functions* $f \in Lip(\Omega)$, $f_\mu^{\star}$ *is locally absolutely continuous, and*

$$\int_0^t ((-f_\mu^{\star})' I(s))^* ds \leq \int_0^t |\nabla f|_\mu^* (s) ds. \tag{4.2.3}$$

(The second rearrangement on the left hand side is with respect to Lebesgue measure).

(iv.) *Bobkov's inequality (cf. [7]): For all bounded* $f \in Lip(\Omega)$ *with* $m(f) = 0$, *and for all* $s > 0$,

$$\int_{\Omega} |f(x)| \, d\mu \leq \beta_1(s) \int_{\Omega} |\nabla f(x)| \, d\mu + s \, Osc_\mu(f), \tag{4.2.4}$$

where $Osc_\mu(f) = ess \sup f - ess \inf f$, *and* $\beta_1(s) = \sup\limits_{s<t\leq 1/2} \dfrac{t-s}{I(t)}$.

Proof. $(i.) \to (ii.)$ By the co-area inequality applied to f (cf. [8, Lemma 3.1]) and the isoperimetric inequality (4.2.1), it follows that

$$\begin{aligned}\int_{\Omega} |\nabla f(x)| \, d\mu &\geq \int_{-\infty}^{\infty} \mu^+(\{f > s\}; \Omega) ds \\ &\geq \int_0^{\infty} I(\mu_f(s)) ds \ .\end{aligned}$$

$(ii.) \to (iii.)$ Let $-\infty < t_1 < t_2 < \infty$. The smooth truncations of f are defined by

$$f_{t_1}^{t_2}(x) = \begin{cases} t_2 - t_1 & \text{if } f(x) \geq t_2, \\ f(x) - t_1 & \text{if } t_1 < f(x) < t_2, \\ 0 & \text{if } f(x) \leq t_1. \end{cases}$$

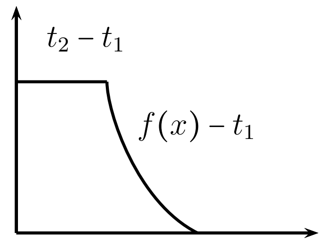

Figure 4.2: $f_{t_1}^{t_2}(x)$

Obviously, $f_{t_1}^{t_2} \in Lip(\Omega)$. Thus (4.2.1) implies

$$\int_{-\infty}^{\infty} I(\mu_{f_{t_1}^{t_2}}(s)) ds \leq \int_{\Omega} \left|\nabla f_{t_1}^{t_2}(x)\right| d\mu.$$

By condition 4.2.1,

$$\left|\nabla f_{t_1}^{t_2}\right| = |\nabla f|_{\chi_{\{t_1<f<t_2\}}},$$

and, moreover,

$$\int_{-\infty}^{\infty} I(\mu_{f_{t_1}^{t_2}}(s))ds = \int_{t_1}^{t_2} I(\mu_{f_{t_1}^{t_2}}(s))ds.$$

Observing that $t_1 < z < t_2$,

$$\mu\{f \geq t_2\} \leq \mu_{f_{t_1}^{t_2}}(z) \leq \mu\{f > t_1\}.$$

Consequently, by the properties of I, we have

$$\int_{t_1}^{t_2} I(\mu_{f_{t_1}^{t_2}}(z))dz \geq (t_2 - t_1)\min\{I(\mu\{f \geq t_2\}), I(\mu\{f > t_1\})\}. \tag{4.2.5}$$

We now show that $f_\mu^{\star}$ is locally absolutely continuous. Indeed, for $s > 0$ and $h > 0$, pick $t_1 = f_\mu^{\star}(s+h)$, $t_2 = f_\mu^{\star}(s)$. Then

$$s \leq \mu\{f(x) \geq f_\mu^{\star}(s)\} \leq \mu_{f_{t_1}^{t_2}}(s) \leq \mu\{f(x) > f_\mu^{\star}(s+h)\} \leq s+h. \tag{4.2.6}$$

Combining (4.2.5) and (4.2.6) yields

$$(f_\mu^{\star}(s) - f_\mu^{\star}(s+h))\min\{I(s+h), I(s)\} \leq \int_{\left\{f_\mu^{\star}(s)<f<f_\mu^{\star}(s+h)\right\}} |\nabla f(x)|\, d\mu \tag{4.2.7}$$

which implies that $f_\mu^{\star}$ is locally absolutely continuous on $[a,b]$ $(0 < a < b < 1)$. Indeed, for any finite family of non-overlapping intervals $\{(a_k, b_k)\}_{k=1}^r$, with $(a_k, b_k) \subset [a,b]$ and $\sum_{k=1}^r (b_k - a_k) \leq \delta$, we have

$$\mu\{\bigcup_{k=1}^r \{f_\mu^{\star}(b_k) < f < f_\mu^{\star}(a_k)\}\} = \sum_{k=1}^r \mu\{f_\mu^{\star}(b_k) < f < f_\mu^{\star}(a_k)\} \leq \sum_{k=1}^r (b_k - a_k) \leq \delta.$$

Therefore, combining this fact with (4.2.7), we have

$$\begin{aligned}
\sum_{k=1}^r (f_\mu^{\star}(a_k) - f_\mu^{\star}(b_k))\min\{I(a), I(b)\} &\leq \sum_{k=1}^r (f_\mu^{\star}(a_k) - f_\mu^{\star}(b_k))\min\{I(a_k), I(b_k)\} \\
&\leq \sum_{k=1}^r \int_{\{f_\mu^{\star}(b_k)<f<f_\mu^{\star}(a_k)\}} |\nabla f(x)|\, d\mu \\
&= \int_{\cup_{k=1}^r \{f_\mu^{\star}(b_k)<f<f_\mu^{\star}(a_k)\}} |\nabla f(x)|\, d\mu \\
&\leq \int_0^{\sum_{k=1}^r (b_k - a_k)} |\nabla f|_\mu^{*}(t)dt \\
&\leq \int_0^{\delta} |\nabla f|_\mu^{*}(t)dt,
\end{aligned}$$

and the local absolute continuity follows.
Now, (4.2.7) implies

$$\frac{(f_\mu^{\star}(s) - f_\mu^{\star}(s+h))}{h}\min(I(s+h), I(s)) \leq \int_{\{f_\mu^{\star}(s+h)<f<f_\mu^{\star}(s)\}} |\nabla f(x)|\, d\mu$$
$$\leq \frac{1}{h}\int_{\{f_\mu^{\star}(s+h)<f\leq f_\mu^{\star}(s)\}} |\nabla f(x)|\, d\mu.$$

