Blenders and non-hyperbolic dynamics arising in generic unfoldings of nilpotent singularities

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Mayo, 2012

A Thesis submitted for the Degree of Philosophiae Doctor at Universidad de Oviedo

Programa de Doctorado en Matemáticas y Estadística

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Universidad de Oviedo

PhD Tesis en Matemáticas y Estadística por la Universidad de Oviedo

Título: *Mezcladores y dinámica no hiperbólica presentes en desdoblamientos genéricos de singularidades nilpotentes*

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Introducción

Los resultados que contiene esta memoria son una contribución al estudio de la dinámica no uniformemente hiperbólica, la cual constituye el escenario donde se plantean muchas cuestiones de actualidad sobre complejidad dinámica. En contraposición, la dinámica uniformemente hiperbólica comenzó a ser bien entendida durante los años sesenta a partir de los primeros trabajos de Anosov [Ano67] y Smale [Sma67]. Tres décadas antes, en un elaborado trabajo [Bir35], Birkhoff probó que en general, cerca de los puntos homoclínicos transversales, introducidos por Poincaré [Poi90], existía un intrincado conjunto de órbitas periódicas, la mayoría con un periodo muy alto. A fin de iluminar este resultado de Birkhoff y otros resultados posteriores sobre la existencia de infinidad de órbitas periódicas en la ecuación de Van der Pol [CL45, Lev49], Smale colocó en un entorno de un punto homoclínico transversal su ingenio geométrico: la aplicación herradura. Esta aplicación, así como los ejemplos propuestos por Anosov sobre el toro, son difeomorfismos cuyo conjunto no errante Ω es hiperbólico y coincide con la clausura de los puntos periódicos. Difeomorfismos con estas propiedades fueron denominados *Axioma A* o *difeomorfismos uniformemente hiperbólicos*, y se planteó el estudio de estos difeomorfismos como un subconjunto del espacio $\mathrm{Diff}^r(M)$ de los difeomorfismos de clase C^r sobre una variedad compacta M.

Un resultado fundamental para el estudio de los difeomorfismos Axioma A en $\mathrm{Diff}^r(M)$ fue el teorema de descomposición espectral dado por Smale [Sma67], según el cual el conjunto no errante Ω se descompone en una unión disjunta y finita de subconjuntos Λ_i llamados *conjuntos básicos*. Cada Λ_i es un conjunto compacto, aislado, invariante y transitivo. Dos puntos periódicos en el mismo conjunto básico Λ_i tienen variedades estables con la misma dimensión (índice de estabilidad) y por consiguiente variedades inestables de la misma dimensión (índice de Morse). Un elocuente y relevante ejemplo de conjunto básico es el conjunto invariante Ω de una aplicación herradura. La dinámica de la restricción de la aplicación herradura sobre este conjunto se sigue de su conjugación con el shift de Bernoulli.

Las aplicaciones herradura asociadas a un punto homoclínico transversal constituyen un hito importante en el estudio de los sistemas dinámicos. Su conjunto invariante Ω aporta un ejemplo de dinámica casi-aleatoria, consecuencia del caracter expansivo de sus órbitas que implica una alta sensibilidad de la dinámica a las condiciones iniciales. Sin embargo, Ω no es un atractor por no tener un recinto de atracción con interior no vacío (o medida positiva) y, por consiguiente, su dinámica interna no es susceptible de ser observable como la dinámica asintótica de un difeomorfismo. Esta deficiencia fue salvada por Smale quien, por analogía con la herradura, construyó el solenoide como un primer ejemplo de *atractor extraño* (atractor con un órbita densa expansiva) que era hiperbólico. Este ejemplo de atractor no periódico con una dinámica interna impredecible, por su sensibilidad exponencial a las variaciones de las condiciones iniciales, inspiró el celebre artículo de Ruelle y Takens [RT71] sobre la naturaleza de la turbulencia.

Una vez visto que los difeomorfismos uniformemente hiperbólicos aportaban nuevas interpretaciones dinámicas, se plantearon en el contexto de la teoría de la bifurcación dos importantes cuestiones: la relación de la hiperbolicidad uniforme con la *estabilidad estructural* y la densidad de los difeomorfismos uniformemente hiperbólicos en el espacio $\mathrm{Diff}^r(M)$, dotado de la C^r-topología. Después de algunos resultados parciales de Robbin [Rob71], de Melo [Mel73] y Robinson [Rob74, Rob76], Mañé [Mañ88] probó que, tal y como habían conjeturado Palis y Smale [PS70], un difeomorfismo $f \in \mathrm{Diff}^1(M)$ es estructuralmente estable si y sólo si es uniformemente hiperbólico y verifica la condición fuerte de transversalidad: todas las variedades invariantes de los puntos del conjunto no errante Ω tienen que intersecarse transversalmente. En relación con la densidad, aunque la hiperbolicidad uniforme se creyó en principio abarcando un subconjunto residual, o al menos denso, de $\mathrm{Diff}^r(M)$, pronto se constató que esto no era cierto. Hay dos configuraciones importantes que fuerzan la persistencia de la no hiperbolicidad uniforme: los *ciclos heterodimensionales* y ciertas *tangencias homoclínicas* (ver ambos conceptos en Definición 1.1). Los primeros fueron usados por Abraham y Smale [AS70] y Simon [Sim72] para construir ejemplos de abiertos de difeomorfismos en $\mathrm{Diff}^1(M)$, con $\dim M \geq 3$, que no son uniformemente hiperbólicos. Las tangencias homoclínicas, cuando se producen entre las variedades invariantes de un punto periódico que pertenece a un conjunto básico no trivial, son el fundamento del bien conocido fenómeno de Newhouse, [New70, New74, New79] para C^2-difeomorfismos sobre superficies. Para pequeñas perturbaciones del difeomorfismo ambas configuraciones fuerzan la persistencia de las tangencias homoclínicas, que implican la presencia de puntos no errantes con diferentes índices de estabilidad y, en definitiva, la persistencia de la no hiperbolicidad uniforme. Por consiguiente, el conjunto $\mathrm{Diff}^r(M)$ es la unión disjunta de dos conjuntos, los uniformemente hiperbólicos y su complementario, que contienen a su vez conjuntos abiertos.

El conjunto de los uniformemente hiperbólicos contiene al abierto de los estructuralmente estables y su dinámica es bastante bien entendida. Por contraposición, los difeomorfismos no uniformemente hiperbólicos, que por ser persistentes son también abundantes, no son estructuralmente estables. Sus dinámicas tienen que comportar infinidad de transiciones y, por consiguiente, pertenecerán a su ámbito las dinámicas que se manifiestan más complicadas. Este es el caso de algunos de los atractores más populares. A partir de su estudio numérico, el atractor de Lorenz [Lor63] parece ser extraño, persistente pero no estructuralmente estable, mientras que la persistencia parece fallar en el caso del atractor de Hénon [Hén76]. Puesto que los atractores hiperbólicos son persistentes y estructuralmente estables, tanto el atractor de Lorenz como el de Hénon no pueden ser atractores hiperbólicos. Pero, ¿existen realmente atractores extraños no hiperbólicos? La primera prueba analítica de la existencia de tales atractores fue dada por Benedicks y Carleson [BC91], quienes probaron que en la familia de Hénon $H_{a,b}(x, y) = (1 - ax^2 + y, bx)$ existían atractores extraños para un conjunto de valores de los parámetros suficientemente próximos a $a = 2$ y $b = 0$ y con medida de Lebesgue positiva (persistencia en el sentido de la medida). Las ideas y las intrincadas técnicas en [BC91] fueron utilizadas por Mora y Viana [MV93] para probar que, tal y como había conjeturado Palis, familias genéricas uniparamétricas de difeomorfismos sobre una superficie desplegando una tangencia homoclínica tienen atractores extraños con probabilidad positiva en el espacio de parámetros. La existencia de tales atractores en familias de campos vectoriales tridimensionales fue probada en [PR97] a partir de la sección transversal a una órbita homoclínica de Shil'nikov [Shi65]. La prueba de la existencia de atractores extraños no hiperbólicos parte en [BC91] de considerar que la familia de Hénon es un despliegue de la familia límite $h_a(x) = 1 - ax^2$ que se obtiene al tomar $b = 0$. Esta familia cuadrática ha sido previamente bien estudiada en [BC85] y su dinámica expansiva se traslada a la variedad inestable del punto de silla de $H_{a,b}(x, y) = (1 - ax^2 + y, bx)$ cuando b es suficientemente pequeño. En [MV93] esta estrategia se aplica después de hacer una adecuada renormalización de la aplicación retorno a un entorno del punto homoclínico. La familia resultante continua siendo un buen despliegue de la familia cuadrática (una familia tipo Hénon) y las ideas y técnicas en [BC91] se pueden adaptar a este caso. En [PR97] se prueba que la familia que se obtiene después de una adecuada renormalización es un buen despliegue de una familia límite, que en este caso es la familia unimodal $f_a(x) = \lambda^{-1} \log a + x + \lambda^{-1} \log \cos x$, y los argumentos en [BC91] continúan siendo válidos.

El objetivo principal en el estudio de los sistemas dinámicos es describir el comportamiento asintótico de las trayectorias de la mayoría de los sistemas. En el fragor del estudio de los sistemas hiperbólicos, Smale conjeturó que el conjunto límite de las trayectorias de un sistema dinámico genérico debería presentar una dinámica interna hiperbólica: incremento y disminución exponencial de las distancias en dimensiones complementarias. Sin embargo, los atractores no hiperbólicos mencionados anteriormente aportaron contraejemplos y plantearon la necesidad de nuevas propuestas. Desde entonces y hasta el presente, la investigación de la dinámica no uniformemente hiperbólica fue principalmente programada por Palis [Pal00a, Pal08], quien propuso un programa de trabajo compuesto de una serie de conjeturas interrelacionadas y encaminadas a describir el comportamiento asintótico de familias genéricas de sistemas dinámicos dependiendo de un número finito de parámetros. Concretamente, conjeturó que, genéricamente, sólo existe un número finito de atractores transitivos donde se pueden acumular casi todas las trayectorias; además, estos atractores deberán ser estocásticamente estables y soportar una medida física. A diferencia del enfoque topológico dado en los años sesenta, el planteamiento ahora es probabilístico y expresado en términos de la medida de Lebesgue, tanto en el espacio de parámetros como en el espacio de fases. A partir del teorema de descomposición espectral y de la teoría de Sinai-Ruelle-Bowen [Sin72, BR75, Rue76], se prueba que para los difeomorfismos uniformemente hiperbólicos de clase C^2 que no tienen ciclos existe a lo sumo un número finito de atractores, que son a su vez estocásticamente estables y soportan una medida física. Entonces, un paso más allá será buscar alguna forma robusta de hiperbolicidad (parcial o descomposición dominada) que esté presente en ausencia de ciclos y donde se pueda probar la conjetura anterior. Esto plantea una dicotomía entre algún conjunto de difeomorfismos hiperbólicos y aquellos que poseen algún tipo de ciclo. Concretamente, Palis conjeturó que cualquier sistema dinámico puede ser C^r aproximado por uno hiperbólico que no tenga ciclos o por uno que presenta alguna tangencia homoclínica o algún ciclo heterodimensional. Una primera respuesta a esta última conjetura fue dada en la topología C^1 por Pujals y Sambarino en [PS00] para difeomorfismos en superficies. Para dimensión superior, Crovisier y Pujals [CP10] probaron que todo difeomorfismo $f \in \mathrm{Diff}^1(M)$ puede ser C^1 aproximado por uno que tiene bien una tangencia homoclina o un ciclo heterodimensional o bien es *esencialemente hiperbólico*, es decir, tiene un número finito de atractores hiperbólicos transitivos tal que la unión de sus recintos de atracción es un abierto y denso en el espacio de fases. En definitiva, las tangencías homoclínica y ciclos heterodimensionales constituyen una *completa obstrución a la hiperbolicidad*.

En relación con las tangencias homoclínicas, y ya en el ámbito de los difeomorfismos de clase C^2 sobre superficies, se han dado resultados notables. Como se ha mencionado más arriba, en familias genéricas desplegando una tangencia homoclínica entre las variedades de un punto periódico hiperbólico aislado aparecen atractores extraños no hiperbólicos y persistentes en el sentido de la medida. Cuando el punto periódico pertenece a un conjunto básico no trivial, la persistencia de las tangencias homoclínicas para un conjunto abierto $\mathcal{U}$ de difeomorfismos se detectada originalmente en [New70]. En un conjunto residual en $\mathcal{U}$, de medida nula, aparecen simultáneamente infinitos atractores periódicos [New74], e incluso infinitos atractores extraños de tipo Hénón [Col98]. Estos resultados se pueden generalizar a situaciones de mayor dimensión [PV94, Via93, Lea08]. El ingrediente geométrico que subyace en la persistencia de las tangencias homoclínicas es la aplicación herradura. Concretamente la espesura de las foliaciones estable e inestable de un conjunto básico Λ, que se prolongan en un entorno del punto homoclínico definiendo, respectivamente, dos conjuntos de Cantor K_s y K_u sobre un determinado segmento. La prevalencia de la hiperbolicidad o de la no hiperbolicidad depende de si la dimensión de Hausdorff $\mathrm{HD}(\Lambda) = \mathrm{HD}(K_s) + \mathrm{HD}(K_u)$ del conjunto básico Λ es menor o mayor que uno [PY94, MPV01].

En relación con los ciclos heterodimensionales, un resultado temprano de Diaz [Día95] implica la existencia de un conjunto abierto no vacío de familias C^∞ de difeomorfismos $(f_t)_{t\in[-1,1]}$ desplegando genéricamente un ciclo heterodimensional de f_0 y tal que para todo $t > 0$ suficientemente pequeño el correspondiente difeomorfismo f_t no es uniformemente hiperbólico: las clases homoclíncias de dos puntos hiperbólicos de diferentes índices de estabilidad coinciden. La prueba de este resultado se puede ilustrar con la elección de un difeomorfismo en $\mathbb{R}^3$ que tenga un ciclo heterodimensional entre dos puntos fijos P y Q. Con hipótesis adicionales de hiperbolicidad parcial, linealización y estructura producto se comprende que el comportamiento de este difeomorfismo en un entorno del ciclo se sigue de la dinámica de un sistema iterado de dos funciones reales (ver una precisa definición de *sistema iterado de funciones*, en la sección §3.1). Esta reducción indica que la persistencia de la hiperbolicidad no uniforme asociada a los ciclos heterodimensionales es de naturaleza diferente a la que se sigue de las bifurcaciones homoclínicas. Si en el fenómeno de Newhouse (persistencia de tangencias homoclínicas) el ingrediente geométrico esencial era la aplicación herradura y la dimensión de Hausdorff de su conjunto básico, ¿qué elemento geométrico subyace en la persistencia de la hiperbolicidad no uniforme al perturbar un ciclo heterodimensional? La respuesta fue dada por Bonatti y Díaz al introducir en [BD96] el concepto de *mezclador* (*blender* en inglés). A modo preliminar, un mezclador puede entenderse como un conjunto hiperbólico Γ suficientemente grueso tal que la clausura de una variedad invariante de dimensión u de un punto silla en Γ contenga una variedad invariante de dimensión $u+1$. Una primera definición precisa de mezclador enfatizando sus aspectos geométricos puede ser encontrada en [BDV05]:

Definición (Mezcladores)**.** *Sea f un C^1 difeomorfismo de una variedad compacta M y $\Gamma \subset M$ un conjunto hiperbólico y transitivo de f con una descomposición dominada de la forma $E^{ss} \oplus E^{cs} \oplus E^u$, donde su fibrado estable $E^s = E^{ss} \oplus E^{cs}$ tiene dimension igual a $s \geq 2$ y E^{cs} es uno dimensional. El conjunto Γ es un cs-*mezclador *si tiene una* region de superposición *C^1-robusta $\mathcal{B}$:*

Existen un C^1-entorno $\mathcal{V}$ of f y un conjunto abierto $\mathcal{B}$ de discos encajados en M de dimension $s-1$ tales que para todo difeomorfismos $g \in \mathcal{V}$, todo disco $D^s \in \mathcal{B}$ interseca la variedad local inestable $W^u_{loc}(\Gamma_g)$ de la continuación Γ_g de Γ para g.

*Un cu-*mezclador *para f es definido como un cs-mezclador para f^{-1}.*

Los mezcladores son el mecanismo subyacente que conduce a la generación de ciclos heterodimensionales robustos [BD08] y a tangencias homoclínicas robustas en la topología C^1 para variedades de dimension mayor o igual que tres [BD11]. Los mezcladores fueron también usados en otras aplicaciones tales como la construcción de difeomorfismos no hiperbólicos robustamente transitivos [BD96], la discontinuidad de la dimensión de conjuntos hiperbólicos [BDV95] y la obtención de resultados sobre ergodicidad estable [RHTU07].

Un mezclador Γ para un C^1 difeomorfismo f no es más que un conjunto hiperbólico, pero su existencia presagia la presencia persistente de no hiperbolicidad uniforme. A modo de ejemplo, esbozaremos como estos conjuntos permiten mezclar puntos de silla de diferentes índices. Supongamos que estamos en dimensión tres, Γ es un cs-mezclador con índice de estabilidad $s = 2$ y $P \in \Gamma$, $Q \notin \Gamma$ son dos puntos periódicos de f, el primero con variedad inestable densa en Γ y el segundo con índice de estabilidad $s - 1 = 1$. Asumimos que la variedad estable $W^s(Q)$ de Q contiene un disco D^s en la región de superposición $\mathcal{B}$ del mezclador Γ. A partir del Lema de Inclinación, la sucesión de preiterados de cualquier disco L de dimensión $s - 1 = 1$ transversal a $W^u(Q)$ convergen a $W^s(Q)$. Por lo tanto, para $n \geq 0$ suficientemente grande $f^{-n}(L)$ contiene un disco en $\mathcal{B}$. En vista de que Γ es un cs-mezclador, $W^u_{loc}(\Gamma)$ interseca a $f^{-n}(L)$ y así por la densidad la variedad inestable de P se sigue que

$$W^u(Q) \subset \overline{W^u(P)}.$$

Esto es, la clausura de la variedad inestable de P (de dimensión uno) contiene a la variedad inestable (bidimensional) de Q. Tenemos por lo tanto que en cierto sentido (digamos topológico) la dimensión de la variedad inestable de P es igual a la variedad inestable de Q (esto es dos). Consecuentemente aumentamos en una unidad la dimensión de la variedad inestable de P, así para efectos prácticos, tendremos que el punto P tiene variedades invariantes de dimensión dos.

La construcción anterior nos permite ver que todos los puntos en la intersección transversal $\gamma = W^s(P) \pitchfork W^u(Q)$ (genéricamente unión de curvas) pertenecen tanto a la clase homoclínica de P como a la de Q. Por lo tanto, γ esta contenida en el conjunto no errante de f. Obviamente los puntos de γ no admiten un descomposición hiperbólica del fibrado tangente y se sigue que f no puede ser uniformente hiperbólico. La persistencia del mezclador hace estos argumentos robustos bajo C^1 perturbaciones y de ahí que se obtiene un abierto de difeomorfismos no uniformente hiperbólicos.

Nótese que el concepto de mezclador está formulado en el contexto más general de los C^1 difeomorfismos. Ya en este contexto se prueba en [DR02] que si f es un C^1 difeomorfismo con un ciclo heterodimensional asociado a dos puntos de silla P y Q con índices s y $s+1$ (es decir, de coíndice uno) y C^1 lejos de tangencias homoclínicas, entonces f pertenece a la clausura de un abierto $\mathcal{U}$ de difeomorfismos no uniformemente hiperbólicos. Posteriormente, en [BD08] se probó que cualquier ciclo heterodimensional de coíndice uno puede ser C^1 aproximado por difeomorfismos teniendo un ciclo heterodimensional C^1-robusto. Uno de los pasos para probar este resultado fue mostrar que aparecen de forma natural *mezcladores tipo herraduras* en el despliegue de ciclos heterodimensionales de coíndice uno.

La representación geométrica más elocuente de un mezclador y su relación con la aplicación herradura se tiene al construir los llamados *mezcladores herradura* [BD11]. Esta construcción involucra un difeomorfismo f definido en un cubo de referencia $C = [-1,1]^{n+1}$, con $n \geq 2$, como un *producto cruzado* de la forma

$$f : C \subset \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}, \quad f(x,y) = (F(x), \phi(x,y)),$$

donde $F : \mathbb{R}^n \to \mathbb{R}^n$ tiene una herradura de Smale $\Lambda \subset [-1,1]^n$ y las aplicaciones sobre las fibras

$$\phi(x,\cdot) : [-1,1] \to [-1,1]$$

son contracciones de clase C^1. Es evidente que si todas las aplicaciones $\phi(x,\cdot)$ son una misma contracción ϕ, entonces la aplicación $f|_C$ es en esencia una aplicación herradura con un conjunto invariante maximal $\Gamma = \Lambda \times \{y_0\}$ donde $y_0 \in [-1,1]$ es el punto fijo de la contracción ϕ. Supongamos ahora que

$$H_1 \cup H_2 = F^{-1}([-1,1]^n) \cap [-1,1]^n$$

es la unión de las dos banda horizontales en la definición de una herradura y que $\phi(x,\cdot) = \phi_i$, según $x \in H_i$ con $i = 1,2$, son dos contracciones diferentes, con puntos fijos $y_1 < y_2$. Los difeomorfismos definidos de esta forma, se les conoce con el nombre de *productos cruzados localmente constantes*. Entonces, en este caso, $\Gamma \subset \Lambda \times [y_1, y_2]$ es de nuevo un conjunto hiperbólico transitivo tal que $f|_\Gamma$ es topológicamente conjugado a $F|_\Lambda$. La proyección de Γ sobre el intervalo $[y_1, y_2]$ viene dada por la dinámica del sistema iterado de funciones generado por las contracciones ϕ_1, ϕ_2. Si existe un abierto $B \subset (y_1, y_2)$ tal que

$$\overline{B} \subset \phi_1(B) \cup \phi_2(B)$$

se prueba que esta proyección contiene a B. Es fácil comprobar que este abierto B sigue estando contenido en la proyección sobre la recta real de la continuación Γ_g de Γ para todo difeomorfismo g producto cruzado localmente contante y C^1 próximo de f. Dicho contenido, es suficiente para mostrar que para todo $(x,y) \in (H_1 \cup H_2) \times B$, la variedad inestable local $W^u_{loc}(\Gamma_g)$ de Γ_g interseca a la variedad estable fuerte local de (x,y) para g. Notando que dichas variedades estables fuertes forman abierto de discos encajados en la variedad, se sigue que Γ satisface la definición de mezclador para todos los difeomorfismos productos cruzados localmente constantes C^1 próximos a f. Esta persistencia es el principal escollo a la hora de probar la existencia de un mezclador porque cualquier difeomorfismo g suficientemente C^1 próximo a f no es necesariamente un producto cruzado y mucho menos, un producto cruzado localmente constante. Esta dificultad se resolverá siguiendo los resultados de hiperbolicidad normal desarrollados en [HPS77]: bajo determinadas hipótesis sobre la descomposición hiperbólica de f se concluye que g es topológicamente conjugado a un producto cruzado. Por consiguiente, a la hora de probar la persistencia de la condición de intersección sera suficiente considerar C^1 perturbaciones de f en la categoría de los difeomorfismos productos cruzados. Por otra parte, puesto que $F|_\Lambda$ es conjugado a un shift de Bernoulli $\tau : \Sigma_2 \to \Sigma_2$ de dos símbolos, los mezcladores-herradura pueden ser estudiados desde el punto de vista simbólico, considerando productos cruzados de la forma

$$\Phi : \Sigma_2 \times \mathbb{R} \to \Sigma_2 \times \mathbb{R}, \quad \Phi(\xi, x) = (\tau(\xi), \phi_\xi(x)),$$

donde cada $\phi_\xi : \mathbb{R} \to \mathbb{R}$ es una contracción de clase C^1.

El concepto de cs-mezclador está asociado a C^1 difeomorfismos que tienen un conjunto parcialmente hiperbólico con un descomposición dominada $E^{ss} \oplus E^{cs} \oplus E^u$, y ha sido siempre definido y manejado en el caso en el que la dirección central E^{cs} es unidimensional. Esto supone un obstáculo en contextos donde se planteen variedades centrales de dimensión $c \geq 2$. Por lo tanto, una cuestión natural es manejar y construir mezcladores cuya dirección central no sea necesariamente unidimensional. Siguiendo la propuesta de Nassiri y Pujals [NP12] un camino para ello es considerar la dinámica simbólica en el contexto de los *productos cruzados simbólicos*

$$\Phi : \Sigma_k \times \mathbb{R}^c \to \Sigma_k \times \mathbb{R}^c, \quad \Phi(\xi, x) = (\tau(\xi), \phi_\xi(x)),$$

donde $\Sigma_k = \{1, \dots, k\}^{\mathbb{Z}}$, $c \geq 1$, y cada $\phi_\xi : \mathbb{R} \to \mathbb{R}$ es de nuevo una contracción de clase C^1. Un mezclador en este contexto será denominado *mezclador herradura simbólico* o simplemente *mezclador simbólico*. El primer objetivo de nuestro trabajo es dar condiciones de existencia de mezclador simbólico en este contexto más general. Estas condiciones se aplicarán para estudiar la génesis de mezcladores en perturbaciones de $\Phi(\xi, x) = (\tau(\xi), x)$ y explicar a partir de ello la presencia de *mezcladores suspendido* para campos de vectores en $\mathbb{R}^4$ arbitrariamente próximos a un campo Hamiltoniano X_H con una órbita homoclínica bifocal no degenerada, donde un difeomorfismo tridimensional se puede definir como la aplicación retorno sobre una sección transversal a la órbita homoclínica. Por una *órbita homoclínica bifocal* de un campo de vectores en $\mathbb{R}^4$ se entiende una conexión homoclínica a un punto de equilibrio foco-foco, esto es, que tiene autovalores $-\rho_1 \pm i\omega_1$, $\rho_2 \pm i\omega_2$. Puesto que en todo despliegue genérico de una singularidad nilpotente de codimensión cuatro en $\mathbb{R}^4$ aparecen órbitas homoclínicas bifocales [BIR11], desplegadas como continuación de similares conexiones homoclínica en familias límite de campos de vectores Hamiltonianos, finalmente mostramos como en estos despliegue de estas singularidades singularidades podrían aparecer mezcladores.

La exposición de esta memória de tesis se organiza en cuatro capítulos autocontenidos. A continuación presentamos algunos de los principales resultados que se recogen en cada uno de ellos:

I – Ciclos robustos y mezcladores – En el primer capítulo de la tesis se introduce con detalle alguno conceptos preliminares, conjeturas y ejemplos que ya han sido invocados a lo largo de esta introducción. El objetivo es llegar a introducir el concepto de mezclador y mezclador-herradura presente en la literatura previa de una forma autocontenida.

II – Mezcladores simbólicos – El segundo capítulo de la memoria de tesis se dedica a estudiar la existencia de mezcladores simbólicos. Los principales resultados de este capítulo son en colaboración con Yuri Ki and Artem Raibekas y recogidos en la prepublicación [BKR12].

Este capítulo se desarrollará en el ámbito de los *productos cruzados simbólicos*

$$\Phi : \Sigma_k \times M \to \Sigma_k \times M, \qquad \Phi(\xi, x) = (\tau(\xi), \phi_\xi(x))$$

donde M es una variedad de Riemann compacta de dimensión $c \geq 1$ y $\phi_\xi : M \to M$ son C^r difeomorfismos, $r \geq 0$, los cuales dependen continuamente con respecto ξ. El primer factor del producto

$\Sigma_k \times M$ es llamado base y al segundo es la fibra. Para destacar el papel de los difeomorfismos de fibras ϕ_ξ usamos la notación $\Phi = \tau \ltimes \phi_\xi$. Al conjunto de estos productos cruzados simbólicos se denotará por $\mathcal{S}_k(M)$. Cuando ϕ_ξ sólo depende de la coordenada ξ_0 de la bisucesión $\xi = (\xi_i)_{i\in\mathbb{Z}}$ se dice que Φ es un *producto cruzado de un solo paso* (o brevemente en inglés *one-step*) y en tal caso se escribe $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k)$ donde $\phi_\xi = \phi_i$ si $\xi_0 = i$. El conjunto de los productos cruzados de un solo paso se denota por $\mathcal{Q}_k(M)$.

Trabajar con productos cruzados simbólicos $\mathcal{S}_k(M)$ es una buena propuesta para estudiar la existencia de mezcladores en difeomorfismos productos cruzados de la forma

$$f : N \times M \to N \times M, \qquad f(x,y) = (F(x), \phi(x,y)),$$

cuando F es un difeomorfismo de una variedad N con una herradura $\Lambda \subset N$. Como ya anticipamos más arriba, las C^1 perturbaciones de estos difeomorfismos no continúan siendo necesariamente productos cruzados. A fin de poder garantizar que las perturbaciones de f son conjugadas con productos cruzados simbólicos se han de imponer a f condiciones de hiperbolicidad parcial y de dominación que son, y esto es muy importante, condiciones abiertas sobre f. Entonces, de acuerdo con los recientes trabajos [Gor06, IN10, PSW11], ver Proposición 2.1, existe $\varepsilon > 0$ y una constante $\alpha \in (0,1]$ que sólo depende del tasa de contracción $\nu \in (0,1)$ de la herradura Λ tal que cualquier pequeña ε-perturbación g de f en la topología C^1 tiene un conjunto Δ_g invariante localmente maximal isomorfo a $\Lambda \times M$, sobre el cual $g|_{\Delta_g}$ es topológicamente conjugado a un producto cruzado $\Phi = \tau \ltimes \phi_\xi$ perteneciente al subconjunto $\mathcal{PHS}^{1,\alpha}_k(M)$ de $\mathcal{S}_k(M)$ de los productos cruzados simbólicos *localmente Hölder continuos* y *parcialmente hiperbólicos*. Este subconjunto se define imponiendo a $\Phi = \tau \ltimes \phi_\xi$ condiciones de regularidad, lipschitzianidad y dominación:

- $\phi_\xi : M \to M$ son difeomorfismos de clase C^1,
- ϕ_ξ dependen *localmente α-Hölder continuamente* de ξ en M: existe $C \geq 0$ tal que

$$d_{C^0}(\phi_\xi^{\pm 1}, \phi_{\xi'}^{\pm 1}) \leq C d_{\Sigma_k}(\xi, \xi')^\alpha, \quad \text{para todo } \xi, \xi' \in \Sigma_k \text{ con } \xi_0 = \xi'_0. \tag{1}$$

  El espacio de símbolos $\Sigma_k = \{1, \dots, k\}^{\mathbb{Z}}$ se dota de la métrica

$$d_{\Sigma_k}(\xi, \xi') \stackrel{\text{def}}{=} \nu^\ell, \quad \ell = \min\{i \in \mathbb{Z}^+ : \xi_i \neq \xi'_i \text{ or } \xi_{-i} \neq \xi'_{-i}\}.$$

  Se denota por C_Φ la constante no negativa más pequeña que verifica (1).

- ϕ_ξ son *biLipschitz* y *parcialmente dominadas*: Existen constantes positivas γ y $\hat{\gamma}$ tales que
  - *s-dominación* y *u-dominación* (hiperbolicidad parcial):

$$\nu^\alpha < \gamma < 1 < \hat{\gamma}^{-1} < \nu^{-\alpha}$$

  - $(\gamma, \hat{\gamma}^{-1})$-Lipschitziadad en M:

$$\gamma \left\| x - x' \right\| < \left\| \phi_\xi(x) - \phi_\xi(x') \right\| < \hat{\gamma}^{-1} \left\| x - x' \right\|,$$

    para cualesquiera $\xi \in \Sigma_k$ y $x,\, x' \in M$. Con $\|x - x'\|$ se denota la distancia en M.

Un mezclador tipo herradura es un conjunto hiperbólico localmente maximal y por lo tanto estará vinculado a un subconjunto D abierto y acotado de M. Esto es, se tratará del conjunto localmente maximal para Φ en $\Sigma_k \times \overline{D}$. De esta forma, podemos imponer condiciones locales adicionales sobre los productos cruzados simbólicos $\Phi = \tau \ltimes \phi_\xi$ con los que estamos trabajando. Por ejemplo, podemos asumir que la restricción de ϕ_ξ al conjunto $\overline{D}$ es una aplicación contractiva o expansiva. Concretamente, trabajaremos con los siguientes conjuntos de productos cruzados simbólicos:

Definición (Conjuntos de productos cruzados simbólicos)**.** *Sea $D \subset M$ un conjunto abierto y acotado y consideremos constantes $0 < \lambda < \beta$ y $0 \leq \alpha \leq 1$. Se define $\mathcal{S}^{r,\alpha}_{k,\lambda,\beta}(D)$, $r \geq 0$, como el conjunto de los productos cruzados simbólicos $\Phi = \tau \ltimes \phi_\xi$ en $\mathcal{S}_k(M)$ tales que*

- *ϕ_ξ es una aplicación C^r-(λ,β)-Lipschitz en $\overline{D}$ para todo ξ en Σ_k, y*
- *ϕ_ξ depende localmente α-Hölder continuamente en $\overline{D}$ con respecto de ξ.*

Adicionalmente si $\beta < 1$ entonces se impone la condición $\phi_\xi(\overline{D}) \subset D$ para todo $\xi \in \Sigma_k$, y, en el caso $1 < \lambda$ se impone $\overline{D} \subset \phi_\xi(D)$ para todo $\xi \in \Sigma_k$. Se dotará al conjunto $\mathcal{S}^{r,\alpha}_{k,\lambda,\beta}(D)$ de la distancia

$$d_{\mathcal{S}}(\Phi,\Psi) = \sup_{\xi \in \Sigma_k} d_{C^r}(\phi_\xi,\psi_\xi) + |C_\Phi - C_\Psi|, \quad \text{con} \;\; \Phi = \tau \ltimes \phi_\xi \;\; \text{y} \;\; \Psi = \tau \ltimes \psi_\xi.$$

Por conveniencia, $\mathcal{S}_{k,\lambda,\beta}(D)$ y $\mathcal{S}^{\alpha}_{k,\lambda,\beta}(D)$ denotará $\mathcal{S}^{0,0}_{k,\lambda,\beta}(D)$ y $\mathcal{S}^{0,\alpha}_{k,\lambda,\beta}(D)$, respectivamente.

Bajo la hipótesis de aplicaciones de fibras contractivas, el siguiente resultado proporciona una descripción del maximal invariante de Φ en $\Sigma_k \times \overline{D}$ y muestra la dependencia respecto a Φ del conjunto

$$K_\Phi \stackrel{\text{def}}{=} \overline{\mathscr{P}(\mathrm{Per}(\Phi)) \cap D}$$

donde $\mathrm{Per}(\Phi)$ es el conjunto de los puntos periódicos de Φ y $\mathscr{P} : \Sigma_k \times M \to M$ es la proyección estándar en M. Aunque este teorema es un caso particular de los resultados de [HPS77], será muy útil disponer de una prueba completa y detallada en el contexto de productos cruzados simbólicos. Se denota por $\mathcal{K}(\overline{D})$ la colección de los subconjuntos compactos de $\overline{D}$ dotada de la métrica de Hausdorff y

$$W^u((\xi,x);\Phi) \stackrel{\text{def}}{=} \{(\zeta,y) \in \Sigma_k \times M : \lim_{n\to\infty} d(\Phi^{-n}(\zeta,y),\Phi^{-n}(\xi,x)) = 0\}$$

es el conjunto inestable de (ξ,x) para Φ.

Teorema A (Geometría del maximal invariante)**.** *Consideremos $\Phi \in \mathcal{S}^{\alpha}_{k,\lambda,\beta}(D)$ con $\beta < 1$ y $\alpha > 0$. Entonces la restricción de Φ a el conjunto*

$$\Gamma_\Phi = \bigcap_{n\in\mathbb{Z}} \Phi^n\big(\Sigma_k \times \overline{D}\big) = \bigcap_{n\in\mathbb{N}} \Phi^n\big(\Sigma_k \times \overline{D}\big)$$

es conjugado con el Bernoulli shift τ de k símbolos. Más aún, $W^u((\xi,x);\Phi) \subset \Gamma_\Phi$ para todo $(\xi,x) \in \Gamma_\Phi$ y existe una única función continua $g_\Phi : \Sigma_k \to \overline{D}$ tal que para todo punto periódico (ϑ,p) de Φ en $\Sigma_k \times D$ se tiene que,

$$\Gamma_\Phi = \overline{W^u((\vartheta,p);\Phi))} = \{(\xi,g_\Phi(\xi)) : \xi \in \Sigma_k\} \quad \text{and} \quad \mathscr{P}(\Gamma_\Phi) = K_\Phi \in \mathcal{K}(D).$$

Finalmente, el mapa $\mathscr{L} : \mathcal{S}^{\alpha}_{k,\lambda,\beta}(D) \to \mathcal{K}(\overline{D})$ dado por $\mathscr{L}(\Phi) = K_\Phi$ es continuo.

Con objeto de introducir un mezclador simbólico, primeramente definiremos una familia de discos casi horizontales los cuales proporcionan la región de superposición del mezclador.

Definición (Discos casi horizontales). *Fijado $\alpha > 0$ y dado un subconjunto abierto $B \subset D$, se dice que D^s es un disco δ-horizontal en $\Sigma_k \times B$ si existen $\zeta \in \Sigma_k$, $z \in B$, alguna constante $C \geq 0$ y una función (α, C)-Hölder continua $h : W^s_{loc}(\zeta, \tau) \to B$ tal que*

$$D^s = \{(\xi, h(\xi)) : \xi \in W^s_{loc}(\zeta; \tau)\}, \quad \|z - h(\xi)\| < \delta \text{ para todo } \xi \in W^s_{loc}(\zeta, \tau) \text{ y } C\nu^a < \delta.$$

Aquí $W^s_{loc}(\zeta; \tau) \stackrel{\text{def}}{=} \{\xi \in \Sigma_k : \xi_i = \zeta_i \text{ para todo } i \geq 0\}$ denota el conjunto estable local de ζ para τ.

La motivación para considerar como región de superposición en la definición de mezclador un conjunto de discos encajados en la variedad es que las variedades locales estables fuertes deben formar parte de este conjunto. Obsérvese que para cualquier $\delta > 0$, el conjunto $W^s_{loc}(\zeta; \tau) \times \{z\}$ con $\zeta \in \Sigma_k$ y $z \in B$, es un disco δ-horizontal en $\Sigma_k \times B$ y en el caso de un producto cruzado de un solo paso coincide con el conjunto estable fuerte local. Ya que nuestra intención es estudiar Hölder perturbaciones de un one-step (o de un producto cruzado próximo a uno de un solo paso), es suficiente considera como región de superposición el conjunto de los discos casi horizontales.

Desde el Teorema A se sigue que $W^u(\Gamma_\Phi) = \Gamma_\Phi$ para todo $\Phi \in \mathcal{S}^\alpha_{k,\lambda,\beta}(D)$ con $\beta < 1$, donde

$$W^u(\Gamma_\Phi) \stackrel{\text{def}}{=} \left\{(\xi, x) \in \Sigma_k \times M : \lim_{n\to\infty} d(\Phi^{-n}(\xi, x), \Gamma_\Phi) = 0\right\}$$

es el conjunto inestable del conjunto maximal invariante Γ_Φ. Por lo tanto, la definición de mezclador en el contexto de mezcladores simbólicos puede ser escrita del siguiente modo:

Definición (Mezcladores simbólicos). *Sea $\Phi \in \mathcal{S}^\alpha_{k,\lambda,\beta}(D)$ con $\alpha > 0$ y $\beta < 1$.*

*Se dice que el conjunto invariante maximal Γ_Φ de Φ en $\Sigma_k \times \overline{D}$ es un cs-*mezclador herradura simbólico *si existe $\delta > 0$, un conjunto no vacío $B \subset D$ y un entorno $\mathcal{V}$ de Φ en $\mathcal{S}^\alpha_{k,\lambda,\beta}(D)$ tal que para todo $\Psi \in \mathcal{V}$ y para cualquier disco δ-horizontal D^s en $\Sigma_k \times B$ se verifica que*

$$\Gamma_\Psi \cap D^s \neq \emptyset, \quad \textit{donde } \Gamma_\Psi \textit{ es la continuación de } \Gamma_\Phi \textit{ para } \Psi.$$

Al conjunto abierto B se le denomina región de superposición del mezclador herradura simbólico*.*

Para definir *cu*-mezclador herradura simbólico, primeramente necesitamos introducir productos cruzados simbólicos inversos. Dado $\Phi = \tau \ltimes \phi_\xi \in \mathcal{S}^\alpha_{k,\lambda,\beta}(D)$, se llama *producto cruzado simbólico inverso asociado con* Φ al producto cruzado simbólico

$$\Phi^* = \tau \ltimes \phi^*_\xi \in \mathcal{S}^\alpha_{k,\,\beta^{-1},\,\lambda^{-1}}(D), \quad \text{donde } \phi^*_\xi : M \to M \text{ dado por } \phi^*_\xi(x) = \phi^{-1}_{\tau^{-1}(\xi^*)}(x).$$

Aquí, $\xi^* = (\ldots\xi_1; \xi_0, \xi_{-1}, \ldots)$ denota la bisucesión conjugada a $\xi = (\ldots\xi_{-1}; \xi_0, \xi_1, \ldots)$. Obsérvese que como $\tau(\xi)^* = \tau^{-1}(\xi^*)$ entonces Φ^* se corresponde con los iterados de Φ^{-1}. Esta observación nos permite definir *cu*-mezcladores simbólicos para un productos cruzados simbólico Φ en $\mathcal{S}^\alpha_{k,\lambda,\beta}(D)$ with $\lambda > 1$. Concretamente, *cu-mezclador herradura simbólico* para Φ se define como un *cs*-mezclador herradura simbólico para Φ^*. En lo que sigue, sólo consideraremos *cs*-mezcladores simbólicos y nos referiremos a ellos preferentemente por brevedad como mezcladores simbólicos.

A partir del Teorema A se tiene también que $\overline{W^u((\vartheta,p);\Phi)} = \Gamma_\Phi$ para todo punto periódico $(\vartheta,p) \in \Sigma_k \times D$ de un producto cruzado simbólico Φ parcialmente hiperbólico. Se probará en la Proposición 2.5 que cada conjunto local estable fuerte $W^{ss}_{loc}((\xi,x);\Phi)$ es un disco casi horizontal. Si Φ es suficientemente próximo a un producto cruzado de un solo paso entonces este disco es una pequeña Hölder perturbación del disco horizontal $W^s_{loc}(\xi;\tau) \times \{x\}$. Por consiguiente, si además Γ_Φ es un *cs*-mezclador simbólico para Φ con región de superposición B, entonces se verifica que

$$\overline{W^u((\vartheta,p_\Psi);\Psi)} \cap W^{ss}_{loc}((\xi,x);\Psi) \neq \emptyset, \quad \text{para todo } (\xi,x) \in \Sigma_k \times B$$

y para toda perturbación $\Psi \in S^\alpha_{k,\lambda\,\beta}(D)$ de Φ, donde (ϑ,p_Ψ) es la continuación de (ϑ,p) por Ψ.

Un conjunto de aplicaciones $\phi_1,\dots,\phi_k$ definidas en $\overline{D}$ se dice que tienen la *propiedad de cobertura* si existe un abierto $B \subset D$ tal que $\overline{B} \subset \phi_1(B) \cup \ldots \cup \phi_k(B)$. Uno de los objetivos es comprender como llevar propiedades robustas de un sistema iterado de funciones generado por $\phi_1,\dots,\phi_k$ a propiedades robustas del producto cruzado $\Phi = \tau \ltimes (\phi_1,\dots,\phi_k)$ bajo Hölder perturbaciones. El siguiente resultado describe como la propiedad de cobertura se traslada a una propiedad robusta en el lenguaje de los productos cruzados localmente Hölder.

Teorema B (Caracterización de la propiedad de cobertura). *Sea $\Phi = \tau \ltimes (\phi_1,\dots,\phi_k) \in \mathcal{S}^\alpha_{k,\lambda,\beta}(D)$ con $\nu^\alpha < \lambda < 1$, $\alpha > 0$ and $B \subset D$ un conjunto abierto. Entonces,*

$$\overline{B} \subset \phi_1(B) \cup \ldots \cup \phi_k(B)$$

si y sólo si existe $\delta > 0$ y un entorno $\mathcal{V}$ de Φ en $\mathcal{S}^\alpha_{k,\lambda,\beta}(D)$ tal que para todo $\Psi \in \mathcal{V}$

$$\Gamma^+_\Psi(B) \cap D^s \neq \emptyset \quad \textit{para todo disco } \delta\textit{-horizontal } D^s \textit{ en } \Sigma_k \times B$$

donde $\Gamma^+_\Psi(B)$ es el conjunto maximal invariante por las iteradas positivas de Ψ en $\Sigma_k \times B$.

Con la hipótesis $\beta < 1$ si $\Phi = \tau \ltimes (\phi_1,\dots,\phi_k) \in \mathcal{S}^\alpha_{k,\lambda,\beta}(D)$ se tiene que $\phi_i(\overline{D}) \subset D$ para $i = 1,\dots,k$. En tal caso, para cualquier pequeña perturbación $\Psi = \tau \ltimes \psi_\xi$ de Φ también se verifica que $\psi_\xi(\overline{D}) \subset D$ y se tiene que

$$\Gamma^+_\Psi(B) \overset{\text{def}}{=} \bigcap_{n\geq 0} \Psi^n(\Sigma_k \times B) \subset \bigcap_{n\in\mathbb{Z}} \Psi^n(\Sigma_k \times \overline{D}) \overset{\text{def}}{=} \Gamma_\Phi.$$

Entonces, combinando el resultado anterior con la definición de mezclador simbólico obtenemos como consecuencia la existencia de mezcladores simbólicos usando la propiedad de cobertura:

Teorema C (Existencia de mezcladores simbólicos). *Sea $\Phi = \tau \ltimes (\phi_1,\dots,\phi_k) \in \mathcal{S}^\alpha_{k,\lambda,\beta}(D)$ con $\alpha > 0$ y $\nu^\alpha < \lambda < \beta < 1$. Supongamos que existe un conjunto abierto $B \subset D$ tal que*

$$\overline{B} \subset \phi_1(B) \cup \ldots \cup \phi_k(B).$$

Entonces el conjunto maximal invariante Γ_Φ de Φ en $\Sigma_k \times \overline{D}$ es un cs-mezclador herradura simbólico de Φ cuya región de superposición contiene a B.

Una parte de este segundo capítulo se dedica al estudio de un subconjunto de producto cruzados simbólicos $\mathcal{S}^+_k(M)$ llamados *productos cruzados simbólicos unilaterales* que generaliza a los one-step. Este conjunto consiste de las aplicaciones $\Phi = \tau \ltimes \phi_\xi$ tales que $\phi_\xi = \phi_{\xi'}$ si $\xi_i = \xi'_i$

para todo $i \geq 0$. En la Proposición 2.3 se muestra que la presencia de una *holonomía estable* (ver Definición 2.3) para una aplicación Φ en $\mathcal{S}_k(M)$ permite conjugar topológicamente Φ con un producto cruzado simbólico unilateral $\tilde{\Phi} \in \mathcal{S}_k^+(M)$. Este hecho, nos permitiría restringir el conjunto de perturbaciones de productos cruzados que tiene que ser considerado e introducir así otra definición (en el contexto unilateral) del concepto de mezclador herradura simbólico (ver Definición 2.11). Concretamente, en la sección §2.4 se estudian perturbaciones dentro del conjunto $\mathcal{S}^+_{k,\lambda,\beta}(D) = \mathcal{S}_k^+(M) \cap \mathcal{S}_{k,\lambda,\beta}(D)$ con $\beta < 1$.

En la Proposición 2.5 se prueba la existencia de holonomía estable para todo producto cruzado simbólico s-dominado, en particular, para $\Phi = \tau \ltimes \phi_\xi \in \mathcal{PHS}_k^{1,\alpha}(M) \cap \mathcal{S}_{k,\lambda,\beta}(D)$. El correspondiente producto cruzado unilateral conjugado con Φ viene dado por

$$\tilde{\Phi} = \tau \ltimes \tilde{\phi}_\xi \in \mathcal{S}_k^+(M) \quad \text{donde} \quad \tilde{\phi}_\xi = h^s_{\tau(\xi),\pi(\tau(\xi))} \circ \phi_\xi \circ h^s_{\pi(\xi),\xi}$$

siendo π la proyección de Σ_k sobre una sección transversal Σ a la partición estable $W^s_{loc}(\xi;\tau)$, $\xi \in \Sigma_k$ y $h^s_{\xi,\xi'} : M \to M$ la familia de aplicaciones que define la holonomía estable. Se prueba en Proposición 2.6 que cada aplicación $h^s_{\xi,\xi'}$ es Hölder continua con constante de Hölder uniforme para todo ξ y ξ'; pero esto no es suficiente para concluir que $\tilde{\Phi} \in \mathcal{S}^+_{k,\lambda,\beta}(D)$. Para garantizar que $\tilde{\Phi} \in \mathcal{S}_{k,\lambda,\beta}(D)$ necesitamos incrementar la regularidad Φ y añadir condiciones de agrupamiento de las fibras. Concretamente, la Proposición 2.9 prueba que las aplicaciones $h^s_{\xi,\xi'}$ que definen la holonomía estable son C^1 difeomorfismos si $\Phi = \tau \ltimes \phi_\xi$ es *fibra agrupado* (ver Definición 2.7) y pertenece al conjunto $\mathcal{PHS}_k^{2,1+\alpha}(M)$ de los productos cruzados parcialmente hiperbólicos cuyos aplicaciones de fibras son de clase C^2 dependiendo *localmente Hölder diferenciablemente* con respecto de la base, es decir,

$$d_{C^1}(\phi_\xi^{\pm 1}, \phi_{\xi'}^{\pm 1}) \leq C d_{\Sigma_k}(\xi,\xi')^\alpha, \quad \text{para todo } \xi, \xi' \in \Sigma_k \text{ con } \xi_0 = \xi'_0.$$

Esta regularidad de la holonomía implica que $\tilde{\Phi} = \tau \ltimes \tilde{\phi}_\xi$ pertence a $\mathcal{S}^+_{k,\lambda,\beta}(D)$. De acuerdo con [Gor06], ver Teorema 2.2, se sigue que estas condiciones adicionales de regularidad y agrupamiento de las fibras pueden ser inferidas para el producto cruzado simbólico $\Psi = \tau \ltimes \psi_\xi$ conjugado a una C^2 perturbación g del C^2 difeomorfismo $f = F \times \mathrm{id}$, donde $F : N \to N$ es una aplicación herradura y $\mathrm{id} : M \to M$ es la aplicación identidad. De está forma se concluye que un mezclador herradura simbólico en el contexto unilateral da lugar a un mezclador para un C^2 difeomorfismo con región de superposición C^2 robusta.

Los resultados anteriores sobre la existencia de mezcladores simbólicos se han dado para productos cruzados simbólicos $\Phi = \tau \ltimes (\phi_1, \ldots, \phi_k)$ de tipo one-step (de un sólo paso). Al margen de las hipótesis de regularidad y dominación impuestas para restringir el espacio de perturbaciones, la condición de existencia de mezclador se reduce a la propiedad de cobertura, que se formula en términos de las contracciones $\phi_1, \ldots, \phi_k$. Esto permite considerar la estructura de mezclador como algo propio de los productos cruzados de un paso que persiste bajo buenas perturbaciones. Parafrasenado lo dicho para la prueba de la existencia de atractores extraños tipo Hénon, los productos cruzados de un solo paso se pueden considerar las aplicaciones límite cuya dinámica hay que comprender, del mismo modo que se necesita entender la dinámica de la familia límite $h_a(x) = 1 - ax^2$ para comprender la existencia de los atractores de Hénon en [BC91]. A partir de esta reflexión, se introduce también en este segundo capítulo la sección §2.3 dedicada al estudio de los mezcladores simbólicos en el contexto de los productos cruzados de un solo paso. Es decir, considerando perturbaciones sólo en el conjunto $\mathcal{Q}_{k,\lambda,\beta}(D)$. Se muestra de qué manera la dinámica de un one-step $\Phi = \tau \ltimes (\phi_1, \ldots, \phi_k)$ viene dada por la dinámica del sistema iterado de funciones generado por $\phi_1, \ldots, \phi_k$ y cómo el concepto de mezclador emerge ya de las propiedades de esta dinámica. Dicha sección es una antesala del siguiente capítulo de la memoria.

III – Sistemas iterados de funciones – El tercer capítulo de la tesis se dedica al estudio de los sistemas iterados, bien definidos sobre un intervalo o sobre la circunferencia S^1. Los principales resultados de este capítulo son en colaboración con Artem Raibekas y se recogen en su tesis doctoral [Rai11] y en la prepublicación [BR].

Un *sistema iterado de funciones* (FS a partir de ahora) generado por la familia de difeomorfismos $\Phi = \{\phi_1, \ldots, \phi_k\}$ de un variedad M es el conjunto $\mathrm{IFS}(\phi_1, \ldots, \phi_k)$ de todas las posibles composiciones de los difeomorfismos $\phi_i \in \Phi$ (incluyendo la identidad). Esto es, el semigrupo con la operación composición generado por $\phi_1, \ldots, \phi_k, \mathrm{id}$. Debido a la estrecha relación entre los productos cruzados de un paso y los sistemas iterados de funciones, escribiremos $\mathrm{IFS}(\Phi) = \mathrm{IFS}(\phi_1, \ldots, \phi_k)$ entendiendo que el IFS es generado por la familia $\Phi = \{\phi_1, \ldots, \phi_k\}$ asociada al one-step $\Phi = \tau \ltimes (\phi_1, \ldots, \phi_k)$ definido sobre $\Sigma_k \times M$.

Como ya anticipamos, la dinámica de un producto cruzado de un paso viene dada por la dinámica de su sistema iterado de funciones asociado. Para poder hablar de dinámica de un IFS es necesario introducir la noción básica de órbita. La *órbita* de un punto $x \in M$ por $\mathrm{IFS}(\Phi)$ es la acción del IFS sobre el punto x, es decir,

$$\mathrm{Orb}_\Phi(x) \stackrel{\text{def}}{=} \{h(x) : \, h \in \mathrm{IFS}(\Phi)\} \subset M.$$

Con esta noción de órbita, algunos conceptos dinámicos conocidos para sistemas dinámicos son traducidos al ámbito de los sistemas iterados. A modo de ejemplo, un conjunto $\Lambda \subset M$ se dice: *invariante* si $\Lambda = \mathrm{Orb}_\Phi(x)$ para todo $x \in \Lambda$; *transitivo* si existe una órbita densa en Λ, es decir,

$$\Lambda \subset \overline{\mathrm{Orb}_\Phi(x)} \quad \text{para algún } x \in \Lambda;$$

y *minimal* si todo punto $x \in \Lambda$ tiene órbita densa en Λ. El *ω-límite de un punto $x \in M$* para el $\mathrm{IFS}(\Phi)$ es el conjunto

$$\omega_\Phi(x) \stackrel{\text{def}}{=} \{y : \text{ existe } (h_n)_n \subset \mathrm{IFS}(\Phi) \setminus \{\mathrm{id}\} \text{ tal que } \lim_{n\to\infty} h_n \circ \cdots \circ h_1(x) = y\}$$

mientras que el *ω-límite de* $\mathrm{IFS}(\Phi)$ es

$$\omega(\mathrm{IFS}(\Phi)) \stackrel{\text{def}}{=} \mathrm{cl}\{y \in M : \text{ existe } x \in M \text{ tal que } \; y \in \omega_\Phi(x)\},$$

donde con "cl" indicamos la clausura del conjunto. Análogamente se define el *α-límite de un punto $x \in M$ y del sistema iterado de funciones* $\mathrm{IFS}(\Phi)$. Finalmente, el *conjunto límite* $L(\mathrm{IFS}(\Phi))$ es la unión de ω-límite y α-límite del $\mathrm{IFS}(\Phi)$. A partir de estas nociones, conocer la dinámica de un IFS implica entender cuales son los posibles conjuntos invariantes para el IFS, describir los ω-límite o α-límite de sus órbitas mostrando, si es posible, una descomposición espectral del conjunto límite de sus órbitas como se hizo en el caso de difeomorfismos para un sistema dinámico hiperbólico.

A fin de encontrar propiedades robustas bajo perturbaciones es importante introducir el concepto de proximidad dentro del conjunto de los IFS. Esto es, $\mathrm{IFS}(\psi_1, \ldots, \psi_k)$ se dice C^1 *próximo* de $\mathrm{IFS}(\phi_1, \ldots, \phi_k)$ si cada uno de los difeomorfismos ψ_i es próximo a ϕ_i en la topología C^1. Como ejemplo de propiedad robusta por perturbaciones se puede pensar en los mezcladores simbólicos definidos en el segundo capítulo y en su traducción al lenguaje de los IFS:

Definición (Región mezcladora)**.** *Un conjunto abierto $B \subset M$ es un* región mezcladora *del sistema iterado de funciones* $\mathrm{IFS}(\Phi)$ *si B es C^1*-robustamente minimal *para* $\mathrm{IFS}(\Phi)$*, es decir,*

$$B \subset \overline{\mathrm{Orb}_\Psi(x)} \quad \text{para todo } x \in B \text{ y todo } \mathrm{IFS}(\Psi) \ C^1 \text{ próximo de } \mathrm{IFS}(\Phi).$$

En el caso de un producto cruzado de un paso con aplicaciones de fibras contractivas, en Proposición 2.21 se prueba que la existencia de una región mezcladora es equivalente a tener un mezclador simbólico en el contexto de productos cruzados tipo one-step. El principal objetivo a lo largo de este tercer capítulo es probar la existencia de regiones mezcladoras para sistema iterados de funciones generados por difeomorfismos genéricos en la recta real $M = \mathbb{R}$ y en el círculo $M = S^1$ próximos a la identidad $\mathrm{id} : M \to M$.

En la sección §3.2 estudiamos regiones mezcladoras en la recta real. Definiremos un intervalo con un tipo de configuración para un par de funciones f_0, f_1 (ver Figura 3.1(a)) que será un candidato a ser región mezcladora para $\mathrm{IFS}(f_0, f_1)$. Denotemos por $\mathrm{Diff}^r_+(\mathbb{R})$ el conjunto de los C^r difeomorfismos en la recta real que preservan la orientación.

Definición (ss-intervalos)**.** *Dado $\Phi = \{f_0, f_1\} \subset \mathrm{Diff}^1_+(\mathbb{R})$, se dice que un intervalo $[p_0, p_1] \subset \mathbb{R}$ es un ss*-intervalo para $\mathrm{IFS}(\Phi)$ *si:*

- $[p_0, p_1] = f_0([p_0, p_1]) \cup f_1([p_0, p_1])$,
- $(p_0, p_1) \cap \mathrm{Fix}(f_i) \neq \emptyset$ *para* $i = 1, 2$, *y* $p_j \notin \mathrm{Fix}(f_i)$ *para* $i \neq j$,
- p_0 *y* p_1 *son puntos fijos atractores de* f_0 *y* f_1 *respectivamente.*

Denotaremos por K^{ss}_Φ a los ss-intervalo $[p_0, p_1]$ para el sistema iterado de funciones $\mathrm{IFS}(\Phi)$.

El siguiente teorema implica que cualquier abierto contenido en un ss-intervalo para un sistema iterado de funciones $\mathrm{IFS}(\Phi)$ con generadores suficiente próximos de la identidad con puntos fijos hiperbólicos es un región mezcladora para $\mathrm{IFS}(\Phi)$. Este teorema es una generalización de un lema debido a Duminy [Dum70] que forma parte de la prueba del llamado Teorema de Duminy (ver Teorema 3.27) relativo a la dinámica de grupos de difeomorfismos en el círculo. La prueba que aquí presentamos es diferente de la prueba original del Lema de Duminy (ver [Nav11] para más detalles) y nos permitirá mejorar ligeramente las conclusiones del Teorema de Duminy. Denotamos por $\mathrm{Per}(\mathrm{IFS}(\Phi))$ al conjunto de los puntos periódicos de $\mathrm{IFS}(\Phi)$, es decir, el conjunto de los puntos $x = h(x)$ para algún $h \neq \mathrm{id}$ en $\mathrm{IFS}(\Phi)$.

Teorema D (Lema de Duminy)**.** *Sea $\Phi = \{f_0, f_1\} \subset \mathrm{Diff}^2_+(\mathbb{R})$ y K^{ss}_Φ un ss-intervalo para* $\mathrm{IFS}(\Phi)$ *con los puntos fijos de f_0 y f_1 en K^{ss}_Φ hiperbólicos. Entonces, existe $\varepsilon \geq 0.17$ tal que si $f_0|_{K^{ss}_\Phi}$, $f_1|_{K^{ss}_\Phi}$ son ε-próximas a la identidad en la topología C^2 se tiene que*

$$K^{ss}_\Psi \subset \overline{\mathrm{Per}(\mathrm{IFS}(\Psi))} \quad y \quad K^{ss}_\Psi = \overline{\mathrm{Orb}_\Psi(x)} \quad \text{para todo } x \in K^{ss}_\Psi$$

y para todo sistema iterado de funciones $\mathrm{IFS}(\Psi)$ C^1 *próximo de* $\mathrm{IFS}(\Phi)$.

En la sección §3.3.2 nos ocupamos de generalizar este teorema para difeomorfismos Morse-Smale en el círculo. Ver Teorema 3.35. Esta generalización, forma parte de la prueba de un resultado para IFS tipo Teorema de Denjoy. Recordar que, teniendo en cuenta el número de rotación de un difeomorfismo f del círculo tenemos que: (i) f tiene un punto periódico, (ii) todas las órbitas (para iterados positivos) de f son densas, y (iii) existe un intervalo errante para f. Los intervalos errantes se tratan de los "gaps" de un conjunto de Cantor $\Lambda \subset S^1$ invariante por f y contenido en el ω-límite para f de todos los puntos de S^1. Estas propiedades dinámicas pueden ser transladadas facilmente para un IFS:

Definición (Cantor invariante minimal)**.** *Sea* $\Phi = \{f_0, f_1\} \subset \mathrm{Diff}^1(S^1)$ *y* $\Lambda \subset S^1$. *Se dice que* Λ *es un* conjunto de Cantor invariant y minimal para $\mathrm{IFS}(\Phi)$ *si*

- Λ *es un conjunto de Cantor,*
- $\Lambda = \overline{\mathrm{Orb}_\Phi(x)}$ *para todo* $x \in \Lambda$.

Del Teorema de Denjoy [Den32] se sigue que este tipo de conjuntos no pueden aparecer para difeomorfismos del círculo con cierta regularidad suficientemente próximos de la identidad. Concretamente, existe $\varepsilon > 0$ tal que si $f \in \mathrm{Diff}^2(S^1)$ y es ε-próximo de la identidad en la topología C^2 entonces no existen conjuntos de Cantor invariantes y minimales. Más aún, son equivalente las siguientes afirmaciones: S^1 es minimal para los sistemas iterados de funciones $\mathrm{IFS}(f)$, y, f no tiene puntos periódicos. Cuando el número de generadores del IFS aumenta los puntos periódicos dejan de ser la obstrucción a la minimalidad. Ahora, ese papel es jugado por los ss-intervalos.

Teorema E (Denjoy para IFS)**.** *Existe* $\varepsilon > 0$ *tal que si* $f_0, f_1 \in \mathrm{Diff}^2(S^1)$ *son difeomorfismos Morse-Smale* ε*-próximos de la identidad en la topología* C^2 *sin puntos periódicos en común, entonces, no existen conjuntos de Cantor invariantes y minimales para el* $\mathrm{IFS}(f_0, f_1)$.

Más aún, denotando n_i *el periodo de* f_i*, son equivalentes:*

- S^1 *es minimal para* $\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})$,
- *no existen* ss*-intervalos para* $\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})$.

A diferencia de lo ocurre para un único difeomorfismo f en el circulo donde S^1 no puede ser C^1-robustamente minimal, en el caso de IFS, esta robustez si que puede llegar a obtenerse. De hecho, hemos de notificar que el teorema anterior es C^1-robusto en el siguiente sentido:

Nota (C^1-robustez)**.** *Las conclusiones del Teorema de Denjoy para IFS son robustas bajo* C^1 *perturbaciones del sistema iterado de funciones* $\mathrm{IFS}(f_0, f_1)$*, es decir, para todo* $\mathrm{IFS}(g_0, g_1)$ *donde* g_0 *y* g_1 *son* C^1*-perturbaciones de* f_0 *y* f_1 *respectivamente.*

Como consecuencia de este teorema tipo Denjoy para IFS, finalizaremos este tercer capítulo de la memoria de tesis mostrando un teorema de descomposición espectral en el círculo. Este teorema afirma que el conjunto límite de $\mathrm{IFS}(\Phi)$ con $\Phi = \{f_0^{n_0}, f_1^{n_1}\}$, donde $f_0, f_1 \in \mathrm{Diff}^2(S^1)$ en la condiciones del teorema anterior, se descompone en unión finita de *conjuntos básicas* disjuntas: aislados y transitivos. Un conjunto A con $A \cap \mathrm{Per}(\mathrm{IFS}(\Phi)) \neq \emptyset$ se dice *aislado* para el $\mathrm{IFS}(\Phi)$ si existe un abierto D tal que $A \subset D$ y $\overline{\mathrm{Per}(\mathrm{IFS}(\Phi)) \cap D} \subset A$.

Teorema F (Descomposición Espectral para IFS)**.** *Existe $\varepsilon > 0$ tal que si $f_0, f_1 \in \mathrm{Diff}^2(S^1)$ son difeomorfismos Morse-Smale de períodos n_0 y n_1, respectivamente, ε-próximos de la identidad en la topología C^2 y sin puntos periódicos en común, entonces, existe un numero finito de intervalos $K_1, \ldots, K_m$ dos a dos disjuntos, aislados y transitivos para $\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})$ tales que*

$$L(\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})) = \bigcup_{i=1}^{m} K_i.$$

Más aún, esta descomposición del conjunto límite de $\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})$ es C^1-robusta.

IV – Ciclos en despliegues de la singularidad nilpotente – En el último capítulo se trasladan al ámbito de los campos vectoriales las conclusiones dinámicas obtenidas en la primera parte de la tesis. El principal resultado de este capítulo es en colaboración con Santiago Ibañéz y J. Ángel Rodríguez y se recoge en [BIR11].

El interés allí por la dinámica asociada a ciclos heterodimensionales obligó a considerar difeomorfismos en dimensión $n \geq 3$. Es bien sabido que estos difeomorfismos se pueden definir como aplicaciones de Poincaré sobre secciones transversales al flujo de un campo en $\mathbb{R}^4$ cerca de un ciclo o de una órbita periódica.

Filosóficamente, las dinámicas posibles en sistemas discretos se elevan al ámbito de los sistemas continuos mediante el proceso de suspensión, que permite definir en un entorno de una órbita periódica un flujo que tiene como sección de Poincaré un difeomorfismo determinado. Sin embargo, el verdadero interés radica en determinar con algún criterio manejable cuando un campo vectorial o, en su defecto una familia de campos vectoriales, posee este o aquel comportamiento dinámico. El estudio de las bifurcaciones globales asociadas a distintos ciclos permitió explicar transiciones dinámicas al tiempo que explicaba la naturaleza del comportamiento. La presencia de infinitas herraduras en un entorno de una órbita de tipo Sil´nikov es un ejemplo. Sin embargo, constatar que en una familia de campos se tiene un determinado ciclo no es tarea fácil, al menos que esa familia sea construida ad hoc. Ese es el caso en [Rod86] para una familia de campos vectoriales cuadráticos que presenta órbitas de Sil´nikov. Como una alternativa a esta búsqueda y captura de ciclos, se puede plantear la prueba de criterios que permitan concluir la presencia de determinada dinámica interesante a partir de los elementos más simples de un campo vectorial: sus singularidades. En estos términos uno puede preguntarse, por ejemplo, cuál es la singularidad de menor codimensión (más frecuente) desde la que se despliegan genéricamente órbitas de tipo Sil´nikov y, consecuentemente, atractores extraños. Una respuesta parcial fue dada en [IR95] al probar que esta configuración se presentaba genéricamente en los despliegues de la singularidad nilpotente de codimensión cuatro en $\mathbb{R}^3$ y posteriormente en [IR05] para la singularidad nilpotente de codimensión tres. Una *singularidad nilpotente* es un campo de vectores C^∞ en $\mathbb{R}^n$ que, en apropiadas coordenadas, en un entorno de origen (punto de equilibrio) puede ser escrito como

$$\sum_{k=1}^{n-1} x_{k+1} \frac{\partial}{\partial x_k} + f(x_1, \ldots, x_n) \frac{\partial}{\partial x_n},$$

con $f(x) = O(\|x\|^2)$ donde $x = (x_1, \ldots, x_n)$. Se dice que es una *singularidad nilpotente de codimensión n* si se cumple la condición genérica $\partial^2 f / \partial x_1^2(0) \neq 0$.

La posible presencia genérica de atractores extraños en el despliegue de singularidades de menor codimensión posible, esto es, en la singularidad Hopf-cero que es de codimensión dos, está siendo analizada en [DIKS]. Los resultados en [IR05] permitieron concluir la presencia de atractores extraños en el acoplamiento de dos Brusselator por difusión lineal [DIR07], avanzando así en una via para generar complejidad por acoplamiento propuesta por Turing [Tur52] y contemplada por Smale en [Sma74]. Por Brusselator se entiende un campo cúbico bidimensional que se propone como modelo de reacción química. El acoplamiento de dos de estos campos conduce a un campo vectorial que tiene una singularidad nilpotente de codimensión cuatro en $\mathbb{R}^4$. El primer objetivo propuesto al inicio de la tesis fue estudiar los despliegues genéricos de esta singularidad para encontrar ciclos que pudieran implicar dinámicas propias de dimensión $n \geq 4$: atractores extraños con más de un exponente de Lyapunov positivo y ciclos heterodimensionales. En [BIR11] se probó la existencia de bifocos homoclínicos en todor despliegue genérico de la singularidad nilpotente de codimensión cuatro en $\mathbb{R}^4$.

Teorema G. *En todo despliegue genérico de una singularidad nilpotente de codimensión cuatro en $\mathbb{R}^4$ hay una hipersuperficie de órbitas homoclínicas bifocales.*

Recuérdese que una órbita homoclínica bifocal es una conexión homoclínica a un punto de equilibrio de un campo de vectores en $\mathbb{R}^4$ con dos pares de autovalores $\rho_k \pm i\omega_k$, con $k = 1, 2$, tales que $\rho_1 < 0 < \rho_2$. La aplicación de Poincaré definida en un entorno de este ciclo será un difeomorfismo tridimensional, susceptible de presentar un mezclador. Se prueba que existen mezcladores suspendidos para campos de vectores arbitrariamente próximos a un campo Hamiltoniano con una órbita homoclínica bifocal no degenerada. Para este campo Hamiltoniano la aplicación retorno se puede escribir con una adecuada elección de coordenadas de la forma

$$f : [-\varepsilon, \varepsilon]^2 \times [-c_0, c_0] \to [-\varepsilon, \varepsilon]^2 \times [-c_0, c_0] \qquad f(x, c) = (F_c(x), c),$$

donde F_c tine un conjunto hiperbólico Λ_c para $0 < c \leq c_0$ conjugado con el shift de Bernoulli sobre $\Sigma_{n(c)}$ (ver Teorema 4.16). Además, la familia de conjuntos $\{\Lambda_c\}_{0<c\leq c_0}$ satisface que $\Lambda_{c-\varepsilon}$ contiene a la continuación dinámica de Λ_c para cualquier $\varepsilon > 0$ suficientemente pequeño. Esta propiedad permite tomar como un subsistema de f a un skew-product de la forma $\Phi = \tau \times \mathrm{id}$ definido sobre $\Sigma_{n(\bar{c})} \times (0, \bar{c})$.

Para probar el teorema anterior, nosotros mostramos que, para algunos valores de los parámetros, la familia límite de un despliegue genérico de la singularidad nilpotente son campos de vectores Hamiltonianos con una órbita homoclínica bifocal no degenerada. Las perturbaciones sobre la hipersuperficie de órbitas homoclínicas bifocales tienen una aplicación de Poincaré conjugada con un producto cruzado simbólico el cual es una perturbación de $\Phi = \tau \times \mathrm{id}$. Como se sigue del tercer capítulo, perturbaciones genéricas de $\Phi = \tau \times \mathrm{id}$ del tipo productos cruzados de un paso, tienen o bien una región mezcladora o bien su dinámica es trivial. Por lo tanto, concluimos el cuarto capítulo estudiando la posible presencia de mezcladores suspendidos y ciclos heterodimensionales en despliegue genéricos de la singularidad nilpotente.

Introduction

The results contained in this thesis are a contribution to the study of non-uniformly hyperbolic dynamics, the context under which there arises many of the current issues concerning dynamical complexity. In contrast, uniformly hyperbolic dynamics began to be well understood in the sixties from the early works of Anosov [Ano67] and Smale [Sma67]. Three decades earlier, in an elaborate work [Bir35], Birkhoff proved that, in general, near a transversal homoclinic point, introduced by Poincaré [Poi90], there exists an intricate set of periodic orbits, most with a large period. In order to explain this Birkhoff's result and other subsequent results on the existence of infinitely many periodic orbits in the equation of Van der Pol [CL45, Lev49], Smale placed his geometric device, the Smale horseshoe map, in a neighborhood of a transversal homoclinic point. This application and the examples given by Anosov on the torus, are diffeomorphisms whose non-wandering set Ω is hyperbolic and coincides with the closure of the periodic points. Diffeomorphisms with these properties were called *Axiom A* or *uniform hyperbolic diffeomorphisms* and their study as a subset of the space $\mathrm{Diff}^r(M)$ of the C^r-diffeomorphisms over a compact manifold was proposed.

A key result for the study of Axiom A diffeomorphisms in $\mathrm{Diff}^r(M)$ was the Spectral Decomposition Theorem by Smale [Sma67]. According to this the non-wandering set Ω is decomposed into a disjoint union of finitely many subsets Λ_i called *basic sets*. Each Λ_i is an invariant isolated transitive hyperbolic compact set. Two periodic points in the same basic set have stable manifolds with the same dimension (*stability index*) and therefore unstable manifolds of the same dimension (*Morse index*). An eloquent and relevant example of a basic set is the invariant set Ω of a horseshoe map. The dynamics of the restriction of a horseshoe map to this set follows from its conjugation to the Bernoulli shift.

Horseshoe maps associated with a transversal homoclinic point are an important landmark in the study of dynamical systems. Its invariant set Ω gives an example of quasi-random dynamics. This randomness is consequence of the expansive character of the orbits in Ω that implies a high sensitivity dynamical to initial conditions. However, Ω is not an attractor because it has no basin of attraction with non-empty interior (or positive measure) and, therefore, its internal dynamics cannot be observable as an asymptotic behavior. This deficiency was solved by Smale who, by analogy with the horseshoe, constructed the solenoid as a first example of *strange attractor* (attractor with a dense expansive orbit) being hyperbolic. This example of non-periodic strange attractor inspired the famous work of Ruelle and Takens [RT71] on the nature of turbulence.

Having seen that uniformly hyperbolic diffeomorphisms provided new dynamical interpretations, two important issues raised in the context of bifurcation theory: the relation between uniform hyperbolicity and *structural stability*, and the density of uniform hyperbolic diffeomorphisms in the space $\mathrm{Diff}^r(M)$, endowed with the C^r-topology. After some partial results by Robbin [Rob71], Melo [Mel73] and Robinson [Rob74, Rob76], Mañé [Mañ88] proved that, as Palis and Smale conjectured in [PS70], a diffeomorphism $f \in \mathrm{Diff}^1(M)$ is structural stable if and only if it is uniformly hyperbolic and satisfies the strong transversality condition: all stable and unstable manifolds of points of the non-wandering set are transversal. Regarding to density , although the uniform hyperbolicity was firstly believed to envolve a residual, or at least dense, subset of $\mathrm{Diff}^r(M)$, it soon emerged that this was not true. There are two important configurations that force the persistence of non-uniform hyperbolicity: *heterodimensional cycles* and certain *homoclinic tangencies* (see both concepts in Definition 1.1). The first was used by Abraham and Smale [AS70] and Simon [Sim72] to construct examples of an open set of non-uniformly hyperbolic diffeomorphisms in $\mathrm{Diff}^1(M)$, with $\dim M \geq 3$. Homoclinic tangencies between the invariant manifolds of a saddle belong to a non-trivial basis set and are the basis of the well-known Newhouse phenomenon [New70, New74, New79] for C^2-diffeomorphisms on surface. For small perturbations of the diffeomorphism both configurations force the persistence of homoclinic tangencies, which imply the presence of non-wandering points with different stability indices and thus the persistence of non-uniform hyperbolicity. Therefore, $\mathrm{Diff}^r(M)$ is disjoint union of two sets, the uniformly hyperbolic diffeomorphisms and its complementary, which contain in turn open sets.

The set of uniformly hyperbolic diffeomorphisms contains the open set of those structurally stable and their dynamics is quite well understood. In contrast, the non-uniformly hyperbolic diffeomorphisms, persistent while abundant, are not structurally stable. Their dynamics must involve a great number of transitions and therefore it belongs to the word of dynamical complexity. This is the case with some of the most popular attractors. From the numerical study, the Lorenz attractor [Lor63] seems to be strange, persistent but not structurally stable, while the persistence seems to fail in the case of the Hénon attractor [Hén76]. Since the hyperbolic attractors are persistent and structurally stable, neither of them can be a hyperbolic attractors. The question is: do non-hyperbolic strange attractors really exist? The first analytic proof of the existence of such attractors was given by Benedicks and Carleson [BC91], who proved that in the Hénon family $H_{a,b}(x, y) = (1 - ax^2 + y, bx)$ there exist strange attractors for a set of parameter values of positive Lebesgue measure (persistent in the sense of measure) close enough to $a = 2$ and $b = 0$. Intricate ideas and techniques in [BC91] were used by Mora and Viana [MV93] to prove that, as conjectured by Palis, generic one-parameter families of diffeomorphisms on a surface unfolding a homoclinic tangency have strange attractors with positive probability in the parameter space. The existence of such attractors in three-dimensional families of vector fields was proven in [PR97] for the cross-section of a Shil'nikov homoclinic orbit [Shi65]. The proof of the existence of non-hyperbolic strange attractors starts in [BC91] considering the Hénon family as an unfolding of the limit family $h_a(x) = 1 - x^2$ which is obtained by taking $b = 0$. This quadratic family had been well studied previously in [BC85] and its expansive dynamics moves into the unstable manifold of a saddle point of $H_{a,b}(x, y) = (1 - x^2 + y, bx)$ when b is small enough. In [MV93] this strategy is then applied to a suitable renormalization of the return map to a neighborhood of a homoclinic point. The resulting family is still a good unfolding of the quadratic family (a Hénon-like family) and the ideas and techniques in [BC91] can be adapted to this case. In [PR97] it is proved that the family obtained after an appropriate renormalization is a good unfolding of a limit family, which in this case is an unimodal family $f_a(x) = \lambda^{-1} \log a + x + \lambda^{-1} \log \cos x$, and arguments in [BC91] remain valid.

The main goal in the study of dynamical systems is to describe the asymptotic behavior of trajectories of most systems. In the heat of the study of hyperbolic systems, Smale conjectured that the limit set of trajectories of a dynamical system should present a generic hyperbolic internal dynamics: exponential increase and decrease of the distances in complementary dimensions. However, non-hyperbolic attractors above provided counterexamples and raised the need for new proposals. From then until the present, investigation of non-uniformly hyperbolic dynamics has been mainly programmed by Palis [Pal00a, Pal08], who proposed a work program consisting of several interrelated conjectures aimed at describing the asymptotic behavior of generic families of dynamical systems depending on a finite number of parameters. In particular, Palis conjectured that, generically, there is only a finite number of transitive attractors which can accumulate almost all trajectories, in addition, these attractors should be stochastically stables and support a physical measure. Unlike the topological approach taken in the sixties, the probabilistic approach is now expressed in terms of the Lebesgue measure, both in the parameter space as in the phase space. From the Spectral Decomposition Theorem and the theory of Sinai-Ruelle-Bowen [Sin72, BR75, Rue76], it is proved that for uniformly hyperbolic C^2-diffeomorphisms while no cycles there exists at most a finite number of attractors, which are in turn stochastically stable and support a physical measure. Then one step further will seek some robust form of hyperbolicity (partial or dominated decomposition) that is present in the absence of cycles, which can prove the above conjecture. This raises a dichotomy between some set of hyperbolic diffeomorphisms and those with some kind of cycles. Namely, Palis conjectured that any dynamical system can be C^r-approximated by a hyperbolic system without cycles or one that has a homoclinic tangency or a heterodimensional cycle. A first answer to this last conjecture was given in the C^1-topology by Pujals and Sambarino in [PS00] for diffeomorphisms on surfaces. For higher dimension, Crovisier and Pujals [CP10] proved that every diffeomorphism $f \in \mathrm{Diff}^1(M)$ can be C^1-approximated by one that has a homoclinic tangency or a heterodimensional cycle, or by an *essentially hyperbolic diffeomorphism*, i.e. one with a finite number of transitive hyperbolic attractors such that the union of their basins of attraction is an open and dense set in the phase space. In short, homoclinic tangencies and heterodimensional cycles are a *complete obstruction to hyperbolicity.*

With regard to homoclinic tangencies, and already in the field of C^2-diffeomorphisms on surfaces, there have been remarkable results. As mentioned above, in generic families unfolding a homoclinic tangency between the invariant manifolds of an isolated hyperbolic periodic point there appear non-hyperbolic strange attractors, persistent in the sense of the measure. When the periodic point belongs to a nontrivial basic set, the persistence of homoclinic tangencies for an open set $\mathcal{U}$ of diffeomorphisms was originally detected in [New70]. On a residual set in $\mathcal{U}$, of measure zero, there are simultaneously infinite periodic attractors [New74] and even infinite Hénon-like strange attractors [Col98]. These results can be generalized to greater dimension [PV94, Via93, Lea08]. The geometric ingredient underlying the persistence of homoclinic tangencies is the horseshoe map. Namely, the thickness of the stable and unstable foliations of a basic set Λ, which extend in a neighborhood of the homoclinic point to define, respectively, two Cantor sets K_s and K_u on a given segment. The prevalence of hyperbolicity or non-hyperbolicity depends on whether the Hausdorff dimension $\mathrm{HD}(\Lambda) = \mathrm{HD}(K_s) + \mathrm{HD}(K_u)$ of Λ is less than or greater than one [PY94, MPV01].

With regard to heterodimensional cycles, an early result by Díaz [Día95] implies the existence of a non-empty open set of parameterized C^∞ families of diffeomorphisms $(f_t)_{t\in[-1,1]}$ unfolding generically a heterodimensional cycle of f_0 and such that for all small positive $t > 0$ the corresponding diffeomorphism f_t is non-uniformly hyperbolic: the homoclinic classes of two saddle points of different indices coincide. The proof of this result can be illustrated with the choice of a diffeomorphism on $\mathbb{R}^3$ which has a heterodimensional cycle between two fixed points P and Q. Under additional hypothesis about partial hyperbolicity, linearization and product structure, it is understood that the behavior of this diffeomorphism in a neighborhood of the cycle follows from the dynamics of an iterated system of two real functions (see a precise definition of *iterated function system* in Section §3.1). This reduction indicates that the persistence of non-uniform hyperbolicity associated with heterodimensional cycles is of different nature from that which follows from the homoclinic bifurcations. In the Newhouse's phenomenon (persistence of homoclinic tangencies) the essential geometric ingredient was the horseshoe map and the Hausdorff dimension of its basic set but, which geometric element lies under the persistence of non-uniform hyperbolicity by perturbing a heterodimensional cycle? The answer was given by Bonatti and Díaz introducing in [BD96] the notion of *blender*. Roughly speaking, a blender can be understood as a sufficiently thick hyperbolic set Γ such that the closure of an invariant manifold of dimension u of a saddle point in Γ contains an invariant manifold of dimension $u+1$. The first precise definition of a blender emphasizing its geometrical aspects can be found in [BDV05]:

Definition (Blenders). *Let f be a C^1-diffeomorphism of a compact manifold M and $\Gamma \subset M$ a transitive hyperbolic set of f with a dominated splitting of the form $E^{ss} \oplus E^{cs} \oplus E^u$, where its stable bundle $E^s = E^{ss} \oplus E^{cs}$ has dimension equal to $s \geq 2$ and E^{cs} is one-dimensional. The set Γ is a* cs-blender *if it has a C^1*-robust superposition region *$\mathcal{B}$:*

there are a C^1-neighborhood $\mathcal{V}$ of f and a C^1-open set $\mathcal{B}$ of embeddings of $s-1$ dimensional disks D^s into M such that for every diffeomorphism $g \in \mathcal{V}$, every disk $D^s \in \mathcal{B}$ intersects the local unstable manifold $W^u_{loc}(\Gamma_g)$ of the continuation Γ_g of Γ for g.

A cu-blender *for f is defined as a cs-blender for f^{-1}.*

Blenders are the subjacent mechanism leading to the generation of robust heterodimensional cycles [BD08] and robust homoclinic tangencies in the C^1-topology for a manifold of dimension greater than or equal to three [BD11]. Blenders were also used in other applications such as the construction of robust non-hyperbolic transitive diffeomorphisms [BD96], the discontinuity of the dimension of hyperbolic sets [BDV95] and to obtain results about stable ergodicity [RHTU07].

A blender Γ for a C^1-diffeomorphism f is just a hyperbolic set, but its existence presages the presence of persistent non-uniform hyperbolicity. By way of example, we will sketch how these sets allow us to mix saddle points of different indices. Suppose we are in dimension three, Γ is a cs-blender with stability index $s = 2$ and $P \in \Gamma$, $Q \notin \Gamma$ are two periodic points of f, the first with unstable manifold dense in Γ and the second with stability index $s-1 = 1$. We assume that the stable manifold $W^s(Q)$ of Q contains a disk D^s in the superposition region $\mathcal{B}$ of the blender Γ. From the Inclination Lemma, the sequence of backward iterates of any disk L of dimension $s-1 = 1$ crossing $W^u(Q)$ converge to $W^s(Q)$. Therefore, for $n \geq 0$ large enough $f^{-n}(L)$ contains a disk in $\mathcal{B}$. Since Γ is a cs-blender, $W^u_{loc}(\Gamma)$ intersects $f^{-n}(L)$ and so because of the density of the unstable manifold of P it follows that

$$W^u(Q) \subset \overline{W^u(P)}.$$

That is, the closure of the unstable manifold of P (of dimension one) contains the unstable manifold (two dimensional) of Q. Therefore we have in some sense (say, topological) that dimension of the unstable manifold of P is equal to dimension of unstable manifold of Q (i.e. two). Consequently, the dimension of the unstable manifold of P is increased in a unit, so for practical purposes, we obtain that P has invariant manifolds of dimension two.

The above construction allows us to see that all points in a transversal intersection $\gamma = W^s(P) \pitchfork W^u(Q)$ (generically an union of curves) belong to both, the homoclinic clase of P and the homoclinic clase of Q. Therefore, γ is contained in the non-wandering set of f. Obviously the points of γ do not admit a hyperbolic splitting and it follows that f cannot be uniformly hyperbolic. The persistence of the blender makes these arguments robust under C^1-perturbations and then we get an open set of non-uniformly hyperbolic diffeomorphisms.

