Realisability problems in Algebraic Topology

Ph. D. Dissertation

David Méndez Martínez

Advisors:
Cristina Costoya & Antonio Viruel

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Programa de Doctorado en Matemáticas
Facultad de Ciencias
Universidad de Málaga

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Por la presente, Dª. María Cristina Costoya Ramos y D. Antonio Ángel Viruel Arbáizar, como directores de la tesis doctoral, y tutor en el segundo caso, de D. David Méndez Martínez,

AUTORIZAMOS la presentación de la misma para que pueda ser defendida como Tesis Doctoral según la legislación vigente.

Además,

INFORMAMOS que el trabajo mencionado es totalmente original y que ha dado lugar a las siguientes publicaciones que lo avalan:

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Hacemos constar que ningún contenido ni resultados de las anteriores publicaciones ha sido utilizado para avalar ninguna otra tesis ni trabajo de investigación.

Para que así conste y surta los efectos oportunos, firmamos el presente escrito en Málaga a 7 de octubre de 2019.

Dª. María Cristina Costoya Ramos D. Antonio Ángel Viruel Arbáizar

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Contents

Introduction

In this thesis we are interested in realisability problems, which are very natural and quite easy to state and generally hard to solve. One such problem is the so-called *inverse Galois problem*, that asks whether any finite group may appear as the Galois group of a finite Galois extension of $\mathbb{Q}$ or not. The first to study this problem in depth was Hilbert in the late 19th century, [52], and to this day it remains open.

In Algebraic Topology, where algebraic structures play a key role, realisability questions have been raised and deeply studied in many settings. A classical example is the problem of *realisability of cohomological algebras*, raised by Steenrod in 1961, [73]. This problem asks for a characterisation of the algebras that arise as the cohomology algebra of a space, and it has been treated in different surveys, [1, 3]. One more example is the *G-Moore space problem*, also raised by Steenrod, [59, Problem 51]. For a given group G, it asks if any $\mathbb{Z}G$-module appears as the homology of a simply connected G-Moore space. It was solved in the negative in 1981, [21], and a characterization of the groups G for which every $\mathbb{Z}G$-module is realisable appeared in 1987, [78].

In this thesis we focus on the so-called *group realisability problem*, which on its most general setting may be stated as follows: given a category $\mathcal{C}$, does every group arise as the group of automorphisms of an object in $\mathcal{C}$? This problem has been studied in different areas of combinatorics, appearing in different surveys, [7, 8, 54], and we make use of their terminology. Namely, if a group G happens to be the automorphism group of an object $X \in \mathrm{Ob}(\mathcal{C})$, i.e., if $G \cong \mathrm{Aut}_{\mathcal{C}}(X)$, we say that X *realises* G, and we say that G is *realisable* in $\mathcal{C}$. A category $\mathcal{C}$ where every finite group is realisable is said to be *finitely universal*, and if every group is realisable in $\mathcal{C}$ we say that it is *universal*.

Our interest in combinatorics goes beyond terminology, since the solution to the historically important *group realisability problem in the category of graphs* is a key element in many of our constructions. König raised this problem as early as 1936, [58], and barely three years later Frucht proved that the category of (finite) graphs is finitely universal, [43]. However, a solution to the general case had to wait for more than twenty additional years, until de Groot, in 1959, [32], and Sabidussi, in 1960, [72], independently proved that the category of graphs is indeed universal.

Nonetheless, the main problem motivating this thesis is the *group realisability problem in the category $\mathcal{H}oTop$*, the homotopy category of pointed topological spaces. It was proposed by Kahn in the sixties and asks if every group appears as the group of automorphisms of an object in $\mathcal{H}oTop$. It has received significant attention, having appeared in different surveys and lists of open problems, [4, 5, 37, 55, 56, 71]. Recall that, given a space X, the group $\mathrm{Aut}_{\mathcal{H}oTop}(X)$ is usually denoted $\mathcal{E}(X)$ and receives the name of *group of self-homotopy equivalences of X*. Its elements are the homotopy classes of continuous self-maps of X that have a homotopy inverse.

Immediate examples that come to mind when trying to realise groups as self-homotopy equivalences of spaces can be found in Eilenberg-MacLane spaces. Indeed, if H is an abelian group and $n \geq 2$, $\mathcal{E}(K(H,n)) \cong \mathrm{Aut}(H)$. However, this procedure does not give a full positive answer to the realisability in the category $\mathcal{H}oTop$, given that not every group G is isomorphic to $\mathrm{Aut}(H)$ for some other group H (for example, $\mathbb{Z}_p$ is not realisable as the automorphism group of any other group, when p is odd).

The difficulty of this problem arises precisely from the fact that, other than making use of Eilenberg-MacLane spaces as described above, there is no obvious procedure to obtain spaces realising a certain given group. Consequently, the tools for seriously digging into Kahn's question were at the beginning insufficient, and for over decades this problem had only been studied using ad-hoc procedures for certain families of groups, [15, 16, 36, 62, 65].

This impasse ended with a general method obtained by Costoya and Viruel, [27], that gives a positive answer to Kahn's realisability problem in the case of finite groups. Namely, for every finite group G, there exists a topological space X (in fact, an infinite number of them) such that $G \cong \mathcal{E}(X)$ ([27, Theorem 1.1]). They provide such a method by combining Frucht's solution for the group realisability problem in the category of graphs with the remarkable computational power of Rational Homotopy Theory.

Our main goal in this thesis is to expand on that sort of techniques to study further realisability problems. On the one hand, we refine the results in [27] (see Chapter 4) and, on the other hand, we extend these techniques to realisability problems in categories other than $\mathcal{H}oTop$ (see Chapter 3). Notice that Rational Homotopy Theory deals with spaces which are not of finite type over $\mathbb{Z}$ (see Definition 1.36), and thus they are not geometrically simple. This fact led us to consider an alternative approach (A_n^2-polyhedra) in order to provide a solution to Kahn's problem in terms of *integral* spaces (see Chapter 6).

Let us start by introducing two generalisations of the group realisability problem which from now on we will refer to as the classical group realisability problem. The first generalisation we consider deals with the so-called *arrow categories*. Recall that the arrow category of a category $\mathcal{C}$, denoted $\mathrm{Arr}(\mathcal{C})$, is the category whose objects are morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(A_1, A_2)$ between any two objects $A_1, A_2 \in \mathrm{Ob}(\mathcal{C})$; and where a morphism between $f \in \mathrm{Hom}_{\mathcal{C}}(A_1, A_2)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B_1, B_2)$ is a pair $(f_1, f_2) \in \mathrm{Hom}_{\mathcal{C}}(A_1, B_1) \times \mathrm{Hom}_{\mathcal{C}}(A_2, B_2)$ such that $g \circ f_1 = f_2 \circ f$. Then, given $f \in \mathrm{Hom}_{\mathcal{C}}(A_1, A_2)$, we see that $\mathrm{Aut}_{\mathrm{Arr}(\mathcal{C})}(f)$, which by abuse of notation we denote $\mathrm{Aut}_{\mathcal{C}}(f)$, is a subgroup of $\mathrm{Aut}_{\mathcal{C}}(A_1) \times \mathrm{Aut}_{\mathcal{C}}(A_2)$. The following problem arises naturally:

Problem 1 (Realisability problem in arrow categories)**.** *Let $\mathcal{C}$ be a category. Can we find, for any groups G_1, G_2 and $H \leq G_1 \times G_2$, an object $f\colon A_1 \to A_2$ in $\mathrm{Arr}(\mathcal{C})$ such that $\mathrm{Aut}_{\mathcal{C}}(A_1) \cong G_1$, $\mathrm{Aut}_{\mathcal{C}}(A_2) \cong G_2$ and $\mathrm{Aut}_{\mathcal{C}}(f) \cong H$?*

The second generalisation of the classical group realisability problem deals with *permutation representations*, that is, actions of a group on a set by permutations. If $\mathcal{C}$ is a category whose objects are sets, we can think of realising a given permutation representation in $\mathcal{C}$ as follows:

Problem 2 (Realisability of permutation representations)**.** *Let $\rho\colon G \to \mathrm{Sym}(V)$ be a permutation representation and let $\mathcal{C}$ be a category whose objects are sets. Is there a (fully) faithful G-object $A \in \mathrm{Ob}(\mathcal{C})$ such that V can be regarded as an $\mathrm{Aut}_{\mathcal{C}}(A)$-invariant subset of A in such a way that the restriction of the G-action to V is ρ?*

We tackle these problems following the techniques of Costoya-Viruel, by first solving them in the category of graphs, $\mathcal{C} = \mathcal{G}raphs$. Then, we will transfer our solution to suitable algebraic frameworks (coalgebras and commutative differential graded algebras). Let us start by discussing our solution to Problem 1 and Problem 2 in $\mathcal{G}raphs$.

We follow Frucht's approach for the classical realisability problem for graphs, [43]. That is, we work in a more general setting: the category of binary relational systems over a set I, see Definition 1.1. Binary relational systems have additional structure since their edges are labelled and directed, so they model in a better way to our purposes the structure of the groups involved. In that context of binary relational systems, we are able to solve Problem 1 (see Theorem 2.16) and also Problem 2 (see Theorem 2.26).

Then, in order to transfer both solutions of Problem 1 and Problem 2 from binary relational systems to $\mathcal{G}raphs$, we use the *arrow replacement procedure*; we replace labelled directed edges by a construction that may be part of a simple graph. Although results in the literature provide an arrow replacement procedure powerful enough to solve the classical group realisability problem, they are not adequate for our purposes. We provide a result tailored to our needs in Theorem 2.33. Combining this result and our solution to Problem 1 in binary relational systems, Theorem 2.16, we are able to positively solve Problem 1 in $\mathcal{C} = \mathcal{G}raphs$:

Theorem 3 (Theorem 2.37)**.** *Let G_1, G_2 and H be groups such that $H \leq G_1 \times G_2$. Then, there exist graphs $\mathcal{G}_1$, $\mathcal{G}_2$ and a morphism of graphs $\varphi\colon \mathcal{G}_1 \to \mathcal{G}_2$ such that $\mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}_1) \cong G_1$, $\mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}_2) \cong G_2$ and $\mathrm{Aut}_{\mathcal{G}raphs}(\varphi) \cong H$.*

Similarly, Theorem 2.33 can be used to transfer the solution to Problem 2 from binary relational systems, Theorem 2.26, to graphs, providing a generalisation of [19, Theorem 1.1]:

Theorem 4 (Theorem 2.41)**.** *Let G be a group, V be a set and $\rho\colon G \to \mathrm{Sym}(V)$ be a permutation representation of G on V. There is a graph $\mathcal{G}$ such that*

(1) $V \subset V(\mathcal{G})$ and each $\psi \in \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G})$ is invariant on V;

(2) $\mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \cong G$;

(3) the restriction $G \cong \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \to \mathrm{Sym}(V)$ is precisely ρ;

(4) there is a faithful action $\bar{\rho}\colon G \cong \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \to \mathrm{Sym}\left(V(\mathcal{G}) \setminus V\right)$ such that the restriction map $G \cong \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \to \mathrm{Sym}\left(V(\mathcal{G})\right)$ is $\rho \oplus \bar{\rho}$.

We highlight the fact that, when the groups and sets involved are finite, Problem 1 and Problem 2 admit a solution where the graphs involved are also finite, see Corollary 2.38 and Corollary 2.42 respectively. This fact will be crucial to us later on, when working with $\mathcal{C} = \mathcal{H}oTop$.

Now that a solution to both problems has been obtained in $\mathcal{G}raphs$, we can move on to algebraic structures. We begin with coalgebras, see Definition 1.22. A lot is known about automorphism groups of rings and algebras (see for example [25, 28, 57] for the associative case, and [44] for the non-associative one), but very little is known about the case of coalgebras, their Eckmann-Hilton dual structure. Moreover, since the dual of an infinite-dimensional algebra may not be a coalgebra, general results on automorphisms of coalgebras cannot be deduced from the pre-existing literature on automorphism groups of rings.

Our aim is then to provide some initial results regarding the classical group realisability problem in the category of coalgebras, $\mathcal{C} = \mathrm{Coalg}_{\Bbbk}$, while also considering Problem 1 and Problem 2. We point out that in this thesis we have not been able to realise an arbitrary given group G as the automorphism group of a coalgebra. However, we are able to provide coalgebras such that G arises as the image of the restriction of the automorphisms of the coalgebra to its set of grouplike elements, see Definition 1.26. We do so by introducing a faithful functor $C\colon \mathcal{D}igraphs \to \mathrm{Coalg}_{\Bbbk}$, Definition 3.4, for which we prove the following:

Theorem 5 (Theorem 3.9). *Let $\Bbbk$ be a field and $\mathcal{G}$ be a digraph. There is a $\Bbbk$-coalgebra $C(\mathcal{G})$ such that $G(C(\mathcal{G})) = V(\mathcal{G})$ and the restriction map $\operatorname{Aut}_{\mathrm{Coalg}_{\Bbbk}}(C(\mathcal{G})) \to \operatorname{Sym}(V(\mathcal{G}))$ induces a split short exact sequence of groups*

$$1 \longrightarrow \prod_{e \in E(\mathcal{G})} (\Bbbk \rtimes \Bbbk^{\times}) \longrightarrow \operatorname{Aut}_{\mathrm{Coalg}_{\Bbbk}}(C(\mathcal{G})) \longrightarrow \operatorname{Aut}_{\mathcal{Digraphs}}(\mathcal{G}) \longrightarrow 1.$$

In particular, since every group is the automorphism group of a (directed) graph, [32, 72], we immediately deduce the following:

Corollary 6 (Corollary 3.11). *Let $\Bbbk$ be a field and G be a group. There is a $\Bbbk$-coalgebra C such that $\operatorname{Aut}_{\mathrm{Coalg}_{\Bbbk}}(C) \cong K \rtimes G$, where K is a direct product of semidirect products of the form $\Bbbk \rtimes \Bbbk^{\times}$. Furthermore, G is the image of the restriction of the automorphisms of C to $\operatorname{Sym}(G(C))$.*

Then, using the functor C previously introduced and its properties, the following result regarding Problem 1 follows from Theorem 3:

Theorem 7 (Theorem 3.12). *Let G_1, G_2 and $H \leq G_1 \times G_2$ be groups. Let $\Bbbk$ be a field. There exist two $\Bbbk$-coalgebras C_1 and C_2 and a morphism $\varphi \in \operatorname{Hom}_{\mathrm{Coalg}_{\Bbbk}}(C_1, C_2)$ such that*

(1) $\operatorname{Aut}_{\mathrm{Coalg}_{\Bbbk}}(C_k) \cong K_k \rtimes G_k$, *where G_k is the image of the restriction* $\operatorname{Aut}_{\mathrm{Coalg}_{\Bbbk}}(C_k) \to \operatorname{Sym}(G(C_k))$ *and K_k is a direct product of factors of the form $\Bbbk \rtimes \Bbbk^{\times}$, $k = 1, 2$;*

(2) $\operatorname{Aut}_{\mathrm{Coalg}_{\Bbbk}}(\varphi) \cong K \rtimes H$, *where H is the image of the restriction map* $\operatorname{Aut}_{\mathrm{Coalg}_{\Bbbk}}(\varphi) \to \operatorname{Sym}(G(C_1)) \times \operatorname{Sym}(G(C_2))$ *and $K \leq K_1 \times K_2$.*

Similarly, regarding Problem 2, the following can be deduced from Theorem 4:

Theorem 8 (Theorem 3.13). *Let G be a group, $\Bbbk$ be a field and $\rho\colon G \to \operatorname{Sym}(V)$ be a permutation representation of G on a set V. There exists a G-coalgebra C such that:*

(1) G acts faithfully on C, that is, there is a group monomorphism $G \hookrightarrow \operatorname{Aut}_{\mathrm{Coalg}_{\Bbbk}}(C)$;

(2) the image of the restriction map $\operatorname{Aut}_{\mathrm{Coalg}_{\Bbbk}}(C) \to \operatorname{Sym}(G(C))$ is G;

(3) there is a subset $V \subset G(C)$ that is invariant through the $\operatorname{Aut}_{\mathrm{Coalg}_{\Bbbk}}(C)$-action on C and such that ρ is the composition of the inclusion $G \hookrightarrow \operatorname{Aut}_{\mathrm{Coalg}_{\Bbbk}}(C)$ with the restriction $\operatorname{Aut}_{\mathrm{Coalg}_{\Bbbk}}(C) \to \operatorname{Sym}(V)$;

(4) there is a faithful action $\bar{\rho}\colon G \to \operatorname{Sym}(G(C) \setminus V)$ such that the composition of the inclusion $G \hookrightarrow \operatorname{Aut}_{\mathrm{Coalg}_{\Bbbk}}(C)$ with the restriction $\operatorname{Aut}_{\mathrm{Coalg}_{\Bbbk}}(C) \to \operatorname{Sym}(G(C))$ is $\rho \oplus \bar{\rho}$.

As a further application of the functor C, Theorem 5 and Corollary 6, we study the isomorphism problem for groups using coalgebra representations. Namely, we want to see if we can distinguish isomorphism classes of groups by looking at their actions on coalgebras.

We are able to provide two results. The first result holds for a large family of groups, co-Hopfian ones, but it requires that we focus on how the action looks like on grouplike elements. Recall that a group is co-Hopfian if it is not isomorphic to any of its proper subgroups. It is a large family of groups, as shown in Example 3.15. In particular, it includes all finite groups. We prove the following result:

Theorem 9 (Theorem 3.16). *Let $\Bbbk$ be a field and let G and H be two co-Hopfian groups. The following statements are equivalent:*

(1) G and H are isomorphic.

(2) For any $\Bbbk$-coalgebra C, there is an action of G on C that restricts to a faithful action on $G(C)$ if and only if there is an action of H on C that restricts to a faithful action on $G(C)$.

For our second result we do not focus on grouplike elements, but we need to further restrict our family of groups. In Definition 3.17 we introduce a family of groups $\mathfrak{G}_{p,n}$ for which we prove:

Theorem 10 (Theorem 3.18)**.** *Let $n \geq 1$ be an integer, p be a prime and $\Bbbk$ be a finite field of order p^n. Let G and H be groups in $\mathfrak{G}_{p,n}$. The following are equivalent:*

(1) G and H are isomorphic.

(2) For every $\Bbbk$-coalgebra C, G acts faithfully on C if and only if H acts faithfully on C.

We remark that, although the family $\mathfrak{G}_{p,n}$ is smaller than the class of co-Hopfian groups, $\mathfrak{G}_{2,1}$ still contains all 2-reduced groups, that is, all groups with no normal 2-subgroups.

We now continue with the study of realisability problems in the category of commutative differential graded algebras, and with the subsequent results in the homotopy category of topological spaces. The works of Costoya and Viruel, [27, 28, 29], constitute our starting point in this framework. The base of their constructions is a non-trivial rational space with trivial group of self-homotopy equivalences, [6, Example 5.1], which they use as a building block for their algebraic models.

Spaces with trivial group of self-homotopy equivalences are called *homotopically rigid spaces*. The first elaborated example of a homotopically rigid space with non-trivial rational homology was constructed by Kahn in 1976, [55]. He expressed the belief that homotopically rigid spaces might play a role in some way of decomposing a space, [56]. Thus, obtaining examples of homotopically rigid spaces becomes of interest, although few examples are known in literature. In Definition 4.1 we build an infinite family of commutative differential graded R-algebras (or CDGA_R for short, see Definition 1.37) $\mathcal{M}_k$, $k \geq 1$ that not only are they homotopically rigid (thus modelling homotopically rigid spaces) but they are rigid as algebras as well. Namely, we prove:

Theorem 11 (Theorem 4.3)**.** *Let R be an integral domain and $k \geq 1$ be an integer. There exists a (k-connected) commutative differential graded R-algebra $\mathcal{M}_k$ that is rigid, that is, such that* $\mathrm{Hom}_{\mathrm{CDGA}_R}(\mathcal{M}_k, \mathcal{M}_k) = \{0, \mathrm{id}\}$.

Following the path laid out by [27], we can use these rigid algebras as building blocks for Sullivan models of spaces that codify the combinatorial data of a graph. Indeed, given an integer $n \geq 1$, to a directed graph $\mathcal{G}$ (without loops and with at least one edge starting at every vertex) we associate an algebra $\mathcal{M}_n(\mathcal{G})$ (see Definition 4.6), and to a morphism between two such digraphs $\sigma\colon \mathcal{G}_1 \to \mathcal{G}_2$ we associate a morphism of algebras $\mathcal{M}_n(\sigma)\colon \mathcal{M}_n(\mathcal{G}_1) \to \mathcal{M}_n(\mathcal{G}_2)$ (see Lemma 4.10).

This association is, in fact, functorial. Indeed, in Definition 4.11 we introduce, for each $n \geq 1$, a functor $\mathcal{M}_n$ from $\mathcal{D}igraphs_+$, the full subcategory of these directed graphs without loops and with at least one edge starting at every vertex (see Definition 4.4), to CDGA_R. Furthermore, if R is an integral domain of characteristic zero or greater than three, $\mathcal{M}_n$ is *almost* fully faithful: for any two digraphs $\mathcal{G}_1$ and $\mathcal{G}_2$ in $\mathcal{D}igraphs_+$, the set $\mathrm{Hom}_{\mathcal{D}igraphs}(\mathcal{G}_1, \mathcal{G}_2)$ is in bijection with $\mathrm{Hom}_{\mathrm{CDGA}_R}(\mathcal{M}_n(\mathcal{G}_1), \mathcal{M}_n(\mathcal{G}_2)) \setminus \{0\}$. In particular, $\mathcal{M}_n$ preserves automorphism groups. Since we know that $\mathcal{G}raphs$ is universal, [32, 72], we are able to enlarge the family of categories that are known to be universal by proving the following result for CDGA_R:

Theorem 12 (Theorem 4.16)**.** *Let R be an integral domain with* $\operatorname{char}(R) = 0$ *or* $\operatorname{char}(R) > 3$, *and $n \geq 1$ be an integer. For every group G, there exists a (n-connected) commutative differential graded R-algebra M such that* $\operatorname{Aut}_{\mathrm{CDGA}_R}(M) \cong G$.

From Theorem 3 and by the properties of the family of functors $\mathcal{M}_n$, Problem 1 is positively solved in $\mathcal{C} = \mathrm{CDGA}_R$:

Theorem 13 (Theorem 4.17)**.** *Let G_1, G_2 and H be groups such that $H \leq G_1 \times G_2$. Let R be an integral domain with* $\operatorname{char}(R) = 0$ *or* $\operatorname{char}(R) > 3$. *For any $n \geq 1$, there are n-connected objects $M_1, M_2 \in \operatorname{Ob}(\mathrm{CDGA}_R)$ and a morphism $\varphi \in \operatorname{Hom}_{\mathrm{CDGA}_R}(M_1, M_2)$ such that* $\operatorname{Aut}_{\mathrm{CDGA}_R}(M_1) \cong G_1$, $\operatorname{Aut}_{\mathrm{CDGA}_R}(M_2) \cong G_2$ *and* $\operatorname{Aut}_{\mathrm{CDGA}_R}(\varphi) \cong H$.

We are now interested in transferring Theorem 13 to $\mathcal{H}oTop$ using tools from Rational Homotopy Theory. To do so, we restrict ourselves to the cases where the algebras from Theorem 13 are of finite type. This will be the case if graphs from Theorem 3 are finite. But, as we previously mentioned, by Corollary 2.38 such graphs can be chosen to be finite if the groups involved are finite. Hence, if we denote $\operatorname{Aut}_{\mathcal{H}oTop}(f)$ as $\mathcal{E}(f)$, we can prove the following:

Theorem 14 (Theorem 4.19)**.** *Let G_1, G_2 be finite groups and H be a subgroup of $G_1 \times G_2$. For any $n \geq 1$, there exist n-connected spaces X_1, X_2 and a continuous map $f\colon X_1 \to X_2$ such that $\mathcal{E}(X_1) \cong G_1$, $\mathcal{E}(X_2) \cong G_2$ and $\mathcal{E}(f) \cong H$.*

We now turn to Problem 2 in this framework. We point out that the results that we obtained for CDGA_R are a particular case of those in [29]. We include them here for the sake of completion and because they follow easily from our results in $\mathcal{G}raphs$. Using Theorem 4 and the properties of the family of functors $\mathcal{M}_n$, the following result can be proven:

Theorem 15 (Theorem 4.20)**.** *Let G be a group, $n \geq 1$ be an integer, R be an integral domain with* $\operatorname{char}(R) = 0$ *or* $\operatorname{char}(R) > 3$ *and $\rho\colon G \to \operatorname{Sym}(V)$ be a permutation representation of G on a set V. There is an n-connected object $A \in \operatorname{Ob}(\mathrm{CDGA}_R)$ such that*

(1) $V \subset A$, and V is invariant through the automorphisms of A;

(2) $\operatorname{Aut}_{\mathrm{CDGA}_R}(A) \cong G$ (and if $R = \mathbb{Q}$, $\mathcal{E}(A) \cong G$ as well);

(3) The restriction map $G \cong \operatorname{Aut}_{\mathrm{CDGA}_R}(A) \to \operatorname{Sym}(V)$ is precisely ρ.

Again, we can transfer this result to $\mathcal{H}oTop$ by restricting ourselves to finite graphs. Using Corollary 2.42 we prove:

Theorem 16 (Theorem 4.22)**.** *Let G be a finite group, V be a finite set, n be a positive integer and $\rho\colon G \to \operatorname{Sym}(V)$ be a permutation representation of G on V. There is an n-connected space X such that*

(1) $V \subset H^{180n^2-142n+28}(X)$, and V is invariant through the maps induced in cohomology by the self-homotopy equivalences of X;

(2) $\mathcal{E}(X) \cong G$;

(3) the map $G \cong \mathcal{E}(X) \to \operatorname{Sym}(V)$ taking $[f] \in \mathcal{E}(X)$ to $H^{180n^2-142n+28}(f)|_V \in \operatorname{Sym}(V)$ is ρ.

Apart from answering positively Problem 1 and Problem 2 in different settings, as we have just explained, other results can be obtained by making use of the properties of our family of functors $\mathcal{M}_n$. For instance, the bijection between sets of morphisms induced by $\mathcal{M}_n$

allows us to interpret $\mathcal{D}igraphs_+$ as a subcategory of CDGA_R. In a sense, we are *representing* $\mathcal{D}igraphs_+$ inside of CDGA_R.

Along this line, we say that a category $\mathcal{C}$ is *representable* in another category $\mathcal{D}$ if there is a fully faithful functor $F\colon \mathcal{C} \to \mathcal{D}$, that is, if $\mathcal{C}$ can be regarded as a full subcategory of $\mathcal{D}$. Recall that a category $\mathcal{C}$ is *concrete* if there is a faithful functor $F\colon \mathcal{C} \to \mathrm{Set}$. Roughly speaking, this means that the objects of $\mathcal{C}$ can be thought of as sets, and morphisms can be regarded as actual maps of sets. In [68, Chapter 4, 1.11], the authors prove that if $\mathcal{C}$ is a concrete small category, it is representable in a certain full subcategory of $\mathcal{D}igraphs_+$ (see Theorem 1.20). If we denote the category of n-connected CDGAs over R by CDGA_R^n, we obtain the following result:

Theorem 17 (Theorem 5.1)**.** *Let R be an integral domain with $\mathrm{char}(R) = 0$ or $\mathrm{char}(R) > 3$. Let $\mathcal{C}$ be a concrete small category. For every $n \geq 1$, there exists a functor $G_n\colon \mathcal{C} \to \mathrm{CDGA}_R^n$ such that $\mathrm{Hom}_{\mathrm{CDGA}_R}(G_n(A), G_n(B)) \setminus \{0\} \cong \mathrm{Hom}_{\mathcal{C}}(A, B)$, for any $A, B \in \mathrm{Ob}(\mathcal{C})$. Furthermore, if $R = \mathbb{Q}$, then $[G_n(A), G_n(B)] \setminus \{[0]\} \cong \mathrm{Hom}_{\mathcal{C}}(A, B)$.*

If we take $\mathbb{Q}$ as the base ring, the functors $\mathcal{M}_n$ arrive at $\mathrm{CDGA}_{\mathbb{Q}}^n$. Furthermore, if $\mathcal{G}$ is a finite graph, $\mathcal{M}_n(\mathcal{G})$ is a commutative graded $\mathbb{Q}$-algebra of finite type, which happens to be a Sullivan model of the rational homotopy type of an n-connected space. This means that the result above can be transferred to spaces, but certain restrictions have to be made first. In [49, Theorem 4.24], the authors prove that if $\mathcal{C}$ is a category with countable objects such that the set of morphisms between any two objects is finite, then $\mathcal{C}$ is representable in $\mathcal{G}raphs_f$ the category of finite graphs (see Theorem 1.21). Thus, we can obtain a similar although less general result for topological spaces. Denote the category of n-connected (pointed) topological spaces by Top^n.

Theorem 18 (Theorem 5.2)**.** *Let $\mathcal{C}$ be a concrete category such that $\mathrm{Ob}(\mathcal{C})$ is countable and $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is finite for any pair of objects $A, B \in \mathrm{Ob}(\mathcal{C})$. For every $n \geq 1$, there exist a functor $F_n\colon \mathcal{C} \to \mathcal{H}oTop^n$ such that $[F_n(A), F_n(B)] \setminus \{[0]\} = \mathrm{Hom}_{\mathcal{C}}(A, B)$, for any $A, B \in \mathrm{Ob}(\mathcal{C})$.*

Our almost fully faithful functors also allow us to study the problem of realising monoids as monoids of endomorphisms of objects. Recall that a monoid M can be regarded as a one object category. When doing so, M is the monoid of endomorphisms of the unique object in the category. Such a category is clearly concrete, and it is finite whenever M is so. Furthermore, adding a zero endomorphism to the unique object in such a category is equivalent to adding a zero element to the monoid M. Thus, if we denote the monoid obtained from M by adding a zero element by M^0, Theorem 17 and Theorem 18 immediately imply the following result:

Corollary 19 (Corollary 5.3)**.** *Let M be a monoid. For every $n \geq 1$, there exists a (n-connected) commutative differential graded R-algebra A_n such that $\mathrm{Hom}_{\mathrm{CDGA}_R}(A_n, A_n) \cong M^0$. If moreover M is finite, there exists a (n-connected) space X_n such that $[X_n, X_n] \cong M^0$.*

In particular, if M is a monoid such that $M \cong N^0$ for some other monoid N, that is, if M has a zero element and no non-trivial zero divisors, we can realise it directly.

Going further in the applications of our algebras, we now consider the isomorphism problem for groups using actions on CDGAs. In [28, Theorem 1.1], the authors prove that isomorphism classes within a certain family of co-Hopfian groups can be distinguished through their faithful representations on CDGAs. It turns out that we can use our algebras from Definition 4.6 to generalise their main result to the entire class of co-Hopfian groups. Namely, we prove:

Theorem 20 (Theorem 5.4)**.** *Let R be an integral domain with $\mathrm{char}(R) = 0$ or $\mathrm{char}(R) > 3$, $n \geq 1$ be an integer and G and H be co-Hopfian groups. The following are equivalent:*

(1) G and H are isomorphic.

(2) For any n-connected commutative differential graded R-algebra (A, d), G acts faithfully on (A, d) if and only if H acts faithfully on (A, d).

We remark that this result actually holds in the broader class of co-Hopfian monoids without zero (see Proposition 5.7), which in particular includes all finite monoids without zero.

Let us now consider some applications to differential geometry. Recall that a manifold M whose cohomology verifies Poincaré duality is said to be *inflexible* if the set of all the possible degrees of its self-maps is finite, that is, if $|\{\deg(f) \mid f\colon M \to M\}| < \infty$. Given that degree is multiplicative, this implies that $\{\deg(f) \mid f\colon M \to M\} \subset \{-1, 0, 1\}$.

Obvious examples of inflexible manifolds include hyperbolic manifolds. However, examples of simply connected inflexible manifolds appear more sparsely in the literature, [2, 27, 31], and they all show low levels of connectivity when observing their minimal Sullivan models. Since the existence of such examples has become important in the theory of functorial seminorms on homology [31, 48], it would be interesting to know if our algebras can produce examples of inflexible manifolds as highly connected as we desire.

It turns out that the cohomologies of our algebras verify Poincaré duality, and they also happen to be inflexible (see Lemma 5.9). Thus, if they were the rational model of a manifold, that manifold would be inflexible. Sullivan, [75], and Barge, [9], studied obstructions for rational homotopy types whose cohomology verifies Poincaré duality to be the rationalisation of a closed, orientable manifolds. By means of that, we prove:

Theorem 21 (Theorem 5.12)**.** *For any finite group G and any integer $n \geq 1$, G is the group of self-homotopy equivalences of the rationalization of an inflexible manifold which is $(30n - 13)$-connected.*

Even more, using a construction in [27], we are also able to provide examples of the so-called *strongly chiral manifolds*, that is, manifolds that do not admit orientation-reversing self-maps. Namely, we prove the following:

Proposition 22 (Proposition 5.14)**.** *For any finite group G and any integer $n \geq 1$, G is the group of self-homotopy equivalences of the rationalization of a strongly chiral manifold which is $(30n - 13)$-connected.*

So far, our realisability problems have been solved by means of Rational Homotopy Theory techniques. As a consequence, the objects that we obtain as an answer to the classical realisability problem in $\mathcal{H}oTop$ (or Kahn's realisability problem) are rational spaces, which are not of finite type over $\mathbb{Z}$. Our purpose now is to find an alternative way of solving this question by means *integral* spaces, that is, spaces of finite type over $\mathbb{Z}$.

One framework where a mostly group-theoretical classification of homotopy types exist is that of A_n^2-polyhedra: $(n-1)$-connected, $(n+2)$-dimensional CW-complexes. In [80], J.H.C. Whitehead classified homotopy types of A_2^2-polyhedra (that is, simply connected 4-dimensional CW-complexes) by means of a certain exact sequence of groups, and Baues later provided a generalisation of such classification to include A_n^2-polyhedra, for all $n \geq 2$, [12, Ch. I, Section 8].

Following the ideas of [17] we introduce a group $\mathcal{B}^{n+2}(X)$ (Definition 1.69) associated to the exact sequence that classifies an A_n^2-polyhedron X, and prove that it is isomorphic to $\mathcal{E}(X)/\mathcal{E}_*(X)$ (Proposition 1.70). Here, the group $\mathcal{E}_*(X)$ is a very well known normal subgroup of $\mathcal{E}(X)$ consisting on those self-homotopy equivalences of X that induce the identity map on homology groups. Of course, the study of $\mathcal{E}(X)/\mathcal{E}_*(X)$ provides us with information regarding which groups may appear as $\mathcal{E}(X)$ for X an A_n^2-polyhedron, but it also makes sense to consider a realisability question directly on the quotient, and it has been raised in [37, Problem 19]. So let us discuss this problem now.

Using Proposition 1.70, we study how the cell structure of an A_n^2-polyhedron X affects its group of self-homotopy equivalences. For instance, we show that under some restrictions on the homology groups of X, $\mathcal{B}^{n+2}(X)$ is infinite, which in particular implies that $\mathcal{E}(X)$ is infinite (see Proposition 6.6 and Proposition 6.9). Or for example, in many situations the existence of cycles of odd order in the homology groups of an A_n^2-polyhedron implies the existence of self-homotopy equivalences of even order (see Lemma 6.4 and Lemma 6.5).

Although we easily obtain that automorphism groups of abelian groups are realisable in our context (see Example 6.2), we also prove the following result:

Theorem 23 (Theorem 6.16)**.** *Let X be a finite type A_n^2-polyhedron, $n \geq 3$. Then either $\mathcal{B}^{n+2}(X)$ is the trivial group or it has elements of even order.*

As an immediate corollary, we obtain the following:

Corollary 24 (Corollary 6.17)**.** *Let G be a non nilpotent finite group of odd order. Then, for any $n \geq 3$ and for any finite type A_n^2-polyhedron X, $G \ncong \mathcal{E}(X)$.*

The case of A_2^2-polyhedra is more complicated. A detailed group-theoretical analysis shows that finite groups of odd order might only be realisable through a finite type A_2^2-polyhedron under very restrictive conditions. Recall that for a group G, $\operatorname{rank} G$ is the smallest cardinal of a set of generators for G, [61, p. 91]. We have the following result:

Theorem 25 (Theorem 6.18)**.** *Suppose that X is a finite type A_2^2-polyhedron with a non-trivial and finite $\mathcal{B}^4(X)$ of odd order. Then the following must hold:*

(1) $\operatorname{rank} H_4(X) \leq 1$*;*

(2) $\pi_3(X)$ *and* $H_3(X)$ *are* 2*-groups, and* $H_2(X)$ *is an elementary abelian* 2*-group;*

(3) $\operatorname{rank} H_3(X) \leq \frac{1}{2} \operatorname{rank} H_2(X)\big(\operatorname{rank} H_2(X) + 1\big) - \operatorname{rank} H_4(X) \leq \operatorname{rank} \pi_3(X)$*;*

(4) the natural action of $\mathcal{B}^4(X)$ *on* $H_2(X)$ *induces a faithful representation* $\mathcal{B}^4(X) \leq \operatorname{Aut}\big(H_2(X)\big)$*.*

Nonetheless, our attempts to find a space satisfying the hypothesis of Theorem 25 were unsuccessful, so we raise the following conjecture:

Conjecture 26 (Conjecture 6.19)**.** *Let X be an A_2^2-polyhedron. If $\mathcal{B}^4(X)$ is a non-trivial finite group, then it necessarily has an element of order* 2*.*

Consequently, this leaves room for further research, both because a negative answer to Kahn's group realisability problem has not yet been obtained and because we are still interested in finding a solution to this problem in terms of *integral* spaces.

Outline of the thesis. Chapter 1 contains the necessary preliminaries for the remainder of the manuscript. Namely, Section 1.2 introduces basic concepts and results that we need from Graph Theory; Section 1.3 introduces the basics on coalgebras; Section 1.4 is devoted to the basics of CDGAs and Rational Homotopy Theory, and Section 1.5 explains the classification of homotopy types of A_n^2-polyhedra.

In Chapter 2 we solve realisability problems in $\mathcal{G}raphs$. Namely, in Section 2.1 we build binary relational systems solving Problem 1. We do the same for Problem 2 in Section 2.2. Then, in Section 2.3 the arrow replacement procedure is studied, and our solutions from binary relational systems to graphs are transferred, thus proving Theorem 3 and Theorem 4.

Chapter 3 is dedicated to the study of realisability problems in the framework of coalgebras. We introduce the functor C in Section 3.1. We also study its properties, Theorem 5 and Corollary 6. In Section 3.2 we apply the results from the previous section to solve Problem 1 and Problem 2, Theorem 7 and Theorem 8. Finally, in Section 3.3 we deal with the group isomorphism problem through coalgebra representations, Theorem 9 and Theorem 10.

Chapter 4 is devoted to the solution of realisability problems in commutative differential graded algebras. Namely, in Section 4.1 we introduce a family of CDGAs which are proven to be rigid in Theorem 11. Then, in Section 4.2 we construct CDGAs associated to digraphs in $\mathcal{D}igraphs_+$, Definition 4.6, to solve the group realisability problem in CDGA_R, Theorem 12. We also show that the association is functorial, thus obtaining for every integer $n \geq 1$ a functor $\mathcal{M}_n \colon \mathcal{D}igraphs_+ \to \mathrm{CDGA}_R$, Definition 4.11. Finally, in Section 4.3 we consider Problem 1 and Problem 2 in both CDGAs and $\mathcal{H}oTop$, Theorem 13, Theorem 14, Theorem 15 and Theorem 16.

Chapter 5 studies further applications of the family of functors $\mathcal{M}_n$ introduced in the previous chapter. It includes both new results and improvements of results in the literature. Namely, Section 5.1 is devoted to the representability of categories, and we prove Theorem 17, Theorem 18 and Corollary 19. Section 5.2 is dedicated to the isomorphism problem for groups and monoids through their actions on CDGAs, Theorem 20. Section 5.3 is devoted to applications to differential geometry, Theorem 21 and Proposition 22. Lastly, in Section 5.4 we use the spaces modelled by our functors to show that two numerical homotopy type invariants (of Lusternik Schnirelmann type) of a finite space can be arbitrarily different.

Finally, Chapter 6 is devoted to the study of self-homotopy equivalences of A_n^2-polyhedra. Namely, Section 6.1 deals with some general results on the group $\mathcal{B}^{n+2}(X)$. Results regarding the finiteness, the realisability of groups that are automorphisms of another group, and the existence of elements of even order in $\mathcal{B}^{n+2}(X)$ are obtained. Then, in Section 6.2 we apply those results to study obstructions to the existence of elements of even order in $\mathcal{B}^{n+2}(X)$. Namely, we prove Theorem 23, Corollary 24 and Theorem 25. We also raise Conjecture 26.

We remark that many of the results included in this thesis can be found in [23, 24, 25, 26].

CHAPTER 1

Preliminaries

In this chapter we summarize all the notation and pre-existing results on which this thesis stands. Namely, Section 1.2 is dedicated to some basics on Graph Theory, along with some statements on the classical group realisability problem in this context, [32, 43, 72]. Section 1.3 introduces coalgebras and some of their properties, following [22]. Then, Section 1.4 deals with the introduction of the basics on CDGAs and Rational Homotopy Theory. We mostly follow [39], our main reference for this subject. Finally, Section 1.5 is dedicated to the classification of the homotopy types of A_n^2-polyhedra due to J.H.C. Whitehead [80] and Baues [11, 12, 13].

1.1 Notation and conventions

If $\mathcal{C}$ is a (locally small) category, we denote its class of objects by $\mathrm{Ob}(\mathcal{C})$, and the set of morphisms between two objects $A, B \in \mathrm{Ob}(\mathcal{C})$ is denoted by $\mathrm{Hom}_{\mathcal{C}}(A, B)$. Similarly, the group of automorphisms of an object $A \in \mathrm{Ob}(\mathcal{C})$ is denoted by $\mathrm{Aut}_{\mathcal{C}}(A)$. If $\mathcal{C}$ admits a model structure, we denote the automorphisms of an object $A \in \mathrm{Ob}\left(\mathcal{H}o(\mathcal{C})\right)$ by $\mathcal{E}(A)$, the *group of self-homotopy equivalences* of A.

A category $\mathcal{C}$ is *universal* if for any group G, there is an object $X \in \mathrm{Ob}(\mathcal{C})$ such that $\mathrm{Aut}_{\mathcal{C}}(X) \cong G$, and it is *finitely universal* if every finite group G is realised as $\mathrm{Aut}_{\mathcal{C}}(X)$ for some $X \in \mathrm{Ob}(\mathcal{C})$.

A *space* will mean a pointed topological space with the homotopy type of a CW-complex, and a *continuous map* between spaces will always preserve the basepoint. We denote the category of such spaces and continuous maps by Top. Following this criteria, given X and Y spaces, $[X, Y]$ denotes the set of homotopy classes of pointed continuous maps from X to Y. We also make use of the following full subcategories of Top: Top^c, Top^{pc} and Top^n, $n \geq 0$, respectively denote the categories of connected, path connected and n-connected spaces, and Top_f denotes the category of spaces of finite type. We will combine the notations above; for instance, Top_f^n denotes the category of n-connected spaces of finite type.

Finally, given a set A, $\mathrm{Sym}(A)$ denotes the group of permutations of the elements of A, and if A has an element denoted 0, we denote $A^* = A \setminus \{0\}$.

1.2 The classical group realisability problem in Graph Theory

Some of the first solutions to the classical group realisability problem appeared in the context of Graph Theory, and these solutions were consequently transferred to other frameworks. The aim of this section is to set the basic notation and terminology for graphs that we will use. We also introduce some pre-existing results regarding the realisability problem for graphs, along with some key ingredients of their proofs that we need in our constructions.

Our main references for Graph Theory are [79] and [49]. We consider [79] as a more classical reference, whereas [49] is more modern and uses categorical language in its notations. Consequently, we will mostly use the notation from [49].

We start by introducing the categories of combinatorial objects that we use: $\mathcal{G}raphs$, $\mathcal{D}igraphs$ and $I\mathcal{R}el$.

Definition 1.1 (categories of combinatorial objects)**.**

- A *graph* is a pair $\mathcal{G} = \big(V(\mathcal{G}), E(\mathcal{G})\big)$ where $V(\mathcal{G})$ is a non-empty set, the set of *vertices*, and $E(\mathcal{G})$ is a set consisting of sets of two vertices, called *edges*.

  A *morphism* of graphs $f\colon \mathcal{G}_1 \to \mathcal{G}_2$ is a map $f\colon V(\mathcal{G}_1) \to V(\mathcal{G}_2)$ such that if $\{v, w\} \in E(\mathcal{G}_1)$, then $\{f(v), f(w)\} \in E(\mathcal{G}_2)$. Graphs, together with this concept of morphism, form a category which we denote by $\mathcal{G}raphs$.

- A *digraph* is a pair $\mathcal{G} = \big(V(\mathcal{G}), E(\mathcal{G})\big)$ where $V(\mathcal{G})$ is a non-empty set, the set of *vertices*, and $E(\mathcal{G})$ is a binary relation on $V(\mathcal{G})$. If $(v, w) \in E(\mathcal{G})$ we also say that (v, w) is an *edge* of $\mathcal{G}$. More precisely, we say that (v, w) is an edge from v to w.

  A digraph is symmetric (respectively reflexive, irreflexive...) if $E(\mathcal{G})$ is a symmetric binary relation (respectively reflexive, irreflexive...).

  A *morphism* of digraphs $f\colon \mathcal{G}_1 \to \mathcal{G}_2$ is a map $f\colon V(\mathcal{G}_1) \to V(\mathcal{G}_2)$ verifying that $\big(f(v), f(w)\big) \in E(\mathcal{G}_2)$, whenever $(v, w) \in E(\mathcal{G}_1)$. $\mathcal{D}igraphs$ denotes the category whose objects are digraphs and whose arrows are morphisms of digraphs.

- A *binary relational system over a set of indices I*, also called *binary I-system*, is a pair $\mathcal{G} = \big(V(\mathcal{G}), \{R_i(\mathcal{G})\}_{i\in I}\big)$ where $V(\mathcal{G})$ is a non-empty set, the set of *vertices*, and where $R_i(\mathcal{G})$ is a binary relation on $V(\mathcal{G})$, for every $i \in I$. Elements (v, w) in $R_i(\mathcal{G})$ receive the name of *edges of colour or label i*.

  A binary I-system is symmetric (respectively reflexive, irreflexive...) if $R_i(\mathcal{G})$ is a symmetric binary relation (respectively reflexive, irreflexive...) for all $i \in I$.

  A *morphism* of binary I-systems $f\colon \mathcal{G}_1 \to \mathcal{G}_2$ is a map $f\colon V(\mathcal{G}_1) \to V(\mathcal{G}_2)$ such that for every $i \in I$ and $(v, w) \in R_i(\mathcal{G}_2)$, $\big(f(v), f(w)\big) \in R_i(\mathcal{G}_2)$. Binary I-systems, together with this concept of morphism, form a category which we denote $I\mathcal{R}el$.

- We say that a graph, digraph or binary I-system $\mathcal{G}$ is *finite* if $V(\mathcal{G})$ is finite. We denote the full subcategories of finite graphs, digraphs and binary relational systems over I by $\mathcal{G}raphs_f$, $\mathcal{D}igraphs_f$ and $I\mathcal{R}el_f$ respectively.

A binary relational system is classically called a labelled pseudograph, [79, Def. 2-4].

Remark 1.2. There are some clear relations between the three categories introduced in Definition 1.1. For instance, when $|I| = 1$, the category $I\mathcal{R}el$ is the category $\mathcal{D}igraphs$. On the other hand, there is a graph underlying any irreflexive binary relational system. Namely, given $\mathcal{G}$ an object in $I\mathcal{R}el$, we can consider a graph $\mathcal{G}'$ such that $V(\mathcal{G}) = V(\mathcal{G}')$ and $\{v, w\} \in E(\mathcal{G})$ if and only if there exists $i \in I$ such that $(v, w) \in R_i(\mathcal{G})$ or $(w, v) \in R_i(\mathcal{G})$. Then $\mathcal{G}'$ is the graph obtained from $\mathcal{G}$ by forgetting labels and directions of edges. On the other hand, a graph can be regarded as a symmetric, irreflexive digraph.

In many cases, our results will require for graphs, digraphs or binary I-relational systems to not have isolated vertices.

Definition 1.3 (isolated vertices)**.**

- Let $\mathcal{G}$ be a graph and let $v \in V(\mathcal{G})$ be a vertex in the graph. We say that v is *isolated* if for every other vertex $w \in V(\mathcal{G})$, $\{v, w\} \notin E(\mathcal{G})$.
- Let I be a set and let $\mathcal{G}$ be a binary I-system. Let $v \in V(\mathcal{G})$ be a vertex. We say that v is *isolated* if for every other vertex $w \in V(\mathcal{G})$ and for every $i \in I$, $(v, w) \notin R_i(\mathcal{G})$ and $(w, v) \notin R_i(\mathcal{G})$.

Clearly, if a vertex is isolated in a digraph or binary I-relational system, it is also isolated in the underlying graph. Similarly, if a vertex is isolated in a graph it is also isolated when regarding it as a symmetric digraph. The notions of connectivity and strong connectivity are also required. In order to introduce them, we first need to define the concept of path.

Definition 1.4 (paths in graphs and digraphs)**.**

- Let $\mathcal{G}$ be a graph. A *path* between two vertices $v \in V(\mathcal{G})$ and $w \in V(\mathcal{G})$ is a finite set of vertices $p = \{v = v_0, v_1, \dots, v_n = w\}$ such that $\{v_{i-1}, v_i\} \in E(\mathcal{G})$, $i = 1, 2, \dots, n$.
- Let $\mathcal{G}$ be a digraph. An *undirected path* between two vertices $v \in V(\mathcal{G})$ and $w \in V(\mathcal{G})$ is a finite set of vertices $p = \{v = v_0, v_1, \dots, v_n = w\}$ such that $(v_{i-1}, v_i) \in E(\mathcal{G})$ or $(v_i, v_{i-1}) \in E(\mathcal{G})$, for each $i = 1, 2, \dots, n$. A *path* between the same two vertices is a finite set of vertices $p = \{v = v_0, v_1, \dots, v_n = w\}$ such that $(v_{i-1}, v_i) \in E(\mathcal{G})$, for each $i = 1, 2, \dots, n$.

We say that a (undirected) path $p = \{v_0, v_1, \dots, v_n\}$ in a (di)graph $\mathcal{G}$ has length n, and we denote $\operatorname{len}(p) = n$. We may regard vertices and edges as paths of length 0 and 1 respectively.

Definition 1.5 (connectivity of graphs and digraphs)**.**

- A graph $\mathcal{G}$ is *connected* if there is a path between any two vertices $v, w \in V(\mathcal{G})$.
- A digraph $\mathcal{G}$ is *weakly connected* or just *connected* if for any two vertices $v, w \in V(\mathcal{G})$ there is an undirected path p between them. On the other hand, we say that $\mathcal{G}$ is *strongly connected* if for any two vertices $v, w \in V(\mathcal{G})$ there is a path p between them.

It is immediate that a strongly connected digraph is (weakly) connected. Furthermore, a graph is connected if and only if it is strongly connected when regarded as a symmetric digraph. We introduce one last definition before discussing automorphisms.

Definition 1.6 (full subgraphs and relational subsystems)**.**

- Let $\mathcal{G}$ be a graph. The *full subgraph* of $\mathcal{G}$ with vertices $V' \subset V(\mathcal{G})$ is a graph $\mathcal{G}'$ whose set of vertices is $V(\mathcal{G}') = V'$ and where two vertices define an edge in $\mathcal{G}'$ if they do so in $\mathcal{G}$. Therefore, if $v, w \in V'$ and $\{v, w\} \in E(\mathcal{G})$, then $\{v, w\} \in E(\mathcal{G}')$. A *full monomorphism* of $\mathcal{G}raphs$ is an inclusion $i \colon \mathcal{G}' \hookrightarrow \mathcal{G}$ of a full subgraph $\mathcal{G}'$ of $\mathcal{G}$ in $\mathcal{G}$.
- Let $\mathcal{G}$ be a binary I-system. The *full binary I-subsystem* of $\mathcal{G}$ with vertices $V' \subset V(\mathcal{G})$ is a binary I-system $\mathcal{G}'$ whose set of vertices is $V(\mathcal{G}') = V'$ and where two vertices define an edge of label i in $\mathcal{G}'$ if they do so in $\mathcal{G}$. Namely, if $v, w \in V'$ and $(v, w) \in R_i(\mathcal{G})$ for some $i \in I$, then $(v, w) \in R_i(\mathcal{G}')$. A *full monomorphism* of $I\mathcal{R}el$ is an inclusion $i \colon \mathcal{G}' \hookrightarrow \mathcal{G}$ of a full binary I-subsystem $\mathcal{G}'$ of $\mathcal{G}$ in $\mathcal{G}$.

Since we are interested in the group realisability problem in the categories we introduced, we need to understand their automorphisms.

Definition 1.7 (automorphisms of graphs and binary I-systems)**.**

- Let $\mathcal{G}$ be a graph. An automorphism of $\mathcal{G}$, $f\colon \mathcal{G} \to \mathcal{G}$, is a bijective map $f\colon V(\mathcal{G}) \to V(\mathcal{G})$ such that $\{v, w\} \in E(\mathcal{G})$ if and only if $\{f(v), f(w)\} \in E(\mathcal{G})$.
- Let $\mathcal{G}$ be a binary I-system. An automorphism of $\mathcal{G}$, $f\colon \mathcal{G} \to \mathcal{G}$, is a bijective map $f\colon V(\mathcal{G}) \to V(\mathcal{G})$ such that for every $i \in I$, $(v, w) \in R_i(\mathcal{G})$ if and only if $(f(v), f(w)) \in R_i(\mathcal{G})$.

One fact that will prove key in many of our graph-theoretical constructions is that automorphisms of graphs and binary I-systems respect vertex degrees. Let us introduce that concept now.

Definition 1.8 (vertex degree)**.**

- Let $\mathcal{G}$ be a graph. The *degree* of a vertex $v \in V(\mathcal{G})$, denoted $\deg(v)$, is the cardinal of the set of vertices that are connected with v through an edge, that is, $\deg(v) = |\{w \in V(\mathcal{G}) \mid \{v, w\} \in E(\mathcal{G})\}|$.
- Let $\mathcal{G}$ be a relational system. The *indegree* of a vertex $v \in V(\mathcal{G})$, denoted $\deg^-(v)$, is the cardinal of the set of edges that arrive at v, that is, $\deg^-(v) = |\sqcup_{i\in I} \{w \in V(\mathcal{G}) \mid (w, v) \in R_i(\mathcal{G})\}|$. The *outdegree* of v, $\deg^+(v)$, is the cardinal of the set of edges that start at v, thus $\deg^+(v) = |\sqcup_{i\in I} \{w \in V(\mathcal{G}) \mid (v, w) \in R_i(\mathcal{G})\}|$. The *degree* of v is then defined as $\deg(v) = \deg^+(v) + \deg^-(v)$.

Notice that in a binary I-system $\mathcal{G}$, an edge $(v, v) \in R_i(\mathcal{G})$ contributes both to the indegree and the outdegree of v.

We now introduce the solution to the group realisability problem in $I\mathcal{R}el$ and review how it was transferred to simple graphs by Frucht, [43], and de Groot, [32]. The starting point is Cayley diagrams:

Definition 1.9. Let G be a group and $S = \{s_i \mid i \in I\}$ be a generating set for G. The *Cayley diagram of G associated to S* is a binary I-system $\mathrm{Cay}(G, S)$ with vertices $V(\mathrm{Cay}(G, S)) = G$ and edges $(g, s_i g) \in R_i(\mathcal{G})$, for $g \in G$ and $i \in I$.

Remark 1.10. Cayley diagrams are clearly dependent on the considered set of generators for the group. However, it is immediate to see that every element $\tilde{g} \in G$ gives raise to an automorphism $\phi_{\tilde{g}}\colon \mathrm{Cay}(G, S) \to \mathrm{Cay}(G, S)$, which is a transitive action on vertices induced by right multiplication by $\tilde{g}^{-1}$, namely $\phi_{\tilde{g}}(g) = g\tilde{g}^{-1}$. In particular, if ϕ_g fixes any vertex then it must be the identity map.

We derive the following, classical result:

Theorem 1.11 ([32, Section 6], [30, Section 3.3])**.** *Let G be a group and let $S = \{s_i \mid i \in I\}$ be a generating set of G. Then,* $\mathrm{Aut}_{I\mathcal{R}el}(\mathrm{Cay}(G, S)) \cong G$.

Therefore, a solution to the group realisability problem can be achieved in graphs if we obtain a procedure to build a graph from a given binary relational system while keeping the same group of automorphisms. This is precisely what Frucht and de Groot did in [43] and [32] respectively.

The key idea for their solution is to associate to each label in a binary I-system an asymmetric graph (that is, a graph whose only automorphism is the identity) so that these graphs are pairwise non-isomorphic. Then, each directed edge with a certain label is replaced by the corresponding asymmetric graph. Should the asymmetric graphs be chosen carefully, we can make sure that the automorphisms of the graph obtained after the replacement operation map each of the asymmetric graphs to copies of themselves. Thus, the asymmetric graphs play the role of the labelled directed edges.

The asymmetric graphs are chosen by ensuring that degrees of their vertices are different to the degrees of vertices in the binary I-system. For that reason, we need to compute the degrees of the vertices in a Cayley diagram.

Remark 1.12. For each label $i \in I$ and vertex $g \in G$, there is exactly one edge arriving at g and another starting at g, $(s_i^{-1}g, g) \in R_i(\operatorname{Cay}(G,S))$ and $(g, s_i g) \in R_i(\operatorname{Cay}(G,S))$ respectively. Thus, $\deg^-(g) = \deg^+(g) = |I|$, for every $g \in G$. Consequently, every vertex in $\operatorname{Cay}(G,S)$ has degree $2|I|$. Moreover, it also becomes clear that if G is a non-trivial group, $\operatorname{Cay}(G,S)$ does not have isolated vertices, for any generating set S.