Letting $h \to 0$,

$$(-f_\mu^{\star})'(s)I(s) \leq \frac{\partial}{\partial s}\int_{\{f>f_\mu^{\star}(s)\}} |\nabla f(x)|\, d\mu.$$

Let us consider a finite family of intervals (a_i, b_i), $i = 1, \ldots, m$, with $0 < a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_m < b_m < 1$. Then

$$\begin{aligned}
\int_{\cup_{1\leq i\leq m}(a_i,b_i)} \left(-f_\mu^{\star}\right)'(s)I(s)ds &\leq \int_{\cup_{1\leq i\leq m}(a_i,b_i)} \left(\frac{\partial}{\partial s}\int_{\left\{|f|>f_\mu^{\star}(s)\right\}} |\nabla f(x)|\, d\mu(x)\right) ds \\
&= \sum_{i=1}^{m}\int_{\left\{f_\mu^{\star}(b_i)<|f|\leq f_\mu^{\star}(a_i)\right\}} |\nabla f(x)|\, d\mu(x) \\
&= \sum_{i=1}^{m}\int_{\left\{f_\mu^{\star}(b_i)<|f|<f_\mu^{\star}(a_i)\right\}} |\nabla f(x)|\, d\mu(x) \text{ (by condition (4.2.1))} \\
&= \int_{\cup_{1\leq i\leq m}\left\{f_\mu^{\star}(b_i)<|f|<f_\mu^{\star}(a_i)\right\}} |\nabla f(x)|\, d\mu(x) \\
&\leq \int_0^{\sum_{i=1}^m(b_i-a_i)} |\nabla f|_\mu^{*}(s)ds.
\end{aligned}$$

Now by a routine limiting process we can show that for any measurable set $E \subset (0,1)$ with Lebesgue measure equal to t we have

$$\int_E (-f_\mu^{\star})'(s)I(s)ds \leq \int_0^{|E|} |\nabla f|_\mu^{*}(s)ds.$$

Therefore

$$\int_0^t ((-f_\mu^{\star})'(\cdot)I(\cdot))^{*}(s)ds \leq \int_0^t \left(|\nabla f|_\mu^{*}(\cdot)\right)^{*}(s)ds, \tag{4.2.8}$$

where the second rearrangement is with respect to Lebesgue measure. Now, since $|\nabla f|_\mu^{*}(s)$ is decreasing, we have

$$\left(|\nabla f|_\mu^{*}(\cdot)\right)^{*}(s) = |\nabla f|_\mu^{*}(s),$$

and thus (4.2.8) yields

$$\int_0^t ((-f_\mu^{\star})'(\cdot)I(\cdot))^{*}(s)ds \leq \int_0^t |\nabla f|_\mu^{*}(s)ds.$$

$(iii.) \to (iv.)$ Assume first that $f \in Lip(\Omega)$ is positive, and bounded, with $m(f) = 0$. By (iii) we have that $f_\mu^{\star} = f_\mu^*$ (since $f \geq 0$) is locally absolutely continuous and $f_\mu^*(1/2) = 0$ (since $m(f) = 0$). Let $0 < s < z \leq 1/2$. Then

$$\begin{aligned}
\int_\Omega |f(x)|\,d\mu &= \int_0^{1/2} f_\mu^*(z)dz = \int_0^{1/2}\int_z^{1/2} (-f_\mu^*)'(x)dxdz = \\
&= \int_0^{1/2} z(-f_\mu^*)'(z)dz - s\int_0^{1/2}(-f_\mu^*)'(z)dz + s\int_0^{1/2}(-f_\mu^*)'(z)dz \\
&= \int_0^{1/2} \frac{z-s}{I(z)}(-f_\mu^*)'(z)I(z)dz + s\int_0^{1/2}(-f_\mu^*)'(z)dz \\
&\leq \sup_{s<z\leq 1/2}\frac{z-s}{I(z)}\int_0^{1/2}(-f_\mu^*)'(z)I(z)dz + s\int_0^{1/2}(-f_\mu^*)'(z)dz \\
&\leq \beta_1(s)\int_0^{1/2}(-f_\mu^*)'(z)I(z)dz + s\int_0^{1/2}(-f_\mu^*)'(z)dz.
\end{aligned}$$

Since

$$s\int_0^{1/2}(-f_\mu^*)'(z) = s(f_\mu^*(0^+) - f_\mu^*(1/2)) \leq s\,Osc_\mu(f),$$

we get

$$\begin{aligned}
\int_\Omega |f(x)|\,d\mu &\leq \beta_1(s)\int_0^t (-f_\mu^*)'(z)I(z)dz + s\,Osc_\mu(f) \\
&\leq \beta_1(s)\int_0^{1/2}\left((-f_\mu^*)'(\cdot)I(\cdot)\right)^*(t)dt + s\,Osc_\mu(f) \\
&\leq \beta_1(s)\int_0^{1/2}|\nabla f|_\mu^*(t)dt + s\,Osc_\mu(f) \quad \text{(by (4.2.3))} \\
&= \beta_1(s)\int_\Omega |\nabla f(x)|\,d\mu + s\,Osc_\mu(f)
\end{aligned}$$

In the general case, we follow [7, Lemma 8.3]. Apply the previous argument to $f^+ = \max(f,0)$ and $f^- = \max(-f,0)$, which are positive, Lipschitz and have median zero, and we obtain

$$\int_{\{f>0\}} |f(x)|\,d\mu \leq \beta_1(s)\int_{\{f>0\}}|\nabla f(x)|\,d\mu + sOsc_\mu(f^+),$$

$$\int_{\{f<0\}} |f(x)|\,d\mu \leq \beta_1(s)\int_{\{f><0\}}|\nabla f(x)|\,d\mu + sOsc_\mu(f^-).$$

Adding the two inequalities and since $Osc_\mu(f^-) + Osc_\mu(f^+) \leq Osc_\mu(f)$, we get (4.2.4).