Note that the notion of blender is formulated in the general context of C^1-diffeomorphisms. Already in this context it is proved in [DR02] that if f is a C^1-diffeomorphism with a heterodimensional cycle associated with two saddle points P and Q with indexes s and $s+1$ (i.e. co-index one) and C^1 away from homoclinic tangencies, then f belongs to the closure of an open $\mathcal{U}$ of non-uniformly hyperbolic diffeomorphisms. Later, in [BD08] it is proved that any heterodimensional cycle of co-index one can be C^1-approximated by diffeomorphisms having a C^1-robust heterodimensional cycle. One of the steps to prove this result was to show that *horseshoes of blender type* occur naturally in an unfolding of a heterodimensional cycle of co-index one.

The most eloquent geometric representation of a blender and its relation to the horseshoe map follows from the construction of the so-called *blender-horseshoe* [BD11]. This construction involves a diffeomorphism f defined in a reference cube $C = [-1,1]^{n+1}$, with $n \geq 2$, as a *skew-product* of the form

$$f : C \subset \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}, \quad f(x,y) = (F(x), \phi(x,y)),$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ has a Smale horseshoe $\Lambda \subset [-1,1]^n$ and the maps $\phi(x,\cdot) : [-1,1] \to [-1,1]$ are C^1-contractions. Clearly, if every application $\phi(x,\cdot)$ is the same contraction ϕ, then the map $f|_C$ is essentially a normally embedded horseshoe map with maximal invariant set $\Gamma = \Lambda \times \{y_0\}$, where $y_0 \in [-1,1]$ is the fixed point of ϕ. Suppose now that

$$H_1 \cup H_2 = F^{-1}([-1,1]^n) \cap [-1,1]^n$$

is the union of two horizontal strings in the definition of the Smale horseshoe and that $\phi(x,\cdot) = \phi_i$ if $x \in H_i$, with $i = 1,2$, are two different contractions, with fixed points $y_1 < y_2$, respectively. The diffeomorphisms defined of this way are called *locally constant skew-product diffeomorphisms*. Hence, in that case, $\Gamma \subset \Lambda \times [y_1, y_2]$ is again a transitive hyperbolic set such that $f|_\Gamma$ is topologically conjugated to $F|_\Lambda$. The projection of Γ on the interval $[y_1, y_2]$ is given by the dynamics of the iterated function system generated by the contractions ϕ_1, ϕ_2. If there exists an open set $B \subset (y_1, y_2)$ such that $\overline{B} \subset \phi_1(B) \cup \phi_2(B)$, it is proved that this projection contains B. It is easy to verify that this open set B is still contained in the projection on the real line of the continuation Γ_g of Γ for every locally constant skew-product diffeomorphism g close to f in the C^1-topology. Such inclusion is sufficient to show that for all $(x,y) \in (H_1 \cup H_2) \times B$, the local unstable manifold $W^u_{loc}(\Gamma_g)$ of Γ_g intersects the local strong stable manifold of (x,y) for g. Notice that these local strong stable manifolds are an open set of embedded disks and thus, it follows that Γ satisfies the definition of blender for all locally constant skew-product diffeomorphisms C^1-close to f. This persistence is the main obstacle to prove the existence of a blender because any diffeomorphism g sufficiently C^1-close to f is not necessarily a skew-product, much less, a locally constant skew-product diffeomorphism. This difficulty is resolved by following the normal hyperbolic result in [HPS77] since with additional assumptions concerning the strength of the hyperbolic splitting for $f|_C$ we conclude that g is topologically conjugate to a skew-product. Thus, in that case, when we try to prove the persistence of the intersection condition it will be sufficient to consider C^1-perturbations of f in the class of skew-product diffeomorphisms. Moreover, since $F|_\Lambda$ is conjugated to a Bernoulli shift $\tau : \Sigma_2 \to \Sigma_2$ of two symbols, the blender-horseshoe can be studied from a symbolic point of view, taking skew-products maps of the form

$$\Phi : \Sigma_2 \times \mathbb{R} \to \Sigma_2 \times \mathbb{R}, \quad \Phi(\xi, x) = (\tau(\xi), \phi_\xi(x)),$$

where each $\phi_\xi : \mathbb{R} \to \mathbb{R}$ is a C^1-contraction.

The notion of *cs*-blender is associated with C^1-diffeomorphisms having a partially hyperbolic dominated splitting $E^{ss} \oplus E^{cs} \oplus E^u$ and it has always been defined and handled in the case where the central bundle E^c is unidimensional. This represents an obstacle in the contexts where central manifolds of dimension $c \geq 2$ are raised. Therefore, a natural question is to handle and to construct blenders whose central bundle is not necessarily one-dimensional. Following the proposal of Nassiri and Pujals [NP12] a way to do this is to consider the symbolic dynamics in the context of the *symbolic skew-products*

$$\Phi : \Sigma_k \times \mathbb{R}^c \to \Sigma_k \times \mathbb{R}^c, \quad \Phi(\xi, x) = (\tau(\xi), \phi_\xi(x)),$$

where $\Sigma_k = \{1, \dots, k\}^{\mathbb{Z}}$, $c \geq 1$, and each $\phi_\xi : \mathbb{R}^c \to \mathbb{R}^c$ is a new C^1-contraction. A blender in this context will be called *symbolic blender-horseshoe* or shortly *symbolic blender*. The first objective of our work is to give conditions for the existence of symbolic blender in this general setting. These conditions will be used to study the genesis of blenders in perturbations of $\Phi(\xi, x) = (\tau(\xi), x)$ and so to explain the presence of *suspended blenders* for vector fields in $\mathbb{R}^4$ arbitrarily close to a Hamiltonian vector field X_H with a non-degenerate bifocal homoclinic orbit, where a three-dimensional diffeomorphism can be defined as the return map on a cross section. A *bifocal homoclinic orbit* of a vector field on $\mathbb{R}^4$ is a homoclinic connection with a focus-focus equilibrium, i.e., having eigenvalues $-\rho_1 \pm i\omega_1$, $\rho_2 \pm i\omega_2$. Since in every generic unfolding of the nilpotent singularity of codimension four in $\mathbb{R}^4$ there are bifocal homoclinic orbits [BIR11], which are unfolded as continuation of similar homoclinic connections for Hamiltonian vector fields in the limit families, we finally show how suspended blenders could appear in these unfoldings.

This thesis is organized into four self-contained chapters. Let us summarize the main results in each chapter.

I – Robust cycles and blenders – In the first chapter of the thesis we introduce in detail preliminary concepts, assumptions and examples which have been mentioned throughout this introduction. The objective is to introduce the notions of blender and blender-horseshoe already present in the literature in a self-contained form.

II – Symbolic blenders – The second chapter of the thesis focuses on the study of the existence of symbolic blenders. The main results of this chapter are in collaboration with Yuri Ki and Artem Raibekas and are collected in prepublication [BKR12].

This chapter is developed in the context of *symbolic skew-products*

$$\Phi : \Sigma_k \times M \to \Sigma_k \times M, \qquad \Phi(\xi, x) = (\tau(\xi), \phi_\xi(x))$$

where M is a compact Riemannian manifold of dimension $c \geq 1$ and $\phi_\xi : M \to M$ is a C^r-diffeomorphism with $r \geq 0$ for all $\xi \in \Sigma_k$, which depends continuously with respect to ξ. The first factor of the product $\Sigma_k \times M$ is called base and the second fiber. To emphasize the role of fiber diffeomorphisms ϕ_ξ we use the notation $\Phi = \tau \ltimes \phi_\xi$. This set of symbolic skew-products will be denoted by $\mathcal{S}_k(M)$. When ϕ_ξ only depends on the coordinate ξ_0 of the bisequence $\xi = (\xi_i)_{i\in\mathbb{Z}}$, we say that Φ is a *one-step skew-product* and, in that case, we write $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k)$ where $\phi_\xi = \phi_i$ if $\xi_0 = i$. The set of one-step skew-products maps is denoted by $\mathcal{Q}_k(M)$.

Working with symbolic skew-products $\mathcal{S}_k(M)$ is a good idea to study the existence of blenders in skew-product diffeomorphisms of the form

$$f : N \times M \to N \times M, \qquad f(x, y) = (F(x), \phi(x, y)),$$

where F is a diffeomorphism of a manifold N with a horseshoe $\Lambda \subset N$. As already mentioned, C^1-perturbations of these diffeomorphisms are not necessarily skew-products. In order to ensure that perturbations of f are conjugated to symbolic skew-products we need to impose on f partial hyperbolicity and dominated conditions which are open conditions. Then, according to recent work [Gor06, IN10, PSW11], see Proposition 2.1, there exist $\varepsilon > 0$ and a constant $\alpha \in (0, 1]$, only depending on the rate of contraction $\nu \in (0, 1)$ of the horseshoe Λ, such that any ε-perturbation g of f in the C^1-topology has a locally maximal invariant set Δ_g isomorphic to $\Lambda \times M$, so that $g|_{\Delta_g}$ is topologically conjugated to a symbolic skew-product $\Phi = \tau \ltimes \phi_\xi$ in the subset $\mathcal{PHS}^{1,\alpha}_k(M)$ of $\mathcal{S}_k(M)$ consisting of *locally Hölder continuous* and *partially hyperbolic* symbolic skew-products. This set is defined by imposing to $\Phi = \tau \ltimes \phi_\xi$ extra conditions of regularity, Lipschitz character and domination:

- $\phi_\xi : M \to M$ is C^1-diffeomorphisms for each ξ.
- ϕ_ξ depends *locally α-Hölder continuously* on ξ in M: there exists $C \geq 0$ such that

$$d_{C^0}(\phi^{\pm 1}_\xi, \phi^{\pm 1}_{\xi'}) \leq C d_{\Sigma_k}(\xi, \xi')^\alpha, \quad \text{for all } \xi, \xi' \in \Sigma_k \text{ with } \xi_0 = \xi'_0. \tag{2}$$

  The space of symbols $\Sigma_k = \{1, \dots, k\}^{\mathbb{Z}}$ is endowed with the distance

$$d_{\Sigma_k}(\xi, \xi') \stackrel{\text{def}}{=} \nu^\ell, \quad \ell = \min\{i \in \mathbb{Z}^+ : \xi_i \neq \xi'_i \text{ or } \xi_{-i} \neq \xi'_{-i}\}.$$

  We will denote by C_Φ the smallest non-negative constant satisfying (2).

- ϕ_ξ is *biLipschitz* and *partially dominates*: There exist positive constants γ and $\hat\gamma$ such that
  - *s-domination* and *u-domination* (partial hyperbolicity): $\nu^\alpha < \gamma < 1 < \hat\gamma^{-1} < \nu^{-\alpha}$
  - $(\gamma, \hat\gamma^{-1})$-Lipschitz in M:

$$\gamma \|x - x'\| < \|\phi_\xi(x) - \phi_\xi(x')\| < \hat\gamma^{-1} \|x - x'\|,$$

    for all $\xi \in \Sigma_k$ and x, $x' \in M$. Here $\|x - x'\|$ denotes the distance in M.

A blender-horseshoe is a locally maximal hyperbolic set and therefore it will be linked to a bounded and open set $D \subset M$. That is, it will be the locally maximal invariant set for Φ in $\Sigma_k \times \overline{D}$. Thus, we can impose additional local conditions for the symbolic skew-products $\Phi = \tau \ltimes \phi_\xi$ with which we work. For instance, we assume that the restriction of ϕ_ξ to the set $\overline{D}$ is a contraction or expansion. Namely, we will work with the following sets of symbolic skew-products:

Definition (Sets of symbolic skew products). *Let $D \subset M$ be a bounded open set and consider constants $0 < \lambda < \beta$ and $0 \leq \alpha \leq 1$. We define $\mathcal{S}^{r,\alpha}_{k,\lambda,\beta}(D)$, $r \geq 0$, as the set of symbolic skew-product maps $\Phi = \tau \ltimes \phi_\xi \in \mathcal{S}_k(M)$ such that*

- *ϕ_ξ is a C^r-(λ, β)-Lipschitz on $\overline{D}$ for all ξ in Σ_k, and*
- *ϕ_ξ depends locally α-Hölder continuously on $\overline{D}$ with respect to ξ.*

Additionally, if $\beta < 1$ we impose the condition $\phi_\xi(\overline{D}) \subset D$ for all $\xi \in \Sigma_k$, and, in the case $1 < \lambda$ the imposed condition is $\overline{D} \subset \phi_\xi(D)$ for all $\xi \in \Sigma_k$. We endow $\mathcal{S}^{r,\alpha}_{k,\lambda,\beta}(D)$ with the distance

$$d_{\mathcal{S}}(\Phi, \Psi) = \sup_{\xi \in \Sigma_k} d_{C^r}(\phi_\xi, \psi_\xi) + |C_\Phi - C_\Psi|, \quad \text{with } \ \Phi = \tau \ltimes \phi_\xi \ \text{ and } \ \Psi = \tau \ltimes \psi_\xi.$$

For notational convenience, $\mathcal{S}_{k,\lambda,\beta}(D)$ and $\mathcal{S}^{\alpha}_{k,\lambda,\beta}(D)$ denote $\mathcal{S}^{0,0}_{k,\lambda,\beta}(D)$ and $\mathcal{S}^{0,\alpha}_{k,\lambda,\beta}(D)$, respectively.

Under the hypothesis of contractive fiber maps, the following result provides a description of the maximal invariant of Φ in $\Sigma_k \times \overline{D}$ and shows the dependence on Φ of the set

$$K_\Phi \overset{\text{def}}{=} \overline{\mathscr{P}(\mathrm{Per}(\Phi)) \cap D}$$

where $\mathrm{Per}(\Phi)$ is the set of periodic points of Φ and $\mathscr{P} : \Sigma_k \times M \to M$ is the standard projection on M. Although this theorem is a special case of the results of [HPS77] it will be very useful to have a complete and detailed proof in the context of symbolic skew-products. It is denoted by $\mathcal{K}(\overline{D})$ the collection of compact subsets of $\overline{D}$ endowed with the Hausdorff metric and

$$W^u((\xi, x); \Phi) \overset{\text{def}}{=} \{(\zeta, y) \in \Sigma_k \times M : \lim_{n\to\infty} d(\Phi^{-n}(\zeta, y), \Phi^{-n}(\xi, x)) = 0\}$$

is the unstable set of (ξ, x) for Φ.

Theorem A (Geometry of the maximal invariant set). *Consider $\Phi \in \mathcal{S}^{\alpha}_{k,\lambda,\beta}(D)$ with $\beta < 1$ and $\alpha > 0$. Then the restriction of Φ to the set*

$$\Gamma_\Phi = \bigcap_{n \in \mathbb{Z}} \Phi^n\big(\Sigma_k \times \overline{D}\big) = \bigcap_{n \in \mathbb{N}} \Phi^n\big(\Sigma_k \times \overline{D}\big)$$

is conjugated to the full shift τ of k symbols. Moreover, $W^u((\xi, x); \Phi) \subset \Gamma_\Phi$ for all $(\xi, x) \in \Gamma_\Phi$ and there exists a unique continuous function $g_\Phi : \Sigma_k \to \overline{D}$ such that for every periodic point (ϑ, p) of Φ in $\Sigma_k \times D$ it holds that,

$$\Gamma_\Phi = \overline{W^u((\vartheta, p); \Phi))} = \{(\xi, g_\Phi(\xi)) : \xi \in \Sigma_k\} \quad \text{and} \quad \mathscr{P}(\Gamma_\Phi) = K_\Phi \in \mathcal{K}(D).$$

Finally, the map $\mathscr{L} : \mathcal{S}^{\alpha}_{k,\lambda,\beta}(D) \to \mathcal{K}(\overline{D})$ given by $\mathscr{L}(\Phi) = K_\Phi$ is continuous.

In order to introduce a symbolic blender, firstly we define a family of almost horizontal disks that provides the superposition region of the blender.

Definition (Almost horizontal disks)**.** *For a fixed $\alpha > 0$ and given an open subset $B \subset D$, we say that D^s is a* δ-horizontal disk *in $\Sigma_k \times B$ if there exist $\zeta \in \Sigma_k$, $z \in B$, a positive constant $C \geq 0$ and a (α, C)-Hölder continuous map $h : W^s_{loc}(\zeta;\tau) \to B$ such that*

$$D^s = \{(\xi, h(\xi)) : \xi \in W^s_{loc}(\zeta;\tau)\}, \ \|z - h(\xi)\| < \delta \ \textit{for all}\ \xi \in W^s_{loc}(\zeta;\tau) \ \textit{and}\ C\nu^\alpha < \delta.$$

Here $W^s_{loc}(\zeta;\tau) = \{\xi \in \Sigma_k : \xi_i = \zeta_i \text{ for all } i \geq 0\}$ denotes the stable set of $\zeta \in \Sigma_k$.

The main reason for considering the set of embedded disks in the definition of blender is that the local strong stable manifolds will be part of this set. Observe that for any $\delta > 0$, the set $W^s_{loc}(\zeta;\tau) \times \{x\}$ is a δ-horizontal disk and, in the case of one-step maps, coincides with the local strong stable set of (ζ, x). Since we want to study Hölder perturbations of a one-step map, it is enough to consider as superposition region the family of almost horizontal disks.

From Theorem A, it follows $W^u(\Gamma_\Phi) = \Gamma_\Phi$ for all $\Phi \in \mathcal{S}^\alpha_{k,\lambda,\beta}(D)$ with $\beta < 1$, where

$$W^u(\Gamma_\Phi) \stackrel{\text{def}}{=} \big\{(\xi, x) \in \Sigma_k \times M : \lim_{n\to\infty} d(\Phi^{-n}(\xi, x), \Gamma_\Phi) = 0\big\}$$

is the unstable set with respect to the maximal invariant set Γ_Φ. Hence, the corresponding definition of *cs*-blender in the context of symbolic skew products can be written as follows:

Definition (Symbolic *cs*-blender-horseshoes)**.** *Let $\Phi \in \mathcal{S}^\alpha_{k,\lambda,\beta}(D)$ with $\beta < 1$ and $\alpha > 0$.*

The maximal invariant set Γ_Φ of Φ in $\Sigma_k \times \overline{D}$ is said to be symbolic *cs*-blender-horseshoe *if there exist $\delta > 0$, a non-empty open set $B \subset D$ and a neighborhood $\mathcal{V}$ of Φ in $\mathcal{S}^\alpha_{k,\lambda,\beta}(D)$ such that for every $\Psi \in \mathcal{V}$ and for any δ-horizontal disk D^s in $\Sigma_k \times B$, it holds that*

$$\Gamma_\Psi \cap D^s \neq \emptyset, \quad \textit{where } \Gamma_\Psi \textit{ is the continuation of } \Gamma_\Phi \textit{ for } \Psi.$$

The open set B is called superposition region *of the symbolic cs-blender-horseshoe.*

To define symbolic *cu*-blenders-horseshoes, firstly we need to introduce the associated inverse symbolic skew product for $\Phi = \tau \ltimes \phi_\xi$. Given $\Phi = \tau \ltimes \phi_\xi \in \mathcal{S}^\alpha_{k,\lambda,\beta}(D)$, the symbolic skew product

$$\Phi^* = \tau \ltimes \phi^*_\xi \in \mathcal{S}^\alpha_{k,\beta^{-1},\lambda^{-1}}(D), \quad \text{where } \phi^*_\xi : M \to M \text{ given by } \phi^*_\xi(x) = \phi^{-1}_{\tau^{-1}(\xi^*)}(x),$$

is called *associated inverse skew product for* Φ. Here $\xi^* = (\dots \xi_1; \xi_0, \xi_{-1}, \dots)$ denotes the conjugate sequence of $\xi = (\dots \xi_{-1}; \xi_0, \xi_1, \dots)$. Note that since $\tau(\xi)^* = \tau^{-1}(\xi^*)$ the iterates of Φ^* correspond with iterates of Φ^{-1}. This observation allows us to define symbolic *cu*-blender-horseshoes for symbolic skew products in $\mathcal{S}^\alpha_{k,\lambda,\beta}(D)$ with $\lambda > 1$ and $\alpha > 0$. Namely, a *symbolic cu-blender-horseshoe* for Φ is defined as a symbolic *cs*-blender-horseshoe for Φ^*. In what follows, we only consider symbolic *cs*-blenders.

From Theorem A, it follows that $\overline{W^{uu}((\vartheta,p);\Phi)} = \Gamma_\Phi$ for every periodic point $(\vartheta, p) \in \Sigma_k \times D$ of a partially hyperbolic symbolic skew-product Φ. In Proposition 2.5, we will prove that each local strong stable set $W^{ss}_{loc}((\xi,x);\Phi)$ is an almost horizontal disk. Hence, if Γ_Φ is close enough to a one-step map then this disk is a small Hölder perturbation of the horizontal disk $W^s_{loc}(\xi;\tau) \times \{x\}$. Therefore, if moreover Γ is a symbolic *cs*-blender for Φ with superposition region B, then it holds

$$\overline{W^u((\vartheta,p_\Psi);\Psi)} \cap W^{ss}_{loc}((\xi,x);\Psi) \neq \emptyset, \quad \text{for all } (\xi,x) \in \Sigma_k \times B$$

and every $\mathcal{S}^\alpha$-perturbation Ψ of Φ where (ϑ, p_Ψ) is the continuation of (ϑ, p) for Ψ.

A set of maps $\phi_1,\dots,\phi_k$ defined on $\overline{D}$ is said to have the *covering property* if there is an open set $B\subset D$ such that $\overline{B}\subset\phi_1(B)\cup\cdots\cup\phi_k(B)$. One of the objectives is to understand how to translate robust dynamical properties of an iterated function system generated by $\phi_1,\dots,\phi_k$ to robust dynamical properties of $\Phi=\tau\ltimes(\phi_1,\dots,\phi_k)$ under $\mathcal{S}^\alpha$-perturbations. The following result describes how the covering property translates to a robust property in the language of Hölder symbolic skew products.

Theorem B (Covering property characterization)**.** *Consider* $\Phi=\tau\ltimes(\phi_1,\dots,\phi_k)\in\mathcal{S}^\alpha_{k,\lambda,\beta}(D)$ *with* $\nu^\alpha<\lambda<1$, $\alpha>0$ *and let* $B\subset D$ *be an open set. Then,*

$$\overline{B}\subset\phi_1(B)\cup\cdots\cup\phi_k(B)$$

if and only if there are $\delta>0$ *and a neighborhood* $\mathcal{V}$ *of* Φ *in* $\mathcal{S}^\alpha_{k,\lambda,\beta}(D)$ *such that for every* $\Psi\in\mathcal{V}$

$$\Gamma^+_\Psi(B)\cap D^s\neq\emptyset\quad\text{for all }\delta\text{-horizontal disk }D^s\text{ in }\Sigma_k\times B$$

where $\Gamma^+_\Psi(B)$ *is the forward maximal invariant set of* Ψ *in* $\Sigma_k\times B$*.*

Under the hypothesis $\beta<1$, if $\Phi=\tau\ltimes(\phi_1,\dots,\phi_k)\in\mathcal{S}^\alpha_{k,\lambda,\beta}(D)$ then $\phi_i(\overline{D})\subset D$ for all $i=1,\dots,k$. In such case, for any small perturbation $\Psi=\tau\ltimes\psi_\xi$ of Φ it holds that $\psi_\xi(\overline{D})\subset D$ and it follows that

$$\Gamma^+_\Psi(B)\overset{\text{def}}{=}\bigcap_{n\geq 0}\Psi^n(\Sigma_k\times B)\subset\bigcap_{n\in\mathbb{Z}}\Psi^n(\Sigma_k\times\overline{D})\overset{\text{def}}{=}\Gamma_\Phi.$$

Therefore, combining the above result with the definition of symbolic blender we obtain the following consequence on the existence of symbolic blenders using the covering property.

Theorem C (Symbolic blender existence)**.** *Consider* $\Phi=\tau\ltimes(\phi_1,\dots,\phi_k)\in\mathcal{S}^\alpha_{k,\lambda,\beta}(D)$ *with* $\nu^\alpha<\lambda<\beta<1$, $\alpha>0$*. Assume that there exists an open set* $B\subset D$ *such that*

$$\overline{B}\subset\phi_1(B)\cup\cdots\cup\phi_k(B)$$

Then the maximal invariant set Γ_Φ *of* Φ *in* $\Sigma_k\times\overline{D}$ *is a symbolic cs-blender-horseshoe for* Φ *whose superposition region contains* B*.*

Part of this second chapter is devoted to the study of a subset of symbolic skew-products $\mathcal{S}^+_k(M)$ called *symbolic unilateral skew-products*, which generalizes the one-step maps. This set consists of the maps $\Phi=\tau\ltimes\phi_\xi$ such that $\phi_\xi=\phi_{\xi'}$ if $\xi_i=\xi'_i$ for all $i\geq 0$. In Proposition 2.3 it is showed that if there exists a *stable holonomy* (see Definition 2.3) for $\Phi\in\mathcal{S}_k(M)$ then Φ is topologically conjugated to a unilateral symbolic skew-product $\tilde{\Phi}\in\mathcal{S}^+_k(M)$. This would allow us to restrict the set of perturbations of skew-products that must be considered and thus introduce another definition (in the unilateral setting) of symbolic blender-horseshoe (see Definition 2.11). Namely, in §2.4 perturbations in the set $\mathcal{S}^+_{k,\lambda,\beta}(D)=\mathcal{S}^+_k(M)\cap\mathcal{S}_{k,\lambda,\beta}(D)$ with $\beta<1$ are studied.

In Proposition 2.5, it is proved the existence of stable holonomy for every s-dominated symbolic skew-products, in particular, for $\Phi=\tau\ltimes\phi_\xi\in\mathcal{PHS}^{1,\alpha}_k(M)\cap\mathcal{S}_{k,\lambda,\beta}(D)$. The corresponding conjugated unilateral skew-product is given by

$$\tilde{\Phi}=\tau\ltimes\tilde{\phi}_\xi\in\mathcal{S}^+_k(M)\quad\text{where}\quad\tilde{\phi}_\xi=h^s_{\tau(\xi),\pi(\tau(\xi))}\circ\phi_\xi\circ h^s_{\pi(\xi),\xi}$$

with π denoting the projection of Σ_k on a transversal section Σ to the stable partition $W^s_{loc}(\xi;\tau)$, $\xi \in \Sigma_k$ and $h^s_{\xi,\xi'} : M \to M$ the family of maps that define the stable holonomy. In Proposition 2.6, it is proved that each $h^s_{\xi,\xi'}$ is a Hölder continuous map with uniform Hölder constant for all ξ and ξ'; but this is not sufficient to conclude that $\tilde{\Phi} \in \mathcal{S}^+_{k,\lambda,\beta}(D)$. To ensure that $\tilde{\Phi} \in \mathcal{S}_{k,\lambda,\beta}(D)$ we need to increase the regularity of Φ and impose additional conditions on fiber maps. Namely, Proposition 2.9 proves that the maps $h^s_{\xi,\xi'}$ are C^1-difeomorfismos if $\Phi = \tau \ltimes \phi_\xi$ is *fiber bunched* (ver Definición 2.7) and belongs to the set $\mathcal{PHS}^{2,1+\alpha}_k(M)$ of the parcial hyperbolic skew-products whose fiber maps are C^2-diffeomorphisms that depend *locally Hölder differenciatiable* with respect to the base point, that is,

$$d_{C^1}(\phi^{\pm 1}_\xi, \phi^{\pm 1}_{\xi'}) \leq C d_{\Sigma_k}(\xi,\xi')^\alpha, \quad \text{for all } \xi,\xi' \in \Sigma_k \text{ with } \xi_0 = \xi'_0.$$

This regularity in the holonomy implies that $\tilde{\Phi} = \tau \ltimes \tilde{\phi}_\xi$ belongs to $\mathcal{S}^+_{k,\lambda,\beta}(D)$. According to [Gor06], see Theorem 2.2, it follows that these additional regularity and fiber bunching conditions can be obtained for the symbolic skew-product $\Psi = \tau \ltimes \psi_\xi$ conjugated to a C^2-perturbation g of the C^2-diffeomorphism $f = F \times \mathrm{id}$ where $F : N \to N$ is a horseshoe map and $\mathrm{id} : M \to M$ is the identity map. In this way, a symbolic blender-horseshoe in the unilateral setting leads a blender for a C^2-diffeomorphism with a C^2-robust superposition region.

The above results about the existence of symbolic blenders are given for one-step symbolic skew-products $\Phi = \tau \ltimes (\phi_1, \ldots, \phi_k)$. Apart from the assumption of regularity and domination imposed to restrict the space of perturbations, the condition of existence of symbolic blender is reduced to the covering property, which is formulated in terms of the contractions $\phi_1, \ldots, \phi_k$. This allows us to consider the structure of a symbolic blender as something specific to the one-step skew-products that persists under good perturbations. By drawing a parallel with the proof of the existence of Hénon-like strange attractors, one-step skew-products could be considered as the limit maps whose dynamics must be understood, just as one needs to understand the dynamics of the limit family $h_a(x) = 1 - x^2$ to understand the existence of Hénon attractors in [BC91]. From this view, Section §2.3 is introduced to study symbolic blenders in the one-step setting, that is, only considering perturbations in the set $\mathcal{Q}_{k,\lambda,\beta}(D)$. It is showed how the dynamics of a one-step map $\Phi = \tau \ltimes (\phi_1, \ldots, \phi_k)$ is given by the dynamics of the iterated function system generated by $\phi_1, \ldots, \phi_k$. In this way the concept of blender emerges as a property of this iteration function system dynamics. This section is a prelude to the next section of the PhD dissertation.

III – Iterated function systems – The third chapter of thesis focuses on the study of iterated function systems, defined on both, an interval or a circle. The main results in this chapter are in collaboration with Artem Raibekas and are collected in the PhD theses [Rai11] and in the prepublication [BR].

By an *iterated function system*, shortly IFS from now on, generated by a family of diffeomorphism $\Phi = \{\phi_1, \ldots, \phi_k\}$ on a manifold M, we mean the set $\mathrm{IFS}(\phi_1, \ldots, \phi_k)$ of all possible compositions of diffeomorphism $\phi_i \in \Phi$ (including the identity map id). That is, the semigroup with identity (a monoid) generated by the compositions of $\phi_1, \ldots, \phi_k$. Because of the close relation between one-step symbolic skew-products and iterated function systems we write $\mathrm{IFS}(\Phi) = \mathrm{IFS}(\phi_1, \ldots, \phi_k)$, meaning that the IFS is generated by the family $\Phi = \{\phi_1, \ldots, \phi_k\}$ associated with the one-step $\Phi = \tau \ltimes (\phi_1, \ldots, \phi_k)$ defined on $\Sigma_k \times M$.

As already mentioned, the dynamics of a one-step skew-product is given by the dynamics of its associated iterated function system. In order to talk about the dynamics of an IFS it is necessary to introduce the basic notion of orbit. The *orbit of* $x \in M$ *for* IFS(Φ) is the action of the IFS over the point x, i.e.,

$$\mathrm{Orb}_\Phi(x) \stackrel{\text{def}}{=} \{h(x) : h \in \mathrm{IFS}(\Phi)\} \subset M.$$

With this notion of orbit, some dynamical concepts known for dynamical system are translated to the field of the iterated function systems. As an example, a set $\Lambda \subset M$ is said to be *invariant* if $\mathrm{Orb}_\Phi(x) \subset \Lambda$ for all $x \in \Lambda$; *transitive* if there exists a dense orbit in Λ, i.e.,

$$\Lambda \subset \overline{\mathrm{Orb}_\Phi(x)} \quad \text{for some } x \in \Lambda;$$

and *minimal* if every $x \in \Lambda$ has a dense orbit in Λ. The *ω-limit of* $x \in M$ *for* IFS(Φ) is the set

$$\omega_\Phi(x) \stackrel{\text{def}}{=} \{y : \text{there exists } (h_n)_n \subset \mathrm{IFS}(\Phi) \setminus \{\mathrm{id}\} \text{ such that } \lim_{n\to\infty} h_n \circ \cdots \circ h_1(x) = y\},$$

while the *ω-limit of* IFS(Φ) is

$$\omega(\mathrm{IFS}(\Phi)) \stackrel{\text{def}}{=} \mathrm{cl}\big(\{y \in M : \text{ there exists } x \in M \text{ such that } \ y \in \omega_\Phi(x)\}\big),$$

where "cl" denotes the closure of a set. Similarly we define the α-limit of both, a point $x \in M$ and the iterated function system IFS(Φ). Finally, the *limit set* L(IFS(Φ)) is the union of ω-limit and α-limit of IFS(Φ). From these concepts, understanding the dynamics of an IFS requires to know the possible invariant sets for the IFS, in order to describe the ω-limit or α-limit of their orbits, and show, if possible, a result on spectral decomposition of the limit set as it was done in the case of a hyperbolic diffeomorphisms.

In order to find robust properties under perturbations it is important to introduce the concept of proximity into the set of the IFS. That is, an iterated function system IFS($\psi_1, \dots, \psi_k$) is said to be C^1-close to IFS($\phi_1, \dots, \phi_k$) if each of the diffeomorphisms ψ_i is close to ϕ_i in the C^1-topology. As an example of a robust property by perturbations one can think in the translation to the language of the IFS of the symbolic blenders defined in the previous chapter:

Definition (Blending region)**.** *An open set* $B \subset M$ *is said to be a* blending region for IFS(Φ) *if* B *is* C^1-robustly minimal *for* IFS(Φ)*, i.e.,*

$$B \subset \overline{\mathrm{Orb}_\Psi(x)} \quad \textit{for all } x \in B \textit{ and every } \mathrm{IFS}(\Psi)\ C^1\textit{-close to } \mathrm{IFS}(\Phi).$$

In the case of a one-step skew-product with contracting fibers, in Proposition 2.21 it is proved that the existence of a blending region is equivalent to have a symbolic blender in the one-step setting. The main goal along this third chapter is to prove the existence of blending regions for IFS generated by generic diffeomorphisms, on both the real line $M = \mathbb{R}$ and the circle $M = S^1$ close to the identity map $\mathrm{id} : M \to M$.

In Section §3.2 we will study blending region on the real line. We will introduce a type of interval with a concrete configuration for a pair of maps f_0, f_1 (see Figure 3.1(a)). It will be a candidate to blending region for IFS(f_0, f_1). Denote by $\mathrm{Diff}^r_+(\mathbb{R})$ the set of orientation preserving C^r-diffeomorphisms on the real line.

Definition (ss-intervals)**.** *Given* $\Phi = \{f_0, f_1\} \subset \mathrm{Diff}^1_+(\mathbb{R})$*, an interval* $[p_0, p_1] \subset \mathbb{R}$ *is called* ss-interval for $\mathrm{IFS}(\Phi)$ *if:*

- $[p_0, p_1] = f_0([p_0, p_1]) \cup f_1([p_0, p_1])$*,*
- $(p_0, p_1) \cap \mathrm{Fix}(f_i) \neq \emptyset$ *for* $i = 1, 2$*, and* $p_j \notin \mathrm{Fix}(f_i)$ *for* $i \neq j$*,*
- p_0 *and* p_1 *are attracting fixed points of* f_0 *and* f_1 *respectively.*

We will denote by K^{ss}_Φ *a ss-interval* $[p_0, p_1]$ *for the iteration function system* $\mathrm{IFS}(\Phi)$*.*

The next theorem implies that any open set contained in a ss-interval for $\mathrm{IFS}(\Phi)$, with generators close enough to the identity and with hyperbolic fixed points, is a blending region for $\mathrm{IFS}(\Phi)$. This theorem is a generalization of a lemma due to Duminy [Dum70], which is part of the proof of the so-called Duminy's Theorem (see Theorem 3.27) about the dynamics of groups of diffeomorphisms in the circle. We will prove this result using some different arguments from the original proof of Duminy's Lemma (see [Nav11] for details) and we will improve slightly the conclusions of Duminy's Theorem. We denote by $\mathrm{Per}(\mathrm{IFS}(\Phi))$ the set of periodic points of $\mathrm{IFS}(\Phi)$, i.e., the set of points $x = h(x)$ for some $h \neq \mathrm{id}$ in $\mathrm{IFS}(\Phi)$.

Theorem D (Duminy's Lemma)**.** *Let* K^{ss}_Φ *be a ss-interval for an iterated function system* $\mathrm{IFS}(\Phi)$ *with* $\Phi = \{f_0, f_1\} \subset \mathrm{Diff}^2_+(\mathbb{R})$ *such that the fixed point of* $f_i|_{K^{ss}_\Phi}$ *is hyperbolic. Then, there exists* $\varepsilon \geq 0.17$ *such that if* $f_0|_{K^{ss}_\Phi}$*,* $f_1|_{K^{ss}_\Phi}$ *are* ε*-close to the identity in the* C^2*-topology, it holds that*

$$K^{ss}_\Psi \subset \overline{\mathrm{Per}(\mathrm{IFS}(\Psi))} \quad \textit{and} \quad K^{ss}_\Psi = \overline{\mathrm{Orb}_\Psi(x)} \quad \textit{for all } x \in K^{ss}_\Psi,$$

for every iterated function system $\mathrm{IFS}(\Psi)$ C^1*-close to* $\mathrm{IFS}(\Phi)$*.*

The Section §3.3.2 is concerned with the generalization of the above theorem for Morse-Smale diffeomorphisms on the circle (see Theorem 3.35). This generalization is part of the proof of a Denjoy type theorem for IFS. Remember that, taking into account the rotation number of a diffeomorphism f of the circle we have three possibilities: (i) f has a periodic point, (ii) all orbits (for forward iterates) of f and f^{-1} are dense, and (iii) there is a wandering interval for f. Wandering intervals are the gaps of a f-invariant Cantor set $\Lambda \subset S^1$, which is contained in the ω-limit for f of all points of S^1. These dynamical properties can be easily translated for IFS:

Definition (Invariant minimal Cantor set)**.** *Let* $\Phi = \{f_0, f_1\} \subset \mathrm{Diff}^1(S^1)$ *and* $\Lambda \subset S^1$*. A subset* $\Lambda \subset S^1$ *is said to be* minimal invariant Cantor set for $\mathrm{IFS}(\Phi)$ *if*

- Λ *is a Cantor set and*
- $\Lambda = \overline{\mathrm{Orb}_\Phi(x)}$ *for all* $x \in \Lambda$*.*

From Denjoy's Theorem [Den32] it follows that these Cantor sets cannot appear for diffeomorphisms on the circle with enough regularity and close to the identity. Namely, there is $\varepsilon > 0$ such that if $f \in \mathrm{Diff}^2(S^1)$ and it is ε-close to identity in the C^2-topology, then there are no minimal invariant Cantor sets. Moreover, the following statements are equivalent: S^1 is minimal for the iterated function systems $\mathrm{IFS}(f)$, and f does not have periodic points. When the number of generators of the IFS increases the periodic points are no longer the obstruction to the minimality. Now, that role is played by the ss-intervals.

Theorem E (Denjoy for IFS). *There exists $\varepsilon > 0$ such that if $f_0, f_1 \in \mathrm{Diff}^2(S^1)$ are Morse-Smale diffeomorphisms ε-close to the identity in the C^2-topology with no periodic point in common then, there are no invariant minimal Cantor sets for* $\mathrm{IFS}(f_0, f_1)$.

Moreover, denoting by n_i of period of f_i, the following conditions are equivalents:

- *S^1 is minimal for* $\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})$,
- *there are no ss-intervals for* $\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})$.

Unlike what occurs for a single diffeomorphism f on the circle where S^1 cannot be C^1-robust minimal, in the case of IFS, the robustness can be obtained. In fact, notice that the above theorem is C^1-robust in the following sense:

Remark (C^1-robustness). *The assertions of the Denjoy's Theorem for IFS are robust under C^1-perturbations of* $\mathrm{IFS}(f_0, f_1)$, *i.e., for every* $\mathrm{IFS}(g_0, g_1)$ *where g_0 and g_1 are C^1-perturbations of f_0 and f_1 respectively.*

As a consequence of this Denjoy's theorem for IFS, we will finish the third chapter of this thesis showing a Spectral Decomposition Theorem on the circle. This theorem states that the limit set of $\mathrm{IFS}(\Phi)$ with $\Phi = \{f_0^{n_0}, f_1^{n_1}\}$, where $f_0, f_1 \in \mathrm{Diff}^2(S^1)$ in the hypothesis of the previous theorem, is decomposed into finite union of disjoint *basic intervals*: isolated and transitive intervals for $\mathrm{IFS}(\Phi)$. A set A with $A \cap \mathrm{Per}(\mathrm{IFS}(\Phi)) \neq \emptyset$ is said *isolated* for $\mathrm{IFS}(\Phi)$ if there exists an open set D such that $A \subset D$ and A contains the closure of $\mathrm{Per}(\mathrm{IFS}(\Phi)) \cap D$.

Theorem F (Spectral decomposition for IFS). *There exists $\varepsilon > 0$ such that if $f_0, f_1 \in \mathrm{Diff}^2(S^1)$ are Morse-Smale diffeomorphisms of periods n_0 and n_1, respectively, ε-close to the identity in the C^2-topology and with no periodic point in common, then there are finitely many isolated, transitive pairwise disjoint intervals $K_1, \ldots, K_m$ for* $\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})$ *such that*

$$L(\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})) = \bigcup_{i=1}^{m} K_i.$$

Moreover, this decomposition of the limit set of $\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})$ *is C^1-robust.*

IV – Cycles in unfoldings of nilpotent singularities – In the last chapter we translate the conclusions obtained in the first part of the thesis to the framework of vector fields . The main result of this chapter is in collaboration with Santiago Ibáñez and J. Ángel Rodríguez and are collected in [BIR11].

The dynamics associated with heterodimensional cycles force us to consider diffeomorphisms in dimension $n \geq 3$. It is well-known that these diffeomorphisms can be defined as Poincaré maps on cross-sections of a vector field in $\mathbb{R}^4$ near a cycle or a periodic orbit. Philosophically, dynamics in discrete systems are lifted to the field of continuous systems by the suspension process. However, the real interest lies in obtaining some manageable criterion to determine when a family of vector fields has this or that dynamical behavior. The study of global bifurcations associated with different cycles explains the dynamical transitions and the nature of the behavior. The presence of infinitely many horseshoes in a neighborhood of an orbit of Sil'nikov type is an example. However, proving that a family of vector fields has a certain cycle is not easy, unless that family is constructed ad hoc. This is the case in [Rod86] for a family of quadratic vector fields having Sil'nikov orbits. As an alternative to this search of cycles, one can consider the proof of criteria that conclude the presence of interesting dynamics determined from the simplest elements of a vector field: its singularities. In these terms one may ask for the singularity of lower codimension (more common) from which Sil'nikov homoclinic orbits (and therefore strange attractors) can be generically unfolded. A partial answer was given in [IR95] where the existence of these configurations was proved in the generic unfoldings of nilpotent singularities of codimension four in $\mathbb{R}^3$. Later, in [IR05] this result was proved for the nilpotent singularity of codimension three. A *nilpotent singularity* is a C^∞ vector field in $\mathbb{R}^n$ such that, in appropriate coordinates, in an neighborhood of the origin (equilibrium point) it can be written as

$$\sum_{k=1}^{n-1} x_{k+1}\frac{\partial}{\partial x_k} + f(x_1,\ldots,x_n)\frac{\partial}{\partial x_n},$$

with $f(x) = O(\|x\|^2)$ where $x = (x_1,\ldots,x_n)$. It is said to be a *nilpotent singularity of codimension n* if it holds the generic condition $\partial^2 f/\partial x_1^2(0) \neq 0$.

Existence of strange attractors in the unfolding of a lower dimensional singularity, a Hopf-cero singularity of codimension two, is studied in [DIKS]. The result in [IR05] allows to conclude the presence of strange attractors in the coupling of two Brusselator by linear diffusion [DIR07]. Hence, complicated dynamics was proved to emerge in couple system, as Turing [Tur52] proposed and Smale [Sma74] completed, regarding the genesis of oscillations. By a Brusselator we mean a cubic bidimensional vector field which is proposed as a model in a chemistry reaction. The coupling of two of these dynamics leads to a vector field having a nilpotent singularity of codimension four in $\mathbb{R}^4$. The first objective proposed at the beginning of this thesis was the study of the generic unfolding of these singularities to find cycles such that they could imply proper dynamics of dimension $n \geq 4$: strange attractors with more than one positive Lyapunov exponent and heterodimensional cycles. In [BIR11] we proved the existence of bifocal homoclinic orbits in every generic unfolding of the nilpotent singularity of codimension four in $\mathbb{R}^4$.

Theorem G. *In every generic unfolding of a four-dimensional nilpotent singularity of codimension four there is a bifurcation hypersurface of bifocal homoclinic orbits.*

Recall that a bifocal homoclinic orbit is a homoclinic connection to an equilibrium point of a vector field on $\mathbb{R}^4$ with two pairs of eigenvalues $\rho_k \pm i\omega_k$ with $k = 1, 2$, such that $\rho_1 < 0 < \rho_2$. The Poincaré map defined in a neighborhood of this cycle will be a three-dimensional diffeomorphism, susceptible to present a blender. We will prove the existence of suspended blenders for vector field arbitrarily close to a Hamiltonian vector fields in $\mathbb{R}^4$ with a non-degenerate bifocal homoclinic orbit. For this Hamiltonian vector field the Poincaré map can be written, with a suitable choice of coordinates, as

$$f : [-\varepsilon,\varepsilon]^2 \times [-c_0, c_0] \to [-\varepsilon,\varepsilon]^2 \times [-c_0, c_0]\,, \qquad f(x,c) = (F_c(x), c),$$

where F_c has a hyperbolic maximal invariant set Λ_c for $|c| \leq c_0$ conjugated to the Bernoulli shift $\Sigma_{n(|c|)}$ (see Teorema 4.16). Moreover, the family of sets $\{\Lambda_c\}_{0\leq c\leq c_0}$ satisfies that $\Lambda_{c-\varepsilon}$ contains the dynamically continuation of Λ_c for every $\varepsilon > 0$ small enough. Similarly for the family $\{\Lambda_c\}_{-c_0\leq c\leq 0}$.

These properties allow us to conjugate a subsystem of f to a symbolic skew-product of the form $\Phi = \tau \times \mathrm{id}$ defined on $\Sigma_{n(|\bar{c}|)} \times [-\bar{c}, \bar{c}]$ with $0 < |\bar{c}| \leq c_0$.

In order to prove the above theorem we will show that, for some parameter values, the limit after rescaling of the generic unfolding of the nilpotent singularities is a family of Hamiltonian vector fields with a non-degenerate bifocal homoclinic orbit. Perturbations on the hypersurface of bifocal homoclinic orbits of each one of these Hamiltonian vector fields have a Poincaré return map conjugated to a symbolic skew-product perturbation of $\Phi = \tau \ltimes \mathrm{id}$. As follows from the third chapter, generic one-step perturbations of $\Phi = \tau \times \mathrm{id}$ has either, a blending region or its dynamics is trivial. Thus, we will conclude the fourth chapter discussing the possible presence of suspended blenders and heterodimensional cycles in the generic unfoldings of nilpotent singularities.

Robust cycles and blenders

One of the basic problems in the study of diffeomorphisms was the characterization of the structurally stable dynamics. This was obtained by means of the hyperbolicity. Then the need arises to know the obstructions to hyperbolicity. Two main mechanisms appear to yield robust non-hyperbolic behavior: homoclinic tangencies and heterodimensional cycles. Blenders are hyperbolic sets that appear as the subjacent mechanism leading to the generation of robust heterodimensional cycles and robust homoclinic tangencies. Blender-horseshoes are examples of blenders which can be constructed by means of skew-product diffeomorphisms called non-normally hyperbolic horseshoes.

1.1 Hyperbolicity and stability

Given a C^r-diffeomorphism f of a Riemannian compact manifold M, we say that a f-invariant compact set $\Lambda \subset M$ is *hyperbolic* if there is a continuos Df-invariant splitting $E^s_\Lambda \oplus E^u_\Lambda$ of the tangent bundle $T_\Lambda M$ and there are constants $C > 0$, $\lambda < 1$, such that

$$\|D_x f^n(v)\| \leq C\lambda^n \|v\| \quad \text{and} \quad \|D_x f^{-n}(w)\| \leq C\lambda^n \|w\|$$

for all $v \in E^s_x$, $w \in E^u_x$, $x \in \Lambda$ and $n \geq 1$. The vector bundle E^s_Λ and E^u_Λ are the *stable* and *unstable* directions of Λ. In particular, when $\Lambda = M$, the diffeomorphisms f is called *Anosov diffeomorphism*.

A hyperbolic set Λ of a diffeomorphism f is called *basic set* if it is *transitive* (i.e. there is a dense orbit of f in Λ), *isolated* (i.e. there is a neighborhood U of Λ such that $\Lambda = \cap_{i\in\mathbb{Z}} f^i(U)$) and contains a dense subset of periodic points. It follows as a consequence of the continuity of the Df-invariant splitting $E^s_\Lambda \oplus E^u_\Lambda$ of a hyperbolic set Λ that the dimension of E^s_x and E^u_x with $x \in \Lambda$ is locally constant. In addition, if Λ is also transitive then these dimensions are constant. In this case, the dimension of the stable bundle E^s_Λ is called *s-index* and is denoted by $\mathrm{ind}^s(\Lambda)$.

A relevant f-invariant compact set is the *non-wandering set*, denoted by $\Omega(f)$, which consists of the points $x \in M$ such that for any neighborhood U of x, there is an integer $n \geq 1$ such that $f^n(U) \cap U \neq \emptyset$. Note that the set of periodic points $\mathrm{Per}(f)$ is contained in $\Omega(f)$. A diffeomorphism f is called *Axiom A* or *uniformly hyperbolic* if $\Omega(f)$ is a hyperbolic set for f and $\mathrm{Per}(f)$ is dense in $\Omega(f)$. In that case, the Spectral Decomposition Theorem due by Smale [Sma67] asserts that the non-wandering set $\Omega(f)$ is decomposed as a finite pairwise disjoint union of hyperbolic basic set which are called *basic pieces* of the spectral decomposition. These basic pieces are hyperbolic isolated homoclinic classes: a *homoclinic class* of a saddle P of f, denoted by $H(P, f)$, is the closure of the transverse intersections of the stable and unstable manifolds of the orbit of P. Notice that a homoclinic class that is not reduced to a saddle contains a horseshoe [Bir27, Sma67]. In that case, we say that the homoclinic class is *non-trivial*.

The homoclinic class $H(P, f)$ can be equivalently defined as the closure of the set of the saddles Q *homoclinically related* to P: the stable manifold of the orbit of Q meets transversely the unstable manifold of the orbit of P and vice-versa. Although all saddles homoclinically related to P have the same s-index as P, the homoclinic class $H(P, f)$ may contain periodic orbits of different s-index from the s-index of P. As a final comment, notice that a homoclinic class is an f-invariant transitive set with dense periodic points.

The following conjecture proposes the relation between the hyperbolicity and the structurally stable dynamical behavior. We recall that a diffeomorphism f is C^r*-structurally stable* if there is a C^r-neighborhood $\mathcal{V}$ of f such that every diffeomorphism $g \in \mathcal{V}$ is C^r-conjugated to f. At the end of the 60's Palis and Smale proposed in [PS70] a complete characterization of the structurally stable systems:

Conjeture (Palis-Smale's Structural Stability Conjecture)**.** *A diffeomorphism f is C^r-structurally stable if and only if it is Axiom A and all the stable and unstable manifolds associated with the points of the non-wandering set are transversal.*

The additional condition about the general position between the stable and unstable manifold is called *strong transversality condition.* Robbin [Rob71] and Robinson [Rob76] showed that Axiom A and strong transversality condition are sufficient to structural stability. In addition, it is also known that in presence of Axioma A, strong tranversality is a necessary condition for stability. The hardest part was to prove that stable system should be uniformly hyperbolic. The key name here is Mañe, who along several works developed new ideas and fundamental techniques that allow him to give a positive answer to the stability conjecture. This result was achieved in [Mañ88], in the C^1 topology:

Theorem 1.1. *A diffeomorphism on a compact manifold is C^1-structurally stable if and only if it is Axiom A and verifies the strong transversality condition.*

A weak property, called C^r Ω*-stability* is defined requiring C^r-conjugacy only restricted to the non-wandering set. Another conjecture in [PS70] proposes a characterization of Ω-stable system:

Conjeture (Palis-Smale's Ω-stability Conjecture)**.** *A diffeomorphism f is C^r Ω-stable if and only if it is Axiom A and there is no basis pieces in their spectral decompositions cyclically related by intersections of the corresponding stable and unstable manifolds.*

The additional condition about the cyclically intersections between the stable and unstable manifolds of basic pieces is called *no-cycle condition.* The Ω-stability theorem of Smale [Sma70] states that the uniform hyperbolicity and no-cycle condition are sufficient in the C^r sense. Palis [Pal70] proved that the no-cycle condition is necessary for Ω-stability in any C^r topology. Recall that if f is a C^1-structurally stable diffeomorphism then f is C^1 Ω-stable. So, the conjecture was proved by Palis [Pal88] in the C^1 setting, based on the ideas of Mañe [Mañ88]:

Theorem 1.2. *A diffeomorphism on a compact manifold is C^1 Ω-stable if and only if it is Axiom A and verifies the no-cycle condition.*

Given $\varepsilon > 0$, a sequence $\{x_n\}_{n\in\mathbb{N}}$ is called ε-*pseudo-orbit* of a diffeomorphism f if the distance between $f(x_n)$ and x_{n+1} is less than ε for all $n \in \mathbb{N}$. A point x is *chain recurrent* if for every $\varepsilon > 0$ there are ε-pseudo orbits starting and ending at x. The set of all chain recurrent points is called *chain recurrent set* and denoted by $\mathcal{R}(f)$. The *chain recurrence class* of x for f, denoted by $C(x, f)$, is the set of points y such that, for every $\varepsilon > 0$, there are ε-pseudo orbits starting at x, passing ε-close to y and ending at x. The diffeomorphism f is called $\mathcal{R}$-*hyperbolic* if its chain recurrent set is hyperbolic. By the Smale spectral theorem [Sma70], this is equivalent to be Axiom A and satisfy the no-cycle condition. Therefore, the C^1 Ω-stable systems are the $\mathcal{R}$-hyperbolic diffeomorphism. Actually, the $\mathcal{R}$-hyperbolic systems coincide with the interior, respect to the C^1 topology, of the set of diffeomorphisms whose all periodic orbits are hyperbolic [Aok92, Hay92].

Although uniform hyperbolicity was originally intended to encompass a residual, or at least dense subset of all dynamical systems, it was soon realized that this is not true. There are two main mechanisms (see the following definition) that yield robustly non-hyperbolic behavior, that is, whole open sets of non-hyperbolic systems. They are at the heart of recent developments that we are going to review in the next sections.

Definition 1.1 (Homoclinic bifurcations). *A diffeomorphism $f : M \to M$ has a*

- homoclinic tangency *associated with a transitive hyperbolic set Λ of f if there is a pair of points $x, y \in \Lambda$ such that the stable manifold $W^s(x)$ of x and the unstable manifold $W^u(y)$ of y have some non-transverse intersection.*

- heterodimensional cycle *associated with transitive hyperbolic sets Λ and Σ of f if these sets have different s-indices and their invariant manifolds meet cyclically, that is, if*

$$W^s(\Lambda) \cap W^u(\Sigma) \neq \emptyset \quad \textit{and} \quad W^u(\Lambda) \cap W^s(\Sigma) \neq \emptyset.$$

  The heterodimensional cycle has coindex $c \geq 1$ *if* $|\mathrm{ind}^s(\Lambda) - \mathrm{ind}^s(\Sigma)| = c$.

The firsts examples of C^1-open subsets of non-hyperbolic diffeomorphism were given by Abraham, Smale [AS70] and Simon [Sim72] for manifolds of dimension $d \geq 4$ and $d = 3$ respectively. These examples arise from heterodimensional cycle. These cycles can only exist in dimension 3 or higher and force the coexistence of periodic points with different s-indices inside the same transitive set. The first robust example of non-hyperbolic diffeomorphisms on surface were constructed by Newhouse [New70], exploiting the homoclinic tangencies. In that work, Newhouse considers a surface C^2-diffeomorphism f with a homoclinic tangency q associated with a hyperbolic periodic point p and a hyperbolic basic set Λ of f containing p (see Figure A). In order to obtain homoclinic tangencies associated with Λ let consider a curve ℓ (called curve of tangencies) containing the initial homoclinic tangency and project the Cantor sets $\Lambda^s = \Lambda \cap W^s_{loc}(p)$ and $\Lambda^u = \Lambda \cap W^u_{loc}(p)$ to ℓ along the stable, respectively unstable, leaves. This gives a pair of Cantor set K^s and K^u in the curve ℓ. Note that $q \in K^s \cap K^u$. By the same construction, for any g close to f, one can obtain two new Cantor sets K^s_g and K^u_g on the curve ℓ from $\Lambda^s_g = \Lambda_g \cap W^s_{loc}(p_g)$ and $\Lambda^u_g = \Lambda_g \cap W^u_{loc}(p_g)$ where Λ_g and p_g are the continuation of Λ and p for g, respectively. Each intersection point between K^s_g and K^u_g corresponds to a homoclinic tangency of Λ_g.

Now, the key ingredient is a kind of fractal dimension called *thickness* $\tau(K)$ of a Cantor set K of the real line (see [PT93] for details). The Gap Lemma [PT93] states that if $\tau(\Lambda^s_g)\tau(\Lambda^u_g) > 1$

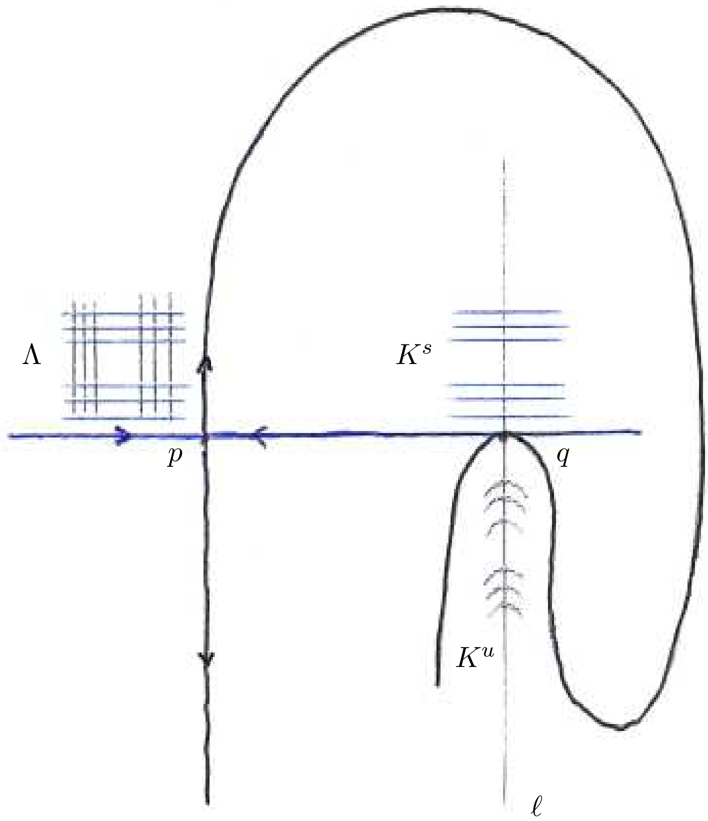

Fig. A: Homoclinic tangency

then either $K_g^s \cap K_g^u \neq \emptyset$ or one of two Cantor sets is contained in a gap of the other one. The geometric position of K_g^s and K_g^u implies that no Cantor set can be contained in a gap of the other one. Hence these sets have non empty intersection.

We say that Λ is a *thick hyperbolic set* if the condition $\tau(\Lambda^s)\tau(\Lambda^u) > 1$ is fulfilled. The final and essential ingredient of Newhouse's construction is that, the property of having a thick hyperbolic set is C^2-open. That is, for every g which is close to f in the C^2-topology, the continuation of Λ_g of Λ is a thick hyperbolic set. This allows us to prove that if Λ is a thick hyperbolic set of f then for any g C^2-close to f, the Cantor set K_g^s intersects K_g^u and one gets C^2-robust homoclinic tangencies:

Definition 1.2 (Robust homoclinic bifurcation)**.** *A C^r-diffeomorphism f has a*

- C^r-robust homoclinic tangency *associated with a hyperbolic basic set Λ of f if there is a C^r-neighborhood $\mathcal{V}$ of f such that for every $g \in \mathcal{V}$ the continuation Λ_g of Λ for g has a homoclinic tangency. The neighborhood $\mathcal{V}$ is called* C^r-open of persistence of homoclinic tangencies.
- C^r-robust heterodimensional cycle *associated with hyperbolic basic sets of f, Λ and Σ, if there is a C^r-neighborhood $\mathcal{V}$ of f such that every diffeomorphism $g \in \mathcal{V}$ has a heterodimensional cycle associated with the continuations Λ_g and Σ_g of Λ and Σ, respectively.*

Note that, by Kupka-Smale theorem [Kup63, Sma63], C^r-generically, invariant manifolds of periodic points are in general position. Hence, generically, the non-transverse intersections in a robust homoclinic intersection (tangency or heterodimensional cycle) involve non-periodic points, i.e., at least a non-trivial hyperbolic set.

Let $\mathcal{V}$ be the *C^r-open of persistence of homoclinic tangencies*, in Definition 1.2. It is well known (see [New74, PT93]) that, in dimension two, there exits a dense subset $\mathcal{D}$ of $\mathcal{V}$ such that each $g \in \mathcal{D}$ exhibits a homoclinic tangency associated with a hyperbolic periodic point.

A *dissipative saddle* is a hyperbolic periodic point p which has the absolute value of the product of the eigenvalues of $Df^n(p)$ less than one, where n is the period of p.

Newhouse in [New79] proved that, in dimension two, homoclinic tangencies associated with a saddle of C^2-diffeomorphisms yield thick horseshoes with C^2-robust homoclinic tangencies. That is, any C^2-diffeomorphism with a hyperbolic periodic point such that both its stable and unstable manifolds have a non-transverse intersection belongs to the closure of a C^2-open of persistence of homoclinic tangencies. With the same regularity assumption, theorems in [PV94, Rom95] extend Newhouse result, proving that homoclinic tangencies in any dimension lead to C^2-robust homoclinic tangencies.

The above construction of thick horseshoes with robust tangencies involves distortion estimates which are typically C^2. The results in [Ure95] present some obstacles for carrying this construction to the C^1-topology: C^1-generic surface diffeomorphisms do not have thick horseshoes. Recent results by Moreira in [Mor11] are a strong indication that there are no surface diffeomorphisms exhibiting C^1-robust homoclinic tangencies:

Theorem 1.3. *There are no C^1-robust homoclinic tangencies associated with hyperbolic basic sets of surface diffeomorphisms.*

If every transitive hyperbolic set of a surface diffeomorphism is contained in a hyperbolic basic set then Moreira's result would imply the non existence of robust tangencies associated with hyperbolic transitive sets of surface diffeomorphisms. This is an important step in direction of the following conjeture:

Conjeture (Smale C^1-density Conjecture)**.** *The uniform hyperbolic diffeomorphisms of a compact surface S are dense in* $\mathrm{Diff}^1(S)$.

Heterodimensional cycles of coindex one yield C^1-robust heterodimensional cycles of coindex one after small C^1-perturbation [BD08]. However, in dimension $d \geq 3$, we do not know when and how homoclinic tangencies may occur in a C^1-robust way. Actually, all the known examples about C^1-robust tangencies also exhibit C^1-robust heterodimensional cycles. Hence, it is natural to expect that robust tangencies lead to heterodimensional cycles (and so C^1-robust heterodimensional cycles) as it is conjectured in [Bon11]:

Conjeture (Bonatti)**.** *Let $\mathcal{U}$ be a C^1-open set of diffeomorphisms f having a hyperbolic basic set Λ_f varying continuously with f and exhibiting a robust tangency. Then there is a C^1-dense open subset $\mathcal{D}$ of $\mathcal{U}$ such that for $f \in \mathcal{D}$ there is a hyperbolic basic set Σ_f of different index as Λ_f and such that f has a C^1-robust heterodimensional cycle associated with Λ_f and Σ_f.*

In dimension two, there are no robust cycles, so this conjecture means that there are no C^1-robust tangencies (see Theorem 1.3 and Smale C^1-density Conjecture).

1.2 Homoclinic bifurcations and blenders

At the end of the 80's, Palis proposed a research line whose main goal was to get a geometrical description about the behavior of most dynamical systems in compact manifold (see [PT93, Pal00a, Pal08]). According to [CP10], this program of research is known as *mechanisms versus phenomena*. *Mechanism* (or *dynamical configuration*) means a simple dynamical configuration for one diffeomorphism (involving for instance few periodic points and their invariant manifolds) that has the following properties: it "creates or destroys" rich and different dynamics for nearby systems and it "generates itself", that is, the system exhibiting this configuration is not isolated. For instance, homoclinic bifurcations (tangencies and heterodimensional cycles) are mechanisms. *Dynamical phenomenon* means any dynamical property which provides a good global description of the system (like hyperbolicity, transitivity, minimality, zero entropy, spectral decomposition) and which happens on a "rather large" subset of systems.

We relate those above notions and say that a *a dynamical configuration is a complete obstruction to a dynamical phenomena*, if it not only prevents the phenomenon to happen but it also generates itself creating rich dynamics. It is common in the complement of the prescribed dynamical phenomenon (see [CP10] for more details).

We recall that *Morse-Smale diffeomorphisms* are those for which the set of non-wandering points is finite and hyperbolic, and the invariant manifolds of the periodic orbits pairwise intersect transversally. Those diffeomorphisms define "simple" dynamics, in particular they have no horseshoes. We say that there is a *transverse homoclinic intersection* if the stable manifold of a hyperbolic periodic point meets transversally its unstable manifold. This implies the existence of horseshoes [Bir27, Sma67] and therefore "complicate" dynamical behavior.

In some sense, the following dichotomy is between simple dynamics (Morse-Smale) and complicate dynamics (horseshoes). That is, the transverse homoclinic bifurcations are a complete obstruction to the Morse-Smale dynamics:

Conjeture (Palis's weak C^r-density Conjecture)**.** *The set of Morse-Smale diffeomorphisms and the set of diffeomorphisms that admit a transverse homoclinic intersection, are two disjoint open sets whose union is dense in* $\mathrm{Diff}^r(M)$*.*

The weak C^1-density Conjecture in 3-dimensional manifolds was proven by Bonatti, Gan and Wen in [BGW07]. Recently, Crovisier [Cro10] proved it in any dimension:

Theorem 1.4. *Any diffeomorphism can be* C^1*-approximated by a Morse-Smale diffeomorphism or by one exhibiting a transverse homoclinic intersection.*

As we mentioned before, there are two main local mechanisms associated with periodic saddles for breaking hyperbolicity of systems: homoclinic tangency and heterodimensional cycle. Palis conjectured that this homoclinic bifurcations are "always" responsible for non-hyperbolicity:

Conjeture (Palis's C^r-density Conjecture)**.** *The union of uniform hyperbolic diffeomorphisms and diffeomorphisms having a homoclinic tangency or a heterodimensional cycle is dense in* $\mathrm{Diff}^r(M)$*.*

It is easy to see that this second conjecture implies the first one since the dynamics at a homoclinic bifurcation can be perturbed in order to create a transverse homoclinic intersection.

In addition, note that these conjectures hold when M is one-dimensional. Actually, in this case Peixoto proved in [Pei62] that Morse-Smale diffeomorphisms are dense in $\mathrm{Diff}^r(M)$ for $r \geq 1$. In dimension two, heterodimensional cycles do not exist and consequently this conjecture can be written as a dichotomy between hyperbolic diffeomorphisms and homoclinic tangencies. In [PS00], Pujals and Sambarino proved the C^1-density Conjecture of Palis for surfaces:

Theorem 1.5. *Any surface diffeomorphism can be C^1-approximated either by uniform hyperbolic diffeomorphisms or by diffeomorphisms exhibiting a homoclinic tangency.*

Crovisier and Pujals in [CP10] proved a slightly modified version of that mentioned Palis's conjecture in the C^1-topology for any dimension introducing the next new weaker notion of hyperbolicity: a diffeomorphism is *essentially hyperbolic* provided that has a finite number of transitive hyperbolic attractors and the union of their basin of attraction is open and dense in the manifold. The essential hyperbolicity recovers the notion of Axiom A: most of the trajectories (in the Baire category) converge to a finite number of transitive attractors that are well described from a both topological and statistical point of view. However, the set of these diffeomorphism is not open a priori.

Theorem 1.6. *Any diffeomorphism can be C^1-approximated either by an essentially hyperbolic diffeomorphisms or by diffeomorphisms exhibiting a homoclinic tangency or heterodimensional cycle.*

The Palis's C^1-density Conjecture is known to be true for diffeomorphisms whose dynamics splits into finitely many pieces only:

Theorem 1.7. *Any diffeomorphism can be C^1-approximated by a diffeomorphism which is $\mathcal{R}$-hyperbolic or has a heterodimensional cycle or has infinitely many chain-recurrence classes.*

The above result was proved on surfaces by Mañé [Mañ82] and for compact manifolds of dimension $d \geq 3$ by Abdenur [Abd03] and Gan-Wen [GW03]. See also [Cro09].

Previous results of Bonatti and Díaz [BD08] proved that if any diffeomorphism having a heterodimensional coindex one cycle associated with a pair of saddles is C^1-approximated by diffeomorphisms having a C^1-robust heterodimensional coindex one cycle. Recently in [BD11] showed that if at least one of the homoclinic classes of these saddles is non-trivial then the C^1-robust heterodimensional coindex one cycles are associated with a hyperbolic basic set containing the continuations of the saddles. Bering in mind this remark, the previous theorem was formulated in [BD08] for tame diffeomorphisms: a diffeomorphism is called *tame* if it has a finitely many chain recurrent classes in a robust way. Let denote by $\mathcal{T}(M)$ the set of tame diffeomorphisms of a manifold M. Note that this set is C^1-open in $\mathrm{Diff}^1(M)$. The set $\mathcal{W}(M) = \mathrm{Diff}^1(M) \setminus \overline{\mathcal{T}(M)}$ is called *wild diffeomorphism set*. Bonatti and Díaz proved:

Theorem 1.8. *There is an open dense subset $\mathcal{D}$ of $\mathcal{T}(M)$ such that every $f \in \mathcal{D}$ is either $\mathcal{R}$-hyperbolic or has a C^1-robust heterodimensional cycle.*

This result also holds in the C^1-settings of the conservative diffeomorphisms (see [Cro09]). These comments and the Bonatti's Conjecture lead to the following strong version of Palis's C^1-density Conjecture:

Conjeture (Bonatti-Díaz)**.** *The union of $\mathcal{R}$-hyperbolic diffeomorphism and diffeomorphisms having a C^1-robust heterodimensional cycle is dense in* $\mathrm{Diff}^1(M)$.

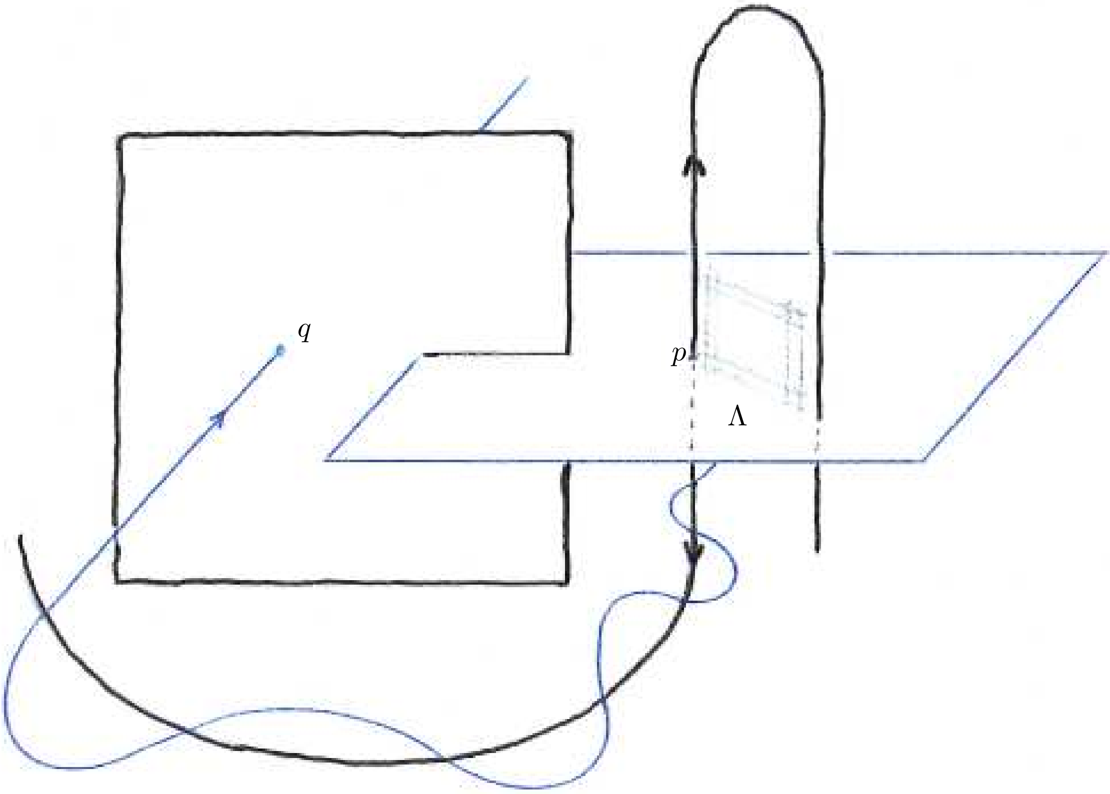

Fig. B: Heterodimensional cycle

A C^1-robust heterodimensional cycle example

We will construct examples of persistent non-hyperbolic diffeomorphisms in the C^1-topology. These examples consist of diffeomorphism exhibiting a C^1-robust heterodimensional cycle. Note again that heterodimensional cycles may only exist in dimension three or higher. In order to simplify the next exposition, we will work in a compact manifold M of dimension three.

A simple example of heterodimensional cycle is associated with periodic points p and q as it is shown in Figure B. Since the one dimensional manifolds of p and q have a quasi-transverse intersection, that is, there is $x \in W^s(q) \cap W^u(p)$ such that

$$T_xW^s(q) + T_xW^u(p) = T_xW^s(q) \oplus T_xW^u(p) \neq T_xM,$$

the heterodimensional cycle associated with p and q is not robust. So, in order to construct a robust heterodimensional cycle, we must involve at least a non-trivial transitive hyperbolic basic set. Thus, we introduce non-trivial transitive hyperbolic set Λ contained p. To obtain a robust cycle it is necessary that for any diffeomorphism g close to f the stable manifold $W^s(q_g)$ of the continuation q_g of q for g meets the local unstable manifold $W^u_{loc}(\Lambda_g)$ of the continuation Λ_g of Λ for g. Recall that

$$W^u_{loc}(\Lambda_g) = \bigcup_{x\in\Lambda_g} W^u_{loc}(x)$$

and therefore the persistence of the heterodimensional cycle is the intersection between $W^s(q_g)$ and $W^u_{loc}(x)$ for some $x \in \Lambda$. We also can assume that q belongs to a non-trivial transitive hyperbolic

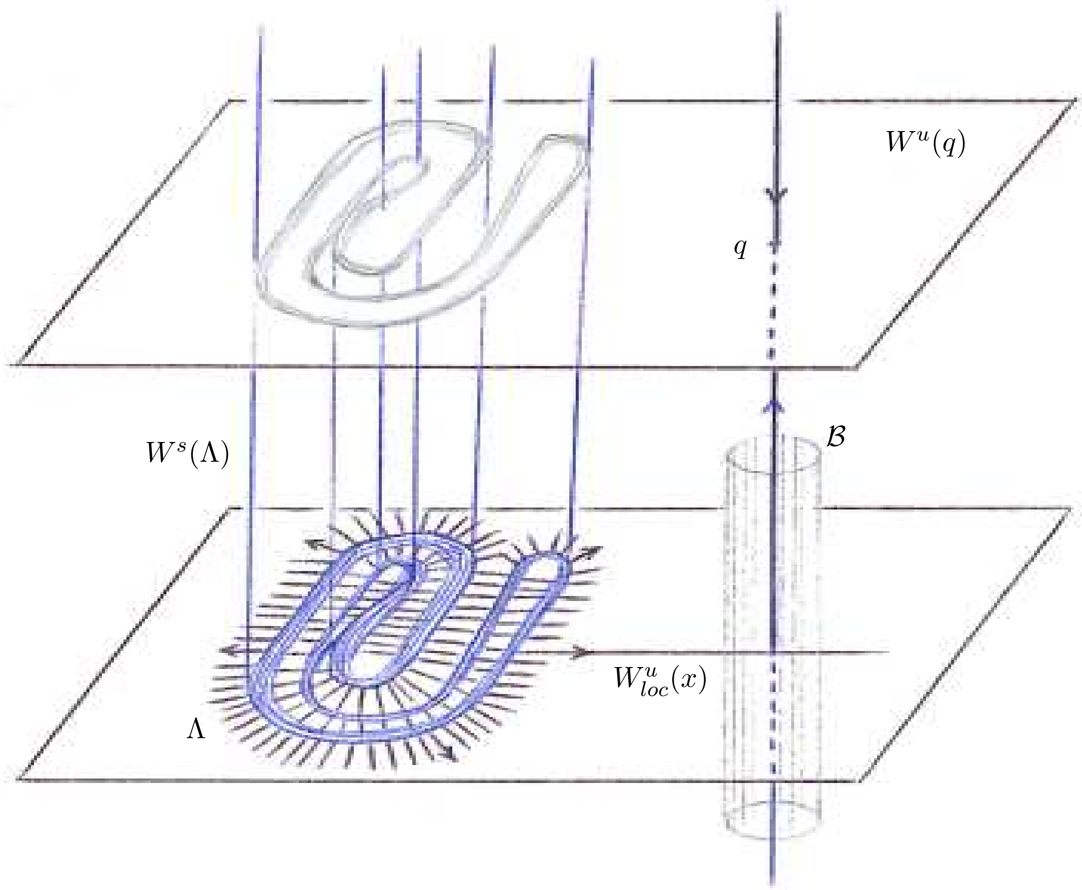

Fig. C: C^1-robust heterodimensional cycles

set Σ so that the robustness of the heterodimensional cycle is reduced to the intersection between the stable leaf of the continuation Σ_g of Σ for g, that is $W^s(x)$ where $x \in \Sigma_g$, and the local unstable leaf $W^u_{loc}(x)$ of Λ_g. In general, if Σ and Λ are Cantor sets then these laminations are a Cantor sets of segments and therefore persistence of the heterodimensional cycle may resemble to the construction of robust tangencies of Newhouse. However, the idea here to obtain the robustness is different. It is about increasing the topological dimension of the local unstable manifold of Λ in a robust sense. That is, for any diffeomorphism g close in the C^1-topology of f, the local unstable manifold $W^u_{loc}(\Lambda_g)$ of the continuation Λ_g of Λ is a topological surface.

The construction we will present here is based in the examples of Abraham-Smale [AS70], Simon [Sim72] or more recently [Asa08]. This construction uses a non-trivial (not a periodic orbit) hyperbolic transitive attractor Λ on surface. Plykin [Ply74] proved that if Λ is not just a periodic orbit, then the trapping region of Λ must have at least three holes removed (see also [Rob99]). Because of this theorem, any of these attractors are called *Plykin attractors.*

We consider a surface diffeomorphism $F : S \to S$ with a Plykin repeller Σ, i.e. a Plykin attractor for F^{-1}. Since Σ is a hyperbolic set then one has that $T_\Sigma S = E^{ss} \oplus E^u$ where E^{ss} and E^u are, respectively, the stable and unstable one-dimensional vector bundles. In particular, for each point $x \in \Sigma$ we follow the existence of one dimensional local unstable manifold $W^u_{loc}(x, F)$. Since Σ is a repeller, its local unstable manifold $W^u_{loc}(\Sigma, F)$ is a local compact neighborhood N of Σ on the surface. Namely, this local unstable manifold is foliated by the one dimensional leaves

$W^u_{loc}(x, F)$, $x \in \Sigma$. One can embed $N \times [-1, 1]$ in a compact three dimensional M and consider the diffeomorphism $f \in \mathrm{Diff}^1(M)$ coinciding with the map

$$f(x, y) = (F(x), \lambda y), \quad (x, y) \in \Sigma \times [-1, 1]$$

where $0 < \lambda < 1$ small enough. The set $\Lambda = \Sigma \times \{0\}$ is a hyperbolic basic set of f. The tangent bundle $T_\Lambda M$ splits in $E^s_\Lambda \oplus E^u_\Lambda$ where $E^s_\Lambda = E^{ss} \oplus E^{cs}$ and E^u_Λ is one-dimensional bundle. However, the unstable manifold of Λ is a topological surface homeomorphic to $N \times \{0\}$. We assume that the two dimensional stable manifold $W^s(\Lambda, f)$ of Λ transversally intersects the unstable manifold of an extra hyperbolic periodic point q of s-index equal to one. Also, we suppose that $W^s(q, f)$ quasi-transversally meets the local unstable manifold $W^u_{loc}((x, 0), f) = W^u_{loc}(x, F) \times \{0\}$ of some point $(x, 0) \in \Lambda$. Hence, f exhibits a heterodimensional cycle associated with Λ and q. Moreover, since the stable manifold of q transversally intersects the embedded repelling region of Σ (see Figure C) then $W^s(q, f)$ and $W^u(\Lambda, f)$ persistently intersect and so the heterodimensional cycle associated with Λ and q is C^1-robust.

The hyperbolic basic set Λ in the above example plays the role of thick horseshoes in Newhouse construction. In fact, the characteristic property of that basis set is that its unstable manifold intersects every one-dimensional disks contained in an open set as shown in Figure C. From this motivation we introduce the general definition of blender. This definition emphasizes the geometric aspects of a blender where the two key ingredients are the existence of a dominated splitting and of a superposition region. A splitting $T_\Lambda M = E_1 \oplus \cdots \oplus E_k$ over a f-invariant compact set Λ of a manifold M it is called *dominated splitting* if it is Df-invariant and there exist constants $C > 0$ and $0 < \lambda < 1$ such that for every $i < j$, every $x \in \Lambda$ and every pair of unit vector $u \in E_i(x)$ and $v \in E_j(x)$, one has $\|D_x f^n(u)\| \le C\lambda^n \|D_x f^{-n}(v)\|$ for all $n \ge 1$ and the dimension of $E_i(x)$ is independent of $x \in \Lambda$ for every $i \in \{1, \dots, k\}$.

Definition 1.3 (Blenders)**.** *Consider f a C^1-diffeomorphism of compact manifold M. Let $\Gamma \subset M$ be a transitive hyperbolic set of f with a dominated splitting of the form $E^{ss} \oplus E^{cs} \oplus E^u$, where its stable bundle $E^s = E^{ss} \oplus E^{cs}$ has dimension equal to $s \ge 2$ and E^{cs} is one-dimensional. We say that the set Γ is a* cs-blender *if has a C^1-*robust superposition region *$\mathcal{B}$:*

There are a C^1-neighborhood $\mathcal{V}$ of f and a C^1-open set $\mathcal{B}$ of embeddings of $s - 1$ dimensional disks D^s into M such that for every diffeomorphism $g \in \mathcal{V}$, every disk $D^s \in \mathcal{B}$ intersects the local unstable manifold $W^u_{loc}(\Gamma_g)$ of the continuation Γ_g of Γ for g.

A cu-blender *for f is defined as a cs-blender for f^{-1}.*

By definition, the property of a diffeomorphism having a *cs*-blender is a C^1-robust property. The notion of a blender was introduced in [BD96] as a class of examples, without a precise and formal definition. The above definition of blenders is given in [BDV05]. Blenders were used to get C^1-robust transitivity, [BD96], and robust heterodimensional cycles, [BD08]. The relevance of blenders comes from their internal geometry and not from their dynamics: a *cs*-blender is a (uniformly) hyperbolic transitive set whose unstable set robustly has Hausdorff dimension greater than its unstable bundle. In [BD11] Bonatti and Díaz defined a special class of blenders which they called *blender-horseshoes*, a sort of hyperbolic basic sets conjugated with a Smale horseshoe with geometrical properties resembling the thick horseshoes introduced by Newhouse. In this work, in the context of critical dynamics (some suitable non-domination property), is showed that the blender-horseshoes yield C^1-robust tangencies.

1.2.1 Non-normally hyperbolic horseshoes

In the example which motivated the definition of blender we consider a hyperbolic transitive basic set which is a normally repeller. That is, a planar repeller embedded in a three dimensional manifold, as an invariant, normally contracting submanifold. This section shows that it is also possible to embed a horseshoe in a higher dimensional manifold increasing the dimension of the unstable manifold. Following the results in [BDV95], we will show as a horseshoe can be perturbed to obtain a *thick* horseshoe where thick means with Hausdorff dimension greater or equal to one. That non-normally hyperbolic horseshoe explains how invariant manifolds (stable or unstable) associated with a hyperbolic bundle of dimension k may behave topologically as a manifold of dimension $k+1$. Firstly, we recall the concept of Hausdorff dimension.

Given $\alpha > 0$, the *Hausdorff α-measure* of a compact space X is

$$m_\alpha(X) = \lim_{\varepsilon\to 0^+} \inf \sum_{U\in\mathcal{U}} \mathrm{diam}(U)^\alpha,$$

where the infimum is taken over all finite coverings $\mathcal{U}$ of X by sets with diameter less than ε. Then there is a unique $d \in [0,\infty]$ such that $m_\alpha(X) = \infty$ if $\alpha < d$ and $m_\alpha(X) = 0$ if $\alpha > d$. We will denote $d = \mathrm{HD}(X)$ and it is said to be *Hausdorff dimension* of X. Here we make used of the fact that Hausdorff dimension is non-increasing under Lipschitz maps.