We now proceed to reviewing the classical results on the group realisability problem in the category of simple graphs. The construction of Frucht, [43], uses the following result:

Proposition 1.13 ([43])**.** *For any positive integer n, there exist n pairwise non-isomorphic finite, connected, asymmetric graphs. Moreover, the highest of the degrees of their vertices is three.*

Proof. This can be deduced from [43, Section 1]. Indeed, one can choose any starlike tree T whose root $v \in V(T)$ has degree 3 and such that the length of the three paths of $T - v$ differ, see [43, Fig. 1]. □

Notice that, when performing a replacement operation on a binary I-system, if neither the binary I-system nor the asymmetric graphs have isolated vertices, the resulting graph does not have isolated vertices either. Moreover, if the starting binary I-system and the asymmetric graphs are connected, the graph obtained after the replacement operation is also connected. Since Cayley diagrams associated to finite groups are always connected, the next result follows from Proposition 1.13:

Theorem 1.14 ([43])**.** *Let G be a finite group. There exists a finite, connected graph $\mathcal{G}$ such that* $\operatorname{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \cong G$.

This proves that $\mathcal{G}raphs$ (in fact $\mathcal{G}raphs_f$) is finitely universal. However, we need a solution that includes infinite groups as well. For that reason, we also work with the solution given by de Groot, [32]. The base of his construction is the following result:

Proposition 1.15 ([32, Section 6])**.** *Let $\mathfrak{m}$ and $\mathfrak{n}$ be two cardinals. There exist $\mathfrak{n}$ pairwise non-isomorphic asymmetric graphs in which, aside from a vertex of degree one, every vertex has degree greater than $\mathfrak{m}$.*

It is worth mentioning that, due to how they are constructed, the graphs obtained in Proposition 1.15 do not have isolated vertices. Using this result, the author is able to prove that $\mathcal{G}raphs$ is universal:

Theorem 1.16 ([32, Theorem 6])**.** *Every group G is isomorphic to the automorphism group of some graph $\mathcal{G}$. Moreover, the graph $\mathcal{G}$ can be built so that it does not have isolated vertices.*

However, these graphs have an important disadvantage that forces us to use the less general result of Frucht, Theorem 1.14, when working with finite groups:

Remark 1.17. The graphs in Proposition 1.15 (and hence the graphs in Theorem 1.16) are always infinite.

We finish this section by introducing some results in [49, 68] regarding the representability of categories in the category of $\mathcal{G}raphs$.

Definition 1.18. We say that a category $\mathcal{C}$ is representable in another category $\mathcal{D}$ if there exists a fully faithful functor $F\colon \mathcal{C} \to \mathcal{D}$.

In particular, this means that $\mathcal{C}$ can be regarded as a full subcategory of $\mathcal{D}$ with objects $F(\operatorname{Ob}(\mathcal{C}))$. It turns out that many categories are representable in a certain subcategory of $\mathcal{D}igraphs$ which we introduce now.

Definition 1.19. We denote by $\mathcal{D}igraphs_0$ the full subcategory of $\mathcal{D}igraphs$ whose objects are irreflexive strongly connected digraphs with more than one vertex.

Recall that a category $\mathcal{C}$ is called *concrete* when it admits a faithful functor to Set the category of sets.

Theorem 1.20 ([68, Chapter 4, 1.11])**.** *If $\mathcal{C}$ is a concrete small category, it is representable in $\mathcal{D}igraphs_0$.*

We will use this result to obtain some consequences on the representability of categories in CDGA. We would also like to obtain results in the same direction in $\mathcal{H}oTop$. However, this will require for the starting graphs to be finite. Thus, we also make use of the following, less general result:

Theorem 1.21 ([49, Theorem 4.24, Proposition 4.25])**.** *Let $\mathcal{C}$ be a concrete category such that $\operatorname{Ob}(\mathcal{C})$ is countable and, for any pair of objects $A, B \in \operatorname{Ob}(\mathcal{C})$, $\operatorname{Hom}_{\mathcal{C}}(A, B)$ is finite. Then, there is a fully faithful functor $F\colon \mathcal{C} \to \mathcal{G}raphs_f$ such that $F(A)$ does not have isolated vertices, for any $A \in \operatorname{Ob}(\mathcal{C})$.*

1.3 Coalgebras

Coalgebras are one of the two kinds of algebraic structures for which we study realisability problems in this thesis. This section is mostly based on [22], since our constructions are built upon path coalgebras. Some standard references on the subject would be [63, 76].

Coalgebras are dual of algebras in an Eckmann-Hilton sense. Namely, an algebra over a field $\Bbbk$ can be pictured as a $\Bbbk$-vector space A together with a multiplication $\mu\colon A \otimes A \to A$ and a unit $\varepsilon\colon \Bbbk \to A$, verifying associativity and unit axioms. These axioms can be interpreted through the commutativity of certain diagrams. Coalgebras are then defined by inverting the arrows on those diagrams. Let us formalise that concept.

Definition 1.22. A *coalgebra* over a field $\Bbbk$, or $\Bbbk$-coalgebra, is a $\Bbbk$-vector space C together with two linear maps, the comultiplication $\Delta\colon C \to C \otimes C$ and the counit $\varepsilon\colon C \to \Bbbk$, verifying the following two properties:

- Coassociativity: $(\Delta \otimes \mathrm{id}_C) \circ \Delta = (\mathrm{id}_C \otimes \Delta) \circ \Delta$.
- Counitary property: $(\mathrm{id}_C \otimes \varepsilon) \circ \Delta = \mathrm{id}_C = (\varepsilon \otimes \mathrm{id}_C) \circ \Delta$.

Such a coalgebra is usually denoted (C, Δ, ε), but in most instances we will just denote it by C. Furthermore, we usually do not notationally distinguish the comultiplication and counit associated to different coalgebras.

A *morphism* of coalgebras $f\colon C_1 \to C_2$ is a linear map verifying that:

- $(f \otimes f) \circ \Delta = \Delta \circ f$.

- $\varepsilon \circ f = \varepsilon$.

Coalgebras over a field $\Bbbk$ together with this notion of morphism form a category which we denote $\mathrm{Coalg}_{\Bbbk}$.

Remark 1.23. Every coalgebra can be dualised to obtain an algebra. However, the dual of an infinite-dimensional algebra may not be a coalgebra. Indeed, if $\mu\colon A \otimes A \to A$ is the multiplication, there may exist elements $a \in A$ such that $\mu^{-1}(A)$ is infinite. Consequently, when dualising μ to a comultiplication, the comultiplication of the dual of a would contain an infinite sum, which is not possible in a coalgebra.

As a consequence, developing a theory of coalgebras is not recursive to the theory of algebras. In fact, the next result, usually referred to as the Fundamental Theorem of Coalgebras, shows that the structure of infinite-dimensional coalgebras is very different to that of algebras.

Theorem 1.24 ([76, Theorem 2.2.1], [63, Section 5.1])**.** *Every coalgebra is the sum of its finite-dimensional subcoalgebras. Equivalently, if C is a coalgebra, every element $c \in C$ is contained in a finite-dimensional subcoalgebra of C.*

We now introduce some distinguished subcoalgebras and elements of a given coalgebra, which will play a key role in our results.

Definition 1.25. A coalgebra C is *simple* if it has no proper non-trivial subcoalgebras.

Notice that, as a consequence of Theorem 1.24, simple coalgebras are always finite-dimensional. And among all of them, it can be expected that simple subcoalgebras of dimension one may play an important role in the structure of a coalgebra. Thus, it is natural to seek conditions in order for an element of a coalgebra to generate a one-dimensional subcoalgebra. The following definition arises naturally:

Definition 1.26. Let C be a coalgebra. An element $c \in C$ is said to be *grouplike* if $\Delta(c) = c \otimes c$. The set of grouplike elements of C is then defined as

$$G(C) = \{c \in C \mid \Delta(c) = c \otimes c\}.$$

Remark 1.27. In some texts, an element $c \in C$ must also verify that $\varepsilon(c) = 1$ in order for it to be considered grouplike. However, this condition is redundant. Indeed, if $\Delta(c) = c \otimes c$, the counitary property implies that $c = \varepsilon(c)c$, thus $\varepsilon(c) = 1$.

Remark 1.28. Coalgebra morphisms must take grouplike elements to grouplike elements. Indeed, if f is a coalgebra morphism and $x \in G(C_1)$, given that $(f \otimes f) \circ \Delta = \Delta \circ f$, it is immediate that $f(x) \in G(C_2)$.

We then have the following result relating one-dimensional subcoalgebras and grouplike elements:

Proposition 1.29. *Let C be a $\Bbbk$-coalgebra. There is a bijective correspondence between one-dimensional simple subcoalgebras of C and grouplike elements of C.*

Proof. Clearly, if $c \in G(C)$, $\Bbbk\{c\}$ is a one-dimensional subcoalgebra of C. Reciprocally, suppose that $\Bbbk\{c\}$ is a simple subcoalgebra of C of dimension one. Then, $\Delta(c) \in \Bbbk\{c\} \otimes \Bbbk\{c\}$, that is, $\Delta(c) = \lambda(c \otimes c)$. Furthermore, $\lambda \neq 0$, for otherwise the counitary property could not be fulfilled for c. But then, $\Delta(\lambda c) = (\lambda c) \otimes (\lambda c)$, thus $\lambda c \in G(C)$ and $\Bbbk\{c\} = \Bbbk\{\lambda c\}$. $\square$

We now introduce the coradical filtration of a coalgebra.

Definition 1.30. Let C be a coalgebra. The *coradical* of C, $\operatorname{corad}(C)$, is the sum of the simple subcoalgebras of C. Now take $C_0 = \operatorname{corad}(C)$. Define C_n inductively as

$$C_n = \Delta^{-1}(C_{n-1} \otimes C + C \otimes C_0).$$

It is easy to see that this yields an increasing sequence of vector subspaces of C,

$$\operatorname{corad}(C) = C_0 \subset C_1 \subset \cdots \subset C_n \subset \cdots \subset C,$$

which is referred to as the *coradical filtration of* C.

Another concept which is of interest to us is that of a pointed coalgebra.

Definition 1.31. A coalgebra C is *pointed* if every simple subcoalgebra of C is of dimension one.

Then, as an immediate consequence of Proposition 1.29, a coalgebra is pointed if and only if its simple subcoalgebras are all generated by a grouplike element, or equivalently, if C_0 is the vector space generated by $G(C)$. In order to study C_1, we need to introduce the skew-primitive elements of a coalgebra.

Definition 1.32. Let C be a coalgebra and take $g, h \in G(C)$. Define

$$P_{g,h}(C) = \{c \in C \mid \Delta(c) = g \otimes c + c \otimes h\}.$$

The elements of $P_{g,h}(C)$ are called (g, h)*-skew primitives* of C.

Now for $g, h \in G(C)$ it is clear that $g - h \in P_{g,h}(C)$, which are the *trivial* elements of $P_{g,h}(C)$. Define $P'_{g,h}(C)$ as a complement of $\Bbbk\{g - h\}$ in $P_{g,h}(C)$, namely, $P_{g,h}(C) = \Bbbk\{g - h\} \oplus P'_{g,h}(C)$. We then have the following result, usually referred to as the Taft-Wilson theorem:

Theorem 1.33 ([63, Theorem 5.4.1])**.** *Let C be a pointed coalgebra. Then,*

$$C_1 = \Bbbk G(C) \oplus \left(\bigoplus_{g,h \in G(C)} P'_{g,h}(C) \right).$$

Finally, let us introduce the path coalgebra associated to a digraph and use the results above to obtain some of its properties. Coalgebras associated to combinatorial objects are usually defined over quivers. However, as our constructions in this thesis are mostly graph-theoretical, we will define them over digraphs. Recall the concept of path in a digraph that was introduced in Definition 1.4.

Definition 1.34. Let $\mathcal{G}$ be a digraph. The path coalgebra of $\mathcal{G}$, $\Bbbk\mathcal{G}$, is the linear span over $\Bbbk$ of all paths of $\mathcal{G}$ with the following coalgebra structure:

$$\Delta(p) = \sum_{p = p_1 p_2} p_1 \otimes p_2; \quad \textit{and} \quad \varepsilon(p) = \begin{cases} 0, & \text{if } \operatorname{len}(p) > 0, \\ 1, & \text{if } \operatorname{len}(p) = 0. \end{cases}$$

Here, $p_1 p_2$ denotes the concatenation of the paths p_1 and p_2, that is, if $p_1 = \{v_0, v_1, \ldots, v_n\}$ and $p_2 = \{v_n, v_{n+1}, \ldots, v_m\}$, then $p_1 p_2 = \{v_0, v_1, \ldots, v_m\}$.

Remark 1.35. Notice that vertices, which as generators in this coalgebra are paths of length 0, are the grouplike elements of $\Bbbk\mathcal{G}$. In fact, $G(\Bbbk\mathcal{G}) = V(\mathcal{G})$. On the other hand, if $e = (v, w) \in E(\mathcal{G})$, then e is (v, w)-skew primitive.

Furthermore, if C is a subcoalgebra of $\Bbbk\mathcal{G}$ and $p \in C$ is a path, for any vertex v in the path p, $v \in C$. Indeed, we may write p as a concatenation of two paths $p = q_1q_2$ such that v is respectively the last and first vertex of q_1 and q_2. Then $q_1 \otimes q_2$ is a summand in $\Delta(p)$, and v appears in a summand in both $\Delta(q_1)$ and $\Delta(q_2)$.

Consequently, $\Bbbk\{v\}$ is a subcoalgebra of C. Namely, simple subcoalgebras of $\Bbbk\mathcal{G}$ are one-dimensional, thus $\Bbbk\mathcal{G}$ is pointed. Then, since $G(\Bbbk\mathcal{G}) = V(\mathcal{G})$, we deduce that $\Bbbk\mathcal{G}_0 = \Bbbk\{V(\mathcal{G})\}$. For the n-th stage of the coradical filtration we immediately see that

$$(\Bbbk\mathcal{G})_n = \Bbbk\{p \in \Bbbk\mathcal{G} \mid |p| \leq n\}.$$

1.4 CDGAs and Rational Homotopy Theory

The second kind of algebraic structures for which we aim to solve realisability problems are commutative differential graded algebras, or CDGAs for short. Moreover, since they are used as algebraic models of homotopy types of spaces in Rational Homotopy Theory, they also provide us with a bridge to transfer our results to topology.

Roughly speaking, Rational Homotopy Theory is the study of homotopy types of spaces modulo torsion. It has its roots in a seminal paper of Quillen, [70], who obtained one of the two classical approaches to the subject. However, we use the other classical approach, which was introduced by Sullivan in [75] and later completed by Bousfield and Guggenheim, [18]. A nice introductory reference is [51], whereas [40, 42, 77] and the one we will mostly follow, [39], are great in-depth references.

We start by introducing rational spaces and rational homotopy types of spaces. In this thesis we will only work with simply connected rational spaces, so most definitions and results in this section will be stated in that framework.

Definition 1.36. A simply connected space X is *rational* if the following equivalent conditions are satisfied.

(1) $\pi_*(X)$ is a $\mathbb{Q}$-vector space.

(2) $\widetilde{H}_*(X; \mathbb{Z})$ is a $\mathbb{Q}$-vector space.

A continuous map $\psi : X \to Y$ between simply connected spaces is a *rational homotopy equivalence* if the following equivalent conditions are satisfied.

(1) $\pi_*(\psi) \otimes \mathbb{Q}$ is an isomorphism.

(2) $\widetilde{H}_*(\psi; \mathbb{Q})$ is an isomorphism.

(3) $\widetilde{H}^*(\psi; \mathbb{Q})$ is an isomorphism.

Then, given a simply connected space X, there exists a rational space X_0 such that there is a rational homotopy equivalence $\psi\colon X \to X_0$. We say that X_0 is a rationalisation of X, and the rational homotopy type of X is the homotopy type of X_0. Also, for simply connected spaces X and Y, we denote $[X, Y]_0 = [X_0, Y_0]$ the set of rational homotopy classes of maps from X to Y. Now, we can formally say that *Rational Homotopy Theory is the study of rational homotopy types of spaces and of the properties of spaces and maps that remain invariant under rational homotopy equivalences.*

We now proceed to summarising the classification of rational homotopy types of simply connected spaces provided by Sullivan. We begin by introducing the algebraic structures that participate in such classification. Let R be a commutative ring.

Definition 1.37 (commutative differential graded algebras)**.**

- A *graded R-algebra* is a graded R-module $A = \{A^n\}_{n\geq 0}$ together with a multiplication, that is, an associative linear map of degree 0, $A \otimes_R A \to A$, that has an identity element $1 \in A^0$. The multiplication of two elements $a, b \in A$ is denoted ab. If R is a field, we say that a graded R-algebra A is of *finite type* if $\dim A^n < \infty$, for all $n \geq 0$.

- A *differential graded R-algebra* is a pair (A, d) where A is a graded R-algebra and d is a linear map of degree 1, $d\colon A \to A$, that is also a derivation, that is, $d(ab) = d(a)b + (-1)^{|a|}d(b)$. A *morphism* of differential graded algebras $f\colon (A, d) \to (B, d)$ is a linear map of degree zero $f\colon A \to B$ such that $df = fd$ and $f(ab) = f(a)f(b)$.

- A *commutative differential graded R-algebra*, R-CDGA or just CDGA for short, is a differential graded R-algebra (A, d) whose multiplication is commutative in a graded sense. Namely, if $a, b \in A$ are homogeneous elements, $ab = (-1)^{|a||b|}ba$. We denote the category whose objects are R-CDGAs and whose morphisms are morphisms of differential graded algebras by CDGA_R.

The following subclasses of CDGAs will be important in the sequel:

Definition 1.38 (subcategories of CDGA_R)**.**

- An R-CDGA (A, d) is *connected* if $A^0 = R$. The full subcategory of CDGA_R whose objects are connected R-CDGAs is denoted by CDGA^c_R.

- An R-CDGA (A, d) is *n-connected*, $n \geq 1$, if it is connected and $A^k = 0$, for $1 \leq k \leq n$. In particular, we say that (A, d) is *simply connected* if it is 1-connected. We denote the full subcategory of CDGA_R whose objects are n-connected R-CDGAs by CDGA^n_R.

- An R-CDGA (A, d) is *homologically connected* if its cohomology algebra $H^*(A, d)$ is connected, and *homologically n-connected*, $n \geq 1$, if its cohomology algebra $H^*(A, d)$ is n-connected. We denote the full subcategory of CDGA_R whose objects are homologically connected and n-connected R-CDGAs by CDGA^{hc}_R and CDGA^{hn}_R respectively.

- If R is a field, the full subcategory of CDGA_R whose objects are R-CDGAs of finite type is denoted by $\mathrm{CDGA}_{R,f}$.

The notations introduced above will be used simultaneously. Thus, if R is a field, $\mathrm{CDGA}^c_{R,f}$, $\mathrm{CDGA}^n_{R,f}$, $\mathrm{CDGA}^{hc}_{R,f}$ and $\mathrm{CDGA}^{hn}_{R,f}$ respectively denote the categories of connected, n-connected, homologically connected and homologically n-connected R-CDGAs of finite type. Finally, when $R = \mathbb{Q}$ we omit the ring from the notation, thus CDGA denotes $\mathrm{CDGA}_{\mathbb{Q}}$, CDGA^c denotes $\mathrm{CDGA}^c_{\mathbb{Q}}$, and so on.

Let us now introduce the equivalence relation that will determine when two CDGAs are associated to the same rational homotopy type. If (A, d) is a differential graded algebra, we can consider its cohomology $H^*(A, d) = \ker d / \operatorname{Im} d$, which is a graded algebra. Furthermore, a morphism of differential graded algebras $f\colon (A, d) \to (B, d)$ induces a morphism of algebras $H^*(f)\colon H^*(A, d) \to H^*(B, d)$ in an obvious way.

Definition 1.39. A morphism of differential graded algebras $f\colon (A, d) \to (B, d)$ is a *quasi-isomorphism* if the morphism it induces on cohomology $H^*(f)\colon H^*(A, d) \to H^*(B, d)$ is an isomorphism. We denote $f\colon (A, d) \xrightarrow{\simeq} (B, d)$. Two differential graded algebras (A, d) and (B, d) are *weakly equivalent* if there is a zig-zag of quasi-isomorphisms

$$(A, d) \xleftarrow{\simeq} \cdots \xrightarrow{\simeq} (B, d).$$

In particular, if (A, d) and (B, d) are weakly equivalent, they have isomorphic cohomology algebras.

The classification of rational homotopy types introduced by Sullivan is provided by a pair of adjoint contravariant functors

$$A_{PL}\colon \mathrm{Top}^c \rightleftarrows \mathrm{CDGA}^{hc}\colon |\ \ |$$

such that two simply connected spaces X and Y are rationally homotopic if and only if the CDGAs $A_{PL}(X)$ and $A_{PL}(Y)$ are weakly equivalent. We will not review how these functors are built, since knowing some of their properties will be enough for our purposes. The construction of the functors A_{PL} and $|\ \ |$ can be found in [39, Chapter 10] and [39, Chapter 17] respectively.

Let us begin by considering the functor $A_{PL}\colon \mathrm{Top}^c \to \mathrm{CDGA}^{hc}$. There are many ways in which the algebraic invariants of a space X are encoded in $A_{PL}(X)$. For example, $H^*(X;\mathbb{Q}) \cong H^*(A_{PL}(X))$. However, $A_{PL}(X)$ is usually a complicated object, thus instead of working directly with it we seek to find a *simpler* CDGA weakly equivalent to $A_{PL}(X)$.

Definition 1.40. Let V be a graded R-module.

- The *tensor algebra* on V, TV, is the graded R-algebra

$$TV = \oplus_{q=0}^{\infty} T^q V, \quad T^q V = V \otimes_R \overset{q}{\cdots} \otimes_R V,$$

  where by convention, we assume that $T^0 V = R$.

- Assume furthermore that $\mathrm{char}(R) \neq 2$. The *free commutative graded R-algebra* on V, ΛV, is the quotient of TV by the bilateral ideal generated by the elements $a \otimes b - (-1)^{|a||b|} b \otimes a$, where a and b are homogeneous elements of TV.

As an algebra, $\Lambda V \cong SV^{\mathrm{even}} \otimes_R EV^{\mathrm{odd}}$, where SV^{even} is the symmetric algebra on V^{even} and EV^{odd} is the exterior algebra on V^{odd}. If $\{v_i \mid i \in I\}$ is a basis of V, we denote $\Lambda V = \Lambda(\{v_i \mid i \in I\})$. We also denote the vector subspace of ΛV generated by the products $v_1 v_2 \dots v_n$, $v_i \in V$, by $\Lambda^n V$. Elements in this space are said to be elements of wordlength n. Similarly, we denote $\Lambda^{\geq n} V = \oplus_{r \geq n} \Lambda^r V$.

Remark 1.41. Let V be a graded R-module, $\mathrm{char}(R) \neq 2$, generated by $\{v_i \mid i \in I\}$. As a consequence of the properties of the free commutative graded R-algebras (cf. [39, Chapter 3(b)]), to have a differential graded algebra $(\Lambda V, d)$ completely determined it is enough to define $d(v_i)$, for every $i \in I$. Similarly, if (A, d) is a differential graded R-algebra, to provide a morphism $f\colon (\Lambda V, d) \to (A, d)$ it is enough to define $f(v_i)$, $i \in I$, as long as $df(v_i) = fd(v_i)$.

From this point on, we assume that $R = \mathbb{Q}$ and omit the base ring from all notations.

Definition 1.42. A *Sullivan algebra* is a CDGA of the form $(\Lambda V, d)$, such that

- V is concentrated in positive degrees, that is, $V = V^{\geq 1}$;
- V admits a decomposition $V = \cup_{k=0}^{\infty} V(k)$, where $V(0) \subset V(1) \subset \cdots$ is an increasing sequence of graded subspaces, $d|_{V(0)} = 0$ and $d\big(V(k)\big) \subset \Lambda V(k-1)$, for all $k \geq 1$.

A Sullivan algebra $(\Lambda V, d)$ is *minimal* if $\operatorname{Im} d \subset \Lambda^{\geq 2} V$.

Notice that Sullivan algebras are always connected CDGAs, and they are n-connected if and only if $V^k = 0$, for $1 \leq k \leq n$.

Definition 1.43. Let (A, d) be a CDGA and X be a connected space.

- A *Sullivan model* for (A, d) is a pair $\big((\Lambda V, d), \varphi\big)$, where $(\Lambda V, d)$ is a Sullivan algebra and $\varphi\colon (\Lambda V, d) \to (A, d)$ is a quasi-isomorphism of differential graded algebras.

- A *Sullivan model* for X is a Sullivan model $((\Lambda V, d), \varphi)$ for the CDGA $A_{PL}(X)$.
- A *Sullivan model* for an algebra or a space is *minimal* if the corresponding Sullivan algebra is minimal.

By abuse of notation, we usually omit the algebra morphism and say that $(\Lambda V, d)$ is a Sullivan model for (A, d) or for X. Sullivan algebras are precisely the simpler CDGAs that we seek to classify rational homotopy types. And luckily for us, every homologically connected CDGA admits a Sullivan model (cf. [39, Proposition 12.1]). This holds in particular for $A_{PL}(X)$ for X a path connected space. However, if X is simply connected, we have a more powerful result:

Proposition 1.44 ([39, Proposition 12.2])**.** *Let X be a simply connected space. Then, X admits a simply connected minimal Sullivan model $(\Lambda V, d)$. Furthermore, if X is n-connected (resp. of finite type), $(\Lambda V, d)$ is n-connected (resp. of finite type).*

Since $(\Lambda V, d)$ is quasi-isomorphic to $A_{PL}(X)$, we know that $H^*(\Lambda V, d) \cong H^*(A_{PL}(X)) \cong H^*(X; \mathbb{Q})$. But furthermore, the generators of the minimal Sullivan model of X are also related to its algebraic invariants:

Proposition 1.45. *Let X be a simply connected space of finite type with minimal Sullivan model $(\Lambda V, d)$. There is a natural isomorphism $V \cong \mathrm{Hom}_{\mathbb{Z}}(\pi_*(X) \otimes \mathbb{Q}, \mathbb{Q})$.*

Regarding the functor $|\ |\colon \mathrm{CDGA}^{hc} \to \mathrm{Top}^c$, which receives the name of Sullivan's spatial realisation functor, we have the following result:

Proposition 1.46 ([39, Theorem 17.10])**.** *Let $(\Lambda V, d)$ be a simply connected Sullivan algebra of finite type. Then, there is a quasi-isomorphism $\varphi\colon (\Lambda V, d) \to A_{PL}(|\Lambda V, d|)$.*

Consequently, if X is a simply connected space of finite type with minimal Sullivan model $(\Lambda V, d)$, and since $A_{PL}(|\Lambda V, d|)$ is quasi-isomorphic to $(\Lambda V, d)$, we deduce that X and $|\Lambda V, d|$ are of the same rational homotopy type. In fact, $|\Lambda V, d|$ is a rationalisation of X.

So far, we have seen how minimal Sullivan algebras classify rational homotopy types of simply connected spaces. However, we are interested in self-homotopy equivalences, thus we need a classification of homotopy classes of maps as well. Such classification is provided by a concept of homotopy between morphisms of CDGAs of the form $(\Lambda V, d) \to (A, d)$, where $(\Lambda V, d)$ is a Sullivan algebra and (A, d) is a CDGA. We introduce it now.

Consider $\Lambda(t, dt)$ the free commutative graded algebra on the basis $\{t, dt\}$ with $|t| = 0$, $|dt| = 1$ and differential $t \mapsto dt$. Define augmentations $\varepsilon_0, \varepsilon_1\colon \Lambda(t, dt) \to \mathbb{Q}$ by $\varepsilon_0(t) = 0$ and $\varepsilon_1(t) = 1$.

Definition 1.47. Two morphisms $\varphi_0, \varphi_1\colon (\Lambda V, d) \to (A, d)$ from a Sullivan algebra to a CDGA are *homotopic* if there is a morphism of differential graded algebras

$$\Phi\colon (\Lambda V, d) \longrightarrow (A, d) \otimes (\Lambda(t, dt), d)$$

such that $(\mathrm{id}_A \otimes \varepsilon_i) \circ \Phi = \varphi_i$, $i = 0, 1$. We say that Φ is a *homotopy* from φ_0 to φ_1, and denote $\varphi_0 \simeq \varphi_1$. We denote the set of homotopy classes of morphisms from $(\Lambda V, d)$ to $(\Lambda W, d)$ by $[(\Lambda V, d), (\Lambda W, d)]$.

Let us now review some important properties of homotopic morphisms. If $\varphi\colon (\Lambda V, d) \to (\Lambda W, d)$ is a morphism between Sullivan algebras, the *linear part of* φ, denoted $Q(\varphi)$, is a linear map $Q(\varphi)\colon V \to W$ defined by $\varphi(v) - Q(\varphi)(v) \in \Lambda^{\geq 2} W$, for all $v \in V$. We have the following result:

Proposition 1.48 ([39, Proposition 12.8]). *Let* $(\Lambda V, d)$ *and* $(\Lambda W, d)$ *be Sullivan algebras, and let* (A, d) *be a* CDGA.

(1) If $\varphi_0 \simeq \varphi_1 \colon (\Lambda V, d) \to (A, d)$*, then* $H(\varphi_0) = H(\varphi_1)$.

(2) Suppose furthermore that $(\Lambda V, d)$ *and* $(\Lambda W, d)$ *are minimal, simply connected Sullivan algebras. If* $\varphi_0 \simeq \varphi_1 \colon (\Lambda V, d) \to (\Lambda W, d)$*, then* $Q(\varphi_0) = Q(\varphi_1)$.

However, the key result regarding homotopy classes of morphisms is the lifting lemma:

Lemma 1.49 (*lifting lemma*, [39, Proposition 12.9]). *Let* $(\Lambda V, d)$ *be a Sullivan algebra and let* (A, d) *and* (C, d) *be* CDGA*s. Let* $\psi \colon (\Lambda V, d) \to (C, d)$ *be a morphism of differential graded algebras and* $\eta \colon (A, d) \to (C, d)$ *be a quasi-isomorphism. There exists a unique homotopy class of morphisms* $\varphi \colon (\Lambda V, d) \to (A, d)$ *such that* $\eta \circ \varphi \simeq \psi$.

$$\begin{tikzcd} & (A,d) \arrow[d, "\eta", "\simeq"'] \\ (\Lambda V, d) \arrow[ur, dashed, "\varphi"] \arrow[r, "\psi"'] & (C,d) \end{tikzcd}$$

This lemma has a clear immediate consequence. Namely, if (A, d) and (C, d) are weakly equivalent homologically connected CDGAs, they have the same Sullivan models. Thus, two homologically connected CDGAs (A, d) and (C, d) are weakly equivalent if and only if there is a zig-zag of quasi-isomorphisms

$$(A, d) \xleftarrow{\simeq} (\Lambda V, d) \xrightarrow{\simeq} (C, d),$$

where $(\Lambda V, d)$ is a Sullivan algebra.

Now that the homotopy relation has been established, we can introduce the models of (rational) homotopy classes of maps between simply connected spaces. In order to do so, from now on we restrict ourselves to simply connected spaces of finite type, which by Proposition 1.44 admit simply connected minimal Sullivan models of finite type.

Definition 1.50. Let $f \colon X \to Y$ be a continuous map between simply connected spaces of finite type, with respective minimal Sullivan models $((\Lambda V, d), \varphi_X)$ and $((\Lambda W, d), \varphi_Y)$. A *Sullivan representative* or *minimal Sullivan model* of f is a morphism $\varphi \colon (\Lambda W, d) \to (\Lambda V, d)$ such that $\varphi \circ \psi_Y \simeq \psi_X \circ A_{PL}(f)$.

We then have the following result:

Proposition 1.51 ([39, Proposition 12.6]). *Any continuous map* $f \colon X \to Y$ *between simply connected spaces of finite type admits a Sullivan representative, unique up to homotopy.*

We also have the following:

Proposition 1.52 ([39, Proposition 17.13, Theorem 17.15]). *Let* X *and* Y *be two simply connected spaces with respective minimal Sullivan models* $(\Lambda V, d)$ *and* $(\Lambda W, d)$.

(1) If $\varphi_0 \simeq \varphi_1 \colon (\Lambda V, d) \to (\Lambda W, d)$ *are homotopic, then* $|\varphi_0| \simeq |\varphi_1| \colon |\Lambda W, d| \to |\Lambda V, d|$.

(2) If $f \colon X \to Y$ *is a continuous map with Sullivan representative* $\varphi \colon (\Lambda W, d) \to (\Lambda V, d)$*, there is a homotopy-commutative diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h_X"', "\simeq"] & Y \arrow[d, "h_Y", "\simeq"'] \\ {|\Lambda V, d|} \arrow[r, "|\varphi|"'] & {|\Lambda W, d|}. \end{tikzcd}$$

Therefore, homotopic maps have homotopic Sullivan representatives, and homotopic morphisms between Sullivan algebras have (rationally) homotopic spatial realisations. We deduce that the functors A_{PL} and $|\;\;|$ induce an equivalence of categories

$$A_{PL} \colon \mathcal{H}o\,\mathrm{Top}^1_{f,\mathbb{Q}} \rightleftarrows \mathcal{H}o\,\mathrm{CDGA}^{h1}_f \colon |\;\;|$$

where $\mathrm{Top}^1_{f,\mathbb{Q}}$ denotes the category of simply connected rational spaces of finite type. Consequently, if X and Y are spaces in $\mathrm{Top}^1_{f,\mathbb{Q}}$ with respective minimal Sullivan models $(\Lambda V, d)$ and $(\Lambda W, d)$, then $[X, Y] \cong [(\Lambda W, d), (\Lambda V, d)] \cong [|\Lambda V, d|, |\Lambda W, d|]$. In particular, $\mathcal{E}(X) \cong \mathcal{E}(\Lambda V, d)$.

This concludes our basic introduction to Rational Homotopy Theory. The remainder of this section is devoted to several miscellaneous results that we need in Chapter 4 and Chapter 5. First, we need to know how Sullivan models of Serre fibrations look like.

Proposition 1.53 ([39, Chapter 15(a)])**.** *Let X, Y and F be simply connected spaces of finite type and suppose that $p\colon X \to Y$ is a Serre fibration with fibre F. Then, if $(\Lambda W, d)$ and $(\Lambda V, \bar{d})$ are minimal Sullivan models for Y and F respectively, X admits a Sullivan model $(\Lambda W \otimes \Lambda V, d)$ such that:*

- *A Sullivan representative of p is the map $i\colon (\Lambda W, d) \hookrightarrow (\Lambda V \otimes \Lambda W, d)$;*
- *$dv - \bar{d}v \in \Lambda^+ W \otimes \Lambda V$, for all $v \in V$.*

We also need several facts regarding rationally elliptic spaces. Recall that a graded vector space $A = \{A^n\}_{n\geq 0}$ is finite-dimensional, denoted $\dim A < \infty$, if $A^n = \{0\}$ for all but a finite amount of degrees and $\dim A^n < \infty$, for all $n \geq 0$.

Definition 1.54. A simply connected space X is *rationally elliptic* if $\dim H_*(X; \mathbb{Q}) < \infty$ and $\dim \pi_*(X) \otimes \mathbb{Q} < \infty$. A minimal Sullivan algebra is *elliptic* if $\dim V < \infty$ and $\dim H^*(\Lambda V, d) < \infty$.

It is clear that X is rationally elliptic if and only if its minimal Sullivan model is elliptic. To determine if a Sullivan algebra is elliptic we use pure Sullivan algebras:

Definition 1.55. A Sullivan algebra $(\Lambda V, d)$ is *pure* if $d|_{V^{\mathrm{even}}} = 0$ and $d(V^{\mathrm{odd}}) \subset \Lambda V^{\mathrm{even}}$.

Then, to a Sullivan algebra $(\Lambda V, d)$ with V finite-dimensional, we associate another Sullivan algebra $(\Lambda V, d_\sigma)$ where d_σ is characterised by $d_\sigma(V^{\mathrm{even}}) = 0$ and $d - d_\sigma(V^{\mathrm{odd}}) \subset \Lambda V^{\mathrm{even}} \otimes \Lambda^{\geq 2} V^{\mathrm{odd}}$. It turns out that $(\Lambda V, d_\sigma)$ is a pure Sullivan algebra which we call the *pure Sullivan algebra associated to* $(\Lambda V, d)$. Moreover, we have the following result:

Proposition 1.56 ([39, Proposition 32.4])**.** *Let $(\Lambda V, d)$ be a simply connected Sullivan algebra with V finite-dimensional. Then, $(\Lambda V, d)$ is elliptic if and only if its associated pure Sullivan algebra $(\Lambda V, d_\sigma)$ is elliptic.*

We are also interested in rational homotopy types of closed, oriented and connected manifolds. The cohomology algebra of such manifolds verifies Poincaré duality, thus the cohomology algebra of their Sullivan models will verify an analogous property. First, we recall what it means for a commutative differential graded algebra to verify Poincaré duality.

Definition 1.57. Let A be a finite-dimensional commutative differential graded algebra such that $A = \{A^i\}_{0\leq i\leq n}$ and $A^0 = \mathbb{Q}$. The algebra A is a *Poincaré duality algebra* if there is an element $\omega_A \in A^n$ such that, for any k, $0 \leq k \leq n$, the pairing

$$\begin{aligned} \langle\, ,\rangle \colon A^k \otimes A^{n-k} &\longrightarrow \mathbb{Q}, \\ a \otimes a' &\longmapsto \langle a, a'\rangle = \omega_A^{\#}(a \otimes a'), \end{aligned}$$

where $\omega_A^{\#} \colon H^n(A) \to \mathbb{Q}$ is the dual of ω_A, is non-degenerate. We say that n is the *formal dimension* of A, and ω_A receives the name of *fundamental class* of A.

Notice that since $A^0 = \mathbb{Q}$, the fact that the pairing corresponding to $k = 0$ is non-degenerate implies that $\omega_A^{\#}(\lambda z) \neq 0$, for any $0 \neq \lambda \in \mathbb{Q}$ and any $0 \neq z \in A^n$. Thus $\omega_A^{\#}$ is an isomorphism, and consequently, $\dim(A^n) = 1$. Then, for $0 \leq k \leq n$, the pairing being non-degenerate means that for any non-trivial element $x \in A^k$, there exists an element $y \in A^{n-k}$ such that $xy = \omega_A$.

We will also make use of the following result:

Proposition 1.58 ([39, Theorem 32.15])**.** *If $(\Lambda V, d)$ is an elliptic Sullivan algebra, then its cohomology algebra verifies Poincaré duality with formal dimension*

$$\sum_{i=1}^{p} |y_i| - \sum_{j=1}^{q} (|x_j| - 1), \tag{1.1}$$

where $\{y_1, y_2, \dots, y_p\}$ is a basis of V^{odd} and $\{x_1, x_2, \dots, x_q\}$ is a basis of V^{even}.

1.5 A classification of the homotopy types of A_n^2-polyhedra

In this section we review an algebraic classification of the homotopy types of A_n^2-polyhedra (that is, $(n-1)$-connected, $(n+2)$ dimensional spaces). Namely, we introduce Whitehead's universal quadratic functor and Whitehead's exact sequence and review how these tools can be used to classify the homotopy types of these spaces. Then, we show how this classification leads to an algebraic description of a distinguished quotient of the group of self-homotopy equivalences of an A_n^2-polyhedron. In Chapter 6, we use this algebraic description to study the possibility of providing a solution to Kanh's group realisability problem in terms of *integral* spaces.

The results in this section are mostly due to J.H.C. Whitehead, who introduced the classification of A_2^2-polyhedra in his celebrated article [80], and to Baues, who later used the basis laid by Whitehead to provide a classification of the homotopy types of A_n^2-polyhedra for every $n \geq 2$, [11, 12, 13].

Let Ab denote the category of abelian groups. In [80], the author introduces a functor $\Gamma\colon$ Ab $\to$ Ab which plays a role in a certain exact sequence that he uses to classify homotopy types of A_2^2-polyhedra. The Γ functor, also called Whitehead's universal quadratic functor, satisfies a universal property in relation with the so-called quadratic maps. Thus, in order to introduce the Γ functor, we first need to review the definition of quadratic maps.

Definition 1.59. Consider $A, B \in \mathrm{Ob}(\mathrm{Ab})$ two abelian groups. A map (of sets) $\eta\colon A \to B$ is *quadratic* if:

(1) $\eta(a) = \eta(-a)$, for all $a \in A$, and;

(2) the map $A \times A \to B$ taking (a, a') to $\eta(a + a') - \eta(a) - \eta(a')$ is bilinear.

Notice that for any $a \in A$, the bilinear map in (2) takes $(a, -a)$ to $2\eta(a)$, whereas for $n \in \mathbb{Z}$, $(na, -na)$ is taken to $2\eta(na)$. By bilinearity, this implies that $\eta(na) = n^2\eta(a)$. Thus, condition (1) could be changed for the following, more restrictive condition.

(1') $\eta(na) = n^2\eta(a)$, for all $a \in A$ and $n \in \mathbb{Z}$.

We can now introduce Whitehead's universal quadratic functor.

Definition 1.60. Let A be an abelian group. Then, $\Gamma(A)$ is the only abelian group such that there exists a quadratic map $\gamma\colon A \to \Gamma(A)$ verifying that every other quadratic map $\eta\colon A \to B$ factors uniquely through γ. This means that there is a unique group homomorphism $\eta^\square\colon \Gamma(A) \to B$ such that $\eta = \eta^\square\gamma$, thus making the following diagram commute.

$$\begin{tikzcd} A \arrow[r, "\eta"] \arrow[d, "\gamma"'] & B \\ \Gamma(A) \arrow[ur, "\eta^\square"'] & \end{tikzcd}$$

The map $\gamma\colon A \to \Gamma(A)$ is unique and receives the name of *universal quadratic map of A*.

Now we describe how Γ acts on morphisms. Let A and B be two abelian groups and $f\colon A \to B$ be a group homomorphism, and consider the respective universal quadratic maps $\gamma\colon A \to \Gamma(A)$ and $\gamma\colon B \to \Gamma(B)$. Then, $\gamma f\colon A \to \Gamma(B)$ is a quadratic map, so there exists a unique group homomorphism $(\gamma f)^\square\colon \Gamma(A) \to \Gamma(B)$ such that $(\gamma f)^\square\gamma = \gamma f$. Define $\Gamma(f) = (\gamma f)^\square$, so $\Gamma(f)$ is the only group homomorphism that makes the following diagram commute:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\gamma"'] & B \arrow[d, "\gamma"] \\ \Gamma(A) \arrow[r, "\Gamma(f)"'] & \Gamma(B). \end{tikzcd}$$

Example 1.61. The squaring map $(-)^2\colon \mathbb{Z} \to \mathbb{Z}$ is a quadratic map. Moreover, $\Gamma(\mathbb{Z}) = \mathbb{Z}$, and the corresponding universal quadratic map is $\gamma = (-)^2$. Indeed, given $\eta\colon \mathbb{Z} \to B$ a quadratic map, the homomorphism defined as $\eta^\square(n) = n\eta(1)$ verifies that $\eta^\square\gamma = \eta$ as a consequence of Definition 1.59.(1').

We now list some properties of the Γ functor that we need in the sequel:

Proposition 1.62 ([13, p. 16–17])**.** *The functor Γ has the following properties:*

(1) $\Gamma(\mathbb{Z}) = \mathbb{Z}$*, as seen in Example 1.61.*

(2) $\Gamma(\mathbb{Z}_n)$ *is* $\mathbb{Z}_{2n}$ *if n is even or* $\mathbb{Z}_n$ *if n is odd.*

(3) Let I be an ordered set and A_i be an abelian group, for each $i \in I$. Then,

$$\Gamma\left(\bigoplus_I A_i\right) = \left(\bigoplus_I \Gamma(A_i)\right) \oplus \left(\bigoplus_{i<j} A_i \otimes A_j\right).$$

The groups $\Gamma(A_i)$ and $A_i \otimes A_j$ are respectively generated by elements $\gamma(a_i)$ and $a_i \otimes a_j$, with $a_i \in A_i$, $a_j \in A_j$, $i < j$. Elements $a_j \otimes a_i$ are identified with $a_i \otimes a_j$, and $a_i \otimes a_i$ is identified with $2\gamma(a_i)$. Moreover, for all $i, j \in I$, $a_i \in A_i$ and $a_j \in A_j$, $\gamma(a_i + a_j) = \gamma(a_i) + \gamma(a_j) + a_i \otimes a_j$, [80, Sections 5 and 7].

We now introduce Whitehead's exact sequence. Let X be a 1-connected space. For $n \geq 1$, the n-th Whitehead Γ-group of X is defined as

$$\Gamma_n(X) = \operatorname{Im}\left(i_*\colon \pi_n(X^{n-1}) \to \pi_n(X^n)\right).$$

Here, $i\colon X^{n-1} \to X^n$ is the inclusion of the $(n-1)$-skeleton of X into its n-skeleton. Then, $\Gamma_n(X)$ is an abelian group for every $n \geq 1$. This group can be embedded in an exact sequence of abelian groups:

Theorem 1.63 ([80, Chapter III])**.** *Let X be a 1-connected space. There is an exact sequence of abelian groups*

$$\cdots \longrightarrow H_{n+1}(X) \xrightarrow{b_{n+1}} \Gamma_n(X) \xrightarrow{i_{n-1}} \pi_n(X) \xrightarrow{h_n} H_n(X) \longrightarrow \cdots \tag{1.2}$$

where h_n is the usual Hurewicz homomorphism and b_{n+1} is a boundary representing the attaching maps.

The final part of Whitehead's exact sequence plays a key role in the classification of the homotopy types of A_n^2-polyhedra attained by Whitehead and Baues. As we want to use that classification to study self-homotopy equivalences of A_n^2-polyhedra, it would be useful to understand the group $\Gamma_n(X)$ for the first index n for which it is non-trivial. For this reason, for each $n \geq 2$ we define a functor $\Gamma_n^1\colon \mathrm{Ab} \to \mathrm{Ab}$ as follows. Define $\Gamma_2^1 = \Gamma$ the universal quadratic functor of Whitehead, and for $n \geq 3$, define $\Gamma_n^1 = - \otimes \mathbb{Z}_2$. Then:

Theorem 1.64 ([13, Theorem 2.1.22])**.** *Let $n \geq 2$ and let X be a $(n-1)$-connected space. There is an isomorphism $\Gamma_n^1(H_n(X)) \cong \Gamma_{n+1}(X)$.*

Consequently, the final part of (1.2) can be written as

$$H_{n+2}(X) \xrightarrow{b_{n+2}} \Gamma_n^1(H_n(X)) \xrightarrow{i_n} \pi_{n+1}(X) \xrightarrow{h_{n+1}} H_{n+1}(X) \longrightarrow 0. \tag{1.3}$$

We now move on to introducing the classification of the homotopy types of A_n^2-polyhedra mentioned above. This classification is provided by a *detecting functor*. Let us introduce this concept.

Definition 1.65 ([80, Section 14])**.** Let $\mathcal{A}$ and $\mathcal{B}$ be two categories. A functor $\lambda\colon \mathcal{A} \to \mathcal{B}$ is said to verify

(1) the *sufficiency condition* if whenever $\lambda(f)$ is an isomorphism, so is f, for f any morphism in $\mathcal{A}$;

(2) the *realisability condition* if λ is full and for each object $B \in \mathrm{Ob}(\mathcal{B})$ there is an object $A \in \mathrm{Ob}(\mathcal{A})$ such that $\lambda(A)$ is isomorphic to B.

We say that λ is a *detecting functor* if it verifies the sufficiency and realisability conditions.

Clearly, a detecting functor $\lambda\colon \mathcal{A} \to \mathcal{B}$ induces a one to one correspondence between isomorphism classes of objects in $\mathcal{A}$ and $\mathcal{B}$. The detecting functors of Whitehead and Baues have the homotopy category of A_n^2-polyhedra as their source category. We now introduce their target categories, that is, the categories whose isomorphism classes classify homotopy types of A_n^2-polyhedra.

Definition 1.66 ([11, Chapter IX, Section 4])**.** Let $n \geq 2$ be an integer. We define the category of Γ-sequences^{n+2} as follows. Objects are exact sequences of abelian groups

$$H_{n+2} \longrightarrow \Gamma_n^1(H_n) \longrightarrow \pi_{n+1} \longrightarrow H_{n+1} \longrightarrow 0$$

where H_{n+2} is free abelian. Morphisms are triples $f = (f_{n+2}, f_{n+1}, f_n)$ of group homomorphisms, $f_i\colon H_i \to H_i'$, such that there exists a group homomorphism $\Omega\colon \pi_{n+1} \to \pi_{n+1}'$ making the following diagram

$$\begin{tikzcd}
H_{n+2} \arrow[r] \arrow[d, "f_{n+2}"] & \Gamma_n^1(H_n) \arrow[r] \arrow[d, "\Gamma_n^1(f_n)"] & \pi_{n+1} \arrow[r] \arrow[d, "\Omega"] & H_{n+1} \arrow[r] \arrow[d, "f_{n+1}"] & 0 \\
H'_{n+2} \arrow[r] & \Gamma_n^1(H'_n) \arrow[r] & \pi'_{n+1} \arrow[r] & H'_{n+1} \arrow[r] & 0
\end{tikzcd}$$

commutative. We say that objects in Γ-sequences^{n+2} are Γ-sequences, and morphisms in the category are called Γ-morphisms.

Notice that given X an A_n^2-polyhedron, the final part of its Whitehead's exact sequence, (1.3), is a Γ-sequence. The next definition follows naturally.

Definition 1.67. Let $n \geq 2$ be an integer. We define a functor $F_n\colon \mathcal{H}oA_n^2$-polyhedra $\to$ Γ-sequences^{n+2} as follows. For an A_n^2-polyhedron X, we define $F_n(X)$ as the final part of the corresponding Whitehead's exact sequence, (1.3). We say that $F_n(X)$ is the Γ-*sequence of* X. On the other hand, given a continuous map $\alpha\colon X \to X'$ of A_n^2-polyhedra, the maps induced by α on homotopy and homology groups provide us with a commutative diagram of group homomorphisms

$$\begin{tikzcd}
H_{n+2}(X) \arrow[r] \arrow[d, "H_{n+2}(\alpha)"] & \Gamma_n^1(H_n(X)) \arrow[r] \arrow[d, "\Gamma_n^1(H_n(\alpha))"] & \pi_{n+1}(X) \arrow[r] \arrow[d, "\pi_{n+1}(\alpha)"] & H_{n+1}(X) \arrow[r] \arrow[d, "H_{n+1}(\alpha)"] & 0 \\
H_{n+2}(X') \arrow[r] & \Gamma_n^1(H_n(X')) \arrow[r] & \pi_{n+1}(X') \arrow[r] & H_{n+1}(X') \arrow[r] & 0,
\end{tikzcd}$$

thus providing a Γ-morphism between the Γ-sequences of X and X'. We can thus define $F_n(\alpha) = (H_{n+2}(\alpha), H_{n+1}(\alpha), H_n(\alpha))$.

Finally, we have the following result:

Theorem 1.68 ([12, Chapter I, Section 8])**.** *Let $n \geq 2$ be an integer. Then, the functor $F_n\colon \mathcal{H}oA_n^2$-polyhedra $\to$ Γ-sequences^{n+2} introduced in Definition 1.67 is a detecting functor.*

Hence, for any object in Γ-sequences^{n+2} there exists an A_n^2-polyhedron whose Γ-sequence is the given object in Γ-sequences^{n+2}. In fact, there is a one to one correspondence between homotopy types of A_n^2-polyhedra and isomorphism classes of Γ-sequences. Then, following the ideas of [17], we introduce the following:

Definition 1.69. For X an A_n^2-polyhedron, we denote the group of Γ-automorphisms of the Γ-sequence of X by $\mathcal{B}^{n+2}(X)$.

Now let $\Psi\colon \mathcal{E}(X) \to \mathcal{B}^{n+2}(X)$ be the map that associates to $\alpha \in \mathcal{E}(X)$ the Γ-isomorphism $\Psi(\alpha) = F_n(\alpha) = (H_{n+2}(\alpha), H_{n+1}(\alpha), H_n(\alpha))$. Then Ψ is a group homomorphism, which is onto as a consequence of Theorem 1.68. Furthermore, its kernel is the subgroup of those self-homotopy equivalences of X that induce the identity map on homology groups, a subgroup of $\mathcal{E}(X)$ usually denoted $\mathcal{E}_*(X)$. The following results follows immediately.

Proposition 1.70. *Let X be an A_n^2-polyhedron, $n \geq 2$. Then $\mathcal{B}^{n+2}(X) \cong \mathcal{E}(X)/\mathcal{E}_*(X)$.*

CHAPTER 2

Realisability problems in Graph Theory

As we discussed in the Introduction, to solve Problems 1 and 2 in different categories, we first tackle them in the category $\mathcal{G}raphs$.

A convenient way to construct objects in $\mathcal{G}raphs$ that serve us to answer in the positive our realisability problems is to first work in the categorical framework of binary relational systems, since we take advantage of the structure given by the labels of the edges. In Section 2.1 we build binary relational systems giving a positive answer to Problem 1, see Theorem 2.16. Similarly, in Section 2.2 we give a solution to Problem 2, see Theorem 2.26.

Then, in Section 2.3 and following ideas from the classical arrow replacement operation [49, Section 4.4], we encode the information contained in the labels and edge directions into simple graphs, see Theorem 2.33. We show that the binary relational systems built in Section 2.1 and Section 2.2 fit the hypothesis of Theorem 2.33, so we can finally transfer the solutions to Problem 1 and Problem 2 from $I\mathcal{R}el$ to $\mathcal{G}raphs$, see Theorem 2.37 and Theorem 2.41.

2.1 Realisability in the arrow category of binary relational systems

In this section, we build two relational systems $\mathcal{G}_1$ and $\mathcal{G}_2$, Definition 2.7, and a morphism between them $\varphi\colon \mathcal{G}_1 \to \mathcal{G}_2$, Definition 2.10. These constructions allow us to positively answer Problem 1 in the arrow category of $I\mathcal{R}el$, as we prove in our main result in this section, Theorem 2.16.

The construction of the binary relational systems involved in Theorem 2.16 is carried out in Section 2.1.1; properties of their automorphism groups are given in Section 2.1.2; and everything is put together to prove Theorem 2.16 in Section 2.1.3. It is worth remarking that the constructions contained in this section are quite technical, so we recommend the reader to work out an example along the way to get a better grasp of the ideas involved. A simple example such as taking $G_1 = \mathbb{Z}_8$, $G_2 = \mathbb{Z}_4$ and $H = \langle(2,2)\rangle \leq \mathbb{Z}_8 \times \mathbb{Z}_4$ is enough to picture every ingredient involved in our construction. We illustrate this example in Section 2.1.4.

2.1.1 Construction of the relational systems involved in Theorem 2.16

Given $H \leq G_1 \times G_2$ groups, we want to build two binary relational systems over a set I, $\mathcal{G}_1$ and $\mathcal{G}_2$, and a morphism $\varphi\colon \mathcal{G}_1 \to \mathcal{G}_2$ between them, so that $\mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}_1) \cong G_1$, $\mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}_2) \cong G_2$ and $\mathrm{Aut}_{I\mathcal{R}el}(\varphi) \cong H$. In order to construct them, we first need a characterisation of the subgroups of a product of two groups. An elementary result, known as Goursat's lemma, is used to that purpose. The basic idea of the lemma's proof can be found in [10, Theorem 2.1 and p. 3].

Lemma 2.1 ([46, Sections 11–12])**.** *Let G_1 and G_2 be arbitrary groups and $H \leq G_1 \times G_2$ be a subgroup. Consider $\iota_j\colon G_j \to G_1 \times G_2$ and $\pi_j\colon G_1 \times G_2 \to G_j$ the respective inclusions and projections, $j = 1, 2$. There exists a group isomorphism*

$$\theta\colon \frac{\pi_1(H)}{\iota_1^{-1}(H)} \longrightarrow \frac{\pi_2(H)}{\iota_2^{-1}(H)},$$

taking a class $[g_1]$ to $\theta([g_1]) = [g_2]$, the class of any element $g_2 \in \pi_2(H)$ such that $(g_1, g_2) \in H$. Moreover,

$$H = \{(g_1, g_2) \in \pi_1(H) \times \pi_2(H) \mid \theta([g_1]) = [g_2]\}.$$

Taking into account the previous lemma, we now proceed with the construction of the binary relational systems in Theorem 2.16. Let G_1 and G_2 be arbitrary groups and $H \leq G_1 \times G_2$.

Definition 2.2 (Generating sets R and S for, respectively, G_1 and G_2)**.**

- Let J_1 be an indexing set for the right cosets of $\iota_1^{-1}(H)$ in G_1. We choose a representative of each right coset, $\{r_j,\, j \in J_1\}$, assuming that $0 \in J_1$ and $r_0 = e_{G_1}$ represents $\iota_1^{-1}(H)$. We fix a generating set $\{r_i \mid i \in I_{\iota_1}\}$ for $\iota_1^{-1}(H)$ and we let $I_1 = I_{\iota_1} \sqcup J_1^*$. Then $R = \{r_i \mid i \in I_1\}$ is a generating set for G_1.

- Let J_2 be an indexing set for the right cosets of $\pi_2(H)$ in G_2. Analogously, we choose a representative of right cosets $\{s_j \mid j \in J_2\}$, assuming that $0 \in J_2$ and $s_0 = e_{G_2}$ represents $\pi_2(H)$. We fix a generating set $\{s_i \mid i \in I_{\pi_2}\}$ for $\pi_2(H)$ and we let $I_2 = I_{\pi_2} \sqcup J_2^*$. Then $S = \{s_i \mid i \in I_2\}$ is a generating set for G_2.