$(iv) \to (i.)$ This part was proved in [7, Lemma 8.3], we include its proof for the sake of completeness. Given a Borel set $A \subset \Omega$ we may approximate the indicator function χ_A by functions with finite Lipschitz seminorm (see [8]) to derive $\mu(A) \leq \beta_1(s)\mu^+(A) + s$. Therefore, if $\mu(A) = t$,

$$t - s \leq \beta_1(s)\mu^+(A)$$

thus the optimal choice should be

$$I(t) = \sup_{0<s<t} \frac{t-s}{\beta_1(s)}.$$

$\square$

4.3 Sobolev–Poincaré and Nash type inequalities

The isoperimetric inequality implies weaker Sobolev–Poincaré and Nash type inequalities. In what follows, we will analyse both.

4.3.1 Sobolev–Poincaré inequalities

The isoperimetric Hardy operator Q_I is the operator defined on Lebesgue measurable functions on $(0,1)$ by

$$Q_I f(t) = \int_t^{1/2} f(s) \frac{ds}{I(s)}, \qquad 0 < t < 1/2,$$

where I is a convex isoperimetric estimator. In this section we consider the possibility of characterizing Sobolev embeddings in terms of the boundedness of Q_I.

Lemma 4.3.1. *Let Y, Z be two q.r.i. spaces on $(0,1)$. Assume that there is a constant $C_0 > 0$ such that*

$$\|Q_I f\|_Y \leq C_0 \|f\|_Z. \tag{4.3.1}$$

Then there exists a constant $C_1 > 0$ such that

$$\left\|\bar{Q}_I f\right\|_Y \leq C_1 \|f\|_Z,$$

where $\bar{Q}_I$ is the operator defined on Lebesgue measurable functions on $(0,1)$ by

$$\bar{Q}_I f(t) = \int_t^{1/2} f(s) \frac{ds}{I(s)}, \qquad 0 < t < 1.$$

Proof. Since

$$\begin{aligned}\bar{Q}_I f(t) &= \chi_{(0,1/2)}(t)\bar{Q}_I f(t) + \chi_{(1/2,1)}(t) \int_t^{1/2} f(s)\frac{ds}{I(s)} \\ &= \chi_{(0,1/2)}(t) Q_I f(t) + \chi_{(1/2,1)}(t) \int_t^{1/2} f(s)\frac{ds}{I(s)},\end{aligned}$$

it is enough to prove the boundedness of $\chi_{(1/2,1)}(t) \int_t^{1/2} f(s) \frac{ds}{I(s)}$.

For $t \in (1/2, 1)$, we have that

$$\int_t^{1/2} f(s)\frac{ds}{I(s)} = -\int_{1/2}^{t} f(s)\frac{ds}{I(s)} = \int_{1/2}^{1-t} f(1-s)\frac{ds}{I(1-s)}$$
$$= -\int_{1-t}^{1/2} f(1-s)\frac{ds}{I(s)} \quad (\text{since } I(s) = I(1-s)).$$

Thus

$$\begin{aligned}\left\|\chi_{(1/2,1)}(t)\int_t^{1/2} f(s)\frac{ds}{I(s)}\right\|_Y &= \left\|\chi_{(1/2,1)}(t)\int_{1-t}^{1/2} f(1-s)\frac{ds}{I(s)}\right\|_Y \\ &= \left\|\chi_{(1/2,1)}(1-t)\int_t^{1/2} f(1-s)\frac{ds}{I(s)}\right\|_Y \quad (\text{since } \|\cdot\|_Y \text{ is r.i}) \\ &= \left\|\chi_{(0,1/2)}(t)\int_t^{1/2} f(1-s)\frac{ds}{I(s)}\right\|_Y \\ &\leq C\left\|\chi_{(0,1/2)}(t)f(1-t)\right\|_Z \\ &\leq C\left\|f(t)(\chi_{(0,1/2)}(1-t))\right\|_Z \quad (\text{since } \|\cdot\|_{\bar{X}} \text{ is r.i}) \\ &\leq C\left\|\chi_{(1/2,1)}(t)f(t)\right\|_Z \\ &\leq C\|f\|_Z .\end{aligned}$$

□

Theorem 16. *Let Y be a q.r.i. space on $(0,1)$, and let X be an r.i. space on Ω. Assume that there is a constant $C > 0$ such that*

$$\|Q_I f\|_Y \leq C\|f\|_{\bar{X}} . \tag{4.3.2}$$

Then, for all $g \in Lip(\Omega)$, we have that

$$\inf_{c\in\mathbb{R}} \left\|(g-c)_\mu^*\right\|_Y \preceq \||\nabla g|\|_X .$$

Proof. Given $g \in Lip(\Omega)$, by part 2 of Theorem 15, $g_\mu^{\star}$ is locally absolutely continuous on $(0,1)$. Thus, for $t \in (0,1)$, we have that

$$\begin{aligned}\left|g_\mu^{\star}(t) - g_\mu^{\star}(1/2)\right| &= \left|\int_t^{1/2}\left(-g_\mu^{\star}\right)'(s)ds\right| = \left|\int_t^{1/2}\left(-g_\mu^{\star}\right)'(s)I(s)\frac{ds}{I(s)}\right| \\ &= \left|\bar{Q}_I\left(\left(-g_\mu^{\star}\right)'(\cdot)I(\cdot)\right)(t)\right|.\end{aligned}$$

Then

$$\begin{aligned}\left\|\left|g_\mu^{\star}(t) - g_\mu^{\star}(1/2)\right|\right\|_Y &= \left\|\bar{Q}_I\left(\left(-g_\mu^{\star}\right)'(\cdot)I(\cdot)\right)(t)\right\|_Y \\ &\preceq \left\|\left(-g_\mu^{\star}\right)'(\cdot)I(\cdot)\right\|_{\bar{X}} \quad (\text{by (4.3.2) and Lemma 4.3.1}) \\ &\preceq \||\nabla g|\|_X \quad (\text{by (4.2.3)}).\end{aligned}$$

Therefore

$$\begin{aligned}\inf_{c\in\mathbb{R}} \left\|(g-c)_{\mu}^{*}(t)\right\|_{Y} &= \inf_{c\in\mathbb{R}} \left\|(g-c)_{\mu}^{\star}(t)\right\|_{Y} \\ &\leq \left\|g_{\mu}^{\star}(t)-g_{\mu}^{\star}(1/2)\right\|_{Y} \\ &\leq \left\||\nabla g|\right\|_{X}.\end{aligned}$$