Since our construction is local, it is not restrictive to consider $M = \mathbb{R}^{n+1}$, $n \geq 2$ and we do so from now on. We begin taking a C^r-diffeomorphism F of $\mathbb{R}^n$ with a basic set Λ (a horseshoe) such that $F|_\Lambda$ is conjugated to the full shift of two symbols. Assume we can split $\mathbb{R}^n$ into stable and unstable variables $\mathbb{R}^n = \mathbb{R}^s \times \mathbb{R}^u$ such that

$$\Lambda = \bigcap_{i\in\mathbb{Z}} F^i(R), \quad \text{with } R = [-1,1]^s \times [-1,1]^u$$

and $F^{-1}(R)\cap R$ consisting of two connected component $\hat{R}_1 = [-1,1]^s \times R_1$ and $\hat{R}_2 = [-1,1]^s \times R_2$. We take F to be affine on each of these components: there are two linear maps

$$S_i\colon \mathbb{R}^s \to \mathbb{R} \quad \text{and} \quad U_i\colon \mathbb{R}^u \to \mathbb{R}^u,\ i = 1,2,$$

such that

$$DF|_{\hat{R}_i} = \begin{pmatrix} S_i & 0 \\ 0 & U_i \end{pmatrix}, \qquad \|S_i\|, \|U_i^{-1}\| < 1/2, \quad i = 1,2.$$

Consider $1/2 < \lambda < 1$ and the diffeomorphism

$$f\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}, \quad f(X,x) = (F(X), \lambda x).$$

Note that f has a horseshoe $\Gamma_f = \Lambda \times \{0\}$ and $\mathrm{HD}(\Gamma_f) = \mathrm{HD}(\Lambda)$. Let $P = (p^s, p^u)$ be some fixed point of F in Λ. Then $p = (P, 0)$ is a hyperbolic fixed point of f and

$$W^s(p,f) = W^s(P,F) \times \mathbb{R} \quad \text{and} \quad W^u(p,f) = W^u(P,F) \times \{0\}.$$

Assume that every contracting eigenvalue of $DF(P)$ is smaller than $1/2$ and then the strong stable manifold of p is

$$W^{ss}(p,f) = W^s(P,F) \times \{0\}.$$

Here, the tangent bundle T_pM splits into $E^s \oplus E^u$ where $E^s = E^{ss} \oplus E^{cs}$ and

$$E^{ss} = (\mathbb{R}^s \times \{0^u\} \times \{0\}), \quad E^{cs} = (\{0^s\} \times \{0^u\} \times \mathbb{R}), \quad E^u = (\{0^s\} \times \mathbb{R}^u \times \{0\}).$$

Since $W^s(P,F)$ and $W^u(P,F)$ meet each other transversely at some $X \in \mathbb{R}^n$ the manifolds $W^u(p,f)$ and $W^{ss}(p,f)$ have a quasi-transverse intersection at $x = (X,0)$. Then we consider an arc of C^r-diffeomorphisms $\{f_\mu\}_{\mu\in[-1,1]}$, with $f_0 = f$, which unfolds generically this intersection. This arc is defined by

$$f_\mu((x^s,x^u),x) = (F(x^s,x^u), \phi_\mu(x^u,x))$$

where

$$\phi_\mu(x^u,x) = \begin{cases} \lambda x & \text{if } x^u \in R_1, \\ \lambda x + \mu & \text{if } x^u \in R_2. \end{cases}$$

The following proposition shows an important geometrical property of this generic unfolding of a normally hyperbolic horseshoe (strong homoclinic intersection). As a consequence of this geometrical property we obtain a thick continuation horseshoes and increasing the topological dimension of the unstable manifold.

Proposition 1.9. *For each $\mu \in [-1,1]$, there is a cube $C_\mu = R \times I_\mu$ where I_μ is the close interval of endpoint 0 and $\mu(1-\lambda)^{-1}$ such that the unstable manifold $W^u(p_\mu, f_\mu)$ of the continuation p_μ of p for f_μ intersects transversely any cs-strip of the form*

$$[-1,1]^s \times \{x^u\} \times J, \quad \text{where } J \subset I_\mu \text{ is an open interval and } x^u \in R_1 \cup R_2.$$

Furthermore, the continuation Γ_μ of $\Gamma_0 = \Gamma_f$ for f_μ satisfies $\mathrm{HD}(\Gamma_\mu) \geq 1$ *for every $\mu \neq 0$.*

Proof. Suppose $\mu > 0$. The case $\mu < 0$ is completely analogous. Set $I_\mu = [0, \mu(1-\lambda)^{-1}]$ and let $p_\mu = (p^s_\mu, p^u_\mu, 0)$ be the continuation of p for f_μ. The following claim is the procedure of the classical blender argument in [BD96].

Claim 1.9.1. *Let $J \subset I_\mu$ be an open interval and $x^u \in R_1 \cup R_2$. Then, either*

- $f_\mu^{-1}([-1,1]^s \times \{x^u\} \times J)$ *intersects* $W^u_{loc}(p_\mu, f_\mu)$,
- *or it contains at least one cs-strip*

  $$[-1,1]^s \times \{\tilde{x}^u\} \times \tilde{J} \text{ with } \tilde{x}^u \in R_1 \cup R_2, \ \ \tilde{J} \subset I_\mu$$

  and the width $|\tilde{J}| = \lambda^{-1}|J| > |J|$.

Proof of the claim. Note that the image by F^{-1} of a s-strip $[-1,1]^s \times \{x^u\} \subset \hat{R}_1 \cup \hat{R}_2$ contains two disjoint s-strip. That is, there exist $x^u_1 \in R_1$ and $x^u_2 \in R_2$ such that

$$F^{-1}([-1,1]^s \times \{x^u\}) \supset ([-1,1]^s \times \{x^u_1\}) \cup ([-1,1]^s \times \{x^u_2\}).$$

Take $\phi_i : I_\mu \to I_\mu$ given by $\phi_i = \phi_\mu|_{R_i \times I_\mu}$, $i = 1,2$. Since $\lambda > 1/2$, $I_\mu = \phi_1(I_\mu) \cup \phi_2(I_\mu)$. Thus either $J \subset \phi_i(I_\mu)$ for some i or $\phi_1(I_\mu) \cap \phi_2(I_\mu) \subset J$. In the first case, we have that $\tilde{J} = \phi_i^{-1}(J) \subset I_\mu$ with the width $|\tilde{J}| = \lambda^{-1}|J|$ and $f_\mu^{-1}([-1,1]^s \times \{x^u\} \times J) \supset [-1,1]^s \times \{x^u_i\} \times \tilde{J}$. In the other case, $0 \in \phi_1^{-1}(J)$ and so $f_\mu^{-1}([-1,1]^s \times \{x^u\} \times J) \supset [-1,1]^s \times \{x^u_1\} \times \{0\}$. Note that

$$W^u_{loc}(p_\mu, f_\mu) = \{p^s_\mu\} \times [-1,1]^u \times \{0\}$$

and therefore the image by f_μ^{-1} of the cs-strip $[-1,1]^s \times \{x^u\} \times J$ intersects $W^u_{loc}(p_\mu, f_\mu)$. $\square$

Repeating this procedure, we get an intersection point between $W^u_{loc}(p_\mu, f_\mu)$ and a backward iterate of the cs-strip $[-1,1]^s \times \{x^u\} \times J$. It gives in turn a transverse intersection point between the initial *cs*-strip and $W^u(p_\mu, f_\mu)$. This ends the proof of the first part of the proposition.

As for the second one, it is now a direct consequence. From the first part of this lemma, for all open interval $J \subset I_\mu$ one has the unstable manifold $W^u(p_\mu, f_\mu)$ of p_μ transversally intersect the *cs*-strip $[-1,1]^s \times \{p^u_\mu\} \times J$. Note that this *cs*-strip is contained in $W^s_{loc}(p_\mu, f_\mu)$ then for every $J \subset I_\mu$ open interval

$$([-1,1]^s \times \{p^u_\mu\} \times J) \cap W^u(p_\mu, f_\mu) \cap W^s(p_\mu, f_\mu) \cap C_\mu \neq \emptyset.$$

Therefore $I_\mu \subset \pi(H(p_\mu, f_\mu))$, where $\pi : R \times \mathbb{R} \to \mathbb{R}$, $\pi(X,x) = x$.

Denote by Γ_μ the continuation for f_μ of the basic set $\Gamma_0 = \Lambda \times \{0\}$ of f_0. Observe that Λ_μ is the maximal invariant set in $C_\mu = R \times I_\mu$ and coincides with the closure of all transverse homoclinic points of p_μ in C_μ. That is, $\Gamma_\mu = H(p_\mu, f_\mu) \cap C_\mu$. Finally, since π is a Lipschitz map, it follows that

$$\mathrm{HD}(\Gamma_\mu) \geq \mathrm{HD}(\pi(\Gamma_\mu)) \geq 1,$$

which proves the lemma. □

1.2.2 Blender-horseshoes

Blender-horseshoe was introduced in [BD11] as a special type of blender. A cs-blender-horseshoe Γ is the maximal invariant set in a cube C and it has a hyperbolic splitting with three non-trivial bundles

$$T_\Gamma M = E^{ss} \oplus E^{cs} \oplus E^{uu},$$

such that the stable bundle of Γ is $E^s = E^{ss} \oplus E^{cs}$ and E^{cs} is one-dimensional. Moreover, the set Γ is conjugated to the complete shift of two symbols (the usual Smale horseshoes). Thus the blender has exactly two fixed points, say P and Q, called *distinguished points of the blender*.

Consider the cube

$$C = [-1,1]^{n+1} = [-1,1]^s \times [-1,1] \times [-1,1]^u$$

with $n \geq 2$. We split the boundary of C into three parts:

$$\begin{aligned}
\partial^s C &= \partial([-1,1]^s) \times [-1,1] \times [-1,1]^u, \\
\partial^c C &= [-1,1]^s \times [-1,1] \times \{-1,1\} \times [-1,-1]^u, \\
\partial^u C &= [-1,1]^s \times [-1,1] \times \partial[-1,1]^u.
\end{aligned}$$

Let us define a local diffeomorphism $f\colon C \to \mathbb{R}^{n+1}$ having a maximal invariant set Γ in the cube C,

$$\Gamma = \bigcap_{i\in\mathbb{Z}} f^i(C)$$

and satisfying the following C^1-robust condition (BH1)-(BH6). Blender-horseshoes will be defined through this local diffeomorphism (see Definition 1.7).

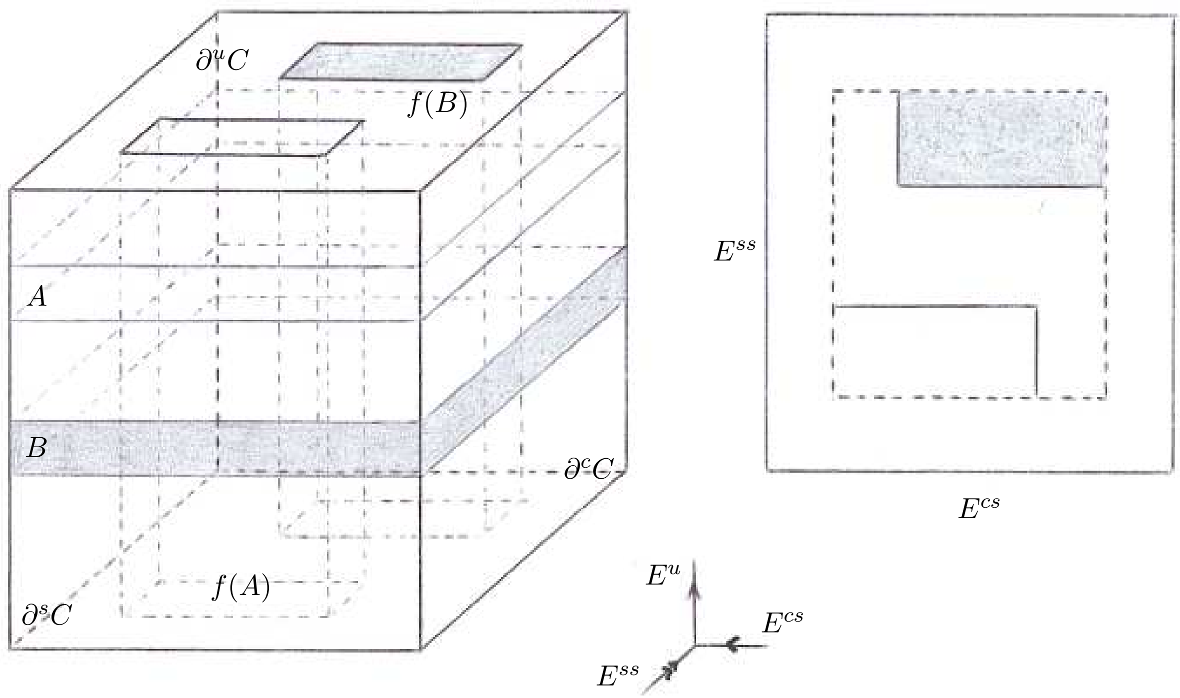

Fig. D: Reference cube of blender-horseshoe

(BH1) *Associated Markov partition*: The intersection $f(C) \cap C$ consist of two connected component, denoted $f(A)$ and $f(B)$. Furthermore,

- The sets A and B are the non-empty connected components of $f^{-1}(C) \cap C$.
- $f(A) \cup f(B)$ is disjoint from $\partial^s C \cup \partial^c C$ and $A \cup B$ is disjoint from $\partial^u C$.

More precisely,

$$f(A) \cup f(B) \subset (-1,1)^s \times (-1,1) \times [-1,1]^u$$
$$A \cup B \subset [-1,1]^s \times [-1,1] \times (-1,1)^u.$$

(BH2) *Cone-fields*: There are families of cones $\mathcal{C}^s_\alpha(x)$, $\mathcal{C}^{ss}_\alpha(x)$, $\mathcal{C}^u_\alpha(x)$ define for each $\alpha \in (0,1)$ and $x \in \mathbb{R}^{n+1}$ as

$$\mathcal{C}^s_\alpha(x) = \{(v^s, v^c, v^u) \in \mathbb{R}^s \oplus \mathbb{R}^c \oplus \mathbb{R}^u = T_x M \;:\; \|v^u\| \le \alpha \|v^s + v^c\|\},$$
$$\mathcal{C}^{ss}_\alpha(x) = \{(v^s, v^c, v^u) \in \mathbb{R}^s \oplus \mathbb{R}^c \oplus \mathbb{R}^u = T_x M \;:\; \|v^c + v^u\| \le \alpha \|v^s\|\},$$
$$\mathcal{C}^u_\alpha(x) = \{(v^s, v^c, v^u) \in \mathbb{R}^s \oplus \mathbb{R}^c \oplus \mathbb{R}^u = T_x M \;:\; \|v^s + v^c\| \le \alpha \|v^u\|\}.$$

These cone-fields satisfice the following properties: there is $0 < \alpha' < \alpha$ such that, for every $x \in f(A) \cup f(B)$

$$Df^{-1}(\mathcal{C}^s_\alpha(x)) \subset \mathcal{C}^s_{\alpha'}(f^{-1}(x)) \quad \text{and} \quad Df^{-1}(\mathcal{C}^{ss}_\alpha(x)) \subset \mathcal{C}^{ss}_{\alpha'}(f^{-1}(x)),$$

and for every $x \in A \cup B$

$$Df(\mathcal{C}^u_\alpha(x)) \subset \mathcal{C}^u_{\alpha'}(f(x)).$$

Moreover, the cones-field C^u_α and C^s_α are uniformly expanding and contracting respectively.

As a consequence of (BH2), the maximal invariant set Γ in the cube C has a hyperbolic splitting $T_\Gamma M = E^s \oplus E^u$ where $E^s = E^{ss} \oplus E^{cs}$. We say that E^{ss} and E^{cs} are the strong unstable bundle and the one-dimensional central-stable bundle of Γ, respectively. Thus, by (BH1), $\{A, B\}$ is a Markov partition generating Λ. Therefore the dynamics of f in Λ is conjugate to the full shift of two symbols. In particular, the hyperbolic set Λ contains exactly two fixed points of f, $P \in A$ and $Q \in B$. The *local invariant manifolds* $W^s_{loc}(P)$, $W^{ss}_{loc}(P)$ and $W^u_{loc}(P)$ are the connected components of the intersections of $W^s(P) \cap C$, $W^{ss}(P) \cap C$ and $W^u(P) \cap C$ containing P, respectively. The definition of the local invariant manifolds of Q is analogous.

Definition 1.4 (ss-disk)**.** *A disk $\Delta \subset [-1,1]^s \times \mathbb{R} \times \mathbb{R}^u$ of dimension s is a ss-disk if $T_x\Delta \subset \mathcal{C}^{ss}_\alpha(x)$ for all $x \in \Delta$ and $\partial\Delta \subset \partial([-1,1]^s) \times \mathbb{R} \times \mathbb{R}^u$.*

(BH3) *ss-disk through the local unstable manifold of P and Q:* Let Δ and Δ' be two different ss-disks such that $\Delta \cap W^u_{loc}(P) \neq \emptyset$ and $\Delta' \cap W^u_{loc}(P) \neq \emptyset$. Then

$$\Delta \cap \partial^c C = \Delta' \cap \partial^c C = \Delta \cap \Delta' = \emptyset.$$

Similar assumption for ss-disks through the local unstable manifold of Q.

There are two different homotopy classes of ss-disks contained in $[-1,1]^s \times \mathbb{R} \times [-1,1]^u$ and disjoint from $W^u_{loc}(P)$. We call these classes *ss-disks at the right* and *at the left* of $W^u_{loc}(P)$. We use the following criterion:

Definition 1.5 (ss-disk at the right and left)**.** *The ss-disks that do not intersect $W^u_{loc}(P)$ in the homotopy class of $W^{ss}_{loc}(Q)$ are at the right of $W^u_{loc}(P)$. The ss-disks disjoint from $W^u_{loc}(P)$ in the other homotopy class are at the left of $W^u_{loc}(P)$. Similarly ss-disks at the left and at the right of $W^u_{loc}(Q)$, where ss-disks at the left of $W^u_{loc}(Q)$ are in the class of $W^{ss}_{loc}(P)$.*

(BH4) *Position of preimages of ss-disk (I)*: Given any ss-disk $\Delta \subset C$, the following holds:

i) if Δ is at the right of $W^u_{loc}(P)$ then $f^{-1}(\Delta \cap f(A))$ is a ss-disk at the right of $W^u_{loc}(P)$,
ii) if Δ is at the left of $W^u_{loc}(P)$ then $f^{-1}(\Delta \cap f(A))$ is a ss-disk at the left of $W^u_{loc}(P)$,
iii) if Δ is at the right of $W^u_{loc}(Q)$ then $f^{-1}(\Delta \cap f(B))$ is a ss-disk at the right of $W^u_{loc}(Q)$,
iv) if Δ is at the left of $W^u_{loc}(Q)$ then $f^{-1}(\Delta \cap f(B))$ is a ss-disk at the left of $W^u_{loc}(P)$,
v) if Δ is at the left of $W^u_{loc}(P)$ or $\Delta \cap W^u_{loc}(P) \neq \emptyset$ then $f^{-1}(\Delta \cap f(B))$ is a ss-disk at the left of $W^u_{loc}(P)$, and
vi) if Δ is at the right of $W^u_{loc}(Q)$ or $\Delta \cap W^s_{loc}(Q) \neq \emptyset$ then $f^{-1}(\Delta \cap f(A))$ is a ss-disk at the right of $W^u_{loc}(Q)$.

Finally, we state the last condition (which will play a key role) in the definition of blender-horseshoes. We need the following concept:

Definition 1.6 (ss-disk in between)**.** *A ss-disk is in between $W^u_{loc}(P)$ and $W^u_{loc}(Q)$ if it is a ss-disk at the right of $W^u_{loc}(P)$ and at the left of $W^u_{loc}(Q)$.*

(BH5) *Position of preimages of ss-disk (II)*: Let Δ be a ss-disk in between $W^u_{loc}(P)$ and $W^u_{loc}(Q)$. Then either $f^{-1}(\Delta \cap f(A))$ or $f^{-1}(\Delta \cap f(B))$ is a ss-disk in between $W^u_{loc}(P)$ and $W^u_{loc}(Q)$.

As a consequence of (BH4)-(BH5) we obtain an open region of ss-disk in between:

Remark 1.10 (C^1-open set of embedding ss-disk)**.** *There is a non-empty open subset U of C such that any ss-disk through a point $x \in U$ is in between $W^u_{loc}(P)$ and $W^u_{loc}(Q)$. In particular, every ss-disk $\Delta \subset [-1,1]^s \times \mathbb{R} \times [-1,1]^u$ in between $W^u_{loc}(P)$ and $W^s_{loc}(Q)$ is contained in C and is also disjoint from $\partial^c C$.*

We are now ready to define blender-horseshoes:

Definition 1.7 (Blender-horseshoes)**.** *Let M be a manifold of dimension $n \geq 3$ and $f\colon M \to M$ be a C^1-differomorphism. A hyperbolic set Γ of f is a* cs-blender-horseshoe *if there are a cube C and families of cone-fields $\mathcal{C}^s$, $\mathcal{C}^{ss}$, and $\mathcal{C}^u$ verifying conditions (BH1)-(BH5).*

We say that C is the reference cube *of the cs-blender-horseshoe Γ and that the saddles P and Q are* distinguished saddles points *of Γ. A cu-blender-horseshoe is a cs-blender-horseshoe for f^{-1}.*

In [BD11] is shown that the conditions (BH1)-(BH5) are C^1-robust. This means that there is a C^1-neighborhood $\mathcal{V}$ of f such that for all $g \in \mathcal{V}$ the continuation Γ_g of Γ for g is a blender-horseshoe with reference cube C and distinguished saddles points P_g and Q_g.

Definition 1.8 (cs-strip)**.** *A cs-strip S through the cube C is the image by a diffeomorphism $\phi\colon [-1,1]^s \times [-1,1] \to C$ such that:*

- *$T_xS \subset \mathcal{C}^s_\alpha(x)$ for all $x \in S$,*
- *for each $t \in [-1,1]$ the curves $S_t = \phi([-1,1]^s, t)$ satisfice that $T_xS_t \subset \mathcal{C}^{ss}_\alpha(x)$ for all $x \in S_t$ and $\partial^s S_t = \phi(\partial([-1,1]^s), t) \subset \partial^s C$.*

The width of S, denoted by $|S|$, is the minimal length of the curves tangents to the central-stable direction E^{cs} contained in S joining $\phi([-1,1]^s, -1)$ and $\phi([-1,1]^s, 1)$.

The following lemma shows that a cs-blender-horseshoe Γ is a cs-blender in the sense of Definition 1.3, where the ss-disks in between $W^s_{loc}(P)$ and $W^s_{loc}(Q)$ define its superposition region. Again, the argument here is the classical blender argument in [BD96].

Lemma 1.11. *Let Γ be a blender-horseshoe of a diffeomorphism f with reference cube C and distinguished saddles P and Q. Then every ss-disk in between $W^u_{loc}(P)$ and $W^u_{loc}(Q)$ intersects $W^u_{loc}(\Gamma)$.*

Proof. Note that a cs-strip S is foliated by the family of ss-disk S_t where $t \in [-1,1]$. We say that a cs-strip S is in between $W^u_{loc}(P)$ and $W^u_{loc}(Q)$ if all of ss-disk S_t are in between $W^u_{loc}(P)$ and $W^u_{loc}(Q)$.

Claim 1.11.1. *Let S be a cs-strip in between $W^u_{loc}(P)$ and $W^u_{loc}(Q)$. Then, there is $\lambda > 1$ such that either*

- *$f^{-1}(S)$ intersects $W^u_{loc}(P)$,*
- *or it contains at least one cs-strip $\tilde{S}$ in between $W^u_{loc}(P)$ and $W^u_{loc}(Q)$ with width $|\tilde{S}| \geq \lambda|S|$.*

Proof. Since $S \subset C$ is a cs-strip foliated by ss-disk S_t, then $f^{-1}(S)$ contains at least the union of two cs-strip $f^{-1}(S\cap f(A))$ and $f^{-1}(S\cap f(B))$ by the hypothesis (B5). Their width is larger than $\lambda|S|$, where $\lambda>1$ is a lower bound of the expansion of Df^{-1} in the central-stable direction E^{cs} inside C. We assume by contradiction that neither of them intersect $W^u_{loc}(P)$ nor it is a cs-strip in between $W^u_{loc}(P)$ and $W^u_{loc}(Q)$. Since S is cs-strip in between $W^u_{loc}(P)$ and $W^u_{loc}(Q)$, in particular, for each $t\in[-1,1]$ one has that S_t is a ss-strip at the right of $W^u_{loc}(P)$ and at the left of $W^u_{loc}(Q)$. Thus, by (BH5) the ss-strip $f^{-1}(S_t\cap f(A))$ is at the right of $W^u_{loc}(P)$. By assumption, it is also at the right of $W^u_{loc}(Q)$. The same argument for the ss-strip $f^{-1}(S_t\cap f(B))$ shows that it is at the left of $W^u_{loc}(P)$. However, this contradicts (BH6) since either $f^{-1}(S_t\cap f(A))$ or $f^{-1}(S_t\cap f(B))$ is a ss-disk in between $W^u_{loc}(P)$ and $W^u_{loc}(Q)$. □

Repeating this procedure, we get an intersection point between $W^u_{loc}(P)$ and a backward iterate of the cs-strip S. It gives in turn a transverse intersection point between the initial cs-strip and $W^u(P)$. Let D^s be a ss-disk in between $W^u_{loc}(P)$ and $W^u_{loc}(Q)$. Consider a nested sequence of cs-strip S^n such that for some $t_n\in[-1,1]$ the ss-disk $S^n_{t_n}=D^s$ for all $n\geq 1$. Then by the above observation $W^u(P)\cap S^n\neq\emptyset$ for all $n\geq 1$. Now, for each $n\in\mathbb{N}$, we consider $z_n\in\Gamma\cap W^u(P)$ such that $W^u_{loc}(z_n)\cap S^n\neq\emptyset$. Let $z\in\Gamma$ be an accumulation point of the sequence $(z_n)_n\subset\Gamma$. Then $z\in\Gamma\cap\overline{W^u(P)}$ and from the election of the nested sequence of cs-strip S^n (that is $D^s\subset S^{n+1}\subset S^n$) it follows $W^u_{loc}(z)\cap D^s\neq\emptyset$. Therefore $W^u_{loc}(\Gamma)$ intersects every ss-disk between $W^u_{loc}(P)$ and $W^u_{loc}(Q)$ and we conclude the proof. □

A blender-horseshoes example: non-normally hyperbolic horseshoes

Now, we will show that the non-normally hyperbolic horseshoes construct in §1.2.1 are really cs-blender-horseshoes. We recall that for each fixed $\lambda\in(1/2,1)$, we constructed an arc of local C^r-diffeomorphism $\{f_{\lambda,\mu}\}_{\mu\in[-1,1]}$ of $\mathbb{R}^{n+1}$ given by

$$f_{\mu,\lambda}(X,x)=(F(X),\phi_{\lambda,\mu}(X,x)) \tag{1.1}$$

where $F\colon\mathbb{R}^n\to\mathbb{R}^n$ has an affine Smale horseshoe Λ with Markov partition $\{\hat{R}_1,\hat{R}_2\}$ and

$$\phi_{\lambda,\mu}(X,x)=\begin{cases}\lambda x & \text{if } X\in\hat{R}_1,\\ \lambda x+\mu & \text{if } X\in\hat{R}_2.\end{cases} \tag{1.2}$$

We denote by $\Gamma_{\lambda,\mu}$ the maximal invariant set of $f_{\lambda,\mu}$ in the cube $C_{\lambda,\mu}=R\times I_{\lambda,\mu}$ given in the Proposition 1.9. Let C be a big cube containing $C_{\lambda,\mu}$ for all $\mu\in[-1,1]$ and $\lambda<1$ close to 1 such that $\Gamma_{\lambda,\mu}$ is also the maximal invariant set in C. Recall that $\Gamma_{\lambda,0}=\Lambda\times\{0\}$ and thus $f_{\lambda,0}$ has two different fixed points $p=(P,0)$ and $q=(Q,0)$ in $\Gamma_{\lambda,0}$. Let $p_{\lambda,\mu}$ and $q_{\lambda,\mu}$ be the continuation points of p and q for $f_{\lambda,\mu}$.

Proposition 1.12. *For every $\lambda<1$ close to 1 and $\mu\in[-1,1]$, the set $\Gamma_{\lambda,\mu}$ is a cs-blender-horseshoe with reference cube C and distinguished saddles point $p_{\lambda,\mu}$ and $q_{\lambda,\mu}$.*

The proof of this proposition can be found in [BD11, Propisition 5.1]. In the next chapter, we will give an estimate on how much should be λ close to 1. Note that $F|_\Lambda$ is conjugated to symbolic dynamics. For this reason, we will focus to understand how the C^1-perturbations of this class of non-normally hyperbolic skew-product diffeomorphisms can be studied by means of skew-products with symbolic dynamics on the base. This allows us to know more about this class of blender-horseshoe examples.

Symbolic blenders

Remarkable partially hyperbolic diffeomorphisms are the skew-products over hyperbolic sets, namely, over horseshoes. These horseshoes in the base are well understood from symbolic dynamics. In fact, it shows that C^1-perturbations of a dominated skew-product diffeomorphisms of this type can be understood through the study of perturbations of symbolic Hölder skew-products. Geometrical properties as the existence of strong stable and unstable sets, holonomies or invariant graphs, are studied for symbolic skew-products. Symbolic cs-blender-horseshoes are introduced as locally maximal invariant sets of symbolic Hölder skew-products with contracting fiber maps. These invariant sets meet, in a robust sense, any almost horizontal disk through an open region and thus, they are understood as blenders with center bundle of any dimension. Symbolic blender-horseshoe examples are constructed from dominated symbolic one-step skew-product maps with covering property in a bounded open set.

2.1 Partial hyperbolicity and skew-product diffeomorphisms

An important restriction in the general definition of cs-blender, Definition 1.3, is that the center direction E^{cs} of the blender is one-dimensional. This is an important constrain for applications in several settings where the center bundle is two-dimensional. Thus, a natural question is to construct blenders where this center bundle has dimension bigger than one. A first approach to this problem was done by Nassiri-Pujals in [NP12] where symbolic blenders were introduced to build robust transitive sets in symplectic diffeomorphisms and Hamiltonian systems.

The construction of blender-horseshoes involves a diffeomorphism f defined in a reference cube $C = [-1,1]^{n+1}$, $n \geq 2$. The blender-horseshoe is the maximal invariant set Γ of f in C which is conjugated to a Smale horseshoe. Blender-horseshoes examples (non-normally hyperbolic horseshoes) are constructed in the context of skew-product C^1-diffeomorphisms

$$f : C \subset \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}, \qquad f(x,y) = \big(F(x), \phi(x,y)\big),$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ has a horseshoe $\Lambda \subset [-1,1]^n$ and $\phi(x,\cdot)\colon \mathbb{R} \to \mathbb{R}$ is a C^1-contraction. A problem in this context (regarding the robustness of the blender) is that the diffeomorphisms g close to f are not necessarily skew-product maps. This difficulty is solved using the normal hyperbolic theory [HPS77] since with additional assumptions concerning the strength of the hyperbolic splitting on C we can conclude that g is conjugated to a skew-product map. In that case, the open set $\mathcal{V}$ in Definition 1.3 can be taken consisting of skew-product maps. Also, since $F|_\Lambda$ is conjugated to the Bernoulli shift, these blender-horseshoes examples can be studied from the symbolic point of view. In what follows of this section we will explain the details of this reduction. Firstly, it is necessarily to introduce partial hyperbolic diffeomorphisms.

2.1.1 Partially hyperbolic diffeomorphisms

We say that a C^1-diffeomorphism $f : M \to M$ of a compact Riemannian manifold M is *partially hyperbolic* if there is a nontrivial Df-invariant splitting of the tangent bundle

$$TM = E^s \oplus E^c \oplus E^u \tag{2.1}$$

and there exists a Riemannian metric for which we can choose continuous positives functions ν, $\hat{\nu}$, γ and $\hat{\gamma}$ with

$$\nu,\, \hat{\nu} < 1 \qquad \text{and} \qquad \nu < \gamma < \hat{\gamma}^{-1} < \hat{\nu}^{-1} \tag{2.2}$$

such that, for any unit vector $v \in T_xM$,

$$\begin{aligned} \|D_xf(v)\| &< \nu(x) && \text{if } v \in E^s_x, && (2.3)\\ \gamma(x) < \|D_xf(v)\| &< \hat{\gamma}(x)^{-1} && \text{if } v \in E^c_x, && (2.4)\\ \hat{\nu}(x)^{-1} < \|D_xf(v)\| & && \text{if } v \in E^u_x. && (2.5)\end{aligned}$$

In another words, $D_xf|_{E^s_x}$ is a uniform contraction, $D_xf|_{E^u_x}$ is a uniform expansion, and, the behavior of $D_xf|_{E^c_x}$ lies in between those two (not quite as contracting nor as expanding, respectively). Partial hyperbolicity is a C^1-open condition: any diffeomorphism sufficiently C^1-close to a partially hyperbolic diffeomorphism is itself partially hyperbolic.

The *stable* and *unstable bundles* E^s and E^u of f are uniquely integrable and their integral manifolds form two transverse (continuous) foliations $\mathcal{W}^s$ and $\mathcal{W}^u$, whose leaves are immersed submanifolds of the same class of differentiability as f. These foliations are referred to as the *strong stable* and *strong unstable* foliations. They are f-invariant, meaning that

$$f(W^{ss}(x)) = W^{ss}(f(x)) \quad \text{and} \quad f^{-1}(W^{uu}(x)) = W^{uu}(f^{-1}(x)),$$

where $W^{ss}(x)$ and $W^{uu}(x)$ denote the leaves of $\mathcal{W}^s$ and $\mathcal{W}^u$, respectively, passing through $x \in M$.

The *center bundle* E^c is not always integrable. An invariant center foliation is obtain assuming that f is dynamical coherent. A partial hyperbolic diffeomorphism f is said to be *dynamically coherent* if there exist f-invariant *center-stable* and *center-unstable foliations* $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$, tangent to the bundles $E^s \oplus E^c$ and $E^c \oplus E^u$, respectively. An invariant *center foliation* $\mathcal{W}^c$ is followed intersecting the leaves of $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ (see [BW08]). It is not known whether every perturbation of a dynamically coherent diffeomorphism is dynamically coherent, but this holds for systems that are plaque expansive. In order to define plaque expansivity, we introduce the notion of central-plaque. A *central-plaque* of a small enough length $\delta > 0$ in $\mathcal{W}^c$ through $x \in M$, denoted by $W^c_\delta(x)$, is the connected component of $W^c(x) \cap B(x, \delta)$ containing x, where $W^c(x)$ is the leaf of $\mathcal{W}^c$ passing through of x and $B(x, \delta)$ denotes the open ball centering in x of radius δ. Then, roughly speaking, f is *plaque expansive* if there exists $\varepsilon > 0$ such that any two ε-pseudo-orbits in different central-plaques will eventually (under forward or backward iterates) be separated by a distance ε. The notion of plaque expansiveness was introduced by Hirsch, Pugh, and Shub [HPS77]. They proved, among other things, that any C^1-perturbation of a plaque expansive partial hyperbolic diffeomorphism is dynamically coherent. Plaque expansiveness holds in a variety of natural settings; in particular if f is dynamically coherent, and either $\mathcal{W}^c$ is a C^1 foliation or the restriction of f to $\mathcal{W}^c$ leaves is an isometry, then f is plaque expansive, and so every C^1-perturbation of f is dynamically coherent.

If f is dynamically coherent, then each leaf of $\mathcal{W}^{cs}$ is simultaneously subfoliated by the leaves of $\mathcal{W}^c$ and by the leaves of $\mathcal{W}^s$. Similarly $\mathcal{W}^{cu}$ is subfoliated by $\mathcal{W}^c$ and $\mathcal{W}^u$. This implies that for any two points $x, y \in M$ with $y \in W^{ss}(x)$ there is a homeomorphism $h^s_{x,y} : W^c_{\delta'}(x) \to W^c_\delta(y)$ between central-plaques $W^c_{\delta'}(x)$ and $W^c_\delta(y)$ with the property that $h^s_{x,y}(x) = y$ and, in general,

$$h^s_{x,y}(z) \in W^{ss}(z) \cap W^c_\delta(y).$$

We refer to $h^s_{x,y}$ as a *(local) stable holonomy map.* We similarly define unstable holonomy maps between local center leaves.

Because of Df restricted to the stable (reps. unstable) bundle is uniformly contracting (resp. expanding), the leaves of strong stable (resp. unstable) foliation are always contractible (resp. expansible). This is not the case for center foliations. We say that f is *center bunched* if the functions ν, $\hat{\nu}$, γ and $\hat{\gamma}$ can be chosen so that can be

$$\nu < \gamma\hat{\gamma} \qquad \text{and} \qquad \hat{\nu} < \gamma\hat{\gamma}. \tag{2.6}$$

It is said that $Df|_{E^c}$ is conformal if $\|D_x f(v)\| = \|D_x f|_{E^c_x}\|$ for any unit vector $v \in E^c_x$. In this case, we can choose both $\gamma(x)$ and $\hat{\gamma}(x)^{-1}$ slightly smaller and bigger than $\|D_x f|_{E^c_x}\|$ respectively. By doing this, we may make the ratio $\gamma(x)/\hat{\gamma}(x)^{-1} = \gamma(x)\hat{\gamma}(x)$ arbitrarily close to 1, and hence, larger than both $\nu(x)$ and $\hat{\nu}(x)$. That is, center bunching always holds when $Df|_{E^c}$ is conformal. In particular, center bunching holds whenever E^c is one-dimensional since in this case $Df|_{E^c}$ is conformal. Center bunching means that the hyperbolicity of f dominates the nonconformality of Df on E^c. Notice that the center bunching property is C^1-open: any sufficiently small C^1-perturbation of a center bunched partial hyperbolic diffeomorphism is center bunched.

According to [PSW97], when f is dynamically coherent, center bunched inequalities (2.6) ensure that the leaves of $\mathcal{W}^{cs}$, $\mathcal{W}^{cu}$, and $\mathcal{W}^c$ are C^1. If f is C^2 and dynamically coherent then these inequalities also imply that the local stable and local unstable holonomies are C^1 local diffeomorphisms. In general, without this additional regularity assumption, in [PSW97, AV10, PSW11, SW00] it is proved that the holonomies maps are only Hölder continuous homeomorphisms.

The Cartesian product of an Anosov diffeomorphism with the identity or with an isometry such as a rotation, provides trivial examples of partially hyperbolic dynamical systems. The second factor is the central direction. The same holds if the second factor is any dynamical system whose maximal expansion is separated from the slowest expansion rate of the Anosov diffeomorphism and likewise for the contraction rates. A slight generalization of this idea is that of skew-product maps. Examples of these can be obtained from an Anosov diffeomorphism F on a manifold X and a family $g_x : Y \to Y$ for $x \in X$ whose rates are again uniformly inside the rate gap of f by setting $f(x, y) = (F(x), g_x(y))$.

As in the case of uniformly hyperbolic dynamical systems, the definition of partial hyperbolic extends readily to compact invariant sets. In general, a partial hyperbolic set is defined to be a compact invariant set Λ of a diffeomorphism f such that the tangent space at every $x \in \Lambda$ admits an invariant splitting as (2.1) that satisfies the contraction and expansion conditions described in (2.3)-(2.5). In what follows, we will restrict our attention to study partially hyperbolic sets for skew-product C^1-diffeomorphisms.

2.1.2 Skew-products over hyperbolic sets

Let X be a compact Riemannian manifold and suppose that $F : X \to X$ is a C^2-diffeomorphism with a locally maximal hyperbolic invariant set $\Lambda \subset X$. Assume that there is a DF-invariant splitting of the tangent bundle

$$T_\Lambda X = E^s_\Lambda \oplus E^u_\Lambda$$

and there exists a Riemannian metric on X for which we can choose real numbers $0 < \mu \leq \nu < 1$ such that

$$\mu \leq \|D_x F(v)\| \leq \nu \quad \text{and} \quad \mu \leq \|D_x F^{-1}(w)\| \leq \nu$$

for all unit vectors $v \in E^s_x$, $w \in E^u_x$ and $x \in \Lambda$. Note that if $\mu = 0$, then we get the standard notion of hyperbolicity. In the sequel, let us consider that the locally maximal hyperbolic invariant set Λ in the above conditions is a horseshoe with k legs. That is, $F|_\Lambda$ is conjugated to the Bernoulli shift $\tau : \Sigma_k \to \Sigma_k$ where $\Sigma_k = \{1, \dots, k\}^{\mathbb{Z}}$ denotes the space of bi-sequences of k symbols.

In order to define a skew-product diffeomorphism over F, we take another compact manifold Y and consider the Cartesian product $X \times Y$. A skew-product diffeomorphism over F is defined as any C^1-diffeomorphism of the form

$$f : X \times Y \to X \times Y, \quad f(x, y) = (F(x), \phi(x, y)) \tag{2.7}$$

where $\phi(x, \cdot) : Y \to Y$ is a family of C^1-diffeomorphisms such that there are positive numbers γ and $\hat{\gamma}$ satisfying

$$\gamma\, d_Y(y, y') < d_Y(\phi(x, y), \phi(x, y')) < \hat{\gamma}^{-1}\, d_Y(y, y'), \tag{2.8}$$

for all y, $y' \in Y$ and $x \in \Lambda$.

Note that, since f is a C^1-diffeomorphism then $d_Y(\phi(x, y), \phi(x', y)) \leq d(f(x, y), f(x', y)) \leq \|f\|\, d_X(x, x')$ for all $y \in Y$ and x, $x' \in X$ where $\|f\|$ denotes the Lipschitz constant of f. Similarly, for each $y \in Y$ the map $\phi^{-1}(\cdot, y) : X \to Y$ is also Lipschitz. Fixed $\delta > 0$ small enough, set

$$L_f = \sup\Big\{ \frac{d_Y\big(\phi^{\pm1}(x, y), \phi^{\pm1}(x', y)\big)}{d_X(x, x')} : \ x, x' \in \Lambda,\ 0 < d_X(x, x') < \delta \text{ and } y \in Y \Big\} \geq 0.$$

We say that L_f is the *local Lipschitz constant* of f (or f^{-1}). Note that $L_f \leq \max\{\|f\|, \|f^{-1}\|\}$ and in general the inequality is strict. For instance, if $\phi(x, \cdot)$ is the identity map id on X then $L_f = 0$ while $\|f\| > 0$.

We will assume that the skew-product (2.7) satisfies that

$$\nu + L_f < \nu^{-1} \quad \text{and} \quad \nu < \gamma < \hat{\gamma}^{-1} < \nu^{-1}. \tag{2.9}$$

These conditions are called *modified dominated splitting condition* in [IN10]. The first inequality is clearly verified if $L_f = 0$. The another inequalities are the dominated conditions (2.2) in the definition of partial hyperbolicity. So, a skew-product as (2.7) satisfying $\nu < \gamma < \hat{\gamma}^{-1} < \nu^{-1}$ is called *partial hyperbolic skew-product.*

Cartesian products (also called *direct-products*) are a special type of skew-product diffeomorphisms where the maps $\phi(x, \cdot) : Y \to Y$ are constantly the same function $\phi : Y \to Y$. As we have already mentioned for $f = F \times \mathrm{id}$, in these trivial cases of direct-products, $L_f = 0$.

Definition 2.1. *A diffeomorphism f as* (2.7) *is called* locally constant skew-product *if $L_f = 0$.*

Partial hyperbolic locally constant skew-products over a horseshoe satisfy the modified dominated splitting condition in [IN10]. Modified dominated splitting condition is a C^1-open condition since the same property is satisfied for any diffeomorphism C^1-close. However, a C^1-close diffeomorphism g of f is a priori not a skew-product. With the additional dominated assumptions (2.9), from Hirsch-Pugh-Shub theory [HPS77] or from the recently work [IN10], it follows that g is topologically conjugated to a skew-product. We explain more about this.

For each $x \in \Lambda$ we consider the fiber $L_x = \{x\} \times Y$. The collection $\mathcal{L}$ of these fibers is an invariant lamination of f. In [HPS77, Theorems 7.1] and also in [IN10, Theorem A] is showed that this lamination is C^1-persistent. The C^1-*persistence* of such lamination means that for any C^1-perturbation of f, there exists a lamination, C^1-close to $\mathcal{L}$, which is preserved by the new dynamics, and such that the dynamics induced on the space of the leaves remains the same. Namely, given $\varepsilon > 0$ small enough we take g a C^1-diffeomorphism ε-close to f in the C^1-topology. Notice that, $g(x,y) = (\tilde{F}(x,y), \tilde{\phi}(x,y))$, where $\tilde{\phi}(x,\cdot) : Y \to Y$ is a C^1-diffeomorphism such that

$$\gamma\, d_Y(y,y') < d_Y(\tilde{\phi}(x,y), \tilde{\phi}(x,y')) < \hat{\gamma}^{-1} d_Y(y,y') \tag{2.10}$$

for all y, $y' \in Y$ and x in a neightbohood of Λ. For each $x \in \Lambda$, the fiber $W_{\sigma(x)}$ continuation of L_x for g is parametrice by the graph of a C^1-map $Q(x,\cdot) : Y \to X$. According to [IN10, Theorem A] and since g is ε-close to f, it follows that

$$d_X(Q(x,y), Q(x,y')) \le O(\varepsilon)\, d_Y(y,y'), \tag{2.11}$$

$$d_{C^0}(Q(x,\cdot), x) \le O(\varepsilon). \tag{2.12}$$

For C^1 maps, the C^0 norm of the first derivative is equal the best Lipschitz constant. Hence, the above inequalities show that $d_{C^1}(Q(x,\cdot), x) \le O(\varepsilon)$.

Let

$$\Delta = \bigcup_{x\in\Lambda} W_{\sigma(x)} \subset X \times Y.$$

Sending $W_{\sigma(x)}$ to x defines a continuous projection $P\colon \Delta \to X$ such that $P(\Delta) = \Lambda$ and $F|_\Lambda \circ P = P \circ g$. Moreover, $h : \Delta \to \Lambda \times Y$, given by $h(x,y) = (P(x,y), y)$ is an homeomorphism whose inverse is $h^{-1}(x,y) = (Q(x,y), y)$. Let $\tilde{g} = h \circ g|_\Delta \circ h^{-1} \colon \Lambda \times Y \to \Lambda \times Y$. Observe that

$$\begin{aligned}\tilde{g}(x,y) &= (P \circ g \circ h^{-1}(x,y), \tilde{\phi} \circ h^{-1}(x,y)) \\ &= (F|_\Lambda \circ P \circ h^{-1}(x,y), \tilde{\phi}(Q(x,y),y)) = (F(x), \psi(x,y)).\end{aligned}$$

Thus, $\tilde{g}$ is a skew-product diffeomorphism defined on $\Lambda \times Y$ which is conjugated to g by means of the conjugation h. Since for each $x \in \Lambda$ the map $\psi(x,\cdot)$ is a composition of C^1-maps then it is a C^1-map. In addition, we can easy check that the rate of contraction and expansion of these maps are uniformly close to $\hat{\gamma}^{-1}$ and γ respectively. Indeed,

$$d_Y(\psi(x,y), \psi(x,y')) \le d_Y(\tilde{\phi}(Q(x,y),y), \tilde{\phi}(Q(x,y),y')) + d_Y(\tilde{\phi}(Q(x,y),y'), \tilde{\phi}(Q(x,y'),y')).$$

By means of (2.10) and (2.11) it follows

$$d_Y(\psi(x,y), \psi(x,y')) < \hat{\gamma}^{-1} d_Y(y,y') + O(\varepsilon)\, d_Y(y,y') \le (\hat{\gamma}^{-1} + O(\varepsilon))\, d_Y(y,y').$$

In the same way, also from (2.10) and (2.11) we obtain

$$\begin{aligned} d_Y(\psi(x,y),\psi(x,y')) &\geq |d_Y(\tilde{\phi}(Q(x,y),y),\tilde{\phi}(Q(x,y),y')) - d_Y(\tilde{\phi}(Q(x,y),y'),\tilde{\phi}(Q(x,y'),y'))| \\ &> |\gamma\, d_Y(y,y') - O(\varepsilon)\, d_Y(y,y')| \geq |\gamma - O(\varepsilon)|\, d_Y(y,y'). \end{aligned}$$

Taken $\varepsilon > 0$ small enough, we can assume that $\hat{\gamma}^{-1}$ and γ remain, respectively, the rates of contraction and expansion for $\psi(x,\cdot)$. This calculation shows that the derivative of $\psi(x,\cdot)$ and $\phi(x,\cdot)$ are $O(\varepsilon)$-close. Moreover, in view of the C^1-closeness of $Q(x,\cdot)$ to the constant function $y \mapsto x$, it follows that $\psi(x,\cdot)$ and $\phi(x,\cdot)$ are $O(\varepsilon)$-close. Consequentely $d_{C^1}(\psi(x,\cdot),\phi(x,\cdot)) \leq O(\varepsilon)$.

Although for each $x \in \Lambda$, the maps $\psi(x,\cdot)$ are C^1-diffeomorphisms, the map $\tilde{g}$ is not a C^1-diffeomorphism since h is not a C^1-conjugation. However, according to [IN10, Theorem A and page 21], it follows that h and h^{-1} are locally α-Hölder continuous maps with $\alpha = \log\nu/\log\mu > 0$. Following [Gor06, Definition 2.3], a map H between metric spaces is called *locally α-Hölder* if there exist $\delta > 0$ and $C \geq 0$ such that if $d(z,w) < \delta$ then $d(H(z),H(w)) \leq C d(z,w)^\alpha$. Thus, the map $\psi = \pi_Y \circ g \circ h^{-1}$ is locally α-Hölder continuos with respect to the base points, i.e., there exsts $\delta > 0$ such that if $d_X(x,x') < \delta$ it holds that

$$d_{C^0}(\psi(x,\cdot),\psi(x',\cdot)) \leq L_g\, d_{C^0}(h^{-1}(x,\cdot),h^{-1}(x',\cdot)) \leq L_g C_h d_X(x,x')^\alpha \tag{2.13}$$

where L_g and C_h are the local Lipschitz and α-Hölder constants of g (or g^{-1}) and h (or h^{-1}) respectively.

Recall that $F|_\Lambda$ is conjugated to the Bernoulli shift $\tau : \Sigma_k \to \Sigma_k$. Let $\ell : \Sigma_k \to \Lambda$ be the topological conjugation: $\tau = \ell^{-1} \circ F|_\Lambda \circ \ell$. From [KH95, Theorem 19.1.2], a topological conjugacy between two locally maximal hyperbolic sets and its inverse are Hölder continuous maps. In [Gor06, Theorem 2.2], this result was generalized to include the conjugation with Bernoulli shifts. This reference also provides an estimate the Hölder exponent of the obtained conjugacy. To calculate this exponent for ℓ, we need to know the Lipschitz constant of τ.

Let $\Sigma_k = \{1,\ldots,k\}^{\mathbb{Z}}$ be the space of the bi-sequences of k symbols endowed with the metric

$$d_{\Sigma_k}(\xi,\xi') = \nu^m, \quad m = \min\{i \in \mathbb{Z}^+ : \xi_i \neq \xi'_i \text{ or } \xi_{-i} \neq \xi'_{-i}\}, \tag{2.14}$$

where $\xi = (\xi_i)_{i\in\mathbb{Z}}$, $\xi' = (\xi'_i)_{i\in\mathbb{Z}} \in \Sigma_k$. Given a bi-sequence $\xi = (\ldots,\xi_{-1};\xi_0,\xi_1,\ldots)$ the symbol at the right of ";" is the "0 coordinate" of the bi-sequence ξ. Define the *Bernoulli shift map* (or *left shift map*) $\tau\colon \Sigma_k \to \Sigma_k$ by $\tau(\xi) = \xi'$, where $\xi'_i = \xi_{i+1}$. The *local unstable* and *stable sets* of a sequence $\xi = (\xi_i)_{i\in\mathbb{Z}}$ are defined by

$$\begin{aligned} W^u_{loc}(\xi;\tau) &= \{\xi' = (\xi'_i)_{i\in\mathbb{Z}} \in \Sigma_k\colon\ \xi'_i = \xi_i \text{ for all } i \leq 0\}, \\ W^s_{loc}(\xi;\tau) &= \{\xi' = (\xi'_i)_{i\in\mathbb{Z}} \in \Sigma_k\colon\ \xi'_i = \xi_i \text{ for all } i \geq 0\}. \end{aligned}$$

Thus, we obtain that

$$\begin{aligned} d_{\Sigma_k}(\tau(\xi),\tau(\xi')) &\leq \nu\, d_{\Sigma_k}(\xi,\xi') \quad \text{for all } \xi,\xi' \in W^s_{loc}(\zeta;\tau), \\ d_{\Sigma_k}(\tau^{-1}(\xi),\tau^{-1}(\xi')) &\leq \nu\, d_{\Sigma_k}(\xi,\xi') \quad \text{for all } \xi,\xi' \in W^u_{loc}(\zeta;\tau). \end{aligned}$$

That is, ν is the contraction rate on both stable and unstable local sets of the Bernoulli shift τ. In addition, $d_{\Sigma_k}(\tau^{\pm1}(\xi),\tau^{\pm1}(\xi')) \leq \nu^{-1}\, d_{\Sigma_k}(\xi,\xi')$, for all $\xi,\xi' \in \Sigma_k$. By [Gor06, Theorem 2.3] the equality[1] $\nu\nu^{-1} = 1$ implies that ℓ is a Lipschitz map. That is, $d_X(\ell(\xi),\ell(\xi')) \leq L_\ell\, d_{\Sigma_k}(\xi,\xi')$.

[1] The condition in Gorodetski's result appears as an strict inequality, but from the proof of this result one can follow that the assertion also holds for the equality.

Therefore, it follows that $\tilde{g}$ is conjugated to the skew-product

$$\Psi : \Sigma_k \times Y \to \Sigma_k \times Y, \qquad \Psi(\xi, y) = (\tau(\xi), \psi_\xi(y)) \tag{2.15}$$

where $\psi_\xi = \psi(\ell(\xi), \cdot) : Y \to Y$ is a C^1-diffeomorphism satisfying

$$\gamma\, d_Y(y, y') < d_Y(\psi_\xi(x, y), \psi_\xi(x, y')) < \hat{\gamma}^{-1}\, d_Y(y, y') \quad \text{for all } \xi \in \Sigma_k, \tag{2.16}$$

$$d_{C^0}(\psi_\xi, \psi_{\xi'}) \leq C_\Psi\, d_{\Sigma_k}(\xi, \xi')^\alpha, \quad \xi,\, \xi' \in \Sigma_k \text{ with } \xi_0 = \xi_0', \tag{2.17}$$

with $C_\Psi = L_g C_h L_\ell^\alpha \geq 0$ and $\alpha = \log \nu / \log \mu > 0$. The last local Hölder condition comes from the imposition in [IN10, *pag.* 21] that $F|_\Lambda$ has local product structure for $\delta > 0$ in (2.13).

Same arguments work to g^{-1} as small perturbation of f^{-1} and therefore we obtain that the inverse map $\Psi^{-1} : \Sigma_k \times Y \to \Sigma_k \times Y$ is also locally Hölder skew-product with the same Hölder exponent α and local Hölder constant C_Ψ.

Summarizing, we have proved the following result.

Proposition 2.1. *Let $f : X \times Y \to X \times Y$ be a C^1-diffeomorphism skew-product of the form of* (2.7) *satisfying* (2.8) *and* (2.9)*. Then given $\varepsilon > 0$ small enough, any ε-perturbation g of f in the C^1-topology has a locally maximal invariant set $\Delta \subset X \times Y$ such that $g|_\Delta$ is conjugated to a skew-product $\Psi : \Sigma_k \times Y \to \Sigma_k \times Y$ of the form* (2.15) *satisfying* (2.16) *and* (2.17)*.*

The restriction of the skew-product f given in (2.7) to the set $\Lambda \times Y$ is conjugated to a *symbolic locally Lipschitz skew-product.* Namely, $f|_{\Lambda\times Y}$ is conjugated to

$$\Phi : \Sigma_k \times Y \to \Sigma_k \times Y, \qquad \Phi(\xi, y) = (\tau(\xi), \phi_\xi(y))$$

where $\phi_\xi = \phi(\ell(\xi), \cdot) : Y \to Y$ is a family of C^1-diffeomorphisms satisfying that

$$d_{C^0}(\phi_\xi^{\pm 1}, \phi_{\xi'}^{\pm 1}) \leq C_\Phi\, d_{\Sigma_k}(\xi, \xi')^\alpha, \quad \text{for } \xi,\, \xi' \in \Sigma_k \text{ with } \xi_0 = \xi_0'$$

with $C_\Phi = C_\ell\, L_f$ where L_f is the local Lipschitz constant of f (or f^{-1}). Under the remaining assumptions in Proposition 2.1, we can identify C^1-perturbations g of f with *symbolic locally Hölder skew-product* perturbations Ψ of Φ with uniform Hölder exponent $\alpha = \log \nu / \log \mu > 0$ and local Hölder constant $C_\Psi = C_\ell\, L_g C_h \geq 0$. Here, L_g is the local Lipschitz constant of g (or g^{-1}) which is close to L_f. Also, the local Hölder constant C_h of h (or h^{-1}) varies continually with respect to h which in turn depends continuously with g. In fact, in view of (2.12) it follows that h and h^{-1} are close to the identity and thus, we obtain that C_Ψ is close to C_Φ.

In the context of C^2-perturbations in [Gor06, Theorem B], it was proved the following result:

Theorem 2.2. *Let $f\colon X \times Y \to X \times Y$, $f = F \times \mathrm{id}$, be a C^2-diffeomorphism. Then for any diffeomorphism g close to f in the C^2-topology, there is an invariant subset Δ_g and homeomorphism $H\colon \Lambda \times Y \to \Delta_g$. Moreover, if $p\colon \Lambda \times Y \to \Lambda$ is the projection in the first factor, then the map $P\colon \Delta_g \to \Lambda$, $P = p \circ H^{-1}$, is a semiconjugacy, and the leaves $P^{-1}(x)$ are C^2-smooth and depend Hölder continuously on a point $x \in \Lambda$ in the C^1-metric. The Hölder exponent and the Hölder constant are uniform in a small C^2-neighborhood of f.*

A similar argument as above using this theorem allows us to conjugate the restriction $g|_{\Delta_g}$ of any perturbation g of $f = F \times \mathrm{id}$ in the C^2-topology to a symbolic skew-product map of the form $\Psi(\xi, y) = (\tau(\xi), \psi_\xi(y))$ where $\psi_\xi : Y \to Y$ are C^2-diffeomorphism satisfying (2.16) and

$$d_{C^1}(\psi_\xi^{\pm 1}, \psi_{\xi'}^{\pm 1}) \leq C_\Psi\, d_{\Sigma_k}(\xi, \xi')^\alpha, \quad \text{for } \xi,\, \xi' \in \Sigma_k \text{ with } \xi_0 = \xi_0'$$

where C_Ψ is a constant close to $C_\Phi = C_\ell L_f$.

2.2 Symbolic skew-products

We consider skew-product maps Φ over the left shift map τ of k symbols of the form

$$\Phi : \Sigma_k \times M \to \Sigma_k \times M, \qquad \Phi(\xi, x) = (\tau(\xi), \phi_\xi(x)) \tag{2.18}$$

where M is a (not necessarily compact) Riemanniana manifold of dimension $c \geq 1$, $\phi_\xi : M \to M$ are homeomorphisms which depend continuously with respect to the base point ξ. These maps are referred to as *symbolic skew-products*. The first factor of the product $\Sigma_k \times M$ is called the *base* and the second one is the *fiber*. To emphasize the role of the fiber maps we write $\Phi = \tau \ltimes \phi_\xi$. We define $\mathcal{S}_k(M)$ as the set of symbolic skew-product maps $\Phi = \tau \ltimes \phi_\xi$ of the form (2.18). A special case of skew-product maps are the one-step ones.

Definition 2.2 (One-step maps)**.** *A symbolic skew-product map $\Phi = \tau \ltimes \phi_\xi$ is* one-step *if the fiber maps ϕ_ξ only depend on the coordinate ξ_0 of the bi-sequence $\xi = (\xi_i)_{i\in\mathbb{Z}} \in \Sigma_k$. In this case, we have $\phi_\xi = \phi_i$ if $\xi_0 = i$, say that Φ is associated with the maps $\phi_1, \ldots, \phi_n$, and write $\Phi = \tau \ltimes (\phi_1, \ldots, \phi_n)$.*

We will denote by $\mathcal{Q}_k(M)$ the subset of $\mathcal{S}_k(M)$ consisting of the one-step maps. An extension of one-step maps are the skew-product maps $\Phi = \tau \ltimes \phi_\xi$ whose fiber maps ϕ_ξ only depend either on the stable sets of ξ or on the unstable sets of ξ. In this case, we say that $\Phi = \tau \ltimes \phi_\xi$ belongs to $\mathcal{S}_k^+(M)$ (resp. $\mathcal{S}_k^-(M)$) if $\Phi \in \mathcal{S}_k(M)$ and $\phi_\xi = \phi_{\xi'}$ if $\xi'_i = \xi_i$ for all $i \geq 0$ (resp. $i \leq 0$).

Definition 2.3. *A stable holonomy, or shortly s-*holonomy*, for $\Phi = \tau \ltimes \phi_\xi$ is a family h^s of homeomorphisms $h^s_{\xi,\xi'} : M \to M$ defined for all ξ and ξ' in the same local stable set of τ and satisfying*

i) $h^s_{\xi',\eta} \circ h^s_{\xi,\xi'} = h^s_{\xi,\eta}$ *and* $h^s_{\xi,\xi} = id$,

ii) $\phi_{\xi'} \circ h^s_{\xi,\xi'} = h^s_{\tau(\xi),\tau(\xi')} \circ \phi_\xi$, *and*

iii) $(\xi, \xi', x) \mapsto h^s_{\xi,\xi'}(x)$ *is continuous.*

In the last condition (ξ, ξ') varies in the space of pairs of points in the same local stable set. Unstable holonomy, or shortly *u-holonomy*, is defined analogously for pairs of points in the same local unstable set of τ. The following result shows that the existence of s-holonomy for a skew-product $\Phi = \tau \ltimes \phi_\xi$ in $\mathcal{S}_k(M)$ implies that Φ is conjugated to a symbolic skew-product in $\mathcal{S}_k^+(M)$. We will denote by Σ a fixed transversal section to the local stable partition $W^s_{loc}(\xi; \tau)$, $\xi \in \Sigma_k$ and we will consider the projection $\pi : \Sigma_k \to \Sigma$ given by $\pi(\xi) = W^s_{loc}(\xi; \tau) \cap \Sigma$.

Proposition 2.3. *Let $\Phi = \tau \ltimes \phi_\xi$ be a symbolic skew-product in $\mathcal{S}_k(M)$. Suppose that there is a s-holonomy h^s for Φ. Then $h : \Sigma_k \times M \to \Sigma_k \times M$, given by $h(\xi, x) = (\xi, h^s_{\pi(\xi),\xi}(x))$ is a homeomorphism and the symbolic skew-product*

$$\tilde{\Phi} : \Sigma_k \times M \to \Sigma_k \times M, \qquad \tilde{\Phi} = h^{-1} \circ \Phi \circ h \in \mathcal{S}_k^+(M)$$

is conjugated to Φ. Moreover, $\tilde{\Phi} = \tau \ltimes \tilde{\phi}_\xi$ with fiber maps $\tilde{\phi}_\xi = h^s_{\tau(\xi),\pi(\tau(\xi))} \circ \phi_\xi \circ h^s_{\pi(\xi),\xi}$.

Proof. Since the projection π is a continuous map and ξ and $\pi(\xi)$ always belong in the same local stable set, it follows from (iii) in the definition of s-holonomy that h is a continuous map. By (i), it follows that the inverse of h is $h^{-1}(\xi,x)=(\xi,h^s_{\xi,\pi(\xi)}(x))$. Then h is a homeomorphism and

$$\tilde{\Phi}(\xi,x)=h^{-1}\circ\Phi\circ h(\xi,x)=\big(\tau(\xi),h^s_{\tau(\xi),\pi(\tau(\xi))}\circ\phi_\xi\circ h^s_{\pi(\xi),\xi}(x)\big).$$

The properties (i) and (ii) in Definition 2.3 provide that

$$\tilde{\phi}_\xi\stackrel{\text{def}}{=}h^s_{\tau(\xi),\pi(\tau(\xi))}\circ\phi_\xi\circ h^s_{\pi(\xi),\xi}=h^s_{\tau(\xi),\pi(\tau(\xi))}\circ h^s_{\tau(\pi(\xi)),\tau(\xi)}\circ\phi_{\pi(\xi)}=h^s_{\tau(\pi(\xi)),\pi(\tau(\xi))}\circ\phi_{\pi(\xi)}.$$

This shows that $\tilde{\phi}_\xi$ is constant on the local stable set of any point ξ. Indeed, since for every $\xi'\in W^s_{loc}(\xi;\tau)$ it holds that $\pi(\xi')=\pi(\xi)$ and $\pi\circ\tau(\xi')=\pi\circ\tau(\xi)$, it follows that $\tilde{\phi}_\xi=\tilde{\phi}_{\xi'}$. Finally, notice that the fiber maps $\tilde{\phi}_\xi$ are homeomorphisms which depend continuously on the base point ξ. Therefore $\tilde{\Phi}=\tau\ltimes\tilde{\phi}_\xi\in\mathcal{S}^+_k(M)$ and we conclude the proof of the proposition. □

Notice that there exists a dual result for skew-product $\Phi=\tau\ltimes\phi_\xi$ in $\mathcal{S}_k(M)$ with u-holonomy provides a conjugation between Φ and a skew-product with constant fiber maps on the unstable local sets. The next step is to investigate whether a skew-product $\Phi=\tau\ltimes\phi_\xi$ has s-holonomy and the regularity of the holonomy maps. In order to do this, we need to impose additional conditions for Φ about regularity and dominated dynamics.

Definition 2.4 (Sets of symbolic skew-products)**.** *Let γ and $\hat{\gamma}$ be positive constants such that $\gamma<\hat{\gamma}^{-1}$. A map $\phi:M\to M$ is called* $(\gamma,\hat{\gamma}^{-1})$-Lipschitz *(in M) if*

$$\gamma\,\|x-x'\|<\|\phi(x)-\phi(x')\|<\hat{\gamma}^{-1}\,\|x-x'\|,\quad \textit{for all } x,\ x'\in M.$$

Here, $\|x-x'\|$ denotes the distance between x and x' in M. Given $\alpha\in(0,1]$, a skew-product $\Phi=\tau\ltimes\phi_\xi\in\mathcal{S}_k(M)$ is said to be locally α-Hölder continuous *(in M), or shortly* Hölder skew-product*, if there is a non-negative constant $C\geq 0$ such that*

$$d_{C^0}(\phi^{\pm1}_\xi,\phi^{\pm1}_{\xi'})\leq C\,d_{\Sigma_k}(\xi,\xi')^\alpha,\quad \textit{for all } \xi,\ \xi'\in\Sigma_k \textit{ with } \xi_0=\xi'_0.$$

We will denote by $S^{r,\alpha}_{k,\gamma,\hat{\gamma}^{-1}}(M)$ the subset of $\mathcal{S}_k(M)$ consistent of locally α-Hölder continuous symbolic skew-products with C^r-fiber maps ($r\geq 0$) which are $(\gamma,\hat{\gamma}^{-1})$-Lipschitz. For notational convenience, we will denote $\mathcal{S}^{0,\alpha}_{k,\gamma,\hat{\gamma}^{-1}}(M)$ by $\mathcal{S}^{\alpha}_{k,\gamma,\hat{\gamma}^{-1}}(M)$.

Sometimes to refer that $\Phi=\tau\ltimes\phi_\xi$ is locally α-Hölder we say that the fiber maps ϕ_ξ of Φ depend locally α-Hölder continuously (in M) with respect to the base points. We will denote

$$C_\Phi\stackrel{\text{def}}{=}\sup\Big\{\frac{\|\phi^{\pm1}_\xi(x)-\phi^{\pm1}_{\xi'}(x)\|}{d_{\Sigma_k}(\xi,\xi')^\alpha}:\ \xi,\ \xi'\in\Sigma_k,\ \text{with } \xi_0=\xi'_0 \text{ and } x\in M\Big\}\geq 0.$$

This constant is called *local α-Hölder (continuous) constant* of $\Phi=\tau\ltimes\phi_\xi$.

Recall that in (2.14) we endowed the space of the bi-sequences of k symbols $\Sigma_k=\{1,\dots,k\}^{\mathbb{Z}}$ with the metric $d_{\Sigma_k}(\xi,\xi')=\nu^m$, where $m=\min\{i\in\mathbb{Z}^+:\ \xi_i\neq\xi'_i \text{ or } \xi_{-i}\neq\xi'_{-i}\}$, and ν is a fixed positive constant less than 1. That is, ν is the contraction rate on both stable and unstable local sets of the Bernoulli shift τ. Since the Bernoulli shift τ on the space of k symbols represents $F|_\Lambda$ in the precious section, we could assume that if the number of symbols k increases then the contraction ν of $F|_\Lambda$ decreases. Therefore, we could expect that the following dominated conditions must be satisfied for a large number k of symbols.

Definition 2.5 (Dominated skew-products). *A symbolic skew-product $\Phi \in \mathcal{S}^{\alpha}_{k,\gamma,\hat{\gamma}^{-1}}(M)$ is said to be* s-dominated *(resp.* u-dominated*) if $\nu^{\alpha} < \gamma$ (resp. $\hat{\gamma}^{-1} < \nu^{-\alpha}$). In particular, if Φ is both, s-dominated and u-dominated, i.e. $\nu^{\alpha} < \gamma < \hat{\gamma}^{-1} < \nu^{-\alpha}$, it said to be* partial hyperbolic.

Let us explain the geometric meaning of the dominated conditions. For each $\xi \in \Sigma_k$ the fiber map ϕ_ξ is a $(\gamma, \hat{\gamma}^{-1})$-Lipschitz diffeomorphism in M. The rate γ is an lower bound for the contraction, and $\hat{\gamma}^{-1}$ is a upper bound for the expansion exhibited by the action of ϕ_ξ on the fiber $\{\xi\} \times M$. First, consider $\alpha = 1$. The s-dominated and u-dominated conditions become $\nu < \gamma$ and $\hat{\gamma}^{-1} < \nu$ respectively. The first condition means the base map τ contracts local stable sets stronger than the skew-product $\Phi = \tau \ltimes \phi_\xi$ contracts fibers; the second one means that the base map expands local unstable sets stronger than $\Phi = \tau \ltimes \phi_\xi$. In other words, these conditions of domination mean that Φ is partially hyperbolic transformation, with the fibers as central leaves. This interpretation extends immediately to the general case $\alpha \in (0, 1]$. It suffices to note that $d_{\Sigma_k}(\cdot, \cdot)^{\alpha}$ is also a metric in Σ_k. With this new metric, the Hölder skew-product Φ has Hölder exponent $\alpha = 1$. This reduces the general case to the previous particular one $\alpha = 1$.

In view of the theory of partial hyperbolic systems [BP74, HPS77, PSW97], one expects that such dominated conditions imply the existence of smooth invariant strong stable an strong unstable foliations for $\Phi = \tau \ltimes \phi_\xi$ in $\Sigma_k \times M$, transverse to the fibers. Theses foliations allow us to find s-holonomy and u-holonomy for a symbolic skew-product Φ. In the next subsection, we will show that this is indeed so.

Notation 2.4. *Given $\Phi = \tau \ltimes \phi_\xi$ for every $n > 0$ and every $(\xi, x) \in \Sigma_k \times M$ we set*

$$\phi^n_\xi(x) \stackrel{\text{def}}{=} \phi_{\tau^{n-1}(\xi)} \circ \cdots \circ \phi_\xi(x) \quad \text{and} \quad \phi^{-n}_\xi(x) \stackrel{\text{def}}{=} \phi^{-1}_{\tau^{-(n-1)}(\xi)} \circ \cdots \circ \phi^{-1}_\xi(x).$$

Note that, for all $n \geq 0$, we have $\Phi^n(\xi, x) = (\tau^n(\xi), \phi^n_\xi(x))$ and $\Phi^{-n}(\xi, x) = (\tau^{-n}(\xi), \phi^{-n}_{\tau^{-1}(\xi)}(x))$.

2.2.1 Strong stable and unstable sets and holonomies

The *stable* and *unstable sets* of a point $(\xi, x) \in \Sigma_k \times M$ for a skew-product map $\Phi = \tau \ltimes \phi_\xi$ as (2.18) are defined by

$$W^s\big((\xi, x); \Phi\big) = \{(\zeta, y) \in \Sigma_k \times M : \lim_{n\to\infty} d(\Phi^n(\zeta, y), \Phi^n(\xi, x)) = 0\},$$
$$W^u\big((\xi, x); \Phi\big) = \{(\zeta, y) \in \Sigma_k \times M : \lim_{n\to\infty} d(\Phi^{-n}(\zeta, y), \Phi^{-n}(\xi, x)) = 0\}.$$

where d denotes the product metric in $\Sigma_k \times M$. In this section we will assume that M is a compact manifold. Under the s-domination condition the usual graph transform argument yields a strong stable lamination for the symbolic skew-product Φ. This strong stable lamination allows us to define a s-holonomy h^s for Φ.

Proposition 2.5 ([AV10, ASV11]). *Consider a s-dominated skew-product $\Phi = \tau \ltimes \phi_\xi \in \mathcal{S}^{\alpha}_{k,\gamma,\hat{\gamma}^{-1}}(M)$ Then, there exists a partition*

$$\mathcal{W}^s = \{W^{ss}_{loc}((\xi, x); \Phi) : (\xi, x) \in \Sigma_k \times M\}$$

of $\Sigma_k \times M$ such that denoting $C = C_\Phi (1 - \gamma^{-1}\nu^{\alpha})^{-1} \geq 0$, it holds that *i) every leaf $W^{ss}_{loc}((\xi,x);\Phi)$ is the graph of α-Hölder function $\gamma^s_{\xi,x}:W^s_{loc}(\xi;\tau)\to M$ with α-Hölder constant less or equal than C (uniform on ξ and x),*

ii) $\Phi(W^{ss}_{loc}((\xi,x));\Phi)\subset W^{ss}_{loc}(\Phi(\xi,x);\Phi)$ for all $(x,\xi)\in\Sigma_k\times M$, and

iii) the family of maps $h^s_{\xi,\xi'}:M\to M$ defined by $h^s_{\xi,\xi'}(x)=\gamma^s_{\xi,x}(\xi')$, for $\xi'\in W^s_{loc}(\xi;\tau)$, is a s-holonomy for Φ. Moreover,

a) $d_{C^0}(h^s_{\xi,\xi'},\mathrm{id})\le C\,d_{\Sigma_k}(\xi,\xi')^\alpha$, and

b) $h^s_{\xi,\xi'}$ coincides with the uniform limit of $(\phi^n_{\xi'})\circ\phi^n_\xi$ as $n\to\infty$.

In [AV10, Proposition 5.2] the continuous dependence of the invariant graphs with respect to Φ is also proved. The partition $\mathcal{W}^s=\mathcal{W}^s(\Phi)$ given by Proposition 2.5 is a *s-lamination* for Φ. That is, Φ sends leaf in leaf of the partition and exponentially contracts points on the same leaf. Indeed, it suffices to show that points in $W^{ss}_{loc}((\xi,x);\Phi)$ are exponentially contracted. This is followed since theses local leaves are α-Hölder graphs with uniform Hölder constant on ξ, and therefore

$$d(\Phi^n(\xi,x),\Phi^n(\xi',x'))\le\nu^n d_{\Sigma_k}(\xi,\xi')+C\nu^{n\alpha}d_{\Sigma_k}(\xi,\xi')^\alpha\le C_0\nu^{n\alpha}d((\xi,x),(\xi',x'))$$

for all $(\xi',x')\in W^{ss}_{loc}((\xi,x);\Phi)$, where $C_0>1$ is a uniform constant. For the above reason, $W^{ss}_{loc}((\xi,x);\Phi)$ is referred to as the *local strong stable set* of the point $(\xi,x)\in\Sigma_k\times M$ for the skew-product Φ. The *strong stable set* of point (ξ,x) for Φ is defined as

$$W^{ss}((\xi,x);\Phi)\overset{\text{def}}{=}\bigcup_{n\ge0}\Phi^{-n}\big(W^{ss}_{loc}(\Phi^n(\xi,x);\Phi)\big)\subset W^s((\xi,x);\Phi).$$

Notice that there is a dual statement of Proposition 2.5 for u-dominated symbolic skew-product maps. Hence, by this dual result, it also follows a partition $\mathcal{W}^u=\mathcal{W}^u(\Phi)$ whose leaves $W^{uu}_{loc}((\xi,x);\Phi)$ are called *local strong unstable sets*. As above, the *strong unstable set* of a point $(\xi,x)\in\Sigma_k\times M$ for Φ is

$$W^{uu}((\xi,x);\Phi)\overset{\text{def}}{=}\bigcup_{n\ge0}\Phi^{n}\big(W^{uu}_{loc}(\Phi^{-n}(\xi,x);\Phi)\big)\subset W^u((\xi,x);\Phi).$$

In addition, we have the *c-lamination* $\mathcal{W}^c(\Phi)=\{W^c_{loc}((\xi,x);\Phi)=\{\xi\}\times M:\ (\xi,x)\in\Sigma_k\times M\}$.

In order to show as s-domination condition are used in the graph transform argument we give the details of the proof of Proposition 2.5.

Proof of Proposition 2.5: Existence (i) and invariance (ii) of the family $\mathcal{W}^s$ follow from a standard application of the graph transform argument [HPS77]. Define for each $(\xi,x)\in\Sigma_k\times M$ and $n>0$,

$$\gamma^{s,n}_{\xi,x}:W^s_{loc}(\xi;\tau)\to M,\quad \gamma^{s,n}_{\xi,x}(\xi')=(\phi^n_{\xi'})^{-1}\circ\phi^n_\xi(x).$$

Then

$$\begin{aligned}\|\gamma^{s,n+1}_{\xi,x}(\xi')-\gamma^{s,n}_{\xi,x}(\xi')\|&\le\gamma^{-n}\|\phi^{-1}_{\tau^n(\xi')}\circ\phi_{\tau^n(\xi)}\circ\phi^n_\xi(x)-\phi^{-1}_{\tau^n(\xi')}\circ\phi_{\tau^n(\xi')}\circ\phi^n_\xi(x)\|\\&\le\gamma^{-n-1}\|\phi_{\tau^n(\xi)}\circ\phi^n_\xi(x)-\phi_{\tau^n(\xi')}\circ\phi^n_\xi(x)\|.\end{aligned}$$

Using that $\Phi=\tau\ltimes\phi_\xi$ is locally α-Hölder skew-product and recalling that ν is the contraction rate of τ on the stable sets we have $\|\gamma^{s,n+1}_{\xi,x}(\xi')-\gamma^{s,n}_{\xi,x}(\xi')\|\le C_\Phi(\gamma^{-1}\nu^\alpha)^{n+1}d_{\Sigma_k}(\xi,\xi')^\alpha$. Hence, since $\nu^\alpha < \gamma$ then the sequence $\{\gamma^{s,n}_{\xi,x}(\xi')\}$ is Cauchy and therefore converges. Denote the limit by $\gamma^s_{\xi,x}(\xi')$. Note that (ii) is consequence of the fact that

$$\begin{aligned} \phi_{\xi'} \circ \gamma^s_{\xi,x}(\xi') &= \lim_{n\to\infty} \phi_{\xi'} \circ (\phi^n_{\xi'})^{-1} \circ \phi^n_\xi(x) \\ &= \lim_{n\to\infty} (\phi^{n-1}_{\tau(\xi')})^{-1} \circ \phi^{n-1}_{\tau(\xi)} \circ \phi_\xi(x) = \gamma^s_{\tau(\xi),\phi_\xi(x)} \circ \tau(\xi'). \end{aligned} \tag{2.19}$$

In order to prove that $\gamma^s_{\xi,x}$ is a α-Hölder map, we require again the s-domination condition. By means of the triangular inequality we get

$$\|\gamma^{s,n}_{\xi,x}(\xi') - x\| = \|\gamma^{s,n}_{\xi,x}(\xi') - \gamma^{s,n}_{\xi,x}(\xi)\| \le \sum_{i=1}^n s_i(\xi') \tag{2.20}$$

where $s_i(\xi')$ is given by

$$\|(\phi^{n-i}_{\xi'})^{-1} \circ \phi^{-1}_{\tau^{n-i}(\xi')} \circ (\phi^{i-1}_{\tau^{n+1-i}(\xi)})^{-1} \circ \phi^n_\xi(x) - (\phi^{n-i}_{\xi'})^{-1} \circ \phi^{-1}_{\tau^{n-i}(\xi)} \circ (\phi^{i-1}_{\tau^{n+1-i}(\xi)})^{-1} \circ \phi^n_\xi(x)\|.$$

With the estimate $s_i(\xi') \le C_\Phi(\gamma^{-1}\nu^\alpha)^{n-i} d_{\Sigma_k}(\xi,\xi')^\alpha$ and taking $n\to\infty$ in the above inequality it follows $\|\gamma^s_{\xi,x}(\xi') - \gamma^s_{\xi,x}(\xi)\| \le C d_{\Sigma_k}(\xi,\xi')^\alpha$ for all $\xi' \in W^s_{loc}(\xi;\tau)$ where $C = C_\Phi(1-\gamma^{-1}\nu^\alpha)^{-1}$. This shows that $\gamma^s_{\xi,x}$ is α-Hölder. Indeed, for every $\xi',\xi'' \in W^u_{loc}(\xi;\tau)$, denoting $x' = \gamma^s_{\xi,x}(\xi')$ and noting that $\gamma^s_{\xi,x}(\xi'') = \gamma^s_{\xi',x'}(\xi'')$, we obtain that

$$\|\gamma^s_{\xi,x}(\xi') - \gamma^s_{\xi,x}(\xi'')\| = \|\gamma^s_{\xi',x'}(\xi') - \gamma^s_{\xi',x'}(\xi'')\| \le C d_{\Sigma_k}(\xi',\xi'')^\alpha.$$

Note that, the Hölder constant obtained is uniform on ξ and x.

For every bi-sequences ξ and ξ' in the same local stable set of τ, we consider the map $h^s_{\xi,\xi'} : M \to M$ given by $h^s_{\xi,\xi'}(x) = \gamma^s_{\xi,x}(\xi')$. Notice that, because of the lamination $\mathcal{W}^s$ is invariant under Φ, i.e. from (2.19), it follows that for every $n \ge 0$

$$h^s_{\xi,\xi'} = (\phi^n_{\xi'})^{-1} \circ h^s_{\tau^n(\xi),\tau^n(\xi')} \circ \phi^n_\xi. \tag{2.21}$$

The estimative calculated for (2.20) allows us to obtain a bounded for the uniform C^0-distance from $h^s_{\xi,\xi'}$ to the identity. Namely,

$$d_{C^0}(h^s_{\tau^n(\xi),\tau^n(\xi')}, \mathrm{id}) \le C d_{\Sigma_k}(\tau^n(\xi),\tau^n(\xi'))^\alpha \le C\nu^{\alpha n} d_{\Sigma_k}(\xi,\xi')^\alpha. \tag{2.22}$$

Putting these two observations together, we find that

$$d_{C^0}(h^s_{\xi,\xi'}, (\phi^n_{\xi'})^{-1} \circ \phi^n_\xi) \le \gamma^{-n} d_{C^0}(h^s_{\tau^n(\xi),\tau^n(\xi')}, \mathrm{id}) \le C(\gamma^{-1}\nu^\alpha)^n d_{\Sigma_k}(\xi,\xi')^\alpha.$$

Hence, $h^s_{\xi,\xi'}$ coincides with the uniform limit of $(\phi^n_{\xi'})^{-1} \circ \phi^n_\xi$ as $n\to\infty$. Notice that by definition of $h^s_{\xi,\xi'}$ we have $(h^s_{\xi,\xi'})^{-1} = h^s_{\xi',\xi}$ and so it follows these inverse maps also as an uniform limit. Therefore, as consequence of the above observations, the family of maps $h^s_{\xi,\xi'}$ is a s-holonomy for $\Phi = \tau \ltimes \phi_\xi$ and we conclude the proof of the proposition. □

We will say that h^s is a *ϱ-Hölder s-holonomy* if the homeomorphisms $h^s_{\xi,\xi'}$ are ϱ-Hölder continuous with uniform Hölder constant in ξ and ξ'. The proof of the following proposition can be found in [AV11].

Proposition 2.6. *Consider a s-dominated skew-product $\Phi = \tau \ltimes \phi_\xi \in \mathcal{S}^{\alpha}_{k,\gamma,\hat{\gamma}^{-1}}(M)$. Then the holonomies maps $h^s_{\xi,\xi'} : M \to M$ are ϱ-Hölder with $\varrho = (1 + 3\log\gamma\hat{\gamma}^{-1}/\log\gamma^{-1}\nu^{\alpha})^{-1} \in (0,1)$ and uniform Hölder constant in ξ and ξ'.*

Proof. We have to check that the maps $h^s_{\xi,\xi'}$ are Hölder maps. Fix two bi-sequence ξ and ξ' in the same local stable set of τ. Recall that $0 < \nu^{\alpha} < \gamma < 1 < \hat{\gamma}^{-1}$. Hence $\theta = 3\log\gamma\hat{\gamma}/\log\gamma^{-1}\nu^{\alpha} > 0$ and $\varrho = (1+\theta)^{-1} < 1$. We will prove that $h^s_{\xi,\xi'}$ is locally ϱ-Hölder with uniform local Hölder constant on ξ, ξ'. Notice that by standard argument this claim also shows that $h^s_{\xi,\xi'}$ is globally ϱ-Hölder (see for instance [Gor06, Proprosition 2.4]).