Remark 2.3. By decomposing $G_1 = \sqcup_{j\in J_1} \iota_1^{-1}(H) r_j$, there exist maps $k_1\colon G_1 \to \iota_1^{-1}(H)$ and $j_1\colon G_1 \to J_1$ such that any $g \in G_1$ can be uniquely expressed as a product $g = k_1(g) r_{j_1(g)}$. By setting $J_{\pi_1} = \{j \in J_1 \mid r_j \in \pi_1(H)\}$, if $g \in \pi_1(H)$ we have that $j_1(g) \in J_{\pi_1}$. Analogously, $G_2 = \sqcup_{j\in J_2} \pi_2(H) s_j$ and there exist maps $k_2\colon G_2 \to \pi_2(H)$ and $j_2\colon G_2 \to J_2$ such that any $g \in G_2$ is uniquely expressed as the product $g = k_2(g) s_{j_2(g)}$.

The maps k_1, j_1, k_2 and j_2 satisfy certain compatibility conditions with the group operation:

Lemma 2.4. *Let $g, g' \in G_1$ (resp. $g, g' \in G_2$). Then,*

(1) $j_1(gg') = j_1(r_{j_1(g)} g')$ (resp. $j_2(gg') = j_2(s_{j_2(g)} g')$).

(2) $k_1(gg') = k_1(g) k_1(r_{j_1(g)} g')$ (resp. $k_2(gg') = k_2(g) k_2(s_{j_2(g)} g')$).

Proof. We only check Lemma 2.4.*(1)* since the proof of Lemma 2.4.*(2)* is analogous. By Remark 2.3, $gg' = k_1(gg') r_{j_1(gg')}$ and also $gg' = k_1(g) r_{j_1(g)} g' = k_1(g) k_1(r_{j_1(g)} g') r_{j_1(r_{j_1(g)} g')}$. Since this decomposition is unique, the result follows immediately. □

The following is an auxiliary binary system that will be used in Definition 2.7.

Definition 2.5 (Auxiliary binary system)**.** Let $I = I_1 \sqcup I_2 \sqcup \{\theta\}$ and $V_1 = \pi_1(H)/\iota_1^{-1}(H)$. We define $\mathcal{G}_{\iota_1}$ to be the binary I-system having as vertices

$$V(\mathcal{G}_{\iota_1}) = \begin{cases} V_1, & \text{if } G_1 = \pi_1(H), \\ V_1 \sqcup \{s\}, & \text{otherwise,} \end{cases}$$

and as edges of label i, with $[g] \in V_1$:

- for $i \in I_{\iota_1}$, $([g],[g]) \in R_i(\mathcal{G}_{\iota_1})$;
- for $i \in J^*_{\pi_1}$, $([g],[r_i g]) \in R_i(\mathcal{G}_{\iota_1})$.

If $G_1 \neq \pi_1(H)$, $\mathcal{G}_{\iota_1}$ also has the edges of label i:

- for $i \in I_1$, $(s,s) \in R_i(\mathcal{G}_{\iota_1})$;
- for $i \in J_1 \setminus J_{\pi_1}$, $([g],s),(s,[g]) \in R_i(\mathcal{G}_{\iota_1})$.

Observe that the set of edges of $\mathcal{G}_{\iota_1}$ corresponding to labels in $I_2 \sqcup \{\theta\}$ is empty.

Remark 2.6. The Cayley diagram $\mathrm{Cay}\left(V_1, \{[r_i] \mid i \in I_{\iota_1} \sqcup J^*_{\pi_1}\}\right)$ is equal to $\mathcal{G}_{\iota_1}$ if $G_1 = \pi_1(H)$ and is a proper full binary relational subsystem of $\mathcal{G}_{\iota_1}$ otherwise.

We are now ready to define the binary I-systems $\mathcal{G}_1$ and $\mathcal{G}_2$ in Theorem 2.16. Recall from Lemma 2.1 that there exists an isomorphism $\theta\colon \pi_1(H)/\iota_1^{-1}(H) \to \pi_2(H)/\iota_2^{-1}(H)$. Also recall that in Definition 2.2 we described two generating sets R and S for, respectively, G_1 and G_2.

Definition 2.7 (Binary relational systems $\mathcal{G}_1$ and $\mathcal{G}_2$ in Theorem 2.16)**.** We define the following binary I-systems:

- $\mathcal{G}_1 = \mathrm{Cay}(G_1, R)$.
- $\mathcal{G}_2$ has vertex set $V(\mathcal{G}_2) = G_2 \sqcup \left(\sqcup_{j\in J_2} V_2^j\right)$ where $V_2^j = \{j\} \times V(\mathcal{G}_{\iota_1})$, and edge set:
  - for $i \in I_2$ and $g \in G_2$, $(g, s_i g) \in R_i(\mathcal{G}_2)$;
  - for $\theta \in I$ and $g \in G_2$, $\left(g, \left(j_2(g), \theta^{-1}[k_2(g)]\right)\right) \in R_\theta(\mathcal{G}_2)$;
  - for $i \in I_1$, $j \in J_2$, if $(v_1,v_2) \in R_i(\mathcal{G}_{\iota_1})$, then $\left((j,v_1),(j,v_2)\right) \in R_i(\mathcal{G}_2)$.

Remark 2.8. Cases of interest for are the following full binary relational subsystems of $\mathcal{G}_2$:

- $\mathcal{G}_2(G_2)$, with vertex set G_2, which is isomorphic to $\mathrm{Cay}(G_2,S)$.
- $\mathcal{G}_2(G_1,j)$, with vertex set V_2^j, which is isomorphic to $\mathcal{G}_{\iota_1}$ for each $j \in J_2$.

The following is an auxiliary construction that will be used in Definition 2.10:

Lemma 2.9 (Auxiliary morphism of binary systems)**.** *The map $\varphi_0\colon \mathcal{G}_1 \to \mathcal{G}_{\iota_1}$ defined by*

$$\varphi_0(g) = \begin{cases} [g], & \textit{if } g \in \pi_1(H), \\ s, & \textit{otherwise,} \end{cases}$$

is a morphism of binary I-systems.

Proof. We need to check that for $i \in I_1$, if $(g, r_i g) \in R_i(\mathcal{G}_1)$ then $\left(\varphi_0(g), \varphi_0(r_i g)\right) \in R_i(\mathcal{G}_{\iota_1})$. We decompose $I_1 = I_{\iota_1} \sqcup J^*_{\pi_1} \sqcup (J_1 \setminus J_{\pi_1})$ and prove it by cases.

For $i \in I_{\iota_1}$, $r_i \in \iota_1^{-1}(H)$. Hence, if $g \in \pi_1(H)$, also $r_i g \in \pi_1(H)$ and $\left(\varphi_0(g), \varphi_0(r_i g)\right) = ([g],[r_i g]) = ([g],[g]) \in R_i(\mathcal{G}_{\iota_1})$. On the other side, if $g \notin \pi_1(H)$, then $r_i g \notin \pi_1(H)$ and by definition, $\left(\varphi_0(g), \varphi_0(r_i g)\right) = (s,s) \in R_i(\mathcal{G}_{i_1})$.

For $i \in J^*_{\pi_1}$, $r_i \in \pi_1(H)$ and the argument goes as previously.

Finally, for $i \in J_1 \setminus J_{\pi_1}$, $r_i \notin \pi_1(H)$. On the one hand, if $g \in \pi_1(H)$ necessarily $r_i g \notin \pi_1(H)$, therefore $\left(\varphi_0(g), \varphi_0(r_i g)\right) = ([g], s) \in R_i(\mathcal{G}_{i_1})$. On the other hand, if $g \notin \pi_1(H)$, we can have either $r_i g \in \pi_1(H)$, in which case $\left(\varphi_0(g), \varphi_0(r_i g)\right) = (s, [r_i g]) \in R_i(\mathcal{G}_{i_1})$, or $r_i g \notin \pi_1(H)$, in which case $\left(\varphi_0(g), \varphi_0(r_i g)\right) = (s,s) \in R_i(\mathcal{G}_{i_1})$. $\square$

Definition 2.10 (Arrow $\varphi\colon \mathcal{G}_1 \to \mathcal{G}_2$ in Theorem 2.16)**.** Let $\varphi\colon \mathcal{G}_1 \to \mathcal{G}_2$ be the composite of the morphism φ_0 from the previous lemma, followed by the inclusion of $\mathcal{G}_{\iota_1} \cong \mathcal{G}_2(G_1, 0)$ (see Remark 2.8) into $\mathcal{G}_2$:

$$\mathcal{G}_1 \xrightarrow[\varphi_0]{} \mathcal{G}_{\iota_1} \cong \mathcal{G}_2(G_1,0) \xhookrightarrow[i_0]{} \mathcal{G}_2.$$

That is, $\varphi(g) = \big(0, \varphi_0(g)\big) \in V_2^0$ for $g \in V(\mathcal{G}_1) = G_1$.

For the sake of clarity, we split the proof of Theorem 2.16 into various intermediate results that we include in the following section.

2.1.2 Properties of the binary relational systems from Definition 2.7

Since $\mathcal{G}_1$ is a Cayley diagram for G_1, we have that $\mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}_1) \cong G_1$ (see Remark 1.10). Proving that $\mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}_2) \cong G_2$ needs further elaboration. The first step is to prove that $\mathcal{G}_{\iota_1}$, the auxiliary binary I-system introduced in Definition 2.5, is sufficiently rigid:

Lemma 2.11. *For a fixed $g \in \pi_1(H)$, there exists a unique $\psi_g \in \mathrm{Aut}_{IRel}(\mathcal{G}_{\iota_1})$ such that $\psi_g([e_{G_1}]) = [g]$.*

Proof. We claim that any automorphism ψ of $\mathcal{G}_{\iota_1}$ maps V_1 to itself. This is clear when $\pi_1(H) = G_1$. Thus we assume that $\pi_1(H) \neq G_1$ (which in particular implies that $|J_1^*| \geq 1$). Notice that then, s is the only vertex connected to itself through an edge $(s,s) \in R_i(\mathcal{G}_{\iota_1})$ of label $i \in J_1^*$. But ψ being a morphism of I-binary systems implies that $\big(\psi(s), \psi(s)\big) \in R_i(\mathcal{G}_{\iota_1})$ for $i \in J_1^*$, which leads to $\psi(s) = s$ and our claim holds.

Now, on the one hand, given $g \in \pi_1(H)$ it is immediate to check that we obtain an automorphism $\psi_g \in \mathrm{Aut}_{IRel}(\mathcal{G}_{\iota_1})$ of binary I-systems by declaring $\psi_g([h]) = [h][g] = \phi_{[g]^{-1}}([h])$ for $[h] \in V_1$, and $\psi_g(s) = s$.

On the other hand, given $\psi \in \mathrm{Aut}_{IRel}(\mathcal{G}_{\iota_1})$ such that $\psi([e_{G_1}]) = [g]$, and bearing in mind Remark 2.6, we can now affirm that $\psi|_{V_1}$ is an automorphism of the full relational subsystem $\mathrm{Cay}\,\big(V_1, \{[r_i] \mid i \in I_{\iota_1} \sqcup J_{\pi_1}^*\}\big)$. Hence, $\psi|_{V_1} = \phi_{[g]^{-1}}$ (see Remark 1.10), the only automorphism sending $[e_{G_1}]$ to $[g]$, and since $\psi(s) = s$, then $\psi = \psi_g$. □

To prove that $G_2 \cong \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}_2)$, we first show that any element $\tilde{g} \in G_2$ induces an automorphism $\Phi_{\tilde{g}}$ on $\mathcal{G}_2$. We now give the construction of $\Phi_{\tilde{g}}$ and then we prove that it is indeed an automorphism of relational systems.

Definition 2.12. Given $\tilde{g} \in G_2$, we define $\Phi_{\tilde{g}} : V(\mathcal{G}_2) = G_2 \sqcup \big(\sqcup_{j \in J_2} V_2^j \big) \to V(\mathcal{G}_2)$ as follows. First, given that $\mathrm{Cay}(G_2, S)$ is a full relational subsystem of $\mathcal{G}_2$ (see Remark 2.8) we define $\Phi_{\tilde{g}}|_{G_2}$ as $\phi_{\tilde{g}}$, the automorphism induced by right multiplication by $\tilde{g}^{-1}$ in $\mathrm{Cay}(G_2, S)$ introduced in Remark 1.10. Thus for $g \in G_2$

$$\Phi_{\tilde{g}}(g) = g\tilde{g}^{-1} \in G_2.$$

Secondly, for $(j, [g]) \in V_2^j$, we define

$$\Phi_{\tilde{g}}(j, [g]) = \big(j_2(s_j\tilde{g}^{-1}), [g]\theta^{-1}[k_2(s_j\tilde{g}^{-1})]\big) \in V_2^{j_2(s_j\tilde{g}^{-1})}.$$

If moreover $\pi_1(H) \neq G_1$, for $(j, s) \in V_2^j$, we finally define

$$\Phi_{\tilde{g}}(j, s) = \big(j_2(s_j\tilde{g}^{-1}), s\big) \in V_2^{j_2(s_j\tilde{g}^{-1})}.$$

The previous self-map of $V(\mathcal{G}_2)$ is indeed a morphism of binary relational systems:

Lemma 2.13. *Given $\tilde{g} \in G_2$, $\Phi_{\tilde{g}} \in \mathrm{Hom}_{I\mathcal{R}el}(\mathcal{G}_2, \mathcal{G}_2)$.*

Proof. We check that $\Phi_{\tilde{g}}$ is a morphism of binary I-systems, that is, $\Phi_{\tilde{g}}$ respects relations $R_i(\mathcal{G}_2)$, $i \in I$. We prove it by cases:

First take $g \in G_2$. Then if $i \in I_2$, we have that $(g, s_i g) \in R_i(\mathcal{G}_2)$ and

$$(\Phi_{\tilde{g}}(g), \Phi_{\tilde{g}}(s_i g)) = (g\tilde{g}^{-1}, s_i g\tilde{g}^{-1}) \in R_i(\mathcal{G}_2).$$

On the other hand, if $\theta \in I$, we have that $\big(g, (j_2(g), \theta^{-1}[k_2(g)])\big) \in R_\theta(\mathcal{G}_2)$ and

$$\begin{aligned}(\Phi_{\tilde{g}}(g), \Phi_{\tilde{g}}(j_2(g), \theta^{-1}[k_2(g)])) &= (g\tilde{g}^{-1}, (j_2(s_{j_2(g)}\tilde{g}^{-1}), \theta^{-1}[k_2(g)]\theta^{-1}[k_2(s_{j_2(g)}\tilde{g}^{-1})])) \\ &= (g\tilde{g}^{-1}, (j_2(g\tilde{g}^{-1}), \theta^{-1}[k_2(g\tilde{g}^{-1})])) \in R_\theta(\mathcal{G}_2),\end{aligned}$$

where the last equality follows from Lemma 2.4.*(1)* and Lemma 2.4.*(2)*, and the fact that θ^{-1} is a group homomorphism.

Now take $g \in \pi_1(H)$. Then if $i \in I_{i_1}$, we have $\big((j,[g]),(j,[g])\big) \in R_i(\mathcal{G}_2)$, $j \in J_2$, and

$$(\Phi_{\tilde{g}}(j,[g]), \Phi_{\tilde{g}}(j,[g])) = ((j_2(s_j\tilde{g}^{-1}), [g]\theta^{-1}[k_2(s_j\tilde{g}^{-1})]), (j_2(s_j\tilde{g}^{-1}), [g]\theta^{-1}[k_2(s_j\tilde{g}^{-1})])),$$

which is an edge in $R_i(\mathcal{G}_2)$. On the other hand, if $i \in J^*_{\pi_1}$, we have $\big((j,[g]),(j,[r_i g])\big) \in R_i(\mathcal{G}_2)$, for $j \in J_2$, and

$$(\Phi_{\tilde{g}}(j,[g]), \Phi_{\tilde{g}}(j,[r_i g])) = ((j_2(s_j\tilde{g}^{-1}), [g]\theta^{-1}[k_2(s_j\tilde{g}^{-1})]), (j_2(s_j\tilde{g}^{-1}), [r_i g]\theta^{-1}[k_2(s_j\tilde{g}^{-1})])),$$

which is an edge in $R_i(\mathcal{G}_2)$.

If moreover $\pi_1(H) \neq G_1$, then for $i \in J_1 \setminus J_{\pi_1}$, we have $\big((j,s),(j,[g])\big)$ and $\big((j,[g]),(j,s)\big)$ in $R_i(\mathcal{G}_2)$, $j \in J_2$. As both are analogous, we only check the first:

$$(\Phi_{\tilde{g}}(j,s), \Phi_{\tilde{g}}(j,[g])) = ((j_2(s_j\tilde{g}^{-1}), s), (j_2(s_j\tilde{g}^{-1}), [g]\theta^{-1}[k_2(s_j\tilde{g}^{-1})])) \in R_i(\mathcal{G}_2).$$

For $i \in I_2$ then $\big((j,s),(j,s)\big) \in R_i(\mathcal{G}_2)$, $j \in J_2$, and

$$(\Phi_{\tilde{g}}(j,s), \Phi_{\tilde{g}}(j,s)) = ((j_2(s_j\tilde{g}^{-1}), s), (j_2(s_j\tilde{g}^{-1}), s)) \in R_i(\mathcal{G}_2).$$

The result is thus proven. □

Indeed, the construction from Definition 2.12 is an automorphism of the relational system and defines a group homomorphism Φ as follows:

Proposition 2.14. *The following hold:*

(1) Given $\tilde{g} \in G_2$, $\Phi_{\tilde{g}} \in \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}_2)$.

(2) The following map is a group homomorphism:

$$\begin{aligned}\Phi\colon G_2 &\longrightarrow \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}_2) \\ \tilde{g} &\longmapsto \Phi_{\tilde{g}}.\end{aligned}$$

Proof. We are going to prove that for $\tilde{g}, \tilde{h} \in G_2$, we have that $\Phi_{\tilde{g}} \circ \Phi_{\tilde{h}} = \Phi_{\tilde{g}\tilde{h}}$. Indeed

$$\begin{aligned}\Phi_{\tilde{g}}(\Phi_{\tilde{h}}(j,[g])) &= \Phi_{\tilde{g}}(j_2(s_j\tilde{h}^{-1}), [g]\theta^{-1}[k_2(s_j\tilde{h}^{-1})]) \\ &= (j_2(s_{j_2(s_j\tilde{h}^{-1})}\tilde{g}^{-1}), [g]\theta^{-1}[k_2(s_j\tilde{h}^{-1})]\theta^{-1}[k_2(s_{j_2(s_j\tilde{h}^{-1})}\tilde{g}^{-1})]),\end{aligned}$$

and,

$$\Phi_{\tilde{g}\tilde{h}}(j,[g])) = (j_2(s_j(\tilde{g}\tilde{h})^{-1}), [g]\theta^{-1}[k_2(s_j(\tilde{g}\tilde{h})^{-1})]) = (j_2(s_j\tilde{h}^{-1}\tilde{g}^{-1}), [g]\theta^{-1}[k_2(s_j\tilde{h}^{-1}\tilde{g}^{-1})]).$$

By Lemma 2.4.*(1)*, $j_2(s_{j_2(s_j\tilde{h}^{-1})}\tilde{g}^{-1}) = j_2(s_j\tilde{h}^{-1}\tilde{g}^{-1})$, and by Lemma 2.4.*(2)*,

$$k_2(s_j\tilde{h}^{-1})k_2(s_{j_2(s_j\tilde{h}^{-1})}\tilde{g}^{-1}) = k_2(s_j\tilde{h}^{-1}\tilde{g}^{-1}).$$

As θ^{-1} is a group isomorphism, it follows that $\Phi_{\tilde{g}}(\Phi_{\tilde{h}}(j,[g])) = \Phi_{\tilde{g}\tilde{h}}(j,[g])$ for $(j,[g]) \in V_2^j$.

Finally, if $\pi_1(H) \neq G_1$, for $(j,s) \in V_2^j$ we have

$$\Phi_{\tilde{g}}(\Phi_{\tilde{h}}(j,s)) = \Phi_{\tilde{g}}(j_2(s_j\tilde{h}^{-1}),s) = (j_2(s_{j_2(s_j\tilde{h}^{-1})}\tilde{g}^{-1}),s) = (j_2(s_j\tilde{h}^{-1}\tilde{g}^{-1}),s) = \Phi_{\tilde{g}\tilde{h}}(j,s),$$

as a consequence of Lemma 2.4.*(1)*. Now, from Definition 2.12 it is clear that $\Phi_{e_{G_2}}$ is the identity map of $\mathcal{G}_2$, and since $\Phi_{\tilde{g}^{-1}} \circ \Phi_{\tilde{g}} = \Phi_{\tilde{g}^{-1}\tilde{g}} = \Phi_{e_{G_2}}$, we obtain that $\Phi_{\tilde{g}}$ and $\Phi_{\tilde{g}^{-1}}$ are inverse maps. Then, by Lemma 2.13, Proposition 2.14.*(1)* is proved.

Now Proposition 2.14.*(2)* follows directly from the fact that Φ is then well-defined, and that we have just proved that $\Phi_{\tilde{g}} \circ \Phi_{\tilde{h}} = \Phi_{\tilde{g}\tilde{h}}$ for $\tilde{g}, \tilde{h} \in G_2$. □

We have all the ingredients to show that $G_2 \cong \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}_2)$.

Lemma 2.15. *The morphism $\Phi : G_2 \to \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}_2)$ from Proposition 2.14 is an isomorphism.*

Proof. It is straightforward to show that Φ is a monomorphism since for any $\tilde{g} \in G_2$, $\Phi_{\tilde{g}}(e_{G_2}) = \tilde{g}^{-1}$. To prove that it is an epimorphism, we need to show that every automorphism of $\mathcal{G}_2$ is of the form $\Phi_{\tilde{g}}$ for some $\tilde{g} \in G_2$.

Let ψ be an automorphism of $\mathcal{G}_2$. Then ψ must respect the edges $R_i(\mathcal{G}_2)$, $i \in I$, which in particular implies that $\psi(G_2)$ is contained in G_2. Moreover, $\psi|_{\mathcal{G}_2(G_2)}$ is an automorphism of the full relational subsystem $\mathcal{G}_2(G_2) = \mathrm{Cay}(G_2, S)$ that must be induced by an element $\tilde{g} \in G_2$. That is, the map $\psi|_{\mathcal{G}_2(G_2)}$ is $\phi_{\tilde{g}}$ introduced in Remark 1.10. We claim that $\psi = \Phi_{\tilde{g}}$.

By construction, we have that $\psi|_{\mathcal{G}_2(G_2)} = \Phi_{\tilde{g}}|_{\mathcal{G}_2(G_2)} \in \mathrm{Aut}_{I\mathcal{R}el}(\mathrm{Cay}(G_2,S))$. Now, the only edge in $R_\theta(\mathcal{G}_2)$ starting at $g \in G_2$ is $(g,(j_2(g),\theta^{-1}[k_2(g)]))$ and, since $\psi(g) = \Phi_{\tilde{g}}(g)$, we have that $\psi(j_2(g),\theta^{-1}[k_2(g)]) = \Phi_{\tilde{g}}(j_2(g),\theta^{-1}[k_2(g)])$. In particular for $g = s_j$, $j \in J_2$ we get that $\psi(j,[e_{G_1}]) = \Phi_{\tilde{g}}(j,[e_{G_1}])$ and therefore for $j \in J_2$, $(\psi^{-1} \circ \Phi_{\tilde{g}})(j,[e_{G_1}]) = (j,[e_{G_1}])$. Using that composition must also preserve edges in $R_i(\mathcal{G}_2)$, for all $i \in I_1$, we obtain that $(\psi^{-1} \circ \Phi_{\tilde{g}})(V_2^j) \subset V_2^j$, for all $j \in J_2$. In fact, $\psi^{-1} \circ \Phi_{\tilde{g}}$ induces an automorphism of the corresponding copy of $\mathcal{G}_{\iota_1}$. Then, by Lemma 2.11, $\psi^{-1} \circ \Phi_{\tilde{g}}$ restricted to V_2^j must be the identity, $j \in J_2$, so we conclude that $\psi = \Phi_{\tilde{g}}$. □

2.1.3 Main theorem in Section 2.1

We now have the necessary ingredients to prove our main theorem in Section 2.1:

Theorem 2.16. *Let G_1 and G_2 be arbitrary groups and $H \leq G_1 \times G_2$. There exists a morphism of binary relational systems over a certain set I, $\varphi\colon \mathcal{G}_1 \to \mathcal{G}_2$, such that $\mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}_1)$, $\mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}_2)$ and $\mathrm{Aut}_{I\mathcal{R}el}(\varphi)$ are respectively isomorphic to G_1, G_2 and H.*

Proof. Consider the morphism $\varphi\colon \mathcal{G}_1 \to \mathcal{G}_2$ from Definition 2.10. As we have mentioned at the beginning of Section 2.1.2, since $\mathcal{G}_1$ is a Cayley diagram for G_1, we have that $\mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}_1) \cong G_1$. Also, from Lemma 2.15, we have that $\mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}_2) \cong G_2$. It only remains to show that $\mathrm{Aut}_{I\mathcal{R}el}(\varphi) \cong H$.

First consider $(\phi_{\tilde{g}_1}, \Phi_{\tilde{g}_2}) \in \mathrm{Aut}_{I\mathcal{R}el}(\varphi)$, where $\phi_{\tilde{g}_1}$ is the automorphism of $\mathcal{G}_1 = \mathrm{Cay}(G_1,S)$ from Remark 1.10, and $\Phi_{\tilde{g}_2}$ the automorphism of $\mathcal{G}_2$ constructed in Definition 2.12 for $\tilde{g}_2 \in G_2$.

We are going to show that $(\tilde{g}_1, \tilde{g}_2) \in H$. Indeed, since $(\phi_{\tilde{g}_1}, \Phi_{\tilde{g}_2}) \in \mathrm{Aut}_{\mathcal{IRel}}(\varphi)$, we have that $\Phi_{\tilde{g}_2} \circ \varphi = \varphi \circ \phi_{\tilde{g}_1}$. Now, by construction

$$\begin{aligned}
&\Phi_{\tilde{g}_2} \circ \varphi(e_{G_1}) = \Phi_{\tilde{g}_2}(0, [e_{G_1}]) = (j_2(\tilde{g}_2^{-1}), \theta^{-1}[k_2(\tilde{g}_2^{-1})]),\\
&\varphi \circ \phi_{\tilde{g}_1}(e_{G_1}) = \varphi(\tilde{g}_1^{-1}) = (0, \varphi_0(\tilde{g}_1^{-1})),
\end{aligned}$$

and therefore $j_2(\tilde{g}_2^{-1}) = 0$. So $\tilde{g}_2^{-1} \in \pi_2(H)$ and $k_2(\tilde{g}_2^{-1}) = \tilde{g}_2^{-1}$, which from the previous equations, leads us to $\varphi_0(\tilde{g}_1^{-1}) = \theta^{-1}[\tilde{g}_2^{-1}]$. This implies that $\tilde{g}_1^{-1} \in \pi_1(H)$ and moreover, $\theta([\tilde{g}_1^{-1}]) = [\tilde{g}_2^{-1}]$. Hence, by Lemma 2.1, we obtain that $(\tilde{g}_1^{-1}, \tilde{g}_2^{-1}) \in H$, and therefore $(\tilde{g}_1, \tilde{g}_2) \in H$.

Now we prove that for $(\tilde{g}_1, \tilde{g}_2) \in H$, the couple $(\phi_{\tilde{g}_1}, \Phi_{\tilde{g}_2}) \in \mathrm{Aut}_{\mathcal{IRel}}(\mathcal{G}_1) \times \mathrm{Aut}_{\mathcal{IRel}}(\mathcal{G}_2)$ is indeed an element in $\mathrm{Aut}_{\mathcal{IRel}}(\varphi)$. For that, we need to show that $(\varphi \circ \phi_{\tilde{g}_1})(g) = (\Phi_{\tilde{g}_2} \circ \varphi)(g)$, for every $g \in G_1$. First observe that since $(\tilde{g}_1, \tilde{g}_2) \in H$, we have that $(\tilde{g}_1^{-1}, \tilde{g}_2^{-1}) \in H$ and therefore $\theta^{-1}[\tilde{g}_2^{-1}] = [\tilde{g}_1^{-1}]$. Now, on the one hand,

$$\varphi \circ \phi_{\tilde{g}_1}(g) = \varphi(g\tilde{g}_1^{-1}) = (0, \varphi_0(g\tilde{g}_1^{-1})) = \begin{cases} (0, [g\tilde{g}_1^{-1}]), & \text{if } g \in \pi_1(H),\\ (0, s), & \text{otherwise.} \end{cases}$$

On the other hand, if $g \in \pi_1(H)$,

$$\Phi_{\tilde{g}_2} \circ \varphi(g) = \Phi_{\tilde{g}_2}(0, [g]) = (0, [g]\theta^{-1}[\tilde{g}_2^{-1}]) = (0, [g\tilde{g}_1^{-1}]),$$

and, if $g \notin \pi_1(H)$, then $\Phi_{\tilde{g}_2} \circ \varphi(g) = \Phi_{\tilde{g}_2}(0, s) = (0, s)$. The result thus follows. $\square$

2.1.4 An illustrative example

Let us illustrate the constructions involved in the proof of Theorem 2.16 with the example mentioned at the beginning of Section 2.1. Let $G_1 = \mathbb{Z}_8$, $G_2 = \mathbb{Z}_4$ and let $H \leq G_1 \times G_2$ be the subgroup generated by the element $(2, 2) \in G_1 \times G_2$, namely, $H = \{(0, 0), (2, 2), (4, 0), (6, 2)\}$. We are now going to describe the isomorphism θ from Lemma 2.1.

First, $\pi_1(H) = \langle 2\rangle \leq \mathbb{Z}_8$, and $\iota_1^{-1}(H) = \langle 4\rangle \leq \mathbb{Z}_8$. The quotient $\pi_1(H)/\iota_1^{-1}(H)$ is isomorphic to $\mathbb{Z}_2$, containing the classes $[0] = \{0, 4\}$ and $[2] = \{2, 6\}$. In a similar way, $\pi_2(H) = \langle 2\rangle \leq \mathbb{Z}_4$, and $\iota_2^{-1}(H)$ is the trivial group, so the quotient $\pi_2(H)/\iota_2^{-1}(H) \cong \pi_2(H)$ contains the classes $[0] = \{0\}$ and $[2] = \{2\}$. Hence, we have:

$$\begin{aligned}
\theta\colon \frac{\pi_1(H)}{\iota_1^{-1}(H)} &\longrightarrow \frac{\pi_2(H)}{\iota_2^{-1}(H)},\\
[0] = \{0, 4\} &\longmapsto [0] = \{0\},\\
[2] = \{2, 6\} &\longmapsto [2] = \{2\}.
\end{aligned}$$

We now describe the generating sets R and S for G_1 and G_2 respectively (see Definition 2.2):

(1) There are four right cosets of $\iota_1^{-1}(H)$ in G_1. Let $J_1 = \{0, 1, 2, 3\}$ and denote $r_j = j \in G_1$, $j \in J_1$. Then the set $\{r_j \mid j \in J_1\}$ contains a representative of each right coset of $\iota_1^{-1}(H)$ in G_1. Moreover, if we denote $I_{\iota_1} = \{4\}$ and $r_4 = 4 \in G_1$, $\{r_i \mid i \in I_{\iota_1}\}$ is a generating set for $\iota_1^{-1}(H)$. Taking $I_1 = I_{\iota_1} \sqcup J_1^* = \{1, 2, 3, 4\}$, the set $R = \{r_i \mid i \in I_1\} = \{1, 2, 3, 4\}$ is our generating set for G_1. We also need to consider the set $J_{\pi_1} = \{j \in J_1 \mid r_j \in \pi_1(H)\} = \{0, 2\}$ introduced in Remark 2.3.

(2) There are two right cosets of $\pi_2(H)$ in G_2, so take $J_2 = \{0, 1\}$. If we denote $s_j = j \in G_2$, $\{s_j \mid j \in J_2\}$ contains a representative of each of the two right cosets. Set $I_{\pi_2} = \{2\}$ and $s_2 = 2 \in G_2$, so $\{s_i \mid i \in I_{\pi_2}\}$ is a generating set for $\pi_2(H)$. Taking $I_2 = I_{\pi_2} \sqcup J_2^* = \{1, 2\}$, the set $S = \{s_i \mid i \in I_2\} = \{1, 2\}$ is our generating set for G_2.

Let $I = I_1 \sqcup I_2 \sqcup \{\theta\}$. We first build the auxiliary I-system $\mathcal{G}_{\iota_1}$ introduced in Definition 2.5. Since $\pi_1(H) \neq G_1$, the set of vertices is $V(\mathcal{G}_{\iota_1}) = V_1 \sqcup \{s\}$, where $V_1 = \pi_1(H)/\iota_1^{-1}(H) = \{[0],[2]\}$, and the labelled edges are shown in Figure 2.1. Note that in the following diagrams, a two-headed arrow means that there is an edge of the corresponding label in each direction.

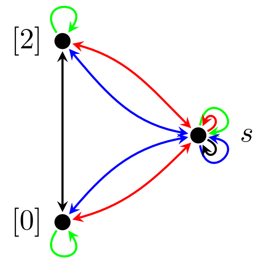

Figure 2.1: $\mathcal{G}_{\iota_1}$

According to Definition 2.7, $\mathcal{G}_1 = \mathrm{Cay}(G_1, R)$. Using the same colours as in Figure 2.1 to represent labels, we get:

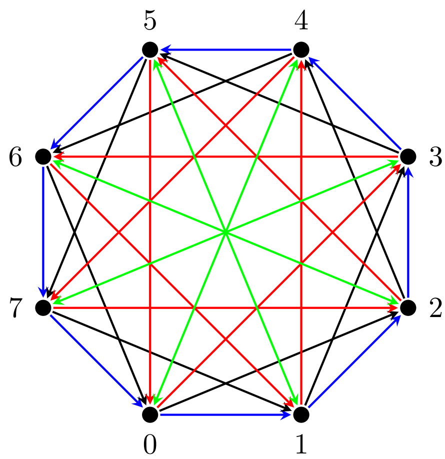

Figure 2.2: $\mathcal{G}_1 = \mathrm{Cay}(G_1, R)$

Finally, $\mathcal{G}_2$ has $\mathrm{Cay}(G_2, S)$ as a full binary I-subsystem and two copies of $\mathcal{G}_{\iota_1}$ (as many as elements in J_2). Moreover, it has edges labelled by θ starting at each vertex in $\mathrm{Cay}(G_2, S)$. The binary relational system $\mathcal{G}_2$ is then as follows:

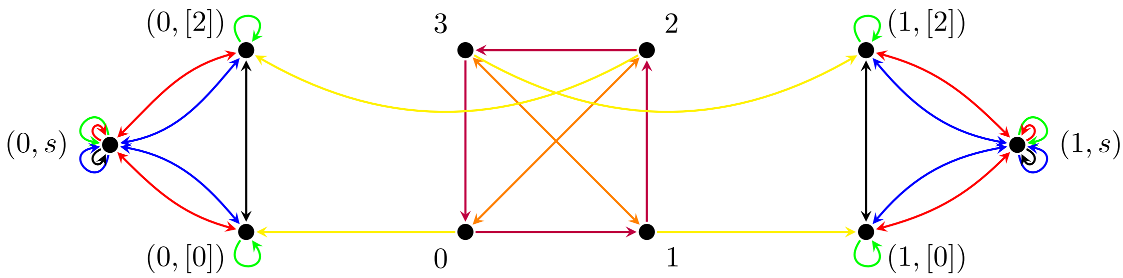

Figure 2.3: $\mathcal{G}_2$

The morphism of binary I-systems $\varphi\colon \mathcal{G}_1 \to \mathcal{G}_2$ is easily obtained from Definition 2.10.

2.2 Realisability of permutation representations in binary relational systems

In [43], Frucht proved that every group is the automorphism group of a graph, but he also proved that the problem of realising permutation groups in the context of graphs has a negative answer. Namely, there are permutation groups $G \hookrightarrow \mathrm{Sym}(V)$ such that a graph $\mathcal{G}$ with $V(\mathcal{G}) = V$ and $\mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \cong G$ does not exist, see [20, Section 4].

However, if we allow the set of vertices to be enlarged, the next result can be proven:

Theorem 2.17 ([19, Theorem 1.1]). *Let $\rho\colon G \hookrightarrow \mathrm{Sym}(V)$ be a finite permutation group. There is a graph $\mathcal{G}$ such that*

(1) $V \subset V(\mathcal{G})$ and V is invariant through the automorphisms of $\mathcal{G}$;

(2) $\mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \cong G$;

(3) the obvious restriction map $G \cong \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \longrightarrow \mathrm{Sym}(V)$ is ρ.

Having in mind the generalisation of Theorem 2.17 to any permutation representation of a group G on a set V, in this section we prove that for any permutation group (and, in fact, for any permutation representation) there is a binary relational system verifying properties akin to the ones listed above, see Theorem 2.26. Then, we will transfer the solution from $I\mathcal{R}el$ to $\mathcal{G}raphs$, providing a generalisation of Theorem 2.17 in Section 2.3.3, see Theorem 2.41.

Throughout this section, we consider $\rho\colon G \to \mathrm{Sym}(V)$ a permutation representation of an arbitrary group G on a set V and $S = \{s_j \mid j \in J\}$ a generating set for G. Let us begin by introducing the binary I-system $\mathcal{G}$ that allows us to generalise Theorem 2.17.

Definition 2.18. Take $I = J \sqcup V$ and define a binary I-system $\mathcal{G}$ with vertex set $V(\mathcal{G}) = G \sqcup V$ and edges:

- for each $j \in J$ and for $g \in G$, $(g, s_j g) \in R_j(\mathcal{G})$.
- for each $v \in V$ and for $g \in G$, $(g, \rho(g^{-1})(v)) \in R_v(\mathcal{G})$.

Remark 2.19. Notice that the full binary I-subsystem of $\mathcal{G}$ with vertex set G is precisely $\mathrm{Cay}(G, S)$, see Definition 1.9. We denote such subsystem by $\mathcal{G}(G)$. Also, notice that the full binary I-subsystem of $\mathcal{G}$ with vertex set V has no edges.

We now proceed to prove that $\mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}) \cong G$. In order to do so, we show that any element $\tilde{g} \in G$ induces an automorphism $\Phi_{\tilde{g}}$ on $\mathcal{G}$. We begin by constructing $\Phi_{\tilde{g}}$.

Definition 2.20. Given $\tilde{g} \in G$, we define $\Phi_{\tilde{g}}\colon V(\mathcal{G}) = G \sqcup V \to V(\mathcal{G})$ as follows. First, given that $\mathrm{Cay}(G, S)$ is a full relational subsystem of $\mathcal{G}$ (see Remark 2.19), we define $\Phi_{\tilde{g}}|_G$ as $\phi_{\tilde{g}}$ the automorphism induced by right multiplication by $\tilde{g}^{-1}$ introduced in Remark 1.10. Thus for $g \in G$,

$$\Phi_{\tilde{g}}(g) = g\tilde{g}^{-1}.$$

On the other hand, for $v \in V$, we define

$$\Phi_{\tilde{g}}(v) = \rho(\tilde{g})(v).$$

The previous self-map of $V(\mathcal{G})$ is then a morphism of binary relational systems:

Lemma 2.21. *For any $g \in G$, $\Phi_{\tilde{g}} \in \mathrm{Hom}_{I\mathcal{R}el}(\mathcal{G}, \mathcal{G})$.*

Proof. We need to prove that $\Phi_{\tilde{g}}$ respects relations $R_i(\mathcal{G})$, $i \in I = J \sqcup V$. First, for $j \in J$, $(g, s_j g) \in R_j(\mathcal{G})$. And $\big(\Phi_{\tilde{g}}(g), \Phi_{\tilde{g}}(s_j g)\big) = (g\tilde{g}^{-1}, s_j g\tilde{g}^{-1}) \in R_j(\mathcal{G})$. On the other hand, for $v \in V$, $(g, \rho(g^{-1})(v)) \in R_v(\mathcal{G})$. And since ρ is a group homomorphism,

$$\big(\Phi_{\tilde{g}}(g), \Phi_{\tilde{g}}(\rho(g^{-1})(v))\big) = \big(g\tilde{g}^{-1}, \rho(\tilde{g})(\rho(g^{-1})(v))\big) = \big(g\tilde{g}^{-1}, \rho(g\tilde{g}^{-1})^{-1}(v)\big) \in R_v(\mathcal{G}).$$

Thus $\Phi_{\tilde{g}}$ is indeed a morphism of binary relational systems. □

We can now prove that the construction from Definition 2.20 is an automorphism of the relational system $\mathcal{G}$ and defines a group homomorphism Φ as follows:

Proposition 2.22. *The following hold:*

(1) Given $\tilde{g} \in G$, $\Phi_{\tilde{g}} \in \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G})$.

(2) The following map is a group homomorphism:

$$\begin{aligned} \Phi\colon G &\longrightarrow \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}) \\ \tilde{g} &\longmapsto \Phi_{\tilde{g}}. \end{aligned}$$

Proof. Let us begin by proving that for $\tilde{g}, \tilde{h} \in G$, $\Phi_{\tilde{g}} \circ \Phi_{\tilde{h}} = \Phi_{\tilde{g}\tilde{h}}$. Indeed, if $g \in G$,

$$(\Phi_{\tilde{g}} \circ \Phi_{\tilde{h}})(g) = \Phi_{\tilde{g}}(\Phi_{\tilde{h}}(g)) = \Phi_{\tilde{g}}(g\tilde{h}^{-1}) = g\tilde{h}^{-1}\tilde{g}^{-1} = g(\tilde{g}\tilde{h})^{-1} = \Phi_{\tilde{g}\tilde{h}}(g).$$

And on the other hand, if $v \in V$ and since ρ is a group homomorphism,

$$(\Phi_{\tilde{g}} \circ \Phi_{\tilde{h}})(v) = \Phi_{\tilde{g}}(\Phi_{\tilde{h}}(v)) = \Phi_{\tilde{g}}(\rho(\tilde{h})(v)) = \rho(\tilde{g})(\rho(\tilde{h})(v)) = \rho(\tilde{g}\tilde{h})(v) = \Phi_{\tilde{g}\tilde{h}}(v).$$

Then $\Phi_{\tilde{g}}$ is bijective, as $\Phi_{\tilde{g}} \circ \Phi_{\tilde{g}^{-1}} = \Phi_{e_G}$ is clearly the identity, thus Proposition 2.22.*(1)* follows. And, since Φ is well-defined and we have proven that $\tilde{g}, \tilde{h} \in G$, $\Phi_{\tilde{g}} \circ \Phi_{\tilde{h}} = \Phi_{\tilde{g}\tilde{h}}$, Φ is a group homomorphism, proving Proposition 2.22.*(2)*. □

To show that $\mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}) \cong G$, it remains to prove that Φ is bijective:

Lemma 2.23. *The morphism $\Phi\colon G \to \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G})$ from Proposition 2.22 is an isomorphism.*

Proof. It is straightforward to show that Φ is a monomorphism since for any $\tilde{g} \in G$, $\Phi_{\tilde{g}}(e_G) = \tilde{g}^{-1}$. To prove that it is an epimorphism, we need to show that every automorphism of $\mathcal{G}$ is of the form $\Phi_{\tilde{g}}$ for some $\tilde{g} \in G$.

Take $\psi \in \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G})$. Notice that the only vertices of $\mathcal{G}$ that are starting vertices of edges labelled v for some $v \in V$ are those in G. Thus, ψ must be invariant on G, so it must induce an automorphism on the full binary I-subsystem with vertex set G, that is, $\psi|_G \in \mathrm{Aut}_{I\mathcal{R}el}\left(\mathcal{G}(G)\right)$. But recall from Remark 2.19 that $\mathcal{G}(G) \cong \mathrm{Cay}(G, S)$. Consequently, by Remark 1.10, there must exist $\tilde{g} \in G$ such that $\psi|_G = \phi_{\tilde{g}}$. We shall prove that, in fact, $\psi = \Phi_{\tilde{g}}$.

We already know that $\psi|_G = \Phi_{\tilde{g}}|_G$. It remains to prove the equality for vertices in V. Thus take $v \in V$. We know that $\big(e_G, \rho(e_G)(v)\big) = (e_G, v) \in R_v(\mathcal{G})$. Then, $\big(\psi(e_G), \psi(v)\big) = \big(\phi_{\tilde{g}}(e_G), \psi(v)\big) = \big(\tilde{g}^{-1}, \psi(v)\big) \in R_v(\mathcal{G})$. But the only edge in $R_v(\mathcal{G})$ starting at $\tilde{g}^{-1}$ is $\big(\tilde{g}^{-1}, \rho(\tilde{g})(v)\big)$. Thus $\psi(v) = \rho(\tilde{g})(v) = \Phi_{\tilde{g}}(v)$, for all $v \in V$. Then $\psi = \Phi_{\tilde{v}}$. □

As a consequence of Proposition 2.22 and Lemma 2.23 we immediately obtain the following:

Corollary 2.24. $\mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}) \cong G$*, and every* $\psi \in \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G})$ *is invariant on* $V \subset V(\mathcal{G})$*.*

We finally need to consider what happens with the restriction of $\mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G})$ to V.

Lemma 2.25. *The restriction map* $G \cong \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}) \to \mathrm{Sym}(V)$ *is* ρ*. Moreover, there is a faithful action* $\bar{\rho}\colon G \cong \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}) \to \mathrm{Sym}\left(V(\mathcal{G}) \setminus V\right)$ *such that the restriction map* $G \cong \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}) \to \mathrm{Sym}\left(V(\mathcal{G})\right)$ *is* $\rho \oplus \bar{\rho}$*.*

Proof. Let $\tilde{g} \in G$. Then $\tilde{g}$ is represented in $\mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G})$ by $\Phi_{\tilde{g}}$, see Proposition 2.22. We first need to consider $\Phi_{\tilde{g}}|_V$. For each $v \in V$, by definition, $\Phi_{\tilde{g}}(v) = \rho(\tilde{g})(v)$. Consequently, $\Phi_{\tilde{g}}|_V = \rho(\tilde{g})$, for all $\tilde{g} \in G$. Thus the restriction map $G \cong \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}) \to \mathrm{Sym}(V)$ is ρ.

On the other hand, consider $\Phi_{\tilde{g}}|_{V(\mathcal{G})\setminus V}$. Since $V(\mathcal{G}) \setminus V = G$, we have $e_G \in V(\mathcal{G}) \setminus V$. Moreover, $\Phi_{\tilde{g}}(e_G) = \tilde{g}^{-1}$, for all $\tilde{g} \in G$. Consequently, the action $\bar{\rho}\colon G \to \mathrm{Sym}\left(V(\mathcal{G}) \setminus V\right)$ taking $\tilde{g} \in G$ to $\Phi_{\tilde{g}}|_{V(\mathcal{G})\setminus V}$ is faithful. Moreover, the restriction map $G \cong \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}) \to \mathrm{Sym}(V)$ is $\rho \oplus \bar{\rho}$, as claimed. □

Finally, summing up Corollary 2.24 and Lemma 2.25, we deduce our main result for this section:

Theorem 2.26. *Let* G *be a group,* V *be a set and take* $\rho\colon G \to \mathrm{Sym}(V)$ *a permutation representation of* G *on* V*. There is a binary relational system* $\mathcal{G}$ *over a set* I *such that*

(1) $V \subset V(\mathcal{G})$ *and each* $\psi \in \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G})$ *is invariant on* V*;*

(2) $\mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}) \cong G$*;*

(3) the restriction $G \cong \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}) \to \mathrm{Sym}(V)$ *is precisely* ρ*;*

(4) there is a faithful action $\bar{\rho}\colon G \cong \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}) \to \mathrm{Sym}\left(V(\mathcal{G})\setminus V\right)$ *such that the restriction map* $\mathcal{G} \cong \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}) \to \mathrm{Sym}\left(V(\mathcal{G})\right)$ *is* $\rho \oplus \bar{\rho}$*.*

2.3 Arrow replacement: from binary relational systems to simple graphs

In this section, we use classical ideas of Frucht [43] and de Groot [32] to transfer the solutions to Problem 1 and Problem 2 from binary relational I-systems to simple graphs.

The idea is to perform a replacement operation [49, Section 4.4] which roughly speaking, consists on assigning an asymmetric graph to each label of the binary I-system in such a way that these asymmetric graphs are pairwise non-isomorphic. Then, every labelled edge is substituted by its corresponding asymmetric graph, thus obtaining simple undirected graphs. If the vertex degrees in the asymmetric graphs are chosen carefully, we can ensure that automorphisms of the resulting graph map each asymmetric graph to a copy of itself, thus the asymmetric graphs play the role of the labelled directed edges.

We want to use this sort of techniques to transfer Theorem 2.16 and Theorem 2.26 from $I\mathcal{R}el$ to $\mathcal{G}raphs$. With that objective in mind, in Section 2.3.1 we give a general arrow replacement result, Theorem 2.33, that allows us to transfer many constructions from $I\mathcal{R}el$ to $\mathcal{G}raphs$. In particular, in Section 2.3.2 we answer positively Problem 1 in $\mathcal{G}raphs$, by transferring Theorem 2.16 into Theorem 2.37. And similarly, in Section 2.3.3, we transfer Theorem 2.26 into Theorem 2.41, obtaining thus a generalisation of Theorem 2.17.

2.3.1 A general arrow replacement result

Recall the definition of degrees of vertices in a graph and in a binary relational system, Definition 1.8. The purpose of this section is to prove an arrow replacement result that is powerful enough to allow us to transfer the solutions to Problems 1 and 2 from $I\mathcal{R}el$ to $\mathcal{G}raphs$.

Thus, assume that $\mathcal{G}_1'$ and $\mathcal{G}_2'$ are two binary I-systems and that $\varphi' \in \mathrm{Hom}_{I\mathcal{R}el}(\mathcal{G}_1', \mathcal{G}_2')$. We want to build two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ such that $\mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}_k) \cong \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}_k')$, $k = 1, 2$, and a morphism between them $\varphi \in \mathrm{Hom}_{\mathcal{G}raphs}(\mathcal{G}_1, \mathcal{G}_2)$ such that $\mathrm{Aut}_{\mathcal{G}raphs}(\varphi) \cong \mathrm{Aut}_{I\mathcal{R}el}(\varphi')$. Not only that, but we also want that $V(\mathcal{G}_k') \subset V(\mathcal{G}_k)$, in such a way that automorphisms of $\mathcal{G}_k$ are automorphisms of $\mathcal{G}_k'$ when restricted to $V(\mathcal{G}_k')$. Furthermore, this restriction should be enough to completely determine the considered automorphism.

Remark 2.27. Throughout this section, we assume that every vertex in both $\mathcal{G}_1'$ and $\mathcal{G}_2'$ has degree greater than three. This can be done with all generality, for otherwise we can add additional labels to I and connect every vertex in $\mathcal{G}_1'$ and $\mathcal{G}_2'$ with each other in both directions by edges of the new labels. This increases the degree of every vertex by twice the number of added labels, and it does not modify neither the automorphisms of $\mathcal{G}_1'$ and $\mathcal{G}_2'$ nor the morphisms between them.

Let us begin by building the graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ from the binary relational I-systems $\mathcal{G}_1'$ and $\mathcal{G}_2'$.

Definition 2.28. Let $\mathcal{G}_1'$ and $\mathcal{G}_2'$ be two binary I-systems for which there is a cardinal α such that $\deg(v) \leq \alpha$, for every $v \in V(\mathcal{G}_1') \cup V(\mathcal{G}_2')$. For every $i \in I$, let R_i be an asymmetric simple graph such that, aside for a vertex of degree one, every vertex has degree greater than α; and such that $R_i \ncong R_j$ if $i \neq j$. Such graphs exist as a consequence of Proposition 1.15. Finally, for $k \in \{1, 2\}$, $i \in I$ and $(v, w) \in R_i(\mathcal{G}_k')$, consider $R_i^{(v,w)}$ a graph isomorphic to R_i, and denote its vertex of degree one by $p_i^{(v,w)}$. We define $\mathcal{G}_k$ a simple graph with vertices and edges:

$$V(\mathcal{G}_k) = V(\mathcal{G}_k') \sqcup \big(\sqcup_{i \in I} \big(\sqcup_{(v,w) \in R_i(\mathcal{G}_k')} \big(V(R_i^{(v,w)}) \sqcup \{r_i^{(v,w)}\} \big) \big) \big),$$
$$E(\mathcal{G}_k) = \sqcup_{i \in I} \big(\sqcup_{(v,w) \in R_i(\mathcal{G}_k')} \big(E(R_i^{(v,w)}) \sqcup \{(v, r_i^{(v,w)}), (r_i^{(v,w)}, p_i^{(v,w)}), (p_i^{(v,w)}, w)\} \big) \big).$$

Notice that $V(\mathcal{G}_k') \subset V(\mathcal{G}_k)$, for $k \in \{1, 2\}$. We begin by proving that $V(\mathcal{G}_k')$ is invariant through the automorphisms of $\mathcal{G}_k$. Not only that, but we also prove that the restriction of the automorphisms of $\mathcal{G}_k$ to $V(\mathcal{G}_k')$ yields an automorphism of $\mathcal{G}_k'$.

Lemma 2.29. *Fix $k \in \{1, 2\}$. For any $\psi \in \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}_k)$, $\psi|_{V(\mathcal{G}_k')} \in \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}_k')$.*

Proof. Let us first prove that $V(\mathcal{G}_k')$ is invariant through the automorphisms of $\mathcal{G}_k$. We do so by computing the degrees of the vertices in $V(\mathcal{G}_k)$. Notice that the degree of $v \in V(\mathcal{G}_k') \subset V(\mathcal{G}_k)$ is the same in both $\mathcal{G}_k'$ and $\mathcal{G}_k$. Indeed, for each $(v, w) \in R_i(\mathcal{G}_k')$, there is an edge $(v, r_i^{(v,w)}) \in E(\mathcal{G}_k)$, and for each $(w, v) \in R_i(\mathcal{G}_k')$, there is another edge $(p_i^{(w,v)}, v) \in E(\mathcal{G}_k)$. Given that these are the only edges in $\mathcal{G}_k$ incident to v, our claim holds. The vertices $r_i^{(v,w)}$ and $p_i^{(v,w)}$ have degree two and three respectively, while the remaining vertices in each of the $R_i^{(v,w)}$, $(v, w) \in R_i(\mathcal{G}_2')$, have the same degree as in R_i, thus greater than α. Consequently, the set $V(\mathcal{G}_k')$ must remain invariant through the automorphisms of $\mathcal{G}_k$.

Now take $\psi \in \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}_k)$ and let us check that $\psi' = \psi|_{V(\mathcal{G}_k')} \in \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}_k')$. Since automorphisms of graphs respect the degrees of vertices, previous considerations on vertex degrees imply that ψ restricts to a bijective map $\psi|_{V(\mathcal{G}_k')} \colon V(\mathcal{G}_k') \to V(\mathcal{G}_k')$. Now, for $(v, w) \in R_i(\mathcal{G}_k')$, we have that $(v, r_i^{(v,w)}) \in E(\mathcal{G}_k)$, thus $(\psi(v), \psi(r_i^{(v,w)})) \in E(\mathcal{G}_k)$. Given that ψ

respects the degree of vertices, $\psi(r_i^{(v,w)}) = r_j^{(\psi(v),u)}$, for some vertex $u \in V(\mathcal{G}'_k)$ and $j \in I$. For the same reason, ψ restricts to an isomorphism $R_i^{(v,w)} \to R_j^{(\psi(v),w)}$. But this implies that $i = j$, so $u = \psi(w)$ and $(\psi(v), \psi(w)) \in R_i(\mathcal{G}'_k)$. Therefore, $\psi|_{V(\mathcal{G}'_k)} \in \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}'_k)$. □

We now prove that, in fact, the restriction map induces an isomorphism between the automorphism groups of $\mathcal{G}_k$ and $\mathcal{G}'_k$.

Lemma 2.30. *Fix $k \in \{1,2\}$. The restriction map $\Psi_k\colon \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}_k) \xrightarrow{\cong} \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}'_k)$ taking $\psi \in \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}_k)$ to $\Psi_k(\psi) = \psi|_{V(\mathcal{G}'_k)}$ is a group isomorphism.*

Proof. First, Ψ_k is clearly a group homomorphism, since the composition of the restriction maps is the restriction of their composition. Let us prove that Ψ_k is bijective by constructing its inverse map $\Phi_k\colon \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}'_k) \to \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}_k)$.

Take $\psi' \in \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}'_k)$. We can naturally define a map $\Phi_k(\psi') = \psi\colon V(\mathcal{G}_k) \to V(\mathcal{G}_k)$ as follows. A vertex $v \in V(\mathcal{G}'_k)$ is taken to $\psi(v) = \psi'(v)$ and, for $(v,w) \in R_i(\mathcal{G}_k)$, $i \in I$, define $\psi(r_i^{(v,w)}) = r_i^{(\psi'(v),\psi'(w))}$. Finally, define $\psi|_{R_i^{(v,w)}}\colon R_i^{(v,w)} \to R_i^{(\psi'(v),\psi'(w))}$ as the identity map between the two copies of R_i. Then it is clear that $\psi \in \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}_k)$. Moreover, $\psi' = \psi|_{V(\mathcal{G}'_j)}$, exhibiting that Φ_k is inverse to Ψ_k. Thus Ψ_k is a group isomorphism. □

We now move on to building $\varphi \in \mathrm{Hom}_{\mathcal{G}raphs}(\mathcal{G}_1, \mathcal{G}_2)$ from $\varphi' \in \mathrm{Hom}_{I\mathcal{R}el}(\mathcal{G}'_1, \mathcal{G}'_2)$. The construction is quite natural and follows the same ideas we used in the proof of Lemma 2.30 to build an automorphism of $\mathcal{G}_k$ from an automorphism of $\mathcal{G}'_k$, $k = 1, 2$.