□

Theorem 17. *Let X be an r.i. space on Ω. Assume that either $\underline{\alpha}_X > 0$ or that there is a $c>0$ such that the convex isoperimetric estimator I satisfies*

$$\int_t^{1/2} \frac{ds}{I(s)} \leq c\frac{t}{I(t)}, \quad 0<t<1/2. \tag{4.3.3}$$

Then, for all $g \in Lip(\Omega)$, we have that

$$\inf_{c\in\mathbb{R}} \left\|(g-c)_{\mu}^{*}(t)\frac{I(t)}{t}\right\|_{\bar{X}} \leq \left\||\nabla g|\right\|_{X}. \tag{4.3.4}$$

Moreover, if Y is a q.r.i. space on $(0,1)$ such that

$$\|Q_I f\|_Y \leq \|f\|_{\bar{X}}, \tag{4.3.5}$$

then for any μ-measurable function g on Ω, we have that

$$\left\|g_{\mu}^{*}\right\|_{Y} \leq \left\|g_{\mu}^{*}(t)\frac{I(t)}{t}\right\|_{\bar{X}}.$$

In particular, for all $g \in Lip(\Omega)$, we get

$$\inf_{c\in\mathbb{R}} \left\|(g-c)_{\mu}^{*}\right\|_{Y} \leq \inf_{c\in\mathbb{R}} \left\|(g-c)_{\mu}^{*}(t)\frac{I(t)}{t}\right\|_{\bar{X}} \leq \left\||\nabla g|\right\|_{X}.$$

Proof. We associate to the r.i. space $\bar{X}$ the weighted q.r.i. space Z on $(0,1)$ whose quasi-norm is defined by

$$\|f\|_Z := \left\|f^{*}(t)\frac{I(t)}{t}\right\|_{\bar{X}}.$$

We claim that there is a $C>0$ such that

$$\|Q_I f\|_Z \leq C\,\|f\|_{\bar{X}},$$

and therefore (4.3.4) follows by Theorem 16.

Case 1: $\underline{\alpha}_X > 0$:

$$
\begin{aligned}
\|Q_I f\|_Z &= \left\| \frac{I(t)}{t} \left(\int_t^{1/2} f(s) \frac{ds}{I(s)} \right)^* \right\|_{\bar{X}} \le \left\| \frac{I(t)}{t} \left(\int_t^{1/2} |f(s)| \frac{ds}{I(s)} \right)^* \right\|_{\bar{X}} \\
&= \left\| \frac{I(t)}{t} \int_t^{1/2} |f(s)| \frac{ds}{I(s)} \right\|_{\bar{X}} \quad \text{(since } Q_I\,|f|\,(t) \text{ is decreasing)} \\
&= \left\| \frac{I(t)}{t} \int_t^{1/2} |f(s)| \frac{s}{I(s)} \frac{ds}{s} \right\|_{\bar{X}} \\
&\le \left\| \int_t^{1/2} |f(s)| \frac{ds}{s} \right\|_{\bar{X}} \quad \text{(since } \frac{s}{I(s)} \text{ decreases)} \\
&\le \|f\|_{\bar{X}} \quad \text{(since } \underline{\alpha}_X > 0\text{)}.
\end{aligned}
$$

Case 2 : The convex isoperimetric estimator satisfies (4.3.3).

Consider $\tilde{Q}_I$ defined by

$$
\tilde{Q}_I f(t) = \frac{I(t)}{t} Q_I f(t).
$$

We claim that $\tilde{Q}_I : L^1(0,1) \to L^1(0,1)$ is bounded, and $\tilde{Q}_I : L^\infty(0,1) \to L^\infty(0,1)$ is bounded, and so by interpolation (see [53]) $\tilde{Q}_I$ will be bounded on $\bar{X}$. Thus

$$
\begin{aligned}
\|Q_I f\|_Z \le \|Q_I\,|f|\|_Z &= \left\| \frac{I(t)}{t} \left(Q_I\,|f|\right)^*(t) \right\|_{\bar{X}} \\
&= \left\| \frac{I(t)}{t} Q_I\,|f|\,(t) \right\|_{\bar{X}} \quad \text{(since } Q_I\,|f|\,(t) \text{ is decreasing)} \\
&= \left\| \tilde{Q}_I\,|f|\,(t) \right\|_{\bar{X}} \\
&\le \|f\|_{\bar{X}}
\end{aligned}
$$

and Theorem 16 applies.

We are now going to prove the claim.

By the convexity of I, $\frac{I(t)}{t}$ is increasing for $0 < t < 1/2$, thus

$$
\int_0^s \frac{I(t)}{t} dt \le I(s),
$$

and therefore

$$\begin{aligned}
\left\|\tilde{Q}_I f\right\|_1 &\leq \int_0^1 \tilde{Q}_I\left(|f|\right)(t)dt \\
&= \int_0^{1/2} \frac{I(t)}{t}\left(\int_t^{1/2} |f(s)| \frac{ds}{I(s)}\right)dt \\
&= \int_0^{1/2} \frac{|f(s)|}{I(s)}\left(\int_0^{s} \frac{I(t)}{t}dt\right)ds \\
&\leq \int_0^{1/2} |f(s)|\, ds \\
&= \|f\|_1 .
\end{aligned}$$

Similarly,

$$\begin{aligned}
\left\|\tilde{Q}_I f\right\|_\infty &\leq \sup_{0<t<1} \tilde{Q}_I\left(|f|\right)(t) \\
&\leq \sup_{0<t<1/2} \frac{I(t)}{t}\int_t^{1/2} |f(s)| \frac{ds}{I(s)} \\
&\leq \|f\|_\infty \sup_{0<t<1/2}\left(\frac{I(t)}{t}\int_t^{1/2} \frac{ds}{I(s)}\right) \\
&\leq c\,\|f\|_\infty \ \text{(by (4.3.3))}.
\end{aligned}$$

To finish the proof of the theorem it remains to show that

$$\|f_\mu^*\|_{\bar{Y}} \preceq \left\|f_\mu^*(t)\frac{I(t)}{t}\right\|_{\bar{X}} . \tag{4.3.6}$$