Let m be a natural number such that $2C\nu^{m\alpha} < 1$ where $C = C_\Phi(1-\gamma^{-1}\nu^{\alpha}) \geq 0$. From the continuity of $h^s_{\xi,\xi'}$ there exists $\delta > 0$ such that if $\|x - x'\| < \delta$ then $\|h^s_{\xi,\xi'}(x) - h^s_{\xi,\xi'}(x')\| < \nu^{m\alpha}$. Fix x and x' in M such that $\|x - x'\| < \delta$ and write $\eta = \|h^s_{\xi,\xi'}(x) - h^s_{\xi,\xi'}(x')\| < \nu^{m\alpha}$. Hence, from the s-domination condition there is $n \in \mathbb{N}$ such that $\eta^3 \leq (\gamma^{-1}\nu^{\alpha})^n \leq \eta^2$. Indeed, it suffices to take n between $2\log\eta/\log\gamma^{-1}\nu^{\alpha}$ and $3\log\eta/\log\gamma^{-1}\nu^{\alpha}$ which it is possible since $\eta < \nu^{m\alpha} \leq \gamma^{-1}\nu^{\alpha}$. Using the above inequality, recalling that the fiber maps ϕ_ξ are $(\gamma, \hat{\gamma}^{-1})$-Lipschitz diffeomorphisms and since by (2.21) it holds that $\phi^n_{\xi'} \circ h^s_{\xi,\xi'} = h^s_{\tau^n(\xi),\tau^n(\xi')} \circ \phi^n_\xi$, we obtain that

$$\begin{aligned} d_{\Sigma_k}(\tau^n(\xi),\tau^n(\xi'))^{\alpha} &\leq \nu^{\alpha n} d_{\Sigma_k}(\xi,\xi')^{\alpha} \leq \gamma^n\eta^2\, d_{\Sigma_k}(\xi,\xi')^{\alpha} < \eta\, \|\phi^n_{\xi'} \circ h^s_{\xi,\xi'}(x) - \phi^n_{\xi'} \circ h^s_{\xi,\xi'}(x')\| \\ &\leq \nu^{m\alpha}\, \|h^s_{\tau^n(\xi),\tau^n(\xi')} \circ \phi^n_\xi(x) - h^s_{\tau^n(\xi),\tau^n(\xi')} \circ \phi^n_\xi(x')\|. \end{aligned}$$

This inequality and the limitation obtained in (2.22) imply that

$$\begin{aligned} &\|h^s_{\tau^n(\xi),\tau^n(\xi')} \circ \phi^n_\xi(x) - h^s_{\tau^n(\xi),\tau^n(\xi')} \circ \phi^n_\xi(x')\| \leq \\ &\quad \leq \|h^s_{\tau^n(\xi),\tau^n(\xi')} \circ \phi^n_\xi(x) - \phi^n_\xi(x)\| + \|\phi^n_\xi(x) - \phi^n_\xi(x')\| + \|h^s_{\tau^n(\xi),\tau^n(\xi')} \circ \phi^n_\xi(x') - \phi^n_\xi(x')\| \\ &\quad \leq 2C d_{\Sigma_k}(\tau^n(\xi),\tau^n(\xi'))^{\alpha} + \|\phi^n_\xi(x) - \phi^n_\xi(x')\| \\ &\quad \leq 2C\nu^{m\alpha}\, \|h^s_{\tau^n(\xi),\tau^n(\xi')} \circ \phi^n_\xi(x) - h^s_{\tau^n(\xi),\tau^n(\xi')} \circ \phi^n_\xi(x')\| + \|\phi^n_\xi(x) - \phi^n_\xi(x')\|. \end{aligned}$$

Since $1 - 2C\nu^{m\alpha} > 0$ then we obtain that

$$\|h^s_{\tau^n(\xi),\tau^n(\xi')} \circ \phi^n_\xi(x) - h^s_{\tau^n(\xi),\tau^n(\xi')} \circ \phi^n_\xi(x')\| \leq (1 - 2C\nu^{m\alpha})^{-1}\|\phi^n_\xi(x) - \phi^n_\xi(x')\|.$$

Finally, this estimate together with (2.21) provides

$$\begin{aligned} \|h^s_{\xi,\xi'}(x) - h^s_{\xi,\xi'}(x')\| &\leq \gamma^{-n}\|h^s_{\tau^n(\xi),\tau^n(\xi')} \circ \phi^n_\xi(x) - h^s_{\tau^n(\xi),\tau^n(\xi')} \circ \phi^n_\xi(x')\| \\ &\leq (1 - 2C\nu^{m\alpha})^{-1}\gamma^{-n}\|\phi^n_\xi(x) - \phi^n_\xi(x')\| \leq (1 - 2C\nu^{m\alpha})^{-1}(\gamma\hat{\gamma})^{-n}\|x - x'\|. \end{aligned}$$

Since $(\gamma^{-1}\nu^{\alpha})^n \geq \eta^3$ then $n \leq 3\log\eta/\log\gamma^{-1}\nu^{\alpha} = \theta\log\eta/\log\gamma\hat{\gamma}$. This implies that $(\gamma\hat{\gamma})^{-n} \leq \eta^{-\theta}$ and therefore $\|h^s_{\xi,\xi'}(x) - h^s_{\xi,\xi'}(x')\|^{1+\theta} \leq (1 - 2C\nu^{m\alpha})^{-1}\|x - x'\|$. This concludes that $h^s_{\xi,\xi'}$ is locally ϱ-Hölder showing our assertion. Moreover, we here observe the uniformity of the Hölder constant. Hence the family of maps $h^s_{\xi,\xi'}$ is a ϱ-Hölder s-holonomy and therefore the proposition is completed. $\square$

Remark 2.7. *The local ϱ-Hölder constant provides by the above proof is $K_{loc} = (1 - 2C\nu^{m\alpha})^{-\varrho}$ where $C = C_\Phi(1 - \gamma^{-1}\nu^{\alpha})^{-1}$. The natural number m was chosen to provide that $K_{loc} > 0$. This choice is not necessary if C_Φ is close enough to zero. In such case, $K = (1 - 2C)^{-\varrho}$ is close to one and this constant can be taken as the global ϱ-Hölder constant of $h^s_{\xi,\xi'}$.*

Remark 2.8. *If the fiber maps ϕ_ξ are isometries then $\gamma\hat{\gamma}$ may be taken arbitrarily close to one and thus the Hölder exponent ϱ is also arbitrarily close to one. The same observation holds when the fiber maps ϕ_ξ of $\Phi = \tau \ltimes \phi_\xi$ are perturbations of the identity* id: $M \to M$.

Under the conditions of the previous proposition the holonomy maps are Hölder continuous. In order to increase this regularity we need to impose additional properties for $\Phi = \tau \ltimes \phi_\xi$. We will need C^2-fiber maps ϕ_ξ whose first derivative depends Hölder continuously with respect to ξ.

Definition 2.6. *Let $\alpha \in (0, 1]$. A skew-product $\Phi = \tau \ltimes \phi_\xi \in \mathcal{S}_k(M)$ with C^1-fiber maps is called* locally α-Hölder differentiable *(in M) if there is a non-negative constant $C \geq 0$ such that*

$$d_{C^1}(\phi_\xi^{\pm 1}, \phi_{\xi'}^{\pm 1}) \leq C\, d_{\Sigma_k}(\xi, \xi')^\alpha, \quad \text{for all } \xi,\, \xi' \in \Sigma_k \text{ with } \xi_0 = \xi_0'.$$

We will denote by $S^{r,1+\alpha}_{k,\gamma,\hat{\gamma}^{-1}}(M)$ the subset of $\mathcal{S}_k(M)$ consistent of symbolic skew-products locally α-Hölder differentiable with C^r-fiber maps ($r \geq 1$) which are $(\gamma, \hat{\gamma}^{-1})$-Lipschitz .

Recall that the center bunched inequalities for partial diffeomorphisms ensure the regularity of the foliations and the holonomies. The following definition introduces the equivalent inequalities for symbolic skew-products.

Definition 2.7 (Fiber bunched skew-products)**.** *A symbolic skew-product $\Phi \in S^{1,1+\alpha}_{k,\gamma,\hat{\gamma}^{-1}}(M)$ is said to be* fiber bunched *if $\nu^\alpha < \gamma\hat{\gamma}$.*

The following proposition states the C^1-regularity of the s-holonomy maps under the above conditions. This result is showed in [AV10, Remark 5.4] although also can be followed from [BGV03, Lemma 1.21] and [ASV11, Proposition 3.4].

Proposition 2.9. *Consider a s-dominated fiber bunched skew-product $\Phi = \tau \ltimes \phi_\xi \in \mathcal{S}^{2,1+\alpha}_{k,\gamma,\hat{\gamma}^{-1}}(M)$. Then, the holonomies maps $h^s_{\xi,\xi'}$ are C^1-diffeomorphisms.*

Proof. We will denote $y = (\xi, x)$ and $A(y) = D\phi_\xi(x)$. Then

$$\|A(y) - A(y')\| \leq \|D\phi_\xi(x) - D\phi_\xi(x')\| + \|D\phi_\xi(x') - D\phi_{\xi'}(x')\|.$$

By assumption, the skew-product Φ is α-Hölder differentiable and its fiber maps ϕ_ξ are C^2-diffeomorphisms. Hence, there are non-negative constants L, $\tilde{C}$ and $\tilde{K}$ such that

$$\|A(y) - A(y')\| \leq L\|x - x'\| + \tilde{C} d_{\Sigma_k}(\xi, \xi')^\alpha \leq \tilde{K} d(y, y')^\alpha. \tag{2.23}$$

Since $\Phi = \tau \ltimes \phi_\xi$ is s-dominated, from Proposition 2.5 it follows a s-lamination $\mathcal{W}^s$ for Φ with $d(\Phi^n(y), \Phi^n(y')) \leq (1 + 2C)\nu^{n\alpha} d(y, y')$ for all y and y' in the same local strong stable leaf where $C = C_\Phi(1 - \gamma^{-1}\nu^\alpha)^{-1} \geq 0$. Since Φ is fiber bunched then there is $n \in \mathbb{N}$ such that

$$\|A^n(y)\|\|A^n(y)^{-1}\|(1 + 2C)\nu^{n\alpha} \leq (\gamma\hat{\gamma})^{-n}(1 + 2C)\nu^{n\alpha} < 1. \tag{2.24}$$

Here, $A^n(y) = A(\Phi^{n-1}(y)) \cdots A(\Phi(y))A(y)$. Consider now, the *linear cocycle* F_A over Φ, given by $F_A(y, v) = (\Phi(y), A(y)v)$ with $y = (\xi, x)$ and $v \in T_xM$. The estimates (2.23) and (2.24) show that this linear cocycle is in the assumptions of Proposition 3.4 in [ASV11]. This result shows the existence of a linear isomorphism $H^s_{y,y'} : T_xM \to T_{x'}M$ such that $H^s_{y,y'}$ is the uniform limit of $A^n(y')^{-1}A^n(y)$. That is, $(\phi_{\xi'})^{-1} \circ \phi^n_\xi$ converges uniformly to $h^s_{\xi,\xi'}$ in the C^1-topology. In particular, in this case the s-holonomy maps are C^1-diffeomorphisms. □

2.2.2 Invariant graph

A closed set Γ of $\Sigma_k \times M$ is called *invariant graph* of a symbolic skew-product $\Phi = \tau \ltimes \phi_\xi$ if $\Phi(\Gamma) = \Gamma$ and there is a function $g\colon \Sigma_k \to M$ such that $\Gamma = \{(\xi, g(\xi))\colon \xi \in \Sigma_k\}$. Bony graphs are a generalization of invariant graphs. A closed set Γ of $\Sigma_k \times M$ is said to be *bony graph* for Φ if $\Phi(\Gamma) = \Gamma$ and it intersects almost every fiber $\{\xi\} \times M$ by a single point, and the rest of the fibers by some compacts, connected and non-empty sets called *bones* or *spines*. Note that a bony graph Γ can be represented as the disjoint union of sets P and Q. The set P is the union of spines $P_\xi = \pi^{-1}(\xi)$ where $\pi\colon \Gamma \to \Sigma_k$ denotes the projection on the base space. The set Q is the graph set of a function $g\colon \Sigma_k \setminus \pi(P) \to M$. Note that $\tau \circ \pi = \pi \circ \Phi|_\Gamma$. If the bony graph Γ has infinitely many spines, the function $g\colon \Sigma_k \setminus \pi(P) \to M$ is continuous and Γ is the maximal invariant set

$$\Gamma = \bigcap_{n\in\mathbb{Z}} \Phi^n(\Sigma_k \times \overline{D})$$

where D is a bounded open set of M, then it is called *porcupine*. A bounded open and connected set D of M is said to be *trapping region* (resp. *inverse trapping region*) for a symbolic skew-product $\Phi = \tau \ltimes \phi_\xi \in \mathcal{S}_k(M)$ if $\phi_\xi(\overline{D}) \subset D$ (resp. $\overline{D} \subset \phi_\xi(D)$) for all $\xi \in \Sigma_k$.

Proposition 2.10. *Let D be a trapping region (resp. inverse trapping region) for $\Phi = \tau \ltimes \phi_\xi$. Then the maximal invariant set Γ in $\Sigma_k \times \overline{D}$ is a bony graph. Moreover, if $\Phi \in \mathcal{S}_k^-(M)$ (resp. $\mathcal{S}_k^+(M)$) and there exists a periodic point $(\vartheta, p) \in \Sigma_k \times \overline{D}$ for Φ of period $s \geq 1$ such that p is a repelling (resp. attracting) fixed point of $\phi_\vartheta^s = \phi_{\tau^{s-1}(\vartheta)} \circ \cdots \circ \phi_\vartheta$ then Γ is a porcupine.*

Proof. Suppose that D is a trapping region for Φ. The case of inverse trapping region is totally analogous. Thus, since $\phi_\xi(\overline{D}) \subset D$ for all $\xi \in \Sigma_k$ then the maximal invariant set Γ intersects the fiber over the bi-sequence ξ in the set

$$D_\xi = \bigcap_{n\geq 1} \phi^n_{\tau^{-n}(\xi)}(\overline{D}) \quad \text{where} \quad \phi^n_{\tau^{-n}(\xi)}(\overline{D}) = \phi_{\tau^{-1}(\xi)} \circ \cdots \circ \phi_{\tau^{-n}(\xi)}(\overline{D}).$$

Indeed, if $(\xi, x) \in \Gamma$ then $(\xi, x) \in \Phi^n(\Sigma_k \times \overline{D})$ for all $n \in \mathbb{Z}$. Hence $x \in \overline{D}$ and for each $n > 0$ we get that x belongs to both $\phi^n_{\tau^{-n}(\xi)}(\overline{D})$ and $\phi^{-n}_{\tau^{n-1}(\xi)}(\overline{D})$. Since D is a trapping region we obtain that $\phi^n_{\tau^{-n}(\xi)}(D) \subset \overline{D}$ and $\overline{D} \subset \phi^{-n}_{\tau^{n-1}(\xi)}(D)$ for all $n > 0$ and therefore

$$x \in \phi^n_{\tau^{-n}(\xi)}(\overline{D}) \cap \phi^{-n}_{\tau^{n-1}(\xi)}(\overline{D}) = \phi^n_{\tau^{-n}(\xi)}(\overline{D}), \quad \text{for all } n > 0.$$

That is, $x \in D_\xi$. Reciprocally, if $x \in D_\xi$ from the above equation x belongs to both $\phi^n_{\tau^{-n}(\xi)}(\overline{D})$ and $\phi^{-n}_{\tau^{n-1}(\xi)}(\overline{D})$ and therefore $(\xi, x) \in \Phi^n(\Sigma_k \times \overline{D})$ for all $n \in \mathbb{Z}$. That is, $(\xi, x) \in \Gamma$.

Note that since $\phi^{n+1}_{\tau^{-(n+1)}(\xi)} = \phi^n_{\tau^{-n}(\xi)} \circ \phi_{\tau^{-(n+1)}(\xi)}$ then the connected compact sets $\phi^n_{\tau^{-n}(\xi)}(\overline{D})$ are nested and hence D_ξ is a single point or a connected compact set (with more than one point). This proves that the maximal invariant set Γ is a bony graph.

Now, assuming that $\Phi \in \mathcal{S}_k^-(M)$, we will prove that the function $g : \Sigma_k \setminus \pi(P) \to M$ is continuous where P is the collection of spines $\pi^{-1}(\xi)$. Take a point (ξ, x) in the graph set $Q = \Gamma \setminus P$ of g and a positive $\varepsilon > 0$. For sufficient large n, the connected compact set $\phi^n_{\tau^{-n}(\xi)}(\overline{D})$ is contained in the open ball $B(x, \varepsilon)$ around of x of radius $\varepsilon > 0$. Hence, since the fiber maps ϕ_ξ only depends on the unstable manifold of τ then $\phi^n_{\tau^{-n}(\xi')}(\overline{D}) \subset B(x, \varepsilon)$ for any sequence ξ'

such that $\xi'_{-i} = \xi_{-i}$ for $i = 1, \dots, n$. That is, the function g is continuous. To prove that Γ is a porcupine only remains to show that it has infinitely many spines $\pi^{-1}(\xi)$. We are assuming that there exist a periodic bi-sequence $\vartheta \in \Sigma_k$ of period s and a repelling fixed point $p \in D$ of ϕ^s_ϑ. Let $U \subset \overline{D}$ be a neighborhood of p such that $\phi^s_\vartheta(U) \supset U$. For any bi-sequence $\xi \in \Sigma_k$ of the form $\xi = (\dots \vartheta_{-2}, \vartheta_{-1}, \xi_{-n}, \dots, \xi_{-1}; \xi_0, \xi_1, \dots)$ the connected compact set D_ξ contains at least the set $\phi^n_{\tau^{-n}(\xi)}(U)$. Note that set of all such bi-sequences is dense and therefore Γ has infinitely many spines. This concludes the prove of the proposition. □

Given a bounded open set D of M, in this section we will study the maximal invariant in $\Sigma_k \times \overline{D}$. The next result claims the existence of a unique invariant attracting graph in $\Sigma_k \times \overline{D}$ for maps $\Phi = \tau \ltimes \phi_\xi$ in $\mathcal{S}_k(M)$ whose fiber maps ϕ_ξ are *locally* (λ, β)*-Lipschitz in* $\overline{D}$ with $\beta < 1$.

Definition 2.8 (Sets of local symbolic skew-products)**.** *Let $D \subset M$ be a bounded open set and consider constants $0 < \lambda < \beta$ and $0 \le \alpha \le 1$. We define $\mathcal{S}^{r,\alpha}_{k,\lambda,\beta}(D)$, $r \ge 0$, as the set of symbolic skew-product maps $\Phi = \tau \ltimes \phi_\xi \in \mathcal{S}_k(M)$ such that*

- *ϕ_ξ is a C^r-diffeomorphism;*
- *$\lambda \|x - x'\| < \|\phi_\xi(x) - \phi_\xi(x')\| < \beta \|x - x'\|$ for all $\xi \in \Sigma_k$ and $x, x' \in \overline{D}$;*
- *$\|\phi_\xi(x) - \phi_{\xi'}(x)\| \le C\, d_{\Sigma_k}(\xi, \xi')^\alpha$ for all $\xi' \in \Sigma_k$ with $\xi'_0 = \xi_0$ and $x \in \overline{D}$;*

We will denote by C_Φ the smallest (uniform) non-negative constants satisfying the above last inequality. Additionally, if $\beta < 1$ we impose the condition $\phi_\xi(\overline{D}) \subset D$ for all $\xi \in \Sigma_k$, and, in the case $1 < \lambda$ the imposed condition is $\overline{D} \subset \phi_\xi(D)$ for all $\xi \in \Sigma_k$. We also set

$$\mathcal{S}^{r,\alpha,+}_{k,\lambda,\beta}(D) = \mathcal{S}^{r,\alpha}_{k,\lambda,\beta}(D) \cap \mathcal{S}^+_k(M) \quad \text{and} \quad \mathcal{S}^{r,\alpha,-}_{k,\lambda,\beta}(D) = \mathcal{S}^{r,\alpha}_{k,\lambda,\beta}(D) \cap \mathcal{S}^-_k(M).$$

For notational convenience, $\mathcal{S}^{0,0}_{k,\lambda,\beta}(D)$, $\mathcal{S}^{0,0,\pm}_{k,\lambda,\beta}(D)$ and $\mathcal{S}^{0,\alpha}_{k,\lambda,\beta}(D)$ denote $\mathcal{S}_{k,\lambda,\beta}(D)$, $\mathcal{S}^{\pm}_{k,\lambda,\beta}(D)$ and $\mathcal{S}^{\alpha}_{k,\lambda,\beta}(D)$ respectively.

We endow $\mathcal{S}^{r,\alpha}_{k,\lambda,\beta}(D)$ with the distance

$$d_{\mathcal{S}}(\Phi, \Psi) = \sup_{\xi \in \Sigma_k} d_{C^r}(\phi_\xi, \psi_\xi) + |C_\Phi - C_\Psi| \quad \text{with} \quad \Phi = \tau \ltimes \phi_\xi \quad \text{and} \quad \Psi = \tau \ltimes \psi_\xi. \tag{2.25}$$

Here, one can see $d_{C^r}(\phi_\xi, \psi_\xi)$ as the C^r-distance between the restriction of ϕ_ξ and ψ_ξ to $\overline{D}$.

In what follows of this subsection, we will consider a fixed bounded open set D in M and unless otherwise stated we will assume that $0 \le \lambda < \beta < 1$.

Theorem 2.11 ([HPS77])**.** *Consider $\Phi = \tau \ltimes \phi_\xi \in \mathcal{S}_{k,\lambda,\beta}(D)$ with $\beta < 1$. Then there exists a unique bounded continuous function $g_\Phi : \Sigma_k \to \overline{D}$ such that*

i) $\Phi\big(\xi, g_\Phi(\xi)\big) = \big(\tau(\xi), g_\Phi(\tau(\xi))\big)$ *for all $\xi \in \Sigma_k$, and*

ii) $\|\phi^n_\xi(x) - g_\Phi(\tau^n(\xi))\| \le \beta^n \|g_\Phi(\xi) - x\|$ *for all $(\xi, x) \in \Sigma_k \times \overline{D}$ and $n \ge 0$.*

In [BHN99, Section 6] the continuous dependence of the invariant graphs with respect to Φ is also proved. On the other hand, notice that the above theorem is a reformulation of the results in [HPS77] which can also be found in [Sta99, Theorem 1.1]. Although the proof is simple we present it here since can be useful to understand better this invariant graph.

Proof. Let $C^0(\Sigma_k, \overline{D})$ be the space of bounded continuous functions $g:\Sigma_k \to \overline{D}$. We define the usual metric of uniform convergence on $C^0(\Sigma_k, \overline{D})$ by

$$d_{C^0}(g,\tilde{g}) = \sup_{\xi\in\Sigma_k} \|g(\xi)-\tilde{g}(\xi)\|,$$

where g and $\tilde{g}$ belong to $C^0(\Sigma_k, \overline{D})$. Note that, $C^0(\Sigma_k, \overline{D})$ with this metric is a complete metric space. Define the usual graph transform Υ for $\Phi = \tau \ltimes \phi_\xi$ by

$$\Upsilon[g](\xi) = \phi_{\tau^{-1}(\xi)} \circ g \circ \tau^{-1}(\xi), \quad \text{for } g \in C^0(\Sigma_k, \overline{D}) \text{ and } \xi \in \Sigma_k.$$

We claim that $\Upsilon[g] \in C^0(\Sigma_k, \overline{D})$. Indeed, we only need to show that $\Upsilon[g](\Sigma_k) \subset \overline{D}$. This follows from the assumption $\phi_\xi(\overline{D}) \subset D$ for all $\xi \in \Sigma_k$ noting that $g:\Sigma_k \to \overline{D}$. Thus, $\Upsilon\colon C^0(\Sigma_k, \overline{D}) \to C^0(\Sigma_k, \overline{D})$. Now, since $\Phi = \tau \ltimes \phi_\xi$ uniformly contracting in $\overline{D}$ with contraction constant $0<\beta<1$, it follows that $\|\Upsilon[g](\xi) - \Upsilon[\tilde{g}](\xi)\| \leq \beta \, \|g(\tau^{-1}(\xi)) - \tilde{g}(\tau^{-1}(\xi))\| \leq \beta \, d_{C^0}(g,\tilde{g})$. Taking the supremum over all $\xi \in \Sigma_k$, we get that $d_{C^0}(\Upsilon[g], \Upsilon[\tilde{g}]) \leq \beta \, d_{C^0}(g,\tilde{g})$. Hence, by the Contraction Mapping Theorem, Υ has a unique attracting fixed point $g_\Phi\colon \Sigma_k \to \overline{D}$.

By definition $g_\Phi \circ \tau(\xi) = \Upsilon[g_\Phi](\tau(\xi)) = \phi_\xi \circ g_\Phi(\xi)$. Hence, as required, the graph of g_Φ is invariant under Φ. Finally, by induction we have $g_\Phi \circ \tau^n(\xi) = \phi^n_\xi \circ g_\Phi(\xi)$ and so for every $x \in \overline{D}$ $\|\phi^n_\xi(x) - g_\Phi(\tau^n(\xi))\| = \|\phi^n_\xi(x) - \phi^n_\xi(g_\Phi(\xi))\| \leq \beta^n \|x - g_\Phi(\xi)\|$. Thus, taking as $n\to\infty$ the graph of g_Φ is attracting under $\Phi = \tau \ltimes \phi_\xi$ and we complete the proof of the theorem. □

Under the additional assumption that $\Phi = \tau \ltimes \phi_\xi$ is locally α-Hölder continuous skew-product for some $0<\alpha\leq 1$, the following result provides the same degree of regularity for the invariant graph g_Φ restricted to local unstable manifolds of τ. The proof of this result can be found in [BHN99, Lemma 2.6]. Here, we give a different proof following the ideas in [Wil98, Theorem 3.3].

Theorem 2.12. *Consider $\Phi = \tau \ltimes \phi_\xi \in \mathcal{S}^\alpha_{k,\lambda,\beta}(D)$ with $\beta < 1$. Then there is a positive constant $K \leq C_\Phi(1-\beta\nu^\alpha)^{-1}$ such that*

$$\|g_\Phi(\xi) - g_\Phi(\xi')\| \leq K \, d_{\Sigma_k}(\xi,\xi')^\alpha, \quad \xi,\, \xi' \in \Sigma_k \text{ with } \xi' \in W^u_{loc}(\xi;\tau).$$

Proof. Let $K = C_\Phi(1-\beta\nu^\alpha)^{-1}$. We define $C^{\alpha,K}$ as the subspace of the bounded continuous function $g:\Sigma_k \to \overline{D}$ such that $\|g(\xi)-g(\xi')\| \leq K\, d_{\Sigma_k}(\xi,\xi')^\alpha$, $\xi,\, \xi' \in \Sigma_k$ with $\xi' \in W^u_{loc}(\xi;\tau)$. Endow this space with the uniform topology. The idea is to show that the graph transformation Υ carries $C^{\alpha,K}$ into itself. Clearly, $C^{\alpha,K}$ is a closed subspace of $C^0(\Sigma_k, \overline{D})$, and hence the unique fixed point of Υ, g_Φ, lies in $C^{\alpha,K}$.

Recall that $\Upsilon[g](\xi) = \phi_{\tau^{-1}(\xi)} \circ g \circ \tau^{-1}(\xi)$. We want to show that $\Upsilon[g] \in C^{\alpha,K}$ if $g \in C^{\alpha,K}$. Indeed, given two bi-sequences ξ and ξ' in the same local unstable set for τ we have that

$$\begin{aligned}
&\|\Upsilon[g](\xi) - \Upsilon[g](\xi')\| \leq \|\phi_{\tau^{-1}(\xi)} \circ g\circ\tau^{-1}(\xi) - \phi_{\tau^{-1}(\xi)} \circ g \circ \tau^{-1}(\xi')\| \\
&+ \|\phi_{\tau^{-1}(\xi)} \circ g \circ \tau^{-1}(\xi') - \phi_{\tau^{-1}(\xi')} \circ g \circ \tau^{-1}(\xi')\| \leq \beta \|g\circ\tau^{-1}(\xi) - g\circ\tau^{-1}(\xi')\| \\
&+ C_\Phi \, d_{\Sigma_k}(\tau^{-1}(\xi),\tau^{-1}(\xi'))^\alpha \leq (\beta K + C_\Phi) d_{\Sigma_k}(\tau^{-1}(\xi),\tau^{-1}(\xi'))^\alpha \leq (\beta K + C_\Phi)\nu^\alpha d_{\Sigma_k}(\xi,\xi')^\alpha.
\end{aligned}$$

From the choice of K, it follows that $\beta K + C_\Phi \leq K$ and thus $\|\Upsilon[g](\xi) - \Upsilon[g](\xi')\| \leq K\, d_{\Sigma_k}(\xi,\xi')^\alpha$. That is, $\Upsilon[g] \in C^{\alpha,K}$ and therefore we conclude the proof of the theorem. □

The *Pointwise Hölder Section Theorem* in [Wil98, Theorem 3.3] states that $\beta < \nu^\alpha$ is a sufficient condition for the invariant graph $g_\Phi : \Sigma_k \to M$ to be α-Hölder. In fact, from [BHN99, Theorem 1.3] it follows that this inequality is generically necessary. The inequality $\beta < \nu^\alpha$ means that Φ contracts the fiber $\{\xi\} \times \overline{D}$ more sharply at Hölder scale α than it contracts the base at ξ. It is exactly the opposite of the s-dominating condition which we need to ensure the existence of strong stable lamination for Φ in Proposition 2.5. Therefore, as we will work in the context of s-dominated skew-products we only obtain that the invariant graph function is α-Hölder along to the unstable manifold, as Theorem 2.12 claims.

Bearing in mind Notation 2.4, Item (i) of Theorem 2.11 implies that for every $n > 0$

$$\phi_\xi^n \circ g_\Phi(\xi) = g_\Phi \circ \tau^n(\xi) \quad \text{and} \quad \phi_{\tau^{-1}(\xi)}^{-n} \circ g_\Phi(\xi) = g_\Phi \circ \tau^{-n}(\xi) \tag{2.26}$$

for all $\xi \in \Sigma_k$. Let $\mathrm{graph}[g_\Phi]\colon \Sigma_k \to \Sigma_k \times M$, $\mathrm{graph}[g_\Phi](\xi) = (\xi, g_\Phi(\xi))$, be the *invariant graph map* and denote the *invariant graph set* by

$$\Gamma_\Phi \stackrel{\text{def}}{=} \{(\xi, g_\Phi(\xi)) \colon \xi \in \Sigma_k\} \subset \Sigma_k \times \overline{D}.$$

The following proposition shows that the invariant graph of Φ is the locally maximal invariant set inside of $\Sigma_k \times \overline{D}$.

Proposition 2.13. *Consider $\Phi = \tau \ltimes \phi_\xi \in \mathcal{S}_{k,\lambda,\beta}(D)$ with $\beta < 1$. Then, the restriction $\Phi|_{\Gamma_\Phi}$ of Φ to the set Γ_Φ is continuously conjugated to τ. Moreover, the invariant graph set*

$$\Gamma_\Phi = \bigcap_{n\in\mathbb{Z}} \Phi^n(\Sigma_k \times \overline{D}) = \bigcap_{n\in\mathbb{N}} \Phi^n(\Sigma_k \times \overline{D})$$

is the maximal invariant set in $\Sigma_k \times \overline{D}$.

Proof. By (i) in Theorem 2.11, it follows $\Phi \circ \mathrm{graph}[g_\Phi] = \mathrm{graph}[g_\Phi] \circ \tau$. Hence $\mathrm{graph}[g_\Phi]$ conjugates the maps $\Phi|_{\Gamma_\Phi}$ and τ. To get the continuity just note that $\mathrm{graph}[g_\Phi]$ is continuous and that $\mathrm{graph}[g_\Phi]^{-1}\colon \Sigma_k \times M \to \Sigma_k$ is the projection on the first coordinate, thus it is also continuous. So, we conclude the first part of the proposition.

Recall that periodic points of the shift map τ are dense in Σ_k, i.e., $\Sigma_k = \overline{\mathrm{Per}(\tau)}$. Conjugation in the first part of this proposition implies that $\Gamma_\Phi = \overline{\mathrm{Per}(\Phi|_{\Gamma_\Phi})}$. Let Γ be the local maximal invariant set of Φ in $\Sigma_k \times D$. Note that $\phi_\xi(\overline{D}) \subset D$ for all $\xi \in \Sigma_k$. Hence $\Gamma = \cap_{n\in\mathbb{Z}} \Phi^n(\Sigma_k \times \overline{D})$ and $\Gamma_\Phi = \overline{\mathrm{Per}(\Phi|_{\Gamma_\Phi})} \subset \Gamma$.

In order to prove that $\Gamma \subset \Gamma_\Phi$ given any $(\xi, x) \in \Gamma$ it suffices to see that $x = g_\Phi(\xi)$. As the set Γ_Φ is bounded, we have that $K = \sup\{d(\gamma, \Gamma_\Phi), \gamma \in \Gamma\} \in [0, +\infty)$. Since the maps ϕ_ξ are contractions with contraction constant $0 < \beta < 1$ we deduce that

$$\begin{aligned} \|x - g_\Phi(\xi)\| &= \|\phi_\xi^n \circ \phi_\xi^{-n}(x) - \phi_\xi^n \circ \phi_\xi^{-n}(g_\Phi(\xi))\| \le \beta^n \|\phi_\xi^{-n}(x) - \phi_\xi^{-n}(g_\Phi(\xi))\| \\ &= \beta^n d\big(\Phi^{-n}(\xi, x), \Phi^{-n}(\xi, g_\Phi(\xi))\big) \le K\beta^n. \end{aligned}$$

Taking $n \to \infty$ we get $x = g_\Phi(\xi)$ and thus $(\xi, x) \in \Gamma_\Phi$, implying that $\Gamma \subset \Gamma_\Phi$. This completes the proof of the proposition. $\square$

Next, we will give more information about the maximal invariant set. In particular, if the symbolic skew-product Φ belongs to $\mathcal{S}_{k,\lambda,\beta}(D)$ with $\beta<1$ then it satisfies the u-dominating condition: $\beta<1<\nu^{-1}$. Thus we could expect the existence of strong unstable lamination through the point of the maximal invariant set Γ_Φ in $\Sigma_k\times\overline{D}$.

Proposition 2.14. *Consider $\Phi=\tau\ltimes\phi_\xi\in\mathcal{S}_{k,\lambda,\beta}(D)$ with $\beta<1$ and let Γ_Φ be the maximal invariant set in $\Sigma_k\times\overline{D}$. Then, there exists a lamination*

$$\mathcal{W}^u_{\Gamma_\Phi}=\{W^{uu}_{loc}((\xi,x);\Phi):\,(\xi,x)\in\Gamma_\Phi\}$$

such that

i) every leaf $W^{uu}_{loc}((\xi,x);\Phi)$ is the graph of a continuous function $\gamma^u_{\xi,x}:W^u_{loc}(\xi;\tau)\to M$,

ii) $\phi^{-1}_{\tau^{-1}(\xi')}\circ\gamma^u_{\xi,x}(\xi')=\gamma^u_{\eta,y}\circ\tau^{-1}(\xi')$ where $\eta=\tau^{-1}(\xi)$, $y=\phi^{-1}_{\tau^{-1}(\xi)}(x)$ and $\xi'\in W^u_{loc}(\xi;\tau)$,

iii) if $(\vartheta,p)\in\Gamma_\Phi$ is a periodic point of Φ then $W^{uu}_{loc}((\vartheta,p);\Phi)\subset W^u((\vartheta,p);\Phi)$.

In addition, if Φ is locally α-Hölder then $\gamma^u_{\xi,x}:W^u_{loc}(\xi;\tau)\to M$ is an α-Hölder function and $W^{uu}_{loc}((\xi,x);\Phi)\subset W^u((\xi,x);\Phi)$ for all $(\xi,x)\in\Gamma_\Phi$.

Proof. Let $(\xi,x)\in\Gamma_\Phi$. Following Proposition 2.5, we define $\gamma^{u,n}_{\xi,x}:W^u_{loc}(\xi;\tau)\to M$ by

$$\gamma^{u,n}_{\xi,x}(\xi')=\phi^n_{\tau^{-n}(\xi')}\circ(\phi^n_{\tau^{-n}(\xi)})^{-1}(x)=\phi^n_{\tau^{-n}(\xi')}\circ\phi^{-n}_{\tau^{-1}(\xi)}(x).$$

Note that since $x=g_\Phi(\xi)$ then $\gamma^{u,n}_{\xi,x}(\xi')=\phi^n_{\tau^{-n}(\xi')}\circ\phi^{-n}_{\tau^{-1}(\xi)}\circ g_\Phi(\xi)$. Thus, for simplicity in the notation we will forget the subindex x and denote this map as $\gamma^{u,n}_\xi$. Observe that $\{\gamma^{u,n}_\xi\}$ is a sequence in the complete metric space $C^0(W^u_{loc}(\xi;\tau),M)$. We will show that this sequence is Cauchy and so converges.

By (2.26) it follows that

$$\phi^{-n}_{\tau^{-1}(\xi)}\circ g_\Phi(\xi)=g_\Phi\circ\tau^{-n}(\xi)\in\overline{D},\quad\text{for all } n\in\mathbb{N}.$$

Then, since $\phi_\xi(\overline{D})\subset D$ for all $\xi\in\Sigma_k$, we have that $\phi^{n-i}_{\tau^{-n}(\xi')}\circ\phi^{-n}_{\tau^{-1}(\xi)}\circ g_\Phi(\xi)\in\overline{D}$ for every $0<i\leq n$. Since $\Phi=\tau\ltimes\phi_\xi$ is in $\mathcal{S}_{k,\lambda,\beta}(D)$ we gett that $\|\gamma^{u,n+1}_\xi(\xi')-\gamma^{u,n}_\xi(\xi')\|$ is less or equal than

$$\begin{aligned}\beta^n\|\phi_{\tau^{-n-1}(\xi')}\circ\phi^{-1}_{\tau^{-n-1}(\xi)}\circ\phi^{-n}_{\tau^{-1}(\xi)}\circ g_\Phi(\xi)-\phi_{\tau^{-n-1}(\xi')}\circ\phi^{-1}_{\tau^{-n-1}(\xi')}\circ\phi^{-n}_{\tau^{-1}(\xi)}\circ g_\Phi(\xi)\|\\ \leq\beta^{n+1}\|\phi^{-1}_{\tau^{-n-1}(\xi)}\circ\phi^{-n}_{\tau^{-1}(\xi)}\circ g_\Phi(\xi)-\phi^{-1}_{\tau^{-n-1}(\xi')}\circ\phi^{-n}_{\tau^{-1}(\xi)}\circ g_\Phi(\xi)\|.\end{aligned}$$

Given any $\varepsilon>0$ and using that ϕ^{-1}_ξ depends uniformly continuously with respect to base point ξ, we find $n_0\in\mathbb{N}$ such that for every $\xi'\in W^u_{loc}(\xi;\tau)$ it holds that $d_{C^0}(\phi^{-1}_{\tau^{-n-1}(\xi)},\phi^{-1}_{\tau^{-n-1}(\xi')})<\varepsilon$ for all $n\geq n_0$. Thus, for $n\geq n_0$ we get $\|\gamma^{u,n+1}_\xi(\xi')-\gamma^{u,n}_\xi(\xi')\|\leq\beta^{n+1}\varepsilon$. Hence,

$$d_{C^0}(\gamma^{u,n+1}_\xi,\gamma^{u,n}_\xi)\leq\beta^{n+1}\varepsilon\quad\text{for all } n\geq n_0.$$

This implies that the sequence $\{\gamma^{u,n}_\xi\}$ is Cauchy and therefore converges. We will denote the limit by $\gamma^u_\xi\in C^0(W^u_{loc}(\xi;\tau),M)$.

Item (ii) is a consequence of the fact that $\phi^{-1}_{\tau^{-1}(\xi)}(x) = \phi^{-1}_{\tau^{-1}(\xi)} \circ g_\Phi(\xi) = g_\Phi \circ \tau^{-1}(\xi)$ and

$$\begin{aligned}\phi^{-1}_{\tau^{-1}(\xi')} \circ \gamma^u_\xi(\xi') &= \lim_{n\to\infty} \phi^{-1}_{\tau^{-1}(\xi')} \circ \phi^n_{\tau^{-n}(\xi')} \circ \phi^{-n}_{\tau^{-1}(\xi)} \circ g_\Phi(\xi)\\ &= \lim_{n\to\infty} \phi^{n-1}_{\tau^{-n}(\xi')} \circ \phi^{-(n-1)}_{\tau^{-2}(\xi)} \circ g_\Phi \circ \tau^{-1}(\xi) = \gamma^u_{\tau^{-1}(\xi)} \circ \tau^{-1}(\xi').\end{aligned}$$

Let $(\vartheta, p) \in \Gamma_\Phi$ be a periodic point of Φ. We will show that $W^{uu}_{loc}((\vartheta,p);\Phi) \subset W^u((\vartheta,p);\Phi)$. Given any point (ξ, x) in $W^{uu}_{loc}((\vartheta,p);\Phi)$ we get $\xi \in W^u_{loc}(\vartheta;\tau)$ and $x = \gamma^u_\vartheta(\xi)$. Thus,

$$\begin{aligned}d(\Phi^{-n}(\xi,x),\Phi^{-n}(\vartheta,p)) &= d_{\Sigma_k}(\tau^{-n}(\xi),\tau^{-n}(\vartheta)) + \|\phi^{-n}_{\tau^{-1}(\xi)}(x) - \phi^{-n}_{\tau^{-1}(\vartheta)}(p)\|\\ &\le \nu^n d_{\Sigma_k}(\xi,\vartheta) + \|\phi^{-n}_{\tau^{-1}(\xi)} \circ \gamma^u_\vartheta(\xi) - \phi^{-n}_{\tau^{-1}(\vartheta)}(p)\|.\end{aligned} \tag{2.27}$$

Now, using that

$$\gamma^u_{\tau^{-n}(\vartheta)} \circ \tau^{-n}(\vartheta) = g_\Phi \circ \tau^{-n}(\vartheta) = \phi^{-n}_{\tau^{-1}(\vartheta)} \circ g_\Phi(\vartheta) = \phi^{-n}_{\tau^{-1}(\vartheta)}(p)$$

and $\phi^{-n}_{\tau^{-1}(\xi)} \circ \gamma^u_\vartheta(\xi) = \gamma^u_{\tau^{-n}(\vartheta)} \circ \tau^{-n}(\xi)$ we infer that

$$\|\phi^{-n}_{\tau^{-1}(\xi)} \circ \gamma^u_\vartheta(\xi) - \phi^{-n}_{\tau^{-1}(\vartheta)}(p)\| = \|\gamma^u_{\tau^{-n}(\vartheta)} \circ \tau^{-n}(\xi) - \gamma^u_{\tau^{-n}(\vartheta)} \circ \tau^{-n}(\vartheta)\|. \tag{2.28}$$

Note that since $\vartheta \in \Sigma_k$ is a periodic bi-sequence then we only have a finite number of functions $\{\gamma^u_{\tau^{-n}(\vartheta)}\}$. Namely $\gamma^u_{\tau^{-i}(\vartheta)}$ for $i = 1, \dots, s$ where s is the period of ϑ. From the uniform continuity of these maps and since $d_{\Sigma_k}(\tau^{-n}(\xi), \tau^{-n}(\vartheta)) \to 0$ then $\|\gamma^u_{\tau^{-n}(\vartheta)} \circ \tau^{-n}(\xi) - \gamma^u_{\tau^{-n}(\vartheta)} \circ \tau^{-n}(\vartheta)\| \to 0$ as $n \to \infty$. This implies that (2.27) goes to zero as n goes to infinity and therefore (ξ, x) belongs to $W^u((\vartheta,p);\Phi)$.

In order to prove that γ^u_ξ is α-Hölder if the skew-product Φ is locally α-Hölder we proceed as in Proposition 2.5 obtaining that

$$\|\gamma^u_\xi(\xi') - \gamma^u_\xi(\xi'')\| \le C d_{\Sigma_k}(\xi',\xi'')^\alpha \quad \text{for all } \xi', \xi'' \in W^u_{loc}(\xi;\tau)$$

where $C = C_\Phi(1-\beta\nu^\alpha)^{-1}$. In particular, using this regularity in (2.28) and substituting in (2.27) it follows that

$$d(\Phi^{-n}(\xi,x),\Phi^{-n}(\vartheta,p)) \le \nu^n d_{\Sigma_k}(\xi,\vartheta) + C\nu^{\alpha n} d_{\Sigma_k}(\xi,\vartheta)^\alpha$$

for all point $(\vartheta, p) \in \Gamma_\Phi$. This shows that $W^{uu}_{loc}((\vartheta,p);\Phi) \subset W^u((\vartheta,p);\Phi)$ for all $(\vartheta,p) \in \Gamma_\Phi$ and we conclude the proof of the Proposition. □

By construction, each one of the leaves of $\mathcal{W}^u_{\Gamma_\Phi}$ is the local strong unstable set $W^{uu}_{loc}((\xi,x);\Phi)$ followed from the dual result of Proposition 2.5 through the point (ξ, x) in Γ_Φ. Recall that these local leaves allow us to define the *strong unstable set of* $(\xi,x) \in \Gamma_\Phi$ *and* Γ_Φ respectively as

$$W^{uu}((\xi,x);\Phi) \stackrel{\text{def}}{=} \bigcup_{n\ge 0} \Phi^n\big(W^{uu}_{loc}(\Phi^{-n}(\xi,x);\Phi)\big) \quad \text{and} \quad W^{uu}(\Gamma_\Phi) \stackrel{\text{def}}{=} \bigcup_{(\xi,x)\in\Gamma_\Phi} W^{uu}((\xi,x);\Phi).$$

The next proposition shows the relation between the invariant graph Γ_Φ, the unstable set and the strong unstable lamination for Γ_Φ. Before that, we introduce some notations. For each $\Phi = \tau \ltimes \phi_\xi$ we denote $\mathrm{Per}(\Phi)$ the set of periodic points of Φ and $\mathscr{P} : \Sigma_k \times M \to M$ is the canonical projection on the fiber space.

Proposition 2.15. *Consider $\Phi = \tau \ltimes \phi_\xi \in \mathcal{S}_{k,\lambda,\beta}(D)$ with $\beta < 1$. Then for every periodic point (ϑ, p) in $\Sigma_k \times \overline{D}$ of Φ it holds*

i) $W^{uu}(\Gamma_\Phi) = \Gamma_\Phi$,

ii) $W^{uu}((\vartheta,p);\Phi) = W^u((\vartheta,p);\Phi)$, *and*

iii) $K_\Phi \overset{\text{def}}{=} \overline{\mathscr{P}\big(\mathrm{Per}(\Phi)\big) \cap D} = \overline{\mathscr{P}\big(W^u((\vartheta,p);\Phi)\big)} = \mathscr{P}(\Gamma_\Phi) = g_\Phi(\Sigma_k)$.

In addition, if Φ is locally α-Hölder then $W^u((\xi,x);\Phi) = W^{uu}((\xi,x);\Phi) \subset \Gamma_\Phi$ for all $(\xi,x) \in \Gamma_\Phi$.

Proof. In order to prove that $W^{uu}((\vartheta,p);\Phi) \subset W^u((\vartheta,p);\Phi)$ for all periodic point (ϑ,p) in Γ_Φ we take $(\xi,x) \in W^{uu}((\vartheta,p);\Phi)$ and we will show that

$$\lim_{n\to\infty} d(\Phi^{-n}(\xi,x), \Phi^{-n}(\vartheta,p)) = 0.$$

Since (ξ,x) belongs to the local strong unstable set, there exist $m \in \mathbb{N}$ and $(\xi',x') \in W^{uu}_{loc}(\Phi^{-m}(\vartheta,p);\Phi)$ such that $(\xi,x) = \Phi^m(\xi',x')$. Let us denote $(\eta,y) = \Phi^{-m}(\vartheta,p)$. Notice that (η,y) is a periodic point in Γ_Φ, $(\xi',x') \in W^{uu}_{loc}((\eta,y);\Phi)$ and

$$d(\Phi^{-n}(\xi,x), \Phi^{-n}(\vartheta,p)) = d(\Phi^{-(n-m)}(\xi',x'), \Phi^{-(n-m)}(\eta,y)).$$

From Proposition 2.14, we have that $W^{uu}_{loc}((\eta,y);\Phi) \subset W^u((\eta,y);\Phi)$ and thus it follows that $d(\Phi^{-n}(\xi,x), \Phi^{-n}(\vartheta,p)) \to 0$ as $n \to \infty$. Note that if Φ is locally α-Hölder the same argue works to show that $W^{uu}((\xi,x);\Phi) \subset W^u((\xi,x);\Phi)$ for all $(\xi,x) \in \Gamma_\Phi$.

We will show that if $(\xi,x) \in \Gamma_\Phi$ then we obtain that the unstable set $W^u((\xi,x);\Phi)$ is contained in the strong unstable set $W^{uu}((\xi,x);\Phi)$. Indeed, take $(\xi',x') \in W^u((\xi,x);\Phi)$. It suffices to show that there is $m \in \mathbb{N}$ such that $\Phi^{-m}(\xi',x') \in W^{uu}_{loc}(\Phi^{-m}(\xi,x);\Phi)$. By definition of unstable set,

$$\lim_{n\to\infty} d_{\Sigma_k}(\tau^{-n}(\xi'), \tau^{-n}(\xi)) = 0 \quad \text{and} \quad \lim_{n\to\infty} \|\phi^{-n}_{\tau^{-1}(\xi')}(x') - \phi^{-n}_{\tau^{-1}(\xi)}(x)\| = 0. \tag{2.29}$$

Since $(\xi,x) \in \Gamma_\Phi$ then $\phi^{-n}_{\tau^{-1}(\xi)}(x) \in D$ for all $n \geq 0$. Thus, there exists $m \in \mathbb{N}$ such that

$$\tau^{-m}(\xi') \in W^u_{loc}(\tau^{-m}(\xi);\tau) \quad \text{and} \quad \phi^{-(n+m)}_{\tau^{-1}(\xi')}(x') \in D$$

for all $n \geq m$. Let us denote $(\eta,y) = \Phi^{-m}(\xi,x)$ and $(\eta',y') = \Phi^{-m}(\xi',x')$. Hence, since $\phi^{n-i}_{\tau^{-n}(\eta)} \circ \phi^{-n}_{\tau^{-1}(\eta)}(y)$ and $\phi^{n-i}_{\tau^{-n}(\eta')} \circ \phi^{-n}_{\tau^{-1}(\eta')}(y')$ belong to D for all $0 < i \leq n$, we get

$$\begin{aligned}\|y' - \gamma^u_{\eta,y}(\eta')\| &= \lim_{n\to\infty} \|\phi^n_{\tau^{-n}(\eta')} \circ \phi^{-n}_{\tau^{-1}(\eta')}(y') - \phi^n_{\tau^{-n}(\eta')} \circ \phi^{-n}_{\tau^{-1}(\eta)}(y)\| \\ &\leq \lim_{n\to\infty} \beta^n \|\phi^{-n}_{\tau^{-1}(\eta')}(y') - \phi^{-n}_{\tau^{-1}(\eta)}(y)\|.\end{aligned}$$

by (2.29) and since $\beta < 1$, it follows that the above limit is equal zero and so $y' = \gamma^u_{\eta,y}(\eta')$. That is, $\Phi^{-m}(\xi',x') \in W^{uu}_{loc}(\Phi^{-m}(\xi,x);\Phi)$ and therefore $(\xi',x') \in W^{uu}((\xi,x);\Phi)$. This concludes, in particular, that $W^{uu}((\vartheta,p);\Phi) = W^u((\vartheta,p);\Phi)$ for all periodic point (ϑ,p) in the locally maximal invariant set Γ_Φ.

Now we will show that $W^{uu}(\Gamma_\Phi) = \Gamma_\Phi$. Observe that by definition $\Gamma_\Phi \subset W^{uu}(\Gamma_\Phi)$. Hence, to complete Item (i) it suffices to see that $W^{uu}_{loc}((\xi,x);\Phi) \subset \Gamma_\Phi$ for all $(\xi,x) \in \Gamma_\Phi$. Consider $(\xi',x') \in W^{uu}_{loc}((\xi,x);\Phi)$. By (2.26) and noting that $x = g_\Phi(\xi)$ we get

$$\|g_\Phi(\xi') - \gamma^u_{\xi,x}(\xi')\| = \lim_{n\to\infty} \|\phi^n_{\tau^{-n}(\xi')} \circ g_\Phi \circ \tau^{-n}(\xi') - \phi^n_{\tau^{-n}(\xi')} \circ g_\Phi \circ \tau^{-n}(\xi)\|.$$

Since $g_\Phi : \Sigma_k \to \overline{D}$ is a continuous bounded function it follows

$$\|g_\Phi(\xi') - \gamma^u_{\xi,x}(\xi')\| \leq \lim_{n\to\infty} \beta^n \|g_\Phi \circ \tau^{-n}(\xi') - g_\Phi \circ \tau^{-n}(\xi)\| \leq \lim_{n\to\infty} \beta^n K$$

where K is a bound of the diameter of $\overline{D}$. Since this limit is equal to zero we have just shown that $g_\Phi(\xi') = \gamma^u_{\xi,x}(\xi') = x'$ and so $(\xi', x') \in \Gamma_\Phi$. Therefore, we conclude the first item of this proposition.

It only remains to prove (iii). Consider any periodic point $\vartheta \in \Sigma_k$ of τ and note that $W^u(\vartheta;\tau)$ and $\mathrm{Per}(\tau)$ are both dense in Σ_k. By means of the conjugation in Lemma 2.13 and both, the first and the second items in this proposition, we get

$$\overline{\mathrm{Per}(\Phi|_{\Gamma_\Phi})} = \Gamma_\Phi = \overline{W^u\big((\vartheta, g_\Phi(\vartheta));\Phi\big)} = \overline{W^{uu}\big((\vartheta, g_\Phi(\vartheta));\Phi\big)}. \tag{2.30}$$

Note that if $(\vartheta, p) \in \Sigma_k \times \overline{D}$ is a periodic point of Φ, from the assumption $\phi_\xi(\overline{D}) \subset D$, it follows that $\Phi^n(\vartheta, p) \in \Sigma_k \times D$ for all $n \in \mathbb{Z}$. Moreover, since g_Φ is the unique invariant graph of Φ restricted to $\Sigma_k \times \overline{D}$, then $p = g_\Phi(\vartheta)$. From this, we have

$$\mathrm{Per}(\Phi|_{\Gamma_\Phi}) = \mathrm{Per}(\Phi|_{\Sigma_k \times \overline{D}}) = \mathrm{Per}(\Phi) \cap (\Sigma_k \times D). \tag{2.31}$$

Thus, recalling that K_Φ is the closure of projection by $\mathscr{P}$ of the periodic points of Φ in M and since the projection $\mathscr{P}$ is a closed map and Σ_k is a compact set, we infer from (2.30) and (2.31) that

$$\mathscr{P}(\Gamma_\Phi) = \overline{\mathscr{P}\big(W^u((\vartheta,p);\Phi)\big)} = \overline{\mathscr{P}\big(\mathrm{Per}(\Phi|_{\Gamma_\Phi})\big)} = \overline{\mathscr{P}\big(\mathrm{Per}(\Phi)\big) \cap D} \overset{\mathrm{def}}{=} K_\Phi.$$

Finally, we note that, from the above equation, $K_\Phi = g_\Phi(\Sigma_k) \subset \overline{D}$. Now, the proof of item (iii) is complete and therefore we conclude the proposition. □

Let $\mathcal{K}(\overline{D})$ denote the complete metric space whose elements are the compact subsets of $\overline{D}$ endowed with the Hausdorff metric d_H given by

$$d_H(A,B) = \sup\{d(a,B), d(b,A) : a \in A, b \in B\}, \quad A,\ B \in \mathcal{K}(\overline{D}).$$

From the above proposition it follows that $K_\Phi \in \mathcal{K}(\overline{D})$ for all $\Phi \in \mathcal{S}_{k,\lambda,\beta}(D)$. Here we gives some properties of the Hausdorff metric which we will use along this chapter. Given a non-empty compact set $C \subset \overline{D}$ and $\delta \geq 0$, the set $C_\delta = \{x \in \overline{D} :$ there is $y \in C$ such that $\|x - y\| < \delta\}$ is called *generalize δ-ball around C*.

Remark 2.16. *Let A, B be non-empty subsets of $\overline{D}$. Then*

$$d_H(A,B) = \inf\{\delta \geq 0 : \ B \subset A_\delta \ \textit{and} \ A \subset B_\delta\},$$

$d_H(\overline{A}, \overline{B}) = d_H(A,B)$ and $d_H(T(A), T(B)) \leq \mathrm{Lip}(T)\, d_H(A,B)$ where $T : \overline{D} \to \overline{D}$ is a Lipschitz map. Also, if A_i and B_i are non-empty subsets for all i in a set of index I then

$$d_H\Big(\bigcup_{i\in I} A_i, \bigcup_{i\in I} B_i\Big) \leq \sup_{i\in I} d_H(A_i, B_i).$$

We can define the map $\mathscr{L} : \mathcal{S}_{k,\lambda,\beta}(D) \to \mathcal{K}(\overline{D})$, given by $\mathscr{L}(\Phi) = K_\Phi$. The following proposition shows that this map is continuous.

Proposition 2.17. *Consider $\Phi = \tau \ltimes \phi_\xi \in \mathcal{S}_{k,\lambda,\beta}(D)$ with $\beta < 1$. Then, for each $\varepsilon > 0$ there is $\delta > 0$ such that for every $\Psi = \tau \ltimes \psi_\xi \in \mathcal{S}_{k,\lambda,\beta}(D)$ with*

$$d_{C^0}(\phi_\xi, \psi_\xi) < \delta \quad \textit{it holds that} \quad d_H(K_\Phi, K_\Psi) = d_H(\mathscr{L}(\Phi), \mathscr{L}(\Psi)) < \varepsilon.$$

Proof. Fixed small $\varepsilon > \epsilon > 0$ let $\delta = \epsilon(1-\beta)/2 > 0$. Take a fixed point $\vartheta \in \Sigma_k$ of τ. As $\phi_\vartheta(\overline{D}) \subset D$, by Brouwer's Fixed Point Theorem, there is $p_\Phi \in D$ such that $\phi_\vartheta(p_\Phi) = p_\Phi$. Thus, (ϑ, p_Φ) is a fixed point of Φ in $\Sigma_k \times D$. Then, if δ is small enough, for every $\Psi = \tau \ltimes \psi_\xi$ with $d_{C^0}(\phi_\xi, \psi_\xi) < \delta$ there is $p_\Psi \in D$ close to p_Φ which is a fixed point of ψ_ϑ. We say $(\vartheta, p_\Psi) \in \Gamma_\Psi$ is the continuation of $(\vartheta, p_\Phi) \in \Gamma_\Phi$ for Ψ where Γ_Ψ and Γ_Φ are the invariant set graph for Ψ and Φ respectively. Take $\Theta = \tau \ltimes \theta \in \{\Phi, \Psi\}$. Since from Proposition 2.15 the strong unstable set and the unstable set coincide, it follows

$$\mathscr{P}\big(W^u((\vartheta, p_\Theta); \Theta)\big) = \bigcup_{n \geq 0} \mathscr{P} \circ \Phi^n\big(W^{uu}_{loc}((\vartheta, p_\Theta); \Theta)\big).$$

By Proposition 2.14, the graph set of the function $\gamma^u_{\vartheta, p_\Theta} : W^u_{loc}(\vartheta; \tau) \to M$ is precisely the local strong set of (ϑ, p_Θ) for Θ. Thus, for each $n \geq 0$, the projection by $\mathscr{P}$ of $\Phi^n(W^{uu}_{loc}((\vartheta, p_\Theta); \Theta))$ is exactly $E_n(\Theta) = \{\theta^n_\xi \circ \gamma^u_{\vartheta, p_\Theta}(\xi) \colon \ \xi \in W^u_{loc}(\vartheta; \tau)\}$. Hence, by Proposition 2.15

$$K_\Theta = \overline{\mathscr{P}\big(W^u\big((\vartheta, p_\Theta); \Theta\big)\big)} = \overline{\bigcup_{n \geq 0} E_n(\Theta)}, \qquad \Theta = \Phi, \ \Psi.$$

According to Remark 2.16

$$d_H\big(K_\Phi, K_\Psi\big) \leq \sup_{n \geq 0} d_H\big(E_n(\Phi), E_n(\Psi)\big).$$

On the other hand, for each $n \geq 0$, we get

$$d_H\big(E_n(\Phi), E_n(\Psi)\big) \leq \sup_{\xi \in W^u_{loc}(\vartheta; \tau)} \|\phi^n_\xi \circ \gamma^u_{\vartheta, p_\Phi}(\xi) - \psi^n_\xi \circ \gamma^u_{\vartheta, p_\Psi}(\xi)\|. \tag{2.32}$$

Fix $\xi \in W^u_{loc}(\vartheta; \tau)$. Firstly we will estimate (2.32) for $n = 0$. From the Item (ii) in Proposition 2.14 we get for every $m \in \mathbb{N}$ that $\gamma^u_{\vartheta, p_\Theta}(\xi) = \theta^m_{\tau^{-m}(\xi)} \circ \gamma^u_{\vartheta, p_\Theta} \circ \tau^{-m}(\xi)$. Thus, from this we infer that $\|\gamma^u_{\vartheta, p_\Phi}(\xi) - \gamma^u_{\vartheta, p_\Psi}(\xi)\|$ is less or equal than

$$\beta^m \|\gamma^u_{\vartheta, p_\Phi} \circ \tau^{-m}(\xi) - \gamma^u_{\vartheta, p_\Psi} \circ \tau^{-m}(\xi)\| + \|\phi^m_{\tau^{-m}(\xi)} \circ \gamma^u_{\vartheta, p_\Psi} \circ \tau^{-m}(\xi) - \psi^m_{\tau^{-m}(\xi)} \circ \gamma^u_{\vartheta, p_\Psi} \circ \tau^{-m}(\xi)\|.$$

By continuity and since ξ is in the local unstable manifold of the fixed point ϑ for τ we get the limit of $\|\gamma^u_{\vartheta, p_\Phi} \circ \tau^{-m}(\xi) - \gamma^u_{\vartheta, p_\Psi} \circ \tau^{-m}(\xi)\|$ as $m \to \infty$ is $\|p_\Phi - p_\Psi\|$. Hence the limit of the first term in the above sum as $m \to \infty$ is equal to zero. Now, we will estimate the limit of the second term:

$$\begin{aligned}
\|\phi_{\tau^{-1}(\xi)} \circ \phi^{m-1}_{\tau^{-m}(\xi)} \circ \gamma^u_{\vartheta, p_\Psi} \circ \tau^{-m}(\xi) \ - \ \psi_{\tau^{-1}(\xi)} \circ \psi^{m-1}_{\tau^{-m}(\xi)} \circ \gamma^u_{\vartheta, p_\Psi} \circ \tau^{-m}(\xi)\| \leq& \\
\leq \|\phi_{\tau^{-1}(\xi)} \circ \phi^{m-1}_{\tau^{-m}(\xi)} \circ \gamma^u_{\vartheta, p_\Psi} \circ \tau^{-m}(\xi) \ - \ \psi_{\tau^{-1}(\xi)} \circ \phi^{m-1}_{\tau^{-m}(\xi)} \circ \gamma^u_{\vartheta, p_\Phi} \circ \tau^{-m}(\xi)\|& \\
+ \|\psi_{\tau^{-1}(\xi)} \circ \phi^{m-1}_{\tau^{-m}(\xi)} \circ \gamma^u_{\vartheta, p_\Phi} \circ \tau^{-m}(\xi) \ - \ \psi_{\tau^{-1}(\xi)} \circ \psi^{m-1}_{\tau^{-m}(\xi)} \circ \gamma^u_{\vartheta, p_\Psi} \circ \tau^{-m}(\xi)\|& \\
\leq \delta + \beta \, \|\phi^{m-1}_{\tau^{-m}(\xi)} \circ \gamma^u_{\vartheta, p_\Phi} \circ \tau^{-m}(\xi) \ - \ \psi_{\tau^{-1}(\xi)} \circ \psi^{m-1}_{\tau^{-m}(\xi)} \circ \gamma^u_{\vartheta, p_\Psi} \circ \tau^{-m}(\xi)\|.&
\end{aligned}$$

Arguing inductively

$$\|\phi^m_{\tau^{-m}(\xi)} \circ \gamma^u_{\vartheta,p_\Psi} \circ \tau^{-m}(\xi) - \psi^m_{\tau^{-m}(\xi)} \circ \gamma^u_{\vartheta,p_\Psi} \circ \tau^{-m}(\xi)\| \leq \delta \sum_{k=0}^{m-1} \beta^k. \tag{2.33}$$

Taking limit as $m \to \infty$ it follows the upper bound $\delta(1-\beta)^{-1}$ for the second term of the previous sum. Putting together this estimates we get $\|\gamma^u_{\vartheta,p_\Phi}(\xi) - \gamma^u_{\vartheta,p_\Psi}(\xi)\| \leq \delta(1-\beta)^{-1} < \epsilon$. Now, for each $n \geq 1$, with a similar calculation we obtain that

$$\|\phi^n_\xi \circ \gamma^u_{\vartheta,p_\Phi}(\xi) - \psi^n_\xi \circ \gamma^u_{\vartheta,p_\Psi}(\xi)\| \leq \beta^n \|\gamma^u_{\vartheta,p_\Phi}(\xi) - \gamma^u_{\vartheta,p_\Psi}(\xi)\| + \|\phi^n_\xi \circ \gamma^u_{\vartheta,p_\Psi}(\xi) - \psi^n_\xi \circ \gamma^u_{\vartheta,p_\Psi}(\xi)\|.$$

Arguing as before when we have estimated (2.33), it follows $\|\phi^n_\xi \circ \gamma^u_{\vartheta,p_\Psi}(\xi) - \psi^n_\xi \circ \gamma^u_{\vartheta,p_\Psi}(\xi)\| \leq \delta(1-\beta)^{-1}$. Since $\beta^n < 1$, the same bounded it is also followed for the first term in the above sum. Hence, $\|\phi^n_\xi \circ \gamma^u_{\vartheta,p_\Phi}(\xi) - \psi^n_\xi \circ \gamma^u_{\vartheta,p_\Psi}(\xi)\| \leq 2\delta(1-\beta)^{-1} = \epsilon$. Therefore, by (2.32), this implies that

$$d_H(K_\Phi, K_\Psi) \leq \sup_{n\in\mathbb{N}} d_H\big(E_n(\Phi), E_n(\Psi)\big) \leq \epsilon < \varepsilon,$$

ending the proof of the proposition. □

Let (ϑ, p) be a periodic point of $\Phi = \tau \ltimes \phi_\xi$ with period n. Then $\tau^n(\vartheta) = \vartheta$ and $\phi^n_\vartheta(p) = p$. In this case we write $\vartheta \in \mathrm{Per}_n(\tau)$ and $p \in \mathrm{Fix}(\phi^n_\vartheta)$. This point (ϑ, p) is called *fiber-hyperbolic periodic point* if p is a hyperbolic fixed point of the map $\phi_{\tau^{n-1}(\vartheta)} \circ \cdots \circ \phi_\vartheta$. If p is an attractor, repeller or saddle point of $\phi_{\tau^{n-1}(\vartheta)} \circ \cdots \circ \phi_\vartheta$ then (ϑ, p) is said to be *fiber-attractor, fiber-repeller or fiber-saddle point* of Φ respectively. In any case, if $\Psi = \tau \ltimes \psi_\xi$ is close to Φ then $\psi_{\tau^{n-1}(\vartheta)} \circ \cdots \circ \psi_\vartheta$ has a fixed point p_Ψ close to p that is also a hyperbolic fixed point (attractor, repeller or saddle respectively). The periodic point (ϑ, p_Ψ) of Ψ is called *continuation of* (ϑ, p) *for* Ψ.

2.2.3 Symbolic blenders-horseshoes

In Proposition 1.12 we showed that non-normally hyperbolic horseshoes are blender-horseshoes. In particular, the maximal invariant set in the blender-horseshoe reference cube is a hyperbolic basic set conjugated to the Bernoulli shift of two symbols. Throughout §2.2.2 we have shown a similar result for symbolic skew-products in $\mathcal{S}_{k,\lambda,\beta}(D)$ with $\beta < 1$ which we summarize here (see Theorem 2.12, Propositions 2.13, 2.15 and 2.17):

Theorem A. *Consider $\Phi \in \mathcal{S}_{k,\lambda,\beta}(D)$ with $\beta < 1$. Then the restriction of Φ to the set*

$$\Gamma_\Phi = \bigcap_{n\in\mathbb{Z}} \Phi^n\big(\Sigma_k \times \overline{D}\big) = \bigcap_{n\in\mathbb{N}} \Phi^n\big(\Sigma_k \times \overline{D}\big)$$

is conjugated to the full shift τ of k symbols. Moreover, $W^{uu}(\Gamma_\Phi) = \Gamma_\Phi$ and there exists a unique continuous function $g_\Phi : \Sigma_k \to \overline{D}$ such that for every periodic point (ϑ, p) of Φ in $\Sigma_k \times D$ it holds that, $W^u((\vartheta, p); \Phi) = W^{uu}((\vartheta, p); \Phi)$ and

$$\Gamma_\Phi = \overline{W^{uu}((\vartheta, p); \Phi))} = \{(\xi, g_\Phi(\xi)) : \xi \in \Sigma_k\} \quad \textit{with} \quad \mathscr{P}(\Gamma_\Phi) = K_\Phi \in \mathcal{K}(D).$$

Finally, the map $\mathscr{L} : \mathcal{S}_{k,\lambda,\beta}(D) \to \mathcal{K}(\overline{D})$ given by $\mathscr{L}(\Phi) = K_\Phi$ is continuous.

In addition, if Φ is locally α-Hölder continuous then $W^u((\xi, x); \Phi) = W^{uu}((\xi, x); \Phi)$ for all $(\xi, x) \in \Gamma_\Phi$ and $g_\Phi : W^u_{loc}(\xi; \tau) \to \overline{D}$ is α-Hölder continuous for all $\xi \in \Sigma_k$.

In order to introduce symbolic *cs*-blender-horseshoes, we firstly define a family of almost horizontal disks which provides the superposition region of the blender.

Definition 2.9 (Almost horizontal disks). *For a fixed $\alpha > 0$ and given an open subset $B \subset D$, we say that D^s is a δ-*horizontal disk *in $\Sigma_k \times B$ if there are $\zeta \in \Sigma_k$, $z \in B$, some positive constant $C \geq 0$ and a (α, C)-Hölder function $h : W^s_{loc}(\zeta;\tau) \to B$ such that*

$$D^s = \{(\xi, h(\xi)) : \xi \in W^s_{loc}(\zeta;\tau)\}, \ \|z - h(\xi)\| < \delta \ \textit{for all}\ \xi \in W^s_{loc}(\zeta;\tau) \ \textit{and}\ C\nu^\alpha < \delta.$$

From Theorem A, it follows $W^{uu}_{loc}(\Gamma_\Phi) = \Gamma_\Phi$ for all $\Phi \in \mathcal{S}^\alpha_{k,\lambda,\beta}(D)$ with $\beta < 1$. Hence, the corresponding definition of *cs*-blender in [BDV95], Definition 1.3, in the context of symbolic skew-product can be written as follows:

Definition 2.10 (Symbolic *cs*-blender-horseshoes). *Let $\Phi \in \mathcal{S}^\alpha_{k,\lambda,\beta}(D)$ with $\beta < 1$, $\alpha > 0$.*

The maximal invariant set Γ_Φ of Φ in $\Sigma_k \times \overline{D}$ is said to be symbolic *cs*-blender-horseshoe *if there are $\delta > 0$, a non-empty open set $B \subset D$ and a neighborhood $\mathcal{V}$ of Φ in $\mathcal{S}^\alpha_{k,\lambda,\beta}(D)$ such that for every $\Psi \in \mathcal{V}$ and for any δ-horizontal disk D^s in $\Sigma_k \times B$, it holds that*

$$\Gamma_\Psi \cap D^s \neq \emptyset, \quad \textit{where } \Gamma_\Psi \textit{ is the continuation of } \Gamma_\Phi \textit{ for } \Psi. \tag{2.34}$$

The open set B is called superposition region *of the symbolic cs-blender-horseshoe.*

From Theorem A, it follows that $\overline{W^{uu}((\vartheta,p);\Phi)} = \Gamma_\Phi$ for every periodic point $(\vartheta,p) \in \Sigma_k \times D$. In Proposition 2.5, we proved that each local strong stable set $W^{ss}_{loc}((\xi,x);\Phi)$ in $\Sigma_k \times B$ is an almost horizontal disk. Hence, if Γ_Φ is a symbolic *cs*-blender-horseshoe for Φ then Equation (2.34) implies that

$$\overline{W^{uu}((\vartheta,p_\Psi);\Psi)} \cap W^{ss}_{loc}((\xi,x);\Psi) \neq \emptyset \tag{2.35}$$

for all $\mathcal{S}^\alpha$-perturbation Ψ of Φ where (ϑ, p_Ψ) is the continuation periodic point of (ϑ,p) for Ψ. Observe that, in particular, if $\Phi \in \mathcal{S}^+_{k,\lambda,\beta}(D)$ then $W^{ss}_{loc}((\xi,x);\Phi) = W^s_{loc}(\xi;\tau) \times \{x\}$. In this case, we infer that $W^{uu}((\vartheta,p);\Phi) \cap (W^s_{loc}(\xi;\tau) \times U) \neq \emptyset$ for all neighborhood U of x. From this fact we introduce the following definition:

Definition 2.11 (Symbolic *cs*-blender-horseshoes in the unilateral setting). *Consider a symbolic skew-product $\Phi \in \mathcal{S}^+_{k,\lambda,\beta}(D)$ with $\beta < 1$.*

The maximal invariant set Γ_Φ of Φ in $\Sigma_k \times \overline{D}$ is said to be symbolic *cs*-blender-horseshoe in the unilateral setting *if there are a non-empty open set $B \subset D$, a fixed point $(\vartheta,p) \in \Sigma_k \times D$ of Φ and a neighborhood $\mathcal{V}$ of Φ in $\mathcal{S}^+_{k,\lambda,\beta}(D)$ such that for every $\Psi \in \mathcal{V}$, it holds that*

$$W^{uu}\big((\vartheta,p_\Psi);\Psi\big) \cap \big(W^s_{loc}(\xi;\tau) \times U\big) \neq \emptyset, \tag{2.36}$$

for every $\xi \in \Sigma_k$ and every non-empty open subset U in B. The open set B is called superposition region *of the symbolic cs-blender-horseshoe in the unilateral setting.*

In [NP12, Definition 3.5], the above definition was given for symbolic skew-product in $\mathcal{S}^-_{k,\lambda,\beta}(D)$. Nevertheless, to get (2.35) from (2.36) we need that $W^{ss}((\xi,x);\Phi) = W^s_{loc}(\xi;\tau) \times \{x\}$. This is only possible if the local strong stable lamination is linear and consequently the skew-products have to belong to $\mathcal{S}^+_{k,\lambda,\beta}(D)$.

To define symbolic cu-blenders-horseshoe, firstly we need to introduce the associated inverse symbolic skew-product for $\Phi = \tau \ltimes \phi_\xi$. Given $\Phi = \tau \ltimes \phi_\xi \in \mathcal{S}^\alpha_{k,\lambda,\beta}(D)$, the symbolic skew-product

$$\Phi^* = \tau \ltimes \phi^*_\xi \in \mathcal{S}^\alpha_{k,\,\beta^{-1},\,\lambda^{-1}}(D), \quad \text{where } \phi^*_\xi : M \to M \text{ given by } \phi^*_\xi(x) = \phi^{-1}_{\tau^{-1}(\xi^*)}(x),$$

is called *associated inverse skew-product for* Φ. Here $\xi^* = (\dots\xi_1;\xi_0,\xi_{-1},\dots)$ denotes the conjugate sequence of $\xi = (\dots\xi_{-1};\xi_0,\xi_1,\dots)$. Note that since $\tau(\xi)^* = \tau^{-1}(\xi^*)$ then iterates of Φ^* are corresponded to iterates of Φ^{-1}. This observation allows us to define symbolic cu-blender-horseshoes for symbolic skew-products in $\mathcal{S}^\alpha_{k,\lambda,\beta}(D)$ with $\lambda > 1$ and $\alpha > 0$. Namely, a *symbolic cu-blender-horseshoe* for Φ is defined as a symbolic cs-blender-horseshoe for Φ^*. Also, observe that if $\Phi \in \mathcal{S}^-_{k,\lambda,\beta}(D)$ then $\Phi^* \in \mathcal{S}^+_{k,\beta^{-1},\lambda^{-1}}(D)$ and thus, analogously *symbolic cu-blender-horseshoes in the unilateral setting* is defined as symbolic cs-blender-horseshoes in the unilateral setting for Φ^*.

Proposition 2.18. *Consider $\Phi \in \mathcal{S}_{k,\lambda,\beta}(D)$ with $\beta < 1$. Let Γ_Φ, (ϑ,p) and B be the maximal invariant set in $\Sigma_k \times \overline{D}$, a fixed point in $\Sigma_k \times D$ of Φ and an open set in D respectively. Then, the following statements are equivalents:*

i) $W^{uu}((\vartheta,p);\Phi) \cap (W^s_{loc}(\xi;\tau) \times U) \neq \emptyset$ for all non-empty open set U in B and $\xi \in \Sigma_k$,

ii) given $(\xi,x) \in \Sigma_k \times B$ there is $(\xi',x') \in \Gamma_\Phi$ such that $W^{uu}_{loc}((\xi',x');\Phi) \cap (W^s_{loc}(\xi;\tau) \times \{x\}) \neq \emptyset$,

iii) $B \subset g_\Phi(W^s_{loc}(\xi;\tau))$ for all $\xi \in \Sigma_k$.

Proof. Let $\{U_n\}_{n\in\mathbb{N}}$ be a sequence of nested open neighborhoods of x whose intersection is the point x. By (i), it follows $W^{uu}((\vartheta,p);\Phi) \cap (W^s_{loc}(\xi;\tau) \times U_n) \neq \emptyset$ for all $n \in \mathbb{N}$. Now, for each $n \in \mathbb{N}$, we consider $(\xi^n,p^n) \in W^{uu}((\vartheta,p);\Phi) \subset \Gamma_\Phi$ such that $W^{uu}_{loc}((\xi^n,p^n);\Phi)$ meets $W^s_{loc}(\xi;\tau) \times U_n$. Let (ξ',x') be an accumulation point of the sequence $\{(\xi^n,p^n)\}_{n\in\mathbb{N}}$. Then this accumulation point belongs to Γ_Φ and from the choice of the nested sequence of neighborhood U_n it follows that $W^{uu}_{loc}((\xi',x');\Psi) \cap (W^s_{loc}(\xi;\tau) \times \{x\}) \neq \emptyset$ and so we obtain (ii). Reciprocally, using that the unstable manifold of a fixed point $W^{uu}((\vartheta,p);\Phi)$ is dense in $\Gamma_\Phi = W^{uu}(\Gamma_\Phi)$ we conclude that (ii) implies (i). Finally, it follows that (ii) is equivalent to $\Gamma \cap (W^s_{loc}(\xi;\tau) \times \{x\}) \neq \emptyset$ for all $x \in B$. Hence, noting that Γ is the graph of a continuous function $g_\Phi : \Sigma_k \to \overline{D}$ we obtain that the above assertion is equivalent to (iii), i.e., $B \subset g_\Phi(W^s_{loc}(\xi;\tau))$. Therefore, we conclude the proposition. □

Remark 2.19. *If $\Phi \in \mathcal{S}^+_{k,\lambda,\beta}(D)$ then $W^{ss}_{loc}((\xi,x);\Phi) = W^s_{loc}(\xi;\tau) \times \{x\}$ and hence (ii) in Proposition 2.18 can be written as*

$$W^{uu}_{loc}(\Gamma_\Phi) \cap W^{ss}_{loc}((\xi,x);\Phi) \neq \emptyset \quad \text{for all } (\xi,x) \in \Sigma_k \times B.$$

In §2.1.2, a skew product C^1-diffeomorphism f over a horseshoe $F : \Lambda \to \Lambda$ was considered. Under the modified dominating splitting condition, in Proposition 2.1 it showed that any C^1-perturbation g close enough to f has a maximal invariant Δ_g such that the restriction of g to this set is conjugated to a α-Hölder continuous symbolic skew-product. Now, we focus our attention in the conjugate Hölder symbolic skew-product Φ of $f|_{\Lambda\times M}$ restricted to the local region $\Sigma_k \times \overline{D}$. Let Γ_Φ be the maximal invariant set in $\Sigma_k \times \overline{D}$ of Φ. Assuming $\beta < 1$, suppose that Γ_Φ is a cs-blender-horseshoe for Φ. By Definition 2.10 it follows that the local strong stable set of any point $(\xi,x) \in \Sigma_k \times B$ meets robustly local strong unstable sets of points in Γ_Φ under $\mathcal{S}^\alpha$-perturbations.

That is, $W^{uu}_{loc}(\Gamma_\Psi)\cap W^{ss}_{loc}((\xi,x);\Psi)\neq\emptyset$ for all $(\xi,x)\in\Sigma_k\times B$ and small enough $\mathcal{S}^\alpha$-perturbation Ψ of Φ. Via conjugation, this assertion is also obtained for f under C^1-perturbations.

In §2.2.1, the conjugation between a symbolic Hölder skew-product and a symbolic unilateral skew-product was studied. Namely, from Propositions 2.3 and 2.9, assuming that the symbolic skew-product Φ is fiber bunched α-Hölder differentiable and has C^2-fiber maps it follows that Φ is conjugated to a unilateral symbolic skew-product in $\mathcal{S}^{1,+}_k(M)$. By Theorem 2.2 it follows that these additional assumptions of regularity to obtain the conjugation can be inferred for the conjugate skew-products of C^2-perturbations of the C^2-diffeomorphism $f=F\times\mathrm{id}$. Hence, if Γ_Φ is a symbolic *cs*-blender-horseshoe in the unilateral setting for a unilateral skew-product Φ conjugated to some partially hyperbolic skew-product C^2-diffeomorphism f restricted to $\Lambda\times M$ then, from Definition 2.11, Remark 2.19 and via conjugation, we only could infer that C^2-robustly $W^{uu}_{loc}(\Gamma_f)\cap W^{ss}_{loc}((z,x);f)\neq\emptyset$ for all $(z,x)\in\Lambda\times B$.

In the rest of this chapter, we will study the existence of symbolic blenders. Namely, given a one-step $\Phi=\tau\ltimes(\phi_1,\dots,\phi_k)$ we will give properties for the maps $\phi_1,\dots,\phi_k$ such a way that Φ has a symbolic *cs*-blender-horseshoe. For instance, the maps $\phi_1,\dots,\phi_k$ defined on $\overline{D}$ must satisfy the *covering property*: there exists an open set $B\subset D$ such that $\overline{B}\subset\phi_1(B)\cup\dots\cup\phi_k(B)$. The following result describes how to the covering property translates to a robust property in the language of Hölder symbolic skew-products:

Theorem B (Covering property characterization)**.** *Consider* $\Phi=\tau\ltimes(\phi_1,\dots,\phi_k)\in\mathcal{S}^\alpha_{k,\lambda,\beta}(D)$ *with* $\nu^\alpha<\lambda<1$ *and let* B *be an open set in* D*. Then,*

$$\overline{B}\subset\phi_1(B)\cup\dots\cup\phi_k(B)\tag{2.37}$$

if and only if there are $\delta>0$ *and a neighborhood* $\mathcal{V}$ *of* Φ *in* $\mathcal{S}^\alpha_{k,\lambda,\beta}(D)$ *such that for every* $\Psi\in\mathcal{V}$

$$\Gamma^+_\Psi(B)\cap D^s\neq\emptyset\quad\text{for all }\delta\text{-horizontal disk }D^s\text{ in }\Sigma_k\times B\tag{2.38}$$

where $\Gamma^+_\Psi(B)$ *is the forward maximal invariant set of* Ψ *in* $\Sigma_k\times B$*.*

Under the additional hypothesis $\beta<1$ if $\Phi=\tau\ltimes(\phi_1,\dots,\phi_k)\in\mathcal{S}^\alpha_{k,\lambda,\beta}(D)$ then $\phi_i(\overline{D})\subset D$ for $i=1,\dots,k$. In such case, for any small perturbation $\Psi=\tau\ltimes\psi_\xi$ of Φ it holds that $\psi_\xi(\overline{D})\subset D$ and it follows

$$\Gamma^+_\Psi(B)\stackrel{\text{def}}{=}\bigcap_{n\geq0}\Psi^n(\Sigma_k\times B)\subset\bigcap_{n\in\mathbb{Z}}\Psi^n(\Sigma_k\times\overline{D})\stackrel{\text{def}}{=}\Gamma_\Phi.$$

Therefore, combining the above result with Definition 2.10, we obtain the following consequence on the existence of symbolic blenders using the covering property.

Theorem C (Symbolic blender-horseshoe existence)**.** *Consider* $\Phi=\tau\ltimes(\phi_1,\dots,\phi_k)\in\mathcal{S}^\alpha_{k,\lambda,\beta}(D)$ *with* $\nu^\alpha<\lambda<\beta<1$, $\alpha>0$*. Assume that there exists an open set* $B\subset D$ *such that*

$$\overline{B}\subset\phi_1(B)\cup\dots\cup\phi_k(B).$$

Then the maximal invariant set Γ_Φ *of* Φ *in* $\Sigma_k\times\overline{D}$ *is a symbolic cs-blender-horseshoe for* Φ *whose superposition region contains* B*.*

Remark that under the covering property assumption we also show the existence of symbolic *cs*-blender-horseshoe in the unilateral setting (see Theorem 2.30). Before showing these theorems, we will studied *symbolic blenders in the one-step setting.* That is, given a one-step map Φ we will study the property (2.36) under $\mathcal{Q}$-perturbations Ψ of Φ. We think that proceeding in this way helps to understand the property (2.36) in the general context of Definition 2.11.

2.3 Symbolic blenders in the one-step setting

Let $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k)$ be a one-step skew-product map. We denote by $\mathrm{IFS}(\phi_1, \dots, \phi_k)$ or $\mathrm{IFS}(\Phi)$ the set of all compositions of the maps $\phi_1, \dots, \phi_k$ (together the identity map id) and we will refer it as the *associated iterated function system* (shortly IFS). For each subset $A \subset M$ let $\mathcal{G}_\Phi(A) = \phi_1(A) \cup \dots \cup \phi_k(A)$. The orbit of a point $x \in M$ for $\mathrm{IFS}(\phi_1, \dots, \phi_k)$, also called $\mathcal{G}_\Phi$-orbit of x, is the set

$$\mathrm{Orb}_\Phi(x) \stackrel{\text{def}}{=} \{\mathcal{G}^n_\Phi(x) : n \geq 0\} = \{\phi(x) : \phi \in \mathrm{IFS}(\phi_1, \dots, \phi_k)\}.$$

The relation between a one-step map and its associated IFS is through the dynamics of a unstable disk D^u. In fact, the first observation is that since $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k)$ is a one-step map then $W^{uu}_{loc}((\xi, x); \Phi) = W^u_{loc}(\xi; \tau) \times \{x\}$. From here, an unstable disk through the point (ξ, x) for a one-step map Φ is

$$D^u(\xi, x) = \Phi(W^{uu}_{loc}(\Phi^{-1}(\xi, x); \Phi)) = \Phi(W^u_{loc}(\tau^{-1}(\xi); \tau) \times \{\phi^{-1}_{\xi_{-1}}(x)\}) = V^u_{loc}(\xi; \tau) \times \{x\}$$

where $V^u_{loc}(\xi; \tau) = \{\xi' \in \Sigma_k : \xi'_i = \xi_i \text{ for all } i < 0\}$. That is, a compact piece of the strong unstable set of (ξ, x) which contains the point (ξ, x). For each $i \in \{1, \dots, k\}$, define the set $U_i = \{\zeta \in \Sigma_k : \zeta_0 = i\}$. These sets U_i form a partition of Σ_k. The unstable disk $D^u(\xi, x)$ intersects every $U_i \times M$, $i \in \{1, \dots, k\}$. Note that if $\zeta \in V^u_{loc}(\xi; \tau) \cap U_i$ then $\phi_\zeta = \phi_i$. Hence, the image of $D^u(\xi, x) \cap (U_i \times M)$ by Φ is the unstable disk $D^u\big(\tau(\xi), \phi_i(x)\big)$. Then

$$\Phi(D^u(\xi, x)) = \Phi\Big(\bigcup_{i=1}^k D^u(\xi, x) \cap (U_i \times M)\Big) = \bigcup_{i=1}^k D^u(\tau(\xi), \phi_i(x)).$$

From a similar argument,

$$\Phi^2(D^u(\xi, x)) = \bigcup_{i=1}^k \bigcup_{j=1}^k D^u(\tau^2(\xi), \phi_j \circ \phi_i(x)).$$

Inductively, we note that the dynamics on the fiber of these new disks is given by

$$\{\phi_{i_n} \circ \dots \circ \phi_{i_1}(x) : \ n \geq 1, \ i_j \in \{1, \dots, k\}\} = \mathrm{Orb}_\Phi(x).$$

The following proposition shows that if (ξ, x) is a fixed point of Φ then the above set is the projection on the fiber space of the strong unstable set of (ξ, x).