Definition 2.31. Let $\varphi' \in \mathrm{Hom}_{I\mathcal{R}el}(\mathcal{G}'_1, \mathcal{G}'_2)$, where $\mathcal{G}'_1$ and $\mathcal{G}'_2$ are two binary I-systems for which there is a cardinal α such that $\deg(v) \leq \alpha$, for every $v \in V(\mathcal{G}'_1) \cup V(\mathcal{G}'_2)$. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the graphs introduced in Definition 2.28. We define a map $\varphi\colon V(\mathcal{G}_1) \to V(\mathcal{G}_2)$ as follows. For $v \in V(\mathcal{G}'_1)$, define $\varphi(v) = \varphi'(v) \in V(\mathcal{G}'_2) \subset V(\mathcal{G}_2)$; and for $i \in I$ and $(v,w) \in R_i(\mathcal{G}'_1)$, define $\varphi(r_i^{(v,w)}) = r_i^{(\varphi'(v),\varphi'(w))}$ and define $\varphi|_{R_i^{(v,w)}}\colon R_i^{(v,w)} \to R_i^{(\varphi'(v),\varphi'(w))}$ as the identity map between the two copies of R_i.

As we defined it, it is clear that $\varphi \in \mathrm{Hom}_{\mathcal{G}raphs}(\mathcal{G}_1, \mathcal{G}_2)$ and that $\varphi|_{V(\mathcal{G}'_1)} = \varphi'$. It only remains to prove the following:

Lemma 2.32. *If $\varphi' \in \mathrm{Hom}_{I\mathcal{R}el}(\mathcal{G}'_1, \mathcal{G}'_2)$ and $\varphi \in \mathrm{Hom}_{\mathcal{G}raphs}(\mathcal{G}_1, \mathcal{G}_2)$ are as introduced in Definition 2.31, then $\mathrm{Aut}_{I\mathcal{R}el}(\varphi') \cong \mathrm{Aut}_{\mathcal{G}raphs}(\varphi)$.*

Proof. Recall from Lemma 2.30 the isomorphisms $\Psi_k\colon \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}_k) \to \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}'_k)$ induced by the restriction maps, $k = 1, 2$. We shall prove that $(\phi_1, \phi_2) \in \mathrm{Aut}_{\mathcal{G}raphs}(\varphi)$ if and only if $(\Psi_1(\phi_1), \Psi_2(\phi_2)) \in \mathrm{Aut}_{I\mathcal{R}el}(\varphi')$.

Take $(\phi_1, \phi_2) \in \mathrm{Aut}_{\mathcal{G}raphs}(\varphi)$. Then, $\varphi \circ \phi_1 = \phi_2 \circ \varphi$. But notice that since $\varphi|_{V(\mathcal{G}'_1)} = \varphi'$ and given that $\Psi_1(\phi_1) = \phi_1|_{V(\mathcal{G}'_1)}$, $\varphi \circ \phi_1|_{V(\mathcal{G}'_1)} = \varphi' \circ \Psi_1(\phi_1)$. For similar reasons, we deduce that $\phi_2 \circ \varphi|_{V(\mathcal{G}'_1)} = \Psi_2(\phi_2) \circ \varphi'$. Therefore $\varphi' \circ \Psi_1(\phi_1) = \Psi_2(\phi_2) \circ \varphi'$, so $(\Psi_1(\phi_1), \Psi_2(\phi_2)) \in \mathrm{Aut}_{I\mathcal{R}el}(\varphi')$.

Reciprocally, take $(\phi'_1, \phi'_2) \in \mathrm{Aut}_{I\mathcal{R}el}(\varphi')$, so $\varphi' \circ \phi'_1 = \phi'_2 \circ \varphi'$. Consider $\phi_k = \Psi_k^{-1}(\phi'_k)$, so that $\phi'_k = \Psi_k(\phi_k) = \phi_k|_{V(\mathcal{G}'_k)}$, $k = 1, 2$. As above, we have that $\varphi \circ \phi_1|_{V(\mathcal{G}'_1)} = \varphi' \circ \phi'_1$ and that $\phi_2 \circ \varphi|_{V(\mathcal{G}'_1)} = \phi'_2 \circ \varphi'$, which by hypothesis are equal. Thus, for $v \in V(\mathcal{G}'_1)$, $(\varphi \circ \phi_1)(v) = (\phi_2 \circ \varphi)(v)$. We need to prove that this also holds for the remaining vertices of $\mathcal{G}_1$.

Take $r_i^{(v,w)} \in V(\mathcal{G}_1)$, for $i \in I$ and for $(v,w) \in R_i(\mathcal{G}'_1)$. On the one hand,

$$(\varphi \circ \phi_1)(r_i^{(v,w)}) = \varphi(r_i^{(\phi'_1(v),\phi'_1(w))}) = r_i^{((\varphi' \circ \phi'_1)(v),(\varphi' \circ \phi'_1)(w))}.$$

On the other hand,

$$(\phi_2 \circ \varphi)(r_i^{(v,w)}) = \phi_2(r_i^{(\varphi'(v),\varphi'(w))}) = r_i^{((\phi_2'\circ\varphi')(v),(\phi_2'\circ\varphi')(w))}.$$

Then, since $\varphi' \circ \phi_1' = \phi_2' \circ \varphi'$, we deduce that $(\varphi \circ \phi_1)(r_i^{(v,w)}) = (\phi_2 \circ \varphi)(r_i^{(v,w)})$.

Only the vertices in the graphs $R_i^{(v,w)}$ remain. But notice that for $\psi \in \{\phi_1, \phi_2, \varphi\}$, $\psi|_{R_i^{(v,w)}}$ is the identity map $\psi|_{R_i^{(v,w)}} \colon R_i^{(v,w)} \to R_i^{(\psi'(v),\psi'(w))}$, where ψ' is the restriction of ψ to $V(\mathcal{G}_1')$, for $\psi \in \{\phi_1, \varphi\}$, or to $V(\mathcal{G}_2')$, for $\psi = \phi_2$. By using that $\varphi' \circ \phi_1' = \phi_2' \circ \varphi'$ we deduce that $\phi_2 \circ \varphi$ and $\varphi \circ \phi_1$ are equal for all vertices in $R_i^{(v,w)}$.

Summing up, $\phi_2 \circ \varphi = \varphi \circ \phi_1$ thus $(\phi_1, \phi_2) \in \mathrm{Aut}_{\mathcal{G}raphs}(\varphi)$. We have thus proven that $\mathrm{Aut}_{\mathcal{G}raphs}(\varphi) \cong \mathrm{Aut}_{I\mathcal{R}el}(\varphi')$. □

Finally, by combining Lemma 2.29, Lemma 2.30 and Lemma 2.32, we deduce our main theorem for this section.

Theorem 2.33. *Let $\mathcal{G}_1'$ and $\mathcal{G}_2'$ be binary relational systems over a set I such that there is a cardinal α for which $\deg(v) \leq \alpha$, for every $v \in V(\mathcal{G}_1') \cup V(\mathcal{G}_2')$. Let $\varphi' \in \mathrm{Hom}_{I\mathcal{R}el}(\mathcal{G}_1', \mathcal{G}_2')$. There are graphs $\mathcal{G}_1, \mathcal{G}_2$ and a morphism of graphs $\varphi \colon \mathcal{G}_1 \to \mathcal{G}_2$ such that:*

(1) there is a subset $V(\mathcal{G}_k') \subset V(\mathcal{G}_k)$ invariant through the automorphisms of $\mathcal{G}_k$, $k = 1, 2$;

(2) if $\psi \in \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}_k)$, the restriction $\psi' = \psi|_{V(\mathcal{G}_k')}$ is in $\mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}_k')$, for $k = 1, 2$;

(3) the restriction map $\Psi_k \colon \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}_k) \xrightarrow{\cong} \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}_k')$ taking $\psi \in \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}_k)$ to $\Psi_k(\psi) = \psi|_{V(\mathcal{G}_k')}$ is a group isomorphism, for $k = 1, 2$;

(4) $\varphi|_{V(\mathcal{G}_1')} = \varphi' \colon V(\mathcal{G}_1') \to V(\mathcal{G}_2')$ and $\mathrm{Aut}_{I\mathcal{R}el}(\varphi') \cong \mathrm{Aut}_{\mathcal{G}raphs}(\varphi)$.

Note that the asymmetric graphs R_i constructed in Theorem 1.16 are infinite, as mentioned in Remark 1.17. Since we use them to build $\mathcal{G}_1$ and $\mathcal{G}_2$ in Definition 2.28, these graphs will also be infinite, even if $\mathcal{G}_1'$ and $\mathcal{G}_2'$ are finite. However, as we will see, we can only transfer the solutions to the realisability problems from graphs to topological spaces if the graphs involved in the solutions are finite. Thus, we are interested in a result similar to Theorem 2.33 building finite graphs from finite binary relational systems. We do so by a proof analogous to that of Theorem 2.33, but using the finite asymmetric graphs from Proposition 1.13, thus obtaining the following result:

Corollary 2.34. *Let $\mathcal{G}_1'$, $\mathcal{G}_2'$ be two finite binary I-systems. Let $\varphi' \colon \mathcal{G}_1' \to \mathcal{G}_2'$ be a morphism of binary I-systems. There are finite graphs $\mathcal{G}_1, \mathcal{G}_2$ and a morphism of graphs $\varphi \colon \mathcal{G}_1 \to \mathcal{G}_2$ such that:*

(1) there is a subset $V(\mathcal{G}_k') \subset V(\mathcal{G}_k)$ invariant through the automorphisms of $\mathcal{G}_k$, $k = 1, 2$;

(2) if $\psi \in \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}_k)$, the restriction $\psi' = \psi|_{V(\mathcal{G}_k')}$ is in $\mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}_k')$, for $k = 1, 2$;

(3) the restriction map $\Psi_k \colon \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}_k) \xrightarrow{\cong} \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}_k')$ taking $\psi \in \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}_k)$ to $\Psi_k(\psi) = \psi|_{V(\mathcal{G}_k')}$ is a group isomorphism, for $k = 1, 2$;

(4) $\varphi|_{V(\mathcal{G}_1')} = \varphi' \colon V(\mathcal{G}_1') \to V(\mathcal{G}_2')$, and $\mathrm{Aut}_{I\mathcal{R}el}(\varphi') \cong \mathrm{Aut}_{\mathcal{G}raphs}(\varphi)$.

Proof. First, recall that by Remark 2.27 we can assume that vertices in $\mathcal{G}_1'$ and $\mathcal{G}_2'$ have degree greater than three. By Proposition 1.13, for any positive integer n there exist n finite asymmetric graphs that can be used in an arrow replacement operation. Moreover, the highest of the degrees of their vertices is three. Since the degrees of the vertices in both $\mathcal{G}_1'$ and $\mathcal{G}_2'$ is at least four, this result follows from a proof analogous to that of Theorem 2.33. □

Remark 2.35. If the binary relational systems $\mathcal{G}_1'$ and $\mathcal{G}_2'$ involved in Theorem 2.33 do not have isolated vertices, the simple graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ introduced in Definition 2.28 do not have isolated vertices either. This is clear from their construction and from the fact that the asymmetric graphs from Proposition 1.15 do not have isolated vertices.

For similar reasons, if the binary I-systems $\mathcal{G}_1'$ and $\mathcal{G}_2'$ from Corollary 2.34 are connected (respectively if they do not have isolated vertices), the obtained graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ are also connected (respectively they do not have isolated vertices).

2.3.2 Realisability in the arrow category of simple graphs

In this section we make use of Theorem 2.33 to transfer the solution to the realisability problem from the arrow category of $I\mathcal{R}el$, Theorem 2.16, to the arrow category of simple graphs, thus proving Theorem 2.37. We also show that if the starting groups are finite, the graphs can also be chosen so that they are finite, Corollary 2.38.

Since Theorem 2.33 requires for the degrees of the vertices in the graphs involved to be bounded, our first step should be to compute the degrees of the vertices in the binary relational systems $\mathcal{G}_1$ and $\mathcal{G}_2$ that we built in Section 2.1.

Lemma 2.36. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the binary I-systems from Definition 2.7. Then:*

(1) Vertices in $\mathcal{G}_1$ have degree $2|I_1|$;

(2) Vertices in $\mathcal{G}_2$ have the following degree:

(a) for $g_2 \in G_2$, $\deg(g_2) = 2|I_2| + 1$;

(b) for $(j, [g_1]) \in V_2^j$, $g_1 \in \pi_1(H)$, $\deg((j, [g_1])) = 2|I_1| + |\iota_2^{-1}(H)|$;

(c) if a vertex $(j, s) \in V_2^j$ exists, then $\deg(j, s) = \deg(s) \geq 2|I_1|$, where $s \in V(\mathcal{G}_{\iota_1})$.

Proof. As $\mathcal{G}_1 = \mathrm{Cay}(G_1, R)$, Lemma 2.36.*(1)* is straightforward from Remark 1.12. To prove Lemma 2.36.*(2)(a)* recall that $\mathcal{G}_2(G_2)$ is a full binary relational subsystem of $\mathcal{G}_2$ isomorphic to $\mathrm{Cay}(G_2, S)$ (see Remark 2.8) and that there exists a unique edge in $R_\theta(\mathcal{G}_2)$ starting at g_2. Therefore $\deg^+(g_2) = |I_2| + 1$, $\deg^-(g_2) = |I_2|$ and $\deg(g_2) = 2|I_2| + 1$.

To prove Lemma 2.36.*(2)(b)*, let $(j, [g_1]) \in V_2^j$, for some $g_1 \in \pi_1(H)$. Recall that the full binary relational subsystem of $\mathcal{G}_2$ with vertices V_2^j and edges with labels in I_1 is isomorphic to $\mathcal{G}_{\iota_1}$. If $G_1 = \pi_1(H)$, $\mathcal{G}_{\iota_1}$ is isomorphic to $\mathrm{Cay}\left(V_1, \{[r_i] \mid i \in I_{\iota_1} \sqcup J_{\pi_1}^*\}\right)$. Hence, as in this case $J_{\pi_1} = J_1$, there are $|I_{\iota_1} \sqcup J_{\pi_1}^*| = |I_1|$ edges with labels in I_1 both starting and arriving at $(j, [g_1])$. If $G_1 \neq \pi_1(H)$, we also have to consider the edges $((j, [g_1]), (j, s))$ and $((j, s), (j, [g_1]))$ for every label in $J_1 \setminus J_{\pi_1}$. Thus, there are a total of $|I_{\iota_1} \sqcup J_{\pi_1}^*| + |J_1 \setminus J_{\pi_1}| = |I_1|$ edges with labels in I_1 both arriving and ending at $(j, [g_1])$. Since no other edges start at $(j, [g_1])$, we obtain that $\deg^+((j, [g_1])) = |I_1|$. To compute the indegree of $(j, [g_1])$ we still have to check how many edges labelled θ arrive at $(j, [g_1])$. Recall that edges in $R_\theta(\mathcal{G}_2)$ are of the form $(g, (j_2(g), \theta^{-1}[k_2(g)]))$, $g \in G_2$. Notice that the uniqueness of the decomposition $g_2 = s_{j_2(g_2)} k_2(g_2)$ (see Remark 2.3), implies that any pair (j, g), $j \in J_2$, $g \in \pi_2(H)$, appears exactly once as $(j_2(g_2), k_2(g_2))$ for some $g_2 \in G_2$. Then, there are as many such edges arriving at $(j, [g_1])$ as elements $g_2 \in \pi_2(H)$ verifying that $\theta^{-1}[g_2] = [g_1]$. Equivalently, there are as many edges labelled θ arriving at $(j, [g_1])$ as elements in the class of $\theta([g_1])$, hence there are $|\iota_2^{-1}(H)|$ such edges. Therefore, $\deg^-((j, [g_1])) = |I_1| + |\iota_2^{-1}(H)|$ and $\deg((j, [g_1])) = 2|I_1| + |\iota_2^{-1}(H)|$, proving Lemma 2.36.*(2)(b)*.

Finally, the degrees of vertices $(j, s) \in V_2^j$ are not entirely determined. However, these vertices only take part in $R_i(\mathcal{G}_2)$, $i \in I_1$, and as we mentioned above, for every $j \in J_2$, the binary relational subsystem with vertices V_2^j and edges with labels in I_1, is isomorphic to $\mathcal{G}_{i_1}$. Hence $\deg(j,s) = \deg(s)$, for $s \in V(\mathcal{G}_{i_1})$ and, as $(s,s) \in R_i(\mathcal{G}_{i_1})$ for every $i \in I_1$, $\deg(s) \geq 2|I_1|$ and Lemma 2.36.*(2)(c)* follows. $\square$

We can now give a positive answer to the realisability problem in the arrow category of simple graphs as mentioned at the beginning of this section.

Theorem 2.37. *Let G_1, G_2 and H be groups such that $H \leq G_1 \times G_2$. Then, there exist graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ and a morphism of graphs $\varphi\colon \mathcal{G}_1 \to \mathcal{G}_2$ such that $\mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}_1) \cong G_1$, $\mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}_2) \cong G_2$ and $\mathrm{Aut}_{\mathcal{G}raphs}(\varphi) \cong H$.*

Proof. By Theorem 2.16, there are two binary I-systems $\mathcal{G}_1'$ and $\mathcal{G}_2'$, introduced in Definition 2.7, and a morphism $\varphi'\colon \mathcal{G}_1' \to \mathcal{G}_2'$, introduced in Definition 2.10, such that $\mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}_1') \cong G_1$, $\mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}_2') \cong G_2$ and $\mathrm{Aut}_{I\mathcal{R}el}(\varphi') \cong H$. Let $\alpha = \max\{2|I_1|, 2|I_2| + |i_2^{-1}(H)|, \deg(s)\}$, $s \in V(\mathcal{G}_{i_1})$. Then every vertex in both $\mathcal{G}_1'$ and $\mathcal{G}_2'$ has degree at most α, as a consequence of Lemma 2.36. Consequently, Theorem 2.33 applies, and there are simple graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ and a morphism of graphs between them $\varphi\colon \mathcal{G}_1 \to \mathcal{G}_2$ verifying Theorem 2.33.*(1)–(4)*. In particular, by Theorem 2.33.*(3)*, $\mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}_k) \cong \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}_k') \cong G_k$, for $k = 1,2$, and by Theorem 2.33.*(4)*, $\mathrm{Aut}_{\mathcal{G}raphs}(\varphi) \cong \mathrm{Aut}_{I\mathcal{R}el}(\varphi') \cong H$. The result follows. $\square$

As mentioned, we will only be able to transfer solutions to realisability problems from $I\mathcal{R}el$ to $\mathcal{H}oTop$ if the graphs involved in the solutions are finite. However, the graphs arising from Theorem 2.33 are never finite, so the graphs obtained in Theorem 2.37 cannot be finite either. Nonetheless, we know from Theorem 1.14 that all finite groups can be realised as the automorphism group of a finite graph. Using this fact, the next result follows from Corollary 2.34 by a proof analogous to that of Theorem 2.37:

Corollary 2.38. *Let G_1 and G_2 be finite groups and $H \leq G_1 \times G_2$. There exist $\mathcal{G}_1$ and $\mathcal{G}_2$ finite objects in $\mathcal{G}raphs$ and $\varphi\colon \mathcal{G}_1 \to \mathcal{G}_2$ an object in $\mathrm{Arr}(\mathcal{G}raphs)$ such that $\mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}_1) \cong G_1$, $\mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}_2) \cong G_2$ and $\mathrm{Aut}_{\mathcal{G}raphs}(\varphi) \cong H$.*

Remark 2.39. As a consequence of Remark 2.35, the graphs in both Theorem 2.37 and Corollary 2.38 do not have isolated vertices.

2.3.3 Realisability of permutation groups in the category of simple graphs

The objective of this section is analogous to that of Section 2.3.2, although in this case we want to transfer the solution to Problem 2 from binary relational systems, Theorem 2.26, to simple graphs, Theorem 2.41.

Again, Theorem 2.33 provides us with the machinery necessary to accomplish the objective of this section. Thus we need to check that the binary relational system $\mathcal{G}$ introduced in Definition 2.18 verifies the requirements for the theorem, that is, we need to compute the degrees of its vertices.

Lemma 2.40. *Let $\mathcal{G}$ be the binary I-system introduced in Definition 2.18.*

(1) Vertices in G have degree $2|S| + |V|$.

(2) Vertices in V have degree $|G|$.

Proof. We begin by proving Lemma 2.40.*(1)*. Fix $g \in G$. First, recall from Remark 2.19 that the full binary subsystem of $\mathcal{G}$ with vertex set G is $\mathcal{G}(G) = \mathrm{Cay}(G,S)$. Thus g is the starting (respectively ending) vertex of exactly $|S|$ edges with labels in S. Furthermore, for each $v \in V$ there is precisely one edge labelled v starting at g, and no more edges start or end in vertices in G. Therefore, $\deg(g) = 2|S| + |V|$.

Now take $v \in V$. Then, for each $g \in G$, $\rho(g^{-1}) \in \mathrm{Sym}(V)$. This implies that each vertex $v \in V$ is connected with g by exactly one edge. As this holds for every $g \in G$, and since there are no other edges starting or ending at $v \in V$, $\deg(v) = |G|$, for all $v \in V$. Thus Lemma 2.40.*(2)* follows. □

We can finally build a simple graph fulfilling the conditions introduced at the beginning of this section.

Theorem 2.41. *Let G be a group, V be a set and $\rho\colon G \to \mathrm{Sym}(V)$ be a permutation representation of G on V. There is a graph $\mathcal{G}$ such that*

(1) $V \subset V(\mathcal{G})$ *and each* $\psi \in \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G})$ *is invariant on* V*;*

(2) $\mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \cong G$*;*

(3) the restriction $G \cong \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \to \mathrm{Sym}(V)$ *is precisely* ρ*;*

(4) there is a faithful action $\bar{\rho}\colon G \cong \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \to \mathrm{Sym}\left(V(\mathcal{G}) \setminus V\right)$ *such that the restriction map* $G \cong \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \to \mathrm{Sym}\left(V(\mathcal{G})\right)$ *is* $\rho \oplus \bar{\rho}$*.*

Proof. Let $\mathcal{G}'$ be the binary I-system introduced in Definition 2.18. As a consequence of Theorem 2.26, $\mathcal{G}'$ verifies properties analogous to Theorem 2.41.*(1)–(4)* in the category $I\mathcal{R}el$. By Lemma 2.40, the degrees of vertices in $\mathcal{G}'$ are bounded. Then, Theorem 2.33 applies, and we can obtain a graph $\mathcal{G}$ verifying Theorem 2.33.*(1)–(3)*. Let us prove that it verifies the required conditions.

First, by Theorem 2.33.*(1)*, $V(\mathcal{G}') \subset V(\mathcal{G})$. In particular, we have $V \subset V(\mathcal{G}') \subset V(\mathcal{G})$. Then, by Theorem 2.33.*(2)*, if $\psi \in \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G})$, then $\psi|_{V(\mathcal{G}')} \in \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}')$, thus ψ is invariant on $V \subset V(\mathcal{G}')$ by Theorem 2.26.*(1)*. Then Theorem 2.41.*(1)* holds.

Moving on, by Theorem 2.33.*(3)*, the restriction map $\mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \to \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}')$ taking $\psi \in \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G})$ to $\psi|_{V(\mathcal{G}')} \in \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}')$ is an isomorphism. Thus $\mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \cong \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}')$, which by Theorem 2.26.*(2)* is isomorphic to G. Consequently, Theorem 2.41.*(2)* holds.

We can now consider the identification $G \cong \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \cong \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}')$. Notice that by Theorem 2.33.*(3)*, the isomorphism $\mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \cong \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}')$ is just the restriction map. Then, since $V \subset V(\mathcal{G}')$, the restriction map $G \cong \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \to \mathrm{Sym}(V)$ is equivalent to the restriction map $G \cong \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}') \to \mathrm{Sym}(V)$, which by Theorem 2.26.*(3)* is precisely ρ. We obtain Theorem 2.41.*(3)*.

Finally, by a similar argument, the restriction map $G \cong \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \to \mathrm{Sym}\left(V(\mathcal{G}') \setminus V\right)$ is equivalent to the restriction map $G \cong \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}') \to \mathrm{Sym}\left(V(\mathcal{G}') \setminus V\right)$, which by Theorem 2.26.*(4)* is injective. Since $V(\mathcal{G}') \setminus V \subset V(\mathcal{G}) \setminus V$, the restriction map $\bar{\rho}\colon G \cong \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \to \mathrm{Sym}\left(V(\mathcal{G}) \setminus V\right)$ must also be injective, or equivalently, $\bar{\rho}$ is a faithful action of G on $V(\mathcal{G}) \setminus V$. And clearly, the restriction map $\mathcal{G} \cong \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \to \mathrm{Sym}\left(V(\mathcal{G})\right)$ is $\rho \oplus \bar{\rho}$, proving Theorem 2.41.*(4)*. □

As with the previous section, we are also interested in obtaining finite graphs in certain situations. Clearly, the binary I-system $\mathcal{G}$ introduced in Definition 2.18 is finite if G and V are both finite. The next result follows from Corollary 2.34 by a proof analogous to that of Theorem 2.41.

Corollary 2.42. *Let G be a finite group, V be a finite set and $\rho\colon G \to \mathrm{Sym}(V)$ be a permutation representation of G on V. There is a finite graph $\mathcal{G}$ such that*

(1) $V \subset V(\mathcal{G})$ *and each* $\psi \in \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G})$ *is invariant on* V*;*

(2) $\mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \cong G$*;*

(3) the restriction $G \cong \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \to \mathrm{Sym}(V)$ *is precisely* ρ*;*

(4) there is a faithful action $\bar{\rho}\colon G \cong \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \to \mathrm{Sym}\left(V(\mathcal{G}) \setminus V\right)$ *such that the restriction map* $\mathcal{G} \cong \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \to \mathrm{Sym}\left(V(\mathcal{G})\right)$ *is* $\rho \oplus \bar{\rho}$*.*

Remark 2.43. As a consequence of Remark 2.35, the graphs in both Theorem 2.41 and Corollary 2.42 do not have isolated vertices.

CHAPTER 3

Realisability problems in coalgebras

Although coalgebras associated to combinatorial objects have been introduced and studied in the past, group realisability problems were not considered in this framework. Thus, our aim here is to provide the first results in the subject. To do so, we define a faithful functor $C\colon \mathcal{D}igraphs \to \mathrm{Coalg}_{\Bbbk}$, Definition 3.4, and use it to prove that every group shows up naturally as a subgroup of the automorphism group of a coalgebra, Corollary 3.11. Moreover, every permutation group arises as the image of the automorphism group of a coalgebra on the permutations of a subset of its group-like elements, Theorem 3.13.

Indeed, the functor C mentioned above is introduced and studied in Section 3.1. We show that if $\mathcal{G}$ is a digraph, $G\big(C(\mathcal{G})\big) = V(\mathcal{G})$. In fact, any morphism $\sigma \in \mathrm{Hom}_{\mathcal{D}igraphs}(\mathcal{G}_1, \mathcal{G}_2)$ induces a morphism $C(\sigma) \in \mathrm{Hom}_{\mathrm{Coalg}_{\Bbbk}}\big(C(\mathcal{G}_1), C(\mathcal{G}_2)\big)$ such that $\sigma = C(\sigma)|_{G(C(\mathcal{G}_1))}$, see Lemma 3.3. Moreover, automorphisms of $C(\mathcal{G})$ yield automorphisms of $\mathcal{G}$ when restricted to the set of grouplike elements, Theorem 3.9. This is the key argument to show that the automorphism group of $C(\mathcal{G})$ has $\mathrm{Aut}_{\mathcal{D}igraphs}(\mathcal{G})$ as a split quotient, see Corollary 3.10. Then, as a consequence of Theorem 1.16, every group G arises as a split quotient of the automorphism group of a faithful G-coalgebra C, Corollary 3.11. In fact, G arises as the image of the restriction of $\mathrm{Aut}_{\mathrm{Coalg}_{\Bbbk}}(C)$ to its set of grouplike elements, $G(C)$.

Then, in Section 3.2 we use these coalgebras to study the generalised realisability problems in this context. Namely, in Theorem 3.12 we show that Problem 1 admits a partial positive solution in $\mathrm{Coalg}_{\Bbbk}$ that follows the same spirit as Corollary 3.11. Then, in Theorem 3.13 we show that permutation representations are realisable as the restriction of the automorphism group of a coalgebra to a certain invariant subset of its set of grouplike elements, providing a partial positive answer to Problem 2.

Finally, in Section 3.3 we gather the results above to study the isomorphism problem for groups through the existence of faithful actions on coalgebras. We prove two results. The first one, Theorem 3.16, shows that the isomorphism type of groups within a large family is determined by the existence of faithful group actions on the group-like elements of coalgebras. We later consider a smaller family of groups to be able to prove Theorem 3.18, a result that does not focus on the restriction of the actions to grouplike elements.

3.1 A faithful functor from digraphs to coalgebras

Let $\Bbbk$ be any field. In this section we build a faithful functor $C\colon \mathcal{D}igraphs \to \mathrm{Coalg}_{\Bbbk}$ such that for any digraph $\mathcal{G}$, $G\big(C(\mathcal{G})\big) = V(\mathcal{G})$, and in such a way that the automorphisms of $C(\mathcal{G})$

induce, when restricted to its set of grouplike elements, an automorphism of $\mathcal{G}$. We do so in Theorem 3.9, the main result in this section. This result allows us to obtain conclusions regarding the classical group realisability problem in the context of coalgebras, see Corollary 3.11.

Let $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ be a digraph. Recall the path coalgebra of $\mathcal{G}$, $\Bbbk\mathcal{G}$, was introduced in Definition 1.34. The coalgebra $C(\mathcal{G})$ is just the first stage of the coradical filtration of $\Bbbk\mathcal{G}$, that is, $C(\mathcal{G})$ is generated by the paths of length 0 (vertices) and 1 (edges). More precisely:

Definition 3.1. Let $\Bbbk$ be a field and $\mathcal{G}$ be a digraph. We define a coalgebra $C(\mathcal{G}) = (C, \Delta, \varepsilon)$ where $C = \Bbbk\{v \mid v \in V(\mathcal{G})\} \oplus \Bbbk\{e \mid e \in E(\mathcal{G})\}$ and where

- for each $v \in V(\mathcal{G})$, $\Delta(v) = v \otimes v$ and $\varepsilon(v) = 1$;
- for each $e = (v, w) \in E(\mathcal{G})$, $\Delta(e) = v \otimes e + e \otimes w$ and $\varepsilon(e) = 0$.

Remark 3.2. Since $C(\mathcal{G})$ is just the first stage of the coradical filtration of $\Bbbk\mathcal{G}$, the grouplike elements of $C(\mathcal{G})$ are precisely those of $\Bbbk\mathcal{G}$, that is, the vertices of the graph. Namely, $G(C(\mathcal{G})) = V(\mathcal{G})$.

We can now easily see that every morphism of digraphs induces a coalgebra morphism between the respective coalgebras associated to the digraphs.

Lemma 3.3. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two digraphs. Every $\sigma \in \mathrm{Hom}_{\mathcal{D}igraphs}(\mathcal{G}_1, \mathcal{G}_2)$ induces a morphism of coalgebras $C(\sigma)\colon C(\mathcal{G}_1) \to C(\mathcal{G}_2)$.*

Proof. Let $\sigma \in \mathrm{Hom}_{\mathcal{D}igraphs}(\mathcal{G}_1, \mathcal{G}_2)$. Define $C(\sigma)\colon C(\mathcal{G}_1) \to C(\mathcal{G}_2)$ as follows.

$$\begin{aligned} &C(\sigma)(v) = \sigma(v), && \text{for } v \in V(\mathcal{G}_1), \\ &C(\sigma)\big((v, w)\big) = \big(\sigma(v), \sigma(w)\big), && \text{for } (v, w) \in E(\mathcal{G}_1). \end{aligned}$$

Then simple computations show that $C(\sigma)$ is a coalgebra morphism. □

Clearly, when looking at the restriction of $C(\sigma)$ to the set of grouplike elements of $C(\mathcal{G}_1)$, we obtain a map $C(\sigma)|_{V(\mathcal{G}_1)}\colon V(\mathcal{G}_1) \to V(\mathcal{G}_2)$ which happens to be precisely σ. In particular, different morphisms of digraphs induce different morphisms of coalgebra between their images. Not only that, but the association made in Lemma 3.3 is clearly functorial, allowing us to define the following functor:

Definition 3.4. Let $\Bbbk$ be a field. We define a faithful functor $C\colon \mathcal{D}igraphs \to \mathrm{Coalg}_{\Bbbk}$ as follows. To a digraph $\mathcal{G}$ we associate $C(\mathcal{G})$ the digraph introduced in Definition 3.1, and to a morphism $\sigma \in \mathrm{Hom}_{\mathcal{D}igraphs}(\mathcal{G}_1, \mathcal{G}_2)$ we associate the coalgebra morphism $C(\sigma)\colon C(\mathcal{G}_1) \to C(\mathcal{G}_2)$ introduced in Lemma 3.3.

We now move on to the computation of the automorphism group of $C(\mathcal{G})$, for $\mathcal{G}$ a digraph. In order to do so, we first introduce a family of linear self-maps of $C(\mathcal{G})$, Definition 3.5, and show that they are in fact automorphisms of $C(\mathcal{G})$, Lemmas 3.6 and 3.7. Then, we show that no other automorphism of $C(\mathcal{G})$ exist, Lemma 3.8. By abuse of notation, given $\sigma \in \mathrm{Aut}_{\mathcal{D}igraphs}(\mathcal{G})$, we write σ also to denote the self-map of $E(\mathcal{G})$ that takes $e = (v, w) \in E(\mathcal{G})$ to $\big(\sigma(v), \sigma(w)\big) \in E(\mathcal{G})$, thus $\sigma(e) = \big(\sigma(v), \sigma(w)\big)$.

Definition 3.5. Let $\mathcal{G}$ be a digraph, $\Bbbk$ be a field and consider $C(\mathcal{G})$ the coalgebra introduced in Definition 3.1. Given $\sigma \in \mathrm{Aut}_{\mathcal{D}igraphs}(\mathcal{G})$ and two maps $\lambda\colon E(\mathcal{G}) \to \Bbbk$ and $\mu\colon E(\mathcal{G}) \to \Bbbk^{\times}$, we introduce a linear map $f^{\sigma}_{\lambda,\mu}\colon C(\mathcal{G}) \to C(\mathcal{G})$ defined as follows:

$$\begin{cases} f^{\sigma}_{\lambda,\mu}(v) = \sigma(v), & \text{for all } v \in V(\mathcal{G}), \\ f^{\sigma}_{\lambda,\mu}(e) = \lambda(e)\big(\sigma(w) - \sigma(v)\big) + \mu(e)\sigma(e), & \text{for all } e = (v, w) \in E(\mathcal{G}). \end{cases}$$

We begin by proving that the linear maps introduced in Definition 3.5 are morphisms of coalgebras.

Lemma 3.6. *Let $\mathcal{G}$ be a digraph, $\Bbbk$ be a field and consider $C(\mathcal{G})$ the coalgebra introduced in Definition 3.1. The linear self-maps $f^{\sigma}_{\lambda,\mu}\colon C(\mathcal{G}) \to C(\mathcal{G})$ introduced in Definition 3.5 are morphisms of coalgebras.*

Proof. We need to check that $\varepsilon \circ f^{\sigma}_{\lambda,\mu} = \varepsilon$ and that $\Delta \circ f^{\sigma}_{\lambda,\mu} = (f^{\sigma}_{\lambda,\mu} \otimes f^{\sigma}_{\lambda,\mu}) \circ \Delta$. We do the computations on the generators of $C(\mathcal{G})$ associated to vertices and edges of $\mathcal{G}$ separately.

Let $v \in V(\mathcal{G})$. Regarding the counit, on the one hand we have that $\varepsilon(v) = 1$, and on the other hand,

$$(\varepsilon \circ f^{\sigma}_{\lambda,\mu})(v) = \varepsilon\big(\sigma(v)\big) = 1.$$

Thus they are equal. Similarly, regarding the comultiplication, on the one hand

$$(\Delta \circ f^{\sigma}_{\lambda,\mu})(v) = \Delta\big(\sigma(v)\big) = \sigma(v) \otimes \sigma(v),$$

and on the other hand,

$$\big((f^{\sigma}_{\lambda,\mu} \otimes f^{\sigma}_{\lambda,\mu}) \circ \Delta\big)(v) = (f^{\sigma}_{\lambda,\mu} \otimes f^{\sigma}_{\lambda,\mu})(v \otimes v) = f^{\sigma}_{\lambda,\mu}(v) \otimes f^{\sigma}_{\lambda,\mu}(v) = \sigma(v) \otimes \sigma(v).$$

Again they are equal.

Now let us take $e = (v, w) \in E(\mathcal{G})$. First, regarding the counit, we know that $\varepsilon(e) = 0$, and on the other hand,

$$(\varepsilon \circ f^{\sigma}_{\lambda,\mu})(e) = \varepsilon\big(\lambda(e)(\sigma(w) - \sigma(v)) + \mu(e)\sigma(e)\big) = \lambda(e)\big(\varepsilon(\sigma(w)) - \varepsilon(\sigma(v))\big) + \mu(e)\varepsilon\big(\sigma(e)\big) = 0.$$

Finally, regarding the comultiplication, on the one hand,

$$\begin{aligned}(\Delta \circ f^{\sigma}_{\lambda,\mu})(e) &= \Delta\big(\lambda(e)(\sigma(w) - \sigma(v)) + \mu(e)\sigma(e)\big)\\ &= \lambda(e)\big(\Delta(\sigma(w)) - \Delta(\sigma(v))\big) + \mu(e)\Delta\big(\sigma(e)\big)\\ &= \lambda(e)\big(\sigma(w) \otimes \sigma(w) - \sigma(v) \otimes \sigma(v)\big) + \mu(e)\big(\sigma(v) \otimes \sigma(e) + \sigma(e) \otimes \sigma(w)\big),\end{aligned}$$

and on the other hand,

$$\begin{aligned}(f^{\sigma}_{\lambda,\mu}{\otimes}f^{\sigma}_{\lambda,\mu})\big(\Delta(e)\big) &= (f^{\sigma}_{\lambda,\mu} \otimes f^{\sigma}_{\lambda,\mu})(v \otimes e + e \otimes w) = f^{\sigma}_{\lambda,\mu}(v) \otimes f^{\sigma}_{\lambda,\mu}(e) + f^{\sigma}_{\lambda,\mu}(e) \otimes f^{\sigma}_{\lambda,\mu}(w)\\ &= \sigma(v) \otimes \big[\lambda(e)\big(\sigma(w) - \sigma(v)\big) + \mu(e)\big(\sigma(e)\big)\big] + \big[\lambda(e)\big(\sigma(w) - \sigma(v)\big) + \mu(e)\big(\sigma(e)\big)\big] \otimes \sigma(w)\\ &= \lambda(e)\big(\sigma(w) \otimes \sigma(w) - \sigma(v) \otimes \sigma(v)\big) + \mu(e)\big(\sigma(v) \otimes \sigma(e) + \sigma(e) \otimes \sigma(w)\big).\end{aligned}$$

Consequently, $f^{\sigma}_{\lambda,\mu}$ is a morphism of coalgebras. $\square$

We now prove that the maps introduced in Definition 3.5 are, in fact, automorphisms of coalgebras.

Lemma 3.7. *Let $\mathcal{G}$ be a digraph, $\Bbbk$ be a field and consider $C(\mathcal{G})$ the coalgebra introduced in Definition 3.1. Then $f^{\sigma}_{\lambda,\mu} \in \mathrm{Aut}_{\mathrm{Coalg}_{\Bbbk}}\big(C(\mathcal{G})\big)$.*

Proof. We proved in Lemma 3.6 that maps $f^{\sigma}_{\lambda,\mu}$ are morphisms of coalgebras. It remains to prove that they are automorphisms. We do so by proving that $f^{\sigma^{-1}}_{-\frac{\lambda}{\mu}\sigma^{-1},\frac{1}{\mu}\sigma^{-1}}$ is inverse to $f^{\sigma}_{\lambda,\mu}$. We first consider the composition $f^{\sigma^{-1}}_{-\frac{\lambda}{\mu}\sigma^{-1},\frac{1}{\mu}\sigma^{-1}} \circ f^{\sigma}_{\lambda,\mu}$. For $v \in V(\mathcal{G})$,

$$\big(f^{\sigma^{-1}}_{-\frac{\lambda}{\mu}\sigma^{-1},\frac{1}{\mu}\sigma^{-1}} \circ f^{\sigma}_{\lambda,\mu}\big)(v) = \big(f^{\sigma^{-1}}_{-\frac{\lambda}{\mu},\frac{1}{\mu}}\big)\big(\sigma(v)\big) = \sigma^{-1}\big(\sigma(v)\big) = v.$$

And, for $e=(v,w)\in E(\mathcal{G})$,

$$\begin{aligned}(f^{\sigma^{-1}}_{-\frac{\lambda}{\mu}\sigma^{-1},\frac{1}{\mu}\sigma^{-1}}\circ f^{\sigma}_{\lambda,\mu})(e) &= (f^{\sigma^{-1}}_{-\frac{\lambda}{\mu},\frac{1}{\mu}})\big(\lambda(e)(\sigma(w)-\sigma(v))+\mu(e)\sigma(e)\big)\\ &= \lambda(e)\big(f^{\sigma^{-1}}_{-\frac{\lambda}{\mu}\sigma^{-1},\frac{1}{\mu}\sigma^{-1}}(\sigma(w))-f^{\sigma^{-1}}_{-\frac{\lambda}{\mu}\sigma^{-1},\frac{1}{\mu}\sigma^{-1}}(\sigma(v))\big)+\mu(e)f^{\sigma^{-1}}_{-\frac{\lambda}{\mu}\sigma^{-1},\frac{1}{\mu}\sigma^{-1}}\sigma(e)\\ &= \lambda(e)(w-v)+\mu(e)\left(-\frac{\lambda(e)}{\mu(e)}(w-v)+\frac{1}{\mu(e)}(e)\right)=e.\end{aligned}$$

We also have to consider the composition $f^{\sigma}_{\lambda,\mu}\circ f^{\sigma^{-1}}_{-\frac{\lambda}{\mu}\sigma^{-1},\frac{1}{\mu}\sigma^{-1}}$. However, notice that $f^{\sigma}_{\lambda,\mu}$ is recovered from $f^{\sigma^{-1}}_{-\frac{\lambda}{\mu}\sigma^{-1},\frac{1}{\mu}\sigma^{-1}}$ by performing on the indexes the same operations that we perform to $f^{\sigma}_{\lambda,\mu}$ to obtain $f^{\sigma^{-1}}_{-\frac{\lambda}{\mu}\sigma^{-1},\frac{1}{\mu}\sigma^{-1}}$. Consequently, the proof above already shows that $f^{\sigma}_{\lambda,\mu}\circ f^{\sigma^{-1}}_{-\frac{\lambda}{\mu}\sigma^{-1},\frac{1}{\mu}\sigma^{-1}}$ is the identity map. Then, $f^{\sigma}_{\lambda,\mu}\in \mathrm{Aut}_{\mathrm{Coalg}_{\Bbbk}}(C(\mathcal{G}))$. □

We now prove that every coalgebra automorphism of $C(\mathcal{G})$ is of this form.

Lemma 3.8. *Let $\mathcal{G}$ be a digraph, $\Bbbk$ be a field and let $C(\mathcal{G})$ be the coalgebra from Definition 3.1. If $f\in \mathrm{Aut}_{\mathrm{Coalg}_{\Bbbk}}(C(\mathcal{G}))$, there exist $\sigma\in \mathrm{Aut}_{\mathcal{D}igraphs}(\mathcal{G})$, $\lambda\colon E(\mathcal{G})\to\Bbbk$ and $\mu\colon E(\mathcal{G})\to\Bbbk^{\times}$ such that f is the coalgebra automorphism $f^{\sigma}_{\lambda,\mu}$ introduced in Definition 3.5.*

Proof. Let $f\in \mathrm{Aut}_{\mathrm{Coalg}_{\Bbbk}}(C(\mathcal{G}))$ be a coalgebra automorphism. First notice that any automorphism of coalgebras must permute grouplike elements. By Remark 3.2, $G\big(C(\mathcal{G})\big)=V(\mathcal{G})$, thus there is a bijective map $\sigma\colon V(\mathcal{G})\to V(\mathcal{G})$ such that $f(v)=\sigma(v)$, for all $v\in V(\mathcal{G})$.

Now take $e\in E(\mathcal{G})$. Then there are, for every $x\in V(\mathcal{G})\cup E(\mathcal{G})$, elements $\gamma(e,x)\in\Bbbk$ such that

$$f(e)=\sum_{x\in V(\mathcal{G})\cup E(\mathcal{G})}\gamma(e,x)x. \tag{3.1}$$

In order for f to be a coalgebra morphism, it needs to verify that $\varepsilon\circ f=\varepsilon$ and that $(f\otimes f)\circ\Delta=\Delta\circ f$. We first consider the equality involving the counit. Recall from Definition 3.1 that $\varepsilon(e)=0$, for $e\in E(\mathcal{G})$. Thus,

$$0=\varepsilon(f(e))=\varepsilon\left(\sum_{x\in V(\mathcal{G})\cup E(\mathcal{G})}\gamma(e,x)x\right)=\sum_{x\in V(\mathcal{G})\cup E(\mathcal{G})}\gamma(e,x)\varepsilon(x)=\sum_{v\in V(\mathcal{G})}\gamma(e,v). \tag{3.2}$$

Now consider the equality regarding the comultiplication. Take $e=(v,w)\in E(\mathcal{G})$. Then, on the one hand,

$$\begin{aligned}(\Delta\circ f)(e)=\Delta\left(\sum_{y\in V(\mathcal{G})\cup E(\mathcal{G})}\gamma(e,y)y\right)&=\sum_{y\in V(\mathcal{G})\cup E(\mathcal{G})}\gamma(e,y)\Delta(y)\\ &=\sum_{u\in V(\mathcal{G})}\gamma(e,u)u\otimes u+\sum_{h=(r,s)\in E(\mathcal{G})}\gamma(e,h)[r\otimes h+h\otimes s].\end{aligned} \tag{3.3}$$

On the other hand,

$$\begin{aligned}((f\otimes f)\circ\Delta)(e)&=(f\otimes f)(v\otimes e+e\otimes w)=f(v)\otimes f(e)+f(e)\otimes f(w)\\ &=\sigma(v)\otimes\left(\sum_{y\in V(\mathcal{G})\cup E(\mathcal{G})}\gamma(e,y)y\right)+\left(\sum_{z\in V(\mathcal{G})\cup E(\mathcal{G})}\gamma(e,z)z\right)\otimes\sigma(w).\end{aligned} \tag{3.4}$$

Equations (3.3) and (3.4) must be equal. First, notice that $\sigma(v)\otimes\sigma(v)$ and $\sigma(w)\otimes\sigma(w)$ are the only summands of the form $u\otimes u$ with $u\in V(\mathcal{G})$ that may arise in Equation (3.4). Thus,

$\gamma(e,u) = 0$ if $u \neq \sigma(v), \sigma(w)$. Regarding the coefficients $\gamma(e,\sigma(v))$ and $\gamma(e,\sigma(w))$, notice that in Equation (3.4) we have the summand

$$[\gamma(e,\sigma(v)) + \gamma(e,\sigma(w))]\sigma(v) \otimes \sigma(w),$$

whereas $\sigma(v) \otimes \sigma(w)$ does not appear in Equation (3.3). Thus, $\gamma(e,\sigma(v)) = -\gamma(e,\sigma(w))$. Moreover, and since no further restrictions exist regarding these coefficients, $\gamma(e,\sigma(w)) \in \Bbbk$.

Finally, regarding the summands $r \otimes (r,s) + (r,s) \otimes s$ arising in Equation (3.3), the only possible non-trivial such summand in Equation (3.4) is $\sigma(v) \otimes (\sigma(v),\sigma(w)) + (\sigma(v),\sigma(w)) \otimes \sigma(w)$. Moreover, the corresponding coefficient $\gamma(e,(\sigma(v),\sigma(w)))$ must be non-trivial, since otherwise f would not be injective. We deduce that $(\sigma(v),\sigma(w)) \in E(\mathcal{G})$, and as a consequence, σ is in fact a morphism of graphs. An analogous reasoning for $f^{-1} \in \mathrm{Aut}_{\mathrm{Coalg}_\Bbbk}(C(\mathcal{G}))$ lets us deduce that σ^{-1} is a morphism of graphs as well, so in fact $\sigma \in \mathrm{Aut}_{\mathcal{D}igraphs}(\mathcal{G})$. Regarding the coefficient, no further restrictions exist, so $\gamma(e,(\sigma(v),\sigma(w))) \in \Bbbk^\times$.

We have thus obtained that there is a graph automorphism $\sigma \in \mathrm{Aut}_{\mathcal{D}igraphs}(\mathcal{G})$ such that

$$\begin{cases} f(v) = \sigma(v), & \text{for all } v \in V(\mathcal{G}), \\ f(e) = \gamma(e,\sigma(w))(\sigma(w) - \sigma(v)) + \gamma(e,\sigma(e))\sigma(e), & \text{for all } e = (v,w) \in E(\mathcal{G}), \end{cases}$$

where $\gamma(e,\sigma(w)) \in \Bbbk$ and $\gamma(e,\sigma(e)) \in \Bbbk^\times$. Consequently, if for every $e = (v,w) \in E(\mathcal{G})$ we define $\lambda(e) = \gamma(e,\sigma(w))$ and $\mu(e) = \gamma(e,\sigma(e))$, we obtain that $f = f^\sigma_{\lambda,\mu}$ as introduced in Lemma 3.7. The result follows. □

Now that we have computed the automorphism group of the coalgebras $C(\mathcal{G})$ introduced in Definition 3.1, we can prove the main result in this section.

Theorem 3.9. *Let $\Bbbk$ be a field and $\mathcal{G}$ be a digraph. There is a $\Bbbk$-coalgebra $C(\mathcal{G})$ such that $G(C(\mathcal{G})) = V(\mathcal{G})$ and the restriction map $\mathrm{Aut}_{\mathrm{Coalg}_\Bbbk}(C(\mathcal{G})) \to \mathrm{Sym}(G(C(\mathcal{G}))) = \mathrm{Sym}(V(\mathcal{G}))$ induces a split short exact sequence of groups*

$$1 \longrightarrow \prod_{e \in E(\mathcal{G})} (\Bbbk \rtimes \Bbbk^\times) \longrightarrow \mathrm{Aut}_{\mathrm{Coalg}_\Bbbk}(C(\mathcal{G})) \longrightarrow \mathrm{Aut}_{\mathcal{D}igraphs}(\mathcal{G}) \longrightarrow 1.$$

Proof. Let $C(\mathcal{G})$ be the coalgebra introduced in Definition 3.1. We shall prove that this is the desired coalgebra. As an immediate consequence of Lemma 3.7 and Lemma 3.8,

$$\mathrm{Aut}_{\mathrm{Coalg}_\Bbbk}(C(\mathcal{G})) = \{f^\sigma_{\lambda,\mu} \mid \sigma \in \mathrm{Aut}_{\mathcal{D}igraphs}(\mathcal{G}), \lambda \colon E(\mathcal{G}) \to \Bbbk, \mu \colon E(\mathcal{G}) \to \Bbbk^\times\}.$$

In particular, the map $\mathrm{Aut}_{\mathrm{Coalg}_\Bbbk}(C(\mathcal{G})) \to \mathrm{Sym}(G(C(\mathcal{G}))) = \mathrm{Sym}(V(\mathcal{G}))$ takes the automorphism $f^\sigma_{\lambda,\mu} \in \mathrm{Aut}_{\mathrm{Coalg}_\Bbbk}(C(\mathcal{G}))$ to $\sigma \in \mathrm{Sym}(V(\mathcal{G}))$. Indeed, for all $v \in V(\mathcal{G})$, $f^\sigma_{\lambda,\mu}(v) = \sigma(v)$. Therefore, the image of the map $\mathrm{Aut}_{\mathrm{Coalg}_\Bbbk}(C(\mathcal{G})) \to \mathrm{Sym}(V(\mathcal{G}))$ is $\mathrm{Aut}_{\mathcal{D}igraphs}(\mathcal{G})$, whereas the kernel is

$$K = \{f^{\mathrm{id}_\mathcal{G}}_{\lambda,\mu} \mid \lambda \colon E(\mathcal{G}) \to \Bbbk, \mu \colon E(\mathcal{G}) \to \Bbbk^\times\}.$$

Let us define $f_{\lambda,\mu} = f^{\mathrm{id}_\mathcal{G}}_{\lambda,\mu}$. We now proceed to prove that $K \cong \prod_{e \in E(\mathcal{G})} (\Bbbk \rtimes \Bbbk^\times)$.

First, let us see how the group operation works in K. Take $f_{\lambda,\mu}, f_{\lambda',\mu'} \in K$. Then, for $v \in V(\mathcal{G})$,

$$(f_{\lambda',\mu'} \circ f_{\lambda,\mu})(v) = f_{\lambda',\mu'}(v) = v,$$

and for $e = (v,w) \in E(\mathcal{G})$,

$$\begin{aligned}(f_{\lambda',\mu'} \circ f_{\lambda,\mu})(e) &= f_{\lambda',\mu'}(\lambda(e)(w-v) + \mu(e)e) \\ &= \lambda(e)(w-v) + \mu(e)(\lambda'(e)(w-v) + \mu'(e)e) \\ &= (\lambda(e) + \mu(e)\lambda'(e))(w-v) + \mu(e)\mu'(e)e.\end{aligned}$$

Consequently, $f_{\lambda',\mu'} \circ f_{\lambda,\mu} = f_{\lambda+\mu\lambda',\mu\mu'}$. Thus, the group operation of K acts independently on each of the elements of $E(\mathcal{G})$. This implies that K can be decomposed as a direct product of groups over $E(\mathcal{G})$. Let us focus on one of the factors, thus pick an edge $e \in E(\mathcal{G})$ and take

$$K_e = \left\{ f_{\lambda,\mu} \;\middle|\; \begin{array}{l} \lambda\colon E(\mathcal{G}) \to \Bbbk \text{ with } \lambda(e') = 0 \text{ for all } e' \neq e, \\ \mu\colon E(\mathcal{G}) \to \Bbbk^\times \text{ with } \mu(e') = 1 \text{ for } e' \neq e \end{array} \right\}.$$

Let us prove that K_e is a semidirect product of the form $\Bbbk \rtimes \Bbbk^\times$.

First, let us denote the maps taking every $e \in E(\mathcal{G})$ to $0_\Bbbk$ and $1_\Bbbk$ by $0\colon E(\mathcal{G}) \to \Bbbk$ and $1\colon E(\mathcal{G}) \to \Bbbk^\times$ respectively. Now consider the subsets of K_e given by $H_e = \{f_{\lambda,\mu} \in K_e \mid \lambda = 0\}$ and $N_e = \{f_{\lambda,\mu} \in K_e \mid \mu = 1\}$. Then, for $f_{0,\mu}, f_{0,\mu'} \in H_e$, $f_{0,\mu'} \circ f_{0,\mu} = f_{0,\mu\mu'}$, so H_e is a subgroup of K isomorphic to $\Bbbk^\times$. Similarly, for $f_{\lambda,1}, f_{\lambda',1} \in N_e$, $f_{\lambda',1} \circ f_{\lambda,1} = f_{\lambda+\lambda',1}$, thus N_e is a subgroup of K isomorphic to $\Bbbk$. Let us now check that $N_e \trianglelefteq K_e$ and that $K_e \cong N_e \rtimes H_e$. Consider the map

$$\begin{aligned} g_e\colon K_e &\longrightarrow H_e \\ f_{\lambda,\mu} &\longmapsto f_{0,\mu}. \end{aligned}$$

Then simple computations show that g_e is a group homomorphism. Moreover, it is clear that $N_e = \ker g_e$, which exhibits that $N_e \trianglelefteq K_e$ and that $K_e \cong N_e \rtimes H_e$. We deduce that

$$K = \prod_{e \in E(\mathcal{G})} K_e = \prod_{e \in E(\mathcal{G})} (N_e \rtimes H_e) \cong \prod_{e \in E(\mathcal{G})} (\Bbbk \rtimes \Bbbk^\times).$$

To finish, let us see that the sequence is split. Consider the map $\mathrm{Aut}_{\mathcal{D}igraphs}(\mathcal{G}) \to \mathrm{Aut}_{\mathrm{Coalg}_\Bbbk}(C(\mathcal{G}))$ taking $\sigma \in \mathrm{Aut}_{\mathcal{D}igraphs}(\mathcal{G})$ to $f^\sigma_{0,1}$. Then, for $\sigma, \tau \in \mathrm{Aut}_{\mathcal{D}igraphs}(\mathcal{G})$, a simple computation shows that $f^\tau_{0,1} \circ f^\sigma_{0,1} = f^{\tau\circ\sigma}_{0,1}$, thus it is a group homomorphism. Moreover, it is clearly a section of the restriction map $\mathrm{Aut}_{\mathrm{Coalg}_\Bbbk}(C(\mathcal{G})) \to \mathrm{Aut}_{\mathcal{D}igraphs}(\mathcal{G})$. The result follows. □

Since we know that the short exact sequence in Theorem 3.9 is split, the next result follows:

Corollary 3.10. *Let $\Bbbk$ be a field and let $\mathcal{G}$ be a digraph. If $C(\mathcal{G})$ is the coalgebra introduced in Definition 3.1, then*

$$\mathrm{Aut}_{\mathrm{Coalg}_\Bbbk}(C(\mathcal{G})) \cong \left(\prod_{e \in E(\mathcal{G})} (\Bbbk \rtimes \Bbbk^\times) \right) \rtimes \mathrm{Aut}_{\mathcal{D}igraphs}(\mathcal{G}).$$

In particular, since by Theorem 1.16 every group G arises as the automorphism group of a graph (which can be regarded as a symmetric digraph), we immediately obtain the following:

Corollary 3.11. *Let $\Bbbk$ be a field and let G be a group. There is a $\Bbbk$-coalgebra C such that $\mathrm{Aut}_{\mathrm{Coalg}_\Bbbk}(C) \cong K \rtimes G$, where K is a direct product of semidirect products of the form $\Bbbk \rtimes \Bbbk^\times$. Furthermore, G is the image of the restriction of the automorphisms of C to* $\mathrm{Sym}(G(C))$.