Let $C_{\bar{Y}}$ be the constant quasi-norm of $\bar{Y}$, then

$$\begin{aligned}
\left\|f_\mu^*\right\|_{\bar{Y}} &= \left\|f_\mu^*(t)\chi_{(0,1/4)}(t) + f_\mu^*(t)\chi_{(1/4,1/2)}(t) + f_\mu^*(t)\chi_{(1/2,3/4)}(t) + f_\mu^*(t)\chi_{(3/4,1)}(t)\right\|_{\bar{Y}} \qquad (4.3.7)\\
&\leq 4C_{\bar{Y}}^2 \left\|f_\mu^*(t)\chi_{(0,1/4)}(t)\right\|_{\bar{Y}} .
\end{aligned}$$

Since f_μ^* is decreasing,

$$f_\mu^*(t)\chi_{(0,1/4)}(t) \leq \frac{1}{\ln 2}\int_{t/2}^{t} f_\mu^*(s)\frac{ds}{s} = \frac{1}{\ln 2}\int_{t/2}^{1/2} f_\mu^*(s)\chi_{(0,1/4)}(s)\frac{I(s)}{s}\frac{ds}{I(s)}.$$

Thus

$$
\begin{aligned}
\left\| f_\mu^*(t)\chi_{(0,1/4)}(t) \right\|_{\bar{Y}} &\leq \left\| Q_I\left(f_\mu^*(\cdot)\chi_{(0,1/4)}(\cdot)\frac{I(\cdot)}{\cdot} \right)(t/2) \right\|_{\bar{Y}} \qquad (4.3.8)\\
&\leq \left\| f_\mu^*(t/2)\chi_{(0,1/4)}(t/2)\frac{I(t/2)}{t/2} \right\|_{\bar{X}} \quad \text{(by (4.3.5))}\\
&\leq \left\| f_\mu^*(t)\chi_{(0,1/2)}(t)\frac{I(t)}{t} \right\|_{\bar{X}}\\
&\leq \left\| f_\mu^*(t)\frac{I(t)}{t} \right\|_{\bar{X}}.
\end{aligned}
$$

Combining (4.3.7) and (4.3.8) we obtain (4.3.6). □

Remark 4.3.1. *If $g \in Lip(\Omega)$ is positive with $m(g) = 0$, then it follows from the previous theorem that*

$$
\left\| g_\mu^*(t)\frac{I(t)}{t} \right\|_{\bar{X}} \leq \| |\nabla g| \|_X .
$$

4.3.2 Nash inequalities

In this section we obtain Nash type inequalities. We will focus on the following type of probability measures.

Definition 4.3.1. *Let μ be a probability measure on Ω which admits a convex isoperimetric estimator I.*

1. *Let $\alpha > 0$. We will say that μ is of α-Cauchy type if*

$$
I(t) = c_\mu \min(t, 1-t)^{1+1/\alpha}.
$$

2. *Let $0 < p < 1$. We will say that μ is of extended p-exponential type if*

$$
I(t) = c_\mu \min(t, 1-t)\left(\log \frac{1}{\min(t, 1-t)} \right)^{1-1/p}.
$$

In both cases c_μ denotes a positive constant.

Theorem 18. *The following Nash inequalities hold:*

1. *Let μ be of α-Cauchy type. Let X be an r.i. space on Ω with $\underline{\alpha}_X > 0$. Let $1 < q \leq \infty$ satisfy $0 \leq 1/q < \underline{\alpha}_X$. Then for all positivef $\in Lip(\Omega)$ with $m(f) = 0$, we have*

$$
\|f\|_X \leq \min_{r>1}\left(r \| |\nabla f| \|_X + \|f\|_{q,\infty} \phi_X(r^{-\alpha}) r^{\alpha/q} \right).
$$

2. *Let μ be of extended p-exponential type. Let X be an r.i. space on Ω . Let $\beta > 0$. Then for all positive $f \in Lip(\Omega)$ with $m(f) = 0$, we have*

$$\|f\|_X \leq \||\nabla f|\|_X^{\frac{\beta}{\beta+1}} \|f\|_{X(\ln(\frac{1}{t})^{\beta(\frac{1}{p}-1)})}^{\frac{1}{\beta+1}}.$$

Proof. Part 1. Let $f \in Lip(\Omega)$ be positive with $m(f) = 0$ and let $\omega(t) = t^{-1/\alpha}$ $(0 < t < 1/2)$. Let $r > 1$ and let $\beta > 0$ be chosen later. Then

$$\begin{aligned}
\|f\|_X = \|f_\mu^*\|_{\bar{X}} &\leq \left\| f_\mu^*(t)\frac{\omega(t)}{\omega(t)}\chi_{\{\omega<r)\}}(t)\right\|_{\bar{X}} + \left\| f_\mu^*(t)\left(\frac{\omega(t)}{\omega(t)}\right)^{\beta}\chi_{\{\omega>r\}}(t)\right\|_{\bar{X}} \qquad (4.3.9)\\
&\leq r\left\|f_\mu^*(t)t^{1/\alpha}\right\|_{\bar{X}} + r^{-\beta}\left\|f_\mu^*(t)t^{-\beta/\alpha}\chi_{(0,r^{-\alpha})}(t)\right\|_{\bar{X}}\\
&= r\left\|f_\mu^*(t)t^{1/\alpha}\right\|_{\bar{X}} + r^{-\beta}\left\|t^{1/q}f_\mu^*(t)t^{-\beta/\alpha-1/q}\chi_{(0,r^{-\alpha})}(t)\right\|_{\bar{X}}\\
&\leq r\left\|f_\mu^*(t)t^{1/\alpha}\right\|_{\bar{X}} + r^{-\beta}\sup_{t>0}(t^{1/q}f_\mu^*(t))\left\|t^{-\beta/\alpha-1/q}\chi_{(0,r^{-\alpha})}(t)\right\|_{\bar{X}}\\
&\leq r\left\|f_\mu^*(t)t^{1/\alpha}\right\|_{\bar{X}} + r^{-\beta}\|f\|_{q,\infty}\left\|t^{-\beta/\alpha-1/q}\chi_{(0,r^{-\alpha})}(t)\right\|_{\Lambda(\bar{X})} \ \text{(by (2.2.8))}\\
&\leq r\left\|f_\mu^*(t)t^{1/\alpha}\right\|_{\bar{X}} + r^{-\beta}\|f\|_{q,\infty}\int_0^{r^{-\alpha}} t^{-\beta/\alpha-1/q}\frac{\phi_X(t)}{t}\\
&= r\left\|f_\mu^*(t)t^{1/\alpha}\right\|_{\bar{X}} + r^{-\beta}\|f\|_{q,\infty}J(r).
\end{aligned}$$