Proposition 2.20. *Consider $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k)$ a one-step map and let (ϑ, p) be a fixed point of Φ. Then*

$$\mathscr{P}(W^{uu}(\vartheta, p); \Phi)) = \{\phi(p) : \ \phi \in \mathrm{IFS}(\phi_1, \dots, \phi_k)\} \stackrel{\text{def}}{=} \mathrm{Orb}_\Phi(p).$$

Proof. Since (ϑ, p) is a fixed point of Φ then

$$W^{uu}(\vartheta, p); \Phi) = \bigcup_{n=0}^\infty \Phi^n(W^{uu}_{loc}((\vartheta, p); \Phi)).$$

On the other hand, for each $n \geq 1$ it holds

$$\Phi^n(W^{uu}_{loc}((\vartheta, p); \Phi) = \{(\tau^n(\zeta), \phi_{\tau^{n-1}(\zeta)} \circ \dots \circ \phi_\zeta(p)) : \zeta \in W^u_{loc}(\vartheta; \tau)\}.$$

Since Φ is a one-step, $\phi_{\tau^i(\zeta)} = \phi_{\zeta_i}$ and noting that $\phi_\zeta(p) = \phi_\vartheta(p) = p$ it follows that

$$\begin{aligned}\mathscr{P}(\Phi^n(W^{uu}_{loc}((\vartheta,p);\Phi)) &= \{\phi_{\tau^{n-1}(\zeta)} \circ \cdots \circ \phi_{\tau(\zeta)}(p) : \zeta \in W^u_{loc}(\vartheta;\tau)\} \\ &= \{\phi_{i_{n-1}} \circ \cdots \circ \phi_{i_1}(p) : i_j \in \{1,\dots,k\},\, 1 \le j < n\}.\end{aligned}$$

Hence this projection on the fiber space is $\mathrm{Orb}_\Phi(p)$. This concludes the proof of the proposition. $\square$

As a consequence of the above proposition, we will show that the density property (2.36) of the strong unstable set in the one-step setting is reduced to a density property of an orbit of the associated IFS. First, recall that by a neighborhood $\mathcal{V}$ of Φ in $\mathcal{Q}_{k,\lambda,\beta}(D)$ we mean a neighborhood in the topology of $\mathcal{S}^\alpha_{k,\lambda,\beta}(D)$ intersection with $\mathcal{Q}_{k,\lambda,\beta}(D)$. Having in mind that the topology of $\mathcal{S}^\alpha_{k,\lambda,\beta}(D)$ is induced by the distance given in (2.25) and noting that for every $\Psi \in \mathcal{Q}_{k,\lambda,\beta}(D)$ the Hölder constant is $C_\Psi = 0$, we get that $\Psi = \tau \ltimes (\psi_1,\dots,\psi_k)$ is δ-close to $\Phi = \tau \ltimes (\phi_1,\dots,\phi_k)$ if $d_{\mathcal{Q}}(\Psi,\Phi) = \max\{d_{C^0}(\psi_i|_D,\phi_i|_D) : i = 1,\dots,k\} < \delta$.

Proposition 2.21. *Consider $\Phi = \tau \ltimes (\phi_1,\dots,\phi_k) \in \mathcal{Q}_{k,\lambda,\beta}(D)$, a non-empty open set $B \subset D$ and a fiber-hyperbolic fixed point $(\vartheta,p) \in \Sigma_k \times D$ of Φ. Then, the following statement are equivalent:*

i) There is a neighborhood $\mathcal{V}$ of Φ in $\mathcal{Q}_{k,\lambda,\beta}(D)$ such that for every $\Psi \in \mathcal{V}$, it holds that

$$W^{uu}\big((\vartheta,p_\Psi);\Psi\big) \cap \big(W^s_{loc}(\xi;\tau) \times U\big) \neq \emptyset,$$

for every $\xi \in \Sigma_k$ and every non-empty open subset U in B;

ii) $B \subset \overline{\mathrm{Orb}_\Psi(p_\Psi)}$ for every $\Psi = \tau \ltimes (\psi_1,\dots,\psi_k) \in \mathcal{Q}_{k,\lambda,\beta}(D)$ close to $\Phi = \tau \ltimes (\phi_1,\dots,\phi_k)$.

Proof. By Proposition 2.20, if (ϑ,p_Ψ) is a fixed point of any one-step $\Psi = \tau \ltimes (\psi_1,\dots,\psi_k)$ then $\mathscr{P}(W^{uu}((\vartheta,p_\Psi);\Psi)) = \mathrm{Orb}_\Psi(p_\Psi)$. Thus, in this case, Item (i) implies that $B \subset \overline{\mathrm{Orb}_\Psi(p_\Psi)}$ for every $\Psi = \tau \ltimes (\psi_1,\dots,\psi_k) \in \mathcal{Q}_{k,\lambda,\beta}(D)$ close to $\Phi = \tau \ltimes (\phi_1,\dots,\phi_k)$. In order to prove the converse, fix $U \subset B$ and $\xi \in \Sigma_k$. By Item (ii), there is $\psi_{i_n} \circ \cdots \circ \psi_{i_1} \in \mathrm{IFS}(\psi_1,\dots\psi_k)$ such that $x = \psi_{i_n} \circ \cdots \circ \psi_{i_1}(p_\Psi) \in U$. Let $\zeta = (\dots\vartheta_{-1}\vartheta_0, i_1,\dots,i_n;\xi_0,\xi_1,\dots)$. Note that $(\zeta,x) \in W^s_{loc}(\xi;\tau) \times U$. We will prove that $(\zeta,x) \in W^{uu}((\vartheta,p_\Psi);\Phi)$. Since (ϑ,p_Ψ) is a fixed point of Ψ,

$$\Psi^{-n-1}(\zeta,x) = \big((\dots,\vartheta_{-1};\vartheta_0,i_1,\dots,i_n,\xi_0,\xi_1,\dots),p_\Psi\big) \in W^u_{loc}(\vartheta;\tau) \times \{p_\Psi\}.$$

Therefore $(\zeta,x) \in \Psi^{n+1}\big(W^u_{loc}(\vartheta;\tau) \times \{p_\Psi\}\big) = \Psi^{n+1}\big(W^{uu}_{loc}((\vartheta,p_\Psi);\Psi)\big) \subset W^{uu}\big((\vartheta,p_\Psi);\Psi\big)$. This implies Item (i) and proves the proposition. $\square$

If (ϑ,p) in the above proposition is a fiber-attractor of $\Phi = \tau \ltimes (\phi_1,\dots,\phi_k)$ with B contained in the attracting region of p for ϕ_ϑ then (ii) is equivalent to

$$B \subset \overline{\mathrm{Orb}_\Psi(x)} \quad \text{for all } x \in B \tag{2.39}$$

for every $\Psi \in \mathcal{Q}_{k,\lambda,\beta}(D)$ close to Φ. Indeed, it suffices to show that Item (ii) in Proposition 2.21 implies (2.39). To do this, let $\Psi = \tau \ltimes (\psi_1,\dots,\psi_k) \in \mathcal{Q}_{k,\lambda,\beta}(D)$ be a $\mathcal{Q}$-perturbation of Φ. Let U be a non-empty open set in B and $x \in B$. By hypotheses, there is $\psi \in \mathrm{IFS}(\psi_1,\dots,\psi_k)$ such that $\psi(p_\Psi) \in U$. Since U is open and ψ is continuous then there exists a neighborhood V of p_Ψ such that $\psi(V) \subset U$. If Ψ is close enough to Φ then B is also in the attracting region of p_Ψ for $\psi_\vartheta = \psi_i$ where $i = \vartheta_0$. Thus there is $n \in \mathbb{N}$ such that $\psi_i^n(x) \in V$ and hence $\psi \circ \psi_i^n(x) \in U$. This shows that $B \subset \overline{\mathrm{Orb}_\Psi(x)}$ for all $x \in B$.

Motivated from (2.39), we give the following definition:

Definition 2.12 (Blending region). *Let $\phi_1, \dots, \phi_k$ be C^1-diffeomorphism of M. A non-empty open set $B \subset M$ is called* blending region *for* $\mathrm{IFS}(\phi_1, \dots, \phi_k)$ *if for every $\Psi = \tau \ltimes (\psi_1, \dots, \psi_k)$ close enough to $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k)$ it holds taht*

$$B \subset \overline{\mathrm{Orb}_\Psi(x)} \quad \text{for all } x \in B.$$

Here, we mean by closeness that the fiber map ψ_i of Ψ is C^1-close to the fiber map ϕ_i of Φ.

There is a similar definition of blending region from the Control Theory [AS91]. A set $A \subset M$ is called *precontrol set* for the $\mathrm{IFS}(\phi_1, \dots, \phi_k)$ if $A \subset \overline{\mathrm{Orb}_\Phi(x)}$ for all $x \in A$ and $\operatorname{int} A \neq \emptyset$. A precontrol set which is maximal with respect to set inclusion is called *control set*. The difference between blending region and (pre)control set is the additional condition of robustness by perturbations of the IFS. Sometimes we will refer to both, item (ii) in Proposition 2.21 and Equation 2.39, saying that B is *robustly transitive* and *robustly minimal* for $\mathrm{IFS}(\phi_1, \dots, \phi_k)$ respectively.

Proposition 2.22. *Let $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k) \in \mathcal{Q}_{k,\lambda,\beta}(D)$. Consider B an open set in D and suppose that there exist a hyperbolic fixed point $p \in D$ of some ϕ_i and a map $\phi \in \mathrm{IFS}(\phi_1, \dots, \phi_k)$ such that $\phi(p) \in B$. Then if B is a blending region for $\mathrm{IFS}(\phi_1, \dots, \phi_k)$ it follows that the maximal invariant set Γ_Φ of Φ in $\Sigma_k \times \overline{D}$ is a symbolic cs-blender in the one-step setting.*

Proof. The proof follows from the equivalence between (i) and (ii) in Proposition 2.21. By hypothesis, there exist a fixed point $p \in D$ of some ϕ_i and a map $\phi_{i_n} \circ \cdots \circ \phi_{i_1} \in \mathrm{IFS}(\phi_1, \dots, \phi_k)$ such that $\phi_{i_n} \circ \cdots \circ \phi_{i_1}(p) \in B$. Since B is an open set, if $\Psi = \tau \ltimes (\psi_1, \dots, \psi_k)$ is close enough to Φ then $\psi_{i_n} \circ \cdots \circ \psi_{i_1}(p_\Psi) \in B$ where p_Ψ is the continuation of p for ψ_i. Now, since B is a blender-like set for $\mathrm{IFS}(\phi_1, \dots, \phi_k)$ it follows that

$$B \subset \overline{\mathrm{Orb}_\Psi(\psi_{i_n} \circ \cdots \circ \psi_{i_1}(p_\Psi))} \subset \overline{\mathrm{Orb}_\Psi(p_\Psi)}.$$

This concludes the proof of the proposition. □

2.3.1 Blending region for contracting IFS

We will work with contracting one-step skew-products maps. Recall that $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k) \in \mathcal{Q}_{k,\lambda,\beta}(D)$ is a contracting one-step skew-product if every ϕ_i is a contraction with contraction constant $0 < \beta < 1$. Here, we will prove the existence of symbolic blender-horseshoes in the one-step setting. Although this has already been proven in [NP12, Proposition 3.6], our approach here is a little bit different. The one-step skew-product maps, or one-step maps for short, $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k)$ considered in [NP12] satisfy the covering property and well-distribution of periodic points. The maps $\phi_1, \dots, \phi_k$ have the *covering property* if there is an open set $B \subset D$ such that

$$\overline{B} \subset \bigcup_{i=1}^{k} \phi_i(B). \tag{2.40}$$

The set of fixed points of $\phi_1, \dots, \phi_k$ is *well-distributed* if any open ball of diameter d and centered in B contains a fixed point of ϕ_i for some $i \in \{1, \dots, k\}$, where

$$d \geq \max\{r > 0\colon \text{for all } x \in B, \text{ there is } i \text{ such that } B(x,r) \subset \phi_i(B)\}.$$

We will see that, in this one-step setting, the well-distribution property is not necessary to obtain symbolic blenders. Indeed, our first approach involves the so-called Hutchinson operator of a contracting IFS.

The Hutchinson attractor

Associated with a one-step map $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k) \in \mathcal{Q}_{k,\lambda,\beta}(D)$ with $0 < \beta < 1$, or with its associated contracting IFS, we define the *Hutchinson's operator* by

$$\mathcal{G}_\Phi \colon \mathcal{K}(\overline{D}) \to \mathcal{K}(\overline{D}), \qquad \mathcal{G}_\Phi(A) = \phi_1(A) \cup \ldots \cup \phi_k(A)$$

where we recall that $\mathcal{K}(\overline{D})$ denotes the complete metric space whose elements are compact subsets of $\overline{D}$ endowed with the Hausdorff metric. Since the maps ϕ_i are contractions, then $\mathcal{G}_\Phi$ is a contracting map. This fact leads to the following result:

Proposition 2.23 ([Wil71, Hut81])**.** *Let $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k) \in \mathcal{Q}_{k,\lambda,\beta}(D)$ with $0 < \beta < 1$. Then there exists a unique compact set $K_{\mathcal{G}_\Phi} \in \mathcal{K}(\overline{D})$ such that*

$$K_{\mathcal{G}_\Phi} = \mathcal{G}_\Phi(K_{\mathcal{G}_\Phi}) = \overline{\mathrm{Per}(\mathrm{IFS}(\Phi)) \cap D} \stackrel{\text{def}}{=} K_\Phi.$$

Moreover, the set $K_{\mathcal{G}_\Phi}$ depends continuously on the one-step map Φ and it is the global attractor of $\mathcal{G}_\Phi$: for every $A \in \mathcal{K}(\overline{D})$ it holds $\lim_{m\to\infty} d_H\big(\mathcal{G}_\Phi^m(A), K_{\mathcal{G}_\Phi}\big) = 0$.

In the above proposition, $\mathrm{Per}(\mathrm{IFS}(\Phi))$ denotes the projection $\mathscr{P}(\mathrm{Per}(\Phi))$. That is, the set of the fixed point of the compositions maps in the associated IFS of $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k)$. The compact set $K_{\mathcal{G}_\Phi}$ (in the sequel denotes by K_Φ) is called *Hutchinson's attractor* of the contracting one-step map Φ restricted to $\Sigma_k \times \overline{D}$ or of its associated IFS on $\overline{D}$.

By Proposition 2.23, $\mathcal{G}_\Phi^m(x) \to K_\Phi$. Thus $K_\Phi \subset \overline{\mathrm{Orb}_\Phi(x)}$, for all $x \in \overline{D}$. We have the following consequences of Proposition 2.23:

Corollary 2.24. *Consider $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k) \in \mathcal{Q}_{k,\lambda,\beta}(D)$ with $0 < \beta < 1$ and let K_Φ be its Hutchinson's attractor. It holds that:*

i) If $A \in \mathcal{K}(\overline{D})$ such that $A \subset \mathcal{G}_\Phi(A)$ then $A \subset K_\Phi \subset \overline{\mathrm{Orb}_\Phi(x)}$ for all $x \in \overline{D}$;

ii) For every $p \in K_\Phi$ there is a sequence $(\sigma_n)_{n\in\mathbb{N}} \in \{1, \dots, k\}^{\mathbb{N}}$ such that

$$\phi_{\sigma_n}^{-1} \circ \dots \circ \phi_{\sigma_1}^{-1}(p) \in K_\Phi \quad \textit{for all } n \in \mathbb{N};$$

iii) For each open set V such that $V \cap K_\Phi \neq \emptyset$, there exist $n \in \mathbb{N}$ and $(i_1, \dots, i_n) \in \{1, \dots, k\}^n$ such that $\phi_{i_n} \circ \dots \circ \phi_{i_1}(K_\Phi) \subset V$.

Proof. In order to prove the first item, note that by hypothesis $A \subset \mathcal{G}_\Phi(A) \subset \ldots \subset \mathcal{G}_\Phi^m(A)$ for all $m \geq 1$. Since $\mathcal{G}_\Phi^m(A) \to K_\Phi$ this implies that $A \subset K_\Phi$. Thus, we obtain that $A \subset K_\Phi \subset \overline{\mathrm{Orb}_\Phi(x)}$ for all $x \in \overline{D}$, and conclude (i).

According to Proposition 2.23, $K_\Phi = \phi_1(K_\Phi) \cup \ldots \cup \phi_k(K_\Phi)$. Thus given $p \in K_\Phi$ there exits $\sigma_1 \in \{1, \dots, k\}$ such that $\phi_{\sigma_1}^{-1}(p) \in K_\Phi$. Arguing inductively, we get a sequence $(\sigma_n)_{n\in\mathbb{N}}$ such that $\phi_{\sigma_n}^{-1} \circ \dots \circ \phi_{\sigma_1}^{-1}(p) \in K_\Phi$ for all $n \in \mathbb{N}$ and therefore we prove Item (ii).

Finally, to prove Item (iii), consider $i \in \{1, \dots, k\}$ and the fixed point s of ϕ_i. By the first item we have $K_\Phi \subset \overline{\mathrm{Orb}_\Phi(s)}$. Hence, there are $m \in \mathbb{N}$ and $(\sigma_1, \dots, \sigma_m) \in \{1, \dots, k\}^m$ such that $\phi_{\sigma_m} \circ \dots \circ \phi_{\sigma_1}(s) \in V$ and thus $\phi_{\sigma_1}^{-1} \circ \dots \circ \phi_{\sigma_m}^{-1}(V)$ is a neighborhood of s. Since ϕ_i^{-1} is an expansion and K_Φ is bounded, there exists $\ell \in \mathbb{N}$ such that $K_\Phi \subset \phi_i^{-\ell} \circ \phi_{\sigma_1}^{-1} \circ \dots \circ \phi_{\sigma_m}^{-1}(V)$. Now it is enough to take $n = \ell + m$ and the sequence $(i, \stackrel{\ell}{\dots}, i, \sigma_1, \dots, \sigma_m)$. This completes the proof of the corollary. $\square$

Remark 2.25. *From the above corollary it follows that if* $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k) \in \mathcal{Q}_{k,\lambda,\beta}(D)$ *with* $0 < \beta < 1$ *and* $B \subset \overline{D}$ *is a non-empty open set such that* $\overline{B} \subset \mathcal{G}_\Phi(B)$*, then*

$$B \subset \overline{B} \subset K_\Phi \subset \overline{\mathrm{Orb}_\Phi(x)}, \quad \textit{for all } x \in \overline{D}.$$

The following corollary shows that B in the above remak is a blending region for $\mathrm{IFS}(\phi_1, \dots, \phi_k)$. Thus, by Proposition 2.22, this result implies the existence of symbolic *cs*-blender-horseshoe in the one-step setting:

Corollary 2.26. *Consider* $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k) \in \mathcal{Q}_{k,\lambda,\beta}(D)$ *with* $0 < \lambda < \beta < 1$*. Let* $B \subset \overline{D}$ *be a non-empty bounded open set with* $\overline{B} \subset \mathcal{G}_\Phi(B)$*. Then for every* $\Psi \in \mathcal{Q}_{k,\lambda,\beta}(D)$ *close enough to* Φ *it holds* $\overline{B} \subset K_\Psi \subset \overline{\mathrm{Orb}_\Phi(x)}$*, for all* $x \in \overline{D}$*.*

Proof. If $\Psi = \tau \ltimes (\psi_1, \dots, \psi_k) \in \mathcal{Q}_{k,\lambda,\beta}(D)$ is close enough to Φ then $d_H(\mathcal{G}_\Psi(\overline{B}), \mathcal{G}_\Phi(\overline{B}))$ is small. From this proximity and since $\mathcal{G}_\Phi(B)$ is open, it follows $\overline{B} \subset \mathcal{G}_\Psi(B) \subset \mathcal{G}_\Psi(\overline{B})$. Inductively, we get $\overline{B} \subset \mathcal{G}^m_\Psi(\overline{B})$ for all $m \geq 0$. Let K_Ψ be the Hutchinson attractor of Ψ restricted to $\Sigma_k \times \overline{D}$. Since K_Ψ is closed and $\lim_{m\to\infty} d_H(\mathcal{G}^m_\Psi(\overline{B}), K_\Psi) = 0$ we obtain $\overline{B} \subset K_\Psi \subset \overline{\mathrm{Orb}_\Phi(x)}$ for all $x \in \overline{D}$. □

Without using the Hutchinson theory

Next, we will show that if $\overline{B} \subset \mathcal{G}_\Phi(B)$ then $B \subset \overline{\mathrm{Orb}_\Phi(x)}$ for all $x \in D$, without to involve the Hutchinson theory. To this end, the following proposition is the key to understood the symbolic *cs*-blender-horseshoes in the one-step setting.

Proposition 2.27. *Let* $\mathrm{IFS}(\phi_1, \dots, \phi_k)$ *be a* (λ, β)*-Lipschitz iterated function system on* $\overline{D} \subset M$ *with* $0 < \lambda < \beta < 1$ *such that* $\phi_i(\overline{D}) \subset D$ *for* $i = 1, \dots, k$*. We assume that there is a no-empty open set* B *satisfying the covering property* (2.40)*. Then there are* C^0*-neighborhood* $\mathcal{U}_i$ *of* ϕ_i*,* $i = 1, \dots, k$ *such that for every family* $\{\psi_1, \dots, \psi_k\}$ *of homeomorphisms with* $\psi_i \in \mathcal{U}_i$ *for* $i = 1, \dots, k$ *and for every* $x \in \overline{B}$ *there is a sequence* $(i_j)_{j>0}$*,* $i_j \in \{1, \dots, k\}$ *such that*

$$x = \lim_{n\to\infty} \psi_{i_1} \circ \dots \circ \psi_{i_n}(y) \quad \textit{for all } y \in B.$$

Proof. Note that since the maps ϕ_i are (λ, β)-Lipschitz on $\overline{D}$ with $0 < \lambda < \beta < 1$ then $\phi_i(B)$ are open sets. Then the covering property $\overline{B} \subset \mathcal{G}_\Phi(B)$ is a robust property. That is, there is small enough C^0-neighborhood $\mathcal{U}_i$ of ϕ_i of homeomorphisms of M for $i = 1, \dots, k$ such that if

$$B_i^* = \bigcap_{\psi \in \mathcal{U}_i} \psi(B) \qquad \text{for } i = 1, \dots, k$$

then $\overline{B} \subset B_1^* \cup \dots \cup B_k^*$. Taking $\mathcal{U}_i$ small enough we can assume that any $\phi \in \mathcal{U}_i$ is also a (λ, β)-Lipschitz on $\overline{D}$ for $i = 1, \dots, k$. Given a family $\{\psi_1, \dots, \psi_k\}$ of maps with $\psi_i \in \mathcal{U}_i$ for $i = 1, \dots, k$, we define recursively for $n > 1$ the sets

$$B^n_{i_1\dots i_n} = \psi_{i_n}(B^{n-1}_{i_1\dots i_{n-1}}) = \psi_{i_n} \circ \dots \circ \psi_{i_1}(B) \qquad \text{for } i_j = 1, \dots, k \text{ and } j = 1, \dots, n.$$

Claim 2.27.1. *For all* $n \in \mathbb{N}$ *it holds*

$$B^n_{i_2\dots i_{n+1}} \subset \bigcup_{i_1=1}^{k} B^{n+1}_{i_1 i_2\dots i_{n+1}} \qquad \textit{and} \qquad \overline{B} \subset \bigcup_{i_1,\dots,i_{n+1}=1}^{k} B^{n+1}_{i_1\dots i_{n+1}}.$$

Proof. The proof is by induction on n. Firstly, we will show that $B^1_{i_2} \subset \cup_{i_1=1}^k B^2_{i_1 i_2}$ and $\overline{B} \subset \cup_{i_1,i_2=1}^k B^2_{i_1 i_2}$. By definition and using that $B^*_i \subset B^1_i$ and $\overline{B} \subset B^*_1 \cup \ldots \cup B^*_k$ it follows

$$\bigcup_{i_1=1}^k B^2_{i_1 i_2} = \bigcup_{i_1=1}^k \psi_{i_2}(B^1_{i_1}) = \psi_{i_2}(\bigcup_{i_1=1}^k B^1_{i_1}) \supset \psi_{i_2}(\bigcup_{i_1=1}^k B^*_{i_1}) \supset \psi_{i_2}(B) = B^1_{i_2}.$$

From this we obtain that $\cup_{i_1,i_2=1}^k B^2_{i_1 i_2} \supset \cup_{i_2=1}^k B^1_{i_2} \supset \overline{B}$. Now, we assume the lemma holds for $n-1$ and we will prove it for n. In the same way as before,

$$\bigcup_{i_1=1}^k B^{n+1}_{i_1\ldots i_{n+1}} = \bigcup_{i_1=1}^k \psi_{i_{n+1}}(B^n_{i_1\ldots i_n}) = \psi_{i_{n+1}}(\bigcup_{i_1=1}^k B^n_{i_1\ldots i_n}).$$

By hypothesis of induction, we have that $B^{n-1}_{i_2\ldots i_n} \subset \cup_{i_1=1}^k B^n_{i_1 i_2\ldots i_n}$ and then

$$\bigcup_{i_1=1}^k B^{n+1}_{i_1\ldots i_{n+1}} \supset \psi_{i_{n+1}}(B^{n-1}_{i_2\ldots i_n}) = B^n_{i_2\ldots i_{n+1}}.$$

Now, note that we have that $B^\ell_{i_2\ldots i_{\ell+1}} \subset \cup_{i_1=1}^k B^{\ell+1}_{i_1 i_2\ldots i_{\ell+1}}$ for every $1 \leq \ell \leq n$ and for all $i_j = 1,\ldots,k$ with $j = 2,\ldots,\ell+1$. Then

$$\bigcup_{i_1,\ldots,i_{n+1}=1}^k B^{n+1}_{i_1\ldots i_{n+1}} \supset \bigcup_{i_2,\ldots,i_{n+1}=1}^k B^n_{i_2\ldots i_{n+1}} \supset \cdots \supset \bigcup_{i_{n+1}=1}^k B^1_{i_{n+1}} \supset \overline{B}$$

and the proof of the claim is completed. □

Since $\overline{B} \subset \cup_{j=1}^k B^1_j$, for each $x \in \overline{B}$ there is $i_1 \in \{1,\ldots,k\}$ such that $x \in B^1_{i_1}$. We now proceed recursively. For $n > 1$ we suppose that we have $i_j \in \{1,\ldots,k\}$ for $j = 1,\ldots,n$ such that $x \in B^n_{i_n\ldots i_1}$. By Claim 2.27.1 we have $B^n_{i_n\ldots i_1} \subset \cup_{j=1}^k B^{n+1}_{j i_n\ldots i_1}$. Then there is $i_{n+1} \in \{1,\ldots,k\}$ such that $x \in B^{n+1}_{i_{n+1} i_n\ldots i_1}$. From this, we construct a positive sequence $i = i_1 i_2 \ldots = (i_j)_{j>0}$ such that $x \in B^n_{i_n\ldots i_1}$ for all $n \geq 1$. Thus, we get

$$x \in \bigcap_{n\geq 1} B^n_{i_n\ldots i_1} = \bigcap_{n\geq 1} \psi_{i_1} \circ \cdots \circ \psi_{i_n}(B) = \bigcap_{n\geq 1} A_n$$

where $A_n = \cap_{\ell=1}^n \psi_{i_1} \circ \cdots \circ \psi_{i_\ell}(B)$ for all $n \in \mathbb{N}$. Note that, since $\phi_i(\overline{D}) \subset D$ for $i = 1,\ldots,k$ then if the neighborhood $\mathcal{U}_i$ are small enough it holds $A_{n+1} \subset A_n \subset \psi_{i_1} \circ \cdots \circ \psi_{i_n}(B) \subset D$. Hence, sice every $\psi \in \cup_{i=1}^k \mathcal{U}_i$ is a (λ,β)-contracting map in D, it follows

$$\mathrm{diam}(A_n) \leq \mathrm{diam}(\psi_{i_1} \circ \cdots \circ \psi_{i_n}(B)) \leq \beta^n \mathrm{diam}(B)$$

where $\mathrm{diam}(A)$ denotes the diameter of a bounded subset A of M. Therefore A_n is a nested sequence of sets whose diameters goes to zero and so $\{x\} = \cap_{n\geq 1} B^n_{i_n\ldots i_1} = \cap_{n\geq 1} \psi_{i_1} \circ \cdots \circ \psi_{i_n}(B)$. Finally, from this we deduce that given $y \in B$,

$$\|\psi_{i_1} \circ \cdots \circ \psi_{i_n}(y) - x\| \leq \mathrm{diam}(\psi_{i_1} \circ \cdots \circ \psi_{i_n}(B)) \leq \beta^n \mathrm{diam}(B)$$

for every $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \beta^n = 0$, then $x = \lim_{n\to\infty} \psi_{i_1} \circ \cdots \circ \psi_{i_n}(y)$ and we conclude the proof of the proposition. □

Recall that, given a one-step $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k)$ or an iterated function system $\mathrm{IFS}(\Phi)$

$$\mathrm{Per}(\mathrm{IFS}(\Phi)) \stackrel{\text{def}}{=} \bigcup_{n\in\mathbb{N}} \{\mathrm{Fix}(\phi_{\tau^{n-1}(\vartheta)} \circ \cdots \circ \phi_\vartheta) \colon \vartheta \in \mathrm{Per}_n(\tau)\} = \mathscr{P}(\mathrm{Per}(\Phi))$$
$$= \{x \in M \colon \ \phi(x) = x \text{ for some } \phi \in \mathrm{IFS}(\phi_1, \dots, \phi_k)\}.$$

The next lemma gives some relations between closure of this set and the closure of some $\mathcal{G}_\Phi$-orbits.

Lemma 2.28. *For every one-step $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k) \in \mathcal{Q}_{k,\lambda,\beta}(D)$ with $0 < \beta < 1$ it holds that*

i) $\overline{\mathrm{Per}(\mathrm{IFS}(\Phi)) \cap D} \subset \overline{\mathrm{Orb}_\Phi(x)}$ *for all* $x \in \overline{D}$,

ii) $\overline{\mathrm{Per}(\mathrm{IFS}(\Phi)) \cap D} = \overline{\mathrm{Orb}_\Phi(p)}$ *for all* $p \in \mathrm{Per}(\mathrm{IFS}(\Phi)) \cap D$.

Proof. Let $x \in \overline{D}$. If $p \in \overline{\mathrm{Per}(\mathrm{IFS}(\Phi)) \cap D}$, then there is a sequence $(p_n)_{n\in\mathbb{N}} \subset D$ and $h_n \in \mathrm{IFS}(\phi_1, \dots, \phi_k)$ such that $\lim_{n\to\infty} p_n = p$ and $h_n(p_n) = p_n$. Since the maps ϕ_i are contracting in $\overline{D}$ and $\phi_i(\overline{D}) \subset D$ for $i = 1, \dots, k$ then h_n are also contracting maps in $\overline{D}$. Thus for all $\varepsilon > 0$ there are m and ℓ_m in $\mathbb{N}$ such that $\|p_m - p\| < \varepsilon/2$ and $\|h_m^{\ell_m}(x) - p_m\| < \varepsilon/2$. Hence, $\|h_m^{\ell_m}(x) - p\| < \varepsilon$. Therefore p is in the closure of $\mathrm{Orb}_\Phi(x)$ for all $x \in \overline{D}$. This proves Item (i).

To prove the second item we only need to show that given $p \in \mathrm{Per}(\mathrm{IFS}(\phi_1, \dots, \phi_k)) \cap D$

$$\overline{\mathrm{Orb}_\Phi(p)} \subset \overline{\mathrm{Per}(\mathrm{IFS}(\Phi)) \cap D}.$$

Since $\phi_i(\overline{D}) \subset D$ for all $i = 1, \dots, k$ then $\mathrm{Orb}_\Phi(p) \subset D$. Let $x \in \overline{\mathrm{Orb}_\Phi(p)} \subset \overline{D}$. Then there is a sequence $(h_n)_n \subset \mathrm{IFS}(\Phi)$ such that $\lim_n h_n(p) = x$. Hence, for all $\varepsilon > 0$ there is $m \in \mathbb{N}$ such that $h_m(p) \in D$ belongs to the open ball $B(x, \varepsilon)$ centered at x and radius ε. From the continuity of h_n, there is $\delta > 0$ such that $h_m(B(p, \delta)) \subset B(x, \varepsilon) \cap D$. In the other hand, since $p \in \mathrm{Per}(\mathrm{IFS}(\Phi))$ then there is $h \in \mathrm{IFS}(\Phi)$ such that $h(p) = p$. Since h is a contracting map in $\overline{D}$ there exists $\ell \in \mathbb{N}$ such that $h^\ell(B(x, \varepsilon) \cap D) \subset B(p, \delta)$. Thus

$$h_m \circ h^\ell(B(x, \varepsilon) \cap D) \subset h_m(B(p, \delta)) \subset B(x, \varepsilon).$$

Therefore, $B(x, \varepsilon)$ meets $\mathrm{Per}(\mathrm{IFS}(\Phi))$ for all $\varepsilon > 0$ and consequently the point x belongs to $\overline{\mathrm{Per}(\mathrm{IFS}(\Phi))}$. This completes the proof of the lemma. □

As a consequence of Proposition 2.27 and Lemma 2.28, recalling that

$$K_\Phi \stackrel{\text{def}}{=} \overline{\mathrm{Per}(\mathrm{IFS}(\Phi)) \cap D} = \overline{\mathscr{P}(\mathrm{Per}(\Phi)) \cap D}, \qquad \Phi = \tau \ltimes (\phi_1, \dots, \phi_k),$$

we reprove Corollary 2.26 without using the Hutchinson Theory. That is, we show the existence of symbolic *cs*-blender-horseshoe in the one-step setting from Proposition 2.27 and Lemma 2.28.

Proof of Corollary 2.26. If $\Psi = \tau \ltimes (\psi_1, \dots, \psi_k) \in \mathcal{Q}_{k,\lambda,\beta}(D)$ is close enough to Φ then $\psi_i \in \mathcal{U}_i$ for $i = 1, \dots, k$, where $\mathcal{U}_i$ are the neighborhoods given in Proposition 2.27. Thus, $B \subset \overline{\mathrm{Orb}_\Psi(x)}$ for all $x \in B$. In particular, fixed $x \in B$, there is $(h_n)_n \subset \mathrm{IFS}(\psi_1, \dots, \psi_k)$ such that $\lim_{n\to\infty} h_n(x) = x$. Since h_n are contractions in $\overline{D}$, for a given $\varepsilon > 0$ small enough there exists $m \in \mathbb{N}$ such that $h_m(B(x, \varepsilon)) \subset B(x, \varepsilon)$ and thus the fixed point p_m of h_m belongs to $B(x, \varepsilon) \subset B$. By Lemma 2.28,

$$\overline{B} \subset \overline{\mathrm{Per}(\mathrm{IFS}(\psi_1, \dots, \psi_k)) \cap D} \subset \overline{\mathrm{Orb}_\Psi(x)} \quad \text{for all } x \in \overline{D}$$

and the corollary is proved. □

2.4 Symbolic blenders in the unilateral setting

Given a bi-sequence $\xi = (\dots,\xi_{-1};\xi_0,\xi_1,\dots) \in \Sigma_k$, the *negative* (resp. *positive*) *tail* of ξ is the sequence $\xi^- = (\xi_i)_{i\leq 0}$ (resp. $\xi^+ = (\xi_i)_{i\geq 0}$). These tails of a bi-sequence can be seen as an unilateral sequence in $\Sigma_k^+ = \{1,\dots,k\}^{\mathbb{Z}^+}$. This space of unilateral sequences is endowed of a topology taken the same metric d_{Σ_k} restricted to Σ_k^+. We will denote by $\sigma : \Sigma_k^+ \to \Sigma_k^+$ the restriction of left shift map $\tau : \Sigma_k \to \Sigma_k$ to the space of the unilateral sequences.

Let us consider skew-product maps H over the unilateral shift map of k symbols of the form

$$H : \Sigma_n^+ \times M \to \Sigma_n^+ \times M, \qquad H(\omega,x) = (\sigma\omega, h_\omega(x)) \tag{2.41}$$

where $h_\omega : M \to M$ are homeomorphisms and the map $\omega \mapsto h_\omega$ is continuous. We will use the notation $H = \sigma \ltimes h_\omega$. This map can be understood by studying forward iterations of the skew-product $\Phi = \tau \ltimes \phi_\xi$ in $\mathcal{S}_k^+(M)$ where $\phi_\xi = h_{\xi^+}$. Reciprocally, for each skew-product $\Phi = \tau \ltimes \phi_\xi$ in $\mathcal{S}_k^+(M)$ we associate the skew-product $\Phi_+ = \sigma \ltimes \phi_{\xi^+}$ of the form of (2.41). In fact, denoting by $\mathscr{P}_+ : \Sigma_k \times M \to \Sigma_k^+ \times M$ the projection given by $\mathscr{P}_+(\xi,x) = (\xi^+,x)$, we have

$$\mathscr{P}_+ \circ \Phi = \Phi_+ \circ \mathscr{P}_+.$$

Thus, we can also see $\mathcal{S}_k^+(M)$ as the set of skew-product maps of the form of (2.41). Similarly, we will understand that $\mathcal{Q}_{k,\lambda,\beta}(D)$ and $\mathcal{S}^+_{k,\lambda,\beta}(D)$ are also sets of symbolic skew-product over unilateral Bernoulli shift σ. So, we extend the previously definitions introduced for bi-lateral symbolic skew-products $\Phi = \tau \ltimes \phi_\xi$ such as fiber-hyperbolic periodic points, continuation point, etc., to unilateral symbolic skew-products $H = \sigma \ltimes h_\omega$ and we will denote

$$h^n_\omega(x) \stackrel{\text{def}}{=} h_{\sigma^{n-1}\omega} \circ \dots \circ h_\omega(x), \qquad H^n(\omega,x) = (\sigma^n\omega, h^n_\omega(x)).$$

Now, we will try to reduce the geometrical property (2.36) in Definition 2.11 for a skew-product $\Phi = \tau \ltimes \phi_\xi \in \mathcal{S}^+_{k,\lambda,\beta}(D)$ to the associated skew-product $\Phi_+ = \sigma \ltimes \phi_{\xi^+}$. Firstly, we introduce some standard definitions.

We define *(strong) unstable disks for* $\Phi = \tau \ltimes \phi_\xi$ *through the point* $(\xi,x) \in \Sigma_k \times M$ as an embedded compact disk D^u in the strong unstable set of (ξ,x) for Φ which contains the point (ξ,x) and intersects every Markov partition element. More precisely, D^u, also denoted $D^u(\xi,x)$, is the graph set of

$$\hat{\gamma}^u_{\xi,x} : V^u_{loc}(\xi;\tau) \to M, \qquad \hat{\gamma}^u_{\xi,x}(\xi') = \phi_{\tau^{-1}(\xi')} \circ \gamma^u_{\tau^{-1}(\xi),\phi^{-1}_{\tau^{-1}(\xi)}(x)} \circ \tau^{-1}(\xi')$$

where $V^u_{loc}(\xi;\tau) = \{\xi' \in \Sigma_k : \ \xi'_i = \xi_i \text{ for all } i<0\}$. Note that, since the unstable lamination is invariant (see for instance Item (ii) in Proposition 2.14) if $\xi' \in W^u_{loc}(\xi;\tau) \subset V^u_{loc}(\xi;\tau)$ then $\hat{\gamma}^u_{\xi,x}(\xi') = \gamma^u_{\xi,x}(\xi')$ and $D^u = \operatorname{graph}[\hat{\gamma}^u_{\xi,x}] \subset W^{uu}((\xi,x);\Phi)$. Recall that the strong unstable set of the point (ξ,x) for Φ is given by

$$W^{uu}((\xi,x);\Phi) = \bigcup_{n\geq 0} \Phi^n\big(W^{uu}_{loc}(\Phi^{-n}(\xi,x);\Phi)\big) = \bigcup_{n\geq 1} \Phi^{n-1}(D^u_n) \tag{2.42}$$

where $D^u_n = \Phi\big(W^{uu}_{loc}(\Phi^{-n}(\xi,x);\Phi)\big)$ is the unstable disk through the point $\Phi^{1-n}(\xi,x)$ for $n \geq 1$. Since the iteration by Φ of an unstable disk D^u provides k new unstable disks it follows that $W^{uu}((\xi,x);\Phi)$ is a numerable union of finitely many unstable disks.

We say that $D^u_+ = D^u_+(\omega, x)$ is a *(strong) unstable disk for* Φ_+ *through* $(\omega, x) \in \Sigma_k^+ \times M$ if there is $\xi \in \Sigma_k$ with positive tail $\xi^+ = \omega$ such that D^u_+ is the projection by $\mathscr{P}_+$ of the unstable disk $D^u = D^u(\xi, x)$ for Φ. Observe that, since $\mathscr{P}_+(V^u_{loc}(\xi;\tau)) = \Sigma_k^+$ then D^u_+ is the graph set of a continuous function $g : \Sigma_k^+ \to M$. Applying the projection $\mathscr{P}_+$ in (2.42) it follows that

$$\mathscr{P}_+\big(W^{uu}((\xi,x);\Phi)\big) = \bigcup_{n\geq 1} \Phi_+^{n-1} \circ \mathscr{P}_+(D^u_n) = \bigcup_{n\geq 1} \Phi_+^{n-1}(D^u_{n+}) \tag{2.43}$$

Similarly, the projection by $\mathscr{P}_+$ of a *cs*-strip $W^s_{loc}(\xi;\tau) \times U$ is the set $\{\xi^+\} \times U$. Therefore, we easily obtain the following result:

Lemma 2.29. *Consider $\Phi = \tau \ltimes \phi_\xi \in \mathcal{S}^+_{k,\lambda,\beta}(D)$ with $\beta < 1$ and let Γ_Φ be the maximal invariant set in $\Sigma_k \times \overline{D}$ of Φ. Then, the following statements are equivalent:*

i) Γ_Φ is a symbolic cs-blender-horseshoe in the unilateral setting with superposition region B;

ii) there is a fixed point $(\vartheta^+, p) \in \Sigma_k^+ \times D$ of $\Phi_+ = \sigma \ltimes \phi_{\xi^+}$ such that for every small enough $\mathcal{S}^+$-perturbation $\Psi_+ = \sigma \times \psi_{\xi^+}$ of Φ_+ it holds

$$\Psi_+^{n-1}(D^u_+) \cap \big(\{\omega\} \times U\big) \neq \emptyset \quad \text{with} \quad D^u_+ = \mathscr{P}_+ \circ \Psi(W^{uu}_{loc}((\vartheta, p_\Psi);\Psi)) \tag{2.44}$$

for all $\omega \in \Sigma_k^+$, non-empty open set $U \subset B$ and for some natural number $n = n(\omega, U)$.

Proof. Let Γ_Φ be a symbolic *cs*-blender-horseshoe in the unilateral setting for Φ with superposition region B. Then there is a fixed point $(\vartheta, p) \in \Sigma_k \times D$ of Φ such that for every $\mathcal{S}^+$-perturbation $\Psi = \tau \ltimes \psi_\xi$ in $\mathcal{S}^+_{k,\lambda,\beta}(D)$ it holds that $W^{uu}((\vartheta,p);\Psi)$ mets any *cs*-strip $W^s_{loc}(\xi;\tau) \times U$ in $\Sigma_k \times B$. Now, we identify these $\mathcal{S}^+$-perturbations $\Psi = \tau \ltimes \psi_\xi$ of $\Phi = \tau \ltimes \phi_\xi$ with unilateral skew-products $\Psi_+ = \sigma \ltimes \psi_{\xi^+}$, that is, with $\mathcal{S}^+$-perturbations of $\Phi_+ = \sigma \ltimes \phi_{\xi^+}$. So, the continuation point $(\vartheta^+, p_{\Psi_+})$ of (ϑ^+, p) for Ψ_+ is well defined and thus, from (2.43), the strong unstable set of (ϑ, p_Ψ) meets any *cs*-strip in $\Sigma_k \times B$, if and only if for every $\omega \in \Sigma_k^+$ and every open set U in B, there is $n \in \mathbb{N}$ such that $\Psi_+^{n-1}(D^u_+) \cap \big(\{\omega\} \times U\big) \neq \emptyset$. This concludes the proof of the lemma. $\square$

The next theorem shows the existence of symbolic *cs*-blender-horseshoe in the unilateral setting according to Definition 2.11. We will consider $\mathcal{S}^+$-perturbations of a contracting one-step satisfying the covering property.

Theorem 2.30. *Consider $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k) \in \mathcal{S}^+_{k,\lambda,\beta}(D)$ with $\lambda < \beta < 1$ and assume that there exists a non-empty open set $B \subset D$ such that*

$$B \subset \phi_1(B) \cup \ldots \cup \phi_k(B).$$

Then the maximal invariant set Γ_Φ in $\Sigma_k \times \overline{D}$ for Φ is a symbolic cs-blender-horseshoe in the unilateral setting for Φ whose superposition region contains B.

The essence of the idea of the proof of this result can be found in [Hom11, Lemma 4.1]. In [HN11] also some relations between robust minimal IFS, robust topologically mixing skew-products and symbolic blenders-horseshoes in the unilateral setting are discussed.

Notation 2.31. *Let $i = i_0 \dots i_{n-1}$ be a finite word in $\{1, \dots, k\}^n$. Given a sequence $\omega \in \Sigma_k^+$, we denote by $i\omega$ the sequence $\xi^+ \in \Sigma_k^+$ such that $\xi_0^+ = i_0, \dots, \xi_{n-1}^+ = i_{n-1}$ and $\xi_{n+j}^+ = \omega_j$ for all $j \geq 0$.*

Let $H = \sigma \ltimes h_\omega$ be an $\mathcal{S}^+$-perturbation of $\Phi_+ = \sigma \ltimes (\phi_1, \dots, \phi_k)$. Suppose that the open set $B \subset D$ in Theorem 2.30 satisfies $B \subset h_{1\omega}(B) \cup \dots \cup h_{k\omega}(B)$ for every $\omega \in \Sigma_k^+$. Then, it follows that for each $\omega \in \Sigma_k^+$ and $n \in \mathbb{N}$,

$$B \subset \bigcup_{|i|=n} h^n_{i\omega}(B) \subset \bigcup_{|i|=n} h^n_{i\omega}(\overline{D}). \tag{2.45}$$

Indeed, (2.45) is immediate for $n = 1$. For $n = 2$, since for each $i \in \{1, \dots, k\}$ we have that $B \subset h_{1i\omega}(B) \cup \ldots \cup h_{ki\omega}(B)$ for all $\omega \in \Sigma_k^+$, it follows that

$$B \subset \bigcup_{|i|=1} h_{i\omega}(B) \subset \bigcup_{|i|=1} h_{i\omega}(h_{1i\omega}(B) \cup \ldots \cup h_{ki\omega}(B)) = \bigcup_{|i|=1} \bigcup_{j=1}^{k} h_{i\omega} \circ h_{ji\omega}(B) = \bigcup_{|i|=2} h^2_{i\omega}(B).$$

Ague similarly by induction we obtain (2.45). Note that $H = \sigma \ltimes h_\omega$ belongs to $\mathcal{S}^+_{k,\lambda,\beta}(D)$ with $\beta < 1$. Thus, h_ω is a contracting map on $\overline{D}$ for every unilateral sequence ω. Hence

$$\lim_{|i|=n\to\infty} \operatorname{diam}\big(h^n_{i\omega}(\overline{D})\big) = 0. \tag{2.46}$$

Now, we can conclude that for every $\omega \in \Sigma_k^+$ and every non-empty open subset U in B there is $n \in \mathbb{N}$ such $H^n(D^u_+) \cap (\{\omega\} \times U) \neq \emptyset$ with D^u_+ any unstable disk contained in $\Sigma_k^+ \times \overline{D}$. Indeed, from (2.46) there is $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ it holds that $\operatorname{diam}(h^n_{i\omega}(\overline{D})) \leq \operatorname{diam}(U)/3$ for all $i \in \{1, \dots, k\}^n$. Since U is a non-empty open subset of B, from (2.45) we find $n \geq n_0$ and $i \in \{1, \dots, k\}^n$ such that $h^n_{i\omega}(\overline{D}) \subset U$. In particular, since the unstable disk D^u_+ is the graph of a continuous function $g : \Sigma_k^+ \to \overline{D}$ then the point $(i\omega, g(i\omega))$ is in the unstable disk D^u_+ and its fiber coordinate $g(i\omega) \in \overline{D}$. Therefore $H^n(i\omega, g(i\omega)) \in \{\omega\} \times h^n_{i\omega}(\overline{D}) \subset \{\omega\} \times U$. So, by Lemma 2.29 we infer that Φ has a symbolic *cs*-blender-horseshoe in the unilateral setting in $\Sigma_k \times \overline{D}$ with superposition region containing B.

Although the above argument can be seen as a proof of Theorem 2.30, we will obtain this result from another similar proof which allows us to give more information about superposition region. Firstly, we need to calculate the iterate $H^n(D^u_+)$ where $H = \sigma \ltimes h_\omega$ and D^u_+ is an unstable disk for H in $\Sigma_k^+ \times \overline{D}$. As above, we can write D^u_+ as a graph of a continuous function g from Σ_k^+ to $\overline{D}$. We consider a thin strip S containing the unstable disk D^u_+. That is

$$S = \bigcup_{\omega \in \Sigma_k^+} \{\omega\} \times I_\omega \quad \text{where } I_\omega = \overline{B}(g(w), \varepsilon_\omega) \subset \overline{D}.$$

Hence,

$$H(S) = \bigcup_{i=1}^{k} \bigcup_{\omega \in \Sigma_k^+} \{\omega\} \times h_{i\omega}(I_{i\omega}) = \bigcup_{\omega \in \Sigma_k^+} \Big(\{\omega\} \times \bigcup_{i=1}^{k} h_{i\omega}(I_{i\omega})\Big).$$

Repeating this reasoning, further iterates $H^n(S)$ is k^n full thin strips in $\Sigma_k^+ \times \overline{D}$ which each one of them contains a new unstable disk in $H^n(D^u_+)$. The collection of these strips is

$$H^n(S) = \bigcup_{\omega \in \Sigma_k^+} \Big(\{\omega\} \times \bigcup_{|i|=n} h^n_{i\omega}(I_{i\omega})\Big).$$

This calculation motives to introduce the following operator. For each $\omega \in \Sigma_k^+$ and $n \in \mathbb{N}$ we define the operator $\mathcal{L}_n(\omega)$ associated with $H = \sigma \ltimes h_\omega$ by

$$\mathcal{L}_n(\omega) : \mathcal{K}(\overline{D}) \times \overset{k^n}{\cdots} \times \mathcal{K}(\overline{D}) \to \mathcal{K}(\overline{D}), \quad \mathcal{L}_n(\omega)[\{A_i\}_{|i|=n}] = \bigcup_{|i|=n} h^n_{i\omega}(A_i). \tag{2.47}$$

Lemma 2.32. *Consider $H = \sigma \ltimes h_\omega \in \mathcal{S}^+_{k,\lambda,\beta}(D)$ with $\beta < 1$. Then, for each $\omega \in \Sigma_k^+$ and $n \in \mathbb{N}$ we obtain the following properties:*

i) $d_H(\mathcal{L}_n(\omega)[\{A_i\}_{|i|=n}], \mathcal{L}_n(\omega)[\{B_i\}_{|i|=n}]) \leq \beta^n \max_{|i|=n} d_H(A_i, B_i)$,

ii) $\mathcal{L}_{n+1}(\omega)[\{A_i\}_{|i|=n+1}] = \mathcal{L}_1(\omega)\big[\big\{\mathcal{L}_k(j\omega)[\{A_{ij}\}_{|i|=n}]\big\}_{j=1}^k\big]$,

iii) $d_H(\mathcal{L}_{n+1}(\omega)[\{A_i\}_{|i|=n+1}], \mathcal{L}_n(\omega)[\{B_i\}_{|i|=n}]) \leq \beta^n \operatorname{diam}(D)$.

Proof. The first item is obtained from the properties of the Hausdorff distance:

$$d_H(\mathcal{L}_n(\omega)[\{A_i\}_{|i|=n}], \mathcal{L}_n(\omega)[\{B_i\}_{|i|=n}]) \leq \max_{|i|=n} d_H(h^n_{i\omega}(A_i), h^n_{i\omega}(B_i)) \leq \beta^n \max_{|i|=n} d_H(A_i, B_i).$$

In order to prove the second item, recall that $h^n_\omega = h_{\sigma^{n-1}\omega} \circ \cdots \circ h_\omega$. Hence,

$$\begin{aligned}\mathcal{L}_{n+1}(\omega)[\{A_i\}_{|i|=n+1}] &= \bigcup_{|i|=n+1} h^{n+1}_{i\omega}(A_i) = \bigcup_{|j|=1} \bigcup_{|i|=n} h_{j\omega} \circ h^n_{ij\omega}(A_{ij}) \\ &= \bigcup_{|j|=1} h_{j\omega}\big(\bigcup_{|i|=n} h^n_{ij\omega}(A_{ij})\big) = \mathcal{L}_1(\omega)\big[\big\{\mathcal{L}_n(j\omega)[\{A_{ij}\}_{|i|=n}]\big\}_{|j|=1}\big].\end{aligned}$$

Using this equality, we obtain that

$$\begin{aligned}&d_H\big(\mathcal{L}_{n+1}(\omega)[\{A_i\}_{|i|=n+1}],\, \mathcal{L}_n(\omega)[\{B_i\}_{|i|=n}]\big) = \\ &\quad = d_H\big(\mathcal{L}_1(\omega)\big[\big\{\mathcal{L}_n(j\omega)[\{A_{ij}\}_{|i|=n}]\big\}_{|j|=1}\big],\, \mathcal{L}_1(\omega)\big[\big\{\mathcal{L}_{n-1}(j\omega)[\{B_{ij}\}_{|i|=n-1}]\big\}_{|j|=1}\big]\big) \\ &\quad \leq \beta \max_{|j|=1} d_H\big(\mathcal{L}_n(j\omega)[\{A_{ij}\}_{|i|=n}],\, \mathcal{L}_{n-1}(j\omega)[\{B_{ij}\}_{|i|=n-1}]\big).\end{aligned}$$

Arguing by induction, we get

$$\begin{aligned}&d_H\big(\mathcal{L}_{n+1}(\omega)[\{A_i\}_{|i|=n+1}],\, \mathcal{L}_n(\omega)[\{B_i\}_{|i|=n}]\big) \leq \\ &\quad \leq \beta^{n-1} \max_{|j|=n-1} d_H\big(\mathcal{L}_2(j\omega)\big[\{A_{ij}\}_{|i|=2}\big],\, \mathcal{L}_1(j\omega)\big[\{B_{ij}\}_{|i|=1}\big]\big) \\ &\quad = \beta^{n-1} \max_{|j|=n-1} d_H\big(\mathcal{L}_1(j\omega)\big[\big\{\mathcal{L}_1(\ell j\omega)[\{A_{i\ell j}\}_{|i|=1}]\big\}_{|\ell|=1}\big],\, \mathcal{L}_1(j\omega)\big[\{B_{ij}\}_{|i|=1}\big]\big) \\ &\quad \leq \beta^n \max_{|j|=n-1} \max_{|\ell|=1} d_H\big(\mathcal{L}_1(\ell j\omega)\big[\{A_{i\ell j}\}_{|i|=1}\big],\, B_{\ell j}\big) = \beta^n \max_{|j|=n} d_H\big(\mathcal{L}_1(j\omega)\big[\{A_{ij}\}_{|i|=1}\big],\, B_j\big).\end{aligned}$$

Since

$$\mathcal{L}_1(j\omega)\big[\{A_{ij}\}_{|i|=1}\big] = \bigcup_{|i|=1} h_{ij\omega}(A_{ij})$$

then $d_H(\mathcal{L}_{n+1}(\omega)[\{A_i\}_{|i|=n+1}],\, \mathcal{L}_n(\omega)[\{B_i\}_{|i|=n}]) \leq \beta^n \max_{|j|=n} \max_{|i|=1} d_H(h_{ij\omega}(A_{ij}), B_j)$. Now, recalling that $H = \sigma \ltimes h_\omega \in \mathcal{S}^+_{k,\lambda,\beta}(D)$ with $\beta < 1$ and hence $h_\omega(\overline{D}) \subset D$ for all $\omega \in \Sigma_k^+$ it follows that $d_H(h_{ij\omega}(A_{ij}), B_j) \leq \operatorname{diam}(D)$. Consequently

$$d_H(\mathcal{L}_{n+1}(\omega)[\{A_i\}_{|i|=n+1}],\, \mathcal{L}_n(\omega)[\{B_i\}_{|i|=n}]) \leq \beta^n \operatorname{diam}(D)$$

and we conclude the proof of the lemma. □

Proposition 2.33. *Consider $H = \sigma \ltimes h_\omega \in \mathcal{S}^+_{k,\lambda,\beta}(D)$ with $\beta < 1$. Then, for each $\omega \in \Sigma_k^+$ there is a compact set K_ω in $\overline{D}$ such that for every sequence of collection $\{A_i\}_{|i|=n}$ of compact sets $A_i \in \mathcal{K}(D)$ with $i \in \{1, \ldots, k\}^n$ it holds*

$$\lim_{n\to\infty} d_H(\mathcal{L}_n(\omega)[\{A_i\}_{|i|=n}], K_\omega) = 0.$$

Moreover, the maps $\mathcal{L} : \Sigma_k^+ \to \mathcal{K}(\overline{D})$ given by $\mathcal{L}(\omega) = K_\omega$ is continuous.

Proof. For each $n \in \mathbb{N}$, let $\{A_i\}_{|i|=n}$ be a collection of compact sets $A_i \in \mathcal{K}(D)$ with $i \in \{1, \dots, k\}^n$. We define $K_n = \mathcal{L}_n(\omega)[\{A_i\}_{|i|=n}] \in \mathcal{K}(D)$. From Item (iii) in Lemma 2.32 it follows that the sequence $\{K_n\}_{n\in\mathbb{N}}$ is Cauchy for the Hausdorff distance. Since $\mathcal{K}(\overline{D})$ is a complete metric space with the Hausdorff distance then $\{K_n\}_{n\in\mathbb{N}}$ converges. We denote the limit by $K(\omega, \{A_i\})$. Now, we will show that this limit is independent of the sequence of compact sets $\{A_i\}$. To do this, take another different collection of compact sets $B_i \in \mathcal{K}(D)$ with $i \in \{1, \dots, k\}^m$ for all $m \in \mathbb{N}$. Hence, for every $n \in \mathbb{N}$,

$$\begin{aligned} d_H\big(K(\omega, \{A_i\}), K(\omega, \{B_i\})\big) &\le d_H\big(K(\omega, \{A_i\}), \mathcal{L}_n(\omega)[\{A_i\}_{|i|=n}]\big) \\ &+ d_H\big(\mathcal{L}_n(\omega)[\{A_i\}_{|i|=n}], \mathcal{L}_n(\omega)[\{B_i\}_{|i|=n}]\big) + d_H\big(\mathcal{L}_n(\omega)[\{B_i\}_{|i|=n}], K(\omega, \{B_i\})\big). \end{aligned}$$

From (i) in Lemma 2.32, noting that $d_H(A_i, B_i) \le \operatorname{diam}(D)$, we have that the second term in the above sum is less or equal than $\beta^n \operatorname{diam}(D)$. Thus, taken limit as $n \to \infty$ it follows that $d_H(K(\omega, \{A_i\}), K(\omega, \{B_i\})) = 0$ and so $K(\omega, \{A_i\}) = K(\omega, \{B_i\})$. We denote this limit by K_ω.

We will show that $\mathcal{L} : \Sigma_k^+ \to \mathcal{K}(\overline{D})$ given by $\mathcal{L}(\omega) = K_\omega$ is continuous. Fix $\varepsilon > 0$ and consider $\varepsilon' = \varepsilon(1-\beta)/3$. Since $\omega \mapsto h_\omega$ is continuous, there exists $\delta > 0$ such that

$$\text{if } d_{\Sigma_k^+}(\omega, \omega') < \delta \text{ then } d_{C^0}(h_\omega, h_{\omega'}) < \varepsilon'. \tag{2.48}$$

We take a compact set $A \in \mathcal{K}(\overline{D})$ and two unilateral sequences ω and ω' such that $d_{\Sigma_k^+}(\omega, \omega') < \delta$. From the first part since $\mathcal{L}_n(\omega)[\{A\}_{|i|=n}]$ and $\mathcal{L}_n(\omega')[\{A\}_{|i|=n}]$ converge to K_ω and $K_{\omega'}$ respectively in the Hausdorff metric we obtain $n \in \mathbb{N}$ such that

$$d_H(\mathcal{L}_n(\omega)[\{A\}_{|i|=n}],\, K_\omega) < \varepsilon/3 \quad \text{and} \quad d_H(\mathcal{L}_n(\omega')[\{A\}_{|i|=n}],\, K_{\omega'}) < \varepsilon/3.$$

Then,

$$\begin{aligned} d_H(K_\omega, K_{\omega'}) &\le d_H\big(K_\omega,\, \mathcal{L}_n(\omega)[\{A\}_{|i|=n}]\big) + d_H\big(\mathcal{L}_n(\omega)[\{A\}_{|i|=n}],\, \mathcal{L}_n(\omega')[\{A\}_{|i|=n}]\big) \\ &+ d_H\big(\mathcal{L}_n(\omega')[\{A\}_{|i|=n}],\, K_{\omega'}\big) < \frac{2}{3}\,\varepsilon + d_H\big(\mathcal{L}_n(\omega)[\{A\}_{|i|=n}],\, \mathcal{L}_n(\omega')[\{A\}_{|i|=n}]\big). \end{aligned} \tag{2.49}$$

Now,

$$\begin{aligned} &d_H\big(\mathcal{L}_n(\omega)[\{A\}_{|i|=n}],\, \mathcal{L}_n(\omega')[\{A\}_{|i|=n}]\big) = \\ &\qquad = d_H\Big(\bigcup_{|i|=n} h^n_{i\omega}(A),\, \bigcup_{|i|=n} h^n_{i\omega'}(A)\Big) \le \max_{|i|=n} d_H(h^n_{i\omega}(A), h^n_{i\omega'}(A)). \end{aligned}$$

Fix $i = i_1 \dots i_n \in \{1, \dots, k\}^n$. Hence, since $d_{\Sigma_k^+}(i_n\omega, i_n\omega') \le \nu d_{\Sigma_k^+}(\omega, \omega') < \nu\delta < \delta$, by (2.48) it follows that

$$\begin{aligned} d_H(h^n_{i\omega}(A), h^n_{i\omega'}(A)) &\le d_H(h_{i_n\omega} \circ h^{n-1}_{i\omega}(A), h_{i_n\omega} \circ h^{n-1}_{i\omega'}(A)) \\ &+ d_H(h_{i_n\omega} \circ h^{n}_{i\omega'}(A), h_{i_n\omega'} \circ h^{n-1}_{i\omega'}(A)) < \beta\, d_H(h^{n-1}_{i\omega}(A), h^{n-1}_{i\omega'}(A)) + \varepsilon'. \end{aligned}$$

Arguing by induction,

$$d_H(h^n_{i\omega}(A), h^n_{i\omega'}(A)) \le \varepsilon' \sum_{j=0}^{n-1} \beta^j \le \frac{\varepsilon'}{1-\beta} = \frac{\varepsilon}{3}.$$

Putting together this inequality and (2.49) we obtain that $d_H(\mathcal{L}(\omega), \mathcal{L}(\omega')) < \varepsilon$ an so we conclude the proof of the proposition. $\square$

Proof of Theorem 2.30. Let $\mathcal{U}_i$ be neighborhood of $\phi_i : \overline{D} \to D$ such that the family

$$B_i = \operatorname{int}\big(\bigcap_{h\in\mathcal{U}_i} h(B)\big), \quad i = 1, \dots, k,$$

is an open covering of $\overline{B}$. By shrinking the size of the sets $\mathcal{U}_i$ we can assume that any $h \in \mathcal{U}_i$ is also (λ, β)-Lipschitz homeomorphism on $\overline{D}$ for all $i = 1, \dots, k$. Consider a $\mathcal{S}^+$-perturbation $H = \sigma \ltimes h_\omega$ of $\Phi_+ = \sigma \ltimes (\phi_1, \dots, \phi_k)$ such that $h_\omega \in \mathcal{U}_i$ if $\omega_0 = i$. For each $n \in \mathbb{N}$, consider $\mathcal{L}_n(\omega)$ the operator associated with H defined in (2.47).

For each $n \in \mathbb{N}$, let $\{B\}_{|i|=n}$ be the collection $\{B_i\}_{|i|=n}$ with $B_i = B$ for all $i \in \{1, \dots, k\}^n$. We claim that for every $\omega \in \Sigma_k^+$ and every $n \in \mathbb{N}$ it holds that $\overline{B} \subset \mathcal{L}_n(\omega)[\{B\}_{|i|=n}]$. The proof of this claim is by induction. For $n = 1$, noting that $h_{i\omega} \in \mathcal{U}_i$ for all $i = 1, \dots, k$ and having in mind the choice of these neighborhoods it follows

$$\mathcal{L}_1(\omega)[\{B\}_{|i|=1}] = \bigcup_{i=1}^{k} h_{i\omega}(B) \supset \bigcup_{i=1}^{k} B_i \supset \overline{B}.$$

We argue inductively. Assuming that the claim holds for n, we see that it also holds for $n+1$. From Item (ii) in Lemma 2.32 we get

$$\mathcal{L}_{n+1}(\omega)[\{B\}_{|i|=n+1}] = \mathcal{L}_1(\omega)[\{\mathcal{L}_n(j\omega)[\{B\}_{|i|=n}]\}_{|j|=1}].$$

By the induction hypothesis it follows that $\mathcal{L}_{n+1}(\omega)[\{B\}_{|i|=n+1}] \supset \mathcal{L}_1(\omega)[\{B\}_{|j|=1}]$. From this, using the first step of the induction, we obtain the desired assertion.

Now, we will conclude the proof of the theorem from Lemma 2.29. Let $D_+^u \subset \Sigma_k^+ \times \overline{D}$ be an unstable disk through a fixed point of H. Recall that D_+^u is the graph of a continuous function g from Σ_k^+ to $\overline{D}$. Fix $\omega \in \Sigma_k^+$ and a non-empty open set U in B. Proposition 2.33 and the above claim imply that $\overline{B} \subset K_\omega$. Note that K_ω is the limit in the Hausdorff metric of $\mathcal{L}_n(\omega)[\{g(i\omega)\}_{|i|=n}]$. Then, since U is non-empty open set in K_ω, there is $n \in \mathbb{N}$ such that $U \cap \mathcal{L}_n(\omega)[\{g(i\omega)\}_{|i|=n}] \neq \emptyset$. This implies that the iterate $H^n(D_+^u)$ meets $\{\omega\} \times U$ and we conclude the proof of the theorem. □

Let $\Phi = \tau \ltimes (\phi_2, \dots, \phi_k)$ be a skew-product in the hypothesis of Theorem 2.30. Consider $\Psi = \tau \ltimes \psi_\xi$ a $\mathcal{S}^+$-perturbation of Φ. For each $\omega \in \Sigma_k^+$ let K_ω be the compact set followed from Proposition 2.33 for $\Psi_+ = \sigma \ltimes \psi_{\xi^+}$. Notice that the above proof of Theorem 2.30 shows that

$$\overline{B} \subset \bigcap_{\omega\in\Sigma_k^+} K_\omega \stackrel{\text{def}}{=} K_\Psi^*.$$

Argue as Proposition 2.15, it is possible to prove that this compact set K_Ψ^* depends continuously with respecto to Ψ. Let $\mathcal{V}$ be a small enough neighborhood of $\mathcal{S}^+$-perturbations Ψ of Φ, and set

$$K^* \stackrel{\text{def}}{=} \bigcap_{\Psi\in\mathcal{V}} K_\Psi^*.$$

Notice that $B \subset K^*$. Also, note that we have precisely proved in the above proof of Theorem 2.30 that if $\Psi \in \mathcal{V}$ and D_+^u is a unstable disk through a fixed point of Ψ_+ then for every open set U in K^* there exists $n \in \mathbb{N}$ such that $\Psi_+^n(D_+^u) \cap (\{\xi^+\} \times U) \neq \emptyset$ for all sequence $\xi^+ \in \Sigma_k^+$. From

Lemma 2.29, this is equivalente to the following: if $(\vartheta, p) \in \Sigma_k \times D$ is a fixed point of Φ then for every $\Psi \in \mathcal{V}$ and every open set $U \subset K^*$ it holds

$$W^{uu}\big((\vartheta, p_\Psi); \Phi\big) \cap (W^s_{loc}(\xi; \tau) \times U) \neq \emptyset \quad \text{for all } \xi \in \Sigma_k.$$

Hence, $B^* = \mathrm{int}(K^*)$ is the superposition region of the symbolic cs-blender-horseshoe Γ_Φ in the unilateral setting for Φ.

2.5 Symbolic blenders in the Hölder setting

In this section we will prove Theorem B. In order to prove this theorem, we need to introduce some notation and preliminary Hölder-like estimates. Given a word $\overline{\omega} = \omega_{-n} \dots \omega_{-1}\, \omega_0\, \omega_1 \dots \omega_n$, we define the *bi-lateral cylinder*

$$\tilde{C}_{\overline{\omega}} \stackrel{\text{def}}{=} \{\xi \in \Sigma_k \colon\ \xi_j = \omega_j,\ -n \leq j \leq n\}.$$

Lemma 2.34. *Consider $\Psi = \tau \ltimes \psi_\xi \in \mathcal{S}^\alpha_{k,\lambda,\beta}(D)$, a word $\overline{\omega} = \omega_{-n} \dots \omega_{-1}\, \omega_0\, \omega_1 \dots \omega_n$ and a point $x \in \overline{D}$ such that for every $\zeta \in \tilde{C}_{\overline{\omega}}$ it holds that $\psi^{-j}_{\tau^{-1}(\zeta)}(x) \in \overline{D}$ for $1 \leq j \leq n$. Then,*

$$\big\|\psi^{-i}_{\tau^{-1}(\xi)}(x) - \psi^{-i}_{\tau^{-1}(\zeta)}(x)\big\| < C_\Psi \nu^{\alpha(n-i)} \sum_{j=0}^{i-1} (\lambda^{-1}\nu^\alpha)^j,$$

for all $1 \leq i \leq n$ and all $\xi, \zeta \in \tilde{C}_{\overline{\omega}}$.

Proof. The proof is by induction. For $i = 1$, the Hölder property and $\xi, \zeta \in \tilde{C}_{\overline{\omega}}$ imply that $\|\psi^{-1}_{\tau^{-1}(\xi)}(x) - \psi^{-1}_{\tau^{-1}(\zeta)}(x)\| \leq C_\Psi \nu^{\alpha(n-1)}$. We argue inductively. Assuming that the lemma holds for $i-1$, $i < n$, we see that it also holds for i. By the triangle inequality, one has that

$$\begin{aligned}\big\|\psi^{-i}_{\tau^{-1}(\xi)}(x) - \psi^{-i}_{\tau^{-1}(\zeta)}(x)\big\| &\leq \big\|\psi^{-i}_{\tau^{-1}(\xi)}(x) - \psi^{-1}_{\tau^{-i}(\xi)} \circ \psi^{-(i-1)}_{\tau^{-1}(\zeta)}(x)\big\| \\ &\quad + \big\|\psi^{-1}_{\tau^{-i}(\xi)} \circ \psi^{-(i-1)}_{\tau^{-1}(\zeta)}(x) - \psi^{-i}_{\tau^{-1}(\zeta)}(x)\big\|.\end{aligned}$$

Let $y \stackrel{\text{def}}{=} \psi^{-(i-1)}_{\tau^{-1}(\zeta)}(x) \in \overline{D}$. Since the inverse of these functions expand at most $1/\lambda$ we obtain that the above equation is less than or equal to

$$\frac{1}{\lambda}\,\big\|\psi^{-(i-1)}_{\tau^{-1}(\xi)}(x) - \psi^{-(i-1)}_{\tau^{-1}(\zeta)}(x)\big\| + \big\|\psi^{-1}_{\tau^{-i}(\xi)}(y) - \psi^{-1}_{\tau^{-i}(\zeta)}(y)\big\|.$$

Since $y \in \overline{D}$, we can apply to the second term the Hölder inequality. Namely, since $\xi, \zeta \in \tilde{C}_{\overline{\omega}}$, we get $\|\psi^{-1}_{\tau^{-i}(\xi)}(y) - \psi^{-1}_{\tau^{-i}(\zeta)}(y)\| \leq C_\Psi\, \nu^{\alpha(n-i)}$. By the induction hypothesis we bound the first term and we get

$$C_\Psi \lambda^{-1} (\nu^\alpha)^{n-i+1} \sum_{j=0}^{i-2} (\lambda^{-1}\nu^\alpha)^j + C_\Psi \nu^{\alpha(n-i)} = C_\Psi \nu^{\alpha(n-i)} \sum_{j=0}^{i-1} (\lambda^{-1}\nu^\alpha)^j,$$

which concludes the proof of the lemma. □

Right now, we will prove Theorem B.

Proof of Theorem B. Assume that the covering property (2.37) is not fulfilled. That is, there exists $x \in \overline{B}$ such that x does not belong to $\phi_i(B)$ for all $i = 1, \dots, k$. Without loss of generality, we can assume that $x \in B$. Otherwise, we can take an arbitrarily small one-step perturbation $\Psi = \tau \ltimes (\psi_1, \dots, \psi_k)$ such that the covering property in B for the IFS$(\psi_1, \dots, \psi_k)$ is not satisfied for a point in B. Then $\Phi^{-1}(\xi, x) \notin \Sigma_k \times \overline{B}$ for all $\xi \in \Sigma_k$ and hence

$$(\xi, x) \notin \bigcap_{n \geq 0} \Phi^n(\Sigma_k \times \overline{B}) \quad \text{for all } \xi \in \Sigma_k.$$

This shows that $\Gamma^+_\Phi(B)$ does not meet any (almost) horizontal disk through $\overline{B}$ of the form $D^s = W^s_{loc}(\xi; \tau) \times \{x\}$ and therefore the intersection property (2.38) is not fulfill.

Now we will prove that the covering property (2.37) implies the intersection property (2.38). Recall that given an open covering $\mathcal{C}$ of a compact set X of a metric space there is a constant $L > 0$, called *Lebesgue number* of $\mathcal{C}$, such that every subset of X with diameter less than L is contained in some member of $\mathcal{C}$. Let $L > 0$ be the Lebesgue number of the open covering (2.37).

There are C^0-neighborhoods $\mathcal{U}_i$ of ϕ_i such that the family

$$B_i = \operatorname{int}\Big(\bigcap_{\psi \in \mathcal{U}_i} \psi(B) \Big), \qquad i = 1, \dots, k,$$

is an open covering of $\overline{B}$. By shrinking the size of the sets $\mathcal{U}_i$ we can assume that the number $L > 0$ is also a Lebesgue number of this covering and in addition any $\psi \in \mathcal{U}_i$ is also a C^0-(λ, β)-Lipschitz map on $\overline{D}$ for all $i = 1, \dots, k$.

Remark 2.35 (Choice of the perturbation I)**.** *Let $\mathcal{V}_1$ be an neighborhood of Φ in $\mathcal{S}^\alpha_{k,\lambda,\beta}(D)$ such that if $\Psi = \tau \ltimes \psi_\xi \in \mathcal{V}_1$ then $\psi_\xi \in \mathcal{U}_i$ where $\xi_0 = i$. In that case, we get that*

$$\psi^{-1}_{\tau^{-1}(\xi)}(\overline{B_i}) \subset B \quad \textit{for all } \xi \in \Sigma_k \textit{ with } \xi_{-1} = i.$$

Note that if Φ is a one-step map then $\phi_\xi = \phi_\zeta$ for every ξ and ζ with $\xi_0 = \zeta_0$. Hence we can take $C_\Phi = 0$. If Ψ is $\mathcal{S}^\alpha$-close to Φ, then from the distance considered in (2.25), it follows that C_Ψ is close to $C_\Phi = 0$. Thus, since the one-step map Φ satisfies the condition $\lambda > \nu^\alpha$, we obtain the following remark:

Remark 2.36 (Choice of the perturbation II)**.** *Let $\mathcal{V}_2$ be an neighborhood of Φ in $\mathcal{S}^\alpha_{k,\lambda,\beta}(D)$ such that if $\Psi = \tau \ltimes \psi_\xi \in \mathcal{V}_2$ then*

$$C_\Psi \sum_{i=0}^{\infty} (\lambda^{-1}\nu^\alpha)^i < L/2. \tag{2.50}$$

In what follows we will consider the neighborhood $\mathcal{V} = \mathcal{V}_1 \cap \mathcal{V}_2$ of Φ in $\mathcal{S}^\alpha_{k,\lambda,\beta}(D)$.

Fix $0 < \delta < L/2$ such that $\lambda^{-1}\delta < L/2$. Consider $V = \mathscr{P}(D^s) \subset B$ where D^s is a δ-horizontal disk in $\Sigma_k \times B$ associated with $W^s_{loc}(\zeta; \tau) \times \{z\}$, $z \in B$, $\zeta \in \Sigma_k$. Note that $\operatorname{diam}(V) \leq 2\delta < L$. Then there is $i_1 \in \{1, \dots, k\}$ such that $V \subset B_{i_1}$. Given a word $\bar\theta = \theta_n \dots \theta_1$, we denote

$$C_{\bar\theta} \stackrel{\text{def}}{=} \{\xi \in W^s_{loc}(\zeta; \tau) : \xi_{-i} = \theta_i \text{ for } i = 1, \dots, n\}.$$

Let $\bar\theta_1 = i_1$ and $V_1 = \mathscr{P}(D^s \cap (C_{\bar\theta_1} \times V))$. Given x and y in V_1, there exist $\xi, \eta \in C_{\bar\theta_1}$ such that $x = h(\xi)$ and $y = h(\eta)$. From the Hölder continuity of h, it follows $\|x - y\| \leq C d_{\Sigma_k}(\xi, \eta)^\alpha \leq C\nu^\alpha$.

Thus, $V_1 \subset V$ and $\operatorname{diam}(V_1) \le C\nu^\alpha \stackrel{\text{def}}{=} \delta_1$. Then, by Remark 2.35, for every $\Psi = \tau \ltimes \psi_\xi \in \mathcal{V}$ we obtain that

$$\psi^{-1}_{\tau^{-1}(\xi)}(V_1) \subset B \quad \text{and} \quad \operatorname{diam}(\psi^{-1}_{\tau^{-1}(\xi)}(V_1)) \le \lambda^{-1}\delta_1 \quad \text{for all } \xi \in C_{\bar\theta_1}.$$

Suppose constructed a word $\bar\theta_n$ and a closed set $V_n \subset V_{n-1}$ with $\operatorname{diam}(V_n) \le C\nu^{n\alpha} \stackrel{\text{def}}{=} \delta_n$ such that for every skew-product $\Psi = \tau \ltimes \psi_\xi \in \mathcal{V}$ it holds

$$\psi^{-n}_{\tau^{-1}(\xi)}(V_n) \subset B \quad \text{and} \quad \operatorname{diam}(\psi^{-n}_{\tau^{-1}(\xi)}(V_n)) \le \lambda^{-n}\delta_n \quad \text{for all } \xi \in C_{\bar\theta_n}.$$

We will construct a word $\bar\theta_{n+1}$ and a closed set $V_{n+1} \subset V_n$ satisfying analogous inclusions and inequalities. Let

$$A_n = \bigcup_{\xi \in C_{\bar\theta_n}} \psi^{-n}_{\tau^{-1}(\xi)}(V_n) \subset B.$$

Given $\bar x$ and $\bar y$ in A_n, there exist $x, y \in V_n$, and $\xi, \eta \in C_{\bar\theta_n}$ such that $\bar x = \psi^{-n}_{\tau^{-1}(\xi)}(x)$ and $\bar y = \psi^{-n}_{\tau^{-1}(\eta)}(y)$. By means of Lemma 2.34, Remark 2.36 and since

$$\lambda^{-n}\delta_n = C(\lambda^{-1}\nu^\alpha)^n \le C\lambda^{-1}\nu^\alpha < \lambda^{-1}\delta < L/2$$

we obtain that

$$\|\bar x - \bar y\| \le \|\psi^{-n}_{\tau^{-1}(\xi)}(x) - \psi^{-n}_{\tau^{-1}(\eta)}(x)\| + \|\psi^{-n}_{\tau^{-1}(\eta)}(x) - \psi^{-n}_{\tau^{-1}(\eta)}(y)\| \le L/2 + \lambda^{-n}\delta_n < L.$$

Hence $\operatorname{diam}(A_n) < L$ and so there is $i_{n+1} \in \{1, \dots, k\}$ such that $A_n \subset B_{i_{n+1}}$. Let

$$\bar\theta_{n+1} = i_{n+1}\bar\theta_n \quad \text{and} \quad V_{n+1} = \mathscr{P}(D^s \cap (C_{\bar\theta_n} \times V_n)).$$

Given x and y in V_{n+1}, there exist $\xi, \eta \in C_{\bar\theta_n}$ such that $x = h(\xi)$ and $y = h(\eta)$. From the (α, C)-Hölder continuity of h, we have that $\|x - y\| \le C d_{\Sigma_k}(\xi, \eta)^\alpha \le C\nu^{(n+1)\alpha}$. Thus, $V_{n+1} \subset V_n$ with $\operatorname{diam}(V_n) \le C\nu^{(n+1)\alpha} \stackrel{\text{def}}{=} \delta_{n+1}$ such that for every $\Psi = \tau \ltimes \psi_\xi \in \mathcal{V}$ it holds

$$\psi^{-(n+1)}_{\tau^{-1}(\xi)}(V_{n+1}) \subset B \quad \text{and} \quad \operatorname{diam}(\psi^{-(n+1)}_{\tau^{-1}(\xi)}(V_{n+1})) \le \lambda^{-(n+1)}\delta_{n+1} \quad \text{for all } \xi \in C_{\bar\theta_{n+1}}.$$

Note that $\{V_n\}$ is a sequence of nested closed sets such that $\lim_{n\to\infty} \operatorname{diam}(V_n) = 0$. Then

$$\{(\xi, x)\} = \bigcap_{n \in \mathbb{N}} (C_{\bar\theta_n} \times V_n) \cap D^s \subset W^s_{loc}(\zeta; \tau) \times B. \tag{2.51}$$

From this, it follows that $\psi^{-n}_{\tau^{-1}(\xi)}(x) \in B$ for all $n \in \mathbb{N}$. So, $\Psi^{-n}(\xi, x) \in \Sigma_k \times B$ for for all $n \in \mathbb{N}$. Therefore, $(\xi, x) \in D^s$ belongs to the maximal forward invariant set $\Gamma^+_\Psi(B)$ in $\Sigma_k \times B$. This concludes the proof of the theorem. □

2.5.1 Symbolic blender-like sets

Let us return to the statement of Theorem B and remember Proposition 2.10. If we do not impose the condition $\beta < 1$ cannot conclude that Γ_Φ is conjugate to the Bernoulli shift of k symbols. Therefore, we cannot talk about symbolic blender-horseshoe. However, according to Proposition 2.10, Γ_Φ can be a porcupine. This fact, gives rise the following notion:

Definition 2.13 (Symbolic blender-like set)**.** *Consider* $\Phi \in \mathcal{S}^{\alpha}_{k,\lambda,\beta}(D)$ *with* $\lambda < 1 < \beta$, $\alpha > 0$.

The maximal invariant set Γ_Φ *of* Φ *in* $\Sigma_k \times \overline{D}$ *is said to be* symbolic cs-blender-like set *with superposition region an open set* B *in* D *if there is* $\delta > 0$ *such that for every* $\mathcal{S}^{\alpha}$*-perturbation* Ψ *of* Φ *it holds*

$$\Gamma_\Psi \cap D^s \neq \emptyset \quad \text{for all } \delta\text{-horizontal disk } D^s \text{ in } \Sigma_k \times B$$

where Γ_Ψ *is the maximal invariant set of* Ψ *in* $\Sigma_k \times \overline{D}$.

The next result is an immediately consequence of Theorem C. Fix $\alpha \in (0,1]$ and recall that D is an open set in the c-dimensional manifold M.

Lemma 2.37. *Let* $\phi_1 : \overline{D} \to D$ *be a* (λ,β)*-Lipschitz map with* $\nu^\alpha < \lambda < \beta < 1$. *Then there are a natural number* k *and translations (in local coordinates)* $\phi_2, \dots, \phi_k$ *of* ϕ_1 *such that the one-step map* $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k) \in \mathcal{S}^{\alpha}_{k,\lambda,\beta}(D)$ *has a symbolic cs-blender-horseshoe with superposition region a neighborhood of the fixed point of* ϕ_1.

Proof. Consider the open ball $B(p,\varepsilon) \subset D$ of radius $\varepsilon > 0$ centered at the fixed point p of ϕ_1. Note that there are $k = k(c,\lambda) > 0$ and points $d_1 = p$ and $d_i \in B(p,\varepsilon)$, $i = 2, \dots, k$, such that

$$\overline{B}(p,\varepsilon) \subset B(d_1, \frac{\lambda}{2}\varepsilon) \cup B(d_2, \frac{\lambda}{2}\varepsilon) \cup \ldots \cup B(d_k, \frac{\lambda}{2}\varepsilon).$$

Consider (in local coordinates) translations ϕ_i of ϕ_1, $i = 2, \dots, k$, such that $B(d_i, \lambda\varepsilon/2) \subset \phi_i(B(p,\varepsilon))$. Then the choice of the points d_i and the inclusion above imply that

$$\overline{B}(p,\varepsilon) \subset \phi_1\big(B(p,\varepsilon)\big) \cup \ldots \cup \phi_k\big(B(p,\varepsilon)\big). \tag{2.52}$$

Consider the contracting iterated function system $\mathrm{IFS}(\phi_1, \dots, \phi_k)$ and its associated one-step skew-product map $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k)$. Then, by Equation (2.52), the covering property is satisfied. Thus, the map Φ satisfies the hypotheses in Theorem C and hence it has symbolic blender-horseshoe with $B(p,\varepsilon)$ contained in its superposition region. □

We observe that the number k of translations of ϕ depends on the dimension of M and the contraction bound λ of ϕ. The following proposition is motivated from [HN11] and shows a construction of a one-step $\Phi = \tau \ltimes (\phi_1, \phi_2)$ on $\Sigma_2 \times M$ such that has a symbolic cs-blender-like set with superposition region an small neighborhood of an ε-weak hyperbolic periodic attractor with period large enough. A periodic point p of diffeomorphism ϕ is said to be ε*-weak hyperbolic periodic attractor* of period n if

$$1 - \varepsilon < m(D\phi^n(p)) < \|D\phi^n(p)\| < 1$$

where $m(A)$ is the *conorm* of a linear operator A, i.e., the infimum of $\|Av\|$ as v vsaries over the unit vectors in the dominie of A.

Theorem 2.38. *Let* $\phi_1 : \overline{D} \to D$ *be a* C^1*-*(λ,β)*-Lipschitz map with* $0 < \lambda < 1 < \beta$ *having an* ε*-weak hyperbolic attracting periodic point* p *with sufficient large period* n *such that* $\nu^\alpha < (1-\varepsilon)^n$. *Then there is* ϕ_2 *arbitrarily* C^1*-close to* ϕ_1 *such that* $\Phi = \tau \ltimes (\phi_1, \phi_2) \in \mathcal{S}^{\alpha}_{2,\lambda,\beta}(D)$ *has a symbolic cs-blender-like set with superposition region a neighborhood of* p.