Namely, we have proven that every group G arises as the permutation group induced by the restriction of the automorphism group of a coalgebra C to its set of grouplike elements $G(C)$. Furthermore, G is a subgroup of $\mathrm{Aut}_{\mathrm{Coalg}_\Bbbk}(C)$, so C is a faithful G-coalgebra. This is as close as we get to a solution to the group realisability problem in the category of coalgebras in this thesis and, in fact, the remaining results regarding the realisability problems in the category of coalgebras will follow the same spirit.

3.2 Generalised realisability problems in coalgebras

In this section, we apply the results proved earlier in this chapter to obtain conclusions regarding the two generalised realisability problems. Let us begin with Problem 1. Using Theorem 3.9 we can prove a result on the realisability problem in the arrow category of coalgebras that follows the spirit of Corollary 3.11.

Theorem 3.12. *Let G_1 and G_2 be groups and take $H \leq G_1 \times G_2$. Let $\Bbbk$ be a field. There exist two $\Bbbk$-coalgebras C_1 and C_2, and a morphism $\varphi \in \operatorname{Hom}_{\mathrm{Coalg}_\Bbbk}(C_1, C_2)$ such that*

(1) $\operatorname{Aut}_{\mathrm{Coalg}_\Bbbk}(C_k) \cong K_k \rtimes G_k$*, where G_k is the image of the restriction* $\operatorname{Aut}_{\mathrm{Coalg}_\Bbbk}(C_k) \to \operatorname{Sym}\left(G(C_i)\right)$ *and K_k is a direct product of factors of the form $\Bbbk \rtimes \Bbbk^\times$, $k = 1, 2$;*

(2) $\operatorname{Aut}_{\mathrm{Coalg}_\Bbbk}(\varphi) \cong K \rtimes H$*, where H is the image of the restriction map* $\operatorname{Aut}_{\mathrm{Coalg}_\Bbbk}(\varphi) \to \operatorname{Sym}\left(G(C_1)\right) \times \operatorname{Sym}\left(G(C_2)\right)$ *and $K \leq K_1 \times K_2$.*

Proof. As a consequence of Theorem 2.37, there are graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, and a morphism of graphs $\psi \colon \mathcal{G}_1 \to \mathcal{G}_2$ such that $\operatorname{Aut}_{\mathcal{G}raphs}(\mathcal{G}_k) \cong G_k$, $k = 1, 2$, and $\operatorname{Aut}_{\mathcal{G}raphs}(\psi) \cong H$. Take $C_k = C(\mathcal{G}_k)$, $k = 1, 2$, and $\varphi = C(\psi)$. We now prove that these are the desired coalgebras and morphism.

By Corollary 3.11, there are groups K_1 and K_2 isomorphic to a direct product of semidirect products of the form $\Bbbk \rtimes \Bbbk^\times$ such that $\operatorname{Aut}_{\mathrm{Coalg}_\Bbbk}(C_k) \cong K_k \rtimes G_k$, $k = 1, 2$. Moreover, G_k is the image of the restriction map $\operatorname{Aut}_{\mathrm{Coalg}_\Bbbk}(C_k) \to \operatorname{Sym}\left(G(C_k)\right)$, for $k = 1, 2$, proving Theorem 3.12.*(1)*.

To prove Theorem 3.12.*(2)*, let us compute $\operatorname{Aut}_{\mathrm{Coalg}_\Bbbk}(\varphi)$. Recall from Lemma 3.7 and Lemma 3.8 that the automorphisms of C_k are the maps $f^{\sigma_k}_{\lambda_k,\mu_k}$ introduced in Definition 3.5, where $\sigma_k \in \operatorname{Aut}_{\mathcal{G}raphs}(\mathcal{G}_k)$, $\lambda_k \colon E(\mathcal{G}_k) \to \Bbbk$ and $\mu_k \colon E(\mathcal{G}_k) \to \Bbbk^\times$, $k = 1, 2$. Let us then take $(f^{\sigma_1}_{\lambda_1,\mu_1}, f^{\sigma_2}_{\lambda_2,\mu_2}) \in \operatorname{Aut}_{\mathrm{Coalg}_\Bbbk}(C_1) \times \operatorname{Aut}_{\mathrm{Coalg}_\Bbbk}(C_2)$ and check when $(f^{\sigma_1}_{\lambda_1,\mu_1}, f^{\sigma_2}_{\lambda_2,\mu_2}) \in \operatorname{Aut}_{\mathrm{Coalg}_\Bbbk}(\varphi)$, that is, when $\varphi \circ f^{\sigma_1}_{\lambda_1,\mu_1} = f^{\sigma_2}_{\lambda_2,\mu_2} \circ \varphi$.

First take $v \in V(\mathcal{G}_1)$. On the one hand, $(\varphi \circ f^{\sigma_1}_{\lambda_1,\mu_1})(v) = \varphi(\sigma_1(v))$. On the other hand, $(f^{\sigma_2}_{\lambda_2,\mu_2} \circ \varphi)(v) = f^{\sigma_2}_{\lambda_2,\mu_2}(\varphi(v)) = \sigma_2(\varphi(v))$. Thus, we need that $\varphi \circ \sigma_1 = \sigma_2 \circ \varphi$ for all $v \in V(\mathcal{G}_1)$, that is, we need that $(\sigma_1, \sigma_2) \in \operatorname{Aut}_{\mathcal{G}raphs}(\psi) \cong H$. Consequently, we can consider the restriction of the automorphisms of φ to $\operatorname{Sym}\left(G(C_1)\right) \times \operatorname{Sym}\left(G(C_2)\right)$ and obtain a map $\operatorname{Aut}_{\mathrm{Coalg}_\Bbbk}(\varphi) \to \operatorname{Aut}_{\mathcal{G}raphs}(\psi) \cong H$. Furthermore, this map is surjective, since for $(\sigma_1, \sigma_2) \in \operatorname{Aut}_{\mathcal{G}raphs}(\psi)$, it is immediate that $(f^{\sigma_1}_{0,1}, f^{\sigma_2}_{0,1}) \in \operatorname{Aut}_{\mathrm{Coalg}_\Bbbk}(\varphi)$ and that its restriction to the sets of grouplike elements is (σ_1, σ_2). It remains to prove that the kernel of the restriction map K is a subgroup of $K_1 \times K_2$.

Let us now consider the images of the edges, thus take $e = (v, w) \in E(\mathcal{G}_1)$. On the one hand,

$$\begin{aligned}(\varphi \circ f^{\sigma_1}_{\lambda_1,\mu_1})(e) &= \varphi(\lambda_1(e)(\sigma_1(w) - \sigma_1(v)) + \mu_1(e)\sigma_1(e)) \\ &= \lambda_1(e)\big((\varphi \circ \sigma_1)(w) - (\varphi \circ \sigma_1)(v)\big) + \mu_1(e)(\varphi \circ \sigma_1)(e).\end{aligned}$$

On the other hand,

$$(f^{\sigma_2}_{\lambda_2,\mu_2} \circ \varphi)(e) = f^{\sigma_2}_{\lambda_2,\mu_2}(\varphi(e)) = \lambda_2(\varphi(e))\big((\sigma_2 \circ \varphi)(w) - (\sigma_2 \circ \varphi)(v)\big) + \mu_2(\varphi(e))(\sigma_2 \circ \varphi)(e).$$

These two expressions should be equal, and since $\varphi \circ \sigma_1 = \sigma_2 \circ \varphi$, we deduce that $\lambda_2(\varphi(e)) = \lambda_1(e)$ and that $\mu_2(\varphi(e)) = \mu_1(e)$, for every $e \in E(\mathcal{G}_1)$. Nonetheless, no further restrictions exist on the automorphisms. In particular, we can compute the kernel of the restriction map and obtain that

$$K = \left\{ \left(f^{\mathrm{id}_{\mathcal{G}_1}}_{\lambda_1,\mu_1}, f^{\mathrm{id}_{\mathcal{G}_2}}_{\lambda_2,\mu_2}\right) \in K_1 \times K_2 \;\middle|\; \begin{array}{l} \lambda_2(\varphi(e)) = \lambda_1(e), \\ \mu_2(\varphi(e)) = \mu_1(e), \end{array} \text{ for all } e \in E(\mathcal{G}_1) \right\} \leq K_1 \times K_2.$$

Then, since the exact sequence induced by the restriction map is split by a proof analogous to that of Theorem 3.9, $\mathrm{Aut}_{\mathrm{Coalg}_{\Bbbk}}(\varphi) \cong K \rtimes H$. Thus Theorem 3.12.*(2)* follows. □

We now consider the realisability of permutation representations, Problem 2, on the category of coalgebras. As a consequence of Theorem 3.9, if $\mathcal{G}$ is any digraph, the permutation group induced by the restriction of the automorphisms of $C(\mathcal{G})$ to its set of grouplike elements is $\mathrm{Aut}_{\mathcal{D}igraphs}(\mathcal{G})$. We now transfer Theorem 2.41 to coalgebras.

Theorem 3.13. *Let G be a group, $\Bbbk$ be a field and $\rho\colon G \to \mathrm{Sym}(V)$ be a permutation representation of G on a set V. There exists a G-coalgebra C such that:*

(1) G acts faithfully on C, that is, there is a group monomorphism $G \hookrightarrow \mathrm{Aut}_{\mathrm{Coalg}_{\Bbbk}}(C)$;

(2) the image of the restriction map $\mathrm{Aut}_{\mathrm{Coalg}_{\Bbbk}}(C) \to \mathrm{Sym}\left(G(C)\right)$ is G;

(3) there is a subset $V \subset G(C)$ that is invariant through the $\mathrm{Aut}_{\mathrm{Coalg}_{\Bbbk}}(C)$-action on C and such that ρ is the composition of the inclusion $G \hookrightarrow \mathrm{Aut}_{\mathrm{Coalg}_{\Bbbk}}(C)$ with the restriction map $\mathrm{Aut}_{\mathrm{Coalg}_{\Bbbk}}(C) \to \mathrm{Sym}(V)$;

(4) there is a faithful action $\bar{\rho}\colon G \to \mathrm{Sym}\left(G(C) \setminus V\right)$ such that the composition of the inclusion $G \hookrightarrow \mathrm{Aut}_{\mathrm{Coalg}_{\Bbbk}}(C)$ with the restriction $\mathrm{Aut}_{\mathrm{Coalg}_{\Bbbk}}(C) \to \mathrm{Sym}\left(G(C)\right)$ is $\rho \oplus \bar{\rho}$.

Proof. By Theorem 2.41, there is a simple graph $\mathcal{G}$ such that $V \subset V(\mathcal{G})$, $\mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \cong G$, the restriction $G \cong \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \to \mathrm{Sym}(V)$ is ρ and there is a faithful action $\bar{\rho}\colon G \cong \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \to \mathrm{Sym}\left(V(\mathcal{G}) \setminus V\right)$ such that the restriction map $\mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \to \mathrm{Sym}(V)$ is $\rho \oplus \bar{\rho}$. Since any simple graph can be regarded as a digraph where every edge is bidirected, we can consider $C = C(\mathcal{G})$ the coalgebra introduced in Definition 3.1. Then, $G(C) = V(\mathcal{G})$. Let us prove that this is the desired coalgebra.

Recall from Lemma 3.7 and Lemma 3.8 that the automorphisms of C are the maps $f^{\sigma}_{\lambda,\mu}$ introduced in Definition 3.5, with $\sigma \in \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G})$, $\lambda\colon E(\mathcal{G}) \to \Bbbk$ and $\mu\colon E(\mathcal{G}) \to \Bbbk^{\times}$. Then since $G \cong \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G})$, G acts on C by taking an element $\sigma \in \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G})$ to $f^{\sigma}_{0,1} \in \mathrm{Aut}_{\mathrm{Coalg}_{\Bbbk}}(C)$, thus C is a G-coalgebra.

On the other hand, for $v \in V(\mathcal{G}) = G(C)$, $f^{\sigma}_{\lambda,\mu}(v) = \sigma(v)$. Namely, the composition of the inclusion $G \cong \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \hookrightarrow \mathrm{Aut}_{\mathrm{Coalg}_{\Bbbk}}(C)$ with the restriction $\mathrm{Aut}_{\mathrm{Coalg}_{\Bbbk}}(C) \to \mathrm{Sym}\left(G(C)\right) = \mathrm{Sym}\left(V(\mathcal{G})\right)$ is precisely the action of G on $\mathcal{G}$ by automorphisms. The result then follows immediately from Theorem 2.41. □

3.3 The isomorphism problem for groups through coalgebra representations

In the last section of this chapter, we review how we can use the results from Section 3.1 to distinguish isomorphism classes of groups through their faithful representations on coalgebras and their restrictions to grouplike elements. All the groups we consider are in the class of co-Hopfian groups, which we introduce now.

Definition 3.14. A group G is said to be co-Hopfian if it does not contain proper subgroups isomorphic to itself. Equivalently, every monomorphism $G \hookrightarrow G$ must be an automorphism.

Clearly, every finite group is co-Hopfian. Several important families of groups are also co-Hopfian, as shown by the following examples.

Example 3.15. The following groups are co-Hopfian.

- Artinian groups. Recall that a group G is Artinian if it satisfies the minimal condition, meaning that any strictly descending chain of subgroups of G, $G_1 > G_2 > G_3 > \cdots > G_k > G_{k+1} > \cdots$, is finite. The fact that G is co-Hopfian follows immediately from the minimal condition.

- Any subgroup of finite index of $\mathrm{Out}(F_n)$ the group of outer automorphisms of the free group on n words, for $n \geq 4$, [35].

- The braid group on n strands B_n modulo its centre, for $n \geq 4$, [14, Main Theorem 3].

- Tarski groups. An infinite group G is a Tarski group for the prime q if all of its proper subgroups are finite and of order q. These groups exist for large enough primes, [66, 67], and are clearly co-Hopfian.

- The special linear group $\mathrm{SL}(n, \mathbb{Z})$, for $n \geq 3$, as can be deduced from [74, Theorem 6].

- Fundamental groups of closed surfaces of genus at least two, [33, p. 58].

- Mapping class groups of compact, connected orientable surfaces of positive genus that are not a torus with at most two holes, [53, Theorem 1].

This exhibits that we are indeed working with a large class of groups. We can now prove our first result regarding the isomorphism problem for groups in this context.

Theorem 3.16. *Let $\Bbbk$ be a field and G and H be two co-Hopfian groups. The following statements are equivalent:*

(1) G and H are isomorphic.

(2) For any $\Bbbk$-coalgebra C, there is an action of G on C that restricts to a faithful action on $G(C)$ if and only if there is an action of H on C that restricts to a faithful action on $G(C)$.

Proof. One implication is obvious. Let us prove the remaining one. Suppose then that G and H are two groups verifying Theorem 3.16.*(2)*. Let us prove that $G \cong H$.

Let $\mathcal{G}$ and $\mathcal{H}$ be graphs such that $\mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \cong G$ and $\mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{H}) \cong H$, which exist as a consequence of Theorem 1.16. Consider the coalgebras $C(\mathcal{G})$ and $C(\mathcal{H})$ introduced in Definition 3.1. As a consequence of Theorem 3.9, G acts faithfully on $C(\mathcal{G})$, and the image of the composition of the inclusion map $G \to \mathrm{Aut}_{\mathrm{Coalg}_\Bbbk}(C(\mathcal{G}))$ with the restriction $\mathrm{Aut}_{\mathrm{Coalg}_\Bbbk}(C(\mathcal{G})) \to \mathrm{Sym}(G(C(\mathcal{G})))$ is G. Therefore, there is an action of G on $C(\mathcal{G})$ that restricts to a faithful action on $G(C(\mathcal{G}))$. By Theorem 3.16.*(2)*, this implies that there is an action of H on $C(\mathcal{G})$ that induces a faithful action on $G(C(\mathcal{G}))$, so we deduce that $H \leq \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \cong G$. Similarly, if there is an action of G on $C(\mathcal{H})$ inducing a faithful action on $G(C(\mathcal{H}))$, then $G \leq \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{H}) \cong H$. Thus $G \leq H \leq G$ and, since G is co-Hopfian, $G \cong H$. □

We now consider the entire action on the coalgebra instead of focusing on its restriction to grouplike elements. To ensure that groups are still distinguished, and since $\mathrm{Aut}_{\mathrm{Coalg}_\Bbbk}(C(\mathcal{G}))$ has subgroups of the form $\Bbbk \rtimes \Bbbk^\times$, we have to further restrict the class of groups we are working with. With such objective in mind, we introduce the following class of groups:

Definition 3.17. Let $\Bbbk$ be a finite field of order p^n, p prime. A group G is in the class $\mathfrak{G}_{p,n}$ if it verifies the following properties:

(1) G is co-Hopfian;

(2) G does not have finite non-trivial normal subgroups whose exponent divides $p^n(p^n-1)$.

Notice that although this class is quite restrictive, it still contains many interesting groups. For example, $\mathfrak{G}_{2,1}$ still contains all 2-reduced groups, that is, all groups with no normal 2-subgroups. We can now prove our last result for this chapter.

Theorem 3.18. *Let $\Bbbk$ be a finite field of order p^n, p prime. Let G and H be groups in $\mathfrak{G}_{p,n}$. The following are equivalent:*

(1) G and H are isomorphic.

(2) For every $\Bbbk$-coalgebra C, G acts faithfully on C if and only if H acts faithfully on C.

Proof. One implication is obvious. Let us prove the remaining one. Thus let G and H be two groups in $\mathfrak{G}_{p,n}$ verifying Theorem 3.18.*(2)* and let us prove that $G \cong H$.

Again, let $\mathcal{G}$ and $\mathcal{H}$ be graphs such that $\mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \cong G$ and $\mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{H}) \cong H$, which exist by Theorem 1.16, and consider $C(\mathcal{G})$ and $C(\mathcal{H})$ the respective coalgebras from Definition 3.1. Then $G \cong \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G})$ acts faithfully on $C(\mathcal{G})$ as an immediate consequence of Corollary 3.10. By the same result, if H acts faithfully on $C(\mathcal{G})$, there is a group monomorphism

$$H \hookrightarrow \mathrm{Aut}_{\mathrm{Coalg}_{\Bbbk}}\left(C(\mathcal{G})\right) \cong \left(\prod_{e\in E(\mathcal{G})} (\Bbbk \rtimes \Bbbk^{\times})\right) \rtimes G.$$

Thus H is isomorphic to a subgroup of $\mathrm{Aut}_{\mathrm{Coalg}_{\Bbbk}}\left(C(\mathcal{G})\right)$, which we also denote by H. We shall see that $H \cap \left(\prod_{e\in E(\mathcal{G})}(\Bbbk \rtimes \Bbbk^{\times})\right) = \{1\}$.

First notice that $\prod_{e\in E(\mathcal{G})}(\Bbbk \rtimes \Bbbk^{\times})$ is normal in $\mathrm{Aut}_{\mathrm{Coalg}_{\Bbbk}}\left(C(\mathcal{G})\right)$, thus $H \cap \left(\prod_{e\in E(\mathcal{G})}(\Bbbk \rtimes \Bbbk^{\times})\right)$ is normal in H. On the other hand, $\Bbbk \rtimes \Bbbk^{\times}$ is a group of order $p^n(p^n-1)$, thus the exponent of $\prod_{e\in E(\mathcal{G})}(\Bbbk \rtimes \Bbbk^{\times})$ divides $p^n(p^n-1)$. Therefore $H \cap \left(\prod_{e\in E(\mathcal{G})}(\Bbbk \rtimes \Bbbk^{\times})\right)$ is a normal subgroup of H whose exponent divides $p^n(p^n-1)$. Hence, since H is in $\mathfrak{G}_{p,n}$, the intersection must be the trivial group. Consequently, the image of H falls in G, so $H \leq G$.

By a similar argument, we deduce that if G acts faithfully on $C(\mathcal{H})$, then $G \leq H$. We then have $G \leq H \leq G$ and, since G is co-Hopfian, $G \cong H$. □

CHAPTER 4

Realisability problems in CDGAs and spaces

In [27], Costoya-Viruel gave the first general solution to the classical group realisability problem in $\mathcal{H}oTop$, also known as Kahn's realisability problem, by proving that every finite group G is the group of self-homotopy equivalences of a rational space X, that is, $G \cong \mathcal{E}(X)$. Their idea is to first go through an intermediate category, $\mathcal{G}raphs$, where as we have explained in Section 1.2, the classical group realisability problem admits a positive answer, i.e., $G \cong \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G})$, for a finite graph $\mathcal{G}$, [43]. Then, by using the computational power of Rational Homotopy Theory, they construct minimal Sullivan algebras $M_{\mathcal{G}}$ encoding the combinatorial data of $\mathcal{G}$, in such a way that $\mathcal{E}(M_{\mathcal{G}}) \cong \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G})$.

Their construction was based on a homotopically rigid Sullivan algebra, that is, a Sullivan algebra whose only self-homotopy equivalence is the class of the identity map. Therefore, the rational space of whom this homotopically rigid algebra is a model inherits the same property, and it is then a homotopically rigid space. Homotopically rigid spaces where supposed to be quite rare, and Kahn expected that they could play a role in some way of decomposing a space. Thus, obtaining examples of homotopically rigid spaces becomes of interest.

In this chapter we construct a uniparametric family of homotopically rigid commutative differential graded algebras with further interesting properties. On the one hand, the connectivity of the algebras increases with the parameter, thus we are able to provide examples of homotopically rigid CDGAs as highly connected as we desire. On the other hand, these CDGAs are, not only homotopically rigid, but (strictly) rigid, which means that their unique endomorphisms are the identity and the trivial one.

The fact that our algebras are (strictly) rigid, which is the main difference between our work and [27], is fundamental to us. We are able to prove that if R is an integral domain of characteristic zero or greater than three, CDGA_R is universal, which means that any group is realisable in $\mathcal{C} = \mathrm{CDGA}_R$. Hence we solve positively the classical realisability problem in the category $\mathcal{C} = \mathrm{CDGA}_R$. Our strategy to prove this result is the same as ever, we construct a functor from a certain subcategory of $\mathcal{D}igraphs$ to CDGA_R that improves the one that Costoya-Viruel introduced in [27].

In first place, Costoya-Viruel's functor can only be defined on a subcategory of $\mathcal{G}raphs$ whose morphisms are full monomorphisms of graphs, whereas our functor will be defined in a full subcategory of graphs. In second place, our functor not only preserves automorphisms, it is *almost* fully faithful. Indeed, the set of morphisms between any two graphs is in bijection with the set of non-trivial morphisms between their associated CDGAs.

This chapter is organised as follows. Section 4.1 is devoted to construct the uniparametric family of rigid algebras, Definition 4.1, and to prove rigidity, Theorem 4.3. Then, in Section 4.2 we obtain a family of functors from a full subcategory of digraphs to CDGA_R, Definition 4.11, we prove that they are almost fully faithful, Theorem 4.14, and we deduce that CDGA_R is universal, Theorem 4.16. Finally, in Section 4.3 we transfer the solutions to Problem 1 and Problem 2 from graphs to CDGAs and spaces by means of that family of functors. We give a complete positive answer to the realisability problem for arrow categories in CDGA_R, Theorem 4.17, and in $\mathcal{H}oTop$ under certain finiteness conditions, Theorem 4.19. We also provide a partial positive answer to the problem of realising permutation representations in CDGA_R, Theorem 4.20, and in $\mathcal{H}oTop$, Theorem 4.22.

Henceforward, R will denote an integral domain.

4.1 Highly connected rigid CDGAs

In this section we introduce a family of highly connected, rigid differential graded R-algebras. We introduced this family (over $\mathbb{Q}$) in [23, Definition 1.1]. In [23, Theorem 1.6] we proved that the monoid of homotopy classes of self-maps of these algebras only contains the classes of the trivial map and the identity map. Here, we go further and prove that, in fact, the identity map and the trivial map are the only endomorphisms of these R-algebras. Hence, not only are they homotopically rigid, they are rigid as R-algebras as well. In particular, their group of automorphisms is trivial.

The starting point for our family of rigid algebras is an example of Arkowitz and Lupton, [6, Example 5.1], which they obtained by modifying an example of Halperin-Oprea. They defined a minimal Sullivan algebra

$$M = (\Lambda(x_1, x_2, y_1, y_2, y_3, z), d)$$

with generators and differentials verifying

$$\begin{array}{ll}
|x_1| = 8, & dx_1 = 0, \\
|x_2| = 10, & dx_2 = 0, \\
|y_1| = 33, & dy_1 = x_1^3 x_2, \\
|y_2| = 35, & dy_2 = x_1^2 x_2^2, \\
|y_3| = 37, & dy_3 = x_1 x_2^3, \\
|z| = 119, & dz = x_1^4(x_2^2 y_1 y_2 - x_1 x_2 y_1 y_3 + x_1^2 y_2 y_3) + x_1^{15} + x_2^{12}.
\end{array}$$

This structure of generators and differentials proves to consistently produce examples of homotopically rigid algebras, as can be deduced from [6, Example 5.2] and [31, Examples 8.1 & 8.2]. However, trying to obtain highly connected rigid algebras by re-scaling the degrees of the generators in M is useless; the differential in M leads to a system of linear equations whose only solution is the one given by M.

We now introduce our family of highly connected rigid R-algebras. These algebras are obtained from the example above by scaling not only the degrees of the generators, but also some of the exponents that appear in the differential of z.

Definition 4.1. Let $k \geq 1$ be an integer. We define the commutative differential graded R-algebra

$$\mathcal{M}_k = (\Lambda(x_1, x_2, y_1, y_2, y_3, z), d)$$

where

$$\begin{aligned}
|x_1| &= 10k+8, & dx_1 &= 0,\\
|x_2| &= 12k+10, & dx_2 &= 0,\\
|y_1| &= 42k+33, & dy_1 &= x_1^3x_2,\\
|y_2| &= 44k+35, & dy_2 &= x_1^2x_2^2,\\
|y_3| &= 46k+37, & dy_3 &= x_1x_2^3,\\
|z| &= 60k^2+98k+39, & dz &= x_1^{6k-6}(x_2^2y_1y_2 - x_1x_2y_1y_3 + x_1^2y_2y_3) + x_1^{6k+5} + x_2^{5k+4}.
\end{aligned}$$

We remark that these algebras are, indeed, the ones introduced in [23, Definition 1.1]. However, they have been reparametrised so that they are indexed over all positive integers. Thus, the algebra we denote by $\mathcal{M}_k$ in Definition 4.1 is the algebra M_{2k+2} in [23, Definition 1.1]. Also, notice that for $R = \mathbb{Q}$, $\mathcal{M}_k$ is a minimal Sullivan algebra.

We start by introducing a technical lemma we need towards the proof of the rigidity of these algebras. The idea behind the lemma is to show that the generators of $\mathcal{M}_k$ are isolated in their respective degrees, thus greatly limiting their possible images through an endomorphism.

Lemma 4.2. *Let $k \geq 1$. For each $u \in \{x_1, x_2, y_1, y_2, y_3, z\}$, $\{u\}$ is a basis of $\mathcal{M}_k^{|w|}$.*

Proof. The following inequalities

$$|x_1| < |x_2| < |y_1| < |y_2| < |y_3| < |x_1y_1| < |x_2y_3| < |z|.$$

are straightforward except perhaps for $|x_2y_3| < |z|$. However, this follows from the fact that x_2y_3 divides a term in dz, and $\mathcal{M}_k$ has no generators in degree one. In view of these inequalities and considering the parity of the degrees of the generators, it becomes clear that the result holds for $u \in \{x_1, x_2, y_1, y_2, y_3\}$.

It remains to prove that a basis of $\mathcal{M}_k^{|z|}$ is $\{z\}$. Since $|z|$ is odd, any monomial in $\mathcal{M}_k^{|z|}$ must be divided by a generator of odd order. Therefore, aside from multiples of z, elements in $\mathcal{M}_k^{|z|}$ must be of the form P_1y_1, P_2y_2, P_3y_3 or $P_{123}y_1y_2y_3$, where $P_1, P_2, P_3, P_{123} \in R[x_1, x_2]$. Consequently, in order to prove the lemma we prove that there is no pair (α, β) of non-negative integers such that $m = |x_1^\alpha x_2^\beta| = \alpha|x_1| + \beta|x_2|$, for $m \in \{|z| - |y_1|, |z| - |y_2|, |z| - |y_3|, |z| - |y_1y_2y_3|\}$.

The linear diophantine equation $m = \alpha|x_1| + \beta|x_2|$ has a solution if and only if m is a multiple of $r = \gcd\big(|x_1|, |x_2|\big)$. In such case, if (α, β) is a particular solution to the equation, the general solution is of the form

$$\left(\alpha + s\frac{|x_2|}{r}, \beta - s\frac{|x_1|}{r}\right), \quad s \in \mathbb{Z}.$$

Considering that $-6|x_1| + 5|x_2| = 2$ and that both $|x_1|$ and $|x_2|$ are even, we deduce that $\gcd\big(|x_1|, |x_2|\big) = 2$. All of the four possible values we are considering for m are even, thus the four linear diophantine equations have solutions. The general solution to the diophantine equation $m = \alpha|x_1| + \beta|x_2|$ is as follows:

m	general solution
$\lvert z\rvert - \lvert y_1\rvert$	$\big(-3 + s(6k+5), 5k+3 - s(5k+4)\big)$
$\lvert z\rvert - \lvert y_2\rvert$	$\big(-2 + s(6k+5), 5k+2 - s(5k+4)\big)$
$\lvert z\rvert - \lvert y_3\rvert$	$\big(-1 + s(6k+5), 5k+1 - s(5k+4)\big)$
$\lvert z\rvert - \lvert y_1y_2y_3\rvert$	$\big(-12 + s(6k+5), 5k+3 - s(5k+4)\big)$.

In the solutions above, in order for α to be non-negative it is necessary that $s > 0$, in which case β is negative. The result follows. $\square$

Using this lemma we can prove the main result for this section.

Theorem 4.3. *Let $k \geq 1$ be an integer. The R-algebra $\mathcal{M}_k$ is rigid, that is, its only endomorphisms are the trivial map and the identity map.*

Proof. Let $f \in \operatorname{Hom}_{\mathrm{CDGA}_R}(\mathcal{M}_k, \mathcal{M}_k)$. As a consequence of Lemma 4.2, there exist a_1, a_2, b_1, b_2, b_3, $c \in R$ such that $f(x_1) = a_1x_1, f(x_2) = a_2x_2, f(y_1) = b_1y_1, f(y_2) = b_2y_2, f(y_3) = b_3y_3$ and $f(z) = cz$. And, as $df = fd$, we immediately obtain that

$$b_1 = a_1^3 a_2, \quad b_2 = a_1^2 a_2^2, \quad b_3 = a_1 a_2^3. \tag{4.1}$$

Now $df(z) = fd(z)$. On the one hand,

$$df(z) = c\big[x_1^{6k-6}(x_2^2y_1y_2 - x_1x_2y_1y_3 + x_1^2y_2y_3) + x_1^{6k+5} + x_2^{5k+4}\big], \tag{4.2}$$

and on the other hand,

$$\begin{aligned} f(dz) = a_1^{6k-6}x_1^{6k-6}(a_2^2b_1b_2x_2^2y_1y_2 - a_1a_2b_1b_3x_1x_2y_1y_3 + a_1^2b_2b_3x_1^2y_2y_3) \\ +a_1^{6k+5}x_1^{6k+5} + a_2^{5k+4}x_2^{5k+4}\big]. \end{aligned} \tag{4.3}$$

Comparing Equations (4.2) and (4.3), we obtain the following equalities:

$$c = a_1^{6k-6}a_2^2b_1b_2 = a_1^{6k-5}a_2b_1b_3 = a_1^{6k-4}b_2b_3 = a_1^{6k+5} = a_2^{5k+4}. \tag{4.4}$$

By replacing b_1, b_2 and b_3 by their values in Equation (4.1), from Equation (4.4) we deduce that

$$c = a_1^{6k+5} = a_2^{5k+4} = a_1^{6k-1}a_2^5.$$

Since R is an integral domain, it has the cancellation property. Thus, from the equality $a_1^{6k+5} = a_1^{6k-1}a_2^5$ we obtain that $a_1^6 = a_2^5$, which implies that $a_1^{6k} = a_2^{5k}$. On the other hand, from the equality $a_2^{5k+4} = a_1^{6k-1}a_2^5$ we deduce that $a_1^{6k-1} = a_2^{5k-1}$. Multiplying by a_1 and using the identity above, $a_1a_2^{5k-1} = a_1^{6k} = a_2^{5k}$, thus we deduce that $a_1 = a_2$. This, together with the equations above implies that $a_1 = a_2 = s$, $s \in \{0,1\}$. It now follows that $a_1 = a_2 = b_1 = b_2 = b_3 = c = s$, $s \in \{0,1\}$. Therefore, f is either the identity map, if $s = 1$, or the trivial map, if $s = 0$. □

4.2 A family of almost fully faithful functors from digraphs to CDGAs

In this section we introduce a family of commutative differential graded R-algebras associated to a given digraph, Definition 4.6. We then use these algebras to define a family of functors between $\mathcal{D}igraphs_+$, a full subcategory of $\mathcal{D}igraphs$ (see Definition 4.4), and CDGA_R. Furthermore, these functors are *almost* fully faithful. Namely, we prove that the set of morphisms between any two digraphs in $\mathcal{D}igraphs_+$ is in bijection with the set of non-trivial morphisms between their images through our functors, Theorem 4.14. Using these results, we obtain a complete solution to the group realisability problem in the category of CDGA_R, Theorem 4.16.

We begin by introducing the category of graphs to which we associate the algebras.

Definition 4.4. We denote by $\mathcal{D}igraphs_+$ the full subcategory of those digraphs $\mathcal{G}$ such that

(1) $\mathcal{G}$ is irreflexive,

(2) $\deg^+(v) > 0$, for all $v \in V(\mathcal{G})$,

which in particular implies that $|V(\mathcal{G})| > 1$.

Remark 4.5. It becomes immediate that $\mathcal{D}igraphs_0$ (see Definition 1.19) is a full subcategory of $\mathcal{D}igraphs_+$. Indeed, in order for an irreflexive digraph with more than one vertex to be strongly connected it is necessary that each vertex is the starting point of at least one edge. It is also clear that when regarding a graph without isolated vertices as an asymmetric digraph, it falls in $\mathcal{D}igraphs_+$.

We now introduce the algebras modelling the behaviour of the digraphs.

Definition 4.6. Let $\mathcal{G}$ be an object in $\mathcal{D}igraphs_+$. For each $n \geq 1$, we associate to $\mathcal{G}$ the commutative differential graded R-algebra

$$\mathcal{M}_n(\mathcal{G}) = (\Lambda(x_1, x_2, y_1, y_2, y_3, z) \otimes_R \Lambda(x_v \mid v \in V(\mathcal{G})) \otimes_R \Lambda(z_{(v,w)} \mid (v,w) \in E(\mathcal{G})), d)$$

where

$$\begin{aligned}
|x_1| &= 30n - 12, & dx_1 &= 0,\\
|x_2| &= 36n - 14, & dx_2 &= 0,\\
|y_1| &= 126n - 51, & dy_1 &= x_1^3 x_2,\\
|y_2| &= 132n - 53, & dy_2 &= x_1^2 x_2^2,\\
|y_3| &= 138n - 55, & dy_3 &= x_1 x_2^3,\\
|x_v| &= 180n^2 - 142n + 28, & dx_v &= 0,\\
|z| &= 540n^2 - 426n + 83, & dz &= x_1^{18n-18}(x_2^2 y_1 y_2 - x_1 x_2 y_1 y_3 + x_1^2 y_2 y_3)\\
& & &\quad + x_1^{18n-7} + x_2^{15n-6},\\
|z_{(v,w)}| &= 540n^2 - 426n + 83, & dz_{(v,w)} &= x_v^3 + x_v x_w x_2^{5n-2} + x_1^{18n-7}.
\end{aligned}$$

Notice that, for any digraph $\mathcal{G}$ in the hypothesis of the definition above, the R-algebra $\mathcal{M}_n(\mathcal{G})$ is $(30n-13)$-connected, thus it is in particular n-connected. We now proceed to give some remarks on these algebras.

Remark 4.7. Note that for any $n \geq 1$, the subalgebra $\Lambda(x_1, x_2, y_1, y_2, y_3, z) \hookrightarrow \mathcal{M}_n(\mathcal{G})$ is precisely the rigid algebra $\mathcal{M}_{3n-2}$ from Definition 4.1. This choice of parameters is made so that $|z_{(v,w)}| + 1 = |z| + 1$ is divisible by three, thus x_v^3 can be a summand of $dz_{(v,w)}$, for $(v,w) \in E(\mathcal{G})$.

In the particular case of $\mathcal{G}$ being a finite digraph and the base ring R being $\mathbb{Q}$, the algebra $\mathcal{M}_n(\mathcal{G})$ is the rational model of a space. Then, as a consequence of Proposition 1.53, it can easily be seen that the inclusion $\mathcal{M}_{3n-2} \hookrightarrow \mathcal{M}_n(\mathcal{G})$ is the Sullivan model of a rational Serre fibration with base the homotopically rigid space modelled by $\mathcal{M}_{3n-2}$. The fibre is modelled by the minimal Sullivan algebra $\Lambda(x_v \mid v \in V(\mathcal{G}), z_{(v,w)} \mid (v,w) \in E(\mathcal{G}), \bar{d})$ with $\bar{d}x_v = 0$, for all $v \in V(\mathcal{G})$, and $\bar{d}(z_{(v,w)}) = x_v^3$, for all $(v,w) \in E(\mathcal{G})$. Thus, what we do when defining $\mathcal{M}_n(\mathcal{G})$ is "gluing together" several copies of a homotopically rigid space following the combinatorial structure of the digraph. Therefore, since the homotopically rigid space does not have any non-trivial self-homotopy equivalence, the self-homotopy equivalences of the total space can be interpreted as permutations of the different copies of the rigid building block that are allowed at the level of the digraph over which we are gluing them.

Remark 4.8. It is worth noting that, although these algebras are introduced using the same ideas as the ones in [23, Definition 2.1], several improvements have been made.

(1) We can use the algebra $\mathcal{M}_1$ as the rigid base for our construction, whereas in [23] we were not able to prove the rigidity of $M_4 = \mathcal{M}_1$. Consequently, the algebra $\mathcal{M}_1(\mathcal{G})$ in Definition 4.6 has both a lower level of connectivity and a lower dimension than the algebra $M_1(\mathcal{G})$ introduced in [23, Definition 2.1].

(2) A monomial in $R[x_1]$ has been added to the differential of the generators in degree $|z|$ associated to the digraph, that is, the generators $z_{(v,w)}$ for $(v,w) \in E(\mathcal{G})$. This subtle difference forces that any morphism that is trivial on the generators related to the digraph must also be trivial on the generators of the rigid algebra, as we see in Theorem 4.14. This fact, together with the rigidity of the algebras $\mathcal{M}_k$ introduced in Definition 4.1 allows us to give a positive answer to the group realisability question in CDGA_R, Theorem 4.16.

(3) Furthermore, notice that the added generators in degree $|z|$, the $z_{(v,w)}$, are associated to edges, whereas in [23, Definition 2.1] the added generators in degree $|z|$, z_v, are associated to vertices of the digraph. Indeed, in the models of [23], the information of the edges is codified in the differentials. This has two important implications.

First, in the differential of z_v in [23, Definition 2.1] there is a summand for each edge in which the vertex v participates. This means that these algebras can only be defined associated to locally finite graphs, that is, graphs in which each vertex participates in a finite number of edges. Otherwise there would be an infinite sum in dz_v. Our algebras do not have such restriction.

Second, in order for a morphism of graphs to induce a morphism of algebras in the models of [23, Definition 2.1], it has to be a full monomorphism, [27, Remark 2.8]. However, any morphism of digraphs induces a morphism between the corresponding algebras in Definition 4.6.

These two facts combined allow us to define a family of functors from the entire category $\mathcal{D}igraphs_+$ to CDGA_R^n, Definition 4.11, that allows us to obtain some interesting applications in the remainder of this thesis.

We will show that, for any $n \geq 1$, if $\mathcal{G}_1$ and $\mathcal{G}_2$ are digraphs in $\mathcal{D}igraphs_+$, there is a bijective correspondence between $\mathrm{Hom}_{\mathcal{D}igraphs}(\mathcal{G}_1, \mathcal{G}_2)$ and $\mathrm{Hom}_{\mathrm{CDGA}_R}\left(\mathcal{M}_n(\mathcal{G}_1), \mathcal{M}_n(\mathcal{G}_2)\right)^*$. In order to do so, we need to prove some lemmas first. We start with a technical lemma in the same spirit as Lemma 4.2. This result extends [23, Lemma 2.5].

Lemma 4.9. *Let $\mathcal{G}$ be a digraph in $\mathcal{D}igraphs_+$ and let $n \geq 1$ be an integer. Then,*

(1) For each $u \in \{x_1, x_2, y_1, y_2, y_3\}$, a basis of $\mathcal{M}_n(\mathcal{G})^{|u|}$ is $\{u\}$.

(2) A basis of $\mathcal{M}_n(\mathcal{G})^{|x_v|}$ is $\{x_2^{5n-2}\} \sqcup \{x_v \mid v \in V(\mathcal{G})\}$.

(3) A basis of $\mathcal{M}_n(\mathcal{G})^{|z|}$ is $\{z\} \sqcup \{z_{(v,w)} \mid (v,w) \in E(\mathcal{G})\}$.

Proof. Recall that the R-subalgebra $\Lambda(x_1, x_2, y_1, y_2, y_3, z) \hookrightarrow \mathcal{M}_n(\mathcal{G})$ is the R-algebra $\mathcal{M}_{3n-2}$ from Definition 4.1. Therefore, the inequalities

$$|x_1| < |x_2| < |y_1| < |y_2| < |y_3| < |x_1y_1| < |x_2y_3| < |z|$$

from Lemma 4.2 still apply. It is also easy to check that $|x_1| < |x_2| < |x_v|$ and $|y_3| < |y_1x_v|$, for every $n \geq 1$. Then Lemma 4.9.*(1)* follows from these inequalities.

We prove the rest of the lemma by using ideas akin to those in the proof of Lemma 4.2. We now consider Lemma 4.9.*(2)*. Elements of degree $|x_v|$, other than $x_v, v \in V(\mathcal{G})$, are of the form P, $P_{12}y_1y_2$, $P_{13}y_1y_3$ and $P_{23}y_2y_3$, where $P, P_{12}, P_{13}, P_{23} \in R[x_1, x_2]$. We have to prove that P can only be a multiple of x_2^{5n-2} and that P_{12}, P_{13} and P_{23} are trivial. Let $m \in \{|x_v|, |x_v| - |y_1y_2|, |x_v| - |y_1y_3|, |x_v| - |y_2y_3|\}$. As in Lemma 4.2, by choosing suitable particular solutions for the diophantine equation $m = \alpha|x_1| + \beta|x_2|$ we obtain its general solution:

m	general solution
$\lvert x_v\rvert$	$(s(18n-7), 5n-2-s(15n-6))$
$\lvert x_v\rvert - \lvert y_1y_2\rvert$	$(-11+s(18n-7), 5n-s(15n-6))$
$\lvert x_v\rvert - \lvert y_1y_3\rvert$	$(-10+s(18n-7), 5n-1-s(15n-6))$
$\lvert x_v\rvert - \lvert y_2y_3\rvert$	$(-9+s(18n-7), 5n-2-s(15n-6))$

In the case $m = |x_v|$, a valid solution $(0, 5n-2)$ exists, thus $x_2^{5n-2} \in \mathcal{M}_n(\mathcal{G})^{|x_v|}$. However, if $s > 0$, $\beta < 0$, whereas if $s < 0$, $\alpha < 0$, for every $n \geq 1$. In the remaining three cases and for every $n \geq 1$, α is negative unless $s > 0$, in which case β is negative. Thus there is no solution where both integers are non-negative, and Lemma 4.9.*(2)* follows.

To prove Lemma 4.9.*(3)* we follow the same approach. We have to consider the product of y_1, y_2, y_3 and $y_1y_2y_3$ with polynomials on the generators of even order. Since $3|x_v| > |z| = |z_{(v,w)}|$, $(v,w) \in E(\mathcal{G})$, such polynomials can only be of the form Q_jy_j, $Q_j(v)y_jx_v$, $Q_j(v,w)y_jx_vx_w$, $Q_{123}y_1y_2y_3$, $Q_{123}(v)y_1y_2y_3x_v$ and $Q_{123}(v,w)y_1y_2y_3x_vx_w$, where Q_j, $Q_j(v)$, $Q_j(v,w)$, Q_{123}, $Q_{123}(v)$, $Q_{123}(v,w) \in R[x_1,x_2]$ for each $j \in \{1,2,3\}$, $v,w \in V(\mathcal{G})$.

First, observe that since $\Lambda(x_1,x_2,y_1,y_2,y_3,z) \leq \mathcal{M}_n(\mathcal{G})$ is the algebra $\mathcal{M}_{3n-2}$ from Definition 4.1, we deduce from Lemma 4.2 that admissible polynomials Q_j, $j \in \{1,2,3\}$ and Q_{123} do not exist. For the remaining polynomials, we have to prove that there is no pair of non-negative integers (α,β) such that $m = \alpha|x_1|+\beta|x_2|$, for $m \in \{|z|-|y_j|-|x_v|, |z|-|y_j|-2|x_v|\}$, $j = 1,2,3$ and for $m \in \{|z|-|y_1y_2y_3|-|x_v|, |z|-|y_1y_2y_3|-2|x_v|\}$. As previously, we obtain the following general solution:

m	general solution
$\lvert z\rvert - \lvert y_1\rvert - \lvert x_v\rvert$	$(-3+s(18n-7), 10n-5-s(15n-6))$
$\lvert z\rvert - \lvert y_2\rvert - \lvert x_v\rvert$	$(-2+s(18n-7), 10n-6-s(15n-6))$
$\lvert z\rvert - \lvert y_3\rvert - \lvert x_v\rvert$	$(-1+s(18n-7), 10n-7-s(15n-6))$
$\lvert z\rvert - \lvert y_1\rvert - 2\lvert x_v\rvert$	$(-3+s(18n-7), 5n-3-s(15n-6))$
$\lvert z\rvert - \lvert y_2\rvert - 2\lvert x_v\rvert$	$(-2+s(18n-7), 5n-4-s(15n-6))$
$\lvert z\rvert - \lvert y_3\rvert - 2\lvert x_v\rvert$	$(-1+s(18n-7), 5n-5-s(15n-6))$
$\lvert z\rvert - \lvert y_1y_2y_3\rvert - \lvert x_v\rvert$	$(-12+s(18n-7), 10n-5-s(15n-6))$
$\lvert z\rvert - \lvert y_1y_2y_3\rvert - 2\lvert x_v\rvert$	$(-12+s(18n-7), 5n-3-s(15n-6))$

Again, it is clear that the first coordinate is non-negative if and only if $s > 0$, in which case the second coordinate is negative. Thus Lemma 4.9.*(3)* follows. □

Henceforward, by abuse of notation, we will use the same letters x_1, x_2, y_1, y_2, y_3 and z, as it will be clear from the context whether we work in $\mathcal{M}_n(\mathcal{G}_1)$ or $\mathcal{M}_n(\mathcal{G}_2)$.

Lemma 4.10. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be objects in $\mathcal{D}igraphs_+$ and let $n \geq 1$ be an integer. Every $\sigma \in \mathrm{Hom}_{\mathcal{D}igraphs}(\mathcal{G}_1,\mathcal{G}_2)$ induces a morphism of commutative differential graded R-algebras $\mathcal{M}_n(\sigma)\colon \mathcal{M}_n(\mathcal{G}_1) \to \mathcal{M}_n(\mathcal{G}_2)$.*

Proof. Since $\sigma \in \mathrm{Hom}_{\mathcal{D}igraphs}(\mathcal{G}_1,\mathcal{G}_2)$, given $(v,w) \in E(\mathcal{G}_1)$, $(\sigma(v),\sigma(w)) \in E(\mathcal{G}_2)$. We can thus define $\mathcal{M}_n(\sigma)\colon \mathcal{M}_n(\mathcal{G}_1) \to \mathcal{M}_n(\mathcal{G}_2)$ as follows:

$$\begin{aligned}
&\mathcal{M}_n(\sigma)(u) = u, && \text{for } u \in \{x_1,x_2,y_1,y_2,y_3,z\},\\
&\mathcal{M}_n(\sigma)(x_v) = x_{\sigma(v)}, && \text{for all } v \in V(\mathcal{G}_1),\\
&\mathcal{M}_n(\sigma)(z_{(v,w)}) = z_{(\sigma(v),\sigma(w))}, && \text{for all } (v,w) \in E(\mathcal{G}_1).
\end{aligned}$$

Simple computations show that $d\mathcal{M}_n(\sigma) = \mathcal{M}_n(\sigma)d$. □

It is clear from the definition that this association takes the identity map to the identity map and behaves well with respect to the composition. Thus, we can introduce the following family of functors.

Definition 4.11. For every $n \geq 1$, we construct a functor $\mathcal{M}_n \colon \mathcal{D}igraphs_+ \to \mathrm{CDGA}_R^n$ as follows. To an object $\mathcal{G}$, the commutative differential graded R-algebra $\mathcal{M}_n(\mathcal{G})$ from Definition 4.6 is associated, and to a morphism $\sigma \in \mathrm{Hom}_{\mathcal{D}igraphs}(\mathcal{G}_1, \mathcal{G}_2)$, the morphism $\mathcal{M}_n(\sigma) \in \mathrm{Hom}_{\mathrm{CDGA}}\left(\mathcal{M}_n(\mathcal{G}_1), \mathcal{M}_n(\mathcal{G}_2)\right)$ from Lemma 4.10 is associated.

We now begin computing the set of morphisms between $\mathcal{M}_n(\mathcal{G}_1)$ and $\mathcal{M}_n(\mathcal{G}_2)$, for $\mathcal{G}_1$ and $\mathcal{G}_2$ any two objects in $\mathcal{D}igraphs_+$, which in particular will allow us to prove that $\mathcal{M}_n$ is indeed *almost* fully faithful. We begin with the following lemma, which is similar to Theorem 4.3.

Lemma 4.12. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two objects in $\mathcal{D}igraphs_+$ and let $f \colon \mathcal{M}_n(\mathcal{G}_1) \to \mathcal{M}_n(\mathcal{G}_2)$ be a morphism of* CDGA*s. There exists $s \in \{0,1\}$ such that $f(x_1) = sx_1$, $f(x_2) = sx_2$, $f(y_1) = sy_1$, $f(y_2) = sy_2$, $f(y_3) = sy_3$ and $f(z) = sz$.*

Proof. By Lemma 4.9.*(1)*, there exist a_1, a_2, b_1, b_2, $b_3 \in R$ such that $f(x_1) = a_1x_1, f(x_2) = a_2x_2, f(y_1) = b_1y_1, f(y_2) = b_2y_2$ and $f(y_3) = b_3y_3$. Since $df = fd$, we immediately obtain

$$b_1 = a_1^3a_2, \quad b_2 = a_1^2a_2^2, \quad b_3 = a_1a_2^3. \tag{4.5}$$

Now, by Lemma 4.9.*(3)*,

$$f(z) = cz + \sum_{(r,s)\in E(\mathcal{G}_2)} c(r,s)z_{(r,s)}, \tag{4.6}$$

where $c(r,s) = 0$ for all but a finite amount of edges $(r,s) \in E(\mathcal{G}_2)$. Then, since $df(z) = f(dz)$, on the one hand,

$$\begin{aligned} df(z) = {} & c\big[x_1^{18n-18}(x_2^2y_1y_2 - x_1x_2y_1y_3 + x_1^2y_2y_3) + x_1^{18n-7} + x_2^{15n-6}\big] \\ & + \sum_{(r,s)\in E(\mathcal{G}_2)} c(r,s)\big[x_r^3 + x_rx_sx_2^{5n-2} + x_1^{18n-7}\big], \end{aligned} \tag{4.7}$$

and on the other hand,

$$\begin{aligned} f(dz) = {} & a_1^{18n-18}x_1^{18n-18}[a_2^2b_1b_2x_2^2y_1y_2 - a_1a_2b_1b_3x_1x_2y_1y_3 + a_1^2b_2b_3x_1^2y_2y_3] \\ & + a_1^{18n-7}x_1^{18n-7} + a_2^{15n-6}x_2^{15n-6}. \end{aligned} \tag{4.8}$$

Comparing Equations (4.7) and (4.8), we immediately see that $c(r,s) = 0$, for all $(r,s) \in E(\mathcal{G}_2)$. We also obtain the following identities.

$$c = a_1^{18n-7} = a_2^{15n-6} = b_1b_2a_1^{18n-18}a_2^2 = b_1b_3a_1^{18n-17}a_2 = b_2b_3a_1^{18n-16}. \tag{4.9}$$

Equations (4.5) and (4.9) are the same as Equations (4.1) and (4.4) with $k = 3n-2$. Since $n \geq 1$, $k \geq 1$ and we deduce from the proof of Theorem 4.3 that there exists $s \in \{0,1\}$ such that $s = a_1 = a_2 = b_1 = b_2 = b_3 = c$. The result follows. □

Now we prove that, in fact, if $s = 0$ then f must be the identity map.

Lemma 4.13. *Under the assumptions of Lemma 4.12, f is the trivial morphism if and only if $s = 0$.*

Proof. One of the implications is trivial. Let us prove the remaining one, thus assume that $f(x_1) = f(x_2) = f(y_1) = f(y_2) = f(y_3) = f(z) = 0$. We still need to compute $f(x_v)$, for $v \in V(\mathcal{G}_1)$, and $f(z_{(v,w)})$, for $(v,w) \in E(\mathcal{G}_1)$. As a consequence of Lemma 4.9.*(2)*,

$$f(x_v) = \sum_{r\in V(\mathcal{G}_2)} a(v,r)x_r + a(v)x_2^{5n-2}, \quad v \in V(\mathcal{G}_1), \tag{4.10}$$

with $a(v,r)=0$ for all but a finite amount of $r\in V(\mathcal{G})$. Furthermore, by Lemma 4.9.*(3)*,

$$f(z_{(v,w)}) = e(v,w)z + \sum_{(r,s)\in E(\mathcal{G}_2)} c\big((v,w),(r,s)\big)z_{(r,s)}, \quad (v,w)\in E(\mathcal{G}_1), \tag{4.11}$$

where $c\big((v,w),(r,s)\big)=0$ for all but a finite amount of $(r,s)\in E(\mathcal{G}_2)$. Now we know that $f(dz_{(v,w)}) = df(z_{(v,w)})$. By Equation (4.11),

$$\begin{aligned} df(z_{(v,w)}) &= e(v,w)\big[x_1^{18n-18}(x_2^2y_1y_2 - x_1x_2y_1y_3 + x_1^2y_2y_3) + x_1^{18n-7} + x_2^{15n-6}\big] \\ &\quad + \sum_{(r,s)\in E(\mathcal{G}_2)} c\big((v,w),(r,s)\big)\big[x_r^3 + x_rx_sx_2^{5n-2} + x_1^{18n-7}\big]. \end{aligned} \tag{4.12}$$

On the other hand, since $s=0$ and using Equation (4.10), we have

$$\begin{aligned} f(dz_{(v,w)}) &= f(x_v^3 + x_vx_wx_2^{5n-2} + x_1^{18n-7}) \\ &= f(x_v^3) = \left[\sum_{r\in V(\mathcal{G}_2)} a(v,r)x_r + a(v)x_2^{5n-2}\right]^3. \end{aligned} \tag{4.13}$$

Comparing Equations (4.12) and (4.13), we immediately obtain that $e(v,w)=0$. Then, a summand containing x_2^{15n-6} does not appear in Equation (4.12), which implies that $a(v)=0$. But this implies that summands containing $x_rx_sx_2^{5n-2}$ do not appear in Equation (4.13). Comparing with Equation (4.12), we see that $c\big((v,w),(r,s)\big)=0$, for all $(r,s)\in E(\mathcal{G}_2)$, which implies that $df(z_{(v,w)})=0$. Consequently, $a(v,r)=0$, for all $r\in V(\mathcal{G}_2)$. Thus $f(x_v)=0$, for all $v\in V(\mathcal{G}_1)$, and $f(z_{(v,w)})=0$, for all $(v,w)\in E(\mathcal{G}_1)$. In other words f is the trivial morphism and we conclude the proof. □

We can finally prove that the functor $\mathcal{M}_n$ introduced in Definition 4.11 is *almost* fully faithful.

Theorem 4.14. *Let R be an integral domain with* $\mathrm{char}(R)>3$ *or* $\mathrm{char}(R)=0$. *For any $n\geq 1$, the functor $\mathcal{M}_n\colon \mathcal{D}igraphs_+ \to$ CDGA induces a bijective correspondence:*

$$\mathrm{Hom}_{\mathcal{D}igraphs}(\mathcal{G}_1,\mathcal{G}_2) \cong \mathrm{Hom}_{\mathrm{CDGA}}\left(\mathcal{M}_n(\mathcal{G}_1),\mathcal{M}_n(\mathcal{G}_2)\right)^*.$$

Proof. Let $f\in \mathrm{Hom}_{\mathrm{CDGA}_R}\left(\mathcal{M}_n(\mathcal{G}_1),\mathcal{M}_n(\mathcal{G}_2)\right)^*$. By Lemma 4.12, there exists $s\in\{0,1\}$ such that $f(x_1)=sx_1, f(x_2)=sx_2, f(y_1)=sy_1, f(y_2)=sy_2$, $f(y_3)=sy_3$ and $f(z)=sz$. Moreover, since f is not the trivial morphism, by Lemma 4.13, $s=1$.