Let $0 \leq 1/q < \gamma < \underline{\alpha}_X$. By Lemma 2.2.1,

$$\int_0^{r^{-\alpha}} t^{-\beta/\alpha-1/q}\frac{\phi_X(t)}{t^{\gamma}t^{1-\gamma}} \leq \frac{\phi_X(r^{-\alpha})}{r^{-\alpha\gamma}}\int_0^{r^{-\alpha}} t^{-\beta/\alpha-1/q+\gamma-1}.$$

At this stage we select $0 < \beta < \alpha(\gamma - 1/q)$. Then

$$\int_0^{r^{-\alpha}} t^{-\beta/\alpha-1/q+\gamma-1} \leq r^{-\alpha(-\beta/\alpha-1/q+\gamma)},$$

and thus

$$J(r) \leq \phi_X(r^{-\alpha})r^{\beta+\alpha/q}.$$

Inserting this information into (4.3.9) and using Remark 4.3.1, we get

$$\begin{aligned}
\|f\|_X &\leq r\left\|f_\mu^*(t)t^{1/\alpha}\right\|_{\bar{X}} + \|f\|_{q,\infty}\phi_X(r^{-\alpha})r^{\alpha/q}\\
&\leq r\||\nabla f|\|_X + \|f\|_{q,\infty}\phi_X(r^{-\alpha})r^{\alpha/q}.
\end{aligned}$$

Part 2. Let $f \in Lip(\Omega)$ be positive with $m(f) = 0$ and let $\omega(t) = \left(\ln \frac{1}{t}\right)^{\frac{1}{p}-1}$ $(0 < t < 1/2)$. Let $r > 1$ and $\beta > 0$.

$$\begin{aligned}
\|f\|_X = \|f_\mu^*\|_{\bar{X}} &\leq \left\| f_\mu^*(t)\frac{\omega(t)}{\omega(t)}\chi_{\{\omega<r)\}}(t)\right\|_{\bar{X}} + \left\| f_\mu^*(t)\left(\frac{\omega(t)}{\omega(t)}\right)^\beta \chi_{\{\omega>r\}}(t)\right\|_{\bar{X}} \\
&\leq r\left\| f_\mu^*(t)\left(\ln\frac{1}{t}\right)^{1-\frac{1}{p}}\right\|_{\bar{X}} + r^{-\beta}\left\| f_\mu^*(t)\left(\ln\frac{1}{t}\right)^{\beta\left(\frac{1}{p}-1\right)}\right\|_{\bar{X}} \\
&\leq r\,\||\nabla f|\|_{X(\log(\frac{1}{t})^{\beta}(\frac{1}{p}-1))} + r^{-\beta}\,\|f\|_X \quad \text{(by Remark 4.3.1)}.
\end{aligned}$$

We finish by taking the inf for $r > 1$. □

Remark 4.3.2. *Let X be an r.i. space on Ω with $\underline{\alpha}_X > 0$. Let $1 < q \leq \infty$ be such that $0 \leq 1/q < \underline{\alpha}_X$. Then*

$$L^{q,\infty}(\Omega) \subset \Lambda(X) \subset X(\Omega).$$

In fact, by Lemma 2.2.1,

$$\|f\|_{\Lambda(X)} = \int_0^1 f^*(t)\frac{\phi_X(t)}{t}dt \leq \|f\|_{q,\infty}\int_0^1 \frac{\phi_X(t)}{t^{1+1/q}}dt.$$

The last integral is finite since taking $0 \leq 1/q < \gamma < \underline{\alpha}_X$, we get

$$\int_0^1 \frac{\phi_X(t)}{t^{1+1/q}}dt = \int_0^1 t^{1/q+\gamma-1}\frac{\phi_X(t)}{t^\gamma}dt \leq \int_0^1 t^{1/q+\gamma-1} < \infty.$$

4.4 Examples and applications

In this section we will apply the previous results to the probability measures introduced in Examples 4.1.2, 4.1.3 and 4.1.4.

4.4.1 Cauchy type laws

Consider the probability measure space $(\mathbb{R}^n, d, \mu)$ where d is the Euclidean distance and μ is the probability measure introduced in Example 4.1.2. Such measures have been introduced by Borell [9] (see also [7]). Prototypes of these probability measures are the generalized Cauchy distributions[5]:

$$d\mu(x) = \frac{1}{Z}\left(\left(1+|x|^2\right)^{1/2}\right)^{-(n+\alpha)}, \quad \alpha > 0.$$

[5]These measures are Barenblatt solutions of the porous medium equations and appear naturally in weighted porous medium equations, giving the decay rate of this nonlinear semigroup towards the equilibrium measure, see [98] and [19].

A convex isoperimetric estimator for these measures is (see [15, Proposition 4.3])

$$I(t) = \min(t, 1-t)^{1+1/\alpha}.$$

Obviously for $0 < t < 1/2$, we have

$$\int_t^{1/2} \frac{ds}{s^{1+1/\alpha}} \preceq \frac{t}{t^{1+1/\alpha}}.$$

Thus by Theorem 17, given an r.i. space X on $\mathbb{R}^n$ we get

$$\inf_{c\in\mathbb{R}} \left\| (g-c)_\mu^* \frac{\min(t,1-t)^{1+1/\alpha}}{t} \right\|_{\bar{X}} \preceq \| |\nabla g| \|_X , \qquad (g \in Lip(\mathbb{R}^n)) .$$

Proposition 4.4.1. *Let $1 \le p < \infty$, $1 \le q \le \infty$. For all $f \in Lip(\mathbb{R}^n)$ positive with $m(f) = 0$, we get*

1. $$\|f\|_{\frac{p\alpha}{p+\alpha},q} \preceq \||\nabla f|\|_{p,q} .$$

2. *For all $s > p$*

$$\|f\|_{p,q} \preceq \||\nabla f|\|_{p,q}^{\frac{\beta}{\beta+1}} \|f\|_{s,\infty}^{\frac{1}{\beta+1}}$$

where $\beta = \alpha(\frac{1}{p} - \frac{1}{s})$.