Proof. Let $D_i = \phi_1^i(D_0)$ be a sufficiently small neighborhoods of $\phi_1^i(p)$, for $i = 0, \dots, n-1$. We consider a local coordinate on each D_i. Denoting T_c the translation map by the vector c in local coordinates, from Lemma 2.37 we find small vectors $c_1 = 0, c_2 \dots, c_k$ (with $k < n$)[2] such that the maps $\phi_1^n = T_{c_1} \circ \phi_1^n$, $T_{c_2} \circ \phi_1^n$, $\dots$, $T_{c_k} \circ \phi_1^n$ from $\overline{D_0}$ to D_0 satisfy the covering property in a neighborhood $B \subset D_0$ of p. Let ϕ_2 be a diffeomorphism C^1-close to ϕ_1 such that on D_{n-i} it is equal to $\phi_1^{1-i} \circ T_{c_i} \circ \phi_1^i$ for $i = 1, \dots, k$. Observe that ϕ_2 is well defined if D_i are disjoint and c_i are sufficiently small. Further, $h_i = \phi_1^{i-1} \circ \phi_2 \circ \phi_1^{n-i}$ on D_0 is equal to $T_{c_i} \circ \phi_1^n$ for $i = 1, \dots, k$. Then the contracting iterated function system $\mathrm{IFS}(h_1, \dots, h_k)$ on $\overline{D_0}$ is a subsystem of $\mathrm{IFS}(\phi_1, \phi_2)$ which satisfies that $B \subset h_1(B) \cup \ldots \cup h_k(B)$. We choose open sets B_i for $i = 1, \dots, k$ such that

$$\overline{B_i} \subset h_i(B) \quad \text{and} \quad B \subset B_1 \cup \ldots \cup B_k.$$

Let $L > 0$ be the Lebesgue number of this above open covering. For each $i = 1, \dots, k$ we define the word $\overline{\omega_i} = \omega_{i1} \dots \omega_{in}$ where $\omega_{ij} = 1$ for all $j \neq n-i+1$ and $\omega_{ij} = 2$ for $j = n-i+1$. With this notation, since p is a ε-weak hyperbolic periodic attractor, it follows that $h_i^{-1} = \phi_{\omega_{in}}^{-1} \circ \dots \circ \phi_{\omega_{i1}}^{-1}$ restricted to $\overline{B_i}$ expands at most $(1-\varepsilon)^{-n}$ for all $i = 1, \dots, k$. Hence, there are C^0-neighborhoods $\mathcal{U}_1$ and $\mathcal{U}_2$ of ϕ_1 and ϕ_2 respectively such that for each $i = 1, \dots, k$, given any map $\psi_j \in \mathcal{U}_{\omega_{ij}}$ for $j = 1, \dots, n$ it holds that $\psi_n^{-1} \circ \dots \circ \psi_1^{-1}(\overline{B}_i) \subset B$ and $\psi_n^{-1} \circ \dots \circ \psi_1^{-1}$ restricted to $\overline{B_i}$ expands at most $(1-\varepsilon)^{-n}$.

The rest of the proof of this proposition is analogous with the proof of Theorem C. We will indicate some modifications in the corresponding choice of the perturbation:

Remark 2.39 (Choice of the perturbation I)**.** *Let $\mathcal{V}_1$ be a neighborhood of $\Phi = \tau \ltimes (\phi_1, \phi_2)$ in $\mathcal{S}^{\alpha}_{2,\lambda,\beta}(D_0)$ such that if $\Phi = \tau \ltimes \psi_\xi \in \mathcal{V}_1$ then $\psi_\xi \in \mathcal{U}_i$ where $i = \xi_0$. In particular, for every $\Psi = \tau \ltimes \psi_\xi \in \mathcal{V}$ it holds that, for every $i = 1, \dots, k$*

$$\psi^{-n}_{\tau^{-1}(\xi)}(\overline{B_i}) \subset B \quad \textit{for all} \ \ \xi \in \Sigma_2 \ \textit{with} \ \ \xi_{-j} = \omega_{ij} \ \ \textit{for} \ \ j = 1, \dots, n$$

and $\psi^{-n}_{\tau^{-1}(\xi)} = \psi^{-1}_{\tau^{-n}(\xi)} \circ \dots \circ \psi^{-1}_{\tau^{-1}(\xi)}$ restricted to $\overline{B_i}$ expands at most $\kappa = (1-\varepsilon)^{-n}$.

Since by hypothesis $\kappa > \nu^\alpha$, we obtain the following remark:

Remark 2.40 (Choice of the perturbation II)**.** *Let $\mathcal{V}_2$ be an neighborhood of $\Phi = \tau \ltimes (\phi_1, \phi_2)$ in $\mathcal{S}^{\alpha}_{2,\lambda,\beta}(D_0)$ such that if $\Psi = \tau \ltimes \psi_\xi \in \mathcal{V}_2$ then one has that*

$$C_\Psi \sum_{i=0}^{\infty} (\kappa^{-1}\nu^\alpha)^i < L/2.$$

Fix $0 < \delta < L/2$ such that $\kappa^{-1}\delta < L/2$. Consider $V = \mathscr{P}(D^s) \subset B$ where D^s is a δ-horizontal disk in B associated with $W^s_{loc}(\zeta; \tau) \times \{z\}$, $z \in B$, $\zeta \in \Sigma_2$. Note that $\mathrm{diam}(V) \leq 2\delta < L$. Then there is $i_1 \in \{1, \dots, k\}$ such that $V \subset B_{i_1}$. Recall that, given a word $\bar{\theta} = \theta_m \dots \theta_1$,

$$C_{\bar{\theta}} = \{\xi \in W^s_{loc}(\zeta; \tau) : \xi_{-i} = \theta_i \text{ for } i = 1, \dots, m\}.$$

Let $\bar{\theta}_1 = \overline{\omega_{i_1}}$ and $V_1 = \mathscr{P}(D^s \cap (C_{\bar{\theta}_1} \times V))$. Given x and y in V_1, there exist $\xi, \eta \in C_{\bar{\theta}_1}$ such that $x = h(\xi)$ and $y = h(\eta)$. From the Hölder continuity of h, it follows

$$\|x - y\| \leq C d_{\Sigma_2}(\xi, \eta)^\alpha \leq C\nu^\alpha.$$

[2]Since p is ε-weak hyperbolic attracting periodic point if ε goes to zero then the number k of translations to obtain the covering property goes to zero.

Thus, $V_1 \subset V$ and $\mathrm{diam}(V_1) \le C\nu^\alpha \stackrel{\text{def}}{=} \delta_1$. Hence, by Remark 2.39, for every $\Psi = \tau \ltimes \psi_\xi \in \mathcal{V}$ we obtain that

$$\psi^{-n}_{\tau^{-1}(\xi)}(V_1) \subset B \quad \text{and} \quad \mathrm{diam}(\psi^{-n}_{\tau^{-1}(\xi)}(V_1)) \le \kappa^{-1}\delta_1 \quad \text{for all } \xi \in C_{\bar\theta_1}.$$

Suppose constructed a word $\overline{\theta}_m$ and a closed set $V_m \subset V_{m-1}$ with $\mathrm{diam}(V_m) \le C\nu^{m\alpha} \stackrel{\text{def}}{=} \delta_m$ such that for every $\Psi = \tau \ltimes \psi_\xi \in \mathcal{V}$ it holds that

$$\psi^{-mn}_{\tau^{-1}(\xi)}(V_m) \subset B \quad \text{and} \quad \mathrm{diam}(\psi^{-mn}_{\tau^{-1}(\xi)}(V_m)) \le \kappa^{-m}\delta_m \quad \text{for all } \xi \in C_{\bar\theta_m}.$$

We will construct a word $\overline{\theta}_{m+1}$ and a closed set $V_{m+1} \subset V_m$ satisfying analogous inclusions and inequalities. Let

$$A_m = \bigcup_{\xi \in C_{\bar\theta_m}} \psi^{-mn}_{\tau^{-1}(\xi)}(V_m) \subset B.$$

Given $\bar x$ and $\bar y$ in A_m, there exit $x, y \in V_m$ and $\xi, \eta \in C_{\bar\theta_m}$ such that $\bar x = \psi^{-mn}_{\tau^{-1}(\xi)}(x)$ and $\bar y = \psi^{-mn}_{\tau^{-1}(\eta)}(y)$. So, from Lemma 2.34, Remark 2.40 and since

$$\kappa^{-m}\delta_m = C(\kappa^{-1}\nu^\alpha)^m \le C\kappa^{-1}\nu^\alpha < \kappa^{-1}\delta < L/2$$

we obtain that

$$\|\bar x - \bar y\| \le \|\psi^{-mn}_{\tau^{-1}(\xi)}(x) - \psi^{-mn}_{\tau^{-1}(\eta)}(x)\| + \|\psi^{-mn}_{\tau^{-1}(\eta)}(x) - \psi^{-mn}_{\tau^{-1}(\eta)}(y)\| \le L/2 + \kappa^{-m}\delta_m < L.$$

Hence $\mathrm{diam}(A_m) < L$ and so there is $i_{m+1} \in \{1, \dots, k\}$ such that $A_m \subset B_{i_{n+1}}$. Let

$$\bar\theta_{m+1} = \overline{\omega_{i_{m+1}}\theta}_m \quad \text{and} \quad V_{m+1} = \mathscr{P}(D^s \cap (C_{\bar\theta_m} \times V_m)).$$

Given x and y in V_{m+1}, there exit $\xi, \eta \in C_{\bar\theta_m}$ such that $x = h(\xi)$ and $y = h(\eta)$. From the (α, C)-Hölder continuity of h, we have that $\|x - y\| \le Cd_{\Sigma_2}(\xi, \eta)^\alpha \le C\nu^{(m+1)\alpha}$. Thus, $V_{m+1} \subset V_m$ with $\mathrm{diam}(V_m) \le C\nu^{(m+1)\alpha} \stackrel{\text{def}}{=} \delta_{m+1}$ such that for every $\Phi = \tau \ltimes \psi_\xi \in \mathcal{V}$ it holds that

$$\psi^{-(m+1)}_{\tau^{-1}(\xi)}(V_{m+1}) \subset B \quad \text{and} \quad \mathrm{diam}(\psi^{-(m+1)}_{\tau^{-1}(\xi)}(V_{m+1})) \le \kappa^{-(m+1)}\delta_{m+1} \quad \text{for all } \xi \in C_{\bar\theta_{m+1}}.$$

Note that $\{V_m\}$ is a sequence of nested close set such that $\lim_{m\to\infty} \mathrm{diam}(V_m) = 0$. Then

$$\{(\xi, x)\} = \bigcap_{n \in \mathbb{N}} (C_{\bar\theta_m} \times V_m) \cap D^s \subset W^s_{loc}(\zeta; \tau) \times B. \tag{2.53}$$

Note that since $\psi_\eta(\overline{D}) \subset D$ for all η then $\Psi^n(\xi, x) \in \Sigma_2 \times D$ for all $n \in \mathbb{N}$. On the other hand, from (2.53) it follows that $\psi^{-m}_{\tau^{-1}(\xi)}(x) \in B$ for all $m \in \mathbb{N}$. So, $\Psi^{-m}(\xi, x) \in \Sigma_2 \times B$ for all $m \in \mathbb{N}$. Therefore, $(\xi, x) \in D^s$ belongs in the maximal invariant set Γ_Ψ in $\Sigma_2 \times \overline{D}$ and we conclude the proof of the proposition. □

Observe that if $\varepsilon \to 0$ in the definition of ε-weak hyperbolic periodic atractor then the periodic point p becomes in non-hyperbolic. Note that since in the special case with one-dimensional fiber the covering property only needs of two maps, then it is not necessary the assumption that the period n of p is sufficient large. In fact, it suffices $n = 1$. Hence, we can consider as limit situation $\Phi = \tau \times \mathrm{id}$ with $\tau : \Sigma_2 \to \Sigma_2$ and the identity map in a one-dimensional manifold. In order to study perturbations of this maps is helpful to understood the dynamics of one-step maps $\Psi = \tau \ltimes (\psi_1, \psi_2)$ with ψ_1 and ψ_2 close enough to the identity. This task can be reduce to understood some dynamical property of IFS(ψ_1, ψ_2). In the next chapter we focus to study iterated function system on dimension one generates by two maps close to the identity.

2.5.2 A blender-horseshoe example: non-normally hyperbolic horseshoes

We will show again, using the theory developed in this chapter, that the non-normally hyperbolic horseshoes introduced in the first chapter are blender-horseshoes. The following proposition, is a slightly generalization of Proposition 1.12. Recall that we mean by a non-normally hyperbolic horseshoes the embedded horseshoes introduced in §1.2.1. That is, a horseshoe for a locally constant skew product diffeomorphism g on $\mathbb{R}^{n+1}$ which is not contained in a hyperplane of the form $\mathbb{R}^n \times \{t\}$ for some $t \in \mathbb{R}$. When g is not a locally constant skew product diffeomorphism but it is conjugated to a symbolic skew product $\Psi = \tau \ltimes \psi_\xi$ on $\Sigma_k \times I$ where I is a close real interval, a horseshoe Γ_g is also said to be *non-normally hyperbolic* for g if there are pairwise disjoint C^1-open sets $\mathcal{U}_1, \dots \mathcal{U}_k$ of diffeomorphisms on I such that $\psi_\xi \in \mathcal{U}_i$ if $\xi_0 = i$ for $i = 1, \dots, k$. Observe that the restriction of Ψ to the corresponding invariant set Γ_Ψ via conjugation to Γ_g must be conjugated to the Bernoulli shift of k symbols and Γ_Ψ cannot be contained in any subset of the form $\Sigma_k \times \{t\}$ with $t \in I$.

Proposition 2.41. *Let $F : N \to N$ be a C^1-diffeomorphisms with a Smale horseshoe*

$$\Lambda_F = \bigcap_{n\in\mathbb{Z}} F^n(\overline{U}), \quad \textit{where } U \subset N \ \textit{ is open set}$$

and consider D an open set in a closed real interval I. Let g be a C^1-diffeomorphism on $N \times I$ close enough in the C^1 topology to $f = F \times \mathrm{id}$ on $\overline{U} \times I$ such that

$$\Gamma_g = \bigcap_{n\in\mathbb{N}} g^n(\overline{U} \times \overline{D})$$

is a non-normally hyperbolic Smale horseshoe for g. Then Γ_g is blender for g.

Proof. Since g is a C^1-perturbation of $f|_{\Lambda_F \times I}$, according to Proposition 2.1 there are a g-invariant set Δ_g in $\overline{U} \times I$ homeomorphic to $\Lambda_F \times I$ and a symbolic locally α-Hölder skew product $\Phi = \tau \ltimes \phi_\xi$ belongs to $\mathcal{S}_2(I)$ such that $g|_{\Delta_g}$ is conjugated to Φ. From the same proposition also it follows that small C^1-perturbations of g should be conjugated to locally α-Hölder skew products close to $\Phi_0 = \tau \times \mathrm{id}$. Notice that $\Gamma_g \subset \Delta_g$. Hence $g|_{\Gamma_g}$ is conjugated to $\Phi|_{\Gamma_\Phi}$ where

$$\Gamma_\Phi = \bigcap_{n\in\mathbb{Z}} \Phi^n(\Sigma_k \times \overline{D}).$$

To prove that Γ_g is a blender for g it suffices to see that Γ_Φ is a symbolic blender-horseshoe for Φ.

Since g is a C^1-perturbation of $f = F \times \mathrm{id}$ it follows that Δ_g is a partial hyperbolic set for g. That is, the tangent bundle of $N \times I$ on Δ_g decomposes into the dominating splitting $E^{ss} \oplus E^c \oplus E^{uu}$ where E^c is a one-dimensional bundle. On the other hand, since Γ_g is a transitive hyperbolic set (it is a horseshoe) then $T_{\Gamma_g}(N \times I) = E^s \oplus E^u$ and the dimension of stable bundle E^s is constant. For this reason, either $E^s = E^{ss} \oplus E^c$ or $E^s = E^{ss}$ on Γ_g. To describe the following arguments we choose $E^s = E^{ss} \oplus E^c$ and $E^u = E^{uu}$ on Γ_g.

By shrinking the size of D if necessary, we assume that the fiber maps $\phi_\xi : \overline{D} \to D$ are (λ, β)-Lipschitz with $0 < \lambda < \beta < 1$ close to the identity map $\mathrm{id} : I \to I$ which depend locally α-Hölder continuously with respect to ξ. Since Γ_g is a non-normally hyperbolic horseshoe for g, there are disjoint small C^1-open sets $\mathcal{U}_i$ of diffeomorphisms on I such that $\phi_\xi \in \mathcal{U}_i$ if $\xi_0 = i$ for

$i = 1, 2$. In order to prove that Γ_Φ is a symbolic cs-blender-horseshoe for $\Phi = \tau \ltimes \phi_\xi$ we will show that the *robust covering property* (Remark 2.35) and the *local constant condition* (Remark 2.36) in the proof of Theorem B are fulfilled. If the symbolic skew product $\Phi = \tau \ltimes \phi_\xi$ satisfies this two remarks then verifies the assertion of Theorem B and so, Γ_Φ is a symbolic cs-blender-horseshoe.

The local constant condition is immediately satisfied since $\Phi = \tau \ltimes \phi_\xi$ is close to $\Phi_0 = \tau \ltimes \mathrm{id}$. Also this proximity implies the robust covering property. Indeed, since Γ_Φ is a non-normally hyperbolic horseshoe then there are fixed points $(\bar{1}, p)$ and $(\bar{2}, q)$ of Φ with $p \neq q$. Let us denote ϕ_1 and ϕ_2 the fiber maps $\phi_{\bar{1}}$ and $\phi_{\bar{2}}$ respectively. Note that since ϕ_1 and ϕ_2 are close to the identity, if K^{ss} denotes the (non-trivial) interval between p and q, then $K^{ss} = \phi_1(K^{ss}) \cup \phi_2(K^{ss})$. Since $\mathcal{U}_i$ are disjoint neighborhoods of ϕ_i it follows that there exists an open interval B in K^{ss} such that $\overline{B} \subset \psi_1(B) \cup \psi_2(B)$ for all $\psi_i \in \mathcal{U}_i$ for $i = 1, 2$. This implies Remark 2.36 in the proof of Theorem B and concludes the proof of this proposition. $\square$

Iterated function systems

Some dynamical properties such as transitivity, minimality, density of periodic orbits, can be also studied for iterated function systems (IFS). Blending regions are introduced as open sets which are minimal sets for an IFS under small C^1-perturbations. Duminy's Lemma shows examples of blending regions for an IFS generated by two maps on the real line close enough to the identity. An extension of this lemma allows us to study the dynamics of IFS of generic diffeomorphisms on the circle close enough to the identity. As in the Denjoy's Theorem, no invariant minimal Cantor sets appear under conditions of regularity in the IFS. In this setting, it is characterized when S^1 is a minimal set of an IFS and it is obtained an spectral decomposition result about of the dynamic of the limit set of an IFS.

3.1 Preliminaries of IFS

Let $\phi_1, \dots, \phi_k$ be continuous selfmaps of a complete metric space X. The *iterated function system*, (IFS for short) of these maps, denoted by $\mathrm{IFS}(\phi_1, \dots, \phi_k)$, is the set of all finite forward composition of these maps. That is, the semigroup generated by the family of maps $\phi_1, \dots, \phi_k$

$$\mathrm{IFS}(\phi_1, \dots, \phi_k) \stackrel{\text{def}}{=} \{h : X \to X : \ h = \phi_{i_n} \circ \dots \circ \phi_{i_1}, \ i_j \in \{1, \dots, k\}\} \cup \{\mathrm{id} : X \to X\}.$$

In similar way it defines the IFS of maps $\phi_i \colon D_i \subset X \to X$. In this case, the possible compositions of ϕ_i's depend on each point: $\phi_i(D_i)$ is not necessarily a subset of D_j and so $\phi_j \circ \phi_i$ is only defined on $D_i \cap \phi_i^{-1}(D_j)$. For simplicity, for the moment, we will consider IFS defined in the whole space X.

Because of the close relationship between IFS and one-step symbolic skew-product maps introduced in the previous chapter, we will write $\mathrm{IFS}(\Phi) = \mathrm{IFS}(\phi_1, \dots, \phi_k)$ meaning that the IFS is generated by the family $\Phi = \{\phi_1, \dots, \phi_k\}$ associated with the one-step map $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k)$ defined on $\Sigma_k \times M$.

Associated with the iterated function system $\mathrm{IFS}(\Phi)$, we define the operator

$$\mathcal{G}_\Phi(A) \stackrel{\text{def}}{=} \phi_1(A) \cup \ldots \cup \phi_k(A)$$

on the subsets $A \subset X$. We define the $\mathcal{G}_\Phi$*-orbit* of a point $x \in X$, also called *orbit of x for* $\mathrm{IFS}(\Phi)$, as the set of the form

$$\mathrm{Orb}_\Phi(x) \stackrel{\text{def}}{=} \{\mathcal{G}^n_\Phi(x) : \ n \geq 0\} = \{h(x) : \ h \in \mathrm{IFS}(\Phi)\} \subset X.$$

The $\mathcal{G}_\Phi$-orbit of a subset $D \subset X$ is defined as the union of all its orbits. The next set of definitions generalizes usual notions of dynamical systems for IFS. Before this, given $h = \phi_{i_n} \circ \dots \circ \phi_{i_1}$ in $\mathrm{IFS}(\Phi)$ we denote by $|h|$ the number of generators in this composition, i.e., $|h| = n$.

Definition 3.1. *Let $\Phi = \{\phi_1, \ldots, \phi_k\}$ be a family of selfmaps of X. A subset $A \subset X$ is said to be*

- invariant *for* $\mathrm{IFS}(\Phi)$ *if* $\mathrm{Orb}_\Phi(x) \subset A$ *for all* $x \in A$*;*
- minimal *for* $\mathrm{IFS}(\Phi)$ *if for every* $x \in A$ *and open set* $U \subset X$ *which has non-empty intersection with* A*, there exists* $h \in \mathrm{IFS}(\Phi)$ *with* $h(x) \in U$*;*
- topologically transitive *for* $\mathrm{IFS}(\Phi)$ *if for any pair of open sets* $U, V \subset X$ *which have non-empty intersection with* A*, there exists* $h \in \mathrm{IFS}(\Phi)$ *such that* $h(V) \cap U \neq \emptyset$*;*
- topologically mixing *for* $\mathrm{IFS}(\Phi)$ *if for any pair of open sets* $U, V \subset X$ *there exists* $n_0 \in \mathbb{N}$ *such that for every* $n \geq n_0$ *there is* $h \in \mathrm{IFS}(\Phi)$ *with* $|h| = n$ *such that* $h(V) \cap U \neq \emptyset$*.*

The next result shows some typically equivalent definition for the above notions:

Proposition 3.1. *Let $\Phi = \{\phi_1, \ldots, \phi_k\}$ be a family of selfmaps of X and consider $A \subset X$. Then*

i) A is a minimal set for $\mathrm{IFS}(\Phi)$ if and only if $A \subset \overline{\mathrm{Orb}_\Phi(x)}$ for all $x \in A$,

ii) if A is minimal for $\mathrm{IFS}(\Phi)$ then it is topologically transitive,

iii) if A is topologically mixing for $\mathrm{IFS}(\Phi)$ then it is topologically transitive,

iv) if there is $x \in A$ such that $A \subset \overline{\mathrm{Orb}_\Phi(x)}$ then A is topologycally transitive for $\mathrm{IFS}(\Phi)$,

v) assuming that X is separable and A with the restricted topology is Baire, it holds that if A topologically transitive for $\mathrm{IFS}(\Phi)$ then there is $x \in A$ such that $A \subset \overline{\mathrm{Orb}_\Phi(x)}$,

vi) assuming invertibility, and denoting $\Phi^{-1} = \{\phi_1^{-1}, \ldots, \phi_k^{-1}\}$ the inverse family, it holds that if A is topologically transitive for $\mathrm{IFS}(\Phi)$ then it is topologically transitive for $\mathrm{IFS}(\Phi^{-1})$.

Proof. Item (i) is followed immediately from definition of a dense set. Items (ii)-(iv) are clear from Definition 3.1. Similarly, the transitivity for the inverse IFS, i.e. item (vi), is immediately obtained again from Definition 3.1. Finally, to prove the proposition only remains to show item (v).

Assume that X is a separable metric space. Hence, there is a numerable base of the restricted topology to A whose open sets are $U_n = V_n \cap A$ where V_n is an open set in the topology of X. We denote by

$$\mathrm{Orb}^-_\Phi(V_n \cap A) = \bigcup_{x \in V_n \cap A} \mathrm{Orb}^-_\Phi(x),$$

where $\mathrm{Orb}^-_\Phi(x)$ denotes the orbit of x for $\mathrm{IFS}(\Phi^{-1})$. Firstly, observe that the negative orbit $\mathrm{Orb}^-_\Phi(V_n \cap A)$ is dense in A. Indeed, given any non-empty set $U \cap A$ where U is an open set of X, by the topological transitivity, there exists $h \in \mathrm{IFS}(\Phi)$ such that $h(U) \cap V_n \neq \emptyset$. Then $U \cap h^{-1}(V_n)$ is a non-empty set contained in $U \cap \mathrm{Orb}^-_\Phi(V_n)$. Suppose now that A is Baire. Hence, we obtain that $Q = \cap_{n\in\mathbb{N}} \mathrm{Orb}^-_\Phi(V_n)$ is a dense set in A. Thus, for every $x \in Q$ it follows that $x \in \mathrm{Orb}^-_\Phi(V_n)$ and hence $V_n \cap \mathrm{Orb}_\Phi(x) \neq \emptyset$ for all $n \in \mathbb{N}$. Therefore, we have proved that the orbit of x for $\mathrm{IFS}(\Phi)$ is dense in A completing the proof of the proposition. □

Notice that in ours definition of minimal set $A \subset X$ for an IFS we do not impose that A needs to be invariant for $\mathrm{IFS}(\Phi)$. Consequently, every single point $A = \{x\} \subset X$ and every subset of a minimal subset for an IFS are minimal set for $\mathrm{IFS}(\Phi)$. If we combine the notions of invariance and minimality, then we obtain that a closed subset $A \subset X$ is invariant and minimal for IFS(Φ) if and only if

$$A = \overline{\mathrm{Orb}_\Phi(x)} \quad \text{for all } x \in A.$$

For short, we will say that A is a *closed invariant minimal set for* IFS(Φ) if it satisfies the above equality. In this case, A is minimal regarding the inclusion, i.e., its only closed invariant subsets for IFS(Φ) are the empty set and A itself.

Set $\Sigma_k^+ = \{1,\dots,k\}^{\mathbb{N}}$ and let $\phi_i \in \mathrm{Hom}(X)$ be homeomorphisms on X for $i = 1,\dots,k$. For every $n \geq 1$ and every $\sigma = (\sigma_i)_{i\in\mathbb{N}} \in \Sigma_k^+$ we will use the notation

$$\phi_\sigma^n \stackrel{\text{def}}{=} \phi_{\sigma_n} \circ \cdots \circ \phi_{\sigma_1} \quad \text{and} \quad \phi_\sigma^{-n} \stackrel{\text{def}}{=} (\phi_\sigma^n)^{-1} = \phi_{\sigma_1}^{-1} \circ \cdots \circ \phi_{\sigma_n}^{-1}.$$

Now, using this above notation, we can extend the definition of limit sets for IFS.

Definition 3.2 (Limit sets for IFS)**.** *Consider $x \in X$, $\Phi = \{\phi_1,\dots,\phi_k\} \subset \mathrm{Hom}(X)$ and $\sigma \in \Sigma_k^+$. We define the* ω-limit set of x with respect to the sequence σ *as the set*

$$\omega_\sigma(x) \stackrel{\text{def}}{=} \{y \in X\colon \textit{there exists } n_i \to \infty \textit{ such that } \lim_{i\to\infty} \phi_\sigma^{n_i}(x) = y\}.$$

The union of the ω-limit sets of x for all sequence $\sigma \in \Sigma_k^+$ is called ω-limit set of x for IFS(Φ) *and we write this set as*

$$\omega_\Phi(x) \stackrel{\text{def}}{=} \{y \in X\colon \textit{there exists } \sigma \in \Sigma_k^+ \textit{ and } n_i \to \infty \textit{ such that } \lim_{i\to\infty} \phi_\sigma^{n_i}(x) = y\}.$$

Finally, we define the forward *or* ω-limit of IFS(Φ) *as*

$$\omega(\mathrm{IFS}(\Phi)) \stackrel{\text{def}}{=} \mathrm{cl}\big(\{y \in X: \ \textit{there exists } x \in X \textit{ such that } \ y \in \omega_\Phi(x)\}\big)$$

where cl *denote the closure of a set.*

Similarly, the backward *or* α-limit of IFS(Φ) *is defined as* $\alpha(\mathrm{IFS}(\Phi)) \stackrel{\text{def}}{=} \omega(\mathrm{IFS}(\Phi^{-1}))$ *where* $\Phi^{-1} = \{\phi_1^{-1},\dots,\phi_k^{-1}\}$*. From the backward and forward limit, we define the* limit set of IFS(Φ) *as*

$$L(\mathrm{IFS}(\Phi)) \stackrel{\text{def}}{=} \omega(\mathrm{IFS}(\Phi)) \cup \alpha(\mathrm{IFS}(\Phi)).$$

We denote by $\mathrm{Orb}_\Phi(x)'$ the *set of accumulation points of* $\mathrm{Orb}_\Phi(x)$. That is, the set of points $y \in X$ such that there exists a sequence $(g_n)_n \subset \mathrm{IFS}(\Phi)$ satisfying that $y = \lim_{n\to\infty} g_n(x)$ and $g_n(x) \neq y$ for all $n \in \mathbb{N}$. Notice that the set of accumulation points is always a closed set. A point $x \in X$ is called *periodic point for* IFS(Φ) if there exists $h \in \mathrm{IFS}(\Phi)$ with $h \neq \mathrm{id}$ such that $h(x) = x$. In this case, we denote the set of periodic points by

$$\mathrm{Per}(\mathrm{IFS}(\Phi)) \stackrel{\text{def}}{=} \{x \in X\colon \ h(x) = x \text{ for some } h \in \mathrm{IFS}(\Phi), \ h \neq \mathrm{id}\}.$$

Observe that $\mathscr{P}(\mathrm{Per}(\Phi)) = \mathrm{Per}(\mathrm{IFS}(\Phi))$ where $\mathrm{Per}(\Phi)$ is the set of periodic points of the symbolic skew-product $\Phi = \tau \ltimes (\phi_1,\dots,\phi_k)$ and $\mathscr{P}$ is the projection on the fiber space.

Definition 3.3. *Let A be a subset of X such that $A \cap \mathrm{Per}(\mathrm{IFS}(\Phi)) \neq \emptyset$. We say that A is* isolated *for* IFS(Φ) *if there exists an open set D of X such that $A \subset D$ and $\overline{\mathrm{Per}(\mathrm{IFS}(\Phi)) \cap D} \subset A$.*

The next lemma shows some properties and relations between the set of periodic points, the ω-limit sets, the accumulation sets and the orbits of an IFS. This properties will be necessary for the proof of some results in the later sections.

Lemma 3.2. *Consider a subset $\Phi = \{\phi_1, \dots, \phi_k\} \subset \mathrm{Hom}(X)$, a non-empty open set $B \subset X$ and a subset K of X. Then it holds that:*

i) $\omega_\Phi(h(x)) \subset \omega_\Phi(x) \subset \overline{\mathrm{Orb}_\Phi(x)}$ *for all* $x \in X$ *and for* $h \in \mathrm{IFS}(\Phi)$*;*

ii) *if* $x \in \mathrm{Per}(\mathrm{IFS}(\Phi))$ *then* $x \in \omega_\Phi(x)$*;*

iii) *if* $K = \overline{\mathrm{Orb}_\Phi(x)}$ *for all* $x \in K$ *then* $K = \overline{\omega_\Phi(x)} = \omega_\Phi(x)$ *for all* $x \in K$*;*

iv) *if* $K \subset \overline{\mathrm{Orb}_\Phi(x)}$ *for all* $x \in X$ *then* $K \subset \omega_\Phi(x)$ *for all* $x \in X$*;*

v) *if* $B \subset \overline{\mathrm{Orb}_\Phi(x)}$ *for all* $x \in B$ *then* $B \subset \omega_\Phi(x)$ *for all* $x \in B$*;*

vi) $\mathrm{Orb}_\Phi(x)' = \phi_1(\mathrm{Orb}_\Phi(x)') \cup \dots \cup \phi_k(\mathrm{Orb}_\Phi(x)')$ *for all* $x \in X$*.*

Proof. It is clear that by definition of ω-limit set of a point $x \in X$ it holds that $\omega_\Phi(h(x)) \subset \omega_\Phi(x)$ for all $h \in \mathrm{IFS}(\Phi)$. On the other hand, since

$$\overline{\mathrm{Orb}_\Phi(x)} = \{y \in X : \text{there exists } (g_n)_n \subset \mathrm{IFS}(\Phi) \text{ such that } y = \lim_{n\to\infty} g_n(x)\}$$

and we can rewrite the ω-limit set of x for $\mathrm{IFS}(\Phi)$ in the form

$$\omega_\Phi(x) = \{y \in X : \text{there exists } (h_n)_n \subset \mathrm{IFS}(\Phi) \setminus \{\mathrm{id}\} \text{ such that } y = \lim_{n\to\infty} h_n \circ \dots \circ h_1(x)\},$$

it follows that $\omega_\Phi(x)$ is a subset of the closure of the orbit of x for $\mathrm{IFS}(\Phi)$. Therefore, we conclude (i). Item (ii) is immediately obtained from the definition of ω-limit set and periodic point for an IFS.

According to the first item, to obtain (iii) it suffices to prove that $K \subset \omega_\Phi(x)$ for all $x \in K$. In order to prove this, we fix $x, y \in K$ and consider a sequence of positive real numbers $\varepsilon_n = 1/n \to 0$. It is not hard to construct by induction a sequence $(h_n)_n \subset \mathrm{IFS}(\Phi) \setminus \{\mathrm{id}\}$ such that the distance $d(y, h_n \circ \dots \circ h_1(x))$ between y and $h_n \circ \dots \circ h_1(x)$ is less than ε_n. Indeed, since the orbit of x for $\mathrm{IFS}(\Phi)$ is dense in K, we find $h_1 \in \mathrm{IFS}(\Phi)$ with $h_1 \neq \mathrm{id}$ such that $d(y, h_1(x)) < \varepsilon_1$. Similarly, since $h_1(x) \in \mathrm{Orb}_\Phi(x) \subset K$ then the orbit of $h_1(x)$ for $\mathrm{IFS}(\Phi)$ is dense in K and by the same density argument we find h_2 such that $d(y, h_2 \circ h_1(x)) < \varepsilon_2$. Argue inductively we obtain the desired sequence $(h_n)_n \subset \mathrm{IFS}(\Phi) \setminus \{\mathrm{id}\}$. Hence, $y = \lim_{n\to\infty} h_n \circ \dots \circ h_1(x)$ and thus $y \in \omega_\Phi(x)$ for all $x, y \in K$. This concludes (iii). A slight modification in this argue allows us to prove (iv) and (v).

We will now prove the last item. That is, we will show that $\mathrm{Orb}_\Phi(x)'$ is a selfsimilar set. Note that $\phi_i(\mathrm{Orb}_\Phi(x)') \subset \mathrm{Orb}_\Phi(x)'$ for all $i = 1, \dots, k$. Indeed, if y is an accumulation point of the orbit of x for $\mathrm{IFS}(\Phi)$, then $\phi_i(y)$ is approximated by points of the form $\phi_i \circ g_n(x) \in \mathrm{Orb}_\Phi(x)$ where $y = \lim_{n\to\infty} g_n(x)$ and $g_n(x) \neq y$ for all $n \in \mathbb{N}$. This implies that $\phi_i(y)$ is also an accumulation point of $\mathrm{Orb}_\Phi(x)$ and so, we conclude one of the inclusions. In order to show the other inclusion $\mathrm{Orb}_\Phi(x)' \subset \phi_1(\mathrm{Orb}_\Phi(x)') \cup \dots \cup \phi_k(\mathrm{Orb}_\Phi(x)')$, we fix any $y \in \mathrm{Orb}_\Phi(x)'$. Hence there exists $(g_n) \subset \mathrm{IFS}(\Phi)$ such that $y = \lim_{n\to\infty} g_n(x)$ and $g_n(x) \neq y$ for all $n \in \mathbb{N}$. Since the semigroup $\mathrm{IFS}(\Phi)$ is finitely generated, taking a subsequence if necessary, we can assume that for some fixed $i \in \{1, \dots, k\}$ we have that $g_n = \phi_i \circ \tilde{g}_n$ with $\tilde{g}_n \in \mathrm{IFS}(\Phi)$. Hence, $\phi_i^{-1}(y) = \lim_{n\to\infty} \tilde{g}_n(x)$ and $\tilde{g}_n(y) \neq \phi_i^{-1}(y)$. Thus, since the accumulation set is a closed set, it holds that $\phi_i^{-1}(y) \in \mathrm{Orb}_\Phi(x)'$. This implies that $y \in \phi_i(\mathrm{Orb}_\Phi(x)')$ obtaining the desired inclusion and therefore (vi). The proof of the lemma is now concluded. □

In what follows, X is a Riemannian manifold and $\phi_1, \dots, \phi_k$ are C^1-diffeomorphisms of X. Let $\Psi = \{\psi_1, \dots, \psi_k\}$ and $\Phi = \{\phi_1, \dots, \phi_k\}$ be two subset of $\mathrm{Diff}^1(X)$. We say that $\mathrm{IFS}(\Psi)$ is *C^1-close* to $\mathrm{IFS}(\Phi)$ if ψ_i is close to ϕ_i in the C^1-topology for $i = 1, \dots, k$. A set $A \subset X$ is *C^1-robustly minimal (topologically transitive)* for $\mathrm{IFS}(\Phi)$ if A is minimal (topologically transitive) for any C^1-close $\mathrm{IFS}(\Psi)$ to $\mathrm{IFS}(\Phi)$. In the case of $A = X$, we also say that $\mathrm{IFS}(\Phi)$ is C^1-robustly minimal (topologically transitive). Note that in this last case, $A = X$ is an open set. With this additional condition in the C^1-robust minimality definition we obtained the notion of blending region (see also Definition 2.22):

Definition 3.4 (Blending region)**.** *An open set B of X is called* blending region *for* $\mathrm{IFS}(\Phi)$ *if*

$$B \subset \overline{\mathrm{Orb}_\Psi(x)} \quad \textit{for all } x \in B \textit{ and every } \mathrm{IFS}(\Psi)\ C^1\textit{-close to } \mathrm{IFS}(\Phi).$$

Blending regions can be constructed for contracting maps as in Section §2.3.1. In the next section we will construct blending regions for not necessarily contracting IFS on the real line.

3.2 Blending region for IFS on the real line

We denotes the orientation preserving C^r-diffeomorphism on the real line by $\mathrm{Diff}^r_+(\mathbb{R})$. Note that if $f \in \mathrm{Diff}^r_+(\mathbb{R})$ then $Df(x) \overset{\text{def}}{=} f'(x) \geq 0$ for all $x \in \mathbb{R}$, $f(x) < f(y)$ if $x < y$ and thus its only periodic points are the fixed points.

Definition 3.5 ($**$-intervals)**.** *Consider $\Phi = \{f_0, f_1\} \subset \mathrm{Diff}^1_+(\mathbb{R})$. Let $[p, q]$ be an interval such that* $\mathrm{Fix}(f_i) \cap (p, q) = \emptyset$ *for $i = 0, 1$ and*

$$[p, q] \subset f_0([p, q]) \cup f_1([p, q]).$$

We say that $[p, q]$ is a $**$-interval *for* $\mathrm{IFS}(\Phi)$ *and write $K^{**}_\Phi = [p, q]$ with $** \in \{ss, su\}$ when p and q satisfy additional properties (see Figure A):*

- K^{ss}_Φ attractor*: $p = f_0(p)$ and $q = f_1(q)$ are both attractors and $f_0(q) \neq q$, $f_1(p) \neq p$,*
- K^{su}_Φ saddle*: p and q are an attractor-repeller pair for the same map say f_0. In this case we ask that $f_1 > id$ in $[p, q]$ and $f_1([p, q]) \cap [p, q] \neq \emptyset$.*

*A uu-*interval *(repeler), denoted by K^{uu}_Φ, is defined as ss-interval for* $\mathrm{IFS}(f_0^{-1}, f_1^{-1})$.

In the Figure A we show an example of a ss-interval and of a su-interval. An example of uu-interval is the inverse of a ss-interval. We will study the $\mathrm{IFS}(f_0, f_1)$ for f_0 and f_1 restricted to a $**$-interval for $** \in \{ss, su\}$. In the case of uu-intervals for f_0 and f_1 it follows the same results for the $\mathrm{IFS}(f_0^{-1}, f_1^{-1})$. Ours goal in the next subsection is to prove that if f_0 and f_1 are close enough to the identity then any $**$-interval for f_0 and f_1 with $** \in \{ss, su\}$ is a minimal set for the $\mathrm{IFS}(f_0, f_1)$. Observe that $K^{ss}_\Phi = f_0(K^{ss}_\Phi) \cup f_1(K^{ss}_\Phi)$ where K^{ss}_Φ is a ss-interval and $\mathrm{Orb}_\Phi(x) \subset K^{ss}_\Phi$ for all $x \in K^{ss}_\Phi$. However the above equality does not follow for a su-interval. In the case of a su-interval notice that one of the endpoint of the interval K^{su}_Φ cannot have dense orbit for the IFS. However, for unify notations, sometimes we say that an $**$-interval is minimal for the IFS or we write that $K^{**}_\Phi \subset \overline{\mathrm{Orb}_\Phi(x)}$ for all $x \in K^{**}_\Phi$ for $** \in \{ss, su\}$.

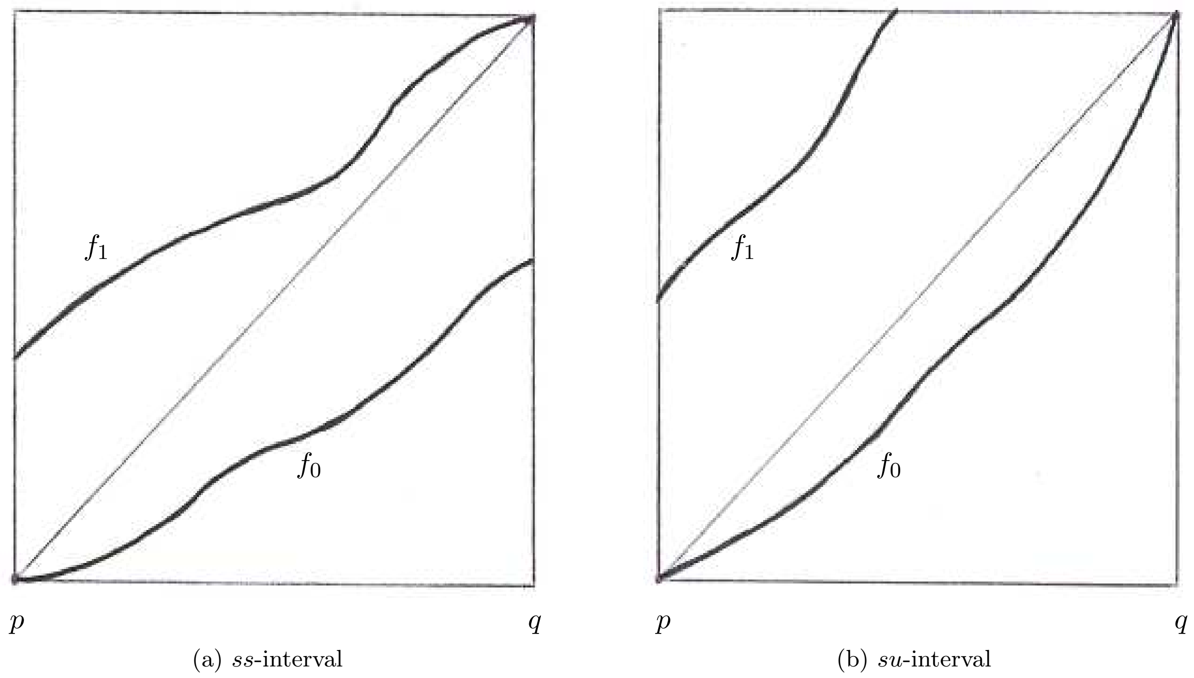

Fig. A: Examples of $**$-intervals

3.2.1 Duminy's Lemma

The next result is a generalization of a lemma that is part of the proof of Duminy's Theorem. Dumniny's Theorem is in an unpublished manuscript [Dum70] and it deals with the dynamics of groups of diffeomorphisms on the circle. We will give more details of Duminy's Theorem in the next section (see Theorem 3.27 and [Nav11]). The following statement is slightly different from the original one by Duminy, and include some improvements about the robustness and the density of periodic points.

Theorem D (Duminy's Lemma)**.** *Consider $\Phi = \{f_0, f_1\} \subset \mathrm{Diff}^2_+(\mathbb{R})$ and let K^{**}_Φ be an $**$-interval for* IFS(Φ) *with $** \in \{ss, su\}$. There exists $\varepsilon \geq 0.17$ such that if $f_0|_{K^{**}_\Phi}$, $f_1|_{K^{**}_\Phi}$ are ε-close to the identity in the C^2-topology then there are open sets $\mathcal{U}_i$ in the C^1-topology for $i = 0, 1$ such that $f_i \in \overline{\mathcal{U}_i}$ and for every* IFS(Ψ) *where $\Psi = \{g_0, g_1\}$ with $g_i \in \overline{\mathcal{U}_i}$ it holds*

$$K^{**}_\Psi \subset \overline{\mathrm{Per}(\mathrm{IFS}(\Psi))} \quad \text{and} \quad K^{**}_\Psi \subset \overline{\mathrm{Orb}_\Psi(x)} \text{ for all } x \in K^{**}_\Psi.$$

Moreover, if the fixed points of f_0 and f_1 in K^{**}_Φ are hyperbolic then $f_i \in \mathcal{U}_i$ for $i = 0, 1$.

We infer from the above theorem the following corollary:

Corollary 3.3. *If $\Phi = \{f_0, f_1\} \subset \mathrm{Diff}^2_+(\mathbb{R})$ are ε-close to the identity in the C^2-topology where $\varepsilon > 0$ is given in Theorem D, and their fixed points are hyperbolic, then every open set B contained in a $**$-interval for* IFS(Φ) *with $** \in \{ss, su\}$ is a blending region for* IFS(Φ)*.*

Now, we will give here a proof of Theorem D which is slightly different from the original proof due to Duminy's. This different proof allows us to improve the result and it will be key to show forthcoming theorems. We must to note that we will actually prove this theorem under more general hypotheses. Namely:

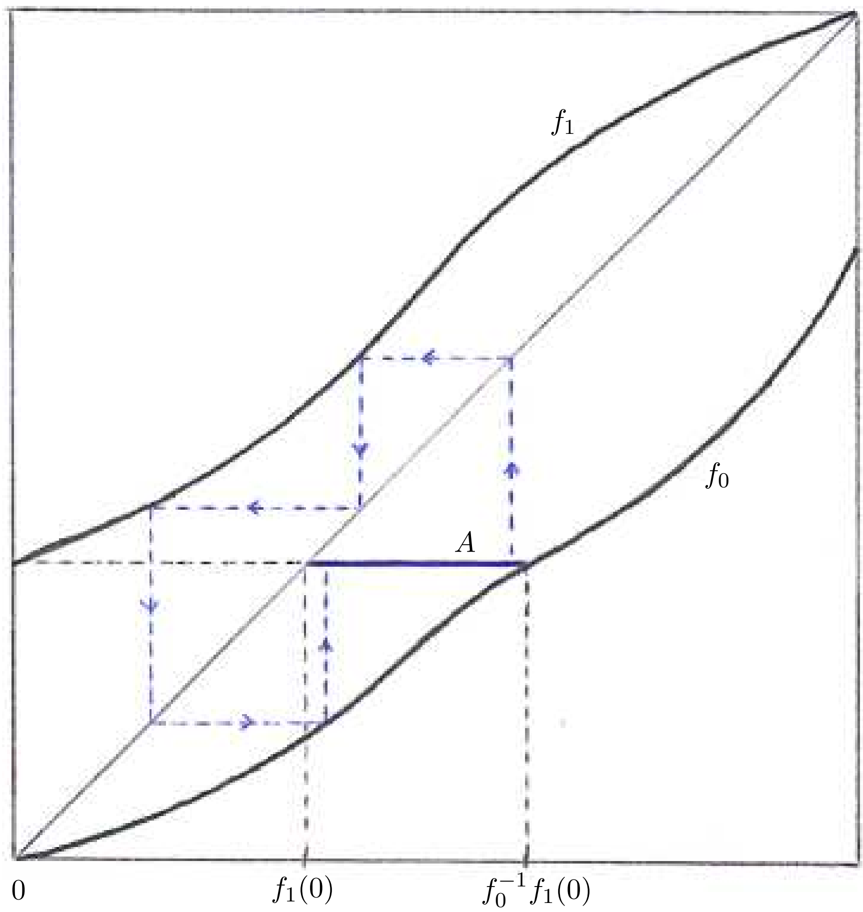

Fig. B: Definition of first return map $\mathcal{R} : A \to A$

Remark 3.4. *Theorem D holds if f_0 and f_1 are in $\mathrm{Diff}^1_+(\mathbb{R})$ such that, setting f_k as the map which has a fixed point in K^{**}_Φ,*

$$\left(\frac{1}{2}\,\frac{\inf Df_k(x)}{\sup|Df_k(x)-1|}\right)^{1/4} > \max_{i=0,1}\sup \frac{Df_i^{-1}(x)}{Df_i^{-1}(y)}$$

where the supremum and infimum are taken in K^{**}_Φ. It is easy to check that this condition is equivalent to the existence of $\varepsilon > 0$ such that

$$|Df_k(x)-1| < \varepsilon \text{ for all } x \in K^{**}_\Phi \quad \text{and} \quad (1-\varepsilon)\varepsilon^{-1}e^{-4C} > 2$$

where $C \geq 0$ is the largest distortion constant of f_0^{-1} and f_1^{-1} in K^{**}_Φ, that is,

$$C = \max_{i=0,1}\sup\{\log \frac{Df_i^{-1}(x)}{Df_i^{-1}(y)} \;:\; x, y \in K^{**}_\Phi\}.$$

For simplicity we assume that $K^{**}_\Phi = [0,1]$, $f_0(0) = 0$ and $f_0 < \mathrm{id}$ and $f_1 > \mathrm{id}$ in $(0,1)$. Note that from definition of $**$-intervals for $** \in \{ss, su\}$, the *overlap condition* is verified, that is, $f_0(K^{**}_\Phi) \cap f_1(K^{**}_\Phi) \neq \emptyset$. This condition implies that $A = (f_1(0), f_0^{-1}(f_1(0))] \subset [0,1]$. Next, we will define a first return map $\mathcal{R} : A \to A$.

Creating a return map

For each $x \in A$ let $m(x) \geq 1$ be the smallest positive number such that $f_1^{-m(x)}(x) \notin A$ and let $k(x)$ be the first time for which $f_1^{-m(x)}(x)$ returns to A by iterations of f_0^{-1}. Then we can define the first return map $\mathcal{R}$ in the following way

$$\mathcal{R}\colon A \to A, \qquad \mathcal{R}(x) = f_0^{-k(x)} \circ f_1^{-m(x)}(x)$$

Note that this map can also be given by $\mathcal{R}(x) = F^{k(x)+m(x)}(x)$ for $x \in A$, where $F : [0,1] \to [0,1]$ is define by $F = f_0^{-1}$ in $[0, f_1(0)]$ and $F = f_1^{-1}$ in $(f_1(0), 1]$. Therefore, for every $x \in [0,1]$ there is a smallest non-negative number $n(x) \geq 0$ such that $F^{n(x)}(x) \in A$ and $\mathcal{R}$ can be extended to the whole interval $[0,1]$ by taking

$$\mathcal{R}\colon [0,1] \to [0,1], \qquad \mathcal{R}(x) = F^{k+m+n}(x),$$

where $n = n(x)$, $m = m(F^n(x))$ and $k = k(F^n(x))$.

A point $d \in A$ is said to be a *discontinuity of* $\mathcal{R}$ if $\mathcal{R}(d) = f_0^{-1} f_1(0)$ or equivalently, if $d = f_1^{m(d)} f_0^{k(d)-1} f_1(0)$. These points define a partition on A . In other to describe this partition we have to consider two cases: $f_1^2(0) \notin A$ and $f_1^2(0) \in A$. In the first case $m(x) = 1$ for all $x \in A$ and we write $I_0 = A$. In the second case, consider $m \in \mathbb{N}$ such that $f_1^m f_1(0) \in A$, but $f_1^{m+1} f_1(0) \notin A$. Then $f_1^j f_1(0)$ for $j = 1, 2, \ldots, m$ define a partition on A given by

$$\begin{aligned} I_0 &= (f_1^m f_1(0),\, f_0^{-1} f_1(0)] \quad \text{and} \\ I_{i_1} &= (f_1^{m-i_1} f_1(0),\, f_1^{m+1-i_1} f_1(0)] \quad \text{for } 0 < i_1 \leq m. \end{aligned}$$

On the other hand, $m(x) = m+1-i_1$ for each $x \in I_{i_1}$ and $f_1^{-m(x)}(I_{i_1}) = (0, f_1(0)]$ for $i_1 = 0, \ldots, m$. At this point, both cases can be studied together assuming $m \geq 0$. Finally, it will be useful to prove the following lemma to note that, the sequence of points $f_0^j f_1(0)$, $j \geq 0$, defines a partition on the interval $(0, f_1(0)]$.

Lemma 3.5 (Return map)**.** *In the above hypothesis, there exist two families of respectively right-closed pairwise disjoint intervals* $I_{i_1} \subset A$ *and* $I_{i_1 i_2} \subset A$, *natural numbers* m_{i_1}, $m_{i_1 i_2}$ *and maps* $h_{i_1 i_2} \in \mathrm{IFS}(f_0, f_1)$ *with* $0 \leq i_1 \leq m$ *for some* $m \geq 0$ *and* $i_2 \geq 0$ *such that*

i) $I_{i_1 i_2} \subset I_{i_1}$ *and* I_{i_1} *is contained in a fundamental domain of* f_1. *Furthermore,*

$$A = \bigcup_{i_1=0}^{m} I_{i_1} = \bigcup_{i_1=0}^{m} \bigcup_{i_2=0}^{\infty} I_{i_1 i_2},$$

ii) $h_{i_1 i_2}^{-1} = f_0^{-m_{i_1 i_2}} \circ f_1^{-m_{i_1}}$, *where* $m_{i_1} = m + 1 - i_1$ *and* $m_{i_1 i_2+1} = m_{i_1 i_2} + 1$ *with*

$$\begin{aligned} m_{i_1 0} &= 1 \quad \textit{if } i_1 > 0 \quad \textit{and} \\ m_{00} &\geq 1 \quad \textit{such that} \quad f_0^{m_{00}}(f_1(0)) < f_1^{-m_0}(f_0^{-1}(f_1(0))) \leq f_0^{m_{00}-1}(f_1(0)), \end{aligned}$$

iii) $\mathcal{R}|_{I_{i_1 i_2}} = h_{i_1 i_2}^{-1}$ *with* $h_{00}^{-1}(I_{00}) \subset A$ *and* $h_{i_1 i_2}^{-1}(I_{i_1 i_2}) = A$ *if otherwise,*

iv) if $d \in A \setminus \{f_0^{-1}(f_1(0))\}$ *is an endpoint of* $I_{i_1 i_2}$ *then it is a discontinuity of* $\mathcal{R}$ *and so, it is in the orbit of* 0 *for* $\mathrm{IFS}(f_0, f_1)$.

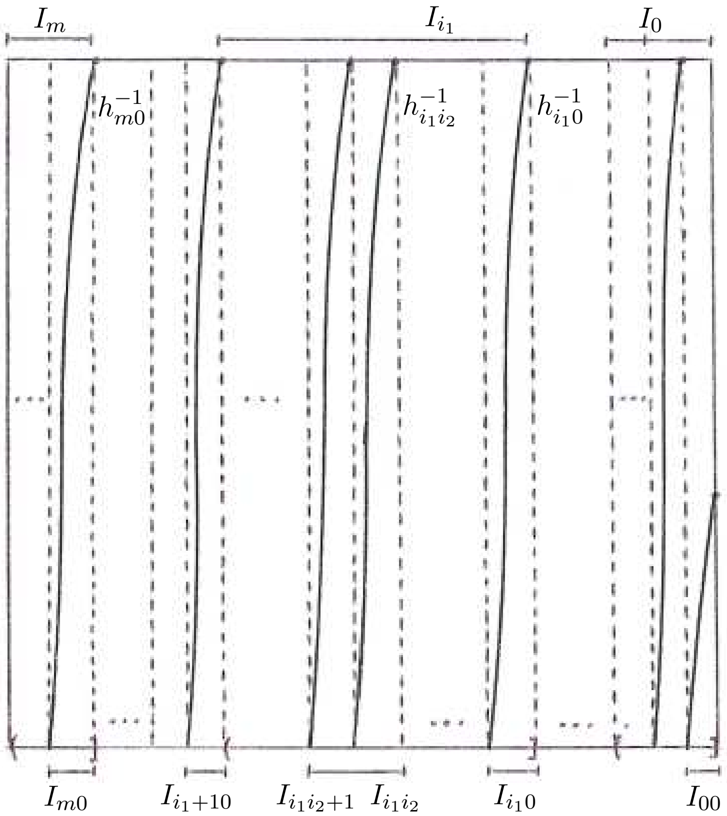

Fig. C: Infinitely many discontinuities of $\mathcal{R}$

Proof. For each $0 < i_1 \leq m$, set

$$d_{i_1 i_2} \stackrel{\text{def}}{=} f_1^{m+1-i_1} \circ f_0^{i_2}(f_1(0)) \in I_{i_1} \quad \text{for all } i_2 \geq 0.$$

Note that $d_{i_1 \ell} < d_{i_1 i}$ if $\ell > i$ and $d_{i_1 i_2} \to f_1^{m-i_1}(f_1(0))$ when i_2 goes to infinity. Moreover, $d_{i_1 0} = f_1^{m+1-i_1}(f_1(0))$ is the right endpoint of the interval I_{i_1}. Hence

$$I_{i_1} = \bigcup_{i_2 \geq 0} I_{i_1 i_2} \quad \text{with} \quad I_{i_1 i_2} \stackrel{\text{def}}{=} (d_{i_1 i_2+1}, d_{i_1 i_2}] = f_1^{m+1-i_1} \circ f_0^{i_2+1}(A).$$

For $i_1 = 0$, we take $c = f_1^{-(m+1)}(f_0^{-1}(f_1(0))) \in (0, f_1(0)]$. As 0 is the unique attractor of f_0 in the interval $[0, f_1(0)]$, there is $j \in \mathbb{N}$ such that $f_0^j(f_1(0)) < c \leq f_0^{j-1}(f_1(0))$. So, we denote

$$d_{00} \stackrel{\text{def}}{=} f_1^{m+1}(c) = f_0^{-1}(f_1(0)) \in I_0 \quad \text{and}$$
$$d_{0 i_2} \stackrel{\text{def}}{=} f_1^{m+1} \circ f_0^{j+i_2}(f_1(0)) \in I_0 \quad \text{for all } i_2 > 0.$$

Note that $d_{0\ell} < d_{0i}$ if $\ell > i$ and $d_{0 i_2} \to f_1^m(f_1(0))$ when i_2 goes to infinity. Hence

$$I_0 = \bigcup_{i_2 \geq 0} I_{0 i_2} \quad \text{with} \quad I_{0 i_2} \stackrel{\text{def}}{=} (d_{0 i_2+1}, d_{0 i_2}].$$

Note that

$$I_{00} = f_1^{m+1} \circ f_0^j((f_1(0), f_0^{-j}(c)]) \quad \text{and} \quad I_{0 i_2} = f_1^{m+1} \circ f_0^{j+i_2}(A) \text{ if } i_2 > 0.$$

Thus, taking the natural numbers $m_{i_1} = m + 1 - i_i$, $m_{0i_2} = j + i_2$ and $m_{i_1 i_2} = i_2 + 1$ if $i_1 > 0$, and writing $h_{i_1 i_2} = f_1^{m_{i_1}} \circ f_0^{m_{i_1 i_2}}$ it follows that

$$\mathcal{R}(x) = f_0^{-m_{i_1 i_2}} \circ f_1^{-m_{i_1}}(x) = h_{i_1 i_1}^{-1}(x) \quad \text{if } x \in I_{i_1 i_2},$$
$$I_{00} = h_{00}((f_1(0), f_0^{-j}(c)]) \quad \text{and} \quad I_{i_1 i_2} = h_{i_1 i_2}(A) \text{ if otherwise.}$$

This concludes the items (ii) and (iii) in the lemma. The items (i) and (iv) are followed from the construction of the intervals $I_{i_1 i_2}$. □

Estimation of the derivative for the return map

Let f be a C^1-map of a compact interval I such that $Df(x) \neq 0$ for all $x \in I$. The non-negative number

$$\mathrm{Dist}(f, I) = \sup_{x,y \in I} \log \left| \frac{Df(x)}{Df(y)} \right|$$

is called *distortion constant* of f in I. Note that $\mathrm{Dist}(f^{-1}, I) = \mathrm{Dist}(f, f^{-1}(I))$.

The main result in this step is the following estimate of the derivative for the return map:

Proposition 3.6. *Let $C > 0$ be the largest distortion constant of f_0^{-1} and f_1^{-1} in $[0, f_0 f_1(0)]$ and $[f_1(0), 1]$ respectively. Consider $\varepsilon > 0$ such that $|Df_0(x) - 1| < \varepsilon$ for all $x \in (0, 1)$. Then*

$$\mathcal{R}'(x) \geq \varepsilon^{-1} e^{-2C} \qquad \text{if} \quad x \in \bigcup_{i_1=0}^{m} \bigcup_{i_2=1}^{\infty} I_{i_1 i_2},$$
$$\mathcal{R}'(x) \geq \frac{1}{2}(1-\varepsilon)\varepsilon^{-1} e^{-4C} \qquad \text{if} \quad x \in I_{00}.$$

In order to estimate the derivative of the map $h_{i_1 i_2}^{-1}$ on the interval $I_{i_1 i_2}$ we would need a bounded distortion estimate. The following standard lemma gives some condition to obtain bounded distortion for the iterates of a map f. Here, we denote by $|J|$ the length of the any interval J.

Lemma 3.7. *Let f be a C^1-map of a compact interval $I \subset [0,1]$ such that $Df(x) \neq 0$ for all $x \in I$ and the map $x \in I \mapsto \log |Df(x)| \in \mathbb{R}$ has Lipschitz constant C. Then*

$$\mathrm{Dist}(f^n, I) \leq C \sum_{i=0}^{n-1} |f^i(I)|.$$

In particular, if $I, f(I), \ldots, f^{n-1}(I)$ are disjoints intervals in $[0,1]$, then $\mathrm{Dist}(f^n, I) \leq C$ and so for every pair of intervals J and L contained in I

$$\frac{|J|}{|L|} e^{-C} \leq \frac{|f^k(J)|}{|f^k(L)|} \leq e^C \frac{|J|}{|L|} \qquad \text{for all } 0 \leq k \leq n.$$

We omit here the proof of this general lemma since it is similar to the proof we will give to obtain the bounded distortion estimate of $h_{i_1 i_2}^{-1}$ in $I_{i_1 i_2}$ (see Lemma 3.9). Before that, it is necessary to show the disjointness of some intervals:

Lemma 3.8 (Disjointness)**.** *Let $i_1 i_2$ be a fixed multi-index. Then*

$$U_\ell \stackrel{\text{def}}{=} f_1^{-\ell}(I_{i_1 i_2}) \quad \text{for } 0 \le \ell < m_{i_1} \text{ and}$$
$$U_{i_1,\ell} \stackrel{\text{def}}{=} f_0^{-\ell} \circ f_1^{-m_{i_1}}(I_{i_1 i_2}) \quad \text{for } 0 \le \ell < m_{i_1 i_2}$$

are right-closed pairwise disjoint intervals in $[0,1]$.

Proof. Note that $I_{i_1 i_2} \subset I_{i_1}$ where I_{i_1} is contained in a fundamental domain of f_1. Thus U_ℓ are pairwise disjoint intervals. Also, from Lemma 3.5, it follows

$$f_0^{-m_{i_1 i_2}} \circ f_1^{-m_{i_1}}(I_{i_1 i_2}) = h_{i_1 i_2}^{-1}(I_{i_1 i_2}) \subset A.$$

Hence $f_1^{-m_{i_1}}(I_{i_1 i_2}) \subset f_0^{m_{i_1 i_2}}(A)$. Since A is a fundamental domain of f_0 then $f_1^{-m_{i_1}}(I_{i_1 i_2})$ is also contained in a fundamental domain of f_0 and thus $U_{i_1,\ell}$ are pairwise disjoint intervals. On the other hand, since m_{i_1} and $m_{i_1 i_2}$ are, respectively, the first time at which the points of I_{i_1} left A by iterations of f_1^{-1} and the first time at which the points of $f_1^{-m_{i_1}}(I_{i_1 i_2})$ come back to A by iterations of f_0^{-1} then

$$f_1^{-\ell}(I_{i_1 i_1}) \subset A \quad \text{for } 0 \le \ell < m_{i_1} \text{ and}$$
$$f_0^{-\ell} \circ f_1^{-m_{i_1}}(I_{i_1 i_2}) \subset (0, f_1(0)] \quad \text{for } 0 \le \ell < m_{i_1 i_2}.$$

Therefore U_ℓ for $0 \le \ell < m_{i_1}$ and $U_{i_1,\ell}$ for $0 \le \ell < m_{i_1 i_2}$ are pairwise disjoint intervals. □

From now on, C_0 and C_1 denotes the distortion constant of f_0^{-1} and f_1^{-1} in $[0, f_1(0)]$ and $[f_1(0), 1]$ respectively. For simplicity we will just say that C_0 and C_1 are the distortion constant of f_0^{-1} and f_1^{-1} respectively omitting the intervals where these are calculated. Also, we will denote by $C > 0$ the largest of these distortion constant. That is, $C = \max\{C_0, C_1\} > 0$.

Lemma 3.9 (Distortion)**.** *Let $C > 0$ be the largest distortion constant of f_0^{-1} and f_1^{-1}. Then*

$$\mathrm{Dist}(h_{i_1 i_2}^{-1}, \overline{I_{i_1 i_2}}) \le C \quad \text{and} \quad \mathrm{Dist}(h_{i_1 i_2+1}^{-1}, \overline{I_{i_1 i_2}}) \le 2C.$$

Consequently, for every pair of intervals J and L contained in $\overline{I_{i_1 i_2}}$

$$\frac{|J|}{|L|}\, e^{-C} \le \frac{|h_{i_1 i_2}^{-1}(J)|}{|h_{i_1 i_2}^{-1}(L)|} \le e^{C}\, \frac{|J|}{|L|} \quad \text{and} \quad \frac{|J|}{|L|}\, e^{-2C} \le \frac{|h_{i_1 i_2+1}^{-1}(J)|}{|h_{i_1 i_2+1}^{-1}(L)|} \le e^{2C}\, \frac{|J|}{|L|}.$$

Moreover, if $I = I_{i_1 i_2+1} \cup I_{i_1 i_2}$ then

$$\frac{|h_{i_1 i_2+1}^{-1}(I)|}{|I|}\, e^{-3C} \le Dh_{i_1 i_2+1}^{-1}(z) \le e^{3C}\, \frac{|h_{i_1 i_2+1}^{-1}(I)|}{|I|} \quad \text{for all } z \in I.$$

Proof. Recall that $h_{i_1 i_2}^{-1} = f_0^{-m_{i_1 i_2}} \circ f_1^{-m_{i_1}}$. Then $Dh_{i_1 i_2}^{-1}(x) = Df_0^{-m_{i_1 i_2}}(f_1^{-m_{i_1}}(x)) Df_1^{-m_{i_1}}(x)$ and, from chain rule

$$Dh_{i_1 i_2}^{-1}(x) = \prod_{\ell=0}^{m_{i_1 i_2}-1} Df_0^{-1}(f_0^{-\ell} \circ f_1^{-m_{i_1}}(x)) \cdot \prod_{\ell=0}^{m_{i_1}-1} Df_1^{-1}(f_1^{-\ell}(x)).$$

Using the distortion control of f_0 and f_1 we obtain that for every $x, y \in \overline{I_{i_1 i_2}}$

$$
\begin{aligned}
|\log \frac{Dh^{-1}_{i_1 i_2}(x)}{Dh^{-1}_{i_1 i_2}(y)}| &= |\log Dh^{-1}_{i_1 i_2}(x) - \log Dh^{-1}_{i_1 i_2}(y)| \\
&\leq C \sum_{\ell=0}^{m_{i_1 i_2}-1} |f_0^{-\ell} \circ f_1^{-m_{i_1}}(x) - f_0^{-\ell} \circ f_1^{-m_{i_1}}(y)| + C \sum_{\ell=0}^{m_{i_1}-1} |f_1^{-\ell}(x) - f_1^{-\ell}(y)| \\
&\leq C \big(\sum_{\ell=0}^{m_{i_1 i_2}-1} |U_{i_1,\ell}| + \sum_{\ell=0}^{m_{i_1}-1} |U_\ell| \big).
\end{aligned}
$$

Similarly, since $m_{i_1 i_2+1} = m_{i_1 i_2} + 1$, denoting $U_{i_1, m_{i_1 i_2}} = f_0^{-m_{i_1 i_2}} \circ f_1^{-m_{i_1}}(I_{i_1 i_2})$,

$$
\begin{aligned}
|\log \frac{Dh^{-1}_{i_1 i_2+1}(x)}{Dh^{-1}_{i_1 i_2+1}(y)}| &\leq C \sum_{\ell=0}^{m_{i_1 i_2+1}-1} |U_{i_1,\ell}| + C \sum_{\ell=0}^{m_{i_1}-1} |U_\ell| \\
&\leq C \big(\sum_{\ell=0}^{m_{i_1 i_2}-1} |U_{i_1,\ell}| + \sum_{\ell=0}^{m_{i_1}-1} |U_\ell| \big) + C\, |U_{i_1, m_{i_1 i_2}}|.
\end{aligned}
$$

Note that $U_{i_1, m_{i_1 i_2}} = h^{-1}_{i_1 i_2}(I_{i_1 i_2}) \subset A$. Finally, the disjointness of U_ℓ, $0 \leq \ell < m_{i_1}$ and $U_{i_1,\ell}$, $0 \leq \ell < m_{i_1 i_2}$ showed in Lemma 3.8 implies that

$$
|\log \frac{Dh^{-1}_{i_1 i_2}(x)}{Dh^{-1}_{i_1 i_2}(y)}| \leq C \quad \text{and} \quad |\log \frac{Dh^{-1}_{i_1 i_2+1}(x)}{Dh^{-1}_{i_1 i_2+1}(y)}| \leq 2C. \tag{3.1}
$$

From here it follows the first part of lemma. To conclude the lema, we will show the last inequality. Let $d_{i_1 i_2} = f_0^{m_1} \circ f_1^{m_{i_1 i_2}}(f_1(0))$ be the right endpoint of $I_{i_1 i_2+1}$ and the left endpoint of $I_{i_1 i_2}$. Then for all $x, y \in I = I_{i_1 i_2+1} \cup I_{i_1 i_2}$

$$
\frac{Dh^{-1}_{i_1 i_2+1}(x)}{Dh^{-1}_{i_1 i_2+1}(y)} = \frac{Dh^{-1}_{i_1 i_2+1}(x)}{Dh^{-1}_{i_1 i_2+1}(d_{i_1 i_2+1})} \cdot \frac{Dh^{-1}_{i_1 i_2+1}(d_{i_1 i_2+1})}{Dh^{-1}_{i_1 i_2+1}(y)}.
$$

From this, and using the estimates (3.1) it follows that

$$
e^{-3C} \leq \frac{Dh^{-1}_{i_1 i_2+1}(x)}{Dh^{-1}_{i_1 i_2+1}(y)} \leq e^{3C} \quad \text{for all } x, y \in I. \tag{3.2}
$$

Now, let J and L be a pair of intervals in I. By Mean Value Theorem, there is $x \in J$ and $y \in L$ such that $|h^{-1}_{i_1 i_2+1}(J)|/|h^{-1}_{i_1 i_2+1}(L)| = Dh^{-1}_{i_1 i_2+1}(x) \cdot |J|/(Dh^{-1}_{i_1 i_2+1}(y) \cdot |L|)$. From this, and using the inequality (3.2) it follows that

$$
\frac{|J|}{|L|}\, e^{-3C} \leq \frac{|h^{-1}_{i_1 i_2+1}(J)|}{|h^{-1}_{i_1 i_2+1}(L)|} \leq e^{3C}\, \frac{|J|}{|L|}. \tag{3.3}
$$

Finally, given $z \in I$, we consider any interval $J \subset I$ such that $z \in J$. Then from (3.3) for the pair of interval J and $L = I$ we obtain that

$$
\frac{|h^{-1}_{i_1 i_2+1}(I)|}{|I|}\, e^{-3C} \leq \frac{|h^{-1}_{i_1 i_2+1}(J)|}{|J|} \leq e^{3C}\, \frac{|h^{-1}_{i_1 i_2+1}(I)|}{|I|}.
$$

Taking the length of J goes to zero we follow the desire inequality and conclude the lemma. $\square$

We will need the following lemma to obtain an estimate of the derivative of $h^{-1}_{i_1 i_2}$ on the interval $I_{i_1 i_1}$. This lemma consists of a lower bounded distortion estimate between the length of I_{i_1} and $I_{i_1 i_2}$.

Lemma 3.10 (Compared intervals)**.** *Let $C_1 > 0$ be the distortion constant of f_1^{-1}. Consider $\varepsilon > 0$ such that $|Df_0(x) - 1| < \varepsilon$ for all $x \in (0,1)$. Then $|I_{i_1}|/|I_{i_1 i_2}| > \varepsilon^{-1} e^{-C_1}$ for all multi-index $i_1 i_2$.*

Proof. Recall that I_{i_1} is contained in a fundamental domain of f_1. Therefore $f_1^{-i}(I_{i_1})$ for $i \geq 0$ are disjoints intervals in $[0,1]$. From Lemma 3.7 it follows that

$$|I_{i_1}|/|I_{i_1 i_2}| \geq e^{-C_1}\, |f_1^{-m_{i_1}}(I_{i_1})|/|f_1^{-m_{i_1}}(I_{i_1 i_2})|.$$

Note that $f_1^{-m_{i_1}}(I_{i_1 k}) = f_0^{m_{i_1 k}}(A)$ for all $i_1 k \neq 00$. By construction in Lemma 3.5,

$$I_{00} = (d_{01}, d_{00}] = (f_1^{m_0} f_0^{m_{01}} f_1(0),\ f_0^{-1} f_1(0)]$$

and thus, $f_1^{-m_0}(I_{00}) = (f_0^{m_{01}} f_1(0),\ f_1^{-m_0} f_0^{-1} f_1(0)]$. Then, since $m_{i_1 k+1} = m_{i_1 k} + 1$ and $I_{i_1} = \cup I_{i_1 k}$ it follows that

$$f_1^{-m_{i_1}}(I_{i_1}) = \bigcup_{k=0}^{\infty} f_1^{-m_{i_1}}(I_{i_1 k}) = \bigcup_{k=1}^{\infty} f_0^{m_{i_1 k}}(A) \cup f_1^{-m_{i_1}}(I_{i_1 0}) = (0,\ f_0^{m_{i_1 0}} f_0^{-1} f_1(0)].$$

Therefore, since 0 is an attractor of f_0 it follows that $f_1^{-m_{i_1}}(I_{i_1}) \supset (0, f_0^{m_{i_1 k}}(f_0^{-1}(f_1(0)))]$ for all $k \geq 0$. From this,

$$\frac{|f_1^{-m_{i_1}}(I_{i_1})|}{|f_1^{-m_{i_1}}(I_{i_1 i_2})|} \geq \frac{f_0^{m_{i_1 i_2}-1} f_1(0)}{|f_0^{m_{i_1 i_2}} f_1(0) - f_0^{m_{i_1 i_2}-1} f_1(0)|}.$$

Since 0 is a fixed point of f_0 then from Mean Value Theorem we can write $f_0^{m_{i_1 i_2}}(f_1(0)) = Df_0(\xi) \cdot f_0^{m_{i_1 i_2}-1}(f_1(0))$ for some ξ. Now, from the assumption we get

$$\frac{|I_{i_1}|}{|I_{i_1 i_2}|} \geq e^{-C_1} \frac{f_0^{m_{i_1 i_2}-1} f_1(0)}{f_0^{m_{i_1 i_2}-1}(f_1(0)) \cdot |Df_0(\xi) - 1|} > \varepsilon^{-1} e^{-C_1}.$$

Therefore, the proof of the lemma is completed. □

Now, we are ready to obtain the estimation desired for the return map in Proposition 3.6.

Proof of Proposition 3.6. Let $x \in A$. Without loss of generality, we assume that x is not a discontinuity point of $\mathcal{R}$. If x is a discontinuity, the first return map only has lateral derivative on this point. A similar argument allows to estimate a bound for its lateral derivative. Hence, since x is not a discontinuity, we find $\eta_0 > 0$ and a unique interval $I_{i_1 i_2}$ such that for every $0 < \eta \leq \eta_0$, the interval $J = (x - \eta, x + \eta)$ satisfies that $J \subset I_{i_1 i_2}$. Notice that $\mathcal{R}(J) = h^{-1}_{i_1 i_2}(J)$.

Suppose that $i_1 i_2 \neq 00$. Then $h^{-1}_{i_1 i_2}(I_{i_1 i_2}) = A \supset I_{i_1} \supset I_{i_1 i_2}$. From Lemma 3.9 we have that

$$|\mathcal{R}(J)| \geq e^{-C}\, \frac{|h^{-1}_{i_1 i_2}(I_{i_1 i_2})|}{|I_{i_1 i_2}|}\, |J| \geq e^{-C}\, \frac{|I_{i_1}|}{|I_{i_1 i_2}|}\, |J|.$$

By Lemma 3.10 and since $C > 0$ is the largest distortion constant of f_0^{-1} and f_1^{-1} then it holds $|\mathcal{R}(J)| > \varepsilon^{-1} e^{-2C}\, |J|$. If $\eta \to 0$ (and so J goes to x) then $\mathcal{R}'(x) = Dh^{-1}_{i_1 i_2}(x) \geq \varepsilon^{-1} e^{-2C}$ for all $x \in I_{i_1 i_2}$ with $i_1 i_2 \neq 00$.

For the case $i_1i_2 = 00$, recalling that $m_{i_1i_2+1} = m_{i_1i_2} + 1$ it follows that

$$h_{00}^{-1} = f_0^{-m_{00}} \circ f_1^{-m_0} = f_0 \circ f_0^{-m_{01}} \circ f_1^{-m_0} = f_0 \circ h_{01}^{-1}.$$

Then, by the Mean Value Theorem, there are $\xi \in h_{01}^{-1}(J)$ and $\zeta \in J$ such that

$$|\mathcal{R}(J)| = |Df_0(\xi)||Dh_{01}^{-1}(\zeta)||J| > (1-\varepsilon)|Dh_{01}^{-1}(\zeta)||J|.$$

In the previous case, we estimate the derivative of h_{01}^{-1} on the interval I_{01}. As $\zeta \in J \subset I_{00}$, we need again to estimate Dh_{01}^{-1} but now on the interval I_{00}. To do this, we will use the estimate of Dh_{01}^{-1} on the $I = I_{01} \cup I_{00}$ obtained in Lemma 3.9. That is, $Dh_{01}^{-1}(\zeta) \geq e^{-3C}|h_{01}^{-1}(I)|/|I|$. As $h_{01}^{-1}(I) \supset A \supset I_0$ then $|h_{01}^{-1}(I)| \geq |I_0|$. Then, by Lemma 3.10, noting again that $C > 0$ is the largest distortion constant of f_0^{-1} and f_1^{-1}, we see that

$$\frac{|h_{01}^{-1}(I)|}{|I|} \geq \frac{|I_0|}{|I|} = \Big(\frac{|I_{01}|}{|I_0|} + \frac{|I_{00}|}{|I_0|}\Big)^{-1} > \frac{1}{2}\,\varepsilon^{-1}e^{-C}.$$

Finally, $|\mathcal{R}(J)| > \frac{1}{2}(1-\varepsilon)\varepsilon^{-1}e^{-4C}\,|J|$. Therefore, if $\eta \to 0$ (and so J goes to x) then it holds that $\mathcal{R}'(x) = Dh_{00}^{-1}(x) \geq (1-\varepsilon)\varepsilon^{-1}e^{-4C}/2$ for all $x \in I_{00}$ and we conclude the proposition. □

End of the proof of Duminy's Lemma for IFS(Φ)

Consider $\Phi = \{f_0, f_1\} \subset \mathrm{Diff}^2_+(\mathbb{R})$ and let K_Φ^{**} be an $**$-interval for $\mathrm{IFS}(\Phi)$ with $** \in \{ss, su\}$. Then, we are now ready to prove Duminy's Lemma for $\mathrm{IFS}(\Phi)$, that is, the first part of Theorem D. Namely, we will prove that there exists $\varepsilon > 0$ such that if $d_{C^2}(f_i|_{K_\Phi^{**}}, \mathrm{id}) < \varepsilon$ for $i = 0, 1$ then

$$K_\Phi^{**} \subset \overline{\mathrm{Per}(\mathrm{IFS}(\Phi))} \quad \text{and} \quad K_\Phi^{**} \subset \overline{\mathrm{Orb}_\Phi(x)} \quad \text{for all } x \in K_\Phi^{**}. \tag{3.4}$$

Note that without losing of generality, and for simplicity we have scaled the $**$-interval assuming that $K_\Phi^{**} = [0,1]$, $f_0(0) = 0$ and $f_0 < \mathrm{id}$ and $f_1 > \mathrm{id}$ in $(0,1)$. The first simplification to prove the minimality property in (3.4), it is note that it is enough to show that the orbit of 0 for $\mathrm{IFS}(\Phi)$ is dense in the interval $[0,1]$.

Lemma 3.11. *Suppose that* $[0,1] \subset \overline{\mathrm{Orb}_\Phi(0)}$. *Then* $[0,1] \subset \overline{\mathrm{Orb}_\Phi(x)}$ *for all* $x \in [0,1]$.

Proof. Consider $x \in [0,1]$ and V any open set in $[0,1]$. From the density of the $\mathcal{G}_\Phi$-orbit of 0, there is $h \in \mathrm{IFS}(\Phi)$ such that $h(0) \in V$. Since h is a continuous map and 0 is a global attractor point of f_0 in $[0,1]$, then exists $\ell \in \mathbb{N}$ such that $f_0^\ell(x)$ is close enough of 0 such that $h \circ f_0^\ell(x) \in V$. Hence, the orbit of x for $\mathrm{IFS}(\Phi)$ is dense in $[0,1]$ and we conclude the proof of the lemma. □

Since we are assuming that f_0 and f_1 are C^2-invertible maps (close to the identity) then the distortion constants of f_0^{-1} and f_1^{-1} can be written

$$C_0 = \max_{x \in [0, f_1(0)]} \left|\frac{D^2 f_0^{-1}(x)}{Df_0^{-1}(x)}\right| > 0, \quad \text{and} \quad C_1 = \max_{x \in [f_1(0), 1]} \left|\frac{D^2 f_1^{-1}(x)}{Df_1^{-1}(x)}\right| > 0.$$

Note that $|D^2 f_i^{-1}(x)|/|Df_i^{-1}(x)| = |D^2 f_i(f_i^{-1}(x)))|/|Df_i(f_i^{-1}(x))|$ and so C_0 and C_1 are also the distortion constant of f_0 and f_1 in $[0, f_0^{-1}f_1(0)]$ and $[0,1]$ respectively. Let $C = \max\{C_0, C_1\} > 0$.

Remark 3.12. *If $\varepsilon > 0$ is such that f_0 and f_1 are ε-close to the identity in the C^2-topology then $0 < C < \varepsilon(1-\varepsilon)^{-1}$. Thus, for $0 < \varepsilon \leq 0.175$, from Proposition 3.6 it follows that*

$$\mathcal{R}'(x) \geq \frac{1}{2}(1-\varepsilon)\varepsilon^{-1}e^{-4\varepsilon(1-\varepsilon)^{-1}} > 1 \quad \textit{for all } x \in A.$$

That is, $\mathcal{R}$ is an expanding return map over the fundamental domain A.

Now, we will prove (3.4). That is, the Duminy's Lemma for the IFS generated by $\Phi = \{f_0, f_1\}$ (see Theorem D). Later, we will prove the robustness of these assertions under C^1-perturbations as Theorem D states.

Proof of Duminy's Lemma for IFS(Φ). Recall that the first return map $\mathcal{R} : A \to A$ can be extended to the interval $[0, 1]$. In particular, this implies that for any interval $I \subset [0, 1]$, there exists $h \in \mathrm{IFS}(\Phi)$ such that $h^{-1}(I) \cap A \neq \emptyset$. From Remark 3.12, for every $0 < \varepsilon \leq 0.175$, the return map $\mathcal{R}$ is expanding map in A. Thus, there is $n \in \mathbb{N}$ such that $\mathcal{R}^n(h^{-1}(I) \cap A)$ contains some discontinuity of $\mathcal{R}$. Recall that the discontinuities $d \in A$ are points in the orbit of 0 for IFS(Φ), i.e., $d = f_1^m \circ f_0^{k-1}(f_1(0))$ for some $m \geq 1$ and $k \geq 1$. Then, one has $h \circ f_1^m \circ f_0^{k-1} \circ f_1(0) \in I$. Therefore, the orbit of 0 for IFS(Φ) is dense. Finally, from Lemma 3.11 we get

$$[0, 1] \subset \overline{\mathrm{Orb}_\Phi(x)} \quad \text{for all } x \in [0, 1].$$

Now, given $x \in [0, 1]$ we will show that $x \in \overline{\mathrm{Per}(\mathrm{IFS}(\Phi))}$. We will use that 0 is a global attractor for f_0 whose orbit for IFS(Φ) is dense in $[0, 1]$. So, let I be any open interval such that $x \in I$. From the density of the orbit of 0, there is $h \in \mathrm{IFS}(\Phi)$ such that $h(0) \in I$. Since h is a continuous map there is $\delta > 0$ such that $h((-\delta, \delta)) \subset I$. Using now that 0 is a global attractor point of f_0 then there is $\ell \geq 0$ such that $f_0^\ell(I) \subset (-\delta, \delta)$. Then $h \circ f_0^\ell(I) \subset I$. By Brouwer's Fixed Point Theorem, $h \circ f_0^\ell$ has a fixed point in I and thus $I \cap \mathrm{Per}(\mathrm{IFS}(\Phi)) \neq \emptyset$. This implies the $[0, 1] \subset \overline{\mathrm{Per}(\mathrm{IFS}(\Phi))}$ and the proof of the theorem for IFS(Φ) is concluded. □

Notice that the density of periodic points is a consequence of the transitivity property of the global attractor of the map f_0 for IFS(Φ). Therefore, this density property is C^1-robust if the transitivity property of the global attractor of f_0 for IFS(Φ) is C^1-robust. The next result yields the robustness of $K_\Phi^{**} \subset \overline{\mathrm{Orb}_\Phi(0)}$ under C^1-perturbations of f_0 and f_1.

Theorem 3.13. *Let $\Phi = \{\phi_1, \ldots, \phi_k\} \subset \mathrm{Diff}^1(M)$. Suppose that there exist $n \in \mathbb{N}$, non-empty bounded open sets B, B_i and maps $h_i \in \mathrm{IFS}(\Phi)$ for $i = 1, \ldots, n$ such that*

i) covering property*: $\overline{B} \subset B_1 \cup \cdots \cup B_n$ with $h_i^{-1}(\overline{B_i}) \subset B$ and the restriction of h_i^{-1} to $\overline{B_i}$ is an expanding map for $i = 1, \ldots, n$.*

ii) density property*: there are a point $p \in M$ and $\ell \in \{1, \ldots, k\}$ such that $B \subset \overline{\mathrm{Orb}_\Phi(p)}$ and p is a hyperbolic fixed point of ϕ_ℓ.*

Then, there is C^1-neighborhood $\mathcal{U}_i$ of ϕ_i such that for every IFS(Ψ) with $\Psi = \{\psi_1, \ldots, \psi_k\}$ and $\psi_i \in \mathcal{U}_i$ for $i = 1, \ldots, k$ it holds $B \subset \overline{\mathrm{Orb}_\Psi(p_\Psi)}$, where p_Ψ is the continuation point of p for ψ_ℓ.