Now notice that the strong connectivity of $\mathcal{G}_1$ implies that for every $v\in V(\mathcal{G}_1)$, v is the starting vertex of an edge $(v,w)\in E(\mathcal{G}_1)$. Therefore the coefficients $a(v,r)$ and $a(v)$ involved in $f(x_v)$ (see Equation (4.10)) can be entirely determined by using that $f(dz_{(v,w)}) = df(z_{(v,w)})$. So, on the one hand, $df(z_{(v,w)})$ is as in Equation (4.12), whereas

$$\begin{aligned} fd(z_{(v,w)}) &= \left[\sum_{r\in V(\mathcal{G}_2)} a(v,r)x_r + a(v)x_2^{5n-2}\right]^3 + x_1^{18n-7} \\ &\quad + \left(\sum_{r\in V(\mathcal{G}_2)} a(v,r)x_r + a(v)x_2^{5n-2}\right)\left(\sum_{s\in V(\mathcal{G}_2)} a(w,s)x_s + a(w)x_2^{5n-2}\right)x_2^{5n-2}. \end{aligned} \tag{4.14}$$

We now compare Equations (4.12) and (4.14). First, no coefficient $x_1^{18n-16}y_2y_3$ exists in Equation (4.14). Thus, $e(v,w)=0$. Now, no summand containing $x_ux_vx_w$ exists in Equation (4.12), if $u\neq v\neq w\neq u$. However, since $\mathrm{char}(R)$ is either zero or greater than three, such a summand would appear in Equation (4.14) if there were three or more non-trivial coefficients $a(v,r)$. We can then assume that there are at most two non-trivial $a(v,r)$. But if there were two non-trivial coefficients, summands containing $x_r x_s^2$ would appear in Equation (4.14). Since they do not appear in Equation (4.12), for each $v \in V(\mathcal{G}_1)$, there is at most one non-trivial coefficient $a(v,r)$. Consequently, for every $(v,w) \in E(\mathcal{G}_1)$, there is also at most one non-trivial coefficient $c\big((v,w),(r,s)\big)$.

Suppose that $c\big((v,w),(r,s)\big) = 0$, for every $(r,s) \in E(\mathcal{G}_2)$. Then, in Equation (4.12) there are no summands containing x_1^{18n-7}. But a summand x_1^{18n-7} appears in Equation (4.14). Consequently, it is not possible that $c\big((v,w),(r,s)\big) = 0$ for every $(r,s) \in E(\mathcal{G}_2)$. Since we obtained earlier that for each $(v,w) \in E(\mathcal{G}_1)$ there is at most one non-trivial coefficient $c\big((v,w),(r,s)\big)$, there exists a unique edge $(r,s) \in E(\mathcal{G}_2)$ such that $c\big((v,w),(r,s)\big) \neq 0$.

We then have that $a(v,r) \neq 0$, that is, there is exactly one non-trivial coefficient $a(v,r)$. Therefore, we can define a map $\sigma\colon V(\mathcal{G}_1) \to V(\mathcal{G}_2)$ that takes $v \in V(\mathcal{G}_1)$ to $\sigma(v)$ the only vertex in $V(\mathcal{G}_2)$ such that $a(v,\sigma(v)) \neq 0$. Moreover, if $(v,w) \in E(\mathcal{G}_1)$, the only non-trivial coefficient $c\big((v,w),(r,s)\big)$ verifies, by comparing Equations (4.12) and (4.14), that $r = \sigma(v)$ and $s = \sigma(w)$. This exhibits that $\big(\sigma(v),\sigma(w)\big) \in E(\mathcal{G}_2)$, so $\sigma \in \mathrm{Hom}_{\mathcal{D}igraphs}(\mathcal{G}_1,\mathcal{G}_2)$.

Furthermore, comparing the coefficient of x_1^{18n-7} in Equations (4.12) and (4.14), we obtain that $c\big((v,w),(\sigma(v),\sigma(w)\big) = 1$. Then, comparing the coefficients of $x^3_{\sigma(v)}$, we see that $a\big(v,\sigma(v)\big) = 1$, for all $v \in V(\mathcal{G}_1)$. Finally, notice that there are no summands $x^2_{\sigma(v)}x_2^{5n-2}$ in Equation (4.12). They would appear in Equation (4.14) if $a(v) \neq 0$, thus we deduce that $a(v) = 0$, for all $v \in V(\mathcal{G}_1)$.

Then, we have proven that there exists $\sigma \in \mathrm{Hom}_{\mathcal{D}igraphs}(\mathcal{G}_1,\mathcal{G}_2)$ such that for every $v \in V(\mathcal{G}_1)$, $f(x_v) = x_{\sigma(v)}$, and for every $(v,w) \in E(\mathcal{G}_1)$, $f(z_{(v,w)}) = z_{(\sigma(v),\sigma(w))}$. This implies that $f = \mathcal{M}_n(\sigma)$, as we claimed. □

The following result is an immediate consequence of Theorem 4.14 by taking $R = \mathbb{Q}$:

Corollary 4.15. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be objects in $\mathcal{D}igraphs_+$. Then, for $n \geq 1$,*

$$\mathrm{Hom}_{\mathrm{CDGA}}\left(\mathcal{M}_n(\mathcal{G}_1),\mathcal{M}_n(\mathcal{G}_2)\right) = [\mathcal{M}_n(\mathcal{G}_1),\mathcal{M}_n(\mathcal{G}_2)].$$

Proof. From our previous result, we deduce that the homotopy equivalence relation is trivial in $\mathrm{Hom}_{\mathrm{CDGA}}\left(\mathcal{M}_n(\mathcal{G}_1),\mathcal{M}_n(\mathcal{G}_2)\right)$. Indeed, for different elements $\sigma,\tau \in \mathrm{Hom}_{\mathcal{D}igraphs}(\mathcal{G}_1,\mathcal{G}_2)$, the induced morphisms $\mathcal{M}_n(\sigma),\mathcal{M}_n(\tau)$ have different linear parts. Hence, by Proposition 1.48, they are not homotopic. Notice also that, by the same argument, the trivial morphism in $\mathrm{Hom}_{\mathrm{CDGA}}\left(\mathcal{M}_n(\mathcal{G}_1),\mathcal{M}_n(\mathcal{G}_2)\right)$ is not homotopic to $\mathcal{M}_n(\sigma)$ for any $\sigma \in \mathrm{Hom}_{\mathcal{D}igraphs}(\mathcal{G}_1,\mathcal{G}_2)$ since the linear part of the former one is never trivial. □

Recall now that any group can be represented as the automorphism group of a graph without isolated vertices, Theorem 1.16. When regarded as a symmetric digraph, this yields an object in $\mathcal{D}igraphs_+$. We deduce our main result for this chapter:

Theorem 4.16. *Let R be an integral domain with* $\mathrm{char}(R) > 3$ *or* $\mathrm{char}(R) = 0$. *For every integer $n \geq 1$, we have the following:*

(1) For $\mathcal{G}$ an object in $\mathcal{D}igraphs_+$, $\mathrm{Aut}_{\mathrm{CDGA}_R}\left(\mathcal{M}_n(\mathcal{G})\right) \cong \mathrm{Aut}_{\mathcal{D}igraphs}(\mathcal{G})$. *Furthermore, if $R = \mathbb{Q}$, then $\mathcal{E}\left(\mathcal{M}_n(\mathcal{G})\right) \cong \mathrm{Aut}_{\mathcal{D}igraphs}(\mathcal{G})$.*

(2) CDGA_R^n *is universal, that is, for G a group, there exists $M_n \in \mathrm{Ob}(\mathrm{CDGA}_R^n)$ such that*

$$G \cong \mathrm{Aut}_{\mathrm{CDGA}_R}(M_n).$$

4.3 Generalised realisability problems in CDGA_R and $\mathcal{H}oTop$

In this section, we use the results obtained so far in this chapter together with the results regarding graphs in Chapter 2 to give a positive answer to the considered generalised realisability problems.

We begin by solving Problem 1, the realisability problem in arrow categories, in both CDGA_R and $\mathcal{H}oTop$. Let us begin with CDGA_R.

Theorem 4.17. *Let G_1, G_2 and H be groups such that $H \leq G_1 \times G_2$. Let R be an integral domain with $\mathrm{char}(R) = 0$ or $\mathrm{char}(R) > 3$. For any $n \geq 1$, there exist $M_1, M_2 \in \mathrm{Ob}(\mathrm{CDGA}_R^n)$ and a morphism $\varphi \in \mathrm{Hom}_{\mathrm{CDGA}_R}(M_1, M_2)$ such that $\mathrm{Aut}_{\mathrm{CDGA}_R}(M_k) \cong G_k$, $k = 1, 2$, and $\mathrm{Aut}_{\mathrm{CDGA}_R}(\varphi) \cong H$.*

Proof. By Theorem 2.37, there exist objects $\mathcal{G}_1$, $\mathcal{G}_2$ in $\mathcal{G}raphs$ and $\psi\colon \mathcal{G}_1 \to \mathcal{G}_2$ an object in $\mathrm{Arr}(\mathcal{G}raphs)$, such that $\mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}_k) \cong G_k$, for $k = 1, 2$, and $\mathrm{Aut}_{\mathcal{G}raphs}(\psi) \cong H$. Now we consider $\mathcal{M}_n : \mathcal{G}raphs \to \mathrm{CDGA}_R^n$ the restriction to simple graphs of the functor from Definition 4.11, and define $M_k = \mathcal{M}_n(\mathcal{G}_k)$, $k = 1, 2$, objects in CDGA_R^n and $\varphi = \mathcal{M}_n(\psi)\colon M_1 \to M_2$ an object in $\mathrm{Arr}(\mathrm{CDGA}_R^n)$. By Theorem 4.16, we obtain that $\mathrm{Aut}_{\mathrm{CDGA}_R}(M_k) \cong \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}_k) \cong G_k$, $k = 1, 2$.

We now prove that $\mathrm{Aut}_{\mathcal{G}raphs}(\psi) \cong \mathrm{Aut}_{\mathrm{CDGA}_R}(\varphi)$. First, take $(\sigma_1, \sigma_2) \in \mathrm{Aut}_{\mathcal{G}raphs}(\psi)$. Then, as $\psi \circ \sigma_1 = \sigma_2 \circ \psi$, by functoriality we also have that $\varphi \circ \mathcal{M}_n(\sigma_1) = \mathcal{M}_n(\sigma_2) \circ \varphi$. Moreover, as $\mathcal{M}_n(\sigma_k) \in \mathrm{Aut}_{\mathrm{CDGA}_R}(M_k)$, we deduce that $\big(\mathcal{M}_n(\sigma_1), \mathcal{M}_n(\sigma_2)\big) \in \mathrm{Aut}_{\mathrm{CDGA}_R}(\varphi)$.

Reciprocally, consider $(f_1, f_2) \in \mathrm{Aut}_{\mathrm{CDGA}_R}(\varphi)$. Then, as f_k is an automorphism of M_k, $k = 1, 2$, by Theorem 4.14 and Corollary 4.15, there exist $\sigma_k \in \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}_k)$ such that $f_k = \mathcal{M}_n(\sigma_k)$, $k = 1, 2$. Now, as $\big(\mathcal{M}_n(\sigma_1), \mathcal{M}_n(\sigma_2)\big) \in \mathrm{Aut}_{\mathrm{CDGA}_R}(\varphi)$, we have that $\mathcal{M}_n(\sigma_2) \circ \varphi = \varphi \circ \mathcal{M}_n(\sigma_1)$. That is, for every $v \in V(\mathcal{G}_1)$,

$$\big(\mathcal{M}_n(\sigma_2) \circ \varphi\big)(x_v) = x_{(\sigma_2 \circ \psi)(v)} = x_{(\psi \circ \sigma_1)(v)} = \big(\varphi \circ \mathcal{M}_n(\sigma_1)\big)(x_v).$$

Hence, $(\sigma_2 \circ \psi)(v) = (\psi \circ \sigma_1)(v)$ for every $v \in V(\mathcal{G}_1)$, so $(\sigma_1, \sigma_2) \in \mathrm{Aut}_{\mathcal{G}raphs}(\psi)$. Then, $\mathrm{Aut}_{\mathrm{CDGA}_R}(\varphi) = \big\{\big(\mathcal{M}_n(\sigma_1), \mathcal{M}_n(\sigma_2)\big) \mid (\sigma_1, \sigma_2) \in \mathrm{Aut}_{\mathcal{G}raphs}(\psi)\big\} \cong H$. □

Then, as a consequence of Corollary 4.15, this result can immediately be transferred to homotopy classes of CDGA morphisms by taking $R = \mathbb{Q}$.

Corollary 4.18. *Let G_1, G_2 and H be groups such that $H \leq G_1 \times G_2$. For any $n \geq 1$, there exist $M_1, M_2 \in \mathrm{Ob}(\mathrm{CDGA}^n)$ and a morphism $\varphi \in \mathrm{Hom}_{\mathrm{CDGA}}(M_1, M_2)$ such that $\mathcal{E}(M_k) \cong G_k$, $k = 1, 2$, and $\mathcal{E}(\varphi) \cong H$.*

Proof. By Theorem 2.37, there exist objects $\mathcal{G}_1$, $\mathcal{G}_2$ in $\mathcal{G}raphs$ and $\psi\colon \mathcal{G}_1 \to \mathcal{G}_2$ object in $\mathrm{Arr}(\mathcal{G}raphs)$, such that $\mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}_k) \cong G_k$, for $k = 1, 2$, and $\mathrm{Aut}_{\mathcal{G}raphs}(\psi) \cong H$. By Theorem 4.17, if we define $M_k = \mathcal{M}_n(\mathcal{G}_k)$ and $\varphi = \mathcal{M}_n(\psi)$ we know that $\mathrm{Aut}_{\mathrm{CDGA}}(M_k) = \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}_k) \cong G_k$ and $\mathrm{Aut}_{\mathrm{CDGA}}(\varphi) = \mathrm{Aut}_{\mathcal{G}raphs}(\psi) \cong H$. We shall see that $\mathcal{E}(M_k) \cong G_k$, $k = 1, 2$, and that $\mathcal{E}(\psi) \cong H$.

First, Theorem 4.16 immediately implies that $\mathcal{E}(M_k) = \mathrm{Aut}_{\mathrm{CDGA}}(M_k) \cong G_k$, $k = 1, 2$. On the other hand, it is clear that if $(\sigma_1, \sigma_2) \in \mathrm{Aut}_{\mathrm{CDGA}}(\varphi)$, then $([\sigma_1], [\sigma_2]) \in \mathcal{E}(\varphi)$. Reciprocally, if $([\sigma_1], [\sigma_2]) \in \mathcal{E}(\varphi)$, then $\sigma_2 \circ \varphi \simeq \varphi \circ \sigma_1$. However, by Corollary 4.15, $[M_1, M_2] \cong \mathrm{Hom}_{\mathrm{CDGA}}(M_1, M_2)$, thus σ_k is the only possible representative of $[\sigma_k]$, $k = 1, 2$, and $\sigma_2 \circ \varphi = \varphi \circ \sigma_1$, that is, $(\sigma_1, \sigma_2) \in \mathrm{Aut}_{\mathrm{CDGA}}(\varphi)$. Consequently, $\mathcal{E}(\varphi) \cong \mathrm{Aut}_{\mathrm{CDGA}}(\varphi) \cong H$. □

Furthermore, if G_1 and G_2 are finite groups, by Corollary 2.38, the objects $\mathcal{G}_1$, $\mathcal{G}_2$ in $\mathcal{G}raphs$ and $\psi\colon \mathcal{G}_1 \to \mathcal{G}_2$ in $\mathrm{Arr}(\mathcal{G}raphs)$ solving Problem 1 can be chosen finite. In such case, the CDGAs $M_k = \mathcal{M}_n(\mathcal{G}_k)$ in Corollary 4.18 are of finite type, allowing us to transfer our solution of Problem 1 to $\mathcal{H}oTop$.

Theorem 4.19. *Let G_1, G_2 be finite groups and let H be a subgroup of $G_1 \times G_2$. For any $n \geq 1$, there exist n-connected spaces X_1, X_2 and a continuous map $f\colon X_1 \to X_2$ such that $\mathcal{E}(X_k) \cong G_k$, $k = 1, 2$, and $\mathcal{E}(f) \cong H$.*

Proof. By Corollary 4.18, there exist n-connected minimal Sullivan algebras M_1 and M_2 and a morphism of algebras $\varphi\colon M_2 \to M_1$ such that $\mathcal{E}(M_k) \cong G_k$, $k = 1, 2$, and $\mathcal{E}(\varphi) \cong H$. Consider $|\quad|$ the spatial realisation functor. We have seen in Section 1.4 that this contravariant functor induces a bijective correspondence between homotopy classes of morphisms between two Sullivan algebras and homotopy classes of maps between their spatial realisations. Furthermore, two morphisms between Sullivan algebras are homotopic if and only if their spatial realisations are homotopic.

Define $X_k = |M_k|$, $k = 1, 2$, and $f = |\varphi|\colon X_1 \to X_2$. Considering the bijection mentioned above, it is immediate that $\mathcal{E}(X_k) = \mathcal{E}(M_k) \cong G_k$, $k = 1, 2$. It remains to see that $\mathcal{E}(f) \cong \mathcal{E}(\varphi) \cong H$.

Assume first that $(f_2, f_1) \in \mathcal{E}(\varphi)$, thus $f_1 \circ \varphi \simeq \varphi \circ f_2$. Then, by the functoriality of $|\quad|$ and Proposition 1.52, $f \circ |f_1| \simeq |f_2| \circ f$. Thus $\big(|f_1|, |f_2|\big) \in \mathcal{E}(f)$. On the other hand, consider $(f_1, f_2) \in \mathcal{E}(f)$. Then, take φ_k a representative of the only homotopy class such that $|\varphi_k| \simeq f_k$, $k = 1, 2$. Since $f \circ f_1 \simeq f_2 \circ f$ with $f = |\varphi|$, we obtain that $\varphi_1 \circ \varphi \simeq \varphi \circ \varphi_2$, thus $(\varphi_2, \varphi_1) \in \mathcal{E}(\varphi)$. Consequently, $\mathcal{E}(f) \cong \mathcal{E}(\varphi) \cong H$. □

We finish this chapter by transferring the solution to the realisability problem for permutation representations, Problem 2, from graphs to CDGAs and $\mathcal{H}oTop$. We remark that the results proved in the remainder of this section are a particular case of those in [29]. Nonetheless, we include them here for the sake of completion and since they can be easily deduced from the results obtained so far. Let us begin with the category of CDGAs.

Theorem 4.20. *Let G be a group, $n \geq 1$ be an integer, R be an integral domain with $\operatorname{char}(R) = 0$ or $\operatorname{char}(R) > 3$ and $\rho\colon G \to \operatorname{Sym}(V)$ be a permutation representation of G on a set V. There is an object $A \in \operatorname{Ob}(\mathrm{CDGA}_R^n)$ such that*

(1) $V \subset A$, and V is invariant through the automorphisms of A;

(2) $\operatorname{Aut}_{\mathrm{CDGA}_R}(A) \cong G$;

(3) the restriction map $G \cong \operatorname{Aut}_{\mathrm{CDGA}_R}(A) \to \operatorname{Sym}(V)$ is precisely ρ.

Proof. By Theorem 2.41, there is a graph $\mathcal{G}$ verifying properties akin to Theorem 4.20.*(1)*–*(3)* in the category of $\mathcal{G}raphs$. Consider $\mathcal{M}_n$ the functor from Definition 4.6 and define $A = \mathcal{M}_n(\mathcal{G})$. We shall prove that this CDGA verifies the desired properties.

First, by Theorem 4.16, $\operatorname{Aut}_{\mathrm{CDGA}_R}(A) \cong \operatorname{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \cong G$, so Theorem 4.20.*(2)* holds. Now, as a consequence of Theorem 2.41.*(1)*, there is a subset $V \subset V(\mathcal{G})$ invariant through the automorphisms of $\mathcal{G}$. Let us identify V with the subset $\{x_v \mid v \in V \subset V(\mathcal{G})\} \subset A$. Since the automorphism of A associated to $\sigma \in \operatorname{Aut}_{\mathcal{G}raphs}(\mathcal{G})$ is the map $\mathcal{M}_n(\sigma)$ introduced in Lemma 4.10, and given that $\mathcal{M}_n(\sigma)(x_v) = x_{\sigma(v)}$, V is invariant through the automorphisms of A, so Theorem 4.20.*(1)* follows. Not only that, but the restriction of $G \cong \operatorname{Aut}_{\mathrm{CDGA}_R}(A)$ to $\operatorname{Sym}(V)$ is equivalent to the restriction of $G \cong \operatorname{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \to \operatorname{Sym}(V)$, thus it is precisely ρ, proving Theorem 4.20.*(3)*. □

Then, as a consequence of Theorem 4.16, the result above can immediately be transferred to homotopy classes of morphisms of CDGA by taking $R = \mathbb{Q}$.

Corollary 4.21. *Let G be a group, $n \geq 1$ be an integer and $\rho\colon G \to \operatorname{Sym}(V)$ be a permutation representation of G on a set V. There is an object $A \in \operatorname{Ob}(\mathrm{CDGA}^n)$ such that*

(1) $V \subset A$*, and* V *is invariant through the automorphisms of* A*;*

(2) $\mathcal{E}(A) \cong G$*;*

(3) the restriction map $G \cong \mathcal{E}(A) \to \mathrm{Sym}(V)$ *is precisely* ρ*.*

Furthermore, if G and V are finite, by Corollary 2.42 the graph $\mathcal{G}$ solving Problem 2 can be built so that it is finite, in which case the CDGA $A = \mathcal{M}_n(\mathcal{G})$ solving Problem 2 in Corollary 4.21 is of finite type. We are then able to obtain the following consequence in the homotopy category of spaces:

Theorem 4.22. *Let* G *be a finite group,* V *be a finite set,* n *be a positive integer and* $\rho\colon G \to \mathrm{Sym}(V)$ *be a permutation representation of* G *on* V*. There is an* n*-connected space* X *such that*

(1) $V \subset H^{180n^2-142n+28}(X)$*, and* V *is invariant through the maps induced in cohomology by the self-homotopy equivalences of* X*;*

(2) $\mathcal{E}(X) \cong G$*;*

(3) the map $G \cong \mathcal{E}(X) \to \mathrm{Sym}(V)$ *taking* $[f] \in \mathcal{E}(X)$ *to* $H^{180n^2-142n+28}(f)|_V \in \mathrm{Sym}(V)$ *is* ρ*.*

Proof. Take $R = \mathbb{Q}$. By Corollary 2.42, there is a finite graph $\mathcal{G}$ verifying properties akin to Theorem 4.22.*(1)–(3)* in the category of $\mathcal{G}raphs$. Consider $\mathcal{M}_n$ the functor from Definition 4.6. Since $\mathcal{G}$ is finite, $\mathcal{M}_n(\mathcal{G})$ is a CDGA of finite type, so we can define $X = |\mathcal{M}_n(\mathcal{G})|$. We shall prove that this is the desired space.

First, as a consequence of Corollary 2.42.*(1)*, V can be identified with a subset $V \subset V(\mathcal{G})$ invariant through the automorphisms of $\mathcal{G}$. Recall that in Theorem 4.20 we identified $V \subset V(\mathcal{G})$ with $\{x_v \mid v \in V \subset V(\mathcal{G})\} \subset \mathcal{M}_n(\mathcal{G})$. Elements x_v, $v \in V(\mathcal{G})$ are clearly independent cocycles. Then, since $H^*(\mathcal{M}_n(\mathcal{G})) \cong H^*(X)$, we may identify $V \equiv \{[x_v] \mid v \in V \subset V(\mathcal{G})\} \subset H^{180n^2-142n+28}(X)$.

Now recall that, as a consequence of the properties of the spatial realisation functor and of Theorem 4.16, $\mathcal{E}(X) \cong \mathcal{E}(\mathcal{M}_n(\mathcal{G})) \cong G$, proving Theorem 4.22.*(2)*. Not only that, but self-homotopy equivalences corresponding through the isomorphism $\mathcal{E}(X) \cong \mathcal{E}(\mathcal{M}_n(\mathcal{G}))$ induced by the spatial realisation functor must induce the same map on cohomology. Then, since all maps $\mathcal{M}_n(\sigma) \in \mathcal{E}(\mathcal{M}_n(\mathcal{G}))$ are invariant on V, so are the maps in $\mathcal{E}(X)$, proving Theorem 4.22.*(1)*.

For the same reason, the map taking $[f] = [|\mathcal{M}_n(\sigma)|] \in \mathcal{E}(X)$ to $H^{180n^2-142n+28}(f)|_V \in \mathrm{Sym}(V)$ is equivalent to the map taking $[\mathcal{M}_n(\sigma)] \in \mathcal{E}(A)$ to $H^{180n^2-142n+28}(\mathcal{M}_n(\sigma))|_V \in \mathrm{Sym}(V)$. Furthermore, $\mathcal{M}_n(\sigma)(x_v) = x_{\sigma(v)}$. Then Theorem 4.22.*(3)* follows from Corollary 2.42.*(3)*. □

CHAPTER 5

Further applications to the family of functors from digraphs to CDGAs

In the previous chapter, we constructed a family of *almost* fully faithful functors $\mathcal{M}_n$ from a full subcategory of digraphs denoted $\mathcal{D}igraphs_+$ (see Definition 4.4) to CDGA_R. The aim of this chapter is to provide further applications to our functors.

In Section 5.1 we make use of our functors $\mathcal{M}_n$ in combination with Theorem 1.20 and Theorem 1.21 to prove results regarding the representability of concrete categories in CDGA_R^n, Theorem 5.1, and in $\mathcal{H}oTop_f^n$, Theorem 5.2. We also obtain results on the realisability of monoids as monoids of endomorphisms of commutative differential graded R-algebras, and as monoids of homotopy classes of self-maps of spaces, Corollary 5.3.

Then, in Section 5.2 we use the minimal Sullivan models introduced in Definition 4.6 to show that, under certain conditions, we can distinguish isomorphism classes of groups by means of the differential graded R-algebras on which they act faithfully, Theorem 5.4. This theorem provides a generalisation of the main result in [28]. Moreover, we are able to extend this result to a certain family of monoids that includes all finite monoids without zero, Proposition 5.7, using our results from Section 5.1.

In Section 5.3 we show that our minimal Sullivan algebras introduced in Section 4.2 provide an infinite amount of examples of highly connected inflexible manifolds, Theorem 5.12. We are also able to produce examples of strongly chiral manifolds, that is, manifolds that do not admit orientation reversing self-maps of degree -1, Proposition 5.14.

Finally, in Section 5.4 we use our minimal Sullivan models models to show that the numerical homotopy invariants involved in a certain lower bound for the Lusternik-Schnirelmann category of a finite dimensional space can be arbitrarily different, Theorem 5.17.

5.1 Representation of categories in CDGA_R and $\mathcal{H}oTop$

In this section, we use the functors $\mathcal{M}_n\colon \mathcal{D}igraphs_+ \to \mathrm{CDGA}_R^n$ introduced in Definition 4.11 to obtain results on the representability of categories in both CDGA_R and $\mathcal{H}oTop$. Recall from Definition 1.18 that a category $\mathcal{C}$ is said to be representable in another category $\mathcal{D}$ if there is a fully faithful functor from $\mathcal{C}$ to $\mathcal{D}$. Thus, by Theorem 4.14 $\mathcal{M}_n$ *almost* induces a representation of $\mathcal{D}igraphs_+$ in CDGA_R^n, but we are quite not there, as the homotopy class of the trivial map is never reached by $\mathcal{M}_n$.

Nonetheless, we can still regard $\mathcal{D}igraphs_+$ as a subcategory of CDGA_R where the objects are all the possible images of digraphs through $\mathcal{M}_n$, and the morphisms between two such objects are the non-trivial morphisms of CDGA_R between them. Therefore, we are still representing $\mathcal{D}igraphs_+$ as an *almost* full subcategory of CDGA_R. And it is as close as we can get to a representation of $\mathcal{D}igraphs_+$ in CDGA_R, since CDGA_R is pointed whereas $\mathcal{D}igraphs_+$ is not. In a similar way, we can also use the family of functors $\mathcal{M}_n$ together with Theorem 1.20 to represent a large family of categories in both CDGA_R and $\mathcal{H}oTop$, as follows:

Theorem 5.1. *Let $\mathcal{C}$ be a concrete small category and R be an integral domain such that* $\mathrm{char}(R) > 3$ *or* $\mathrm{char}(R) = 0$. *For every $n \geq 1$, there is a functor $G_n\colon \mathcal{C} \to \mathrm{CDGA}_R^n$ verifying that* $\mathrm{Hom}_{\mathrm{CDGA}_R}\left(G_n(A), G_n(B)\right)^* \cong \mathrm{Hom}_{\mathcal{C}}(A, B)$*, for any $A, B \in \mathrm{Ob}(\mathcal{C})$. Furthermore, if $R = \mathbb{Q}$, then $[G_n(A), G_n(B)]^* \cong \mathrm{Hom}_{\mathcal{C}}(A, B)$, for all $A, B \in \mathrm{Ob}(\mathcal{C})$.*

Proof. As a consequence of Theorem 1.20, for any concrete small category $\mathcal{C}$ there exists a fully faithful functor $G\colon \mathcal{C} \to \mathcal{D}igraphs_+$. Thus, if we define $G_n = \mathcal{M}_n \circ G\colon \mathcal{C} \to \mathrm{CDGA}_R$, by Theorem 4.14 we obtain that

$$\mathrm{Hom}_{\mathrm{CDGA}_R}\left(G_n(A), G_n(B)\right)^* = \mathrm{Hom}_{\mathcal{D}igraphs}\left(G(A), G(B)\right) = \mathrm{Hom}_{\mathcal{C}}(A, B).$$

Furthermore, if $R = \mathbb{Q}$, by Corollary 4.15,

$$[G_n(A), G_n(B)]^* = \mathrm{Hom}_{\mathrm{CDGA}}\left(G_n(A), G_n(B)\right)^* = \mathrm{Hom}_{\mathcal{C}}(A, B).$$ □

A similar result can be achieved in $\mathcal{H}oTop$. In order to have a bijection between homotopy classes of morphisms of CDGA and homotopy classes of continuous maps between their Sullivan spatial realisations, we have to restrict ourselves to CDGAs of finite type. This forces, by the way we have constructed $\mathcal{M}_n$, our graphs to be finite, so we can apply Theorem 1.21.

Theorem 5.2. *Let $\mathcal{C}$ be a concrete category such that* $\mathrm{Ob}(\mathcal{C})$ *is countable and* $\mathrm{Hom}_{\mathcal{C}}(A, B)$ *is finite for any pair of objects $A, B \in \mathrm{Ob}(\mathcal{C})$. For every $n \geq 1$, there exist a functor $F_n\colon \mathcal{C} \to \mathcal{H}oTop^n$ such that $[F_n(A), F_n(B)]^* = \mathrm{Hom}_{\mathcal{C}}(A, B)$, for any $A, B \in \mathrm{Ob}(\mathcal{C})$.*

Proof. First, notice that the category $\mathrm{Set}^{\mathrm{op}}$ is concrete, since the functor $\mathrm{Hom}(-, \mathbf{2})\colon \mathrm{Set}^{\mathrm{op}} \to \mathrm{Set}$, where $\mathbf{2}$ denotes a set of two elements, is clearly faithful. Given that the composition of faithful functors and the dual of a faithful functor are both faithful, we deduce that if $\mathcal{C}$ is concrete, so is $\mathcal{C}^{\mathrm{op}}$. Then, as a consequence of Theorem 1.21 there is a fully faithful functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{G}raphs_f$ such that $F(A)$ does not have isolated vertices, for any $A \in \mathrm{Ob}(\mathcal{C})$. By regarding F as a contravariant functor, we can define a functor $F_n = |\quad| \circ \mathcal{M}_n \circ F\colon \mathcal{C} \to \mathcal{H}oTop_*$ which takes any object of $\mathcal{C}$ into a Sullivan algebra of finite type. Using the properties of the spatial realisation functor and applying Theorem 4.14 and Corollary 4.15, $\left[F_n(A), F_n(B)\right]^* = [\mathcal{M}_n(F(B)), \mathcal{M}_n(F(A))]^* = \mathrm{Hom}_{\mathcal{G}raphs}\left(F(B), F(A)\right) = \mathrm{Hom}_{\mathcal{C}}(A, B)$ for any $A, B \in \mathcal{C}$. □

Recall now that a monoid M can be regarded as a one object category. When doing so, M is the monoid of endomorphisms of the single object of the category. Such a category is clearly small and concrete, and it is finite whenever M is so, in which case it fits in the hypothesis of Theorem 5.2. Also notice that if M has a zero element, it becomes a zero endomorphism of the only object in the category associated to M. If we denote the monoid obtained from M by adjoining a zero element by M^0, the following result is an immediate consequence of Theorem 5.1 and Theorem 5.2:

Corollary 5.3. *Let M be a monoid. For every $n \geq 1$, there exists an n-connected commutative differential graded R-algebra A_n such that* $\mathrm{Hom}_{\mathrm{CDGA}_R}(A_n, A_n) \cong M^0$*. If moreover M is finite, there exists an n-connected space X_n such that $[X_n, X_n] \cong M^0$.*

In particular, if $M \cong N^0$ for some other monoid N, that is, if M has a zero element and no non-trivial zero divisors, we can realise it directly.

5.2 The isomorphism problem for groups through CDGA representations

In this section we study the isomorphism problem for groups using group representations on CDGAs. We generalise the main result in [28], proving that co-Hopfian groups (see Definition 3.14) can be distinguished by the commutative differential graded R-algebras on which they act faithfully, Theorem 5.4. We are also able to extend this result to an analogous class of monoids without zero, Proposition 5.7. Let us begin by proving Theorem 5.4.

Theorem 5.4. *Let R be an integral domain with $\operatorname{char}(R) > 3$ or $\operatorname{char}(R) = 0$. Let $n \geq 1$ be an integer and let G and H be co-Hopfian groups. The following are equivalent:*

(1) G and H are isomorphic.

(2) For any n-connected commutative differential graded R-algebra (A, d), G acts faithfully on (A, d) if and only if H acts faithfully on (A, d).

Proof. One of the two implications is immediate. We prove the remaining one. Thus, assume that Theorem 5.4.*(2)* holds. As a consequence of Theorem 1.16, there exist graphs without isolated vertices $\mathcal{G}$ and $\mathcal{H}$ such that $\operatorname{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \cong G$ and $\operatorname{Aut}_{\mathcal{G}raphs}(\mathcal{H}) \cong H$. Consider the n-connected commutative differential graded R-algebras $\mathcal{M}_n(\mathcal{G})$ and $\mathcal{M}_n(\mathcal{H})$ introduced in Definition 4.6. As a consequence of Theorem 4.16, $\operatorname{Aut}_{\mathrm{CDGA}_R}(\mathcal{M}_n(\mathcal{G})) \cong G$ and $\operatorname{Aut}_{\mathrm{CDGA}_R}(\mathcal{M}_n(\mathcal{H})) \cong H$.

Clearly, G acts faithfully on $\mathcal{M}_n(\mathcal{G})$. Then by Theorem 5.4.*(2)*, H acts faithfully on $\mathcal{M}_n(\mathcal{G})$, which implies that there is a group monomorphism $H \hookrightarrow \operatorname{Aut}_{\mathrm{CDGA}_R}(\mathcal{M}_n(\mathcal{G})) \cong G$, thus $H \leq G$. Similarly, H acts faithfully on $\mathcal{M}_n(\mathcal{H})$, which by Theorem 5.4.*(2)* implies that G acts faithfully on $\mathcal{M}_n(\mathcal{H})$, thus there is a group monomorphism $G \hookrightarrow \operatorname{Aut}_{\mathrm{CDGA}_R}(\mathcal{M}_n(\mathcal{H})) \cong H$. Therefore, $G \leq H$. We then have that $G \leq H \leq G$ and, since G is co-Hopfian, $G \cong H$. $\square$

In comparison to [28, Theorem 1.1], Theorem 5.4 requires less restrictions on both the ring R and the groups G and H. Indeed, [28, Theorem 1.1] requires for R to not to have primitive third roots of the unity, whereas in Theorem 5.4 the base ring R does not have such a requirement. Furthermore, we only require for G and H to be co-Hopfian, whereas [28, Theorem 1.1] requires that both groups are isomorphic to the automorphism group of a locally finite graph and that all of their abelian normal subgroups have p-torsion with $p \neq \operatorname{char}(R)$.

This result can also be extended to actions of co-Hopfian monoids, that we now introduce.

Definition 5.5. A monoid M is *co-Hopfian* if any monomorphism $G \hookrightarrow G$ is an isomorphism.

Although the literature on co-Hopfian monoids is sparse, this class contains all finite monoids. Moreover, co-Hopfian groups are, in particular, co-Hopfian monoids.

We can then think about proving an analogous to Theorem 5.4. However, there is a slight difference. Indeed, we have not proven that every monoid is realisable as the monoid of endomorphisms of a commutative differential graded R-algebra, since the monoids we are realising have an added zero element. Thus, we need the following result:

Lemma 5.6. *Let M and N be monoids and suppose that N has a zero element, 0_N. Then, if there exists a monomorphism of monoids $f\colon M \hookrightarrow N$ such that $0_N \in \operatorname{Im}(f)$, M has a zero element.*

Proof. Assume that $f\colon M \to N$ is a monomorphism of monoids. In the category of monoids, monomorphisms are precisely the injective maps, thus we can assume that f is injective. Suppose that there exists $m \in M$ such that $f(m) = 0_N$. Then, for every $m' \in M$, $0_N = f(m')f(m) = f(m'm)$. Since f is injective, this means that $m'm = m$. Similarly, we obtain that $mm' = m$, for all $m' \in M$. But by definition this means that m is a zero element in M. The result follows. □

We can now prove the following result:

Proposition 5.7. *Let R be an integral domain with $\operatorname{char}(R) > 3$ or $\operatorname{char}(R) = 0$. Let $n \geq 1$ be an integer and let M and N be co-Hopfian monoids without zero. The following are equivalent.*

(1) M and N are isomorphic.

(2) For any n-connected commutative differential graded R-algebra (A, d), M acts faithfully on (A, d) if and only if N acts faithfully on (A, d).

Proof. One of the two implications is immediate. We prove the remaining one. Thus, assume that Proposition 5.7.*(2)* holds. As a consequence of Corollary 5.3, there exist n-connected commutative differential graded R-algebras (A, d) and (B, d) such that $\operatorname{Hom}_{\mathrm{CDGA}_R}(A, A) \cong M^0$ and $\operatorname{Hom}_{\mathrm{CDGA}_R}(B, B) \cong N^0$.

Clearly, M acts faithfully on (A, d). Then by Proposition 5.7.*(2)*, N acts faithfully on (A, d) as well. Thus, there is a monomorphism $N \hookrightarrow \operatorname{Hom}_{\mathrm{CDGA}_R}(A, A) \cong M^0$. As a consequence of Lemma 5.6 and since N does not have a zero element, the zero of M^0 is not in the image of the injection. We deduce that $N \leq M$.

Similarly, N acts faithfully on (B, d), which by Proposition 5.7.*(2)* implies that M acts faithfully on (B, d). Thus, there is a monomorphism $M \hookrightarrow \operatorname{Hom}_{\mathrm{CDGA}_R}(B, B) \cong N^0$. Therefore, $M \leq N^0$, which in a similar manner implies that $M \leq N$. We then have that $M \leq N \leq M$ and, since M is co-Hopfian, $M \cong N$. □

5.3 Highly connected inflexible and strongly chiral manifolds

A closed, oriented and connected manifold M is said to be inflexible if the set of all the possible degrees of its continuous self-maps is finite. As the degree is multiplicative, this condition is equivalent to asking for the set of all the possible degrees to be a subset of $\{-1, 0, 1\}$. On the other hand, a strongly chiral manifold is a manifold that does not admit orientation-reversing self-maps, that is, self-maps of degree -1. In this section we show how we can produce examples of both inflexible and strongly chiral manifolds from the algebras introduced in Section 4.2.

Inflexible manifolds naturally appear within the framework of functorial seminorms on singular homology developed by Gromov [47, 48] and derived degree theorems (e.g. [31, Remark 2.6]): let M be a closed, oriented and connected manifold with fundamental class c_M. If there exists a functorial seminorm on singular homology $|\cdot|$ such that $|c_M| \neq 0$, then M is inflexible. In this way, oriented closed connected hyperbolic manifolds are shown to be inflexible; they do have non-trivial simplicial volume, the value of the ℓ^1-seminorm applied to the fundamental class [47, Section 0.3]. But the ℓ^1-seminorm is trivial on simply connected manifolds [47, Section 3.1], which led Gromov to raise the question of whether every functorial seminorm on singular homology is trivial on all simply connected spaces [48, Remark (b) in 5.35]. This question is solved in the negative in [31] by constructing functorial seminorms associated to simply connected inflexible manifolds. Therefore, inflexible manifolds are extraordinary objects and still not many examples are known. Indeed, all the examples found in the literature show low levels of connectivity when observing their minimal Sullivan models [2, 6, 27, 31].

In this section we work with rational models of spaces, thus we assume that $R = \mathbb{Q}$. We closely follow the lines of [27] and provide new examples of inflexible manifolds whose Sullivan models are as highly connected as desired (see Corollary 5.13). For such reason, we need to define inflexible CDGAs, in such a way that when there is a manifold whose rational homotopy type is represented by an inflexible CDGA, then the manifold is also inflexible.

Definition 5.8. Let M be a CDGA whose cohomology is a Poincaré duality algebra of formal dimension m (see Definition 1.57). Take $\omega_M \in H^m(M)$ the fundamental class of $H^*(M)$. Given $f \in \mathrm{Hom}_{\mathrm{CDGA}}(M, M)$, we say that *the degree of* f is $k \in \mathbb{Q}$, denoted $\deg(f) = k$, if $H^*(f)(c_M) = kc_M$. We say that M is *inflexible* if the set of all possible degrees of its self-maps is finite.

Since the degree of a morphism is clearly multiplicative, M is inflexible if and only if $\{\deg(f) \mid f \in \mathrm{Hom}_{\mathrm{CDGA}}(M, M)\} \subset \{-1, 0, 1\}$. Our first step will be to prove that our algebras are inflexible.

Lemma 5.9. *Let $\mathcal{G}$ be a finite object in $\mathcal{D}igraphs_+$ and let $n, k \geq 1$ be integers.*

(1) The minimal Sullivan algebra $\mathcal{M}_k$ from Definition 4.1 is an inflexible elliptic Sullivan algebra of formal dimension $(6k + 16)|x_1|$.

(2) The minimal Sullivan algebra $\mathcal{M}_n(\mathcal{G})$ from Definition 4.6 is an inflexible elliptic Sullivan algebra of formal dimension $(18n + 4)|x_1| + |E(\mathcal{G})||z| - |V(\mathcal{G})|\frac{|z|-2}{3}$.

Proof. In both cases, we use Proposition 1.56 to prove the ellipticity of the algebras by proving the ellipticity of their associated pure Sullivan algebras. Then, by Proposition 1.58, their cohomologies are Poincaré duality algebras and their formal dimensions follow from a straight computation.

(1) Denote $\mathcal{M}_k = (\Lambda V, d)$ and consider $(\Lambda V, d_\sigma)$ its associated pure Sullivan algebra. The differential d_σ is defined as

$$\begin{aligned} &d_\sigma(x_1) = 0, && d_\sigma(x_2) = 0, && d_\sigma(z) = x_1^{6k+5} + x_2^{5k+4}, \\ &d_\sigma(y_1) = x_1^3 x_2, && d_\sigma(y_2) = x_1^2 x_2^2, && d_\sigma(y_3) = x_1 x_2^3. \end{aligned}$$

The cohomology of $(\Lambda V, d_\sigma)$ is finite-dimensional since

$$d_\sigma(zx_1 + y_3 x_2^{5k+1}) = x_1^{6k+6}, \quad d_\sigma(zx_2 - y_1 x_1^{6k+2}) = x_2^{5k+5}.$$

Thus $\mathcal{M}_k$ is elliptic. Moreover, by Theorem 4.3, $\mathrm{Hom}_{\mathrm{CDGA}}(\mathcal{M}_k, \mathcal{M}_k) = \{\mathrm{id}, 0\}$. These two maps have respective degrees 1 and 0. Therefore, $\mathcal{M}_k$ is inflexible.

(2) Since $\mathcal{G}$ is finite, $\mathcal{M}_n(\mathcal{G})$ is of finite type, so we can apply Proposition 1.56. Denote $\mathcal{M}_n(\mathcal{G}) = (\Lambda V, d)$ and consider $(\Lambda V, d_\sigma)$ the associated pure Sullivan algebra. The differential d_σ is then defined as

$$\begin{aligned} &d_\sigma(x_1) = 0, && d_\sigma(y_1) = x_1^3 x_2, && d_\sigma(z) = x_1^{18n-7} + x_2^{15n-6}, \\ &d_\sigma(x_2) = 0, && d_\sigma(y_2) = x_1^2 x_2^2, && d_\sigma(z_{(v,w)}) = x_v^3 + x_v x_w x_2^{5n-2} + x_1^{18n-7}, \\ &d_\sigma(x_v) = 0, && d_\sigma(y_3) = x_1 x_2^3. \end{aligned}$$

Then, on the one hand,

$$d_\sigma(zx_1 - y_3 x_2^{15n-9}) = x_1^{18n-6}, \qquad d_\sigma(zx_2 - y_1 x_1^{18n-10}) = x_2^{15n-5}.$$

On the other hand, given $v \in V(\mathcal{G})$, the strong connectivity of $\mathcal{G}$ implies that v is the starting vertex of at least one edge $(v,w) \in E(\mathcal{G})$, and from $dz_{(v,w)}$ we obtain that

$$[x_v^3]^4 = [-x_v x_w x_2^{5n-2} - x_1^{18n-7}]^4 = 0.$$

This proves the ellipticity of the algebra. To prove that it is inflexible notice that, as a consequence of Theorem 4.14, $\operatorname{Hom}_{\mathrm{CDGA}}\left(\mathcal{M}_n(\mathcal{G}), \mathcal{M}_n(\mathcal{G})\right)^* = \operatorname{Hom}_{\mathcal{D}igraphs}(\mathcal{G}, \mathcal{G})$ is finite, thus the set of possible degrees of self-maps must be finite.

We have thus proven the result. □

Remark 5.10. Note that when applying $\mathcal{M}_n$ to a graph $\mathcal{G}$ without isolated vertices, we are regarding it as a symmetric digraph. Therefore, every edge in the graph gives raise to two directed edges between the same pair of vertices. Then, as a consequence of Lemma 5.9, the formal dimension of $\mathcal{M}_n(\mathcal{G})$ is $(18n+4)|x_1| + 2|E(\mathcal{G})||z| - |V(\mathcal{G})|\frac{|z|-2}{3}$.

Now, we recall that for elliptic Sullivan algebras, the Barge and Sullivan obstruction theory ([9, 75]) decides if there exists a manifold (over $\mathbb{Z}$) of the same rational homotopy type of that algebra. Roughly speaking, the obstruction theory is trivial when the formal dimension of the algebra is not divisible by four. We shall avoid this situation with our examples.

Take G a finite group and consider $\mathcal{G}$ a finite graph such that $\operatorname{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \cong G$, see Theorem 1.16. It follows from Lemma 5.9.*(2)* that the formal dimension of $\mathcal{M}_n(\mathcal{G})$ is odd if and only if $\mathcal{G}$ has an odd number of vertices. We shall prove that we can choose $\mathcal{G}$ so that it verifies this property.

Lemma 5.11. *Let G be a finite group. There exists a finite graph $\mathcal{G}$ without isolated vertices and with an odd (respectively even) number of vertices such that $\operatorname{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \cong G$.*

Proof. As a consequence of Theorem 1.16, there exists a finite graph without isolated vertices $\tilde{\mathcal{G}}$ with $\operatorname{Aut}_{\mathcal{G}raphs}(\tilde{\mathcal{G}}) \cong G$. If $|V(\tilde{\mathcal{G}})|$ is odd (respectively even), define $\mathcal{G} = \tilde{\mathcal{G}}$. Otherwise, consider $\mathcal{G}$ a graph with vertices $V(\tilde{\mathcal{G}}) \sqcup \{v\}$ and edges $E(\mathcal{G}) = E(\tilde{\mathcal{G}}) \sqcup \{\{v,w\} \mid w \in V(\tilde{\mathcal{G}})\}$. Clearly, $\mathcal{G}$ has an odd (respectively even) number of vertices. We shall prove that $\operatorname{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \cong \operatorname{Aut}_{\mathcal{G}raphs}(\tilde{\mathcal{G}})$.

First, suppose that $\tilde{\varphi} \in \operatorname{Aut}_{\mathcal{G}raphs}(\tilde{\mathcal{G}})$. We define a map $\varphi\colon V(\mathcal{G}) \to V(\mathcal{G})$ as follows. Define $\varphi(v) = v$, and for $w \in V(\tilde{\mathcal{G}})$, define $\varphi(w) = \tilde{\varphi}(w)$. It is then immediate that φ is a morphism of graphs and, moreover, it is an automorphism; its inverse is the map φ^{-1} arising from $\tilde{\varphi}^{-1}$ using the same construction.

Then, to prove our lemma it is enough to show that every automorphism of $\mathcal{G}$ arises this way. Take $\varphi \in \operatorname{Aut}_{\mathcal{G}raphs}(\mathcal{G})$. Notice that the graph $\tilde{\mathcal{G}}$ is obtained from a Cayley colour diagram of G by replacing each directed edge by a certain construction containing several vertices and edges. This implies that no vertex in the graph $\tilde{\mathcal{G}}$ may be connected to every other vertex of $\tilde{\mathcal{G}}$. Thus, for each $w \in V(\tilde{\mathcal{G}})$, the degree of w in $\mathcal{G}$ is strictly lower than that of v. We deduce that $\varphi(v) = v$. Then, φ restricts to a map $\varphi|_{V(\tilde{\mathcal{G}})}\colon V(\tilde{\mathcal{G}}) \to V(\tilde{\mathcal{G}})$. Since the full subgraph of $\mathcal{G}$ with vertices $V(\tilde{\mathcal{G}})$ is precisely $\tilde{\mathcal{G}}$, $\varphi|_{V(\tilde{\mathcal{G}})} \in \operatorname{Aut}_{\mathcal{G}raphs}\left(V(\tilde{\mathcal{G}})\right)$. We now deduce that $\operatorname{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \cong \operatorname{Aut}_{\mathcal{G}raphs}(\tilde{\mathcal{G}})$. □

We can now prove the main theorem in this section:

Theorem 5.12. *For any finite group G and any integer $n \geq 1$, G is the group of self-homotopy equivalences of the rationalization of an inflexible manifold which is $(30n-13)$-connected.*

Proof. As a consequence of Lemma 5.11, there exists a finite graph $\mathcal{G}$ without isolated vertices and with an odd number of vertices such that $\mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \cong G$. Then, from Lemma 5.9.*(2)*, we deduce that the formal dimension of $\mathcal{M}_n(\mathcal{G})$ is odd. Thus, the obstruction theory of Barge, [9], and Sullivan, [75], is trivial, so there exists a $(30n-13)$-connected manifold $M_n(\mathcal{G})$ whose rationalisation is of the homotopy type of $\mathcal{M}_n(\mathcal{G})$.

Now consider $f\colon M_n(\mathcal{G}) \to M_n(\mathcal{G})$ and take $\varphi\colon \mathcal{M}_n(\mathcal{G}) \to \mathcal{M}_n(\mathcal{G})$ a Sullivan representative of f. Then $H^*(\varphi) = H^*(f)\colon H^*(M_n(\mathcal{G}), \mathbb{Q}) \to H^*(M_n(\mathcal{G}), \mathbb{Q})$. Thus, $\deg(f) = \deg(\varphi)$. In particular, since $\mathcal{M}_n(\mathcal{G})$ is inflexible, so is $M_n(\mathcal{G})$. Furthermore,

$$G \cong \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \cong \mathcal{E}(\mathcal{M}_n(\mathcal{G})) \cong \mathcal{E}(M_n(\mathcal{G})_0),$$

thus $M_n(\mathcal{G})$ is the desired manifold. □

In particular, we immediately obtain the following:

Corollary 5.13. *There exist infinitely many non-homotopically equivalent inflexible manifolds as highly connected as desired.*

We now turn to the existence of strongly chiral manifolds, that is, manifolds that do not admit orientation-reversing self-maps of degree -1 (see [2], [27] or [69]). We know that our algebras are inflexible. However, seeing that they do not admit self-maps of degree -1 is more involved. Instead, we will use a construction in [27] to obtain strongly chiral algebras from our algebras. This construction requires for the formal dimension of the algebra to be even. Otherwise, we could end up with a Sullivan algebra whose formal dimension is divisible by four, and the obstruction theory of Sullivan and Barge would not be trivial. However, we can make use of Lemma 5.11 to ensure that the graphs we work with have an even number of vertices, thus $\mathcal{M}_n(\mathcal{G})$ has even formal dimension as a consequence of Lemma 5.9.*(2)*.

We begin by recalling the construction from [27, Proposition 3.1, Lemma 3.2]. Let A be a 1-connected elliptic Sullivan algebra of formal dimension $2m$. Then, we can construct a 1-connected elliptic Sullivan algebra of formal dimension $4m-1$ as follows: we choose a representative of the fundamental class of A, let us say x with $|x| = 2m$ and we define $\tilde{A} = (A \otimes \Lambda(z), dz = x)$. The algebra $\tilde{A}$ inherits some of the properties of A: it is elliptic, the monoid of self-maps of $\tilde{A}$ coincides with the monoid of self-maps of A, and finally, the connectivity is preserved. Moreover, by choosing an element $y \in \tilde{A}^{4m-1}$ such that $dy = x^2$, $xz - y$ is a representative of the fundamental class in $H^{4m-1}(A)$. Using this construction, we are able to prove the following:

Proposition 5.14. *For any finite group G and any integer $n \geq 1$, G is the group of self-homotopy equivalences of the rationalization of a strongly chiral manifold which is* $(30n-13)$*-connected.*

Proof. As a consequence of Lemma 5.11, there exists a finite graph $\mathcal{G}$ without isolated vertices and with an even number of vertices such that $\mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \cong G$. Consider $\mathcal{M}_n(\mathcal{G})$ the $(30n-13)$-connected algebra from Definition 4.6. We proved in Lemma 5.9.*(2)* that $\mathcal{M}_n(\mathcal{G})$ is inflexible. Moreover, since $|V(\mathcal{G})|$ is even, we deduce from the same result that $\mathcal{M}_n(\mathcal{G})$ has even formal dimension.

Consider $\tilde{\mathcal{M}}_n(\mathcal{G})$ the algebra obtained from $\mathcal{M}_n(\mathcal{G})$ by the process explained above. It is still a $(30n-13)$-connected algebra whose cohomology is a Poincaré duality algebra. Its formal dimension is odd, thus the obstruction theory of Barge, [9], and Sullivan, [75], is trivial. Thus, there exists a $(30n-13)$-connected manifold $M_n(\mathcal{G})$ whose Sullivan model is $\tilde{\mathcal{M}}_n(\mathcal{G})$.

The algebra $\tilde{\mathcal{M}}_n(\mathcal{G})$ is inflexible as a consequence of $\mathcal{M}_n(\mathcal{G})$ being inflexible. Moreover, any self-map $\tilde{f}$ of $\tilde{\mathcal{M}}_n(\mathcal{G})$ is shown in [27, Lemma 3.2] to verify that $\deg(\tilde{f}) = \deg(\tilde{f}|_{\mathcal{M}_n(\mathcal{G})})^2$, thus $\deg(\tilde{f}) \in \{0,1\}$. By an argument analogous to that of Theorem 5.12, degrees of self-maps of $M_n(\mathcal{G})$ are equal to those of their respective Sullivan representatives, thus $M_n(\mathcal{G})$ is an strongly chiral manifold. Furthermore,

$$G \cong \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \cong \mathcal{E}(\mathcal{M}_n(\mathcal{G})) \cong \mathcal{E}(M_n(\mathcal{G})_0),$$

thus $M_n(\mathcal{G})$ is the desired manifold. □

Remark 5.15. For each $k \geq 1$, the rigid Sullivan algebra $\mathcal{M}_k$ introduced in Definition 4.1 is inflexible and has even formal dimension, as proven in Lemma 5.9.*(1)*. Thus, these algebras can also be used in the construction of Proposition 5.14 to obtain, for each $k \geq 1$, a $(10k+7)$-connected strongly chiral manifold whose rationalisation has a trivial group of self-homotopy equivalences.

Manifolds provided by Proposition 5.14 can be used to construct inflexible and strongly chiral product manifolds by exploiting techniques from [64]. In [64, Example 3.7], the author shows that if N is a simply connected inflexible manifold and M is a closed oriented inflexible manifold of dimension $\dim N < \dim M$ that is not simply connected and such that it does not admit maps of non-zero degree from direct products, then $M \times N$ is inflexible. Here we prove an "inverse":

Corollary 5.16. *Let M be a non necessarily simply connected closed oriented inflexible (resp. strongly chiral) manifold that does not admit maps of non-zero degree from direct products. Then, there exists a simply connected strongly chiral manifold N such that* $\dim N > \dim M$ *and $M \times N$ is inflexible (resp. strongly chiral).*

Proof. Let $m = \dim M$, and $n \geq 1$ be an integer such that $m < 30n-13$. Let N be any of the $(30n-13)$-connected strongly chiral manifolds from Proposition 5.14. Since $H^m(N;\mathbb{Q}) = 0$, any continuous map $f\colon N \to M$ maps $H^m(M;\mathbb{Q})$ to 0. Thus the pair (M,N) is under the assumptions of [64, Theorem 1.4], and the result follows from [64, Corollary 1.5(c)] (resp. [64, Corollary 1.5(a)]). □

We finish this section by remarking that as a consequence of a result of Lambrechts and Stanley, [23, Proposition A.1], the manifolds obtained in Theorem 5.12, Proposition 5.14 and Corollary 5.16 can actually be chosen so that they are smooth.

5.4 On a lower bound for the LS-category

Let X be a finite dimensional space. The Lusternik-Schnirelmann category or LS-category of X, denoted $\mathrm{cat}(X)$, is the least integer n such that X admits a covering by $n+1$ contractible open subspaces of X. It was first introduced in [60], where the authors show that if X is a smooth manifold, $\mathrm{cat}(X)+1$ is a lower bound for the number of critical points of a smooth function on X. It is a homotopy invariant of X that has been extensively studied.