Proof. 1) By Theorem 17 we get

$$\left\| f_\mu^* t^{\frac{1}{\alpha}} \right\|_{p,q} \preceq \||\nabla f|\|_{p,q} .$$

Now by [53, Page 76] we have that

$$\left\| f_\mu^* t^{\frac{1}{\alpha}} \right\|_{p,q}^q = \int_0^1 \left[\left(t^{\frac{1}{\alpha}} f_\mu^*(t) \right)^* t^{\frac{1}{p}} \right]^q \frac{dt}{t} \simeq \int_0^1 \left(t^{\frac{1}{\alpha}+\frac{1}{p}} f_\mu^*(t) \right)^q \frac{dt}{t} = \|f\|_{\frac{p\alpha}{p+\alpha},q}^q .$$

2) is a direct application of Theorem 18. □

Remark 4.4.1. *If in the previous proposition we take $p = q = 1$, we obtain*

$$\|f\|_{\frac{\alpha}{\alpha+1},1} \preceq \||\nabla f|\|_1 . \tag{4.4.1}$$

If $\frac{1}{q} = \frac{1}{p} + \frac{1}{\alpha}$, then we get

$$\|f\|_{p,\frac{p(1+\alpha)}{\alpha}} \preceq \||\nabla f|\|_{q,\frac{p(1+\alpha)}{\alpha}} . \tag{4.4.2}$$

For $p \ge 1$ and $s = \infty$, we have that

$$\|f\|_p \preceq \||\nabla f|\|_p^{\frac{\beta}{\beta+1}} \|f\|_\infty^{\frac{1}{\beta+1}} . \tag{4.4.3}$$

Inequalities 4.4.1 and 4.4.2 were proved in [72, Proposition 5.13]. Inequality 4.4.3 was obtained in [72, Proposition 5.15].

We close this section with the following optimality result:

Theorem 19. *Let $\alpha > 0$. Let $\bar{X}$ be an r.i. space on $(0,1)$ and let Z be a q.r.i. space on $(0,1)$. Assume that for any probability measure μ of α-Cauchy type in $\mathbb{R}^n$, there is a $C_\mu > 0$ such that for all positive $f \in Lip(\mathbb{R}^n)$ with $m(f) = 0$, we get*

$$\left\| f_\mu^* \right\|_Z \leq C_\mu \left\| |\nabla f|_\mu^* \right\|_{\bar{X}} .$$

Then for all $g \in Lip(\mathbb{R}^n)$

$$\left\| g_\mu^* \right\|_Z \preceq \left\| g_\mu^*(t) \frac{I(t)}{t} \right\|_{\bar{X}}$$

Proof. Let μ be the Cauchy probability measure on $\mathbb{R}$ defined by

$$d\mu(s) = \frac{\alpha}{2\left(1 + |s|^2\right)^{\frac{1+\alpha}{2}}} ds = \varphi(s) dx, \qquad s \in \mathbb{R}.$$

It is known (see [15, Proposition 5.27]) that its isoperimetric profile is given by

$$I_\mu(t) = \varphi\left(H^{-1}(t)\right) = \alpha 2^{1/\alpha} \min(t, 1-t)^{1+1/\alpha}, \qquad t \in [0,1],$$

where H is the distribution function of μ, i.e. $H : \mathbb{R} \to (0,1)$ is the increasing function given by

$$H(r) = \int_{-\infty}^{r} \varphi(t) dt.$$

Consider the product measure μ^n on $\mathbb{R}^n$. By Proposition 5.27 of [15] the function

$$I(t) = \frac{c_\alpha}{n^{1/\alpha}} \min(t, 1-t)^{1+1/\alpha}$$

is a convex isoperimetric estimator of μ^n (c_α denotes a positive constant depending only on α).

Given a positive measurable function f with *suppf* $\subset (0, 1/2)$, consider

$$F(t) = \int_t^1 f(s) \frac{ds}{I_{\mu_\alpha}(s)}, \qquad t \in (0,1),$$

and define

$$u(x) = F(H(x_1)), \qquad x \in \mathbb{R}^n.$$

Then,

$$|\nabla u(x)| = \left| \frac{\partial}{\partial x_1} u(x) \right| = \left| -f(H(x_1)) \frac{H'(x_1)}{I_\mu(H(x_1))} \right| = f(H(x_1)).$$

Let A be a Young's function and let $s = H(x_1)$. Then,

$$\begin{aligned} \int_{\mathbb{R}^n} A(f(H(x_1))) d\mu^n(x) &= \int_{\mathbb{R}} A(f(H(x_1))) d\mu(x_1) \\ &= \int_0^1 A(f(s)) ds. \end{aligned}$$

Therefore, by [6, exercise 5 p. 88]

$$|\nabla u|^*_{\mu^n}(t) = f^*(t).$$

Similarly

$$u^*_{\mu^n}(t) = \int_t^1 f(s)\frac{ds}{I_\mu(s)}.$$

Since $m(u) = 0$, by hypothesis we get

$$\begin{aligned}\left\| \int_t^1 f(s)\frac{ds}{I_\mu(s)} \right\|_Z &= \left\| u^*_{\mu^n} \right\|_Z \\ &\leq C_{\mu^n} \left\| |\nabla f|^*_{\mu^n} \right\|_{\bar{X}} \\ &= C_{\mu^n} \left\| f^*(t) \right\|_{\bar{X}} \\ &= C_{\mu^n} \left\| f \right\|_{\bar{X}}.\end{aligned}$$

Finally, from

$$I_\mu(t) = \frac{\alpha 2^{1/\alpha} n^{1/\alpha}}{c_\alpha} I(t)$$

we have that

$$\left\| Q_I f \right\|_Z \leq \frac{c_\alpha C_{\mu^n}}{\alpha 2^{1/\alpha} n^{1/\alpha}} \left\| f \right\|_{\bar{X}}$$

and the results follow from Theorem 17. □

4.4.2 Extended sub-exponential law

Consider the probability measure on $\mathbb{R}^n$ defined by

$$d\mu_p(x) = \frac{1}{Z_p} e^{-V(x)^p} dx = \varphi(x)dx$$

for some positive convex function $V : \mathbb{R}^n \to (0, \infty)$ and $p \in (0, 1)$.

A typical example is $V(x) = |x|^p$, and $0 < p < 1$, which yields to sub-exponential type law.