Let $\mathrm{IFS}(\phi_1,\dots,\phi_k)$ be an iterated function system satisfying the covering property (i). Since h_i^{-1} restricted to $\overline{B_i}$ is an expanding map then there exists $\kappa>1$ such that

$$\kappa\,\|x-y\| < \|h_i^{-1}(x)-h_i^{-1}(y)\|$$

for all $x,y\in\overline{B_i}$ and for each $i=1,\dots,n$. Furthermore, this inequality is persistent:

Remark 3.14 (Choice of perturbation I)**.** *There are C^1-neighborhoods $\mathcal{U}_i^1$ of ϕ_i such that for every $\mathrm{IFS}(\psi_1,\dots,\psi_k)$ with $\psi_i\in\mathcal{U}_i^1$ for $i=1,\dots,k$ it holds that there are maps $\tilde h_i\in\mathrm{IFS}(\psi_1,\dots,\psi_k)$ for $i=1,\dots,n$ such that $\tilde h_i^{-1}(\overline{B_i})\subset B$ and $\tilde h_i^{-1}|_{\overline{B}_i}$ is expanding with expansion at least $\kappa>1$.*

Recall that given an open covering of a compact set X of a metric space there is a constant $L>0$, called Lebesgue number of the covering, such that every subset of X with diameter less than L is contained in some open set of the covering. Let L be the Lebesgue number of the open cover in the assumption of covering property (i) and suppose that there is a hyperbolic fixed point $p\in M$ of some ϕ_ℓ such that $B\subset\overline{\mathrm{Orb}_\Phi(p)}$. Since the $\mathcal{G}_\Phi$-orbit of p is dense in B, there exists $m\in\mathbb{N}$ and maps $g_i\in\mathrm{IFS}(\phi_1,\dots,\phi_k)$ for $i=1,\dots,m$ such that the set of point $\{g_i(p):i=1,\dots,m\}$ is $L/3$-dense in B. That is, for every open ball V in B of radius greater than $L/3$ there is $i\in\{1,\dots,m\}$ such that $g_i(p)\in V$. Thus, we obtain the following remak:

Remark 3.15 (Choice of perturbation II)**.** *There are C^1-neighborhoods $\mathcal{U}_i^2$ of ϕ_i such that for every $\mathrm{IFS}(\psi_1,\dots,\psi_k)$ with $\psi_i\in\mathcal{U}_i^2$ for $i=1,\dots,k$ it holds that there are maps $\tilde g_i\in\mathrm{IFS}(\psi_1,\dots,\psi_k)$ for $i=1,\dots,m$ such that the set of point $\{\tilde g_i(p_\Psi):i=1,\dots,m\}$ is $L/3$-dense in B where p_Ψ is the continuation point of p for ψ_ℓ.*

In order to prove the above theorem we will show the following lemma:

Lemma 3.16. *Consider $0<r<L/2$ and $x\in B$ such that $B(x,r)\subset B$. Then, for every $\mathrm{IFS}(\psi_1,\dots,\psi_k)$ with $\psi_i\in\mathcal{U}_i^1$ for $i=1,\dots,k$ there is $\tilde h\in\mathrm{IFS}(\psi_1,\dots,\psi_k)$ such that*

$$\tilde h^{-1}\big(B(x,r)\big)\subset B, \quad \textit{and} \quad \operatorname{diam}\big(\tilde h^{-1}\big(B(x,r)\big)\big)>L.$$

Proof. Let $\mathrm{IFS}(\psi_1,\dots,\psi_k)$ be an iterated function system with $\psi_i\in\mathcal{U}_i^1$. Since $r<L/2$ and L is the Lebesgue number of the open cover then $B(x,r)\subset B_{i_1}$ for some $i_1\in\{1,\dots,k\}$. According to Remark 3.14, it follows that there is $\tilde h_{i_1}\in\mathrm{IFS}(\psi_1,\dots,\psi_k)$ such that $\tilde h_{i_1}^{-1}(B(x,r))\subset B$ and $\operatorname{diam}\big(\tilde h_{i_1}^{-1}(B(x,r))\big)\geq 2\kappa r$. If $\kappa r\leq L/2$, then we find $i_2\in\{1,\dots,k\}$ such that $\tilde h_{i_1}^{-1}(B(x,r))\subset B_{i_2}$ and again from Remark 3.14, there exists $\tilde h_{i_2}\in\mathrm{IFS}(\Psi)$ such that

$$\tilde h_{i_2}^{-1}\circ\tilde h_{i_1}^{-1}(B(x,r))\subset B \quad \text{and} \quad \operatorname{diam}\big(\tilde h_{i_2}^{-1}\circ\tilde h_{i_1}^{-1}(B(x,r))\big)\geq\kappa\operatorname{diam}\big(\tilde h_{i_1}^{-1}(B(x,r))\big)\geq 2\kappa^2 r.$$

Since $\kappa>1$, there exists $m\in\mathbb{N}$ such that $\kappa^m r>L/2$. Thus, by repeating the above procedure m times, we find $\tilde h=\tilde h_{i_1}\circ\cdots\circ\tilde h_{i_m}\in\mathrm{IFS}(\psi_1,\dots,\psi_k)$ such that $\tilde h^{-1}(B(x,r))\subset B$, and $\operatorname{diam}(\tilde h^{-1}(B(x,r)))\geq 2\kappa^m r>L$. This concludes the proof of the lemma. □

Proof of Theorem 3.13. We will take the C^1-neighborhoods $\mathcal{U}_i$ of ϕ_i given by $\mathcal{U}_i=\mathcal{U}_i^1\cap\mathcal{U}_i^2$ for all $i=1,\dots,k$. Consider $x\in B$ and let $r>0$ be any positive number such that $B(x,r)\subset B$. Let $\mathrm{IFS}(\Psi)$ be an iterated function system with $\Psi=\{\psi_1,\dots,\psi_k\}$ and $\psi_i\in\mathcal{U}_i$ for $i=1,\dots,k$. To prove the theorem we need to show that there is $\psi\in\mathrm{IFS}(\Psi)$ such that $\psi(p_\Psi)\in B(x,r)$.

By Remark 3.15, if $r > L/3$ then there exists $\tilde{g}_i \in \mathrm{IFS}(\Psi)$ such that $\tilde{g}_i(p_\Psi) \in B(x,r)$. Therefore, in this case, it is enough to take $\psi = \tilde{g}_i$. If $0 < r \leq L/3 < L/2$ then from Lemma 3.16 there is $\tilde{h} \in \mathrm{IFS}(\psi_1, \ldots, \psi_k)$ such that $\tilde{h}^{-1}(B(x,r)) \subset B$ and $\mathrm{diam}(\tilde{h}^{-1}(B(x,r))) > L$. This implies that there are $z \in \tilde{h}^{-1}(B(x,r))$ and a $\rho > L/3$ such that $B(z,\rho) \subset \tilde{h}^{-1}(B(x,r)) \subset B$. Again from Remark 3.15 there exists $\tilde{g}_i \in \mathrm{IFS}(\Psi)$ such that $\tilde{g}_i(p_\Psi) \in B(z,\rho)$. Hence, for $\psi = \tilde{h} \circ \tilde{g}_i \in \mathrm{IFS}(\Psi)$ it follows that $\psi(p_\Psi) \in B(x,r)$. Therefore,

$$B \subset \overline{\mathrm{Orb}_\Psi(p_\Psi)} \quad \text{for all IFS}(\Psi)\ C^1\text{-close enough to IFS}(\Phi)$$

and the proof of the theorem is completed. □

In the next subsection we will show that IFS(Φ) satisfies the assumptions in Theorem 3.13 and thus it follows the C^1-robustness assertions in Theorem D.

Robustness of the minimality property and density of periodic points

Consider $\Phi = \{f_0, f_1\} \subset \mathrm{Diff}^2_+(\mathbb{R})$ and let K^{**}_Φ be an $**$-interval for IFS(Φ) with $** \in \{ss, su\}$. If f_0 and f_1 are in the assumptions of Theorem D then we have proved that $K^{**}_\Phi \subset \overline{\mathrm{Orb}_\Phi(x)}$ for all $x \in K^{**}_\Phi$. Note that, at least one of the endpoints of K^{**}_Φ is an global attracting fixed point of either f_0 or f_1. Thus, the $\mathcal{G}_\Phi$-orbit of this endpoint is dense in any open interval contained in K^{**}_Φ. In particular, this implies the density property in Theorem 3.13. In what follows, we will show that also the covering property is satisfied: *there are $n \in \mathbb{N}$, non-empty bounded open intervals B, $B_i \subset K^{**}_\Phi$ and maps $h_i \in \mathrm{IFS}(\Phi)$ such that $\overline{B} \subset B_0 \cup \ldots \cup B_n$ with $h_i^{-1}(\overline{B_i}) \subset B$ and $Dh_i^{-1}(x) > 1$ for all $x \in \overline{B}_i$ and for $i = 0, \ldots, n$.*

Recall that we had show the Duminy's Lemma for IFS(Φ) constructing an expanding first return map $\mathcal{R}$ over a fundamental domain $A = (f_1(0), f_0^{-1}f_1(0)]$ of f_0. As shown in Figure C, this return map has infinite expanding branches or discontinuities. In the following lemma, we will show that generically we can define a new expanding return map $\tilde{\mathcal{R}}$ over A with only a finite number of discontinuities as shown in Figure D.

Lemma 3.17. *Consider f_0, f_1 in $\mathrm{Diff}^1_+(\mathbb{R})$ and let $A = (f_j(b), b]$ be a fundamental domain of f_j for some $j \in \{0,1\}$. Suppose that there are $\varepsilon > 0$, a families of maps $h_i \in \mathrm{IFS}(f_0, f_1)$ and pairwise disjoint right-closed intervals I_i for $i \geq 0$ such that*

(a) $A = I_0 \cup \ldots \cup I_n \cup \ldots$, with $b \in I_0$ and $h_i^{-1}(I_i) \subset A$ for all $i \geq 0$,

(b) $h_0 = h \circ f_j^m$ with $m \geq 1$ and $h \in \mathrm{IFS}(f_0, f_1)$,

(c) $1 - \varepsilon < Df_j(b) < 1 + \varepsilon$ and $Dh_i^{-1}(x) > 1$ for all $x \in I_i$ and $i \geq 0$ with $Dh_0^{-1}(b) > 1 + \varepsilon$.

Then there exists an interval $A^* = (a^*, b^*]$, maps $g_i \in \mathrm{IFS}(f_0, f_1)$ and close intervals $J_i = [t_{i+1}, t_i]$ for $i = 0, \ldots, M$ with $a^* = t_{M+1} < t_M < \ldots < t_1 < t_0 = b^*$ such that

i) $A \subset \overline{A^*} = J_0 \cup \ldots \cup J_M$,

ii) $g_i^{-1}(J_i) \subset A^*$ for all $i = 0, \ldots, M$ with $g_0^{-1}(b^*)$, $g_M^{-1}(a^*) \in \mathrm{int}\, A^*$, and

iii) $Dg_i^{-1}(x) > 1$ for all $x \in J_i$ and every $i = 0, \ldots, M$.

Moreover, with the aditional generic condiction $h_0^{-1}(b) \in \mathrm{int}\, A$ it follows $A^* = A$.

Proof. Let $d \in A$ be an accumulation point of endpoints of intervals I_i. Hence, since A is union of pairwise right-closed intervals I_i, then there exists $i > 0$ such that d is the right-endpoint of I_i. If $h_i^{-1}(d) \in \operatorname{int} A$ then, as h_i^{-1} is an expanding map, there is a closed interval $\tilde{J}$ such that $d \in \tilde{J}$, $h_i^{-1}(\tilde{J}) \subset A$ and $Dh_i^{-1}(x) > 1$ for all $x \in \tilde{J}$. So, we replace an infinite number of expanding by taking $g^{-1} = h_i^{-1}$ on $J = \overline{I_i} \cup \tilde{J}$. Let us suppose that $h_i^{-1}(d) \notin \operatorname{int} A$, that is, $h_i^{-1}(d) = b$. Set $g = h_i \circ h_0 \in \mathrm{IFS}(f_0, f_1)$. Hence $g^{-1}(d) = h_0^{-1}(b) \in A$ and

$$Dg^{-1}(d) = Dh_0^{-1}(b)Dh_i^{-1}(d) > (1 + \varepsilon) > 1.$$

Firstly, we assume the generic condition $h_0^{-1}(b) \in \operatorname{int} A$ and we will prove the lemma for $A^* = A$. In this case, by continuity we find a closed interval J such that $d \in \operatorname{int} J$, $g^{-1}(J) \subset A$ and $Dg^{-1}(x) > 1$ for all $x \in J$. Thus, again we replace an infinite number of expanding branches defined on intervals I_i by the expanding branch g on J. This process lets remove all of accumulating points of endpoints of intervals I_i in A. Therefore, now, the only point accumulated by endpoints of these intervals could be $f_j(b)$. Relabeling the intervals if necessary we suppose $f_j(b)$ is the only point in such a condition and we will apply the above argument once more. Let us take $g = f_j \circ h_0 \in \mathrm{IFS}(f_0, f_1)$ then $g^{-1}(f_j(b)) = h_0^{-1}(b) \in \operatorname{int} A$ and from assumption (c)

$$Dg^{-1}(f_j(b)) = Dh_0^{-1}(b)Df_j^{-1}(f_j(b)) > (1 + \varepsilon)(1 + \varepsilon)^{-1} = 1.$$

Hence, we find a closed interval J with $f_j(b) \in J$ such that $g^{-1}(J) \subset A$ and $Dg^{-1}(x) > 1$ for all $x \in J$. So, we obtain that there are $M \in \mathbb{N}$, maps $g_i = h_i \in \mathrm{IFS}(f_0, f_1)$ and close intervals $J_i = \overline{I}_i$ for $i = 0, \ldots, M - 1$ such that $\overline{A} = J_0 \cup \ldots \cup J_M$ and $g_0^{-1}(b)$, $g_M^{-1}(f_j(b)) \in \operatorname{int} A$ with $J_M = J$ and $g_M = g$. Moreover, $g_i^{-1}(J_i) \subset A$ and $Dg_i^{-1}(x) > 1$ for all $x \in J_i$.

Let us now suppose that $h_0^{-1}(b) \notin \operatorname{int} A$. Hence $h_0(b) = b$. Set $h_\infty = f_j \circ h_0 \in \mathrm{IFS}(f_0, f_1)$ and $h_{-1} = h_0^n \circ f_j^{-1}$ for n large enough. From assumption (b), it follows that $h_{-1} \in \mathrm{IFS}(f_0, f_1)$. Moreover, we have that $h_\infty^{-1}(f_j(b)) = h_0^{-1}(b) = b$, $h_{-1}^{-1}(b) = f_j(b)$ and

$$Dh_\infty^{-1}(f_j(b)) = Dh_0^{-1}(b)Df_j^{-1}(f_j(b)) > (1 + \varepsilon)(1 + \varepsilon)^{-1} = 1$$

and

$$Dh_{-1}^{-1}(b) = Df_j(b)Dh_0^{-n}(b) > (1 - \varepsilon)(1 + \varepsilon)^n \geq 1.$$

By continuity we find closed intervals $I_\infty = [a^*, f_j(b)]$ and $I_{-1} = [b, b^*]$ such that $h_\infty^{-1}(a^*)$ and $h_{-1}^{-1}(b^*)$ belong to $\operatorname{int} A$ and where h_∞^{-1} restricted to I_∞ and h_{-1}^{-1} restricted to I_{-1} are both expanding maps. Set $A^* = (a^*, b^*]$. Just as the previous case, we can increase a little bit I_∞ on the right and so replace the infinite number of expanding branches which are accumulated in $f_j(b)$. Therefore, there is $M \in \mathbb{N}$, maps $g_{i+1} = h_i \in \mathrm{IFS}(f_0, f_1)$ and close intervals J_{i+1} for $i = 0, \ldots, M - 1$ such that $\overline{A^*} = J_0 \cup \ldots \cup J_M$ and $g_0^{-1}(b^*)$, $g_M^{-1}(a^*) \in \operatorname{int} A^*$ with $I_\infty \subset J_M$, $J_0 = I_{-1}$ and $g_M = h_\infty$, $g_0 = h_{-1}$. Note that, also $g_i^{-1}(J_i) \subset A^*$ and $Dg_i^{-1}(x) > 1$ for all $x \in J_i$. Therefore, the proof of the lemma is concluded. □

As already mentioned, the first return map $\mathcal{R} : A \to A$ where $A = (f_1(0), f_0^{-1}(f_1(0))] = (f_0(b), b]$ constructed in Lemma 3.5 to prove Duminy's Lemma for $\mathrm{IFS}(\Phi)$ satisfies the above lemma. Therefore, we obtain the following remak:

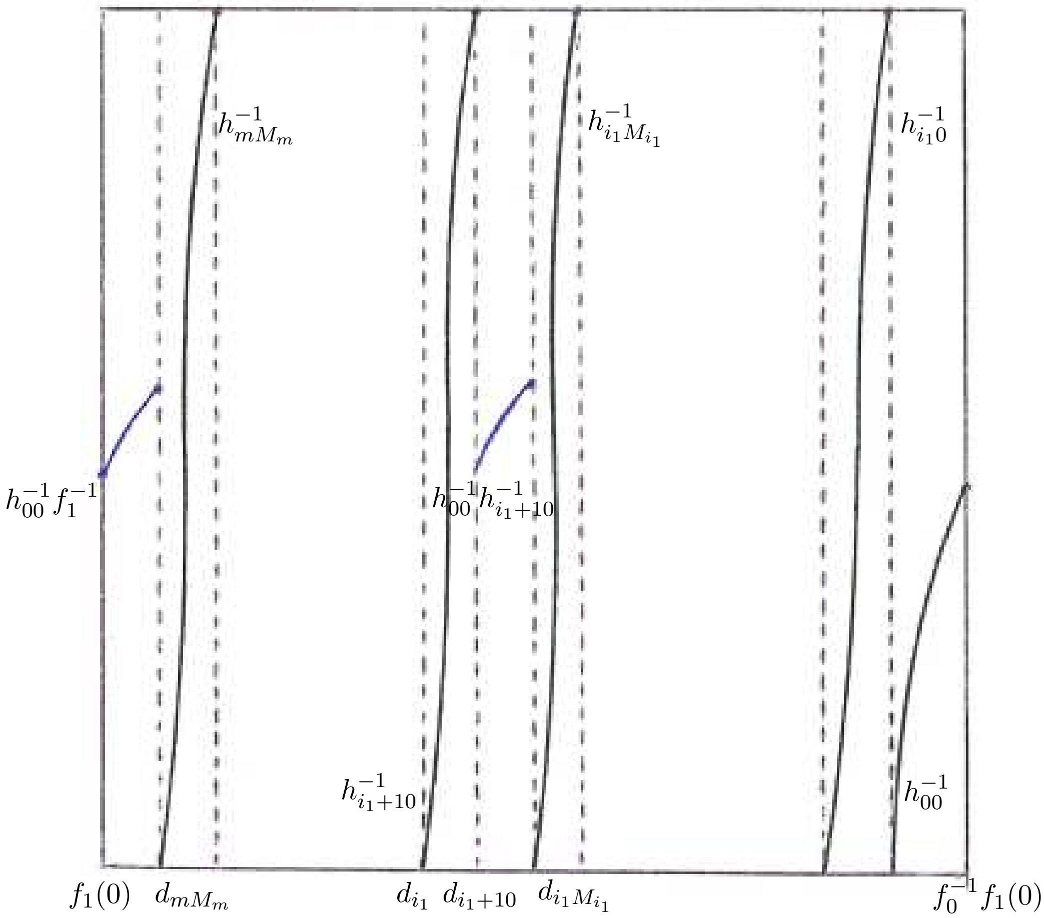

Fig. D: Modified return map over A with a finite number of discontinuities

Remark 3.18. *Let $\mathcal{R}: A \to A$ be an expanding return map for an IFS of two maps f_0 and f_1 with an infinite number of expanding branches (or discontinuities) as the one in Lemma 3.5. Assuming the generic condition $\mathcal{R}(b) \in \operatorname{int} A$, there exists a new expanding return map over $\overline{A}$*

$$\tilde{\mathcal{R}}\colon \overline{A} \to \overline{A}, \qquad \tilde{\mathcal{R}}|_{J_i} = g_i^{-1} \quad \textit{for } i = 0, \dots, M.$$

which only has a finite number M of discontinuities.

The following lemma shows that we can always construct an expanding return map $\mathcal{R} : A \to A$, with $A = (f_j(b), b]$ a subset of a su-interval for IFS(f_0, f_1), satisfying the above generic condition $\mathcal{R}(b) \in \operatorname{int} A$.

Lemma 3.19. *Every su-interval for f_0 and f_1 close enough to the identity in the C^2-topology has an expanding return map satisfying the generic condition in Remark 3.18.*

Proof. Let K^{su}_Φ be a su-interval for IFS(f_0, f_1) with f_i close enough to the identity. Without loss of generality, we suppose $K^{**}_\Phi = [0, 1]$, $b = f_0^{-1}f_1(0)$ and $A = (f_0(b), b]$. We will show that some expanding branches of $\mathcal{R} : A \to A$ can be modified to obtain this generic condition. Suppose that $h^{-1}_{00}(b) = b$ and recall that $Dh^{-1}_{00}(b) > 1$. We need the following claim:

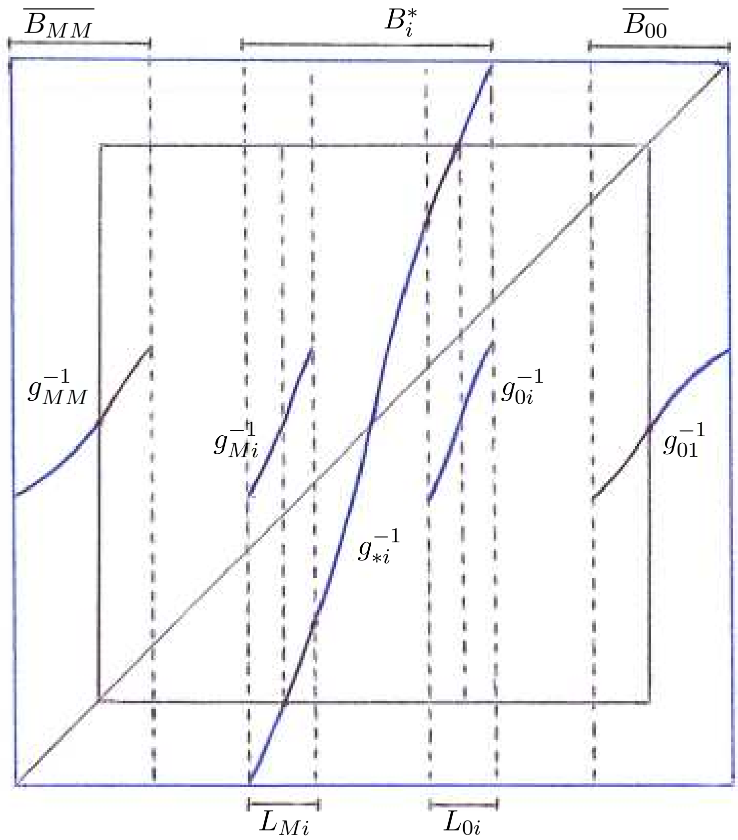

Fig. E: Covering property on $B = \operatorname{int} A$ for the modified expanding return map over A.

Claim 3.19.1. *Let (a, b) be a non-empty interval on the real line. Suppose that there are maps $h, g \in \mathrm{IFS}(f_0, f_1)$ such that $g^{-1}(b) \in (a, b)$, $h^{-1}(b) = b$ and $Dh^{-1}(b) > 1$. Then there is*

$$f \in \mathrm{IFS}(f_0, f_1) \quad \text{such that} \quad f^{-1}(b) \in (a, b) \text{ and } Df^{-1}(b) > 1.$$

Proof of the claim. Since b is a repeler fixed point of h^{-1} there is $n \in \mathbb{N}$ such that

$$D(g^{-1} \circ h^{-n})(b) = Dg^{-1}(b) Dh^{-n}(b) > 1.$$

Set $f = h^n \circ g \in \mathrm{IFS}(f_0, f_1)$. Then $f^{-1}(b) = g^{-1} \circ h^{-n}(b) = g^{-1}(b) \in (a, b)$ and $Dh^{-1}(b) > 1$. □

According to Duminy's Lemma for IFS(Φ) and since K_Φ^{su} is an su-interval for both, IFS(Φ) and IFS(Φ^{-1}) = IFS(f_0^{-1}, f_1^{-1}), it follows that K_Φ^{su} is a minimal set for IFS(Φ^{-1}). Thus, there is

$$g^{-1} \in \mathrm{IFS}(\Phi^{-1}) \text{ such that } g^{-1}(b) \in \operatorname{int} A.$$

From Claim 3.19.1 it gets that we can replace the expanding branch h_{00}^{-1} by a new expanding branch f^{-1} on a small interval $J = [b - \delta, b]$ with $f^{-1}(b) \in \operatorname{int} A$. So, we conclude the proof of this lemma. □

Let B be the interior of the set A^* in Lemma 3.17. By means of a slight restructuring of this new expanding return map for the IFS(f_0, f_1), the following result shows that IFS(f_0, f_1) satisfies the covering property in Theorem 3.13 for the open set B.

Lemma 3.20. *Consider f_0, f_1 in* $\mathrm{Diff}^1_+(\mathbb{R})$ *and assume that there exists a non-empty interval* $A^* = (a^*, b^*]$, *maps* $g_i \in \mathrm{IFS}(f_0, f_1)$ *and closed intervals* $J_i = [t_{i+1}, t_i]$ *for* $i = 0, \dots, M$ *with* $a^* = t_{M+1} < t_M < \dots < t_1 < t_0 = b^*$ *such that*

(i) $\overline{A^*} = J_0 \cup \dots \cup J_M$,

(ii) $g_i^{-1}(J_i) \subset A^*$ *for all* $i = 0, \dots, M$ *with* $g_0^{-1}(b^*)$, $g_M^{-1}(a^*) \in \operatorname{int} A^*$, *and*

(iii) $Dg_i^{-1}(x) > 1$ *for all* $x \in J_i$ *and every* $i = 0, \dots, M$.

Then there exist $n \in \mathbb{N}$, *non-empty bounded open intervals* B_i *and maps* $h_i \in \mathrm{IFS}(f_0, f_1)$ *for* $i = 0, \dots, n$ *such that*

$$\overline{A^*} \subset B_0 \cup \dots \cup B_n$$

with $h_i^{-1}(\overline{B_i}) \subset \operatorname{int} A^*$ *and* $Dh_i^{-1}(x) > 1$ *for all* $x \in \overline{B_i}$ *and for* $i = 0, \dots, n$.

Proof. By assumption, g_0^{-1} and g_M^{-1} are expanding maps in J_0 and J_M such that $g_0^{-1}(b^*)$ and $g_M^{-1}(a^*)$ belong to the interior of A. Thus, there is closed intervals $[t, t_0] \subset J_0$ and $[t_{M+1}, s] \subset J_M$ such that $g_0^{-1}([t, t_0])$ and $g_M^{-1}([t_{M+1}, s])$ are contained in the interior of A^*. For simplicity of notation, we denote these two closed intervals by J_0 and J_M.

Now, we extend the closed intervals J_0 and J_M to open intervals B_{00} and B_{MM} such that for each $k \in \{0, M\}$,

$$g_k^{-1}(\overline{B_{kk}}) \subset \operatorname{int} A^* \quad \text{and} \quad Dg_k^{-1}(x) > 1 \text{ for all } x \in \overline{B_{kk}}.$$

Similarly, as $g_i^{-1}(J_i) \subset A^*$, we can find open intervals B_i^* such that $J_i \subset B_i^*$ and

$$g_i^{-1}(\overline{B_i^*}) \subset \overline{B_{00}} \cup A^* \cup \overline{B_{MM}} \quad \text{for } i = 1, \dots, M-1.$$

For each $k \in \{0, M\}$ and for every $i = 1, \dots, M-1$, we denote $L_{ki} = \overline{B_i^*} \cap g_i(\overline{B_{kk}})$ and set $g_{ki} = g_i \circ g_k \in \mathrm{IFS}(f_0, f_1)$. Then for each $k \in \{0, M\}$ and for every $i = 1, \dots, M-1$

$$g_{ki}^{-1}(L_{ki}) = g_k^{-1} \circ g_i^{-1}(\overline{B_i^*}) \cap g_k^{-1}(\overline{B_{kk}}) \subset \operatorname{int} A^*.$$

Let L_{*i} be the closure of $B_i^* \setminus (L_{0i} \cup L_{Mi})$. Observe that $L_{*i} \subset \operatorname{int} J_i$. Hence, writing $g_{*i} = g_i$, it follows that $g_{*i}^{-1}(L_{*i}) \subset \operatorname{int} g_i^{-1}(J_i) \subset \operatorname{int} A^*$ for every $i = 1, \dots, M-1$. Therefore, briefly, it holds that for each $k \in \{0, M, *\}$ and for every $i = 0, \dots, M$

$$g_{ki}^{-1}(L_{ki}) \subset \operatorname{int} A^* \quad \text{and} \quad Dg_{ki}^{-1}(x) > 1 \text{ for all } x \in \overline{L_{ki}}.$$

Finally, from here, taking open intervals B_{ki} such that $L_{ki} \subset B_{ki}$, $g_{ki}^{-1}(\overline{B_{ki}}) \subset \operatorname{int} A^*$ and $Dg_{ki}^{-1}(x) > 1$ for all $x \in \overline{B_{ki}}$ it follows that

$$\bigcup_{k=0,M,*} \bigcup_{i=0}^{M} B_{ki} \supset \bigcup_{i=0}^{M} J_i = \overline{A^*}.$$

Renaming the opens intervals and the return maps we complete the lemma. $\square$

Now, we are ready to prove the C^1-robustness in Theorem D.

Proof of C^1-robustness in Duminy's Lemma. For simplicity, we can assume that the fixed point of f_i in K^{**}_Φ are hyperbolic. As we have already noted, Lemma 3.20 implies the covering property of Theorem 3.13 taking B the interior of $A^* \subset K^{**}_\Phi$ in this lemma. From the first part of Duminy's Lemma we also have $\mathcal{G}_\Phi$-orbit of p is dense in B where p is the global attractor of f_j in K^{**}_Φ. Therefore, Theorem 3.13 implies the existence C^1-neighborhood $\mathcal{U}_i$ of f_i for $i = 0, 1$ such that the $\mathcal{G}_\Psi$-orbit of p_Ψ is dense in B for all $\mathrm{IFS}(\Psi)$ with $\Psi = \{g_0, g_1\}$, $g_i \in \mathcal{U}_i$ and where p_Ψ is the continuation point of p.

The argument to show the minimality is standard. Firstly, note that by Lemma 3.11 it suffices to show the $\mathcal{G}_\Psi$-orbit of p_Ψ is dense in K^{**}_Ψ. Let I be any interval in K^{**}_Ψ. Since K^{**}_Ψ is a $**$-interval there is a first return map over B which can be extend to the interval K^{**}_Ψ. In particular this implies that there exists $h_0 \in \mathrm{IFS}(\Psi)$ such that $h_0^{-1}(I) \cap B \neq \emptyset$. Since the $\mathcal{G}_\Psi$-orbit of p_Ψ is dense in B then there is $h_1 \in \mathrm{IFS}(\Psi)$ such that $h_1(p_\Psi) \in h_0^{-1}(I) \cap B$ and so $h_0 \circ h_1(p_\Psi) \in I$. This concludes the density and therefore also the minimality. Finally, notice that the robustness of the density of periodic points is followed from this minimality property as previously notified in the proof of Dumniny's Lemma for $\mathrm{IFS}(\Phi)$. Therefore, the proof of Theorem D is concluded. $\square$

3.2.2 Spectral decomposition

Let f be a diffeomorphism on the real line. We say that f is *Morse-Smale diffeomorphism on the real line* if it has countable non-empty set of fixed points all of them hyperbolic. The next theorem gives a completely description the global topological dynamics of a IFS of two Morse-Smale diffeomorphisms on the real line close to the identity. In oder to state the theorem we have to enlarge the set of different types of $**$-intervals for an IFS generated by a pair of diffeomorphisms on the real line. Now the $**$ can also be s or u and in this case K^{**}_Φ denotes an unbounded interval. Namely,

- $K^s_\Phi = [p, \infty)$ *semi-attractor*: if p is an attracted fixed point of a map, say f_0, satisfying $f_0 < \mathrm{id}$ in (p, ∞) and $f_1 > \mathrm{id}$ in $[p, \infty)$. The case $K^s_\Phi = (-\infty, q]$ is defined analogously.
- K^u_Φ *semi-repeler*: if it is a s-interval for f_0^{-1} and f_1^{-1}.

The proof of Duminy's Lemma (Theorem D) is exactly the same for s-intervals and so they are minimality sets and have dense periodic point for $\mathrm{IFS}(f_0, f_1)$ if f_0 and f_1 are C^2-close to the identity. This implies that a u-interval for $\mathrm{IFS}(f_0, f_1)$ is transitive and has also dense set of periodic points.

Theorem 3.21 (Spectral decomposition on the real line)**.** *Let f_0 and f_1 be Morse-Smale diffeomorphisms of the real line with no fixed points in common. Then, there exists $\varepsilon \geq 0.17$ such that if $d_{C^2}(f_i, \mathrm{id}) < \varepsilon$ for $i = 0, 1$ then there are $m \in \mathbb{N} \cup \{\infty\}$ and pairwise disjoint isolated topologically transitive intervals K_i for $\mathrm{IFS}(f_0, f_1)$, for $i = 1, \dots, m$ such that*

$$L(\mathrm{IFS}(f_0, f_1)) \setminus \{\pm\infty\} = \bigcup_{i=1}^{m} K_i.$$

Moreover, each K_i is either a $**$-interval for $\mathrm{IFS}(f_0, f_1)$ with $** \in \{ss, su, uu, s, u\}$, or a single fixed point of f_0 or f_1.

Proof. Consider $z \in L(\mathrm{IFS}(f_0, f_1)) \setminus \{\pm\infty\}$. We can assume that $z \in \omega(\mathrm{IFS}(f_0, f_1))$ since the situation for the closure of α-limit of $\mathrm{IFS}(f_0, f_1)$ is followed by a similar arguments. Then, by definition of ω-limit of $\mathrm{IFS}(f_0, f_1)$, the point z is approximated by points of the form $y_m \in \omega_\Phi(x_m)$. Each of these points $y = y_m$ are again approximated by points of the form $f_\xi^{n_k}(x)$ with $x = x_m$, $\xi = \xi(m) \in \Sigma_k^+$ and $n_k = n_k(m) \to \infty$ when $k \to \infty$. We claim that if $y = \lim_{k\to\infty} f_\xi^{n_k}(x)$, then either y belongs to some $**$-interval for $\mathrm{IFS}(f_0, f_1)$ with $** \in \{ss, su, uu, s, u\}$ or it is a fixed point of f_0 or f_1. This claim concludes the theorem since either, z is a fixed point of f_0 or f_1, or then for m_0 large enough y_m belongs in the same $**$-interval for $\mathrm{IFS}(f_0, f_1)$ for all $m \geq m_0$ and thus z is also in this $**$-interval.

In order to prove the above claim, let $\{p_i\}$ be the ordered set of fixed pints of both maps f_0 and f_1. Without loos of generality, we suppose that $y \geq 0$. We can assume that $y \in [p_i, p_{i+1}]$. Otherwise, there is a fixed point p_j such that $p_i \leq p_j$ for all i and $y \in [p_j, \infty)$. It is not hard to check, via the geometry of the functions, that in this case is not possible that $f_0, f_1 > \mathrm{id}$ in $(p_j, \infty]$ since then $y = \infty$. In other case, $[p_j, \infty)$ is s or u-interval or $y = p_j$.

If p_i and p_{i+1} are both attractors or repellers but for different maps, from the closeness of f_0 and f_1 to the identity it follows that $[p_i, p_{i+1}]$ is a ss or uu-interval. So, we may assume that p_i and p_{i+1} are an attractor-repeler pair for the same maps, say f_0. Note that in this case $f_0 < \mathrm{id}$ in (p_i, p_{i+1}). We have two options: $f_1 < \mathrm{id}$ or $f_1 > \mathrm{id}$ in $[p_i, p_{i+1}]$. In the first case, both maps are below to the identity and then if $f_\xi^{n_k}(x) \in [p_i, p_{i+1}]$ for all k large enough implies that $\xi_{n_k} = 0$ and so $y = \lim_{k\to\infty} f_\xi^{n_k}(x) = p_i$. For the second case, $f_1 > \mathrm{id}$ in (p_1, p_{i+1}) and we have again two options: $f_1([p_1, p_{i+1}]) \cap [p_1, p_{i+1}] \neq \emptyset$ or $f_1([p_i, p_{i+1}]) \cap [p_i, p_{i+1}] = \emptyset$. In the first option, $[p_i, p_{i+1}]$ is a su-interval for $\mathrm{IFS}(f_0, f_1)$. For the other option, it follows as before that $y = p_i$. Therefore y belongs to a $**$-interval for $\mathrm{IFS}(f_0, f_1)$ with $** \in \{ss, su, uu, s, u\}$ or it is a fixed point of f_0 or f_1.

Finally, note that from Duminy's Lemma (Theorem D), ss, su and s-intervals are minimal set for the $\mathrm{IFS}(f_0, f_1)$. In particular are transitive set and thus uu and u-intervals are also transitive set for the $\mathrm{IFS}(f_0, f_1)$. Similarly, again from Duminy's Lemma (Theorem D) it follows that are isolated sets. This concludes the proof of the theorem. □

The above Spectral Decomposition Theorem can be extended for an IFS generated by a pair of diffeomorphisms on a compact interval I. In order to do this, we understand the compact interval I like the compactified real line $[-\infty, \infty]$. So, the endpoints of the interval I became in $\pm\infty$ respectively. In this way, we will understand a $**$-interval $K^{**} \subset I$ for $** \in \{s, u\}$ as a $**$-interval for the IFS defined on the real line. Therefore, Theorem 3.21 concludes the following remark. Here, by a *Morse Smale diffeomorphism on a compact interval* I we mean a diffeomorphism f with a non-empty finite set of fixed points in the interior of I and all of them hyperbolic.

Remark 3.22. *If f_0 and f_1 are Morse Smale diffeomorphisms on a compact interval I close enough to the identity map in the C^2-topology and with no periodic points in common in the interior of I, then $L(\mathrm{IFS}(f_0, f_1))$ is finite union of pairwise disjoint intervals. Namely each interval is either a $**$-interval for $\mathrm{IFS}(f_0, f_1)$ with $** \in \{ss, su, uu, s, u\}$ or a single fixed point of f_0 or f_1.*

We want to note that these results about the spectral decomposition of the limit set of the $\mathrm{IFS}(f_0, f_1)$ are C^1-robust. That is, the same property holds for any IFS generates by C^1-perturbations of f_0 and f_1. This is followed from the fact of all of fixed point are hyperbolic and from the C^1-robustness of the Duminy's Lemma (Theorem D).

3.3 Blending regions for IFS on the circle

Let f be an orientation preserving circle homeomorphism. Taking into account the rotation number of f, this maps can have either rational rotation number or irrational rotation number. With rational rotation number f has at least a periodic point while with irrational rotation number either each orbit of f is dense in S^1 or there is a wandering interval for f. Let us consider the group of orientation preserving circle homeomorphisms $\mathrm{Hom}_+(S^1)$. Note that, the forward and backward iterations of f are a particular case of a subgroup $\mathrm{G}(f)$ of $\mathrm{Hom}_+(S^1)$ finitely generated. In order to extend the above classification for general subgroup of orientation preserving homeomorphisms, we need to explain the notion of invariant subset of S^1 by a subgroup.

Let $\mathrm{G}(\Phi)$ be a subgroup of $\mathrm{Hom}_+(S^1)$ generated by a family Φ of homeomorphisms on the circle and let Λ be a subset of S^1. The orbit of a point $x \in S^1$ for $\mathrm{G}(\Phi)$ is the set of elements of S^1 to which x can be moved by the elements of $\mathrm{G}(\Phi)$. Following the notation for IFS, when no confusion can arise, the *orbit of x for* $\mathrm{G}(\Phi)$ is denoted as

$$\mathrm{Orb}_\Phi(x) \stackrel{\text{def}}{=} \{g(x) \colon g \in \mathrm{G}(\Phi)\}.$$

We say that Λ is *invariant for* $\mathrm{G}(\Phi)$ if $\mathrm{Orb}_\Phi(x) \subset \Lambda$ for all $x \in \Lambda$. Assuming that Λ is also compact, it is said to be *closed invariant minimal set for* $\mathrm{G}(\Phi)$ if its only closed invariant subsets for $\mathrm{G}(\Phi)$ are the empty set and Λ itself, or equivalently if

$$\Lambda = \overline{\mathrm{Orb}_\Phi(x)} \quad \text{for all } x \in \Lambda.$$

Now, we introduce a special type of minimal set:

Definition 3.6 (Exceptional minimal set)**.** *Let* $\mathrm{G}(\Phi)$ *be group generated by a family* $\Phi \subset \mathrm{Hom}_+(S^1)$*. A subset* $\Lambda \subset S^1$ *is said to be an* exceptional minimal set for $\mathrm{G}(\Phi)$ *if*

- Λ *is a Cantor set,*
- $\Lambda = \overline{\mathrm{Orb}_\Phi(x)}$ *for all* $x \in \Lambda$*,*
- $\Lambda \subset \overline{\mathrm{Orb}_\Phi(x)}$ *for all* $x \in S^1$*.*

Notice that we can define ω-limit set $\omega_\Phi(x)$ of $x \in S^1$ for the action of group $\mathrm{G}(\Phi)$ in a manner similar to the action of IFS. Hence, Item (iii) in Lemma 3.2 can also be shown for $\mathrm{G}(\Phi)$ with the same proof that for IFS. Consequently, it follows that, the second and third properties in the above definition are, respectively, equivalent to $\Lambda = \omega_\Phi(x)$ for $x \in \Lambda$ and $\Lambda \subset \omega_\Phi(x)$ for all $x \in S^1$. Moreover, as an immediately consequence of these two properties, we obtain the following remark:

Remark 3.23. *If* $\Lambda \subset S^1$ *is an exceptional minimal for action of a group* $\mathcal{G}(\Phi)$ *of circle homeomorphisms then* Λ *is the unique closed minimal invariant set for* $\mathrm{G}(\Phi)$*.*

Notice that in the particular case of a unique map f, the above definition is closely related to the notion of wandering interval. In fact, the wandering intervals are the gaps of the exceptional minimal set. In this particular case of a single generator, the exceptional minimal set Λ is invariant for f and for f^{-1}.

The following proposition is a classic result that can be find for instance in [Ghy01].

Proposition 3.24. *Let* $\mathrm{G}(\Phi)$ *be a subgroup of* $\mathrm{Hom}_+(S^1)$. *Then one (and only one) of the following possibilities occurs:*

i) there exists a finite orbit,

ii) S^1 *is a minimal for* $\mathrm{G}(\Phi)$ *(i.e., all orbits are dense),*

iii) there exists an exceptional minimal set for $\mathrm{G}(\Phi)$.

Proof. The family of non-empty closed invariant subsets of S^1 is ordered by inclusion. Since the intersection of nested compact sets is a non-empty compact set, the Zorn Lemma allows us to conclude the existence of a minimal (regarding the inclusion) non-empty closed invariant set Λ. Namely, if K is a non-empty close invariant set for $\mathrm{G}(\Phi)$ contained in Λ then $K = \Lambda$. Observe that this is equivalent to that Λ is an closed invariant minimal set for $\mathrm{G}(\Phi)$. The boundary $\partial\Lambda$ and the set Λ' of the accumulation points of Λ are closed invariant sets contained in Λ. By the minimality (regarding the inclusion) of Λ, one of the following possibilities occurs:

(i) Λ' is empty: in this case Λ is a finite orbit,

(ii) $\partial\Lambda$ is empty: in this case $\Lambda = S^1$, and therefore all the orbits are dense,

(iii) $\Lambda = \Lambda' = \partial\Lambda$: in this case Λ is a closed set with empty interior and having no isolated point. In other words, Λ is homeomorphic to the Cantor set.

We will show that, in the last case, Λ is an exceptional minimal set. Namely, we will prove that

$$\Lambda \subset \overline{\mathrm{Orb}_\Phi(x)} \quad \text{for all } x \in S^1,$$

which clearly together with the invariant minimality implies its uniqueness. For $x \in \Lambda$, this follows since Λ is an invariant minimal set for $\mathrm{G}(\Phi)$. Let x and y be arbitrary points in $S^1 \setminus \Lambda$ and Λ, respectively. We need to show that there exists a sequence $(g_n)_n \subset \mathrm{G}(\Phi)$ such that $g_n(x)$ converges to y. In order to prove this, let us consider the interval $I = (a, b)$ contained in $S^1 \setminus \Lambda$ such that both a, b belong to Λ and $x \in I$.

Claim 3.24.1. *There exists* $(g_n)_n \subset \mathrm{G}(\Phi)$ *such that* $g_n(a) \to y$ *and* $g_n(I)$ *are two-by-two disjoint.*

Proof. Since Λ is the closure of the orbit of a for $\mathrm{G}(\Phi)$ and $\Lambda = \Lambda'$ it follows that $\Lambda = \mathrm{Orb}_\Phi(a)'$ and thus there exists $(h_n)_n \subset \mathrm{G}(\Phi)$ such that $h_n(a)$ tends to y with $h_n(a) \neq y$ for all $n \in \mathbb{N}$. The collection of intervals $\{h_n(I) : n \in \mathbb{N}\}$ cannot be finite otherwise, $h_n(a)$ only takes a finite number of values (the endpoints of this intervals) and then it could not tend to y at least that $h_n(a) = y$, but this is not possible from the choice of $(h_n)_n$. Moreover, either $h_n(I) \cap h_m(I) \neq \emptyset$ or $h_n(I) = h_m(I)$. Indeed, if $h_n(I)$ meets $h_m(I)$ but $h_n(I) \neq h_m(I)$ then at least one of the endpoints of $h_m(I)$, say $h_m(a)$, belongs to $h_n(I)$ and so, $(h_n)^{-1} \circ h_m(a) \in \Lambda \cap I$ since $a \in \Lambda$ and Λ is invariance for $\mathrm{G}(\Phi)$. However, this is not possible since I is a gap of the Cantor set Λ. Therefore, we can choice a subsequence of $(g_n)_n$ of $(h_n)_n$ such that $g_n(I)$ are two-by-two disjoint since in otherwise we have a finite collection of intervals $h_n(I)$. □

From the above claim, we observe that the length of the intervals $g_n(I)$ must go to zero, and thus $g_n(x)$ converges to y (since $g_n(a)$ tends to y). This concludes the proof. □

The next result provides a similar classification as above for IFS. Notice that this classification describes the shape of possible attractors of an IFS generated by homeomorphisms on an arbitrarily compact manifold M.

Theorem 3.25. *Consider* $\mathrm{IFS}(\Phi)$ *generated by a family* $\Phi = \{f_1, \dots, f_N\} \subset \mathrm{Hom}(M)$ *of homeomorphisms of a compact manifold* M*. Then exists a non-empty closed subset* K *of* M *such that*

$$K = \bigcup_{i=1}^{N} f_i(K) \quad \text{and} \quad K = \overline{\mathrm{Orb}_\Phi(x)} = w_\Phi(x) \text{ for all } x \in K.$$

Moreover, one (and only one) of the following possibilities occurs:

i) K is a finite orbit,

ii) K have non-empty interior,

iii) K is a Cantor set.

Proof. The family of non-empty closed subset Λ of M such that $\Lambda = f_1(\Lambda) \cup \cdots \cup f_N(\Lambda)$ is ordered by inclusion. Since the intersection of nested compact sets is compact and non-empty, the Zorn Lemma allows us to conclude the existence of a minimal (regarding the inclusion) non-empty closed set K such that $K = \cup f_i(K)$. We will show that K is an invariant minimal set for $\mathrm{IFS}(\Phi)$

$$K = \overline{\mathrm{Orb}_\Phi(x)} \quad \text{for all } x \in K. \tag{3.5}$$

Then, according to Item (iii) in Lemma 3.2, we have $K = \overline{\mathrm{Orb}_\Phi(x)} = \omega_\Phi(x)$ for all $x \in K$.

In order to prove (3.5), since K is an closed invariant set we get

$$\mathrm{Orb}_\Phi(x)' \subset \overline{\mathrm{Orb}_\Phi(x)} \subset K \quad \text{for all } x \in K.$$

On other hand, according to (vi) in Lemma 3.2, for each $x \in K$ the set of accumulation points $\mathrm{Orb}_\Phi(x)'$ is a closed selfsimilar set. Since K is minimal (regarding the inclusion) then either, $\mathrm{Orb}_\Phi(x)'$ is an empty set or $K = \mathrm{Orb}_\Phi(x)'$. We have two possibilities:

(i) there is $x \in K$ such that $\mathrm{Orb}_\Phi(x)'$ is empty;

(ii) for all $x \in K$, it holds that $K = \mathrm{Orb}_\Phi(x)'$.

In the first case, it follows that $\mathrm{Orb}_\Phi(x)$ is finite set, and therefore, it is a non-empty closed selfsimilar set contained in K. This implies that $K = \mathrm{Orb}_\Phi(x) = \omega_\Phi(x)$. In the second case, we obtain that K is an invariant minimal set for $\mathrm{IFS}(\Phi)$. Note that in this case $K' = K$. Moreover, we also have two options: K has non-empty interior or the interior of K is empty and thus K is a Cantor set. This concludes the proof of the theorem. □

An invariant minimal Cantor set $K \subset M$ for $\mathrm{IFS}(\Phi)$ is said to be *exceptional minimal set for* $\mathrm{IFS}(\Phi)$ if it is the unique attractor of the IFS or, equivalently, if the orbit of any point of M is dense in K. Notice that an invariant minimal Cantor set for $\mathrm{IFS}(\Phi)$ obtained in the above theorem is not necessarily an exceptional minimal set. An example of an IFS with at least two attractors, being one of them a Cantor set, is constructed by taking two disjoints regions $D_1, D_2 \subset M$. Each one of the generators $f_j \in \Phi$ of $\mathrm{IFS}(\Phi)$ is a contracting map on D_i and satisfies $f_j(D_i) \subset D_i$, for all $i = 1, 2$; but, for D_1 at least, the overlapping condition fails to verify, i.e. $f_k(D_1) \cap f_j(D_1) = \emptyset$ for all $k \neq j$.

3.3.1 Duminy's Theorem

An example of a circle homeomorphism with irrational rotation number admitting an invariant minimal Cantor set for f is a Denjoy diffeomorphism. Notice that, in this case, this Cantor set is an "exceptional" minimal set for both $\mathrm{IFS}(f)$ and $\mathrm{IFS}(f^{-1})$ and consequently it is also an exceptional minimal set for the group generated by f. These examples are only possible for diffeomorphisms of class C^1 at the most. Indeed, from a classical theorem due to Denjoy, C^2-diffeomorphisms have not invariant minimal Cantor sets. Namely, if f is a orientation preserving circle C^2-diffeomorphism with irrational rotation number $\rho(f)$, then f is topologically conjugate to the rotation of angle $\rho(f)$. In fact, since every diffeomorphism close enough to the identity is orientation preserving, we can obtain the following implication of Denjoy's Theorem:

Theorem 3.26 (Denjoy)**.** *There exists $\varepsilon > 0$ such that if $f \in \mathrm{Diff}^2(S^1)$ is ε-close to the identity in the C^2-topology then there are no invariant minimal Cantor sets for neither $\mathrm{IFS}(f)$ nor $\mathrm{IFS}(f^{-1})$. Moreover, the following conditions are equivalents:*

i) S^1 is minimal for $\mathrm{IFS}(f)$,

ii) S^1 is minimal for $\mathrm{IFS}(f^{-1})$,

iii) there are no periodic points for f.

As a consequence of the above equivalences, if any of the theses conditions is satisfied then S^1 is also minimal for $\mathrm{G}(f)$. For action groups Duminy, at the end of the seventies in an unpublished work [Dum70], proved that there is no an exceptional minimal set for a group generated by circle diffeomorphisms with certain regularity. The key idea is to create a ss-interval for an IFS generated by two maps in the group $\mathrm{G}(f_0, f_1)$ which are obtained making the necessary compositions by means of the inverse of f_0 or f_1. Then Duminy's Lemma (Theorem D) implies the minimality of this interval for the IFS, and thus, for $\mathrm{G}(f_0, f_1)$. With the help of same inverse map this minimality is moved around the whole circle.

Theorem 3.27 (Duminy)**.** *There exists $\varepsilon > 0$ such that if f_0 and f_1 in $\mathrm{Diff}^2(S^1)$ are ε-close to the identity in the C^2-topology and at least one of them, say f_0, has finitely many periodic points, then there is no exceptional minimal set for $\mathrm{G}(f_0, f_1)$. Moreover, the following conditions are equivalents[1]:*

i) S^1 is minimal for $\mathrm{G}(f_0, f_1)$,

ii) $f_1(\mathrm{Per}(f_0)) \neq \mathrm{Per}(f_0)$.

The condition about the regularity in Denjoy's Theorem as well as Duminy's Theorem can be improved. In fact, Duminy's Theorem can be proved for a group $\mathrm{G}(f_0, f_1)$ of orientation preserving circle diffeomorphisms where the distortion constant of f_0 and f_1 is bounded by a positive universal constant. This condition means that f_0 and f_1 are close to rotations: the equality $\mathrm{Dist}(f_i) = 0$ is satisfied if and only if f_i is a rotation. Concerning the hypothesis of existence of at least a generator

[1]Condition (ii) is satisfies if f_0 and f_1 have not periodic points in common.

with isolated periodic points, let us point out that it is generically satisfied. See [Nav04, Nav11] for more details about Duminy's result (non-existence of exceptional minimal set for action group). We round off Duminy's result by adding the second part in Theorem 3.27, which states the equivalence between minimality of the action group $\mathrm{G}(f_0, f_1)$ and condition (ii) about the common periodic points. This equivalence together with the genericity (open and dense) of the set of Morse-Smale diffeomorphisms in $\mathrm{Diff}^2(S^1)$ implies the following remark:

Remark 3.28. *While the periodic dynamics are generic (open and dense) in* $\mathrm{Diff}^2(S^1)$*, even close to the identity, generically* S^1 *is minimal for action groups on the circle with at least two generators close enough to the identity (open an dense set of a neighborhood* $\mathcal{U} \subset \mathrm{Diff}^2(S^1)$ *of* id*).*

We will now give a proof of Duminy's Theorem slightly different from the proof in [Nav11].

Proof of Theorem 3.27. Set $\Phi = \{f_0, f_1\}$. According to Denjoy's Theorem, the set of periodic points of each element in $\mathrm{G}(\Phi)$ is non-empty, otherwise S^1 is minimal for $\mathrm{G}(\Phi)$ and we conclude the theorem. By hypothesis, we assume that the periodic points of f_0 are isolated. Let us denote by $\mathrm{Per}(f_0)$ the set of these points.

Claim 3.28.1. *If either there exists exceptional minimal set for* $\mathrm{G}(\Phi)$ *or* f_0 *and* f_1 *have not periodic points in common, then there exists* $p \in \mathrm{Per}(f_0)$ *such that* $f_1(p)$ *or* $f_1^{-1}(p)$ *is in* $S^1 \backslash \mathrm{Per}(f_0)$*.*

Proof. Firstly, we assume that f_0 and f_1 do not have periodic points in common. If for every $p \in \mathrm{Per}(f_0)$ one has that $f_1(p)$ and $f_1^{-1}(p)$ are in $\mathrm{Per}(f_0)$ then $f_1(\mathrm{Per}(f_0)) = \mathrm{Per}(f_0)$. As $\mathrm{Per}(f_0)$ is a finite set then there are $m \in \mathbb{N}$ and $p \in \mathrm{Per}(f_0)$ such that $f_1^m(p) = p$ contradicting that f_0 and f_1 do not have periodic points in common.

We now assume that there exists Λ exceptional minimal set for $\mathrm{G}(\Phi)$ and denote

$$\mathrm{P}(f_0) = \mathrm{Per}(f_0) \cap \Lambda.$$

Notice that $\mathrm{P}(f_0)$ is non-empty. Indeed, if the period of the periodic points of f_0 is k and $p \in \Lambda$ is not a fixed point of f_0^k, then

$$\lim_{i \to \pm\infty} f_0^{ik}(p)$$

are fixed points of f_0^k contained in Λ. Just like the previous case, there exists $p \in \mathrm{P}(f_0)$ such that $f_1(p)$ or $f_1^{-1}(p)$ is in $S^1 \setminus \mathrm{P}(f_0)$. Otherwise, the finite set $\mathrm{P}(f_0)$ would be invariant for $\mathrm{G}(f_0, f_1)$, thus contradicting the minimality of Λ. $\square$

In what follows, we will assume that there exists $p \in \mathrm{Per}(f_0)$ such that $f_1(p)$ or $f_1^{-1}(p)$ belongs to $S^1 \backslash \mathrm{Per}(f_0)$. Otherwise, according to the above claim the first part of the theorem follows. Under this assumption we will show that S^1 is minimal for $\mathrm{G}(\Phi)$. This makes impossible the existence of an exceptional minimal set for $\mathrm{G}(\Phi)$ and therefore we again obtain the first part of the theorem. Whit regard to the second part of the theorem, since (ii) implies that there exists $p \in \mathrm{Per}(f_0)$ such that $f_1(p) \in S^1 \setminus \mathrm{Per}(f_0)$, once proved the minimality of S^1 under this assumption, we obtain that (ii) implies (i). In order to prove that (i) implies (ii), according to Proposition 3.24, if S^1 is minimal for $\mathrm{G}(\Phi)$ then there is no finite orbit and this implies (ii). Indeed, suppose that $f_1(\mathrm{Per}(f_0)) = \mathrm{Per}(f_0)$ then this finite set is invariant by f_0 and f_1 and therefore it is a finite orbit, which is a contradiction.

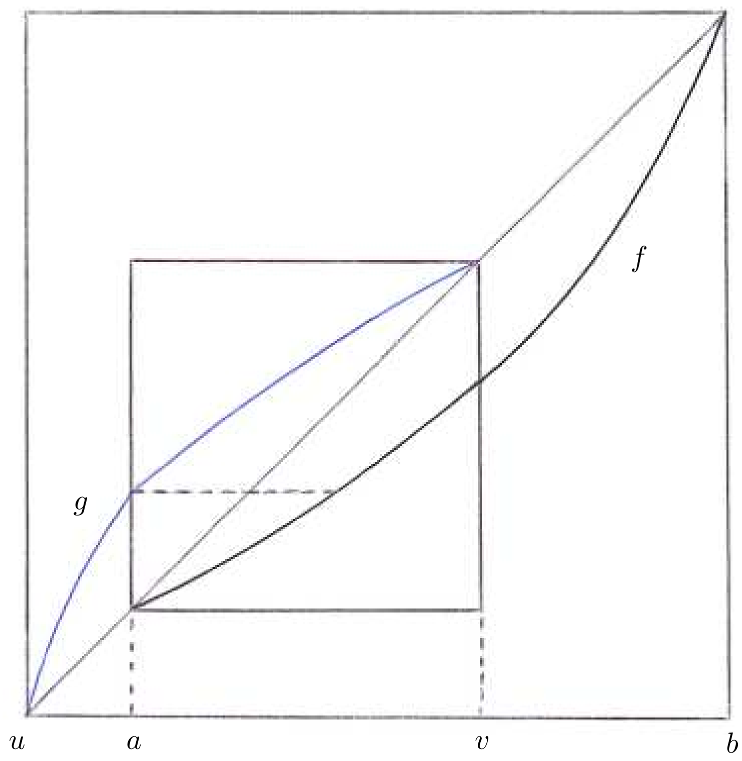

Fig. F: Configuration in the proof of Duminy's theorem.

Let us now prove that S^1 is minimal for $\mathrm{G}(\Phi)$. Since $\mathrm{G}(\Phi) = \mathrm{G}(f_0, f_1^{-1})$, without loss of generality, we will suppose that $f_1(p) \in S^1 \setminus P(f_0)$. Let k be the period of the periodic points of f_0. Let $g = f_0^k \in \mathrm{G}(f_0, f_1)$ and let us denote by u and v the periodic points of f_0 immediately to the left and to the right of $f_1(p)$, respectively.[2] The map $f = f_1 \circ g \circ f_1^{-1}$ has a fixed point in $[u, v]$, namely $f_1(p)$. Let a be the first fixed point of this map to the left of v, and let b be the first fixed point to the right of a. Replacing g by g^{-1} and/or f by f^{-1} if necessary, we may suppose that $f(x) < x$ and $g(x) > x$ for every $x \in (a, v)$. See Figure F.

We now claim that $[a, v]$ is a ss-interval for $\mathrm{IFS}(f, g)$. We only need to show the overlap condition: $f^{-1}(g(a)) \in (a, v)$. To show this, we first notice that

$$\begin{aligned}\mathrm{Dist}(f, [a, b]) &\leq \sum_{i=0}^{k-1} \mathrm{Dist}(f_1 \circ f_0 \circ f_1^{-1}, f_1 \circ f_0^i \circ f_1^{-1}([a, b])) \\ &\leq \mathrm{Dist}(f_1 \circ f_0 \circ f_1^{-1}, S^1) \leq 3C\end{aligned}$$

where C is the largest distortion constant of f_0 and f_1 in S^1. In the same way one obtains $\mathrm{Dist}(g, [u, v]) \leq C$. Let $x_0 \in (a, v)$ and $y_0 \in (a, b)$ be such that

$$Df(x_0) = \frac{f(v) - a}{v - a} \quad \text{and} \quad Df(y_0) = 1.$$

Clearly, we have $|\log Df(y_0) - \log Df(x_0)| \leq \mathrm{Dist}(f, [a, b])$, and hence $f(v) - a \geq e^{-3C}(v - a)$. By a similar argument it follows that $v - g(a) \geq e^{-C}(v - a)$. If $f^{-1}(g(a))$ were not contained in the

[2] We work with the lift map of g for which the existence of at least two fixed points of g is guarantied.

interval (a,v), then $f(v)\leq g(a)$ and hence, from the above inequalities,

$$v-a\geq f(v)-a+v-g(a)\geq (e^{-3C}+e^{-C})(v-a).$$

Therefore, $e^{-3C}+e^{-C}\leq 1$ which is imposible if f_0 and f_1 are ε-close to the identity in the C^2-topology for $\varepsilon>0$ small enough, because of $0<C<\varepsilon(1-\varepsilon)^{-1}$.

The elements f and g in $\mathrm{G}(\Phi)$ are thus as in Figure F over the interval $[a,v]$ and therefore this interval is a ss-interval for f and g. We will show that this interval is minimal for $\mathrm{IFS}(f,g)$. In order to prove this, we apply the Duminy's Lemma showing that f and g satisfy the assumptions in Remark 3.4:

$$1-\varepsilon<Df(x)<1+\varepsilon,\quad 1-\varepsilon<Dg(x)<1+\varepsilon\quad\text{and}\quad (1-\varepsilon)\varepsilon^{-1}e^{-4\tilde{C}}>2$$

where $\tilde{C}$ is the largest distortion constant of f^{-1} and g^{-1} in $[a,v]$. Observe that, the distortion constant of f^{-1} in $[a,b]$ and the distortion constant of g^{-1} in $[u,v]$ coincide with $\mathrm{Dist}(f,[a,b])$ and $\mathrm{Dist}(g,[u,v])$ respectively. Thus, as we have noticed, this constants are less than or equal to $3C$ and C, respectively, and so $\tilde{C}\leq 3C$.

Now, notice that for every $x\in(a,b)$ it holds $|\log Df(x)|=|\log Df(x)-\log Df(y_0)|\leq \mathrm{Dist}(f,[a,b])\leq 3C$ and hence

$$e^{-3C}\leq Df(x)\leq e^{3C}\quad\text{for all } x\in[a,v].$$

Similarly, we follow an analogous inequality for the absolute value of the derivative of the logarithm in $[u,v]$ and we obtain

$$e^{-3C}<e^{-C}\leq Dg(x)\leq e^{C}<e^{3C}\quad\text{for all } x\in[a,v].$$

Recall that as f_0 and f_1 are ε-close to the identity we have $C<\varepsilon(1-\varepsilon)^{-1}$. Then since $e^{\pm 3\varepsilon(1-\varepsilon)^{-1}}$ and $1\pm\varepsilon$ are equivalent infinitesimals, it follows

$$1-\varepsilon\sim e^{-3\varepsilon(1-\varepsilon)^{-1}}<Df(x),\,Dg(x)<e^{3\varepsilon(1-\varepsilon)^{-1}}\sim 1+\varepsilon\quad\text{for all } x\in[a,v]. \tag{3.6}$$

Finally, for $\varepsilon>0$ small enough we get

$$(1-\varepsilon)\varepsilon^{-1}e^{-4\tilde{C}}\geq(1-\varepsilon)\varepsilon^{-1}e^{-12\varepsilon(1-\varepsilon)^{-1}}>2. \tag{3.7}$$

Therefore, since Equations (3.6) and (3.7) are the desired assumptions to apply the Duminy's Lemma we obtain that $[a,v]$ is minimal for $\mathrm{IFS}(f,g)$. Now, we will move the minimality of this interval along the whole circle, again by means of the inverse maps. Let I be and open interval in S^1 and let x be any point in S^1. Since $[a,v]$ contains at least a fundamental domain of f and g we can find[3] F and H in $\mathrm{G}(f,g)$ such that

$$F(I)\cap[a,v]\neq\emptyset\quad\text{and}\quad H(x)\in[a,v].$$

By the minimality of this interval there is $h\in\mathrm{IFS}(f,g)$ such that $h\circ H(x)\in F(I)$. Therefore,

$$F^{-1}\circ h\circ H(x)\in I$$

with $F^{-1}\circ h\circ H\in\mathrm{G}(f_0,f_1)$. This shows the minimality of S^1 for $\mathrm{G}(f_0,f_1)$ and the proof of the theorem is completed. □

[3] More details of this claim can be understood in Section §3.3.3 where we study cycles for IFS.

The following theorem shows a generalization of Denjoy's Theorem for IFS. Namely, this result shows that there are no invariant minimal Cantor sets for any IFS generated by a family of circle diffeomorphisms with at least two generators C^2-close to the identity satisfying certain generic conditions.

Theorem E (Denjoy for IFS)**.** *There exists $\varepsilon > 0$ such that if $f_0, f_1 \in \mathrm{Diff}^2(S^1)$ are ε-close to the identity in the C^2-top. with no periodic points in common and both maps have finitely many periodic points then there are no invariant minimal Cantor sets for neither $\mathrm{IFS}(f_0, f_1)$ nor $\mathrm{IFS}(f_0^{-1}, f_1^{-1})$.*

Moreover, if n_i is the periodic of f_i, then there are no invariant minimal Cantor sets for neither $\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})$ nor $\mathrm{IFS}(f_0^{-n_0}, f_1^{-n_1})$ and the following conditions are equivalents:

i) S^1 is minimal for $\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})$,

ii) S^1 is minimal for $\mathrm{IFS}(f_0^{-n_0}, f_1^{-n_1})$,

iii) there are no ss-intervals for $\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})$,

iv) there are no uu-intervals for $\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})$.

If the periodic points of two maps in the hypothesis of the above theorem are hyperbolic then both diffeomorphisms are Morse-Smale. Recall that, the set of Morse-Smale diffeomorphisms is open and dense in $\mathrm{Diff}^2(S^1)$. On the other hand, the non-existence of common periodic points is also a generic condition. Therefore, this subset of diffeomorphisms in $\mathrm{Diff}^2(S^1)$ satisfying the hypothesis of Theorem E, is open and dense in a neighborhood of the identity.

Remark 3.29. *Under the additional generic assumption of hyperbolic periodic points, the assertions in Theorem 3.27 and Theorem E are C^1-robust. That is, if f_0 and f_1 are C^2-diffeomorphisms with hyperbolic fixed points in the corresponding assumptions of Theorem 3.27 and Theorem E then there exist C^1-neighborhoods $\mathcal{U}_i$ of f_i such that for every pair $g_0 \in \mathcal{U}_0$ and $g_1 \in \mathcal{U}_1$ the assertions in both theorems are fulfilled.*

In order to prove the Theorem E, we can assume that the set of periodic points of f_0 and f_1 is non-empty, otherwise, by Denjoy's Theorem, S^1 is minimal for both $\mathrm{IFS}(f_0, f_1)$ and $\mathrm{IFS}(f_0^{-1}, f_1^{-1})$. By the assumption of finiteness of periodic points, f_0 and f_1 cannot be rational rotations. Although this periodic points are not necessarily hyperbolic, we will assume that f_0 and f_1 are Morse-Smale diffeomorphisms but we will never use along the proof of the above theorem the hyperbolic character of its periodic points. This proof is given in the following two sections. We will find $\varepsilon > 0$ such that for every pair of Morse-Smale diffeomorphisms, f_0, f_1, in the hypothesis of Theorem E with a ss-interval for $\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})$, there are no invariant minimal Cantor sets for none of them: $\mathrm{IFS}(f_0, f_1)$, $\mathrm{IFS}(f_0^{-1}, f_1^{-1})$, $\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})$ and $\mathrm{IFS}(f_0^{-n_0}, f_1^{-n_1})$. See Proposition 3.36 and Remark 3.37. Previously, we need to generalize Duminy's Lemma for Morse-Smale diffeomorphisms. See Theorem 3.35. Then, in Section §3.3.3, under the assumption that there are no ss-intervals for $\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})$, we will prove the equivalences in the statement of Theorem E according to the following scheme:

$$(i) \to (iii)$$
$$(ii) \to (iv) \Leftrightarrow (iii)$$
$$(iii) \Rightarrow (i) \text{ and } (ii).$$

Here, "$\to$" means an immediately implication. The other two implications "$\Leftrightarrow$" and "$\Rightarrow$" will be shown in Proposition 3.38 and Theorem 3.53 (see also Proposition 3.39) respectively. Therefore, it follows from this equivalences that, in this case, there can be no invariant minimal Cantor sets for neither $\mathrm{IFS}(f_0, f_1)$ nor $\mathrm{IFS}(f_0^{-1}, f_1^{-1})$.

3.3.2 Duminy's Lemma for Morse-Smale diffeomorphisms

Let f_0 and f_1 be two C^2-diffeomorphisms on the circle. We want to show that if f_0 and f_1 are close enough to the identity then there are no invariant Cantor sets for $\mathrm{IFS}(f_0, f_1)$. According to Denjoy's Theorem (Theorem 3.26) we must assume that f_0 and f_1 have periodic points. In addition, we will suppose that both maps, f_0 and f_1, have finitely many periodic points all of them different (i.e., with no periodic points in common). Duminy's Lemma (Theorem D) provides a neighborhood of the identity such that if f_0 and f_1 belong to this neighborhood and there exists a ss-interval for $\mathrm{IFS}(f_0, f_1)$ then there is no exceptional minimal set. This claim follows from the uniqueness of the exceptional minimal set for an IFS since a ss-interval for an IFS in this hypothesis is an invariant minimal set. When f_0 and f_1 have periodic points of period, respectively, n_0 and n_1 larger than one, we could consider $\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})$, which is called *periodic IFS*. The distortion constant of this periodic maps $f_i^{n_i}$ is $n_i C_i$ where C_i is the distortion constant of f_i. If we try to apply Duminy's Lemma for the periodic IFS, then the estimate for the return map derivative obtained in Proposition 3.6 (see also Remarks 3.4 and 3.18) will be

$$\mathcal{R}'(x) \geq \frac{1}{2}(1-\varepsilon)\varepsilon^{-1}e^{4nC}, \quad \text{for all } x \in A$$

where $C = \max\{C_0, C_1\}$ and $n = \max\{n_0, n_1\}$. That is, the estimate depends on the period. When the period increases this estimate goes to zero and we need to reduce the size of $\varepsilon > 0$ to obtain an expanding first (periodic) return map. Notice that this is a problem if we are looking for a uniform neighborhood of the identity.

The first goal of this section is to show that actually we can obtain a new bound for the periodic return map derivative independent of the periods n_i. In the sequel we will assume that f_0 and f_1 are Morse-Smale diffeomorphisms on the circle of period n_0 and n_1, respectively, and such that there exists a $**$-interval for $\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})$ with $** \in \{ss, su\}$. For simplicity, we scale the $**$-interval into the interval $[0,1]$ and assume that $f_0^{n_0}(0) = 0$, $f_0^{n_0} < \mathrm{id}$ and $f_1^{n_1} > \mathrm{id}$ in $(0,1)$. As in the proof of Duminy's Lemma we can construct a first (periodic) return map $\mathcal{R}$ on a fundamental domain

$$A = (f_1^{n_1}(0), f_0^{-n_0}(f_1^{n_1}(0))] \subset [0,1]$$

of $f_0^{n_0}$ (see Lemma 3.5). Note that now the return maps $h_{i_1 i_2}$ correspond to $\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})$. It will be helpful to write $g_0 = f_0^{n_0}$ and $g_1 = f_1^{n_1}$. We will indicate the modifications required while estimating the return map derivative in Section §3.2.1.

In order to extending Duminy's Lemma for Morse-Smale diffeomorphisms with arbitrarily large period we will need the following properties of Morse-Smale dynamics on the circle.

Lemma 3.30. *Let $f \in \mathrm{Diff}^2(S^1)$ be a Morse-Smale diffeomorphism with period n. Set*

$$C = \max\{\frac{D^2 f(x)}{Df(x)} : x \in S^1\}$$

the distortion constant of f in S^1 and let I be a fundamental domain of f^n. Then

i) $e^{-C} \leq Df^n(x) \leq e^C$ *for all* $x \in S^1$,

ii) $f^i(I) \cap f^j(I) = \emptyset$ *for all* $i \neq j$ *(not necessarily multiples of n).*

Proof. Consider $J = [p, q]$ any interval in S^1 where p and q are two consecutive fixed points of f^n. We will show that

$$e^{-C} \leq Df^n(x) \leq e^C \quad \text{for all } x \in J. \tag{3.8}$$

Since S^1 is finite union of theses intervals we obtain (i). Since the interval J always contains a point with derivative equals one, according to Lemma 3.7, we only need to show the disjointness of $J, f(J), \dots, f^{n-1}(J)$ to prove (3.8). In order to prove this, suppose that $f^i(J) \cap f^j(J) \neq \emptyset$ for $0 \leq i < j \leq n-1$. Then $f^{j-i}(J)$ meets J. Since n is the period of f and $0 < j - i < n$ then this two closed intervals cannot be the same. Thus, either $f^{j-i}(p)$ or $f^{j-i}(q)$ belongs to the interior of J. Since any of these points are fixed points of f^n and f^n has not fixed point into the interior of J, we find a contradiction.

In a similar way, in order to prove the statement (ii), we consider a fundamental domain I for f^n. Then I is contained in some interval J of consecutive fixed points of f^n. Suppose that $f^i(I) \cap f^j(I) \neq \emptyset$ with $i < j$. Then $f^{j-i}(J) \cap J \neq \emptyset$. This two intervals must be the same since on the contrary arguing as above we obtain a contradiction. Thus $j - i$ must be a multiple of the period n. We write $j - i = kn$ for $k > 0$. So, $f^{kn}(I) \cap I \neq \emptyset$ which is a contradiction since I is a fundamental domain for f^n. Therefore, the proof of the lemma is now concluded. □

As a consequence of the above lemma, we modify Disjointness Lemma (Lemma 3.8) as follows:

Lemma 3.31 (Disjointness)**.** *Let $i_1 i_2$ be a fixed multi-index. Then,*

i) $U_\ell \stackrel{\text{def}}{=} f_1^{-\ell}(I_{i_1 i_2})$ for $\ell \geq 0$ are pairwise disjoint right-closed intervals of S^1;

ii) $U_{i_1,\ell} \stackrel{\text{def}}{=} f_0^{-\ell} \circ f_1^{-n_1 m_{i_1}}(I_{i_1 i_2})$ for $\ell \geq 0$ are pairwise disjoint right-closed intervals of S^1.

Proof. Note that $I_{i_1 i_2} \subset I_{i_1}$ where I_{i_1} is contained in a fundamental domain of $g_1 = f_1^{n_1}$. Thus, by Lemma 3.30, it follows that the intervals $U_\ell \subset S^1$ are right-closed disjoint with respect to each other. Also, according to Lemma 3.5, we have that

$$g_0^{-m_{i_1 i_2}} \circ g_1^{-m_{i_1}}(I_{i_1 i_2}) = h_{i_1 i_2}^{-1}(I_{i_1 i_2}) \subset A.$$

Hence $f_1^{-n_1 m_{i_1}}(I_{i_1 i_2}) = g_1^{-m_{i_1}}(I_{i_1 i_2}) \subset g_0^{m_{i_1 i_2}}(A)$. Since A is a fundamental domain of $g_0 = f_0^{n_0}$ then $f_1^{-n_1 m_{i_1}}(I_{i_1 i_2})$ is contained in a fundamental domain of $f_0^{n_0}$ and thus, by Lemma 3.30, it follows that the intervals $U_{i_1,\ell} \subset S^1$ are pairwise disjoint. This concludes the lemma. □

The above lemma allows us to obtain a new estimate for the distortion of $h_{i_1 i_2}^{-1}$ which is independently of the period n_i. Namely, the statement of Distortion Lemma (Lemma 3.9) takes the following form:

Lemma 3.32 (Distortion)**.** *Let $C > 0$ be the largest distortion constant of f_0 and f_1 in S^1. Then for every $j \geq 0$ it follows that* $\mathrm{Dist}(h_{i_1 j}^{-1}, \overline{I_{i_1 i_2}}) \leq 2C$. *Consequently, for every pair of intervals J and L contained in $\overline{I_{i_1 i_2}}$ and for $j \geq 0$, it holds that*

$$\frac{|J|}{|L|}\, e^{-2C} \leq \frac{|h_{i_1 j}^{-1}(J)|}{|h_{i_1 j}^{-1}(L)|} \leq e^{2C}\, \frac{|J|}{|L|}.$$

Moreover, if $I = I_{i_1 j+1} \cup I_{i_1 j}$ then for every $j \geq 0$, it holds that

$$\frac{|h_{i_1 j+1}^{-1}(I)|}{|I|}\, e^{-4C} \leq Dh_{i_1 j+1}^{-1}(z) \leq e^{4C}\, \frac{|h_{i_1 j+1}^{-1}(I)|}{|I|} \quad \text{for all } z \in I.$$

Proof. Recall that $h_{i_1j}^{-1} = g_0^{-m_{i_1j}} \circ g_1^{-m_{i_1}} = f_0^{-n_0 m_{i_1j}} \circ f_1^{-n_1 m_{i_1}}$. Then

$$Dh_{i_1j}^{-1}(x) = \prod_{\ell=0}^{n_0 m_{i_1j}-1} Df_0^{-1}(f_0^{-\ell} \circ f_1^{-n_1 m_{i_1}}(x)) \cdot \prod_{\ell=0}^{n_1 m_{i_1}-1} Df_1^{-1}(f_1^{-\ell}(x)).$$

By means of the distortion control of f_0 and f_1, we obtain that for every $x, y \in \overline{I_{i_1 i_2}}$

$$|\log \frac{Dh_{i_1j}^{-1}(x)}{Dh_{i_1j}^{-1}(y)}| \leq C\,\Big(\sum_{\ell=0}^{n_0 m_{i_1j}-1} |U_{i_1,\ell}| + \sum_{\ell=0}^{n_1 m_{i_1}-1} |U_\ell|\,\Big).$$

The disjointness of each families of intervals U_ℓ and $U_{i_1,\ell}$ for $\ell \geq 0$ showed in Lemma 3.31 implies that $\mathrm{Dist}(h_{i_1j}^{-1}, \overline{I_{i_1i_2}}) \leq 2C$. The rest of the proof of this lemma is analogous of second part in Lemma 3.9. □

As in Lemma 3.10 we can obtain the same lower bounded distortion estimate between the length of I_{i_1} and $I_{i_1i_2}$.

Lemma 3.33 (Compared intervals)**.** *Let $C_1 > 0$ be the distortion constant of f_1 in S^1. Consider $\delta > 0$ such that $|Df_0^{n_0}(x) - 1| < \delta$ for all $x \in (0, 1)$. Then*

$$\frac{|I_{i_1}|}{|I_{i_1i_2}|} > \delta^{-1} e^{-C_1}$$

for all multi-index $i_1 i_2$.

Proof. Recall that I_{i_1} is contained in a fundamental domain of $g_1 = f_1^{n_1}$. Therefore, from Lemma 3.30 we see that $f_1^{-i}(I_{i_1})$ for $i \geq 0$ are disjoints intervals in S^1. Using Lemma 3.7 we still have that

$$\frac{|I_{i_1}|}{|I_{i_1i_2}|} \geq e^{-C_1} \frac{|f_1^{-n_1 m_{i_1}}(I_{i_1})|}{|f_1^{-n_1 m_{i_1}}(I_{i_1i_2})|} = e^{-C_1} \frac{|g_1^{-m_{i_1}}(I_{i_1})|}{|g_1^{-m_{i_1}}(I_{i_1i_2})|}.$$

The rest of the proof is the same that Lemma 3.10. Observe that we only need to use the bounded distortion estimate from above and the assumption $|Dg_0(x) - 1| < \delta$ for all $x \in (0, 1)$. Therefore, it follows the desired result. □

Now, we are ready to obtain an estimation for the derivative of the periodic return map.

Proposition 3.34. *Let $C > 0$ be the largest distortion constant of f_0 and f_1. Consider $\delta > 0$ such that $|Df_0^{n_0}(x) - 1| < \delta$ for all $x \in (0, 1)$. Then*

$$\begin{aligned} &\mathcal{R}'(x) \geq \delta^{-1} e^{-3C} && \text{if} \quad x \in \bigcup_{i_1=0}^{m} \bigcup_{i_2=1}^{\infty} I_{i_1i_2}, \\ &\mathcal{R}'(x) \geq \frac{1}{2}\,(1-\delta)\delta^{-1} e^{-5C} && \text{if} \quad x \in I_{00}. \end{aligned}$$

Proof. With the help of the above lemmas the proof of these estimates in the case of Morse-Smale diffeomorphisms turns into the proof of Proposition 3.6. We only indicate how these lemmas are used to obtain the new estimates.

Let x be a interior point of $I_{i_1i_2}$. Take an arbitrarily small open interval J such that $x \in J \subset I_{i_1i_2}$. Notice that $\mathcal{R}(J) = h^{-1}_{i_1i_2}(J)$. Suppose that $i_1i_2 \neq 00$. Then $h^{-1}_{i_1i_2}(I_{i_1i_2}) = A \supset I_{i_1} \supset I_{i_1i_2}$. By Lemma 3.32, it follows that

$$|\mathcal{R}(J)| \geq e^{-2C}\,\frac{|h^{-1}_{i_1i_2}(I_{i_1i_2})|}{|I_{i_1i_2}|}\,|J| \geq e^{-2C}\,\frac{|I_{i_1}|}{|I_{i_1i_2}|}\,|J|.$$

By Lemma 3.33, since $C > 0$ is the largest distortion constant of f_0 and f_1 then $|\mathcal{R}(J)| > \delta^{-1}e^{-3C}\,|J|$. The above inequality implies that $\mathcal{R}'(x) = Dh^{-1}_{i_1i_2}(x) \geq \delta^{-1}e^{-2C}$ for all $x \in I_{i_1i_2}$ with $i_1i_2 \neq 00$. For the case $i_1i_2 = 00$, recalling that $m_{i_1i_2+1} = m_{i_1i_2} + 1$ it follows $h^{-1}_{00} = f_0^{n_0} \circ h^{-1}_{01}$. Then, by Mean Value Theorem, there are $\xi \in h^{-1}_{01}(J)$ and $\zeta \in J$ such that

$$|\mathcal{R}(J)| = |Df_0^{n_0}(\xi)||Dh^{-1}_{01}(\zeta)||J| > (1-\delta)|Dh^{-1}_{01}(\zeta)||J|.$$

From the estimate of Dh^{-1}_{01} on the $I = I_{01} \cup I_{00}$ obtained in Lemma 3.32 it follows that, $Dh^{-1}_{01}(\zeta) \geq e^{-4C}\,|h^{-1}_{01}(I)|/|I|$. As $h^{-1}_{01}(I) \supset A \supset I_0$ then $|h^{-1}_{01}(I)| \geq |I_0|$. Then, by Lemma 3.33, taking into account again that $C > 0$ is the largest distortion constant of f_0 and f_1, we see that

$$\frac{|h^{-1}_{01}(I)|}{|I|} \geq \frac{|I_0|}{|I|} = \Big(\frac{|I_{01}|}{|I_0|} + \frac{|I_{00}|}{|I_0|}\Big)^{-1} > \frac{1}{2}\,\delta^{-1}e^{-C}.$$

Finally, $|\mathcal{R}(J)| > \frac{1}{2}(1-\delta)\delta^{-1}e^{-5C}\,|J|$. This implies that $\mathcal{R}'(x) = Dh^{-1}_{00}(x) \geq (1-\delta)\delta^{-1}e^{-5C}/2$ for all $x \in I_{00}$ and we conclude the proposition. $\square$

Now, we are ready to extend the Duminy's Lemma for Morse-Smale diffeomorphisms:

Theorem 3.35 (Duminy's Lemma for Morse-Smale diffeomorphisms)**.** *There exists $\varepsilon \geq 0.13$ such that if $f_0, f_1 \in \mathrm{Diff}^2(S^1)$ are Morse-Smale diffeomorphisms of period n_0 and n_1, respectively, and ε-close to the identity in the C^2-topology, then for any $**$-interval K^{**}_Φ for $\mathrm{IFS}(\Phi)$ with $** \in \{ss, su\}$ and $\Phi = \{f_0^{n_0}, f_1^{n_1}\}$, there are neighborhoods $\mathcal{U}_i$ of f_i in the C^1-topology such that*

$$K^{**}_\Psi \subset \overline{\mathrm{Per}(\mathrm{IFS}(\Psi))} \quad \textit{and} \quad K^{**}_\Psi \subset \overline{\mathrm{Orb}_\Psi(x)} \textit{ for all } x \in K^{**}_\Psi$$

for every $\Psi = \{g_0^{n_0}, g_1^{n_1}\}$ with $g_i \in \mathcal{U}_i$.

Proof. Take $\delta = 0.17 > 0$. Then $(1-\delta)\delta^{-1}e^{-5\delta} > 2$. Let $\varepsilon > 0$ small enough such that

$$1 - \delta < e^{-\varepsilon(1-\varepsilon)^{-1}} < e^{\varepsilon(1-\varepsilon)^{-1}} < 1 + \delta \quad \text{and} \quad \varepsilon(1-\varepsilon)^{-1} < \delta. \tag{3.9}$$

Note that these conditions are satisfies for every positive $\varepsilon \leq 0.13$.