Then, the rational LS-category of a space X, denoted $\mathrm{cat}_0(X)$, is defined as the LS-category of its rationalisation, $\mathrm{cat}_0(X) = \mathrm{cat}(X_0)$. The computational power of the algebraic tools used in Rational Homotopy Theory allows for the proof of different characterizations and bounds for the rational LS-category of a space, as can be seen in the results by different authors gathered in [39, Part V]. Furthermore, the study of the rational LS-category is also useful towards the study of LS-category itself, since $\mathrm{cat}(X) \geq \mathrm{cat}_0(X)$, [39, Lemma 28.2]. Here, we briefly consider a lower bound of the category of a space, and using the functor $\mathcal{M}_n$ introduced in Definition 4.11 we show that the two numerical invariants involved can be arbitrarily different.

Let X be a finite dimensional space. Let $\mathcal{E}_\sharp^m(X)$ be the subgroup of $\mathcal{E}(X)$ consisting of homotopy classes of self-homotopy equivalences of X that induce the identity on the homotopy groups $\pi_i(X)$ for $i \leq m$. Let $\mathcal{E}_\sharp(X)$ be the subgroup of $\mathcal{E}(X)$ of those homotopy classes of self-homotopy equivalences that induce the identity map on all homotopy groups. The very well-known result of Dror-Zabrodsky [34] shows that $\mathcal{E}_\sharp(X)$ and $\mathcal{E}_\sharp^m(X)$, $m \geq \dim(X)$, are nilpotent groups. Thus, the nilpotency class of these groups, denoted by $\operatorname{nil}\left(\mathcal{E}_\sharp^m(X)\right)$ and $\operatorname{nil}\left(\mathcal{E}_\sharp(X)\right)$ respectively, are numerical homotopy invariants of X that one can compare to other classical invariants [5, Problem (9)]: Félix and Murillo proved that if X is a finite dimensional space, then $\operatorname{nil}\left(\mathcal{E}_\sharp^m(X_0)\right) \leq \operatorname{cat}(X)-1$ for $m \geq \dim(X)$ and $\operatorname{nil}\left(\mathcal{E}_\sharp(X_0)\right) \leq \operatorname{cat}(X)-1$, [41, Theorem 1]. Therefore, it is natural to ask how different these invariants can be.

It turns out that we can use the algebras associated to graphs we introduced in Definition 4.6 to prove that these invariants are arbitrarily different. We do so now.

Theorem 5.17. *Given any integer $k > 1$ there exists a finite dimensional space X such that $\operatorname{cat}(X) - \operatorname{nil}\left(\mathcal{E}_\sharp^m(X_0)\right) \geq k$, for $m \geq \dim(X)$. In particular, $\operatorname{cat}(X) - \operatorname{nil}(\mathcal{E}_\sharp(X_0)) \geq k$.*

Proof. Take $\mathcal{G} = C_k$ the directed cycle of k vertices, that is, a digraph with $V(\mathcal{G}) = \{v_1, v_2, \ldots, v_k\}$ and $E(\mathcal{G}) = \{(v_1,v_2),(v_2,v_3),\ldots,(v_{k-1},v_k),(v_k,v_1)\}$. It is a strongly connected digraph with $|V(\mathcal{G})| = |E(\mathcal{G})| = k$. Thus, it is in $\mathcal{D}igraphs_+$ (see Definition 4.4) whenever $k \geq 1$. Consider any integer $n \geq 1$. By Lemma 5.9.*(2)*, the Sullivan model $\mathcal{M}_n(\mathcal{G})$ is elliptic of formal dimension $\delta = (18n+4)|x_1|+k|z|-k\frac{|z|-2}{3} = k(900n^2-710n+139)+540n^2-96n-48$. Thus, there exists X a δ-dimensional space whose minimal model is $\mathcal{M}_n(\mathcal{G})$, [38, Theorem A]. Then, $\mathcal{E}_\sharp^m(X_0) = \mathcal{E}_\sharp^m(\mathcal{M}_n(\mathcal{G}))$ for any m, and $\operatorname{cat}(X) \geq \operatorname{cat}(X_0) = \operatorname{cat}\left(\mathcal{M}_n(\mathcal{G})\right)$.

Now note that since $\mathcal{M}_n(\mathcal{G})$ is a minimal Sullivan algebra of a finite dimensional space, its generators are dual to $\pi_*(X)\otimes\mathbb{Q}$, Proposition 1.45. Consequently, $\mathcal{E}_\sharp^m(\mathcal{M}_n(\mathcal{G}))$ is the group of those self-homotopy equivalences of $\mathcal{M}_n(\mathcal{G})$ whose linear part is the identity on indecomposables of $\mathcal{M}_n(\mathcal{G})$ up to degree m. Similarly, $\mathcal{E}_\sharp(\mathcal{M}_n(\mathcal{G}))$ is the subgroup of those self-homotopy equivalences of $\mathcal{M}_n(\mathcal{G})$ whose linear part is the identity map.

According to Theorem 4.14, every automorphism of $\mathcal{M}_n(\mathcal{G})$ is completely determined by the image of the generators x_v, that is, by the morphism induced on the module of indecomposables in degree $180n^2-142n+28$. Hence, for $m > 180n^2-142n+28$, every element in $\mathcal{E}_\sharp^m(M_n(\mathcal{G}))$ induces the identity on the indecomposables in degree $180n^2-142n+28$, thus it must be the identity. In other words, if $m > 180n^2-142n+28$, then $\mathcal{E}_\sharp^m(\mathcal{M}_n(\mathcal{G})) = \{1\}$, whose nilpotency class is 0. For the same reason, $\mathcal{E}_\sharp(\mathcal{M}_n(\mathcal{G})) = \{1\}$.

Now, since $\mathcal{M}_n(\mathcal{G})$ is elliptic, $\operatorname{cat}\left(\mathcal{M}_n(\mathcal{G})\right)$ is bounded below by the number of generators in odd degree [39, Theorem 32.6*(iv)*], so $\operatorname{cat}\left(\mathcal{M}_n(\mathcal{G})\right) \geq k+4$. Hence, we conclude that

$$\operatorname{cat}(X) - \operatorname{nil}\left(\mathcal{E}_\sharp^m(X_0)\right) \geq k+4-0 \geq k,$$

for $m \geq \dim(X)$. Similarly, $\operatorname{cat}(X) - \operatorname{nil}\left(\mathcal{E}_\sharp(X_0)\right) \geq k$. $\square$

CHAPTER 6

A NON RATIONAL APPROACH

So far, our realisability problems have been solved by using Rational Homotopy Theory techniques. As a consequence, the objects that we obtain as an answer to the classical realisability problem in $\mathcal{H}oTop$ (or Kahn's realisability problem) are rational spaces, which are not of finite type over $\mathbb{Z}$. Our purpose in this chapter is to find an alternative way of solving this question by means of *integral* spaces, that is, spaces of finite type over $\mathbb{Z}$.

With that objective in mind, in Section 1.5 we introduced a framework where the homotopy types of a family of *integral* spaces are classified using group-theoretical tools: the A_n^2-polyhedra classified by Whitehead and Baues. This chapter is devoted to the study of self-homotopy equivalences of A_n^2-polyhedra, as a way of exploring the classical realisability problem for groups via spaces of finite type.

We focus on the group $\mathcal{B}^{n+2}(X)$ introduced in Definition 1.69, which by Proposition 1.70 is isomorphic to the group $\mathcal{E}(X)/\mathcal{E}_*(X)$, one of the distinguished quotients of $\mathcal{E}(X)$ for which a group realisability question has been raised in [37, Problem 19].

In Section 6.1, we study how the cell structure of X is carried onto the group $\mathcal{B}^{n+2}(X)$. Mainly, we obtain results regarding the finiteness, the realisability of groups that are automorphisms of another group, and the existence of elements of even order in $\mathcal{B}^{n+2}(X)$. We observe that, unless strong restrictions are imposed on an A_n^2-polyhedron X, it must have self-equivalences of even order.

Along those lines, in Section 6.2 we study obstructions to the existence of elements of even order in $\mathcal{B}^{n+2}(X)$. Here we prove our two main results for this chapter. Namely, in Theorem 6.16 we show that if X is a finite type A_n^2-polyhedron, $n \geq 3$, then either $\mathcal{B}^{n+2}(X)$ is trivial or it has elements of even order. This allows us to deduce in Corollary 6.17 that not every group can appear as $\mathcal{E}(X)$ for X a finite type A_n^2-polyhedron with $n \geq 3$.

Then, in Theorem 6.18 we show that if X is a finite type A_2^2-polyhedron for which $\mathcal{B}^4(X)$ is non-trivial and finite, very strong restrictions have to be imposed on X for it not to have self-homotopy equivalences of even order. However, at the present we do not know if there exists an A_2^2-polyhedra satisfying these restrictions, and since all our attempts to provide an example were unsuccessful, we raise Conjecture 6.19.

All things considered, the results from this chapter make us think that A_n^2-polyhedra might not be the best context for our purposes.

6.1 Some general results on self-homotopy equivalences of A_n^2-polyhedra

The Γ-sequence tool introduced in Section 1.5 will help us to illustrate, from an algebraic point of view, how different restrictions on an A_n^2-polyhedron X affect the quotient group $\mathcal{E}(X)/\mathcal{E}_*(X)$. We devote this section to that matter. We also obtain several results that will be needed in the proof of Theorem 6.16 and Theorem 6.18. The following result is a generalisation of [17, Theorem 4.5].

Proposition 6.1. *Let X be an A_n^2-polyhedron and suppose that the Hurewicz homomorphism $h_{n+2}\colon \pi_{n+2}(X) \to H_{n+2}(X)$ is onto. Then, every automorphism of $H_{n+2}(X)$ is realised by a self-homotopy equivalence of X.*

Proof. As part of the exact sequence (1.2) for X we have:

$$\cdots \longrightarrow \pi_{n+2}(X) \xrightarrow{h_{n+2}} H_{n+2}(X) \xrightarrow{b_{n+2}} \Gamma_n^1(H_n(X)) \longrightarrow \pi_{n+1}(X) \longrightarrow \cdots$$

Then, since h_{n+2} is onto by hypothesis, b_{n+2} is the trivial homomorphism. Thus, for every $f_{n+2} \in \operatorname{Aut}\left(H_{n+2}(X)\right)$, $b_{n+2} \circ f_{n+2} = b_{n+2} = 0$, so if we take $\Omega = \mathrm{id}$, $(f_{n+2}, \mathrm{id}, \mathrm{id}) \in \mathcal{B}^{n+2}(X)$. Then there exists $f \in \mathcal{E}(X)$ such that $H_{n+2}(f) = f_{n+2}$, $H_{n+1}(f) = \mathrm{id}$ and $H_n(f) = \mathrm{id}$. □

We can easily prove that automorphism groups can be realised, a result that can also be obtained as a consequence of [71, Theorem 2.1]:

Example 6.2. Let G be a group isomorphic to $\operatorname{Aut}(H)$ for some abelian group H. Then, for any integer $n \geq 2$, there exists an A_n^2-polyhedron X such that $G \cong \mathcal{B}^{n+2}(X)$: take the Moore space $X = M(H, n+1)$, which in particular is an A_n^2-polyhedron. The Γ-sequence of X, (1.3), is

$$H_{n+2}(X) = 0 \longrightarrow \Gamma_n^1(H_n(X)) = 0 \longrightarrow H \xrightarrow{=} H \longrightarrow 0.$$

Then, for every $f \in \operatorname{Aut}(H)$, by taking $\Omega = f$ we see that $(\mathrm{id}, f, \mathrm{id}) \in \mathcal{B}^{n+2}(X)$, and those are the only possible Γ-isomorphisms of the Γ-sequence of X. Thus $\mathcal{B}^{n+2}(X) \cong \operatorname{Aut}(H) \cong G$.

The use of Moore spaces is not required in the $n = 2$ case:

Example 6.3. Let G be a group isomorphic to $\operatorname{Aut}(H)$ for some abelian group H. Consider the following object in Γ-sequences[4]:

$$\mathbb{Z} \xrightarrow{b_4} \Gamma(\mathbb{Z}_2) = \mathbb{Z}_4 \longrightarrow H \xrightarrow{=} H \longrightarrow 0. \tag{6.1}$$

By Theorem 1.68, there exists an A_2^2-polyhedron X realising this object. In particular, $H_4(X) = \mathbb{Z}$, $H_3(X) = \pi_3(X) = H$ and $H_2(X) = \mathbb{Z}_2$. It is clear from (6.1) that $(\mathrm{id}, f, \mathrm{id})$ is a Γ-isomorphism for every $f \in \operatorname{Aut}(H)$. Now $\operatorname{Aut}(\mathbb{Z}_2)$ is the trivial group while $\operatorname{Aut}(\mathbb{Z}) = \{-\mathrm{id}, \mathrm{id}\}$. It is immediate to check that $(-\mathrm{id}, f, \mathrm{id})$ is not a Γ-isomorphism of (6.1) since $\mathrm{id} \circ b_4 \neq b_4 \circ (-\mathrm{id})$. Then, we obtain that $\mathcal{B}^4(X) \cong \operatorname{Aut}(H)$.

As not every group G is isomorphic to the automorphism group of an abelian group, the examples above only provide a partial positive answer to the realisability problem for $\mathcal{B}^{n+2}(X)$. Indeed, the automorphism group of an abelian group (other than $\mathbb{Z}_2$) has elements of even order. The following results go in that direction:

Lemma 6.4. *Let X be an A_n^2-polyhedron, $n \geq 2$. If $H_n(X)$ is not an elementary abelian 2-group, then $\mathcal{B}^{n+2}(X)$ has an element of order 2.*

Proof. Since $H_n(X)$ is not an elementary abelian 2-group, it admits a non-trivial involution $-\operatorname{id}\colon H_n(X) \to H_n(X)$. But $\Gamma_n^1(-\operatorname{id}) = \operatorname{id}$ for every $n \geq 2$, so $(\operatorname{id}, \operatorname{id}, -\operatorname{id}) \in \mathcal{B}^{n+2}(X)$ and the result follows. $\square$

We point out a key difference between the $n = 2$ and the $n \geq 3$ cases: $\Gamma_2^1(A) = \Gamma(A)$ is never an elementary abelian 2-group when A is a finitely generated abelian group, as it can be deduced from Proposition 1.62. However, for $n \geq 3$, $\Gamma_n^1(A) = A \otimes \mathbb{Z}_2$ is always an elementary abelian 2-group. Taking advantage of this fact we can prove the following result:

Lemma 6.5. *Let X be an A_n^2-polyhedron, $n \geq 3$. If any of the homology groups of X is not an elementary abelian 2-group (in particular, if $H_{n+2}(X) \neq 0$), then $\mathcal{B}^{n+2}(X)$ contains a non-trivial element of order 2.*

Proof. Under our assumptions, $\Gamma_n^1(H_n(X))$ is an elementary abelian 2-group. For $\Omega = -\operatorname{id}$, the triple $(-\operatorname{id}, -\operatorname{id}, -\operatorname{id})$ is a Γ-isomorphism of order 2 unless it is trivial, that is, unless $H_{n+2}(X)$, $H_{n+1}(X)$ and $H_n(X)$ are all elementary abelian 2-groups. $\square$

We remark that this result does not hold for A_2^2-polyhedra. Indeed, if we consider the construction in Example 6.3 for $H = \mathbb{Z}_2$, then $\mathcal{B}^4(X) \cong \operatorname{Aut}(\mathbb{Z}_2) = \{*\}$ does not contain a non-trivial element of order 2 although $H_4(X) = \mathbb{Z}$ is not an elementary abelian 2-group.

We now prove some results regarding the finiteness of $\mathcal{B}^{n+2}(X)$. Recall that if X is a simply connected finite type space, the homotopy and homology groups $H_n(X)$ and $\pi_n(X)$ are finitely generated and abelian for $n \geq 1$.

Proposition 6.6. *Let X be a finite type A_n^2-polyhedron, $n \geq 2$, with* $\operatorname{rank} H_{n+2}(X) \geq 2$ *and every element of $\Gamma_n^1(H_n(X))$ of finite order. Then $\mathcal{B}^{n+2}(X)$ is an infinite group.*

Proof. Since $\operatorname{rank} H_{n+2}(X) \geq 2$, we may write $H_{n+2}(X) = \mathbb{Z}^2 \oplus G$, G a (possibly trivial) free abelian group. Consider the Γ-sequence of X:

$$\mathbb{Z}^2 \oplus G \xrightarrow{b_{n+2}} \Gamma_n^1(H_n(X)) \xrightarrow{i_n} \pi_{n+1}(X) \xrightarrow{h_{n+1}} H_{n+1}(X) \longrightarrow 0.$$

Since $b_{n+2}(\mathbb{Z}^2) \leq \Gamma_n^1(H_n(X))$ is a finitely generated $\mathbb{Z}$-module with finite order generators, it is a finite group. Define $k = \exp\left(b_{n+2}(\mathbb{Z}^2)\right)$ and consider the automorphism of $\mathbb{Z}^2$ given by the matrix

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}),$$

which is of infinite order. If we take $f \oplus \operatorname{id}_G \in \operatorname{Aut}(\mathbb{Z}^2 \oplus G)$, then $b_{n+2} \circ (f \oplus \operatorname{id}) = b_{n+2}$, thus $(f \oplus \operatorname{id}_G, \operatorname{id}, \operatorname{id}) \in \mathcal{B}^{n+2}(X)$, which is an element of infinite order. $\square$

As we have previously mentioned, $\Gamma_n^1(H_n(X))$ is an elementary abelian 2-group, for $n \geq 3$. Hence, from Proposition 6.6 we get:

Corollary 6.7. *Let X be a finite type A_n^2-polyhedron, $n \geq 3$, with* $\operatorname{rank} H_{n+2}(X) \geq 2$. *Then $\mathcal{B}^{n+2}(X)$ is an infinite group.*

This result does not hold, in general, for $n = 2$. However, if A is a finite group, Proposition 1.62 implies that $\Gamma(A)$ is finite as well so from Proposition 6.6 we get:

Corollary 6.8. *Let X be a finite type A_2^2-polyhedron with* $\operatorname{rank} H_4(X) \geq 2$ *and $H_2(X)$ finite. Then $\mathcal{B}^4(X)$ is an infinite group.*

We end this section with one more result on the finiteness of $\mathcal{B}^{n+2}(X)$:

Proposition 6.9. *Let X be an A_n^2-polyhedron, $n \geq 3$. If $H_n(X) = \mathbb{Z}^2 \oplus G$ for a certain abelian group G, then $\mathcal{B}^{n+2}(X)$ is an infinite group.*

Proof. If $H_n(X) = \mathbb{Z}^2 \oplus G$, then $\Gamma_n^1(H_n(X)) = H_n(X) \otimes \mathbb{Z}_2 = \mathbb{Z}_2^2 \oplus (G \otimes \mathbb{Z}_2)$. Hence $\mathrm{GL}_2(\mathbb{Z}) \leq \mathrm{Aut}\left(H_n(X)\right)$ and $\mathrm{GL}_2(\mathbb{Z}_2) \leq \mathrm{Aut}\left(H_n(X) \otimes \mathbb{Z}_2\right)$. Moreover, for every $f \in \mathrm{GL}_2(\mathbb{Z})$ we have $f \oplus \mathrm{id}_G \in \mathrm{Aut}\left(H_n(X)\right)$ which yields, through Γ_n^1, an automorphism $(f \oplus \mathrm{id}_G) \otimes \mathbb{Z}_2 = (f \otimes \mathbb{Z}_2) \oplus \mathrm{id}_{G\otimes\mathbb{Z}_2} \in \mathrm{Aut}\left(H_n(X) \otimes \mathbb{Z}_2\right)$. This means that the functor Γ_n^1 restricts to $\mathrm{GL}_2(\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}_2)$, giving us the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Aut}\left(H_n(X)\right) \arrow[r, "-\otimes\mathbb{Z}_2"] & \mathrm{Aut}\left(H_n(X)\otimes\mathbb{Z}_2\right) \\
\mathrm{GL}_2(\mathbb{Z}) \arrow[r, "-\otimes\mathbb{Z}_2"] \arrow[u, hook] & \mathrm{GL}_2(\mathbb{Z}_2). \arrow[u, hook]
\end{tikzcd}$$

Moreover, $-\otimes\mathbb{Z}_2\colon \mathrm{GL}_2(\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}_2)$ has an infinite kernel. Hence, there are infinitely many morphisms $f \in \mathrm{Aut}\left(H_n(X)\right)$ such that $f \otimes \mathbb{Z}_2 = \mathrm{id}$. For any such a morphism f, $(\mathrm{id}, \mathrm{id}, f)$ is an element of $\mathcal{B}^{n+2}(X)$. Therefore $\mathcal{B}^{n+2}(X)$ is infinite. □

6.2 Obstructions to the realisability of groups

We have seen in Section 6.1 that $\mathcal{B}^{n+2}(X)$ contains elements of even order unless strong restrictions are imposed on the homology groups of the A_n^2-polyhedron X. Since we are interested in realising an arbitrary group G as $\mathcal{B}^{n+2}(X)$ for X a finite type A_n^2-polyhedron, in this section we focus our attention in the remaining situations and prove Theorem 6.16 and Theorem 6.18.

As we shown in Lemma 6.4, if we do not want $\mathcal{B}^{n+2}(X)$ to have elements of order 2 we have to assume that $H_n(X)$ is an elementary abelian 2-group. To get a better grasp of the situation, we begin with some previous results regarding the Γ functor in this particular case.

Lemma 6.10. *For G an elementary abelian 2-group, $\Gamma(-)\colon \mathrm{Aut}(G) \to \mathrm{Aut}\left(\Gamma(G)\right)$ is injective.*

Proof. Let us show that the kernel of $\Gamma(-)$ is trivial. Assume that G is generated by $\{e_j \mid j \in J\}$, J an ordered set. If $f \in \mathrm{Aut}(G)$ is in the kernel of $\Gamma(-)$, then for each $j \in J$, there exists a finite subset $I_j \subset J$ such that $f(e_j) = \sum_{i\in I_j} e_i$, and

$$\gamma(e_j) = \Gamma(f)\gamma(e_j) = \gamma f(e_j) = \gamma\left(\sum_{i\in I_j} e_i\right) = \sum_{i\in I_j}\gamma(e_i) + \sum_{i<k} e_i \otimes e_k,$$

as a consequence of Proposition 1.62.*(3)*, so $I_j = \{j\}$ and $f(e_j) = e_j$ for every $j \in J$. □

Definition 6.11. Let G be a p-group. The subgroup of G generated by the elements whose order divide p^i is denoted by $\Omega_i(G)$.

Lemma 6.12. *Let $H_2 = \oplus_{i=1}^n \mathbb{Z}_2$ and $\chi \in \Gamma(H_2)$ be an element of order 4. If there exists a non-trivial automorphism of odd order $f \in \mathrm{Aut}(H_2)$ such that $\Gamma(f)(\chi) = \chi$, then there exists $g \in \mathrm{Aut}(H_2)$ of order 2 such that $\Gamma(g)(\chi) = \chi$.*

Proof. Recall from Proposition 1.62.*(3)* that we can write $h \otimes h = 2\gamma(h)$, for any element $h \in H_2$. Therefore, given a basis $\{h_1, h_2, \dots, h_n\}$ of H_2, and replacing $3\gamma(h_i)$ by $\gamma(h_i)+h_i\otimes h_i$ if needed, we can write

$$\chi = \sum_{i=1}^{n} a(i)\gamma(h_i) + \sum_{i,j=1}^{n} a(i,j) h_i \otimes h_j,$$

where every coefficient $a(i)$, $a(i,j)$ is either 0 or 1. We now construct inductively a basis $\{e_1, e_2, \dots, e_n\}$ of H_2 as follows. Without loss of generality, assume $a(1) = 1$ and define $e_1 = \sum_{i=1}^{n} a(i)h_i$. Then $\{e_1, h_2, \dots, h_n\}$ is again a basis of H_2 and

$$\chi = \gamma(e_1) + \alpha_1 e_1 \otimes e_1 + \beta_1 e_1 \otimes \left(\sum_{s=2}^{n} b(1,s)h_s\right) + \sum_{i,j>1}^{n} a_1(i,j) h_i \otimes h_j,$$

where every coefficient in the equation is either 0 or 1. Assume that we have constructed a basis $\{e_1, \dots, e_r, h_{r+1}, \dots, h_n\}$ such that

$$\begin{aligned}\chi = \gamma(e_1) + \sum_{j=1}^{r} \alpha_j e_j \otimes e_j \ &+ \sum_{j=1}^{r-1} \beta_j e_j \otimes e_{j+1} \\ &+ \beta_r e_r \otimes \left(\sum_{s=r+1}^{n} b(r,s)h_s\right) + \sum_{i,j>r}^{n} a_r(i,j) h_i \otimes h_j,\end{aligned}$$

where every coefficient in the equation is either 0 or 1. We may assume $b(r, r+1) = 1$ and define $e_{r+1} = \sum_{s=r+1}^{n} b(r,s)h_s$. Thus $\{e_1, \dots, e_{r+1}, h_{r+2}, \dots, h_n\}$ is again a basis of H_2 and

$$\begin{aligned}\chi = \gamma(e_1) + \sum_{j=1}^{r+1} \alpha_j e_j \otimes e_j \ &+ \sum_{j=1}^{r} \beta_j e_j \otimes e_{j+1} \\ &+ \beta_{r+1} e_{r+1} \otimes \left(\sum_{s=r+2}^{n} b(r+1,s)h_s\right) + \sum_{i,j>r+1}^{n} a_{r+1}(i,j) h_i \otimes h_j.\end{aligned}$$

Finally, we obtain a basis $\{e_1, e_2, \dots, e_n\}$ of H_2 such that

$$\chi = \gamma(e_1) + \sum_{j=1}^{n} \alpha_j e_j \otimes e_j + \sum_{j=1}^{n-1} \beta_j e_j \otimes e_{j+1}, \tag{6.2}$$

for some coefficients $\alpha_j \in \{0,1\}$, $j = 1, 2, \dots, n$, and $\beta_j \in \{0,1\}$, $j = 1, 2, \dots, n-1$.

Now, for $n = 1$, $H_2 = \mathbb{Z}_2$ has a trivial group of automorphisms, so the result holds. For $n = 2$, assume that there exists $f \in \mathrm{Aut}(H_2)$ such that $\Gamma(f)(\chi) = \chi$. From Equation (6.2), $\chi = \Gamma(f)\big(\gamma(e_1)\big) + \Gamma(f)(P)$, where $P \in \Omega_1\big(\Gamma(H_2)\big)$. Then $\Gamma(f)\big(\gamma(e_1)\big)$ has a multiple of $\gamma(e_1)$ as its only summand of order 4, which implies that $f(e_1) = e_1$. Then either $f(e_2) = e_2$, so f is trivial, or $f(e_2) = e_1 + e_2$, so f has order 2.

Henceforth, assume that $n \geq 3$. We shall prove that there always exists an automorphism $g \in \mathrm{Aut}(H_2)$ of order 2 such that $\Gamma(g)(\chi) = \chi$. In order to do so, we discuss the existence of g in terms of the possible values of the coefficients α_{n-j} and β_{n-j-1}, for $j = 0, 1$. We will only define $g(e_{n-1})$ and $g(e_n)$, since in all cases $g(e_j) = e_j$, for $j \in \{1, 2, \dots, n-2\}$. To perform the necessary computations, observe that as a consequence of Proposition 1.62.*(3)*,

$$\begin{aligned}\Gamma(g)(e_i \otimes e_j) &= \Gamma(g)\big(\gamma(e_i + e_j) - \gamma(e_i) - \gamma(e_j)\big) \\ &= \gamma\big(g(e_i) + g(e_j)\big) - \gamma\big(g(e_i)\big) - \gamma\big(g(e_j)\big) = g(e_i) \otimes g(e_j).\end{aligned}$$

Furthermore, if $g(e_1) = e_1$, then $\Gamma(g)(e_1) = \gamma(g(e_1)) = \gamma(e_1)$.

First notice that if $\beta_{n-1} = \alpha_n = 0$, e_n does not appear in any of the summands of χ. Then, if we define $g(e_{n-1}) = e_{n-1}$ and $g(e_n) = e_{n-1} + e_n$, it is clear that $\Gamma(g)(\chi) = \chi$. Furthermore, g is clearly of order 2. On the other hand, if $\beta_{n-1} = \alpha_n = 1$, this same g verifies that

$$\begin{aligned} g(e_{n-1} \otimes e_n + e_n \otimes e_n) &= g(e_{n-1}) \otimes g(e_n) + g(e_n) \otimes g(e_n) \\ &= e_{n-1} \otimes (e_{n-1} + e_n) + (e_{n-1} + e_n) \otimes (e_{n-1} + e_n) = e_{n-1} \otimes e_n + e_n \otimes e_n. \end{aligned}$$

Since e_n does not appear in any of the remaining summands of χ, we have that $\Gamma(g)(\chi) = \chi$.

We move on to the case $\beta_{n-1} \neq \alpha_n$. Assume first that $\alpha_n = 1$ and $\beta_{n-1} = 0$. Then for $\beta_{n-2} = \alpha_{n-1}$, by defining $g(e_{n-1}) = e_{n-2} + e_{n-1}$ and $g(e_n) = e_n$ we have an automorphism of order 2, and computations analogous to the ones above show that $\Gamma(g)(\chi) = \chi$.

On the other hand, if $\beta_{n-2} = 0$ and $\alpha_{n-1} = 1$, e_{n-1} only appears in the summand $e_{n-1} \otimes e_{n-1}$, and e_n only appears in $e_n \otimes e_n$. Thus, if we define $g(e_{n-1}) = e_n$ and $g(e_n) = e_{n-1}$, g is an automorphism of order 2 such that $\Gamma(g)(\chi) = \chi$. Finally, if $\beta_{n-2} = 1$ and $\alpha_{n-1} = 0$, define $g(e_{n-1}) = e_{n-2} + e_{n-1}$ and $g(e_n) = e_{n-2} + e_n$. Then g has order 2, and the only summands of χ not fixed by g are $e_{n-2} \otimes e_{n-1}$ and $e_n \otimes e_n$. Nonetheless,

$$\begin{aligned} g(e_{n-2} \otimes e_{n-1} + e_n \otimes e_n) &= g(e_{n-2}) \otimes g(e_{n-1}) + g(e_n) \otimes g(e_n) \\ &= e_{n-2} \otimes (e_{n-2} + e_{n-1}) + (e_{n-2} + e_n) \otimes (e_{n-2} + e_n) = e_{n-2} \otimes e_{n-1} + e_n \otimes e_n, \end{aligned}$$

so $\Gamma(g)(\chi) = \chi$.

It remains to consider the cases where $\alpha_n = 0$ and $\beta_{n-1} = 1$. We consider all of the possible values for α_{n-1} and β_{n-2} separately. First, if $\alpha_{n-1} = \beta_{n-2} = 0$, e_{n-1} and e_n only appear in the summand $e_{n-1} \otimes e_n$. Then, if we define $g(e_{n-1}) = e_n$ and $g(e_n) = e_{n-1}$, it is clear that g has order 2 and $\Gamma(g)(\chi) = \chi$.

If $\alpha_{n-1} = 1$ and $\beta_{n-2} = 0$, define $g(e_{n-1}) = e_{n-1} + e_n$ and $g(e_n) = e_n$. Then g has order 2 and it fixes all summands except those involving e_{n-1}, which are $e_{n-1} \otimes e_{n-1}$ and $e_{n-1} \otimes e_n$. Nonetheless, $\Gamma(g)(\chi) = \chi$ since

$$\begin{aligned} g(e_{n-1} \otimes e_{n-1} + e_{n-1} \otimes e_n) &= g(e_{n-1}) \otimes g(e_{n-1}) + g(e_{n-1}) \otimes g(e_n) \\ &= (e_{n-1} + e_n) \otimes (e_{n-1} + e_n) + (e_{n-1} + e_n) \otimes e_n = e_{n-1} \otimes e_{n-1} + e_{n-1} \otimes e_n. \end{aligned}$$

Moving on, if $\alpha_{n-1} = 0$ and $\beta_{n-2} = 1$, define $g(e_{n-1}) = e_{n-2} + e_n$ and $g(e_n) = e_{n-2} + e_{n-1}$. Then g has order 2. Moreover, all summands of χ but those involving e_{n-1} and e_n, which are $e_{n-2} \otimes e_{n-1}$ and $e_{n-1} \otimes e_n$, are fixed by g. But $\Gamma(g)(\chi) = \chi$ since

$$\begin{aligned} g(e_{n-2} \otimes e_{n-1} + e_{n-1} \otimes e_n) &= g(e_{n-2}) \otimes g(e_{n-1}) + g(e_{n-1}) \otimes g(e_n) \\ &= e_{n-2} \otimes (e_{n-2} + e_n) + (e_{n-2} + e_n) \otimes (e_{n-2} + e_{n-1}) = e_{n-2} \otimes e_{n-1} + e_{n-1} \otimes e_n. \end{aligned}$$

Finally, we have the case where $\alpha_{n-1} = \beta_{n-2} = 1$. Define $g(e_{n-1}) = e_{n-2} + e_{n-1} + e_n$, and $g(e_n) = e_n$, so g is an automorphism of order 2. The only summands not fixed by g are those involving e_{n-1}, which are $e_{n-2} \otimes e_{n-1}$, $e_{n-1} \otimes e_{n-1}$ and $e_{n-1} \otimes e_n$. Nonetheless,

$$\begin{aligned} g(e_{n-2} \otimes e_{n-1} + e_{n-1} \otimes e_{n-1} + e_{n-1} \otimes e_n) &= g(e_{n-1} \otimes (e_{n-2} + e_{n-1} + e_n)) \\ &= (e_{n-2} + e_{n-1} + e_n) \otimes e_{n-1} = e_{n-2} \otimes e_{n-1} + e_{n-1} \otimes e_{n-1} + e_{n-1} \otimes e_n, \end{aligned}$$

so again $\Gamma(g)(\chi) = \chi$. The result follows. □

Definition 6.13. Let $f\colon H \to K$ be a morphism of abelian groups. We say that a non-trivial subgroup $A \leq K$ is f-split if there exist groups $B \leq H$ and $C \leq K$ such that $H \cong A \oplus B$, $K = A \oplus C$ and f can be written as $\mathrm{id}_A \oplus g\colon A \oplus B \to A \oplus C$ for some $g\colon B \to C$.

Henceforward we will make extensive use of this notation applied to the Hurewicz morphism
$$h_{n+1}\colon \pi_{n+1}(X) \longrightarrow H_{n+1}(X).$$
We prove the following:

Lemma 6.14. *Let X be an A_n^2-polyhedron, $n \geq 2$. Let $A \leq H_{n+1}(X)$ be an h_{n+1}-split subgroup, thus $H_{n+1}(X) = A \oplus C$ for some abelian group C. Then, for every $f_A \in \operatorname{Aut}(A)$ there exists $f \in \mathcal{E}(X)$ inducing* $(\mathrm{id}, f_A \oplus \mathrm{id}_C, \mathrm{id}) \in \mathcal{B}^{n+2}(X)$.

Proof. By hypothesis $H_{n+1}(X) = A \oplus C$, $\pi_{n+1}(X) \cong A \oplus B$, for some abelian group B, and h_{n+1} can be written as $\mathrm{id}_A \oplus g$ for some morphism $g\colon B \to C$. Thus, for every $f_A \in \operatorname{Aut}(A)$ we have a commutative diagram

$$\begin{tikzcd}
H_{n+2}(X) \arrow[r, "b_{n+2}"'] \arrow[d, "\mathrm{id}"] & \Gamma_n^1(H_n(X)) \arrow[r] \arrow[d, "\mathrm{id}"] & A \oplus B \arrow[r, "h_{n+1}"'] \arrow[d, "f_A \oplus \mathrm{id}_B"] & A \oplus C \arrow[r] \arrow[d, "f_A \oplus \mathrm{id}_C"] & 0 \\
H_{n+2}(X) \arrow[r, "b_{n+2}"] & \Gamma_n^1(H_n(X)) \arrow[r] & A \oplus B \arrow[r, "h_{n+1}"] & A \oplus C \arrow[r] & 0.
\end{tikzcd}$$

Hence $(\mathrm{id}, f_A \oplus \mathrm{id}_C, \mathrm{id}) \in \mathcal{B}^{n+2}(X)$, and by Theorem 1.68 there exists $f \in \mathcal{E}(X)$ such that $H_{n+1}(f) = f_A \oplus \mathrm{id}_C$, $H_{n+2}(f) = \mathrm{id}$ and $H_n(f) = \mathrm{id}$. $\square$

The following lemma is crucial in the proof of Theorem 6.16 and Theorem 6.18:

Lemma 6.15. *Let X be a finite type A_n^2-polyhedron, $n \geq 2$. Suppose that there exist h_{n+1}-split subgroups of $H_{n+1}(X)$. Then:*

(1) If $n \geq 3$, $\mathcal{B}^{n+2}(X)$ is either trivial or it has elements of even order.

(2) If $\mathcal{B}^4(X)$ is finite and non-trivial, then it has elements of even order.

Proof. First of all, observe that we just need to consider when $H_n(X)$ is an elementary abelian 2-group. In other case, the result is a consequence of Lemma 6.4.

Let A be an arbitrary h_{n+1}-split subgroup of $H_{n+1}(X)$. If $A \neq \mathbb{Z}_2$, there is an involution $\iota \in \operatorname{Aut}(A)$ that induces, by Lemma 6.14, an element $(\mathrm{id}, \iota \oplus \mathrm{id}, \mathrm{id}) \in \mathcal{B}^{n+2}(X)$ of order 2, and the result follows. Hence we can assume that every h_{n+1}-split subgroup of $H_{n+1}(X)$ is $\mathbb{Z}_2$.

Both assumptions, $H_n(X)$ being an elementary abelian 2-group and every h_{n+1}-split subgroup of $H_{n+1}(X)$ being $\mathbb{Z}_2$, imply that $H_{n+1}(X)$ is a finite 2-group. Indeed, since $H_n(X)$ is finitely generated, $\Gamma_n^1(H_n(X))$ is a finite 2-group and so is $\operatorname{coker} b_{n+2}$. Then, since $H_{n+1}(X)$ is also finitely generated, any direct summand of $H_{n+1}(X)$ which is not a 2-group would be h_{n+1}-split, contradicting our assumption that every h_{n+1}-split subgroup of $H_{n+1}(X)$ is $\mathbb{Z}_2$.

To prove our lemma, we start with the case $A = H_{n+1}(X)$ is h_{n+1}-split.

When $H_{n+2}(X) = 0$, the Γ-sequence of X becomes then the short exact sequence
$$0 \to \Gamma_n^1(H_n(X)) \to \Gamma_n^1(H_n(X)) \oplus \mathbb{Z}_2 \to \mathbb{Z}_2 \to 0.$$
Notice that any automorphism of order 2 in $H_n(X)$ yields an automorphism of order 2 in $\Gamma_n^1(H_n(X))$ since Γ_n^1 is injective on morphisms: it is immediate for $n \geq 3$, and for $n = 2$ we apply Lemma 6.10. As our sequence is split, any $f \in \operatorname{Aut}(H_n(X))$ induces the Γ-isomorphism $(\mathrm{id}, \mathrm{id}, f)$ of the same order. Hence, for $H_n(X) \neq \mathbb{Z}_2$ it suffices to consider an involution. For

$H_n(X) = \mathbb{Z}_2$, since by hypothesis $H_{n+1}(X) = \mathbb{Z}_2$ and $H_{n+2}(X) = 0$, the only Γ-isomorphism is $(\mathrm{id}, \mathrm{id}, \mathrm{id})$ and therefore $\mathcal{B}^{n+2}(X)$ is trivial as claimed.

When $H_{n+2}(X) \neq 0$, for $n \geq 3$ the result follows directly from Lemma 6.5. For $n = 2$ we also assume that $\mathcal{B}^4(X)$ is finite and non-trivial. Hence, since $H_2(X)$ is an elementary abelian 2-group, Proposition 6.6 implies that $H_4(X) = \mathbb{Z}$. Then, if a Γ-isomorphism of the form $(-\mathrm{id}, f, \mathrm{id})$ exists, it is of even order. In particular, if $\operatorname{Im} b_4$ is a subgroup of $\Gamma\big(H_2(X)\big)$ of order 2, $(-\mathrm{id}, \mathrm{id}, \mathrm{id})$ is a Γ-isomorphism of even order.

Assume otherwise that $\operatorname{Im} b_4$ is a group of order 4. If a Γ-isomorphism $(\mathrm{id}, f, \mathrm{id})$ of odd order exists, then $\Gamma(f) \circ b_4 = b_4$. In this situation, by Lemma 6.12 for $\chi = b_4(1)$, there exists $g \in \operatorname{Aut}\big(H_2(X)\big)$ an automorphism of order 2 such that $\Gamma(g)b_4(1) = b_4(1)$. Moreover, as we are in the case $A = H_3(X)$ being h_3-split, $(\mathrm{id}, g, \mathrm{id}) \in \mathcal{B}^4(X)$ is a Γ-isomorphism of order 2.

We deal now with the case $A \subsetneqq H_{n+1}(X)$. Since $A = \mathbb{Z}_2$ is a proper h_{n+1}-split subgroup of $H_{n+1}(X)$, there exist non-trivial groups B and C such that

$$\begin{aligned} \pi_{n+1}(X) = \mathbb{Z}_2 \oplus B &\xrightarrow{h_{n+1}} \mathbb{Z}_2 \oplus C = H_{n+1}(X) \\ (t, b) &\longmapsto (t, g(b)) \end{aligned}$$

for some group morphism $g\colon B \to C$. Moreover, $H_{n+1}(X)$ is a finite 2-group, thus C is a (non-trivial) finite 2-group and there exists an epimorphism $\tau\colon C \to \mathbb{Z}_2$.

Define $f \in \operatorname{Aut}(\mathbb{Z}_2 \oplus C) = \operatorname{Aut}\big(H_{n+1}(X)\big)$, and $\Omega \in \operatorname{Aut}\big(\mathbb{Z}_2 \oplus B\big) = \operatorname{Aut}\big(\pi_{n+1}(X)\big)$ to be the non-trivial involutions given by $f(t, c) = \big(t + \tau(c), c\big)$ and $\Omega(t, b) = \big(t + \tau(g(b)), b\big)$. By construction, $h_{n+1}\Omega = f h_{n+1}$, and if $(t, b) \in \operatorname{coker} b_{n+2} = \ker h_{n+1}$ (thus $g(b) = 0$), then $\Omega(t, b) = (t, b)$. In other words, $(\mathrm{id}, f, \mathrm{id}) \in \mathcal{B}^{n+2}(X)$ and it has order 2. $\square$

We now prove our main results.

Theorem 6.16. *Let X be a finite type A_n^2-polyhedron, $n \geq 3$. Then $\mathcal{B}^{n+2}(X)$ is either the trivial group or it has elements of even order.*

Proof. Assume that $H_n(X)$ and $H_{n+1}(X)$ are elementary abelian 2-groups, and $H_{n+2}(X) = 0$. Otherwise, there would already be elements of order 2 in $\mathcal{B}^{n+2}(X)$ as a consequence of Lemma 6.5.

Write $H_n(X) = \oplus_I \mathbb{Z}_2$, I an ordered set. Since $n \geq 3$, $\Gamma_n^1 = - \otimes \mathbb{Z}_2$, so $\Gamma_n^1\big(H_n(X)\big) = H_n(X)$. We can also assume that there are no subgroups in $H_{n+1}(X)$ that are h_{n+1}-split. In other case, we would deduce from Lemma 6.15 that there are elements of order 2 in $\mathcal{B}^{n+2}(X)$. Thus $H_{n+1}(X) = \oplus_J \mathbb{Z}_2$ with $J \subset I$, and the Γ-sequence corresponding to X is

$$0 \longrightarrow \bigoplus_I \mathbb{Z}_2 \xrightarrow{b} \left(\bigoplus_{I-J} \mathbb{Z}_2\right) \oplus \left(\bigoplus_J \mathbb{Z}_4\right) \xrightarrow{h} \bigoplus_J \mathbb{Z}_2 \longrightarrow 0.$$

We may rewrite the sequence as

$$0 \longrightarrow \left(\bigoplus_{I-J} \mathbb{Z}_2\right) \oplus \left(\bigoplus_J \mathbb{Z}_2\right) \xrightarrow{b} \left(\bigoplus_{I-J} \mathbb{Z}_2\right) \oplus \left(\bigoplus_J \mathbb{Z}_4\right) \xrightarrow{h} \bigoplus_J \mathbb{Z}_2 \longrightarrow 0$$

and assume that $b(x, y) = (x, 2y)$ and $h(x, y) = y \bmod 2$. Clearly any $f \in \operatorname{Aut}\big(\bigoplus_{I-J} \mathbb{Z}_2\big)$ induces a Γ-isomorphism $(0, \mathrm{id}, f \oplus \mathrm{id})$ of the same order.

On the one hand, for $|I - J| \geq 2$, $\bigoplus_{I-J} \mathbb{Z}_2$ has an involution and therefore $\mathcal{B}^{n+2}(X)$ has elements of even order. On the other hand, for $|I - J| < 2$, we consider the remaining possibilities.

Suppose that $|I - J| = 1$. Then, $\pi_{n+1}(X) = \mathbb{Z}_2 \oplus (\oplus_J \mathbb{Z}_4)$. If J is trivial, $\mathcal{B}^{n+2}(X)$ is clearly trivial as well. Otherwise, suppose that $I - J = \{i\}$ and choose $j \in J$. Define $f \in$

$\mathrm{Aut}\,(\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus(\oplus_{I-\{i,j\}}\mathbb{Z}_2))$ by $f(x,y,z)=(x,x+y,z)$ and $g\in\mathrm{Aut}\,(\mathbb{Z}_2\oplus\mathbb{Z}_4\oplus(\oplus_{I-\{i,j\}}\mathbb{Z}_4))$ by $g(x,y,z)=(x,2x+y,z)$. Then $(\mathrm{id},\mathrm{id},f)$ is a Γ-isomorphism of order 2 since we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_2\oplus\mathbb{Z}_2\oplus(\oplus_{I-\{i,j\}}\mathbb{Z}_2) \arrow[r]\arrow[d, "f"] & \mathbb{Z}_2\oplus\mathbb{Z}_4\oplus(\oplus_{I-\{i,j\}}\mathbb{Z}_4) \arrow[r]\arrow[d, "g"] & \mathbb{Z}_2\oplus(\oplus_{J-\{j\}}\mathbb{Z}_2) \arrow[r]\arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \mathbb{Z}_2\oplus\mathbb{Z}_2\oplus(\oplus_{I-\{i,j\}}\mathbb{Z}_2) \arrow[r] & \mathbb{Z}_2\oplus\mathbb{Z}_4\oplus(\oplus_{I-\{i,j\}}\mathbb{Z}_4) \arrow[r] & \mathbb{Z}_2\oplus(\oplus_{J-\{j\}}\mathbb{Z}_2) \arrow[r] & 0.
\end{tikzcd}$$

Suppose that $I=J$. If $H_n(X)=H_{n+1}(X)=\mathbb{Z}_2$, $\mathcal{B}^{n+2}(X)$ is trivial. If not, choose $i,j\in I$ and define maps $f\in\mathrm{Aut}\,(\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus(\oplus_{I-\{i,j\}}\mathbb{Z}_2))$ by $f(x,y,z)=(y,x,z)$, and $g\in\mathrm{Aut}\,(\mathbb{Z}_4\oplus\mathbb{Z}_4\oplus(\oplus_{I-\{i,j\}}\mathbb{Z}_4))$ by $g(x,y,z)=(y,x,z)$. We have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_2\oplus\mathbb{Z}_2\oplus(\oplus_{I-\{i,j\}}\mathbb{Z}_2) \arrow[r]\arrow[d, "f"] & \mathbb{Z}_4\oplus\mathbb{Z}_4\oplus(\oplus_{I-\{i,j\}}\mathbb{Z}_4) \arrow[r]\arrow[d, "g"] & \mathbb{Z}_2\oplus\mathbb{Z}_2\oplus(\oplus_{I-\{i,j\}}\mathbb{Z}_2) \arrow[r]\arrow[d, "f"] & 0 \\
0 \arrow[r] & \mathbb{Z}_2\oplus\mathbb{Z}_2\oplus(\oplus_{I-\{i,j\}}\mathbb{Z}_2) \arrow[r] & \mathbb{Z}_4\oplus\mathbb{Z}_4\oplus(\oplus_{I-\{i,j\}}\mathbb{Z}_4) \arrow[r] & \mathbb{Z}_2\oplus\mathbb{Z}_2\oplus(\oplus_{I-\{i,j\}}\mathbb{Z}_2) \arrow[r] & 0.
\end{tikzcd}$$

Then, $(0,f,f)$ is a Γ-isomorphism of order 2. □

As a consequence, we obtain a negative answer to the problem of realising groups as self-homotopy equivalences of finite type A_n^2-polyhedra:

Corollary 6.17. *Let G be a non nilpotent finite group of odd order. Then, for any $n\geq 3$ and for any finite type A_n^2-polyhedron X, $G\ncong\mathcal{E}(X)$.*

Proof. Assume that there exists a finite type A_n^2-polyhedron X such that $\mathcal{E}(X)\cong G$. Then, if $\mathcal{E}(X)\neq\mathcal{E}_*(X)$, the quotient $\mathcal{E}(X)/\mathcal{E}_*(X)$ is a finite group of odd order, which contradicts Theorem 6.16. Thus $G\cong\mathcal{E}(X)=\mathcal{E}_*(X)$. However, since X is a 1-connected and finite-dimensional space, $\mathcal{E}_*(X)$ is a nilpotent group, [34, Theorem D], which contradicts the fact that G is non nilpotent. □

We now prove our second main result for this chapter. Recall that for a group G, $\mathrm{rank}\,G$ is the smallest cardinal of a set of generators for G, [61, p. 91]. Then:

Theorem 6.18. *Suppose that X is a finite type A_2^2-polyhedron with a non-trivial and finite $\mathcal{B}^4(X)$ of odd order. Then the following must hold:*

(1) $\mathrm{rank}\,H_4(X)\leq 1$;

(2) $\pi_3(X)$ *and* $H_3(X)$ *are 2-groups, and* $H_2(X)$ *is an elementary abelian 2-group;*

(3) $\mathrm{rank}\,H_3(X)\leq\frac{1}{2}\mathrm{rank}\,H_2(X)\big(\mathrm{rank}\,H_2(X)+1\big)-\mathrm{rank}\,H_4(X)\leq\mathrm{rank}\,\pi_3(X)$;

(4) the natural action of $\mathcal{B}^4(X)$ on $H_2(X)$ induces a faithful representation $\mathcal{B}^4(X)\leq$ $\mathrm{Aut}\,(H_2(X))$.

Proof. By hypothesis $\mathcal{B}^4(X)$ is a finite group of odd order. From Lemma 6.4 we deduce that $H_2(X)$ is an elementary abelian 2-group and from Proposition 1.62 that $\Gamma(H_2(X))$ is a 2-group. In particular, every element of $\Gamma(H_2(X))$ is of finite order, and therefore, by Proposition 6.6, $\operatorname{rank} H_4(X) \leq 1$ so we have Theorem 6.18.*(1)*. Now, any element in $\mathcal{B}^4(X)$ is of the form $(0, f_2, f_3)$ if $H_4(X) = 0$ or (id, f_2, f_3) if $H_4(X) = \mathbb{Z}$. Notice that a Γ-morphism of the form $(-\mathrm{id}, f_2, f_3)$ has even order thus it cannot be a Γ-isomorphism under our hypothesis. Therefore, if $H_4(X) = \mathbb{Z}$, then $b_4(1)$ generates a $\mathbb{Z}_4$ factor in $\Gamma(H_2(X))$, and under our hypothesis the equation

$$\operatorname{rank}\Gamma(H_2(X)) = \operatorname{rank} H_4(X) + \operatorname{rank}(\operatorname{coker} b_4)$$

holds for $\operatorname{rank} H_4(X) \leq 1$.

Observe that any Γ-isomorphism of X induces a chain morphism of the short exact sequence

$$0 \to \operatorname{coker} b_4 \to \pi_3(X) \xrightarrow{h_3} H_3(X) \to 0.$$

We will draw our conclusions from this induced morphism, which can be seen as an automorphism of $\pi_3(X)$ that maps the subgroup $i_2(\operatorname{coker} b_4)$ to itself, thus inducing an isomorphism on the quotient, $H_3(X)$.

As we mentioned above, $\Gamma(H_2(X))$ is a 2-group. Then $\operatorname{coker} b_4$ is a quotient of a 2-group so a 2-group itself. We claim that $H_3(X)$ is also a 2-group: otherwise, $H_3(X)$ has a summand whose order is either infinite or odd and therefore this summand would be h_3-split, which from Lemma 6.15 implies that $\mathcal{B}^4(X)$ has elements of even order, leading to a contradiction. Since $\operatorname{coker} b_4$ and $H_3(X)$ are 2-groups, so is $\pi_3(X)$, proving thus Theorem 6.18.*(2)*.

Moreover, no subgroup of $H_3(X)$ can be h_3-split as a consequence of Lemma 6.15, and thus, $\operatorname{rank} H_3(X) \leq \operatorname{rank}(\operatorname{coker} b_4) = \operatorname{rank}\Gamma(H_2(X)) - \operatorname{rank} H_4(X)$. We can compute $\operatorname{rank}\Gamma(H_2(X))$ using Proposition 1.62 and immediately obtain Theorem 6.18.*(3)*.

Now one can easily check that $\Omega_1(\pi_3(X)) \leq i_2(\operatorname{coker} b_4)$ and, from [45, Ch. 5, Theorem 2.4], we obtain that any automorphism of odd order of $\pi_3(X)$ acting as the identity on $i_2(\operatorname{coker} b_4)$ must be the identity.

Then, if $(\mathrm{id}, f_3, f_2) \in \mathcal{B}^4(X)$ is a Γ-morphism with f_3 non-trivial, f_3 has odd order, so we may assume that $\Omega\colon \pi_3(X) \to \pi_3(X)$ (see Definition 1.66) has odd order too. By the argument above, it must induce a non-trivial homomorphism on $i_2(\operatorname{coker} b_4)$ and therefore f_2 is non-trivial as well. Thus, the natural action of $\mathcal{B}^4(X)$ on $H_2(X)$ must be faithful, since any Γ-automorphism $(\mathrm{id}, f_3, f_2) \in \mathcal{B}^4(X)$ induces a non-trivial $f_2 \in \operatorname{Aut}(H_2(X))$. Then, Theorem 6.18.*(4)* follows. $\square$

However, it is worth mentioning that our attempts to find a space satisfying the hypothesis of Theorem 6.18 have been unsuccessful. We therefore raise the following conjecture:

Conjecture 6.19. *Let X be an A_2^2-polyhedron. If $\mathcal{B}^4(X)$ is a non-trivial finite group, then it necessarily has an element of order 2.*

Resumen en español

En esta tesis nos interesamos por los problemas de realizabilidad, que si bien surgen de forma natural y tienen un planteamiento muy sencillo, no son por lo general fáciles de resolver. Un ejemplo de un problema de este tipo es el denominado *problema de Galois inverso*, que pregunta si todo grupo aparece como grupo de Galois de una extensión finita de $\mathbb{Q}$. Hilbert fue el primero en estudiar este problema en profundidad a finales del siglo XIX, [52], y a día de hoy continúa abierto.

En la Topología Algebraica, donde las estructuras algebraicas juegan un papel fundamental, distintos problemas de realizabilidad han sido planteados y estudiados. Un problema de realizabilidad clásico es el *problema de realizabilidad para álgebras de cohomología*, propuesto por Steenrod en 1961, [73]. Este problema consiste en determinar qué álgebras aparecen como álgebra de cohomología de un espacio topológico, y ha sido bastante estudiado, [1, 3]. Otro problema propuesto por Steenrod es el *problema de los G-espacios de Moore*, [59, Problem 51]. Consiste en determinar si, dado un grupo G, todo $\mathbb{Z}G$-módulo aparece como homología de algún G-espacio de Moore simplemente conexo. Una respuesta negativa fue obtenida en 1981, [21], y una caracterización de aquellos grupos G para los que todo $\mathbb{Z}G$-módulo es realizable apareció en 1987, [78].

Sin embargo, nuestra atención se centra en el llamado *problema de realizabilidad de grupos*, que pregunta lo siguiente: dada una categoría $\mathcal{C}$, ¿aparece todo grupo como grupo de automorfismos de un objeto de $\mathcal{C}$? Este problema ha sido estudiado en distintas áreas de la combinatoria, como se deduce de [7, 8], de donde obtenemos la siguiente terminología: si un grupo G es el grupo de automorfismos de un objeto $X \in \mathrm{Ob}(\mathcal{C})$, es decir, si $G \cong \mathrm{Aut}_{\mathcal{C}}(X)$, diremos que X *realiza* a G, y que G es *realizable* en $\mathcal{C}$. Una categoría $\mathcal{C}$ en la que todo grupo finito es realizable se dice *finitamente universal*, mientras que una categoría en la que todo grupo es realizable se dice *universal*.

Pero nuestro interés en la combinatoria transciende a la terminología, ya que la solución del históricamente importante *problema de realizabilidad de grupos en la categoría de grafos* supone un elemento clave en buena parte de nuestras construcciones. Este problema fue propuesto por König ya en 1936, [58], y apenas tres años después Frucht demostró que la categoría de grafos (finitos) es finitamente universal, [43]. Sin embargo, el problema general permaneció abierto durante más de 20 años, hasta que de Groot, en 1959, [32], y Sabidussi, en 1960, [72], demostraron de manera independiente que la categoría de grafos es universal.

No obstante, el problema que motiva esta tesis es el denominado *problema de realizabilidad de grupos en la categoría $\mathcal{H}oTop$*, la categoría de homotopía de espacios topológicos punteados. Fue propuesto por Kahn en los años 60 y pregunta si todo grupo aparece como grupo de automorfismos de un espacio en $\mathcal{H}oTop$. Es un problema que ha recibido una atención considerable; prueba de ello es su aparición en distintos sondeos y listas de problemas abiertos, [4, 5, 37, 55, 56, 71]. Recuérdese que, dado un espacio X, el grupo $\mathrm{Aut}_{\mathcal{H}oTop}(X)$ suele denotarse por $\mathcal{E}(X)$, y es el denominado *grupo de auto-equivalencias de homotopía de*

X; sus elementos son aquellas clases de homotopía de auto-aplicaciones continuas de X que tienen inversa homotópica.