A convex isoperimetric estimator for this type of measure is (see [15, Proposition 4.5]):

$$I(t) = c_p \min(t, 1-t) \left(\log \frac{1}{\min(t, 1-t)} \right)^{1-1/p}.$$

By Theorem 17, given an r.i. space X on $\mathbb{R}^n$ with $\underline{\alpha}_X > 0$, we get

$$\inf_{c\in\mathbb{R}} \left\| (g-c)^*_\mu \frac{c_p \min(t, 1-t) \left(\log \frac{1}{\min(t,1-t)} \right)^{1-1/p}}{t} \right\|_{\bar{X}} \leq \left\| |\nabla g| \right\|_X, \qquad (g \in Lip(\mathbb{R}^n)).$$

In the particular case that $X = L^{r,q}$ we obtain the following proposition.

Proposition 4.4.2. *Let $1 \le r < \infty$, $1 \le q < \infty$. For all positive $f \in Lip(\mathbb{R}^n)$ with $m(f) = 0$,*

1.
$$\|f\|_{L^{r,q}(\log L)^{1-1/p}} \preceq \||\nabla f|\|_{r,q}.$$

2. *For all $\beta > 0$*
$$\|f\|_{r,q} \preceq \||\nabla f|\|_{r,q}^{\frac{\beta}{\beta+1}} \|f\|_{L^{r,q}(\log L)^{\beta(1-1/p)}}^{\frac{1}{\beta+1}}$$

Theorem 20. *Let $p \in (0,1)$. Let $\bar{X}$ be an r.i. space on $(0,1)$ and let Z be a q.r.i. space on $(0,1)$. Assume that for any p-extended sub-exponential law μ in $\mathbb{R}^n$ there is a $C_\mu > 0$ such that for all positive $f \in Lip(\mathbb{R}^n)$ with $m(f) = 0$,*
$$\|f_\mu^*\|_Z \le C_\mu \||\nabla f|_\mu^*\|_{\bar{X}}.$$
Then, for all $g \in Lip(\mathbb{R}^n)$,
$$\|g_\mu^*\|_Z \preceq \left\| g_\mu^*(t) \frac{I(t)}{t} \right\|_{\bar{X}}.$$

Proof. Let μ be a probability measure on $\mathbb{R}$ with density
$$d\mu_p(s) = \frac{e^{-|s|^p}}{Z_p} ds = \varphi(s) ds, \qquad s \in \mathbb{R}.$$
Its isoperimetric profile is (see [15, Proposition 5.25])
$$I_{\mu_p}(t) = \varphi\left(H^{-1}(t)\right) = c_p \min(t, 1-t) \left(\log \frac{1}{\min(t,1-t)}\right)^{1-1/p}, \qquad t \in [0,1],$$
where H is the distribution function of μ, i.e. $H : \mathbb{R} \to (0,1)$ is defined by
$$H(r) = \int_{-\infty}^{r} \varphi(t) dt.$$
Consider the product measure μ^n on $\mathbb{R}^n$. By proposition 5.25 of [15], there exists a positive constant c such that the function
$$I(t) = c \min(t, 1-t) \left(\log \frac{n}{\min(t,1-t)}\right)^{1-1/p}$$
is a convex isoperimetric estimator of μ_p^n.

Let f be a positive measurable function f with $\mathrm{supp}(f) \subset (0, 1/2)$. Consider
$$F(t) = \int_t^1 f(s) \frac{ds}{I_{\mu_p}(s)}, \qquad t \in (0,1),$$

and define

$$u(x) = F(H(x_1)), \qquad x \in \mathbb{R}^n.$$

Using the same method as used in Theorem 19, we obtain

$$|\nabla u|^*_{\mu_p^n}(t) = f^*(t) \text{ and } u^*_{\mu_p^n}(t) = \int_t^1 f(s)\frac{ds}{I_{\mu_p}(s)}.$$

Since $m(u) = 0$, by hypothesis we get

$$\begin{aligned}
\left\| \int_t^1 f(s)\frac{ds}{I_{\mu_p}(s)} \right\|_Z &= \left\| u^*_{\mu_p^n} \right\|_Z \\
&\leq C_{\mu_p^n} \left\| |\nabla f|^*_{\mu_p^n} \right\|_{\bar{X}} \\
&= C_{\mu_p^n} \left\| f^*(t) \right\|_{\bar{X}} \\
&= C_{\mu_p^n} \left\| f \right\|_{\bar{X}}.
\end{aligned}$$

Finally, from

$$I_{\mu_p}(t) \simeq I(t)$$

we have that

$$\|Q_I f\|_Z \preceq \|f\|_{\bar{X}}.$$

and Theorem 17 applies. □

4.4.3 Weighted Riemannian manifolds with negative dimension

Let (M^n, g, μ) be an n-dimensional weighted Riemannian manifold ($n \geq 2$) that satisfies the $CD(0, N)$ curvature condition with $N < 0$. (See [72, Secction 5.4].)

A convex isoperimetric estimator is given by

$$I(t) = \min(t, 1-t)^{-1/N}.$$

Obviously for $0 < t < 1/2$ we have

$$\int_t^{1/2} \frac{ds}{s^{-1/N}} \preceq \frac{t}{t^{-1/N}}.$$

Thus by Theorem 17, given an r.i. space X on $\mathbb{R}^n$ we get

$$\inf_{c \in \mathbb{R}} \left\| (g-c)^*_\mu \frac{\min(t, 1-t)^{-1/N}}{t} \right\|_{\bar{X}} \preceq \| |\nabla g| \|_X, \qquad (g \in Lip(\mathbb{R}^n)).$$

In particular, if $1 \leq p < \infty$, $1 \leq q \leq \infty$) and $X = L^{p,q}$, then for all positive $f \in Lip(\mathbb{R}^n)$ with $m(f) = 0$, ($1 \leq p < \infty$, $1 \leq q \leq \infty$),

$$\|f\|_{\gamma,q} \preceq \||\nabla f|\|_{p,q}\,,$$

where $\gamma = \frac{Np}{N-p(N+1)}$ for any p, q satisfying $\frac{N}{N-1} \leq p \leq -N$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{N} - 1$.

Now by Theorem 18, we have that

$$\|f\|_{p,q} \preceq \||\nabla f|\|_{p,q}^{\frac{\beta}{\beta+1}} \, \|f\|_{s,\infty}^{\frac{1}{\beta+1}}$$

where $s > p$ and $\beta = \alpha(\frac{1}{p} - \frac{1}{s})$.

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