Un primer ejemplo que se nos viene a la cabeza a la hora de realizar grupos como auto-equivalencias de homotopía de espacios son los espacios de Eilenberg-MacLane. En efecto, si H es un grupo abeliano y $n \geq 2$ es un entero, $\mathcal{E}\big(K(H,n)\big) \cong \mathrm{Aut}(H)$. Sin embargo, esto no proporciona una solución completa a este problema, pues no todo grupo G es isomorfo a $\mathrm{Aut}(H)$ para algún H (así sucede, por ejemplo, con $\mathbb{Z}_p$ para p impar).

El mayor escollo a la resolución de este problema radica precisamente en que, a excepción del uso de espacios de Eilenberg-MacLane, no existe una forma obvia de obtener espacios que realicen a un grupo dado. En consecuencia, las herramientas con las que se contaba en aquel momento para tratar seriamente el problema de Kahn eran insuficientes, y durante décadas este problema solo fue estudiado utilizando técnicas ad-hoc para ciertas familias de grupos, [15, 16, 36, 62, 65].

Este escollo fue superado gracias a un método general obtenido por Costoya y Viruel, [27], que proporciona una solución positiva al problema de realizabilidad de Kahn en el caso de grupos finitos: para todo grupo finito G, existe un espacio topológico X (de hecho, una cantidad infinita de ellos) de modo que $G \cong \mathcal{E}(X)$ ([27, Theorem 1.1]). Son capaces de proporcionar este método combinando la solución de Frucht al problema de realizabilidad de grupos en la categoría de grafos con las notables características computacionales de las herramientas algebraicas involucradas en la Teoría de Homotopía Racional.

Nuestro objetivo principal en esta tesis es el de extender este tipo de técnicas al estudio de otros problemas de realizabilidad. Por un lado, mejoramos algunos de los resultados en [27], y por otro, estudiamos problemas de realizabilidad en otras categorías además de $\mathcal{H}oTop$. Nótese que la Teoría de Homotopía Racional estudia espacios que no son de tipo finito sobre $\mathbb{Z}$ (véase la Definición 1.36) y que por tanto no son geométricamente sencillos. Esto nos lleva a considerar también un marco alternativo (A_n^2-poliedros) con el objetivo de proporcionar una solución al problema de Kahn en términos de espacios *enteros*.

Comenzamos introduciendo dos generalizaciones del problema de realizabilidad de grupos, al que nos referiremos a partir de ahora como problema de realizabilidad de grupos clásico. La primera generalización que vamos a considerar se desarrolla en el ámbito de las *categorías de flechas*. Recuérdese que dada una categoría $\mathcal{C}$, su categoría de flechas $\mathrm{Arr}(\mathcal{C})$ es una categoría cuyos objetos son los morfismos $f \in \mathrm{Hom}_{\mathcal{C}}(A_1, A_2)$, y donde un morfismo entre dos objetos $f \in \mathrm{Hom}_{\mathcal{C}}(A_1, A_2)$ y $g \in \mathrm{Hom}_{\mathcal{C}}(B_1, B_2)$ es un par de morfismos $(f_1, f_2) \in \mathrm{Hom}_{\mathcal{C}}(A_1, B_1) \times \mathrm{Hom}_{\mathcal{C}}(A_2, B_2)$ tales que $g \circ f_1 = f_2 \circ f$. Entonces, cuando consideramos $f \in \mathrm{Hom}_{\mathcal{C}}(A_1, A_2)$ vemos que $\mathrm{Aut}_{\mathrm{Arr}(\mathcal{C})}(f)$, que por abuso de notación denotamos simplemente por $\mathrm{Aut}_{\mathcal{C}}(f)$, es un subgrupo de $\mathrm{Aut}_{\mathcal{C}}(A_1) \times \mathrm{Aut}_{\mathcal{C}}(A_2)$. Por consiguiente, surge naturalmente el siguiente problema.

Problema 1. *Sea $\mathcal{C}$ una categoría. Dados grupos G_1, G_2 y $H \leq G_1 \times G_2$, ¿existe $f\colon A_1 \to A_2$ un objeto en $\mathrm{Arr}(\mathcal{C})$ tal que $\mathrm{Aut}_{\mathcal{C}}(A_1) \cong G_1$, $\mathrm{Aut}_{\mathcal{C}}(A_2) \cong G_2$, y $\mathrm{Aut}_{\mathcal{C}}(f) \cong H$?*

La segunda generalización del problema de realizabilidad de grupos clásico trata de realizar *representaciones por permutaciones*, es decir, acciones por permutaciones de un grupo en un conjunto. Si $\mathcal{C}$ es una categoría cuyos objetos pueden verse como conjuntos, podemos pensar en realizar en $\mathcal{C}$ una representación por permutaciones de la siguiente manera:

Problema 2. *Sea $\rho\colon G \to \mathrm{Sym}(V)$ una representación por permutaciones y sea $\mathcal{C}$ una categoría cuyos objetos son conjuntos. ¿Existe un G-objeto (fiel) $A \in \mathrm{Ob}(\mathcal{C})$ de modo que V puede verse como un subconjunto $\mathrm{Aut}_{\mathcal{C}}(A)$-invariante de A y tal que la restricción de la G-acción a V sea ρ?*

Pretendemos resolver estos problemas siguiendo las técnicas de Costoya-Viruel, es decir, resolviéndolos en primer lugar en la categoría $\mathcal{C} = \mathcal{G}raphs$ para posteriormente trasladar esa solución a un marco algebraico apropiado (coálgebras y álgebras graduadas conmutativas diferenciales). Empecemos introduciendo nuestra solución a los Problemas 1 y 2 en $\mathcal{G}raphs$.

Siguiendo el esquema de la solución al problema clásico proporcionada por Frucht, [43], empezamos trabajando en un marco más general que el de grafos: la categoría de sistemas relacionales binarios sobre un conjunto I, véase la Definición 1.1. La ventaja que proporcionan estos objetos combinatorios más generales es la posibilidad de utilizar etiquetas en las aristas para codificar información que proviene del grupo. Así pues, partimos de la solución al problema de realizabilidad clásico en $I\mathcal{R}el$, los diagramas de Cayley, y la modificamos para codificar la información adicional necesaria para dar solución a los problemas planteados.

La base de nuestra solución al Problema 1 en $I\mathcal{R}el$ es un resultado clásico y elemental de teoría de grupos que caracteriza a los subgrupos de un producto directo de dos grupos, el Lema de Goursat (véase el Lema 2.1). Utilizando esta caracterización somos capaces de construir, asociados a dos grupos cualesquiera G_1, G_2 y a un subgrupo $H \leq G_1 \times G_2$, dos sistemas relacionales binarios $\mathcal{G}_1$ y $\mathcal{G}_2$ sobre un cierto conjunto I (Definición 2.7) y un morfismo entre ellos $\varphi\colon \mathcal{G}_1 \to \mathcal{G}_2$ (Definición 2.10) que nos permiten deducir el siguiente resultado.

Teorema 2.16. *Sean G_1 y G_2 dos grupos y sea $H \leq G_1 \times G_2$. Existe un morfismo de sistemas binarios relacionales sobre un cierto conjunto I, $\varphi\colon \mathcal{G}_1 \to \mathcal{G}_2$, tal que $\mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}_1)$, $\mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}_2)$ y $\mathrm{Aut}_{I\mathcal{R}el}(\varphi)$ son respectivamente isomorfos a G_1, G_2 y H.*

Respecto al Problema 2, en la Definición 2.18 introducimos, asociado a una representación por permutaciones $\rho\colon G \to \mathrm{Sym}(V)$, un sistema relacional binario $\mathcal{G}$ sobre un cierto conjunto I para el que demostramos el siguiente resultado.

Teorema 2.26. *Sea G un grupo y V un conjunto. Sea $\rho\colon G \to \mathrm{Sym}(V)$ una representación por permutaciones de G en V. Existe un sistema relacional binario $\mathcal{G}$ sobre un cierto conjunto I tal que*

(1) $V \subset V(\mathcal{G})$ y cada $\psi \in \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G})$ deja V invariante;

(2) $\mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}) \cong G$;

(3) la aplicación restricción $G \cong \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}) \to \mathrm{Sym}(V)$ es ρ;

(4) existe una acción fiel $\bar{\rho}\colon G \cong \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}) \to \mathrm{Sym}\left(V(\mathcal{G}) \setminus V\right)$ tal que la aplicación restricción $\mathcal{G} \cong \mathrm{Aut}_{I\mathcal{R}el}(\mathcal{G}) \to \mathrm{Sym}\left(V(\mathcal{G})\right)$ es $\rho \oplus \bar{\rho}$.

Una vez resueltos ambos problemas en $I\mathcal{R}el$, el siguiente objetivo es trasladar estas soluciones a $\mathcal{G}raphs$. Para ello utilizamos el *reemplazamiento de flechas*, es decir, asociamos a cada etiqueta de nuestros sistemas I-relacionales un cierto grafo asimétrico (un grafo cuyo único automorfismo es la identidad) de modo que grafos asociados a etiquetas distintas no sean isomorfos, y sustituimos las aristas etiquetadas del sistema relacional por estos grafos. Si elegimos los grafos asimétricos con cuidado, podemos asegurar que a través de un automorfismo del grafo resultante tras el reemplazamiento de flechas solo pueden ir a copias de sí mismos, es decir, juegan el mismo papel que la arista dirigida y etiquetada a la que sustituyen.

Si bien se pueden encontrar resultados de esta índole en la literatura, estos se centran exclusivamente en preservar los grupos de automorfismos, por lo que no son suficientes para nuestros propósitos. No obstante, utilizando este tipo de técnicas, somos capaces de obtener un resultado lo bastante potente como para permitirnos traducir las soluciones de los Problemas 1 y 2 a la categoría de grafos:

Teorema 2.33. *Sean $\mathcal{G}_1'$, $\mathcal{G}_2'$ dos sistemas relacionales binarios sobre un cierto conjunto I para los que existe un cardinal α de modo que $\deg(v) \leq \alpha$ (véase la Definición 1.8), para todo $v \in V(\mathcal{G}_1') \cup V(\mathcal{G}_2')$. Sea $\varphi'\colon \mathcal{G}_1' \to \mathcal{G}_2'$ un morfismo de sistemas I-relacionales binarios. Existen grafos $\mathcal{G}_1, \mathcal{G}_2$ y un morfismo entre ellos $\varphi\colon \mathcal{G}_1 \to \mathcal{G}_2$ de manera que:*

(1) *existe un subconjunto* $V(\mathcal{G}'_k) \subset V(\mathcal{G}_k)$ *que queda invariante a través de los automorfismos de* $\mathcal{G}_k$*,* $k = 1, 2$*;*

(2) *si* $\psi \in \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}_k)$*, la restricción* $\psi' = \psi|_{V(\mathcal{G}'_k)}$ *está en* $\mathrm{Aut}_{\mathcal{IR}el}(\mathcal{G}'_k)$*, para* $k = 1, 2$*;*

(3) *la aplicación* $\Psi_k \colon \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}_k) \xrightarrow{\cong} \mathrm{Aut}_{\mathcal{IR}el}(\mathcal{G}'_k)$ *que lleva* $\psi \in \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}_k)$ *a la restricción* $\Psi_k(\psi) = \psi|_{V(\mathcal{G}'_k)}$ *es un isomorfismo de grupos, para* $k = 1, 2$*;*

(4) $\varphi|_{V(\mathcal{G}'_1)} = \varphi' \colon V(\mathcal{G}'_1) \to V(\mathcal{G}'_2)$ *y* $\mathrm{Aut}_{\mathcal{IR}el}(\varphi') \cong \mathrm{Aut}_{\mathcal{G}raphs}(\varphi)$*.*

Una vez probado este resultado, resulta sencillo trasladar el Teorema 2.16 a la categoría $\mathcal{C} = \mathcal{G}raphs$, proporcionando una solución afirmativa al Problema 1 en este ámbito:

Teorema 2.37. *Sean* G_1*,* G_2 *y* $H \leq G_1 \times G_2$ *grupos. Existen grafos* $\mathcal{G}_1$*,* $\mathcal{G}_2$ *y un morfismo entre ellos* $\varphi \colon \mathcal{G}_1 \to \mathcal{G}_2$ *de modo que* $\mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}_k) \cong G_k$*,* $k = 1, 2$*, y* $\mathrm{Aut}_{\mathcal{G}raphs}(\varphi) \cong H$*.*

De manera análoga, observamos que el Teorema 2.33 puede utilizarse para trasladar la solución del Problema 2 en $\mathcal{IR}el$, Teorema 2.26, a $\mathcal{G}raphs$. Obtenemos así una generalización de [19, Theorem 1.1]:

Teorema 2.41. *Sea* G *un grupo,* V *un conjunto y* $\rho \colon G \to \mathrm{Sym}(V)$ *una representación por permutaciones de* G *en* V*. Existe un grafo* $\mathcal{G}$ *tal que*

(1) $V \subset V(\mathcal{G})$ *y todo* $\psi \in \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G})$ *deja invariante a* V*;*

(2) $\mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \cong G$*;*

(3) *la restricción* $G \cong \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \to \mathrm{Sym}(V)$ *es precisamente* ρ*;*

(4) *existe una acción fiel* $\bar{\rho} \colon G \cong \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \to \mathrm{Sym}\left(V(\mathcal{G}) \setminus V\right)$ *tal que la aplicación restricción* $\mathcal{G} \cong \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \to \mathrm{Sym}\left(V(\mathcal{G})\right)$ *es* $\rho \oplus \bar{\rho}$*.*

Además, observamos que si los grupos y conjuntos involucrados son finitos, los Problemas 1 y 2 admiten una solución en términos de grafos finitos, véase el Corolario 2.38 y el Corolario 2.42. Este hecho resulta de suma importancia a la hora de resolver estos problemas en $\mathcal{H}oTop$.

Ahora que hemos obtenido una solución a los problemas de realizabilidad generalizados en la categoría de grafos, podemos proceder a tratar estos problemas en estructuras algebraicas. Empezamos con las coálgebras, véase la Definición 1.22. Nos interesa este marco pues si bien se sabe mucho acerca de los grupos de automorfismos de álgebras (véanse, por ejemplo, las referencias [25, 28, 57] para el caso asociativo y [44] para el caso no asociativo), en el caso de sus estructuras duales, las coálgebras, se sabe muy poco. Además, y dado que el dual de un álgebra infinito-dimensional no es en general una coálgebra, no se pueden deducir resultados en este sentido por dualización de la literatura existente en el caso de las álgebras.

En consecuencia, nuestro objetivo en este ámbito es proporcionar los primeros resultados respecto al problema de realizabilidad de grupos clásico en la categoría de coálgebras, $\mathcal{C} = \mathrm{Coalg}_{\Bbbk}$, además de considerar los Problemas 1 y 2. En esta tesis no somos capaces de obtener coálgebras cuyo grupo de automorfismos es un grupo dado G; en su lugar, obtenemos coálgebras para las que G aparece como la imagen de la restricción de sus automorfismos a su conjunto de elementos de grupo, véase la Definición 1.26.

Nuestro punto de partida es $\Bbbk\mathcal{G}$ la $\Bbbk$-coálgebra de caminos asociada al grafo dirigido $\mathcal{G}$, Definición 1.34. Es sencillo comprobar que el conjunto de elementos de grupo de $\Bbbk\mathcal{G}$ es precisamente $V(\mathcal{G})$. Es más, todo morfismo de digrafos $\sigma \colon \mathcal{G}_1 \to \mathcal{G}_2$ induce un morfismo de coálgebras $\Bbbk\mathcal{G}_1 \to \Bbbk\mathcal{G}_2$ cuya restricción a los elementos grupo de $\Bbbk\mathcal{G}_1$ es precisamente σ. En consecuencia, no es difícil observar que $\mathrm{Aut}_{\mathcal{D}igraphs}(\mathcal{G}) \leq \mathrm{Aut}_{\mathrm{Coalg}_{\Bbbk}}(\Bbbk\mathcal{G})$. Sin embargo, el grupo $\mathrm{Aut}_{\mathrm{Coalg}_{\Bbbk}}(\Bbbk\mathcal{G})$ resulta ser demasiado grande.

Así pues, nuestra manera de enfrentarnos a este problema es quedarnos exclusivamente con la información mínima estrictamente necesaria en la coálgebra de caminos para que esta siga modelando al grafo. Es decir, nos quedamos únicamente con los generadores asociados a caminos de longitud 0 (vértices) y de longitud 1 (aristas):

Definición 3.1. Sea $\Bbbk$ un cuerpo y $\mathcal{G}$ un digrafo. Definimos una coálgebra $C(\mathcal{G}) = (C, \Delta, \varepsilon)$ con $C = \Bbbk\{v \mid v \in V(\mathcal{G})\} \oplus \Bbbk\{e \mid e \in E(\mathcal{G})\}$ y donde

- para cada $v \in V(\mathcal{G})$, $\Delta(v) = v \otimes v$ y $\varepsilon(v) = 1$;
- para cada $e = (v_1, v_2) \in E(\mathcal{G})$, $\Delta(e) = v_1 \otimes e + e \otimes v_2$ y $\varepsilon(e) = 0$.

No es difícil ver que la coálgebra $C(\mathcal{G})$ verifica las propiedades que nos interesaban de $\Bbbk\mathcal{G}$, y en consecuencia tenemos un funtor fiel $C\colon \mathcal{Digraphs} \to \mathrm{Coalg}_{\Bbbk}$, Definición 3.4. Si calculamos los automorfismos de $C(\mathcal{G})$ para un digrafo $\mathcal{G}$ fijado podemos observar lo siguiente:

Teorema 3.9. *Sea $\Bbbk$ un cuerpo y $\mathcal{G}$ un digrafo. La $\Bbbk$-coálgebra $C(\mathcal{G})$ introducida en la Definición 2.18 es tal que $G(C(\mathcal{G})) = V(\mathcal{G})$ y la aplicación restricción $\mathrm{Aut}_{\mathrm{Coalg}_{\Bbbk}}(C(\mathcal{G})) \to \mathrm{Sym}\,(G(C(\mathcal{G}))) = \mathrm{Sym}\,(V(\mathcal{G}))$ induce una sucesión exacta corta escindida*

$$1 \longrightarrow \prod_{e \in E(\mathcal{G})} (\Bbbk \rtimes \Bbbk^{\times}) \longrightarrow \mathrm{Aut}_{\mathrm{Coalg}_{\Bbbk}}(C(\mathcal{G})) \longrightarrow \mathrm{Aut}_{\mathcal{Digraphs}}(\mathcal{G}) \longrightarrow 1.$$

Como consecuencia inmediata, y dado que todo grupo es realizable como grupo de automorfismos de un grafo, [32, 72], obtenemos el siguiente resultado:

Corolario 3.11. *Sea $\Bbbk$ un cuerpo y G un grupo. Existe una $\Bbbk$-coálgebra C verificando que $\mathrm{Aut}_{\mathrm{Coalg}_{\Bbbk}}(C) \cong K \rtimes G$, donde K es un producto directo de grupos de la forma $\Bbbk \rtimes \Bbbk^{\times}$. Es más, G es la imagen de la restricción de los automorfismos de C a $\mathrm{Sym}\,(G(C))$.*

Así, utilizando el funtor C y sus propiedades, del Teorema 3.9 se deduce el siguiente resultado respecto al Problema 1:

Teorema 3.12. *Sean G_1, G_2 y $H \leq G_1 \times G_2$ grupos. Sea $\Bbbk$ un cuerpo. Existen $\Bbbk$-coálgebras C_1 y C_2 y un morfismo entre ellas $\varphi \in \mathrm{Hom}_{\mathrm{Coalg}_{\Bbbk}}(C_1, C_2)$ de modo que*

(1) $\mathrm{Aut}_{\mathrm{Coalg}_{\Bbbk}}(C_k) \cong K_k \rtimes G_k$, donde G_k es la imagen de la restricción $\mathrm{Aut}_{\mathrm{Coalg}_{\Bbbk}}(C_k) \to \mathrm{Sym}\,(G(C_k))$ y K_k es producto directo de factores isomorfos a $\Bbbk \rtimes \Bbbk^{\times}$, $k = 1, 2$;

(2) $\mathrm{Aut}_{\mathrm{Coalg}_{\Bbbk}}(\varphi) \cong K \rtimes H$, donde H es imagen de la aplicación restricción $\mathrm{Aut}_{\mathrm{Coalg}_{\Bbbk}}(\varphi) \to \mathrm{Sym}\,(G(C_1)) \times \mathrm{Sym}\,(G(C_2))$ y $K \leq K_1 \times K_2$.

Del mismo modo, respecto al Problema 2 probamos lo siguiente:

Teorema 3.13. *Sea G un grupo, $\Bbbk$ un cuerpo y $\rho\colon G \to \mathrm{Sym}(V)$ una representación por permutaciones de G en un conjunto V. Existe una G-coálgebra C tal que:*

(1) G actúa fielmente en C, es decir, la G-acción induce un monomorfismo de grupos $G \hookrightarrow \mathrm{Aut}_{\mathrm{Coalg}_{\Bbbk}}(C)$;

(2) la imagen de la aplicación restricción $\mathrm{Aut}_{\mathrm{Coalg}_{\Bbbk}}(C) \to \mathrm{Sym}\,(G(C))$ es G;

(3) existe un subconjunto $V \subset G(C)$ invariante a través de la $\mathrm{Aut}_{\mathrm{Coalg}_{\Bbbk}}(C)$-acción en C verificando que ρ es la composición de la inclusión $G \hookrightarrow \mathrm{Aut}_{\mathrm{Coalg}_{\Bbbk}}(C)$ con la restricción $\mathrm{Aut}_{\mathrm{Coalg}_{\Bbbk}}(C) \to \mathrm{Sym}(V)$;

(4) existe una acción fiel $\bar{\rho}\colon G \to \operatorname{Sym}(G(C) \setminus V)$ *tal que la composición de la inclusión* $G \hookrightarrow \operatorname{Aut}_{\mathrm{Coalg}_{\Bbbk}}(C)$ *con la restricción* $\operatorname{Aut}_{\mathrm{Coalg}_{\Bbbk}}(C) \to \operatorname{Sym}(G(C))$ *es* $\rho \oplus \bar{\rho}$.

Como aplicación adicional del funtor C, del Teorema 3.9 y del Corolario 3.11, estudiamos el problema de isomorfía de grupos utilizando representaciones en coálgebras. El *problema de isomorfía de grupos* consiste en estudiar si una determinada teoría o contexto permite discernir los tipos de isomorfía de grupos. Por ejemplo, se ha estudiado en multitud de contextos si las representaciones de grupos se pueden utilizar para distinguir sus clases de isomorfía, interesando particularmente el caso de los grupos finitos. En este ámbito, Hertweck obtuvo una respuesta negativa, demostrando que existen dos grupos finitos no isomorfos G y H, ambos de orden $2^{21}97^{28}$, con anillos grupo integrales isomorfos, [50], lo que en particular implica que G y H tienen la misma teoría de representaciones lineales.

Nuestros resultados de realizabilidad nos permiten estudiar el problema de isomorfía de grupos utilizando representaciones de grupos co-Hopfianos en coálgebras. Recuérdese que un grupo G es co-Hopfiano si no es isomorfo a ninguno de sus subgrupos propios o, dicho de otro modo, si todo monomorfismo $G \hookrightarrow G$ es un isomorfismo. La clase de grupos co-Hopfianos incluye de manera obvia a los grupos finitos, pero también a los grupos artinianos, grupos de Tarski, grupos lineales especiales de $\mathbb{Z}$, grupos fundamentales de superficies cerradas de género mayor que dos... Se trata, por tanto, de una clase de grupos significativa, véase el Ejemplo 3.15.

En este ámbito obtenemos dos resultados. El primero es cierto para toda la clase de grupos co-Hopfianos, pero requiere que nos centremos en la restricción de las acciones a los conjuntos de elementos grupo:

Teorema 3.16. *Sea* $\Bbbk$ *un cuerpo y sean* G *y* H *dos grupos co-Hopfianos. Son equivalentes:*

(1) G *y* H *son isomorfos.*

(2) Para toda $\Bbbk$*-coálgebra* C*, existe una* G*-acción en* C *que restringe a una acción por permutaciones fiel en* $G(C)$ *si y solo si existe una* H*-acción en* C *que restringe a una acción por permutaciones fiel en* $G(C)$.

El segundo resultado se formula directamente en base a la acción sobre toda la coálgebra, pero como contrapartida debemos restringir la clase de grupos a estudiar:

Definición 3.17. Sea $\Bbbk$ un cuerpo finito de orden p^n, p primo. Un grupo G está en la clase $\mathfrak{G}_{p,n}$ si verifica las siguientes propiedades:

(1) G es co-Hopfiano.

(2) G no admite subgrupos normales finitos no triviales cuyo exponente divide a $p^n(p^n-1)$.

Esta clase todavía incluye grupos interesantes. Por ejemplo $\mathfrak{G}_{2,1}$ incluye a los grupos 2-reducidos, es decir, grupos que no contienen 2-subgrupos normales. Para esta familia, demostramos:

Teorema 3.18. *Sea* $\Bbbk$ *un cuerpo finito de orden* p^n*,* p *primo. Sean* G *y* H *dos grupos en* $\mathfrak{G}_{p,n}$*. Son equivalentes:*

(1) G *y* H *son isomorfos.*

(2) Para toda $\Bbbk$*-coálgebra* C*,* G *actúa fielmente en* C *si y solo si* H *actúa fielmente en* C.

Continuamos ahora con el estudio de los problemas de realizabilidad en la categoría de álgebras graduadas conmutativas diferenciales o CDGAs (véase la Definición 1.37) y con los resultados que se pueden deducir en la categoría de homotopía de espacios topológicos utilizando las técnicas de homotopía racional. Los trabajos de Costoya-Viruel, [27, 28, 29], son nuestro punto de partida en este marco. La base de sus construcciones es un espacio racional no trivial cuyo grupo de autoequivalencias de homotopía es trivial, [6, Example 5.1].

Los espacios cuyo grupo de auto-equivalencias de homotopía es trivial reciben el nombre de *espacios homotópicamente rígidos*. El primer ejemplo no trivial de un espacio homotópicamente rígido con cohomología racional no trivial fue obtenido por Kahn, [55], quien creía que este tipo de espacios podrían jugar un papel fundamental en alguna forma de descomponer espacios, [56]. En consecuencia, la obtención de nuevos ejemplos de espacios homotópicamente rígidos resulta de interés. Y es de destacar que los ejemplos de tales espacios que se pueden encontrar en la literatura, [6, Example 5.2], [31, Examples 8.1 & 8.2], tienen su nivel de conectividad acotado superiormente. No obstante, todos estos ejemplos tienen una estructura de generadores y diferenciales muy determinada, y su estudio nos permite introducir una familia monoparamétrica de CDGAs candidatas a ser homotópicamente rígidas.

Definición 4.1. Sea R un dominio de integridad y $k \geq 1$ un entero. Definimos la R-CDGA

$$\mathcal{M}_k = (\Lambda(x_1, x_2, y_1, y_2, y_3, z), d)$$

donde

$$\begin{aligned}
&|x_1| = 10k + 8, && dx_1 = 0,\\
&|x_2| = 12k + 10, && dx_2 = 0,\\
&|y_1| = 42k + 33, && dy_1 = x_1^3 x_2,\\
&|y_2| = 44k + 35, && dy_2 = x_1^2 x_2^2,\\
&|y_3| = 46k + 37, && dy_3 = x_1 x_2^3,\\
&|z| = 60k^2 + 98k + 39, && dz = x_1^{6k-6}(x_2^2 y_1 y_2 - x_1 x_2 y_1 y_3 + x_1^2 y_2 y_3) + x_1^{6k+5} + x_2^{5k+4}.
\end{aligned}$$

Un análisis en profundidad de la estructura de los generadores de estas álgebras (véase el Lema 4.2) permite concluir que cada uno de ellos se encuentra aislado en su grado, por lo que los endomorfismos de estas álgebras deben llevar cada generador a un múltiplo de sí mismo. Utilizando este hecho podemos comprobar que estas álgebras no solo son homotópicamente rígidas, sino que también son rígidas como álgebras. Es decir, probamos el siguiente resultado:

Teorema 4.3. *Sea $k \geq 1$ un entero. La R-CDGA $\mathcal{M}_k$ es rígida, es decir, sus únicos endomorfismos son la aplicación trivial y la identidad.*

Ahora que ya tenemos las álgebras rígidas, deberemos estudiar cómo utilizarlas para resolver los problemas de realizabilidad. Utilizaremos estas álgebras para construir espacios asociados a digrafos. Así pues, sea $\mathcal{D}igraphs_+$ la subcategoría plena de la categoría de digrafos cuyos objetos son los grafos dirigidos sin bucles tales que todo vértice es el vértice de inicio de al menos una arista (véase la Definición 4.4). Entonces, a cada objeto de $\mathcal{D}igraphs_+$ le asociamos una R-CDGA como sigue:

Definición 4.6. Sea $\mathcal{G}$ un objeto en $\mathcal{D}igraphs_+$ y sea $n \geq 1$ un entero. Considérese la R-CDGA

$$\mathcal{M}_n(\mathcal{G}) = (\Lambda(x_1, x_2, y_1, y_2, y_3, z) \otimes_R \Lambda(x_v \mid v \in V(\mathcal{G})) \otimes_R (z_{(v,w)} \mid (v, w) \in E(\mathcal{G})), d)$$

donde

$$\begin{array}{ll}
|x_1| = 30n - 12, & dx_1 = 0, \\
|x_2| = 36n - 14, & dx_2 = 0, \\
|y_1| = 126n - 51, & dy_1 = x_1^3 x_2, \\
|y_2| = 132n - 53, & dy_2 = x_1^2 x_2^2, \\
|y_3| = 138n - 55, & dy_3 = x_1 x_2^3, \\
|x_v| = 180n^2 - 142n + 28, & dx_v = 0, \\
|z| = 540n^2 - 426n + 83, & dz = x_1^{18n-18}(x_2^2 y_1 y_2 - x_1 x_2 y_1 y_3 + x_1^2 y_2 y_3) \\
 & \quad + x_1^{18n-7} + x_2^{15n-6}, \\
|z_{(v,w)}| = 540n^2 - 426n + 83, & dz_{(v,w)} = x_v^3 + x_v x_w x_2^{5n-2} + x_1^{18n-7}.
\end{array}$$

Si $\mathcal{G}_1$ y $\mathcal{G}_2$ son objetos de $\mathcal{D}igraphs_+$, existe una forma obvia de asociar a un morfismo de grafos $\sigma\colon \mathcal{G}_1 \to \mathcal{G}_2$ un morfismo de R-CDGAs $\mathcal{M}_n(\sigma)\colon \mathcal{M}_n(\mathcal{G}_1) \to \mathcal{M}_n(\mathcal{G}_2)$. Efectivamente, en el Lema 4.10 se demuestra que la aplicación $\mathcal{M}_n(\sigma)\colon \mathcal{M}_n(\mathcal{G}_1) \to \mathcal{M}_n(\mathcal{G}_2)$ definida como

$$\begin{array}{ll}
\mathcal{M}_n(\sigma)(w) = w, & \text{si } w \in \{x_1, x_2, y_1, y_2, y_3, z\}, \\
\mathcal{M}_n(\sigma)(x_v) = x_{\sigma(v)}, & \text{si } v \in V(\mathcal{G}_1), \\
\mathcal{M}_n(\sigma)(z_{(v,w)}) = z_{(\sigma(v),\sigma(w))}, & \text{si } (v,w) \in E(\mathcal{G}_1),
\end{array}$$

es un morfismo de R-CDGA (nótese que por abuso de notación estamos denotando de la misma manera a los generadores de la parte rígida en CDGAs asociadas a grafos distintos). Además, es evidente que $\mathcal{M}_n$ lleva la identidad de un grafo a la identidad de su álgebra asociada y que respeta la composición. Es decir, si denotamos a la categoría de CDGA sobre el anillo R como CDGA_R y consideramos un entero cualquiera $n \geq 1$, $\mathcal{M}_n$ es un funtor $\mathcal{M}_n\colon \mathcal{D}igraphs_+ \to \mathrm{CDGA}_R$ (véase la Definición 4.11). Es más, si $n \geq 1$ es un entero y R es un dominio de integridad tal que $\mathrm{char}(R)$ no es 2 ni 3, el funtor $\mathcal{M}_n$ es *casi* plenamente fiel: dados $\mathcal{G}_1, \mathcal{G}_2 \in \mathrm{Ob}(\mathcal{D}igraphs_+)$, los conjuntos $\mathrm{Hom}_{\mathcal{D}igraphs}(\mathcal{G}_1, \mathcal{G}_2)$ y $\mathrm{Hom}_{\mathrm{CDGA}_R}(\mathcal{M}_n(\mathcal{G}_1), \mathcal{M}_n(\mathcal{G}_2)) \setminus \{0\}$ son biyectivos. En particular, y dado que sabemos que la categoría de grafos es universal, obtenemos nuestro primer resultado de realizabilidad.

Teorema 4.16. *Sea G un grupo. Para cada $n \geq 1$, existe una R-CDGA n-conexa M_n tal que $\mathrm{Aut}_{\mathrm{CDGA}_R}(M_n) \cong G$.*

En consecuencia, hemos probado que la categoría CDGA_R es universal. Además, podemos utilizar el Teorema 2.37 junto con las propiedades de la familia de funtores $\mathcal{M}_n$ para resolver el Problema 1 en la categoría $\mathcal{C} = \mathrm{CDGA}_R$:

Teorema 4.17. *Sean G_1, G_2 grupos y sea $H \leq G_1 \times G_2$. Sea $n \geq 1$ un entero. Existen álgebras $M_1, M_2 \in \mathrm{Ob}(\mathrm{CDGA}_R^n)$ y un morfismo entre ellas $\varphi \in \mathrm{Hom}_{\mathrm{CDGA}_R}(M_1, M_2)$ de modo que $\mathrm{Aut}_{\mathrm{CDGA}_R}(M_1) \cong G_1$, $\mathrm{Aut}_{\mathrm{CDGA}_R}(M_2) \cong G_2$ y $\mathrm{Aut}_{\mathrm{CDGA}_R}(\varphi) \cong H$.*

Para traducir este resultado a $\mathcal{H}oTop$, observamos en el Corolario 2.38 que si G_1 y G_2 son grupos finitos, los grafos $\mathcal{G}_1$ y $\mathcal{G}_2$ que proporcionan una solución al Problema 1 también pueden ser construidos finitos. En consecuencia, si $R = \mathbb{Q}$, las CDGA $\mathcal{M}_n(\mathcal{G}_1)$ y $\mathcal{M}_n(\mathcal{G}_2)$ son $\mathbb{Q}$-álgebras simplemente conexas de tipo finito y por tanto modelos de espacios racionales. Así, si por abuso de notación denotamos $\mathcal{E}(f) = \mathrm{Aut}_{\mathcal{H}oTop}(f)$, somos capaces de demostrar lo siguiente:

Teorema 4.19. *Sean G_1, G_2 grupos finitos y sea $H \leq G_1 \times G_2$. Para todo $n \geq 1$, existen espacios n-conexos X_1, X_2 y una aplicación continua entre ellos $f\colon X_1 \to X_2$ de forma que $\mathcal{E}(X_1) \cong G_1$, $\mathcal{E}(X_2) \cong G_2$ y $\mathcal{E}(f) \cong H$.*

Con el Problema 2 podemos realizar un procedimiento análogo para obtener resultados que, si bien son casos particulares de los probados en [29], se incluyen al ser consecuencias sencillas de las propiedades de nuestros funtores. Así, del Teorema 2.41 y de las propiedades de la familia de funtores $\mathcal{M}_n$ deducimos lo siguiente:

Teorema 4.20. *Sea G un grupo, $n \geq 1$ un entero, R un dominio de integridad tal que $\operatorname{char}(R) = 0$ o $\operatorname{char}(R) > 3$ y $\rho\colon G \to \operatorname{Sym}(V)$ una representación por permutaciones de G en un conjunto V. Existe un objeto $A \in \operatorname{Ob}(\mathrm{CDGA}_R^n)$ tal que*

(1) $V \subset A$, y V permanece invariante a través de los automorfismos de A;

(2) $\operatorname{Aut}_{\mathrm{CDGA}_R}(A) \cong G$ (y si $R = \mathbb{Q}$, $\mathcal{E}(A) \cong G$);

(3) la aplicación de restricción $G \cong \operatorname{Aut}_{\mathrm{CDGA}_R}(A) \to \operatorname{Sym}(V)$ es ρ.

Y de nuevo, podemos traducir este resultado a $\mathcal{H}oTop$ restringiéndonos a los casos en los que el grafo que resuelve el problema es finito. Así, como consecuencia del Corolario 2.42 obtenemos el siguiente resultado:

Teorema 4.22. *Sea G un grupo finito, V un conjunto finito, $n \geq 1$ un entero y $\rho\colon G \to \operatorname{Sym}(V)$ una representación por permutaciones. Existe un espacio n-conexo X tal que*

(1) V se puede identificar como subconjunto $V \subset H^{180n^2-142n+28}(X)$ que permanece invariante a través de las aplicaciones inducidas en cohomología por las auto-equivalencias de homotopía de X;

(2) $\mathcal{E}(X) \cong G$;

(3) la aplicación $G \cong \mathcal{E}(X) \to \operatorname{Sym}(V)$ que lleva $[f] \in \mathcal{E}(X)$ a $H^{180n^2-142n+28}(f)|_V \in \operatorname{Sym}(V)$ es ρ.

Además de para resolver los Problemas 1 y 2, podemos utilizar la familia de funtores $\mathcal{M}_n$ para deducir más resultados. Por ejemplo, como consecuencia de la biyección entre morfismos inducida por $\mathcal{M}_n$, podemos interpretar $\mathcal{D}igraphs_+$ como una subcategoría de CDGA_R. En cierto sentido, estamos *representando* la categoría de $\mathcal{D}igraphs_+$ dentro de CDGA_R.

En general, se dice que un funtor $F\colon \mathcal{C} \to \mathcal{D}$ induce una representación de $\mathcal{C}$ en $\mathcal{D}$ si F es plenamente fiel, es decir, se puede ver $\mathcal{C}$ como una subcategoría plena de $\mathcal{D}$. Recordemos que una categoría $\mathcal{C}$ se dice *concreta* si existe un funtor fiel $F\colon \mathcal{C} \to \mathrm{Set}$. Intuitivamente, esto significa que podemos interpretar los objetos de $\mathcal{C}$ como conjuntos, y los morfismos de $\mathcal{C}$ como aplicaciones de conjuntos. En [68, Chapter 4, 1.11] los autores prueban que toda categoría pequeña y concreta es representable en una subcategoría plena de $\mathcal{D}igraphs_+$ (véase el Teorema 1.20). Si denotamos a la categoría de R-CDGAs n-conexas por CDGA_R^n, obtenemos el siguiente resultado:

Teorema 5.1. *Sea $\mathcal{C}$ una categoría pequeña concreta. Para todo $n \geq 1$, existe un funtor $G_n\colon \mathcal{C} \to \mathrm{CDGA}_R^n$ tal que $\operatorname{Hom}_{\mathrm{CDGA}_R}\big(G_n(A), G_n(B)\big) \setminus \{0\} = \operatorname{Hom}_{\mathcal{C}}(A, B)$, para todo $A, B \in \operatorname{Ob}(\mathcal{C})$. Además, si $R = \mathbb{Q}$, $[G_n(A), G_n(B)] \setminus \{[0]\} = \operatorname{Hom}_{\mathcal{C}}(A, B)$, para todo $A, B \in \operatorname{Ob}(\mathcal{C})$.*

Si tomamos $\mathbb{Q}$ como anillo base, los funtores $\mathcal{M}_n$ tienen como rango a la categoría $\mathrm{CDGA}_{\mathbb{Q}}^n$. Si además $\mathcal{G}$ es un grafo finito, $\mathcal{M}_n(\mathcal{G})$ es una $\mathbb{Q}$-CDGA n-conexa, y por tanto es modelo de Sullivan del tipo de homotopía racional de un espacio n-conexo. En consecuencia, se puede trasladar el resultado anterior a $\mathcal{H}oTop$, siempre y cuando impongamos restricciones adicionales sobre las categorías que queremos representar. En [49, Theorem 4.24, Proposition 4.25] los autores prueban que si $\mathcal{C}$ es una categoría concreta con objetos numerables y de forma que el conjunto de morfismos entre dos objetos cualesquiera es finito, $\mathcal{C}$

es representable en una subcategoría de $\mathcal{D}igraphs_+$ donde todos los grafos son finitos. Así, si denotamos a la categoría de homotopía de espacios topológicos n-conexos por $\mathcal{H}oTop^n$, obtenemos el siguiente resultado:

Teorema 5.2. *Sea $\mathcal{C}$ una categoría concreta tal que* $\mathrm{Ob}(\mathcal{C})$ *es numerable y* $\mathrm{Hom}_{\mathcal{C}}(A, B)$ *es finito para cualesquiera dos objetos $A, B \in \mathrm{Ob}(\mathcal{C})$. Para todo $n \geq 1$, existe un funtor $F_n \colon \mathcal{C} \to \mathcal{H}oTop^n$ tal que $[F_n(A), F_n(B)] \setminus \{[0]\} = \mathrm{Hom}_{\mathcal{C}}(A, B)$, para todo $A, B \in \mathrm{Ob}(\mathcal{C})$.*

Nuestros funtores casi plenamente fieles también nos permiten estudiar el problema de realizabilidad de monoides como monoides de endomorfismos de objetos en una categoría. Recuérdese que todo monoide M puede ser interpretado como una categoría con un único objeto cuyo monoide de endomorfismos es M. Una sencilla comprobación muestra que esta categoría es concreta, y es finita si M es finito. Además, añadir un endomorfismo cero al único objeto de esta categoría es equivalente a añadir un elemento 0 al monoide M. Así, si denotamos al monoide resultante de añadir a M un elemento 0 como M^0, el siguiente resultado se deduce inmediatamente de los Teoremas 5.1 y 5.2:

Corolario 5.3. *Sea M un monoide. Para todo $n \geq 1$, existe una R-*CDGA *n-conexa A_n tal que* $\mathrm{Hom}_{\mathrm{CDGA}_R}(A_n, A_n) \cong M^0$. *Si además M es finito, existe un espacio n-conexo X_n tal que $[X_n, X_n] \cong M^0$.*

Nótese que si M es un monoide para el que existe otro monoide N verificando que $N^0 \cong M$, es decir, si M es un monoide con cero y sin divisores de cero propios, entonces M es realizable.

Continuando con las aplicaciones de nuestras álgebras, consideramos ahora el problema de isomorfía de grupos utilizando acciones sobre CDGAs. En [28, Theorem 1.1], los autores demuestran que se pueden utilizar las acciones fieles sobre CDGAs para distinguir las clases de isomorfía de grupos de una familia estrictamente más pequeña que la de los grupos co-Hopfianos. Y resulta que podemos utilizar nuestros funtores para extender este resultado a toda la clase de grupos co-Hopfianos, demostrando lo siguiente:

Teorema 5.4. *Sea $n \geq 1$ un entero y sean G y H grupos co-Hopfianos. Son equivalentes:*

(1) G y H son isomorfos.

*(2) Para toda R-*CDGA *n-conexa (A, d), G actúa fielmente en (A, d) si y solo si H actúa fielmente en (A, d).*

Cabe destacar que el resultado es también cierto si consideramos acciones de monoides y nos restringimos a monoides co-Hopfianos sin cero, como vemos en la Proposición 5.7. Esta clase de monoides incluye a todos los monoides finitos sin cero.

Nuestros modelos encuentran también aplicaciones a la geometría diferencial. En particular, permiten proporcionar ejemplos nuevos de las denominadas *variedades inflexibles* y *variedades fuertemente quirales*. Una variedad cerrada, orientada y conexa M se dice inflexible si el conjunto de los posibles grados de sus auto-aplicaciones continuas es finito, es decir, si $|\{\deg(f) \mid f \colon M \to M\}| < \infty$. Dado que el grado es multiplicativo, esto implica que $\{\deg(f) \mid f \colon M \to M\} \subset \{-1, 0, 1\}$. Por otro lado, M se dice fuertemente quiral si no admite auto-aplicaciones de grado -1.

La importancia de las variedades inflexibles radica en el papel que estas juegan en el marco de las seminormas funtoriales en homología singular introducidas por Gromov, [47, 48], y en los teoremas en cuanto a grados de aplicaciones que se derivan de esta teoría (véase, por ejemplo, [31, Remark 2.6]): sea M una variedad cerrada, orientada y conexa con clase fundamental c_M. Si existe una seminorma funtorial en homología singular $|\cdot|$ tal que $|c_M| \neq 0$, entonces M es inflexible. En particular, las variedades hiperbólicas cerradas, orientadas y conexas son inflexibles; su volumen simplicial, que no es más que la ℓ^1-seminorma aplicada a la clase fundamental, es no trivial, [47, Section 0.3]. No obstante, la ℓ^1-seminorma es trivial en variedades simplemente conexas, [47, Section 3.1], lo que llevó a Gromov a preguntarse si toda seminorma funtorial en homología singular se anula en espacios simplemente conexos, [48, Remark (b) en 5.35]. A esta cuestión se le dio una respuesta negativa en [31] construyendo seminormas funtoriales asociadas a variedades inflexibles simplemente conexas. Así, este tipo de variedades son objetos extraordinarios de los que se conocen pocos ejemplos y que además presentan niveles de conectividad acotados, [2, 6, 27, 31].

Ahora bien, si la cohomología de una CDGA (A, d) verifica dualidad de Poincaré, puede definirse el grado de un endomorfismo de manera análoga a como se define el grado de una aplicación entre variedades cerradas, orientadas y conexas. Entonces, (A, d) será una CDGA inflexible si el conjunto de los grados de todos sus posibles endomorfismos es finito. Y resulta que una variedad cerrada, orientada y conexa es inflexible si y solo si su modelo racional lo es. Nuestras álgebras tienen muy pocos endomorfismos, así que son candidatas a ser inflexibles. Por tanto, si podemos comprobar que son inflexibles y que son modelos racionales de variedades, las variedades que modelan serán inflexibles. Además, como el nivel de conectividad de nuestros modelos aumenta con el parámetro asociado, esto permitiría obtener ejemplos de variedades inflexibles cuyo nivel de conectividad es arbitrariamente alto.

Resulta que tanto las álgebras rígidas $\mathcal{M}_k$, Definición 4.1, como las asociadas a digrafos finitos $\mathcal{M}_n(\mathcal{G})$, Definición 4.6, tienen cohomologías verificando dualidad de Poincaré y son inflexibles, tal y como se demuestra en el Lema 5.9. Entonces, utilizando la teoría de obstrucciones a la existencia de variedades con el tipo de homotopía racional de una CDGA dada desarrollada por Sullivan, [75], y Barge, [9], somos capaces de demostrar el siguiente resultado:

Teorema 5.12. *Dado G un grupo finito y $n \geq 1$ un entero, existe una variedad inflexible $(30n - 13)$-conexa cuya racionalización tiene a G como grupo de autoequivalencias de homotopía.*

Y en particular, obtenemos inmediatamente el siguiente corolario:

Corolario 5.13. *Existe una cantidad infinita de variedades inflexibles no homótopas entre sí y con un nivel de conexidad tan alto como se desee.*

Pasamos ahora a las variedades fuertemente quirales (véase [2], [27] y [69]). Ya hemos comprobado que nuestras CDGAs son inflexibles. Sin embargo, comprobar que no admiten endomorfismos de grado -1 conlleva un mayor nivel de dificultad. Así pues, en lugar de demostrar esto directamente, utilizamos una construcción de [27] para construir álgebras fuertemente quirales a partir de nuestras álgebras inflexibles y obtener el siguiente resultado:

Proposición 5.14. *Dado un grupo finito G y un entero $n \geq 1$, existe una variedad fuertemente quiral $(30n - 13)$-conexa cuya racionalización tiene a G como grupo de autoequivalencias de homotopía.*

Cabe destacar que las variedades obtenidas en la Proposición 5.14 pueden ser utilizadas para construir nuevas variedades inflexibles y fuertemente quirales mediante el producto de variedades, utilizando técnicas de [64]. En particular, en [64, Example 3.7] se demuestra que si M es una variedad cerrada, orientada, inflexible, no simplemente conexa y que no admite aplicaciones de grado no trivial desde productos directos y N es otra variedad inflexible simplemente conexa tal que $\dim N < \dim M$, entonces $M \times N$ es inflexible. Somos capaces de demostrar un resultado «inverso».

Corolario 5.16. *Sea M una variedad cerrada, orientada, conexa e inflexible (respectivamente fuertemente quiral) que no admite aplicaciones de grado no trivial desde productos directos. Entonces existe una variedad simplemente conexa y fuertemente quiral N tal que* $\dim N > \dim M$ *y $M \times N$ es inflexible (respectivamente fuertemente quiral).*

Para finalizar con las aplicaciones a la geometría diferencial, cabe mencionar que todas las variedades construidas en el Teorema 5.13, la Proposición 5.14 y el Corolario 5.16 pueden ser elegidas de forma que sean diferenciables, como consecuencia de un resultado de Lambrechts y Stanley, [23, Proposition A.1].

Continuamos con una última aplicación de nuestros funtores $\mathcal{M}_n$. Recuérdese que dado un espacio X, la categoría de Lusternik-Schnirelmann o categoría LS de X, denotada $\mathrm{cat}(X)$, es el menor entero n tal que X admite un recubrimiento por $n+1$ abiertos contráctiles. Se trata de un invariante homotópico de X que ha sido ampliamente estudiado y cuya importancia ya fue constatada cuando se introdujo en [60]. Efectivamente, Lusternik y Schinerlmann muestran que si X es una variedad diferenciable, $\mathrm{cat}(X)+1$ es una cota inferior del número de puntos críticos de una función diferenciable en X.

Para la categoría LS se conocen multitud de cotas en relación con otros invariantes homotópicos, y nuestra contribución aquí es mostrar que nuestras álgebras proporcionan un ejemplo de que la diferencia entre los dos términos involucrados en una cierta cota es arbitrariamente grande. Así, dado un espacio X, denótese por $\mathcal{E}_{\#}^{m}(X)$ al subgrupo de $\mathcal{E}(X)$ formado por aquellas autoequivalencias de homotopía que inducen la identidad en los grupos de homotopía $\pi_i(X)$, para $i \leq m$. Análogamente, sea $\mathcal{E}_{\#}(X)$ el grupo de aquellas autoequivalencias de homotopía de X que inducen la identidad en $\pi_i(X)$, para todo i.

Dror y Zabrodsky demostraron que tanto $\mathcal{E}_{\#}(X)$ como $\mathcal{E}_{\#}^{m}(X)$ para $m \geq \dim(X)$ son grupos nilpotentes, [34]. Sus clases de nilpotencia, denotadas $\mathrm{nil}\left(\mathcal{E}_{\#}(X)\right)$ y $\mathrm{nil}\left(\mathcal{E}_{\#}^{m}(X)\right)$ respectivamente, son entonces invariantes homotópicos de X que pueden ser comparados con los invariantes clásicos. Así, Félix y Murillo prueban en [41] que $\mathrm{nil}\left(\mathcal{E}_{\#}^{m}(X_0)\right) \leq \mathrm{cat}(X)-1$, si $m \geq \dim(X)$, y también que $\mathrm{nil}\left(\mathcal{E}_{\#}(X_0)\right) \leq \mathrm{cat}(X)-1$. Recuérdese que X_0 denota al racionalizado de X, es decir, a un espacio racional que es racionalmente homótopo a X. Resulta natural preguntarse hasta qué punto pueden diferir los términos de estas desigualdades. Así pues, probamos el siguiente resultado:

Teorema 5.17. *Dado un entero $k > 1$ cualquiera, existe un espacio finito-dimensional X tal que* $\mathrm{cat}(X) - \mathrm{nil}\left(\mathcal{E}_{\sharp}^{m}(X_0)\right) \geq k$, *para $m \geq \dim(X)$. En particular,* $\mathrm{cat}(X) - \mathrm{nil}(\mathcal{E}_{\sharp}(X_0)) \geq k$.

Las soluciones a los problemas de realizabilidad que hemos obtenido hasta el momento utilizan herramientas de la Teoría de Homotopía Racional. En consecuencia, los objetos que construimos respondiendo al problema de realizabilidad clásico en $\mathcal{H}oTop$ (o problema de realizabilidad de Kahn) son espacios racionales, y por tanto no son de tipo finito sobre $\mathbb{Z}$. Nos proponemos ahora encontrar una forma alternativa de resolver este problema utilizando espacios *enteros*, es decir, espacios de tipo finito sobre $\mathbb{Z}$.

Un contexto en el que existe una clasificación de tipos de homotopía basada fundamentalmente en herramientas de teoría de grupos es el de los A_n^2-poliedros: CW-complejos $(n-1)$-conexos $(n+2)$-dimensionales. En [80], J.H.C. Whitehead clasificó los tipos de homotopía de A_2^2-poliedros (Es decir, CW-complejos 4-dimensionales simplemente conexos) utilizando una cierta sucesión exacta de grupos, y posteriormente Baues obtuvo una generalización de esta clasificación a A_n^2-poliedros para cualquier $n \geq 2$, [12, Ch. I, Section 8].

Siguiendo ideas de [17] definimos, asociado a la sucesión exacta de grupos utilizada para clasificar el tipo de homotopía de un A_n^2-poliedro X, un grupo $\mathcal{B}^{n+2}(X)$ (véase la Definición 1.69) que probamos isomorfo a $\mathcal{E}(X)/\mathcal{E}_*(X)$ en la Proposición 1.70. Aquí, $\mathcal{E}_*(X)$ es un subgrupo normal de $\mathcal{E}(X)$ cuyos elementos son aquellas autoequivalencias de homotopía de X

que inducen la identidad en grupos de homología. Por supuesto, el estudio de este cociente proporciona información sobre los grupos que pueden aparecer como $\mathcal{E}(X)$ siendo X un A_n^2-poliedro, pero también tiene sentido plantearse un problema de realizabilidad directamente sobre el cociente, y es uno de los cocientes distinguidos de $\mathcal{E}(X)$ para los que se propuso un problema de realizabilidad de grupos en [37, Problem 19]. Así pues, veamos qué resultados hemos sido capaces de obtener en ese ámbito.

Utilizando la Proposición 1.70 somos capaces de estudiar cómo la estructura celular de un A_n^2-poliedro se manifiesta en su grupo de autoequivalencias de homotopía. Por ejemplo, demostramos que si se imponen ciertas restricciones sobre los grupos de homología de X, $\mathcal{B}^{n+2}(X)$ es infinito, de lo que se deduce que $\mathcal{E}(X)$ también lo es (véanse las Proposiciones 6.6 y 6.9). También podemos demostrar que en muchas ocasiones la existencia de ciclos de orden impar en los grupos de homología de un A_n^2-poliedro X fuerzan la existencia de elementos de orden par en $\mathcal{E}(X)$ (véanse los Lemas 6.4 y 6.5).

Sin embargo, estos dos últimos resultados nos llevan a pensar que quizá este contexto no sea adecuado para resolver los problemas de realizabilidad planteados. Y en efecto, aunque comprobamos fácilmente que todo grupo de automorfismos de un grupo abeliano es realizable en este ámbito (véase el Ejemplo 6.2), también obtenemos el siguiente resultado:

Teorema 6.16. *Sea X un A_n^2-poliedro de tipo finito, $n \geq 3$. Entonces, si $\mathcal{B}^{n+2}(X)$ no es trivial, tiene elementos de orden par.*

Y como corolario inmediato, obtenemos lo siguiente:

Corolario 6.17. *Sea G un grupo finito no nilpotente de orden impar. Entonces, para todo $n \geq 3$, si X es un A_n^2-poliedro, $G \not\cong \mathcal{E}(X)$.*

Tratar el caso de los A_2^2-poliedros es más complejo. No obstante, un análisis detallado nos permite deducir que los grupos finitos de orden impar solo pueden ser realizados a través de un A_2^2-poliedro de tipo finito bajo condiciones muy restrictivas. Recuérdese que dado un grupo G, $\operatorname{rank} G$ es el menor cardinal de un conjunto de generadores de G, [61, p. 91]. Entonces:

Teorema 6.18. *Supóngase que X es un A_2^2-poliedro de tipo finito tal que $\mathcal{B}^4(X)$ es un grupo no trivial de orden impar. Entonces:*

(1) $\operatorname{rank} H_4(X) \leq 1$*;*

(2) $\pi_3(X)$ *y* $H_3(X)$ *son* 2*-grupos, y* $H_2(X)$ *es un* 2*-grupo abeliano elemental;*

(3) $\operatorname{rank} H_3(X) \leq \frac{1}{2} \operatorname{rank} H_2(X)\big(\operatorname{rank} H_2(X) + 1\big) - \operatorname{rank} H_4(X) \leq \operatorname{rank} \pi_3(X)$*;*

(4) la representación $\mathcal{B}^4(X) \leq \operatorname{Aut}\big(H_2(X)\big)$ inducida por la acción natural de $\mathcal{B}^4(X)$ en $H_2(X)$ es fiel.

Sin embargo, ninguno de nuestros intentos de proporcionar un espacio satisfaciendo las hipótesis del Teorema 6.18 ha tenido éxito, lo que nos lleva a proponer la siguiente conjetura:

Conjetura 6.19. *Sea X un A_2^2-poliedro. Si $\mathcal{B}^4(X)$ es un grupo finito no trivial, entonces tiene elementos de orden par.*

Así pues, todavía queda investigación por hacer en el futuro, pues por un lado todavía no se ha obtenido una solución negativa al problema de realizabilidad de grupos de Kahn, y por otro lado, todavía estamos interesados en encontrar un marco en el que se pueda dar una solución afirmativa a este problema en términos de espacios *enteros*.

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