As we are assuming that f_0 and f_1 are C^2-diffeomorphisms then the distortion constants of f_0 and f_1 can be written

$$C_0 = \max_{x\in S^1}\left|\frac{D^2 f_0(x)}{Df_0^{-1}(x)}\right| > 0, \quad \text{and} \quad C_1 = \max_{x\in S^1}\left|\frac{D^2 f_1(x)}{Df_1^{-1}(x)}\right| > 0.$$

Note that

$$|D^2 f_i(x)|/|Df_i(x)| = |D^2 f_i^{-1}(x)|/|Df_i^{-1}(x)|$$

and so the constant C_0 and C_1 are also the distortion constant of f_0^{-1} and f_1^{-1}. Set $C = \max\{C_0, C_1\} > 0$. Since f_i are ε-close to the identity in the C^2-topology and by choosing of $\varepsilon > 0$ in (3.9) it follows that $0 < C < \varepsilon(1-\varepsilon)^{-1} < \delta$.

We will show that $|Df_0^{n_0}(x)-1|<\delta$ for all $x\in K_\Phi^{**}=[0,1]$. By Lemma 3.30 it follows that for every $x\in S^1$,

$$e^{-\varepsilon(1-\varepsilon)^{-1}}<e^{-C}\leq Df_0^{n_0}(x)\leq e^{C}<e^{\varepsilon(1-\varepsilon)^{-1}}.$$

Substituting (3.9) in this inequality we conclude that $|Df_0^{n_0}(x)-1|<\delta$ for all $x\in S^1$.

Proposition 3.6 and the above estimates calculated for both, the distortion constant C and the derivative of f_0, imply that

$$\mathcal{R}'(x)\geq\frac{1}{2}(1-\delta)\delta^{-1}e^{-5C}>\frac{1}{2}(1-\delta)\delta^{-1}e^{-5\delta}>1\quad\text{for all } x\in A.$$

That is, $\mathcal{R}$ is an expanding return map over the fundamental domain A.

The rest of the proof of this theorem and the proof of Duminy's Lemma (Theorem D) are totally analogous. With regard to the C^1-robustness, again we can use the same argument of Duminy's Lemma and thus this part follows from Theorem 3.13. So, the proof of this result is completed. □

The following result shows that in the presence of a ss-interval for the periodic IFS there are no invariant minimal Cantor sets.

Proposition 3.36. *Let $\varepsilon>0$ be in Theorem 3.35. Let $f_0,f_1\in\mathrm{Diff}^2(S^1)$ be a pair of Morse-Smale diffeomorphisms of period n_0 and n_1, respectively, and ε-close to the identity in the C^2-topology. Assume that they have no common periodic points and there exists a ss-interval for* $\mathrm{IFS}(f_0^{n_0},f_1^{n_1})$. *Then there are no minimal invariant Cantor sets for neither* $\mathrm{IFS}(f_0^{n_0},f_1^{n_1})$ *nor* $\mathrm{IFS}(f_0,f_1)$.

Proof. Observe that, by Theorem 3.35, any ss-interval K_Φ^{ss} for $\mathrm{IFS}(\Phi)$ satisfies that

$$K_\Phi^{ss}=\overline{\mathrm{Orb}_\Phi(x)}\quad\text{for all } x\in K_\Phi^{ss}$$

where $\Phi=\{f_0^{n_0},f_1^{n_1}\}$. Moreover, note that K_Φ^{ss} is also an invariant minimal set for $\mathrm{IFS}(f_0,f_1)$ since $\mathrm{IFS}(\Phi)$ is contained in $\mathrm{IFS}(f_0,f_1)$. Let us denote by $\mathcal{F}$ either, $\mathrm{IFS}(\Phi)$ or $\mathrm{IFS}(f_0,f_1)$ since the argument to exclude the invariant minimal Cantor sets is analogous for both of them.

Suppose that Λ is an invariant minimal Cantor set for $\mathcal{F}$. The first observation is that there are attracting fixed points of $f_i^{n_i}$ in Λ for $i=1,2$. In order to prove this, we fix any point $p\in\Lambda$. Hence, since $f_i^{n_i}$ is a Morse-Smale diffeomorphism then $q=\lim_{k\to\infty}f_i^{kn_i}(p)$ is an attracting fixed point of $f_i^{n_i}$ and since Λ is closed invariant for $\mathcal{F}$ then $q\in\Lambda$. We claim[4] that one of these fixed points is the endpoint of a ss-interval for $\mathrm{IFS}(\Phi)$. Indeed, since there are no common periodic points, the same above argument to find periodic points in Λ allows us to move out through the basin of the attracting fixed point of $f_i^{n_i}$ to an endpoint of some ss-interval K_Φ^{ss} for $\mathrm{IFS}(\Phi)$. Finally, since Λ is an invariant minimal set for $\mathcal{F}$, the closure of the orbit of this endpoint of K_Φ^{ss} is Λ. However, as already mentioned, the closure of orbit of this endpoint for $\mathcal{F}$ is the K_Φ^{ss} and hence $K_\Phi^{ss}=\Lambda$ which is a contradiction since Λ is a Cantor set and K_Φ^{ss} has not empty interior. Therefore, it follows that there is no exceptional minimal set for $\mathcal{F}$, that is, for both $\mathrm{IFS}(\Phi)$ and $\mathrm{IFS}(f_0,f_1)$, and so, we conclude the proof of the proposition. □

[4]More details of this claim can be understood in the following Section §3.3.3 where we study cycles for IFS.

In the following section, we will show that the existence of a ss-interval for $\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})$ implies the existence of a uu-interval for the same IFS (see Proposition 3.38) and thus, the existence of a ss-interval for $\mathrm{IFS}(f_0^{-n_0}, f_1^{-n_1})$. If necessary, we reduce the size of the neighborhood of the identity to obtain that for every f_0 and f_1 in this new neighborhood, f_0^{-1} and f_1^{-1} are ε-close to the identity. So, from the above proposition it follows that:

Remark 3.37. *Under the assumptions of Proposition 3.36, it holds that there are no invariant minimal Cantor sets for both* $\mathrm{IFS}(f_0^{-n_0}, f_1^{-n_0})$ *and* $\mathrm{IFS}(f_0^{-1}, f_1^{-1})$.

3.3.3 Cycles for IFS on the circle

Recall that a ss-interval for $\mathrm{IFS}(f_0, f_1)$ close enough to the identity map is an interval define by a pair of consecutive attractors each from a different diffeomorphism f_i, $i = 0, 1$. In a similar way, a uu-interval for $\mathrm{IFS}(f_0, f_1)$ is define as a ss-interval for $\mathrm{IFS}(f_0^{-1}, f_1^{-1})$.

Proposition 3.38. *Let f_0, f_1 be Morse-Smale circle diffeomorphisms with no fixed points in common. Then, there is a ss-interval for* $\mathrm{IFS}(f_0, f_1)$ *if and only if there is a uu-interval for* $\mathrm{IFS}(f_0, f_1)$.

Proof. We only need to prove that if $\mathrm{IFS}(f_0, f_1)$ has an interval K^{uu}_Φ then also has an interval K^{ss}_Φ. Consider S^1 parametrice by $[0, 1] \bmod 1$. By abuse of notation, we continue to write f_i for the lift map. Note that since f_0 is a circle diffeomorphism then the number of fixed point is even. Thus, we may denote by p_i and $\tilde{p}_i$ for $i = 1, \dots n$ the attractor and repeler fixed points of f_0 respectively ordered in the real order on $[0, 1]$. In a similar way, q_j and $\tilde{q}_j$ for $j = 1, \dots, m$ denote the attractor and repeler points of f_1 respectively ordered in the real order on $[0, 1]$. We may assume without loss of generality that $0 = \tilde{p}_1 < \tilde{q}_1 < p_1 < q_2$ and that $p_n < q_m < 1$. Thus $K^{uu}_\Phi = [\tilde{p}_1, \tilde{q}_1]$ and the rest of the fixed points of f_0 and f_1 belong to the interval $[p_1, q_m]$. Note that, since K^{uu}_Φ has only two fixed points of f_0 and f_1 then, in $[p_1, q_m]$ there is an even number of fixed points. We assume that there are no ss-intervals. Then, since there are no fixed points in common, we may construct a sequence of attractor $p_1 < q_1 < \ldots < p_{i_k} < q_{j_k} < p_{i_{k+1}} < \ldots < p_n < q_m$ where p_{i_k} is in the basin of attraction of q_{j_k} and q_{j_k} is in the basin $p_{i_{k+1}}$. From this, we have the partition

$$(p_1, q_m] = (p_1, q_1] \cup \ldots \cup (p_{i_k}, q_{j_k}] \cup (q_{j_k}, p_{i_{k+1}}] \cup \ldots \cup (p_n, q_n].$$

Since there are no ss-intervals then in each interval of the above partition there are an even number of fixed points of f_0 and f_1. Hence, there is an even number of fixed point in $(p_1, q_m]$. This leads to a contradiction with the number of fixed points in $[p_1, q_m]$ and it proves the proposition. $\square$

We introduce the notion of cycle for an IFS of two diffeomorphisms on the circle.

Definition 3.7 (Cycle). *Let f_0 and f_1 be two circle diffeomorphisms. Denote by p_i the attractors of f_0 and by q_i the attractors of f_1. Define a partial order on the attracting points by $p_i \prec q_j$ if and only if p_i belongs to the basin of attraction of q_j for f_1. Similar definitions for $q_i \prec p_j$. A sequence of attractors forms a* cycle of length n for $\mathrm{IFS}(f_0, f_1)$ *if we have*

$$p_{i_1} \prec q_{i_2} \prec p_{i_3} \prec \ldots \prec q_{i_n} \prec p_{i_{n+1}} \quad \textit{and} \quad p_{i_1} = p_{i_{n+1}}.$$

The cycle is said to be minimal *if it has no sub-cycles, that is, there are no $1 \le j < k \le n$ such that $p_{i_j} = p_{i_k}$ or $q_{i_j} = q_{i_k}$.*

A consecutive pair of attractors for two different diffeomorphisms f_0 and f_1 close enough to the identity is a minimal cycle of length 2. Note that this type of cycle of length 2 define an ss-interval for $\mathrm{IFS}(f_0, f_1)$. The following result shows the existence of a minimal cycle for an IFS of Morse-Smale diffeomorphisms with fixed points on the circle.

Proposition 3.39. *Let f_0 and f_1 are Morse-Smale diffeomorphisms on S^1 of period one with no fixed points in common. Then there exists at least one minimal cycle for $\mathrm{IFS}(f_0, f_1)$.*

Proof. If the IFS has a ss-interval then has a minimal cycle of length 2. Suppose that there are no ss-intervals. By Proposition 3.38, there are no uu-intervals. In the proof of that result we actually construct a cycle for $\mathrm{IFS}(f_0, f_1)$. Hence, argue in a similar way, we find a minimal cycle for it follows $\mathrm{IFS}(f_0, f_1)$ and therefore we conclude the proof of the proposition. $\square$

In the sequel, we assume that f_0 and f_1 are Morse-Smale diffeomorphisms on S^1 of period one ($n_i = 1$) and with no fixed points in common. Let $\mathcal{C}_n = \{p_{j_1} \prec q_{j_2} \prec \ldots \prec q_{j_n} \prec p_{j_1}\}$ be a minimal cycle for $\mathrm{IFS}(f_0, f_1)$. Note that the length n of the minimal cycle is even. We will use the symbols s_k for $k = 0, \ldots, n$ to denote of cycle elements. In particular, $s_{2i-1} = q_{j_{n-(2i-1)+1}}$ and $s_{2\ell} = p_{j_{n-2i+1}}$ for $i = 1, \ldots, n/2$. Note that with this new notation

$$s_n \prec s_{n-1} \prec \ldots \prec s_2 \prec s_1 \prec s_0 \quad \text{and} \quad s_0 = s_n.$$

Note that s_k is an attracting fixed point for $f_{k \bmod 2}$. That is, if k is an even number then s_k is a fixed point of f_0 and if k is an odd number then of f_1. We consider S^1 parametrice by $[s_0, s_0 + 1] \bmod 1$. Then $s_0 < s_k < s_0 + 1$ for $k = 1, \ldots, n-1$ in the real order on the interval $[s_0, s_0 + 1]$. We denote by s_k^- and s_k^+ the repelling points of $f_{k \bmod 2}$ closest from the left and right respectively to s_k. For the special case $k = 0$ note that the ordered on the interval $[s_0, s_0 + 1]$ is $s_0 < s_0^+ \leq s_0^- = s_n^- < s_n = s_0 + 1$. Since $s_1 \prec s_0$ then $s_1 \in (s_0, s_0^+) \cup (s_0^-, s_0 + 1)$. We may suppose, without losing generality, that $s_1 \in (s_0, s_0^+)$.

Lemma 3.40. *If there are no ss-intervals for $\mathrm{IFS}(f_0, f_1)$ then*

$$s_k < s_{k+1}^- < s_{k+1} < s_k^+ \quad \textit{for } k = 0, \ldots, n-1.$$

Proof. Since there are no ss-intervals from the geometry of the functions, we cannot have an attractor-attractor pair for $\mathrm{IFS}(f_0, f_1)$. Hence, since $s_1 \prec s_0$ and $s_1 \in (s_0, s_0^+)$, the fixed point of f_1 in this interval closest to s_0 is a repeller s_1^-. Therefore $s_0 < s_1^- < s_1 < s_0^+$. Note that this shows the claim for $k = 0$. Inductively suppose that $s_{k-1} < s_k^- < s_k < s_{k-1}^+$. Since $s_{k+1} \prec s_k$ then $s_{k+1} \in (s_k^-, s_k) \cup (s_k, s_k^+)$. If $s_{k+1} \in (s_k^-, s_k)$ then $s_{k-1} < s_k^- < s_{k+1} < s_k < s_{k-1}^+$ contradicting that $s_k \prec s_{k-1}$. Thus $s_{k+1} \in (s_k, s_k^+)$. As there are no ss-intervals and $s_{k+1} \prec s_k$ then the fixed point of $f_{k+1 \bmod 2}$ in (s_k, s_k^+) closest to s_k is a repeller s_{k+1}^-. Therefore $s_{k+1} < s_{k+1}^- < s_{k+1} < s_k^+$ and in $n-1$ steps of this induction we conclude the proof of the lemma. $\square$

If $\mathrm{IFS}(f_0, f_1)$ has a minimal cycle of length 2 defining a ss-interval then, according to Proposition 3.38, it has a uu-interval. Note that, now, this uu-interval defines a minimal cycle of length 2 for $\mathrm{IFS}(f_0^{-1}, f_1^{-1})$. We will show the same observation for a general cycle $\mathcal{C}_n$. Before that, we denote by $\tilde{s}_k$ the repeler of $f_{k \bmod 2}$ closest of s_k. This means that $\tilde{s}_k$ satisfies that there are no repeler $\tilde{s}$ of $f_{k \bmod 2}$ with the order $s_{k-1} < \tilde{s} < \tilde{s}_k$ on the interval $[s_0, s_0 + 1]$.

Lemma 3.41. *If there are no ss-intervals for* $\mathrm{IFS}(f_0, f_1)$ *then*

$$\tilde{s}_1 \prec \ldots \prec \tilde{s}_n \prec \tilde{s}_{n+1} \quad \text{where } \tilde{s}_{n+1} = \tilde{s}_1$$

is a minimal cycle for $\mathrm{IFS}(f_0^{-1}, f_1^{-1})$.

Proof. We will show that $\tilde{s}_k$ belongs to the basin of attraction of $\tilde{s}_{k+1}$. Note that by definition of $\tilde{s}_k$ and from Lemma 3.40 we have $\tilde{s}_k \leq s_k^-$ for $k = 1, \ldots, n$. So, we have the following order $s_{k-1} < \tilde{s}_k \leq s_k^- < s_k < \tilde{s}_{k+1}$. Hence, as $s_k \prec s_{k-1}$ it follows that the interval $(s_{k-1}, s_k]$ has no fixed point of $f_{k+1 \bmod 2}$. By definition, $\tilde{s}_{k+1}$ is the closest fixed points of $f_{k+1 \bmod 2}$ from the right to s_k. Thus, the interval $(s_{k-1}, \tilde{s}_{k+1}]$ is contained in the basin of attraction of $\tilde{s}_{k+1}$ for $f^{-1}_{k+1 \bmod 2}$ and therefore also $\tilde{s}_k$. The minimality of this cycle is followed from the minimality of $\mathcal{C}_n$. $\square$

According to Proposition 3.39, we always have a minimal cycle $\mathcal{C}_n$ for $\mathrm{IFS}(f_0, f_1)$. If the cycle is a ss-interval for $\mathrm{IFS}(f_0, f_1)$ then, according to Proposition 3.38, this IFS has also a uu-interval. Under the assumption that f_0 and f_1 are close enough to the identity map, Theorem D implies that theses intervals are minimal sets for $\mathrm{IFS}(f_0, f_1)$ and $\mathrm{IFS}(f_0^{-1}, f_1^{-1})$ respectively. If the cycle is not a ss-interval for $\mathrm{IFS}(f_0, f_1)$, Proposition 3.41 shows the existence of a minimal cycle different of a uu-interval for $\mathrm{IFS}(f_0^{-1}, f_1^{-1})$. The following theorem shows that in this case, S^1 is minimal for both $\mathrm{IFS}(f_0, f_1)$ and $\mathrm{IFS}(f_0^{-1}, f_1^{-1})$.

Theorem 3.42 (Cycle)**.** *There exists* $\varepsilon \geq 0.23$ *such that if* $f_0, f_1 \in \mathrm{Diff}^2(S^1)$ *are Morse-Smale diffeomorphisms* ε*-close to the identity in the* C^2*-topology with a minimal cycle for* $\mathrm{IFS}(f_0, f_1)$ *different of a ss-interval, then* S^1 *is* C^1*-robustly minimal for* $\mathrm{IFS}(f_0, f_1)$ *and* $\mathrm{IFS}(f_0^{-1}, f_1^{-1})$.

The proof will be very similar to the local case in the Duminy's Lemma (see Theorem D). Here the expanding return map will be of global character.

Creating a return map

Lemma 3.43 (Creating a return map)**.** *In the hypothesis of Theorem 3.42, there exist families of right-closed pairwise disjoint intervals* $I_{i_1 \ldots i_n} \subset S^1$ *and maps* $h_{i_i \ldots i_n} \in \mathrm{IFS}(f_0, f_1)$ *with* $i_j \geq 0$ *for* $j = 1, \ldots, n$ *such that*

i) $A = (s_0, f_1^{-1}(s_0)] = \bigcup I_{i_1 \ldots i_n}$,

ii) $h^{-1}_{i_1 \ldots i_n}(I_{i_1 \ldots i_n}) \subset A$ *if* $i_n = 0$ *and* $h^{-1}_{i_1 \ldots i_n}(I_{i_1 \ldots i_n}) = A$ *if* $i_n > 0$,

iii) if $c \in A \setminus \{f_1^{-1}(s_0)\}$ *is an endpoint of* $I_{i_1 \ldots i_n}$ *then there exist* $h \in \mathrm{IFS}(f_0, f_1)$ *and* $s \in \mathcal{C}_n$ *such that* $h(s) = c$*. That is, it is in the orbit of the cycle for* $\mathrm{IFS}(f_0, f_1)$.

Proof. By Lemma 3.40 we have $s_{k-1} < s_k^- < s_k < s_{k+1}^-$. Then $s_k < f^{-1}_{k+1 \bmod 2}(s_k) < s_{k+1}^-$. Thus, we can define a non-empty fundamental domain of $f_{k+1 \bmod 2}$, $A_k = (s_k, f^{-1}_{k+1 \bmod 2}(s_k)]$ for $k = 0 \ldots, n$. As on the circle $s_0 = s_n$ then we also identify on S^1 the intervals in the real line $A_0 = (s_0, f_1^{-1}(s_0)]$ and $A_n = (s_n, f_1^{-1}(s_n)]$. We denote this interval by A. In order to create the expanding return map we will divide this fundamental domain inductively.

As $s_1 \prec s_0$, there exists $j \in \mathbb{N}$ such that $s_0 < f_0^j(s_1) < f_1^{-1}(s_0) \leq f_0^{j-1}(s_1) < s_1$. Then

$$A = (s_0, f_1^{-1}(s_0)] = \bigcup_{i_1=0}^{\infty} I_{i_1},$$

with $I_0 = (f_0^j(s_1), f_1^{-1}(s_0)]$ and $I_{i_1} = (f_0^{j+i_1}(s_1), f_0^{j+i_1-1}(s_1)]$ if $i_1 > 0$. Let $h_{i_1} = f_0^{j+i_1}$, we have that $h_{i_1}^{-1}(I_{i_1}) = (s_1, c_{i_1}]$ where $c_0 = f_0^{-j} \circ f_1^{-1}(s_0) \in (s_1, f_0^{-1}(s_1)]$ and $c_{i_1} = f_0^{-1}(s_1)$ if $i_1 > 0$. Therefore, $h_0^{-1}(I_0) \subset A_1$ and $h_{i_1}^{-1}(I_1) = A_1$ if $i_1 > 0$. Let $c \in A \setminus \{f_1^{-1}(s_0)\}$ be an endpoint of I_{i_1}. Then either $c = f_0^j(s_1)$ if $i_1 = 0$ or $c \in \{f_0^{j+i_1}(s_1), f_0^{j+i_1-1}(s_1)\}$ if $i_1 > 0$. In any case, c belongs to the orbit of the cycle. This completes the first step of the induction and now we proceed with the inductive hypothesis. Suppose that we have families of right-closed pairwise disjoint intervals $I_{i_1\dots i_k} \subset S^1$ and maps $h_{i_1\dots i_k} \in \mathrm{IFS}(f_0, f_1)$ with $i_j \geq 0$ for $j = 1, \dots, k$ such that

(i) $A = \bigcup I_{i_1\dots i_k}$,

(ii) $h^{-1}_{i_1\dots i_k}(I_{i_1\dots i_k}) = (s_k, c_{i_1\dots i_k}]$ where $c_{i_1\dots i_{k-1}0} \in (s_k, f^{-1}_{k+1 \bmod 2}(s_k)]$ and $c_{i_1\dots i_k} = f^{-1}_{k+1 \bmod 2}(s_k)$ if $i_k > 0$. Therefore, $h^{-1}_{i_1\dots i_{k-1}0}(I_{i_1\dots i_{k-1}0}) \subset A_k$ and $h^{-1}_{i_1\dots i_k}(I_{i_1\dots i_k}) = A_k$ if $i_k > 0$.

(iii) if $c \in A \setminus \{f_1^{-1}(s_0)\}$ is an endpoint of $I_{i_1\dots i_k}$ then there exist $h \in \mathrm{IFS}(f_0, f_1)$ and $s \in \mathcal{C}_n$ such that $h(s) = c$.

Recall that by Lemma 3.40 it follows that $s_k < f^{-1}_{k+1 \bmod 2}(s_k) < s^-_{k+1}$. Hence, from the inductive hypothesis we have $s_k < c_{i_1\dots i_k} \leq f^{-1}_{k+1 \bmod 2}(s_k) < s_{k+1}$. Now, since $s_{k+1} \prec s_k$, for each multi-index $i_1 \dots i_k$ there exists $j_{i_1\dots i_k} \in \mathbb{N}$ such that

$$s_k < f^{j_{i_1\dots i_k}}_{k \bmod 2}(s_{k+1}) \leq c_{i_1\dots i_k} < f^{j_{i_1\dots i_k}-1}_{k \bmod 2}(s_{k+1}) < s_{k+1}.$$

Then

$$h^{-1}_{i_1\dots i_k}(I_{i_1\dots i_k}) = (s_k, c_{i_1\dots i_k}] = \bigcup_{\ell=0}^{\infty} J_{i_1\dots i_k \ell}$$

with

$$J_{i_1\dots i_k 0} = (f^{j_{i_1\dots i_k}}_{k \bmod 2}(s_{k+1}), c_{i_1\dots i_k}] \quad \text{and} \quad J_{i_1\dots i_k \ell} = (f^{j_{i_1\dots i_k}+\ell}_{k \bmod 2}(s_{k+1}), f^{j_{i_1\dots i_k}+(\ell-1)}_{k \bmod 2}(s_{k+1})] \ \text{ if } \ell > 0.$$

By construction the intervals $J_{i_1\dots i_k \ell}$ for $\ell \geq 0$ are pairwise disjoint. Define $I_{i_1\dots i_k \ell} \subset I_{i_1\dots i_k}$ by $I_{i_1\dots i_k \ell} = h_{i_1\dots i_k}(J_{i_1\dots i_k \ell})$. By definition, fixed a multi-index $i_1 \dots i_k$, the intervals $I_{i_i\dots i_k \ell}$ for $\ell \geq 0$ are also pairwise disjoint. Since $I_{i_1\dots i_k \ell} \subset I_{i_1\dots i_k}$ and as by induction hypothesis the intervals $I_{i_1\dots i_k}$ are pairwise disjoint then $\{I_{i_1\dots i_{k+1}} : \ i_j \geq 0 \text{ for } j = 1, \dots k+1\}$ is a family of right-closed pairwise disjoint intervals. Note that each right-closed interval $I_{i_1\dots i_k}$ is union of the intervals $I_{i_1\dots i_k \ell}$ for $\ell \geq 0$. Then, by the induction hypothesis it also follows that $A = \cup I_{i_1\dots i_{k+1}}$.

In order to prove the third item, we fix an interval $I_{i_1\dots i_k \ell}$. For every $\ell \geq 0$ the left endpoints of this interval is $h_{i_1\dots i_k} \circ f^{j_{i_1\dots i_k}+\ell}_{k \bmod 2}(s_{k+1})$. So, it belongs to the orbit of the cycle. The right is

$$h_{i_1\dots i_k}(c_{i_1\dots i_k}) \text{ if } \ell = 0 \quad \text{and} \quad h_{i_1\dots i_k} \circ f^{j_{i_1\dots i_k}+\ell-1}_{k \bmod 2}(s_{k+1}) \text{ if } \ell > 0.$$

Note that $h_{i_1\dots i_k}(c_{i_1\dots i_k})$ is the endpoint of $I_{i_1\dots i_k}$. Therefore, by the inductive hypothesis, either $h_{i_1\dots i_k}(c_{i_1\dots i_k}) = f_1^{-1}(s_0)$ or $h_{i_1\dots i_k}(c_{i_1\dots i_k}) = h(s)$ for some $h \in \mathrm{IFS}(f_0, f_1)$ and $s \in \mathcal{C}_n$.

Set $h_{i_1\dots i_k i_{k+1}} = h_{i_1\dots i_k} \circ f_{k \bmod 2}^{j_{i_1\dots i_k}+i_{k+1}}$. By construction

$$h^{-1}_{i_1\dots i_{k+1}}(I_{i_1,\dots i_{k+1}}) = f_{k \bmod 2}^{-j_{i_1\dots i_k}-i_{k+1}}(J_{i_1\dots i_k i_{k+1}}) = (s_{k+1}, c_{i_1\dots i_{k+1}}]$$

where $c_{i_1\dots i_k i_{k+1}} = f^{-1}_{k \bmod 2}(s_{k+1})$ if $i_{k+1} > 0$ and

$$c_{i_1\dots i_k 0} = h^{-1}_{i_1\dots i_k 0} \circ h_{i_1\dots i_k}(c_{i_1\dots i_k}) = f_{k \bmod 2}^{-j_{i_1\dots i_k}}(c_{i_1\dots i_k}) \in (s_{k+1}, f^{-1}_{k \bmod 2}(s_{k+1})].$$

Therefore, $h^{-1}_{i_1\dots i_k,0}(I_{i_1\dots i_k,0}) \subset A_{k+1}$ and $h^{-1}_{i_1\dots i_{k+1}}(I_{i_1\dots i_{k+1}}) = A_{k+1}$ if $i_{k+1} > 0$.

Going through the n steps of the cycle we conclude the lemma. □

Remark 3.44. *From the above lemma we define the return map over $A = (s_0, f_1^{-1}(s_0)]$ as*

$$\mathcal{R}\colon A \to A, \qquad \mathcal{R}|_{I_{i_1\dots i_n}} = h^{-1}_{i_1\dots i_n}.$$

The endpoint of the intervals $I_{i_1\dots i_n}$ are called discontinuities *of $\mathcal{R}$. Note that this discontinuities points are in the orbit of the cycle for the* IFS(f_0, f_1).

Addemdum to Lemma 1. *For each $k = 1, \dots, n$, there exist a family*

$$\{(I_{i_1\dots i_k}, h_{i_1\dots i_k}, m_{i_1\dots i_k}) : i_j \geq 0 \ j = 1, \dots k\}$$

with $I_{i_1\dots i_k}$ pairwise disjoint right-closed intervals of S^1, $h_{i_1\dots i_k} \in$ IFS(f_0, f_1) and $m_{i_1\dots i_k}$ natural numbers such that $A = (s_0, f_1^{-1}(s_0)] = \cup I_{i_1\dots i_k}$, and for $k = 1, \dots, n-1$

i) $I_{i_1\dots i_k i_{k+1}} \subset I_{i_1\dots i_k}$ *and I_{i_1} is contained in a fundamental domain of f_0,*

ii) $h^{-1}_{i_1\dots i_k i_{k+1}} = f_{k \bmod 2}^{-m_{i_1\dots i_{k+1}}} \circ h^{-1}_{i_1\dots i_k} = f_{k \bmod 2}^{-m_{i_1\dots i_{k+1}}} \circ f_{k-1 \bmod 2}^{-m_{i_1\dots i_k}} \circ \dots \circ f_1^{-m_{i_1 i_2}} \circ f_0^{-m_{i_1}}$,

iii) $h^{-1}_{i_1\dots i_{k-1}0}(I_{i_1\dots i_{k-1}0}) = (s_k, c_{i_1\dots i_k}]$ *and* $h^{-1}_{i_1\dots i_k}(I_{i_1\dots i_k}) = A_k$ *if $i_k > 0$ with*

$$s_k < c_{i_1\dots i_k} \leq f^{-1}_{k+1 \bmod 2}(s_k) \quad \text{and} \quad A_k = (s_k, f^{-1}_{k+1 \bmod 2}(s_k)],$$

iv) $m_{i_1\dots i_k+1} = m_{i_1\dots i_k} + 1$ *and $m_{i_1\dots i_{k-1}0} \geq 1$ satisfies that*

$$f_{k \bmod 2}^{m_{i_1\dots i_{k-1}0}}(s_{k+1}) < c_{i_1\dots c_{i_k}} \leq f_{k \bmod 2}^{m_{i_1\dots i_{k-1}0}-1}(s_{k+1}).$$

Proof. We only need to show the second item. Let $m_{i_1} = j + i_1$ and $m_{i_1\dots i_k} = j_{i_1\dots i_k} + i_{k+1}$. Note that $h^{-1}_{i_1} = f_0^{-m_{i_1}}$. From the construction in the inductive process in proof of Lemma 3.43,

$$h^{-1}_{i_1\dots i_k i_{k+1}} = f_{k \bmod 2}^{-j_{i_1\dots i_k}-i_{k+1}} \circ h^{-1}_{i_1\dots i_k} = f_{k \bmod 2}^{-m_{i_1\dots i_{k+1}}} \circ h^{-1}_{i_1\dots i_k}.$$

Hence by the induction hypothesis we have $h^{-1}_{i_1\dots i_k i_{k+1}} = f_{k \bmod 2}^{-m_{i_1\dots i_{k+1}}} \circ f_{k-1 \bmod 2}^{-m_{i_1\dots i_k}} \circ \dots \circ f_1^{-m_{i_1 i_2}} \circ f_0^{-m_{i_1}}$. Therefore, in n step of this induction it follows the addendum. □

Estimation of the derivative for the return map

The main result in this step is the following estimate of the derivative for the return maps:

Proposition 3.45. *Let $C>0$ the largest distortion constant of f_0^{-1} and f_1^{-1}. Consider $\varepsilon>0$ such that $|Df_i(x)-id|<\varepsilon$ for all $x\in(0,1)$ and $i=0,1$. Then*

$$\mathcal{R}'(x) > (1-\varepsilon)\varepsilon^{-1}e^{-C} \qquad \text{if} \quad x\in\bigcup_{\ell=1}^{\infty} I_{i_1\dots i_{n-1}\ell},$$
$$\mathcal{R}'(x) > (1-\varepsilon)^2\varepsilon^{-1}e^{-3C} \qquad \text{if} \quad x\in\bigcup I_{i_1\dots i_{n-1}0}.$$

As in the case of Proposition 3.6 in the proof of Duminy's Lemma, we need of some preliminar lemmas to obtain the bound distortion estimate of $h^{-1}_{i_1\dots i_n}$ in $I_{i_1\dots i_n}$.

Lemma 3.46 (Disjointness)**.** *Let $i_1\dots i_n$ be a fixed multi-index. Then*

$$U_\ell \stackrel{\text{def}}{=} f_0^{-\ell}(I_{i_1\dots i_n}) \quad \text{for } 0\le \ell < m_{i_1} \text{ and}$$
$$U_{i_1,\dots i_k,\ell} \stackrel{\text{def}}{=} f^{-\ell}_{k \bmod 2}\circ h^{-1}_{i_1\dots i_k}(I_{i_1\dots i_n}) \quad \text{for } k=1,\dots,n-1 \text{ and } 0\le\ell<m_{i_1\dots i_{k+1}}.$$

are right-closed pairwise disjoints intervals in S^1.

Proof. By construction $I_{i_1\dots i_{k+1}} \subset h_{i_1\dots i_k}\circ f^{m_{i_1\dots i_{k+1}}}_{k \bmod 2}(s_{k+1}, f^{-1}_{k \bmod 2}(s_{k+1})]$. As $I_{i_1\dots i_n}\subset I_{i_1\dots i_{k+1}}$ then

$$U_{i_1\dots i_k,\ell}\subset f^{-\ell}_{k \bmod 2}\circ h^{-1}_{i_1\dots i_k}(I_{i_1\dots i_{k+1}})\subset f^{m_{i_1\dots i_{k+1}}-\ell}_{k \bmod 2}(s_{k+1}, f^{-1}_{k \bmod 2}(s_{k+1})].$$

Since $0\le \ell < m_{i_1\dots i_{k+1}}$ then $f^{m_{i_1\dots i_{k+1}}-\ell}_{k \bmod 2}(s_{k+1}, f^{-1}_{k \bmod 2}(s_{k+1})]\subset(s_k,s_{k+1}]$. This implies that $U_{i_1\dots i_k,\ell}$ is contained in a fundamental domain of $f_{k \bmod 2}$ in the interval $(s_k,s_{k+1}]$. Suppose that $U_{i_1\dots i_k,\ell}\cap U_{i_1\dots i_m,r}\neq\emptyset$. Then $(s_k,s_{k+1}]\cap(s_m,s_{m+1}]\neq\emptyset$. Hence $k=m$. Now, it follows that

$$f^{m_{i_1\dots i_{k+1}}-\ell}_{k \bmod 2}(s_{k+1}, f^{-1}_{k \bmod 2}(s_{k+1})]\cap f^{m_{i_1\dots i_{k+1}}-r}_{k \bmod 2}(s_{k+1}, f^{-1}_{k \bmod 2}(s_{k+1})]\neq\emptyset$$

where $0\le r,\ell<m_{i_1\dots i_{k+1}}$. This it is only possible if $r=\ell$. □

Lemma 3.47 (Distortion)**.** *Let $C>0$ be the largest distortion constant of f_0 and f_1. Then*

$$\mathrm{Dist}(h^{-1}_{i_1\dots i_n},\overline{I_{i_1\dots i_n}})\le C \quad \text{and} \quad \mathrm{Dist}(h^{-1}_{i_1\dots i_n+1},\overline{I_{i_1\dots i_n}})\le 2C.$$

Consequently, for every pair of intervals J and L contained in $\overline{I_{i_1\dots i_n}}$

$$\frac{|J|}{|L|}e^{-C}\le\frac{|h^{-1}_{i_1\dots i_n}(J)|}{|h^{-1}_{i_1\dots i_n}(L)|}\le e^{C}\frac{|J|}{|L|} \quad \text{and} \quad \frac{|J|}{|L|}e^{-2C}\le\frac{|h^{-1}_{i_1\dots i_n+1}(J)|}{|h^{-1}_{i_1\dots i_n+1}(L)|}\le e^{2C}\frac{|J|}{|L|}.$$

Moreover, if $I=I_{i_1\dots i_n+1}\cup I_{i_1\dots i_n}$ then

$$\frac{|h^{-1}_{i_1\dots i_n+1}(I)|}{|I|}e^{-3C}\le Dh^{-1}_{i_1\dots i_n+1}(z)\le e^{3C}\frac{|h^{-1}_{i_1\dots i_n+1}(I)|}{|I|} \quad \text{for all } z\in I.$$

Proof. Recall that for every $k = 1, \dots, n-1$

$$h^{-1}_{i_1\dots i_{k+1}} = f^{-m_{i_1\dots i_{k+1}}}_{k \bmod 2} \circ f^{-m_{i_1\dots i_k}}_{k-1 \bmod 2} \circ \dots \circ f^{-m_{i_1 i_2}}_1 \circ f^{-m_{i_1}}_0 = f^{-m_{i_1\dots i_{k+1}}}_{k \bmod 2} \circ h^{-1}_{i_1\dots i_k}.$$

For simplicity of notation, we mean for $k = 0$ that map $h^{-1}_{i_1\dots i_k}$ is the identity map. Then

$$Dh^{-1}_{i_1\dots i_n}(x) = \prod_{k=0}^{n-1} Df^{-m_{i_1\dots i_{k+1}}}_{k \bmod 2}(h^{-1}_{i_1\dots i_k}(x)) = \prod_{k=0}^{n-1} \prod_{\ell=0}^{m_{i_1\dots i_{k+1}}-1} Df^{-1}_{k \bmod 2}(f^{-\ell}_{k \bmod 2} \circ h^{-1}_{i_1\dots i_k}(x)).$$

By means of the distortion control of f_0 and f_1, for every $x, y \in \overline{I_{i_1\dots i_n}}$

$$\begin{aligned} |\log \frac{Dh^{-1}_{i_1\dots i_n}(x)}{Dh^{-1}_{i_1\dots i_n}(y)}| &= |\log Dh^{-1}_{i_1\dots i_n}(x) - \log Dh^{-1}_{i_1\dots i_n}(y)| \\ &\le C \sum_{k=0}^{n-1} \sum_{\ell=0}^{m_{i_1\dots i_{k+1}}-1} |f^{-\ell}_{k \bmod 2} \circ h^{-1}_{i_1\dots i_k}(x) - f^{-\ell}_{k \bmod 2} \circ h^{-1}_{i_1\dots i_k}(y)| \\ &\le C \sum_{k=0}^{n-1} \sum_{\ell=0}^{m_{i_1\dots i_{k+1}}-1} |U_{i_1\dots i_k, \ell}|. \end{aligned}$$

The disjointness of $U_{i_1\dots i_k,\ell}$ showed in Lemma 3.46, implies that $\mathrm{Dist}(h^{-1}_{i_1\dots i_n}, \overline{I_{i_1\dots i_n}}) \le C$. Similarly, as $m_{i_1\dots i_n+1} = m_{i_1\dots i_n} + 1$, we have that

$$\begin{aligned} Dh^{-1}_{i_1\dots i_n+1}(x) &= Df^{-m_{i_1\dots i_n+1}}_{n-1 \bmod 2}(h^{-1}_{i_1\dots i_{n-1}}(x)) \cdot \prod_{k=0}^{n-2} \prod_{\ell=0}^{m_{i_1\dots i_{k+1}}-1} Df^{-1}_{k \bmod 2}(f^{-\ell}_{k \bmod 2} \circ h^{-1}_{i_1\dots i_k}(x)) \\ &= Df^{-1}_{n-1 \bmod 2}(f^{-m_{i_1\dots i_n}}_{n-1 \bmod 2} \circ h^{-1}_{i_1\dots i_{n-1}}(x)) \prod_{k=0}^{n-1} \prod_{\ell=0}^{m_{i_1\dots i_{k+1}}-1} Df^{-1}_{k \bmod 2}(f^{-\ell}_{k \bmod 2} \circ h^{-1}_{i_1\dots i_k}(x)). \end{aligned}$$

Denoting $U_{i_1\dots i_n, m_{i_1\dots i_n}} = f^{-m_{i_1\dots i_n}}_{n-1 \bmod 2} \circ h^{-1}_{i_1\dots i_{n-1}}(I_{i_1\dots i_n})$, we have that for every x and y in $\overline{I_{i_1\dots i_n}}$,

$$|\log \frac{Dh^{-1}_{i_1\dots i_n+1}(x)}{Dh^{-1}_{i_1\dots i_n+1}(y)}| \le C \sum_{k=0}^{n-1} \sum_{\ell=0}^{m_{i_1\dots i_{k+1}}-1} |U_{i_1\dots i_k,\ell}| + C|U_{i_1\dots i_n, m_{i_1\dots i_n}}| \le 2C.$$

From this $\mathrm{Dist}(h^{-1}_{i_1\dots i_n+1}, \overline{I_{i_1\dots i_n}}) \le 2C$ and we conclude the first part of the lemma. The rest of the assertions of this lemma are followed analogously as in Lemma 3.9. Therefore the proof of the lemma is concluded. □

Proof of Proposition 3.45. Let $x \in A$. Without loss of generality, we assume that x is not a discontinuity point of $\mathcal{R}$. If x is a discontinuity, the first return map only has lateral derivative on this point. A similar argument allows to estimate a bound for its lateral derivative. Hence, since x is not a discontinuity, we find $\eta_0 > 0$ and a unique interval $I_{i_1\dots i_n}$ such that for every $0 < \eta \le \eta_0$, the interval $J = (x - \eta, x + \eta)$ satisfies that $J \subset I_{i_1\dots i_n}$. Notice that $\mathcal{R}(J) = h^{-1}_{i_1\dots i_n}(J)$.

From Lemma 3.46 we have that

$$|\mathcal{R}(J)| \ge e^{-C} \frac{|h^{-1}_{i_1\dots i_n}(I_{i_1\dots i_n})|}{|I_{i_1\dots i_n}|} |J|. \tag{3.10}$$

By construction $I_{i_1\dots i_n} \subset I_{i_1} \subset (f_0^{m_{i_1}}(s_1), f_0^{m_{i_1}-1}(s_1)]$. Then $|I_{i_1\dots i_n}| < |f_0^{m_{i_1}}(s_1) - f_0^{m_{i_1}-1}(s_1)|$. We write this bounded as $|f_0^{m_{i_1}}(s_1) - f_0^{m_{i_1}-1}(s_1)| = |s_0 - f_0^{m_{i_1}-1}(s_1)| - |s_0 - f_0^{m_{i_1}}(s_1)|$. Since s_0 is a fixed point of f_0 then, from Mean Value Theorem, there exits $\zeta \in (s_0, f_0^{m_{i_1}-1}(s_1))$ such that $|s_0 - f_0^{m_{i_1}}(s_1)| = |Df_0(\zeta)||s_0 - f_0^{m_{i_1}-1}(s_1)|$. Therefore,

$$|f_0^{m_{i_1}}(s_1) - f_0^{m_{i_1}-1}(s_1)| = (1 - |Df_0(\zeta)|)|s_0 - f_0^{m_{i_1}-1}(s_1)| \le \varepsilon |f_0^{m_{i_1}-1}(s_1) - s_0|.$$

Finally,

$$|I_{i_1\dots i_n}| < |f_0^{m_{i_1}}(s_1) - f_0^{m_{i_1}-1}(s_1)| \le \varepsilon |f_0^{m_{i_1}-1}(s_1) - s_0|. \tag{3.11}$$

In order to obtain a bounded for $|h^{-1}_{i_1\dots i_n}(I_{i_1\dots i_n})|$, we divide in two different cases:

i) $h^{-1}_{i_1\dots i_n}(I_{i_1\dots i_n}) \cap I_{i_1\dots i_n} \neq \emptyset$.

Note that if $i_n > 0$ by Lemma 3.43 $h^{-1}_{i_1\dots i_n}(I_{i_1\dots i_n}) = A$ and then we are in this case.

We write $h^{-1}_{i_1\dots i_n}(I_{i_1\dots i_n}) = (s_0, c_{i_1\dots i_n}]$. Recall that $I_{i_1\dots i_n} \subset (f_0^{m_{i_1}}(s_1), f_0^{m_{i_1}-1}(s_1)]$. Since $I_{i_1\dots i_n} \cap (s_0, c_{i_1\dots i_n}] \neq \emptyset$ then $s_0 < f^{m_{i_1}}(s_1) \le c_{i_1\dots i_n}$. Let $k \in \mathbb{N}$ be the first time such that $f_0^k(s_1) \in (s_0, c_{i_1\dots i_n}]$. Then $s_0 < f_0^{m_{i_1}}(s_1) \le f_0^k(s_1) \le c_{i_1\dots i_n} < f_0^{k-1}(s_1)$. Hence,

$$|h^{-1}_{i_1\dots i_n}(I_{i_1\dots i_n})| = |c_{i_1\dots i_n} - s_0| \ge |f_0^k(s_1) - s_0|.$$

Again, since s_0 is a fixed point of f_0 and from Mean Value Theorem we have $\xi \in (s_0, f_0^{k-1}(s_0))$ such that $|f_0^k(s_0) - s_0| = |Df_0(\xi)||f_0^{k-1}(s_1) - s_0| \ge (1-\varepsilon)|f_0^{k-1}(s_0) - s_0|$. Therefore, since that $f_0^{m_{i_1}-1}(s_1) \le f_0^{k-1}(s_1)$, it follows that

$$|h^{-1}_{i_1\dots i_n}(I_{i_1\dots i_n})| \ge (1-\varepsilon)|f_0^{k-1}(s_1) - s_0| \ge (1-\varepsilon)|f_0^{m_{i_1}-1}(s_1) - s_0|. \tag{3.12}$$

Now, substituting this bounded and the inequality (3.11) in the equation (3.10) we obtain $|\mathcal{R}(J)| > e^{-C}(1-\varepsilon)\varepsilon^{-1}|J|$. Since this holds for all intervals $J \subset I_{i_1\dots i_n}$ contained x, then we have the same bound for the derivative of $\mathcal{R}$ at the point x.

ii) $h^{-1}_{i_1\dots i_n}(I_{i_1\dots i_n}) \cap I_{i_1\dots i_n} = \emptyset$.

Observe that this case only is possible if $i_n = 0$. Note that $h^{-1}_{i_1\dots i_{n-1}0} = f_1 \circ h^{-1}_{i_1\dots i_{n-1}1}$. By means of Mean Value Theorem $|\mathcal{R}(J)| = |Df_1(\xi_0)||Dh^{-1}_{i_1\dots i_{n-1}1}(\xi_1)||J| \ge (1-\varepsilon)|Dh^{-1}_{i_1\dots i_{n-1}1}(\xi_1)||J|$ for some $\xi_0 \in h^{-1}_{i_1\dots i_{n-1}1}(J)$ and $\xi_1 \in J$. From Lemma 3.47 we have estimate $Dh^{-1}_{i_1\dots i_{n-1}1}$ on the interval $I = I_{i_1\dots i_{n-1}1} \cup I_{i_1\dots i_{n-1}0}$ and so $|Dh^{-1}_{i_1\dots i_{n-1}1}(\xi_1)| \ge e^{-3C}|h^{-1}_{i_1\dots i_{n-1}1}(I)|/|I|$. Note that $h^{-1}_{i_1\dots i_{n-1}1}(I_{i_1\dots i_{n-1}1}) \subset h^{-1}_{i_1\dots i_{n-1}1}(I)$. Then, from the bounded obtain in the equation (3.12) of the previous case it follows

$$|h^{-1}_{i_1\dots i_{n-1}1}(I)| \ge |h^{-1}_{i_1\dots i_{n-1}1}(I_{i_1\dots i_{n-1}1})| \ge (1-\varepsilon)|f_0^{m_{i_1}-1}(s_1) - s_0|.$$

Note that $I \subset I_{i_1\dots i_{n-1}} \subset I_{i_1} \subset (f_0^{m_{i_1}}(s_1), f_0^{m_{i_1}-1}(s_1)]$. Then, as in the equation (3.11), we have $|I| < |f_0^{m_{i_1}}(s_1) - f_0^{m_{i_1}-1}(s_1)| \le \varepsilon |f_0^{m_{i_1}-1}(s_1) - s_0|$. Thus, $|Dh^{-1}_{i_1\dots i_{n-1}1}(y)| > e^{-3C}(1-\varepsilon)\varepsilon^{-1}$ for all $y \in I$. Therefore, $|\mathcal{R}(J)| > e^{-3C}(1-\varepsilon)^2\varepsilon^{-1}$. Finally, as $J = (x-\eta, x+\eta)$ for all $0 < \eta < \eta_0$, we take $\eta \to 0$ and conclude the same bounded for the derivative at x. □

End of the proof of Cycle Theorem

Now, we will conclude Theorem 3.42 for IFS(f_0, f_1). Recall that $C = \max\{C_0, C_1\}$ where C_i is the distortion constant of f_i. If f_0 and f_1 are ε-close to the identity in the C^2 topology then

$$C_i = \max\{|f_i''(x)||f_i'(x)|^{-1} : x \in S^1\} \leq \varepsilon(1-\varepsilon)^{-1}$$

for $i = 1, 2$. Therefore $C \leq \varepsilon(1-\varepsilon)^{-1}$. For $0 < \varepsilon \leq 0.23$, if f_0 and f_1 are ε-close to the identity in the C^2 topology then there exists $\lambda > 1$ such that $\mathcal{R}'(x) \geq \lambda$ for all $x \in A$. That is,

Remark 3.48. *There is $\varepsilon \geq 0.23$ such that $\mathcal{R}$ is an expanding return map over A.*

Proof of cycle Theorem for IFS(f_0, f_1). Let $I \subset S^1$ be an interval. Fixed $x \in S^1$. In order to prove the minimality of S^1 for IFS(f_0, f_1), we should show that there is a map $h \in$ IFS(f_0, f_1) such that $h(x) \in I$. Let $\mathcal{C}_n$ be a minimal cycle. We follow the notation and the assumptions for the cycle used in the above lemmas. So, we have an expanding return map $\mathcal{R}\colon A \to A$ where $A = (s_0, f_1^{-1}(s_0)]$ with $s_0 \in \mathcal{C}_n$. Let $y \in I$. By means of the cycle for IFS(f_0^{-1}, f_1^{-1}) constructed in Lemma 3.41, there is $g \in$ IFS(f_1^{-1}, f_0^{-1}) such that $g(y) \in (s_{n-1}, \tilde{s}_n)$. Note that $\tilde{s}_n < s_n$ where $s_n = s_0 + 1$ on the lift. However, on S^1 we have that $s_n = s_0$. Hence, we also write that A is the fundamental domain of f_1 parametrice by $(s_n, f_1^{-1}(s_n)]$. There is $k \in \mathbb{N}$ such that $f_1^{-k} \circ g(y) \in A$. That is, there exists $g_0 \in$ IFS(f_0, f_1) such that $g_0^{-1}(I) \cap A \neq \emptyset$. From Remark 3.48 the return map $\mathcal{R}$ is expanding map in A. Thus, there is $n \geq 0$ such that $\mathcal{R}^n(f_\sigma^{-1}(I) \cap A)$ contains some discontinuity of $\mathcal{R}$. Note that $\mathcal{R}^n(f_\sigma^{-1}(I) \cap A) = g_1^{-1}(f_\sigma^{-1}(I) \cap A)$ for some $g_1 \in$ IFS(f_0, f_1). Recall that the discontinuities are the endpoint of $I_{i_1 \dots i_n}$. That points are in the orbit of the cycle $\mathcal{C}_n$. Therefore, there is $g_2 \in$ IFS(f_0, f_1) and $s \in \mathcal{C}_n$ such that $g_2(s) \in g_1^{-1}(g_0^{-1}(I) \cap A)$. From the continuity of g_2 it follows $\delta > 0$ such that $g_2((s-\delta, s+\delta)) \subset g_1^{-1}(g_0^{-1}(I) \cap A)$. As the union of the basin of the attractor points in the cycle $\mathcal{C}_n$ is S^1, then there is $g_3 \in$ IFS(f_0, f_1) such that $g_3(x) \in (s-\delta, s+\delta)$. Therefore, taken $h = g_0 \circ g_1 \circ g_2 \circ g_3 \in$ IFS(f_0, f_1) it follows $h(x) \in I$. Finally, by Lemma 3.41, we have again a minimal cycle of fixed point different of a ss-interval for IFS(f_0^{-1}, f_1^{-1}). Therefore, the same proof works to prove that S^1 is minimal for IFS(f_0^{-1}, f_1^{-1}) and so, we conclude the proposition. □

Robustness of Cycle Theorem

We have prove the first part in the Cycle Theorem (Theorem 3.42) for IFS(f_0, f_1) constructing a return map $\mathcal{R} : A \to A$ with an infinite number of expanding branches. Each expanding branches is a different map in IFS(f_0, f_1). This expansivity of $\mathcal{R}$ allows us to show that A is minimal for IFS(f_0, f_1) and so, using the cycle, we move this minimality property throughout the whole circle. Any C^1-close iterated function system IFS(g_0, g_1) to IFS(f_0, f_1) has a minimal cycle different of an ss-interval. However, since return map $\mathcal{R}$ involves an infinite number of composition of f_0 and f_1 then we cannot guarantee that corresponding analogous return map for IFS(g_0, g_1) is expansive. In order to show this expansiveness it suffices, as in Section 3.2.1 was done (see Remark 3.18), modify the expanding return map $\mathcal{R}$ over $\overline{A}$ to obtain a new return map $\tilde{\mathcal{R}}$ which only has a finite number of expanding branches (or discontinuities). Thus, now it follows that any IFS C^1-close to IFS(f_0, f_1) has also and expanding return maps close to $\tilde{\mathcal{R}}$ and a minimal cycle different of an ss-interval. With these two ingredients, repeating the same proof of the Cycle Theorem for IFS(f_0, f_1) it follows that S^1 is minimal for any C^1-close IFS.

3.3.4 Periodic cycles for IFS on the circle

Let f_0 and f_1 be Morse-Smale diffeomorphism on the circle of period n_0 and n_1 respectively. Assume that there is a cycle

$$\mathcal{C}_n = \{s_n \prec s_{n-1} \prec \cdots \prec s_1 \prec s_0 = s_n\}$$

different of a ss-interval for $\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})$. We say that $\mathcal{C}_n$ is a *periodic cycle* for $\mathrm{IFS}(f_0, f_1)$. As we noted in Section §3.3.2 for a periodic $**$-interval, if we use directly the estimates calculated obtained in Proposition 3.45 for the derivative of the corresponding return map for $\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})$ then this bound depends on the period $m = \max\{n_0, n_1\}$. In fact, when the period increasing this bounded estimate goes to zero, and thus, we need to reduce the size of the neighborhood. This is again a problem if we are looking for a uniform neighborhood of the identity.

In this subsection we will show a new estimative for the (periodic) return map derivative independent of the period m. As in the proof of Cycle Theorem we can construct a first return map $\mathcal{R}$ on a fundamental domain $A = (s_0, f_1^{-n_1}(s_0)]$ of $f_1^{n_1}$ (see Lemma 3.43). Note that now the return maps $h_{i_1\dots i_n}$ are with respect to the system $\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})$. It will be helpful to write $g_0 = f_0^{n_0}$ and $g_1 = f_1^{n_1}$. We will indicate the modifications required in estimating the return map derivative in Subsection 3.2.1.

Lemma 3.49 (Disjointness). *Let $i_1 \dots i_n$ be a fixed multi-index. Then the right-closed intervals*

i) $U_\ell \overset{\text{def}}{=} f_0^{-\ell}(I_{i_1\dots i_n})$ for $\ell \geq 0$ are disjoint with respect to each other, and

ii) $U_{i_1,\dots i_k,\ell} \overset{\text{def}}{=} f_{k \bmod 2}^{-\ell} \circ h_{i_1\dots i_k}^{-1}(I_{i_1\dots i_n})$ for $k = 1, \dots, n-1$ and $\ell \geq 0$ are also pairwise disjoint.

Proof. Note that from the Addendum of Lemma 3.43, $I_{i_1\dots i_n} \subset I_{i_1}$ where I_{i_1} is contained in a fundamental domain of $g_0 = f_0^{n_0}$. Thus, by Lemma 3.30, it follows that U_ℓ are right-closed intervals in S^1 disjoint with respect to each other. Also, by Lemma 3.43,

$$g_{k \bmod 2}^{-m_{i_1\dots i_{k+1}}} \circ h_{i_1\dots i_k}^{-1}(I_{i_1\dots i_n}) \subset h_{i_1\dots i_{k+1}}^{-1}(I_{i_1\dots i_{k+1}}) \subset A_{k+1} = (s_{k+1}, g_{k \bmod 2}^{-1}(s_{k+1})].$$

Hence $h_{i_1\dots i_k}^{-1}(I_{i_1\dots i_n}) \subset g_{k \bmod 2}^{m_{i_1\dots i_{k+1}}}(A_{k+1})$. Since A_{k+1} is a fundamental domain of $g_{k \bmod 2}$ then $h_{i_1\dots i_k}^{-1}(I_{i_1\dots i_n})$ is also contained in a fundamental domain of $g_{k \bmod 2}$ and thus, again by Lemma 3.30, it follows that $U_{i_1\dots i_k,\ell}$ are pairwise disjoint intervals in S^1. □

Lemma 3.50 (Distortion). *Let $C > 0$ be the largest distortion constant of f_0 and f_1. Then for every $j \geq 0$ it holds that $\mathrm{Dist}(h_{i_1\dots i_{n-1}j}^{-1}, \overline{I_{i_1\dots i_n}}) \leq nC$. Consequently, for every pair of intervals J and L contained in $\overline{I_{i_1\dots i_n}}$ and for $j \geq 0$*

$$\frac{|J|}{|L|}\, e^{-nC} \leq \frac{|h_{i_1\dots i_{n-1}j}^{-1}(J)|}{|h_{i_1\dots i_{n-1}j}^{-1}(L)|} \leq e^{nC}\, \frac{|J|}{|L|}$$

Moreover, if $I = I_{i_1\dots i_{n-1}j+1} \cup I_{i_1\dots i_{n-1}j}$ then

$$\frac{|h_{i_1\dots i_{n-1}j+1}^{-1}(I)|}{|I|}\, e^{-2nC} \leq Dh_{i_1\dots i_{n-1}j+1}^{-1}(z) \leq e^{2nC}\, \frac{|h_{i_1\dots i_{n-1}j+1}^{-1}(I)|}{|I|} \quad \textit{for all } z \in I.$$

Proof. Recall that $h^{-1}_{i_1\dots i_{k+1}} = g^{-m_{i_1\dots i_{k+1}}}_{k \bmod 2} \circ h^{-1}_{i_1\dots i_k}$ where for simplicity of notation, $h^{-1}_{i_1\dots i_k}$ for $k=0$ is the identity map. Then, denoting by n_k the period of $f_{k \bmod 2}$, that is, $g_{k \bmod 2} = f^{n_k}_{k \bmod 2}$, and to shorten writing $m_{i_1\dots i_{n-1}i_n}$ instead of $m_{i_1\dots i_{n-1}j}$, it holds that

$$Dh^{-1}_{i_1\dots i_{n-1}j}(x) = \prod_{k=0}^{n-1} \prod_{\ell=0}^{n_k m_{i_1\dots i_{k+1}}-1} Df^{-1}_{k \bmod 2}(f^{-\ell}_{k \bmod 2} \circ h^{-1}_{i_1\dots i_k}(x)).$$

By means of the distortion control of f_0 and f_1, for every $x, y \in \overline{I_{i_1\dots i_n}}$

$$\begin{aligned} |\log \frac{Dh^{-1}_{i_1\dots i_{n-1}j}(x)}{Dh^{-1}_{i_1\dots i_{n-1}j}(y)}| &\leq C \sum_{k=0}^{n-1} \sum_{\ell=0}^{n_k m_{i_1\dots i_{k+1}}-1} |f^{-\ell}_{k \bmod 2} \circ h^{-1}_{i_1\dots i_k}(x) - f^{-\ell}_{k \bmod 2} \circ h^{-1}_{i_1\dots i_k}(y)| \\ &\leq C \sum_{k=0}^{n-1} \sum_{\ell=0}^{n_k m_{i_1\dots i_{k+1}}-1} |U_{i_1\dots i_k,\ell}|. \end{aligned}$$

The disjointness of each families of intervals $U_{i_1\dots i_k,\ell}$ for $\ell \geq 0$ showed in Lemma 3.50 implies that $\mathrm{Dist}(h^{-1}_{i_1\dots i_{n-1}j}, \overline{I_{i_1\dots i_n}}) \leq nC$. The rest of the proof of this lemma is analogous of second part in Lemma 3.47 and 3.9. Therefore the proof of the lemma is concluded. □

Lemma 3.51 (Compared intervals). *Let $C>0$ be the largest distortion constant of f_0 and f_1. Consider $\delta>0$ such that $|Df_i^{n_i}(x)-1|<\delta$ for all $x \in S^1$ and for $i=0,1$. Then*

$$\frac{|I_{i_1}|}{|I_{i_1\dots i_n}|} > \big((1-\delta)\delta^{-1}e^{-C}\big)^{n-1}$$

for all multi-index $i_1\dots i_n$.

Proof. Recall that $h^{-1}_{i_1\dots i_k} = g^{-m_{i_1\dots i_k}}_{k-1 \bmod 2} \circ h^{-1}_{i_1\dots i_{k-1}}$ where for simplicity of notation, $h^{-1}_{i_1\dots i_{k-1}}$ for $k=1$ is the identity map. Argue as in the proof of Lemma 3.50, we have that $h^{-1}_{i_1\dots i_{k-1}}(I_{i_1\dots i_n})$ as well as $h^{-1}_{i_1\dots i_{k-1}}(I_{i_1\dots i_k})$ are contained in $I = g^{m_{i_1\dots i_k}}_{k-1 \bmod 2}(A_k)$. Note that $A_k = (s_k, g^{-1}_{k-1 \bmod 2}(s_k)]$ is a fundamental domain of $g_{k-1 \bmod 2}$. Hence, $g^{-\ell}_{k-1 \bmod 2}(I)$ for $0 \leq \ell < m_{i_1\dots i_k}$ are pairwise disjoint intervals. Thus, from the classical distortion lemma (see Lemma 3.7) it follows that $\mathrm{Dist}(g^{-m_{i_1\dots i_k}}_{k-1 \bmod 2}, \overline{I}) \leq C$ and consequently, for every $k=1,\dots,n-1$ one has that

$$\frac{|h^{-1}_{i_1\dots i_{k-1}}(I_{i_1\dots i_k})|}{|h^{-1}_{i_1\dots i_{k-1}}(I_{i_1\dots i_n})|} \geq e^{-C} \frac{|h^{-1}_{i_1\dots i_k}(I_{i_1\dots i_k})|}{|h^{-1}_{i_1\dots i_k}(I_{i_1\dots i_n})|}. \tag{3.13}$$

In the particular case of $k=1$ we have that $|I_{i_1}|/|I_{i_1\dots i_n}| \geq e^{-C}|h^{-1}_{i_1}(I_{i_1})|/|h^{-1}_{i_1}(I_{i_1\dots i_n})|$. Multiplying by $|h^{-1}_{i_1}(I_{i_1 i_2})|/|h^{-1}_{i_1}(I_{i_1 i_2})|$ and using (3.13) gives

$$\frac{|I_{i_1}|}{|I_{i_1\dots i_n}|} \geq e^{-2C} \frac{|h^{-1}_{i_1}(I_{i_1})|}{|h^{-1}_{i_1}(I_{i_1 i_2})|} \cdot \frac{|h^{-1}_{i_1 i_2}(I_{i_1 i_2})|}{|h^{-1}_{i_1 i_2}(I_{i_1\dots i_n})|}.$$

Repeating this argue $n-1$ times we obtain

$$\frac{|I_{i_1}|}{|I_{i_1\dots i_n}|} \geq e^{-(n-1)C} \frac{|h^{-1}_{i_1}(I_{i_1})|}{|h^{-1}_{i_1}(I_{i_1 i_2})|} \frac{|h^{-1}_{i_1 i_2}(I_{i_1 i_2})|}{|h^{-1}_{i_1 i_2}(I_{i_1 i_2 i_3})|} \cdots \frac{|h^{-1}_{i_1\dots i_{n-1}}(I_{i_1\dots i_{n-1}})|}{|h^{-1}_{i_1\dots i_{n-1}}(I_{i_1\dots i_n})|}. \tag{3.14}$$

Claim 3.51.1. $|h^{-1}_{i_1\dots i_k}(I_{i_1\dots i_k})|/|h^{-1}_{i_1\dots i_k}(I_{i_1\dots i_{k+1}})| > (1-\delta)\delta^{-1}$ *for all $k=1,\dots,n-1$.*

Proof of the claim. We have that $h^{-1}_{i_1\dots i_k}(I_{i_1\dots i_k}) = (s_k, c_{i_1\dots i_k}]$ where $c_{i_1\dots i_{k-1}0} \in (s_k, g^{-1}_{k+1 \bmod 2}(s_k)]$ and $c_{i_1\dots i_k} = g^{-1}_{k+1 \bmod 2}(s_k)$ if $i_k > 0$. On the other hand, from the construction of the return map in Lemma 3.43

$$h^{-1}_{i_1\dots i_k}(I_{i_1\dots i_k 0}) = J_{i_1\dots i_k 0} = (g^{m_{i_1\dots i_k 0}}_{k \bmod 2}(s_{k+1}), c_{i_1\dots i_k}] \quad \text{and}$$
$$h^{-1}_{i_1\dots i_k}(I_{i_1\dots i_{k+1}}) = J_{i_1\dots i_{k+1}} = (g^{m_{i_1\dots i_{k+1}}}_{k \bmod 2}(s_{k+1}), g^{m_{i_1\dots i_{k+1}}-1}_{k \bmod 2}(s_{k+1})] \quad \text{if } i_{k+1} > 0.$$

In the case $i_{k+1} > 0$, as $s_k < g^{m_{i_1\dots i_{k+1}}-1}_{k \bmod 2}(s_{k+1}) < c_{i_1\dots i_k}$ we obtain

$$\frac{|h^{-1}_{i_1\dots i_k}(I_{i_1\dots i_k})|}{|h^{-1}_{i_1\dots i_k}(I_{i_1\dots i_{k+1}})|} = \frac{|s_k - c_{i_1\dots i_k}|}{|g^{m_{i_1\dots i_{k+1}}}_{k \bmod 2}(s_{k+1}) - g^{m_{i_1\dots i_{k+1}}-1}_{k \bmod 2}(s_{k+1})|}$$
$$\geq \frac{|s_k - g^{m_{i_1\dots i_{k+1}}-1}_{k \bmod 2}(s_{k+1})|}{|g^{m_{i_1\dots i_{k+1}}}_{k \bmod 2}(s_{k+1}) - g^{m_{i_1\dots i_{k+1}}-1}_{k \bmod 2}(s_{k+1})|} \geq \frac{|s_k - q|}{|g_{k \bmod 2}(q) - q|}$$

where $q = g^{m_{i_1\dots i_{k+1}}-1}_{k \bmod 2}(s_{k+1})$. Using the mean value theorem it follows that

$$|g_{k \bmod 2}(q) - q| = |(s_k - q) - (g_{k \bmod 2}(q) - s_k)| \geq |s_k - q|\,|1 - Dg_{k \bmod 2}(\xi)|$$

for some $\xi \in S^1$. Since $|Dg_i(x) - 1| < \delta$ where $g_i = f_i^{n_i}$ for $x \in S^1$ and $i = 0,1$ we have that $|h^{-1}_{i_1\dots i_k}(I_{i_1\dots i_k})|/|h^{-1}_{i_1\dots i_k}(I_{i_1\dots i_{k+1}})| > \delta^{-1}$.

In the case $i_{k+1} = 0$, as $s_k < g^{m_{i_1\dots i_k 0}}_{k \bmod 2}(s_{k+1}) < c_{i_1\dots i_k} \leq g^{m_{i_1\dots i_k 0}-1}_{k \bmod 2}(s_{k+1})$ we obtain

$$\frac{|h^{-1}_{i_1\dots i_k}(I_{i_1\dots i_k})|}{|h^{-1}_{i_1\dots i_k}(I_{i_1\dots i_k 0})|} = \frac{|s_k - c_{i_1\dots i_k}|}{|g^{m_{i_1\dots i_k 0}}_{k \bmod 2}(s_{k+1}) - c_{i_1\dots i_k}|}$$
$$\geq \frac{|s_k - g^{m_{i_1\dots i_k 0}}_{k \bmod 2}(s_{k+1})|}{|g^{m_{i_1\dots i_k 0}}_{k \bmod 2}(s_{k+1}) - g^{m_{i_1\dots i_k 0}-1}_{k+1 \bmod 2}(s_{k+1})|} \geq \frac{|s_k - g_{k \bmod 2}(q)|}{|g_{k \bmod 2}(q) - q|}$$

where q now denotes $g^{m_{i_1\dots i_k 0}-1}_{k+1 \bmod 2}(s_{k+1})$. Since $|s_k - g_{k \bmod 2}(q)| = Dg_{k \bmod 2}(\xi)|s_k - q| > (1-\delta)|s_k - q|$ then $|h^{-1}_{i_1\dots i_k}(I_{i_1\dots i_k})|/|h^{-1}_{i_1\dots i_k}(I_{i_1\dots i_k 0})| > (1-\delta)|s_k - q|/|g_{k \bmod 2}(q) - q|$. Arguing as above, it follows that $|h^{-1}_{i_1\dots i_k}(I_{i_1\dots i_k})|/|h^{-1}_{i_1\dots i_k}(I_{i_1\dots i_k 0})| > (1-\delta)\delta^{-1}$. $\square$

Finally, using the above claim in (3.14), it follows $|I_{i_1}|/|I_{i_1\dots i_n}| \geq e^{-(n-1)C}(1-\delta)^{n-1}\delta^{-(n-1)}$ and therefore the proof of the lemma is concluded. $\square$

Proposition 3.52. *Let $C > 0$ be the largest distortion constant of f_0 and f_1 and n the length of the minimal cycle for* IFS(f_0, f_1)*. Consider $\delta > 0$ such that $|Df_i^{n_i}(x) - 1| < \delta$ for all $x \in (0,1)$ and for $i = 0,1$. Then*

$$\mathcal{R}'(x) \geq \big((1-\delta)\delta^{-1}e^{-2C}\big)^{n-1}e^{-C} \qquad \text{if} \quad x \in \bigcup_{\ell=1}^{\infty} I_{i_1\dots i_{n-1}\ell},$$
$$\mathcal{R}'(x) \geq \frac{1}{2}\,\delta e^{C}\big((1-\delta)\delta^{-1}e^{-3C}\big)^{n} \qquad \text{if} \quad x \in \bigcup I_{i_1\dots i_{n-1}0}.$$

Proof. Let x be a interior point of $I_{i_1\dots i_n}$. Take an arbitrarily small open interval J such that $x \in J \subset I_{i_1\dots i_n}$. Notice that $\mathcal{R}(J) = h^{-1}_{i_1\dots i_n}(J)$. Suppose that $i_n > 0$. Then $h^{-1}_{i_1\dots i_n}(I_{i_1\dots i_n}) = A \supset I_{i_1} \supset I_{i_1\dots i_n}$. From Lemma 3.50 we have that

$$|\mathcal{R}(J)| \geq e^{-nC}\,\frac{|h^{-1}_{i_1\dots i_n}(I_{i_1\dots i_n})|}{|I_{i_1\dots i_n}|}\,|J| \geq e^{-nC}\,\frac{|I_{i_1}|}{|I_{i_1\dots i_n}|}\,|J|.$$

By Lemma 3.51 it follows $|\mathcal{R}(J)| > ((1-\delta)\delta^{-1}e^{-C})^{n-1}e^{-nC}\,|J|$. The above inequality implies that

$$\mathcal{R}'(x) = Dh^{-1}_{i_1\dots i_n}(x) \geq \big((1-\delta)\delta^{-1}e^{-2C}\big)^{n-1}e^{-C} \quad \text{for all } x \in I_{i_1\dots i_n} \text{ with } i_n > 0.$$

For the case $i_n = 0$, recalling that $m_{i_1\dots i_n+1} = m_{i_1\dots i_n} + 1$ it follows $h^{-1}_{i_1\dots i_{n-1}0} = g_{n-1 \bmod 2} \circ h^{-1}_{i_1\dots i_{n-1}1}$. Then, by the mean value theorem, there are $\xi \in h^{-1}_{i_1\dots i_{n-1}1}(J)$ and $\zeta \in J$ such that

$$|\mathcal{R}(J)| = |Dg_{n-1 \bmod 2}(\xi)||Dh^{-1}_{i_1\dots i_{n-1}1}(\zeta)||J| > (1-\delta)|Dh^{-1}_{i_1\dots i_{n-1}1}(\zeta)||J|.$$

From the estimate of $Dh^{-1}_{i_1\dots i_{n-1}1}$ on the $I = I_{i_1\dots i_{n-1}1} \cup I_{i_1\dots i_{n-1}0}$ obtained in Lemma 3.50 it follows that, $Dh^{-1}_{i_1\dots i_{n-1}1}(\zeta) \geq e^{-2nC}\,|h^{-1}_{i_1\dots i_{n-1}1}(I)|/|I|$. As $h^{-1}_{i_1\dots i_{n-1}1}(I) \supset A \supset I_0$ then $|h^{-1}_{i_1\dots i_{n-1}1}(I)| \geq |I_0|$. Then, by Lemma 3.51, we see that

$$\frac{|h^{-1}_{i_1\dots i_{n-1}1}(I)|}{|I|} \geq \frac{|I_0|}{|I|} = \Big(\frac{|I_{i_1\dots i_{n-1}1}|}{|I_0|} + \frac{|I_{i_1\dots i_{n-1}0}|}{|I_0|}\Big)^{-1} > \frac{1}{2}\,\big((1-\delta)\delta^{-1}e^{-C}\big)^{n-1}.$$

Finally, $|\mathcal{R}(J)| > (1-\delta)e^{-2nC}((1-\delta)\delta^{-1}e^{-C})^{n-1}|J|/2$. This implies that

$$\mathcal{R}'(x) = Dh^{-1}_{i_1\dots i_n 0}(x) \geq \frac{1}{2}\,\delta e^{C}\big((1-\delta)\delta^{-1}e^{-3C}\big)^{n} \quad \text{for all } x \in I_{i_1\dots i_{n-1}0}$$

and we conclude the proposition. $\square$

Now, we are ready to extend the Cycle Theorem (Theorem 3.42) for Morse-Smale diffeomorphisms with arbitrarily large period:

Theorem 3.53 (Periodic Cycle)**.** *There exists $\varepsilon \geq 0.12$ such that if $f_0, f_1 \in \mathrm{Diff}^2(S^1)$ are Morse-Smale diffeomorphisms of period n_0 and n_1, respectively, and ε-close to the identity in the C^2-topology with a minimal cycle for* $\mathrm{IFS}(f_0^{n_1}, f_1^{n_0})$ *different of a ss-interval, then S^1 is C^1-robustly minimal for* $\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})$ *and* $\mathrm{IFS}(f_0^{-n_0}, f_1^{-n_1})$*.*

Proof. Take $\delta = 0.15 > 0$. Hence $(1-\delta)^2\delta^{-1}e^{-5\delta} > 2$. Let $\varepsilon > 0$ small enough such that

$$1-\delta < e^{-\varepsilon(1-\varepsilon)^{-1}} < e^{\varepsilon(1-\varepsilon)^{-1}} < 1+\delta \quad \text{and} \quad \varepsilon(1-\varepsilon)^{-1} < \delta. \tag{3.15}$$

Note that these condition are satisfies for every positive $\varepsilon \leq 0.12$. We are assuming that f_0 and f_1 are C^2-diffeomorphisms ε-close to the identity. Thus, the largest distortion constants of f_0 and f_1 is $0 < C < \varepsilon(1-\varepsilon)^{-1} < \delta$. From Lemma 3.30 it follows that $e^{-\varepsilon(1-\varepsilon)^{-1}} < e^{-C} \leq Df_i^{n_i}(x) \leq e^{C} < e^{\varepsilon(1-\varepsilon)^{-1}}$ for all $x \in S^1$ and for $i = 0, 1$. Using Equation (3.15) in this inequality we conclude that $|Df_i^{n_i}(x) - 1| < \delta$ for all $x \in S^1$. Note that $n \geq 2$ and $(1-\delta)\delta^{-1}e^{-3C} \geq (1-\delta)\delta^{-1}e^{-3\delta} > 1$. Hence, Proposition 3.6 implies that

$$\mathcal{R}'(x) \geq \frac{1}{2}\,\delta e^{C}\big((1-\delta)\delta^{-1}e^{-3C}\big)^{n} \geq \frac{1}{2}(1-\delta)^2\delta^{-1}e^{-5\delta} > 1 \quad \text{for all } x \in A.$$

That is, $\mathcal{R}$ is an expanding return map over the fundamental domain A.

The rest of the proof of this theorem and the proof of Cycle Theorem are totally analogous. See end of the proof of cycle Theorem and robustness in Subsection 3.3.3. So, finally, the proof of the this theorem is completed. $\square$

3.3.5 Spectral decomposition

We finish this chapter with the following theorem:

Theorem F (Spectral decomposition on the circle)**.** *There exists $\varepsilon > 0$ such that if $f_0, f_1 \in \mathrm{Diff}^2(S^1)$ are two Morse-Smale diffeomorphisms of period n_0 and n_1, respectively, ε-close to the identity in the C^2-topology and with no fixed points in common, then, there is finitely many intervals $K_1, \dots, K_m$ pairwise disjoints, isolated and transitive for* $\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})$ *such that*

$$L(\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})) = \bigcup_{i=1}^{m} K_i.$$

Moreover, each K_i is either a $**$-interval for $\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})$*, or a single fixed point of f_0 of f_1.*

We want to remark that the above decomposition of the limite set of $\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})$ is C^1-robust. This means that the same assertion it holds for every $\mathrm{IFS}(g_0^{n_0}, g_1^{n_1})$ where g_0 and g_1 are C^1-close enough to f_0 and f_1 respectively.

Proof. This result is immediately followed from the Theorem 3.21, Theorem 3.35 and Theorem 3.53. Indeed, consider $\tilde{f}_0^{n_0}$ and $\tilde{f}_1^{n_1}$ the lift on the real line of $f_0^{n_1}$ and $f_1^{n_1}$. Note that $\tilde{f}_0^{n_0}$ and $\tilde{f}_1^{n_1}$ are periodic function. Arguing as in Theorem 3.21 it follows a decomposition in pairwise disjoints intervals of $L(\mathrm{IFS}(\tilde{f}_0^{n_0}, \tilde{f}_1^{n_1}))\backslash\{\pm\infty\}$ on the real line. From the periodicity, it follows that this intervals project on the circle in a finitely many pairwise disjoint $**$-intervals for $** \in \{ss, su, uu\}$. Also, notice that from Theorem 3.35 these intervals are isolates and topologically transitive. We only need to study the limit set of point whose ω-limit (or α-limit) contains $\pm\infty$. This only can be happened if there is a cycle for $\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})$ different of an ss-interval. In this case, Theorem 3.53 implies that S^1 is minimal for $\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})$ and $\mathrm{IFS}(f_0^{-n_0}, f_1^{-n_1})$. Therefore, we obtain a decomposition of the limit set and conclude the proof of the theorem. $\square$

Cycles in unfoldings of nilpotent singularities

Singularities of a vector field are simplest elements from which interesting dynamics may emerge. For instance, it is proved that any generic nilpotent singularity of codimension four in $\mathbb{R}^4$ unfolds a bifurcation hypersurface of bifocal homoclinic orbits, that is, homoclinic orbits to equilibrium points with two pairs of complex eigenvalues. All return map defined over a transversal section to this homoclinic orbit is a diffeomorphism in $\mathbb{R}^3$ and thus, susceptible to exhibit heterodimensional cycles. We will approach the study of the existence of these cycles showing how suspended blenders could appear in the generic unfoldings of these nilpotent singularities.

4.1 Nilpotent singularity

The relationship between dynamic complexity and the presence of homoclinic orbits was discovered by Poincaré more than a century ago. In his famous essay on the stability of the solar system [Poi90], Poincaré showed that the invariant manifolds of a hyperbolic fixed point of a diffeomorphism could cut each other at points, called homoclinics, which yield the existence of more and more points of this type and consequently, a very complicated configuration of the manifolds. Many years later, Birkhoff [Bir35] showed that, in general, near a homoclinic point there exists an extremely intrincated set of periodic orbits, mostly with a very long period. By the mid 60's, Smale [Sma67] placed his geometrical device, the Smale horseshoe, in a neighborhood of a transversal homoclinic point. The horseshoes explained the Birkhoff's result and arranged the complicated dynamics that occur near a homoclinic orbit by means of a conjugation to the Bernoulli's shift. In [MV93] it is proved the appearance of strange attractors during the process of creation or destruction of the Smale horseshoes. These attractors are like those shown in [BC91] for the Hénon family, that is, they are nonhyperbolic and persistent in the sense of measure.

In the framework of vector field, Shil'nikov [Shi65] proved that in every neighborhood of a homoclinic orbit to a hyperbolic equilibrium point of an analytical vector field on $\mathbb{R}^3$, with eigenvalues λ and $-\varrho \pm \omega i$ such that $0 < \varrho < \lambda$, that is, the so-called *Shil'nikov homoclinic orbit*, there exists a countable set of periodic orbits. This result is similar to that found by Birkhoff for diffeomorphisms and thus, it should be understood in a manner similar to that devised by Smale. Indeed, Tresser [Tre84] showed that in every neighborhood of such a homoclinic orbit, an infinity of linked horseshoes can be defined in such a way that the dynamics is conjugated to a subshift of finite type on an infinite number of symbols. Once again, these horseshoes appear and disappear by means of generic homoclinic bifurcations leading to persistent in the sense of the measure non hyperbolic strange attractors like those in [MV93, BC91].

As follows from [OS86], nonhyperbolic dynamics is dense in the space $\mathcal{X}$ of vector fields with a Shil'nikov homoclinic orbit. In particular, for each $\varepsilon > 0$, the subset of vector fields with a homoclinic tangency to a hyperbolic periodic orbit in an ε-neighbourhood of the homoclinic orbit is dense in $\mathcal{X}$. These tangencies give rise to suspended Hénon-like strange attractors. In [PR97, PR01] it was proved that infinitely many of these strange attractors can coexist in non generic families of vector fields with a Shil'nikov homoclinic orbit, for parameter values in a set of positive Lebesgue measure. Later [Hom02], it was proved that an infinity of such attractors can coexist in a more general context. For an extensive study of the phenomena accompanying homoclinic bifurcations, see [BDV05, HS10, PT93].

Because of the importance of homoclinic orbits in Dynamics, many papers were devoted to prove their existence. A seminal work was due to Melnikov [Mel63], who introduced original ideas to prove the existence of transversal homoclinic orbits in non-autonomous perturbations of a planar hamiltonian vector field. These ideas were developed in [CHM80] in order to determine both, homoclinic bifurcation curves and the existence of subharmonics in two-parameter families of non-autonomous second order differential equations. In [Pal84], Palmer developed a theory involving transversal homoclinic points and exponential dichotomies that was very useful for the study of homoclinic bifurcations in higher dimensions.

Since Shil'nikov homoclinic orbits are not transversal, Melnikov's techniques had to be modified in order to prove their existence in families of vector fields. In [Rod86], generic families of quadratic three dimensional vector fields with Shil'nikov homoclinic orbits were given. Putting together ideas from [Rod86, CHM80, Pal84], it was proved in [IR95] that Shil'nikov homoclinic orbits appear in generic unfoldings of a nilpotent singularity of codimension four in $\mathbb{R}^3$. A *nilpotent singularity* is a C^∞ vector field on $\mathbb{R}^n$ which in appropriate coordinates in a neighborhood of the origin can be written as

$$\sum_{k=1}^{n-1} x_{k+1}\frac{\partial}{\partial x_k} + f(x_1,\ldots,x_n)\frac{\partial}{\partial x_n},$$

with $f(x) = O(\|x\|^2)$ where $x = (x_1,\ldots,x_n)$. It is said that X is a *nilpotent singularity of codimension n* if the generic condition $\partial^2 f/\partial x_1^2(0) \neq 0$ is fulfilled. Since singularities (non-hyperbolic equilibrium points) are the simplest elements to be found in phase portraits of vector fields, arguing the existence of homoclinic orbits from the presence of singularities is a highly relevant task. Nevertheless, in order to get the greatest interest in applications, such singularities should be of codimension as low as possible. With this in mind, the result obtained in [IR95] was improved in [IR05, BIR11], where it was showed that Shil'nikov homoclinic orbits appear in every generic unfolding of the nilpotent singularity of codimension three in $\mathbb{R}^3$. Proving that Shil'nikov homoclinic orbits can be unfolded generically from a singularity of codimension less than three is currently a very interesting open problem. The dimension of the corresponding center manifold should be at least three. The lowest codimension singularities in $\mathbb{R}^3$ with a three-dimensional center manifold are the Hopf-zero singularities which have codimension two [GH02]. The difficulties that appear on studying the existence of Shil'nikov homoclinic orbits in generic unfoldings of Hopf-zero singularities are discussed in [DIKS].

The above result about Shil'nikov homoclinic orbits and nilpotent singularities was essential in [DIR07] to prove the existence of persistent strange attractors in the four parametric family of vector fields obtained when two Brusselators are linearly coupled by diffusion. Indeed, this family is a generic unfolding of three-dimensional nilpotent singularities of codimension three. Therefore it displays Shil'nikov homoclinic orbits and, consequently, persistent strange attractors. Nevertheless, this family may display a richer dynamics. Three-dimensional nilpotent singularities appear along two bifurcation curves which emerge from a bifurcation point corresponding to a four-dimensional nilpotent singularity of codimension four, for which the family is also a generic unfolding. Therefore, one should wonder whether a different class of homoclinic orbits can take place from this four-dimensional nilpotent singularity. In this chapter, we will prove the following result which is collected in [BIR11]:

Theorem G. *In every generic unfolding of a four-dimensional nilpotent singularity of codimension four there is a bifurcation hypersurface of bifocal homoclinic orbits.*

Bifocal homoclinic orbits are homoclinic orbits to equilibrium points with two pairs of eigenvalues $\rho_k \pm \omega_k i$, with $k = 1, 2$, such that $\rho_1 < 0 < \rho_2$. Shil'nikov [Shi67] was again the first one in studying the dynamics associated with them. He proved, as in [Shi65], the existence of a countable set of periodic orbits in the non-resonant case $-\rho_1 \neq \rho_2$. Subsequent works [Dev76, FS91, LG97, Här98] were devoted to analyze the formation and bifurcations of these periodic orbits by studying the Poincaré map associated with the flow in a neighborhood of the bifocal homoclinic orbit. Devaney [Dev76] considers the hamiltonian case, hence with $-\rho_1 = \rho_2$. He proves that for any local transverse section to the homoclinic orbit, and for any positive integer N, there is a compact invariant hyperbolic set on which the Poincaré map is topologically conjugate to the Bernoulli shift on N symbols. In seeking to determine the invariant set of this Poincaré map in the general case, it is shown in [FS91] that this set is contained in a neighborhood of a spiral sheet (shaped like a scroll). In fact, the invariant set is a neighborhood of the intersection of this scroll and its image under the map, which is another scroll, in general skewed and offset from the original. In [LG97] the authors extend the known theory regarding bifocal homoclinic bifurcations and present numerical verification of the more interesting theoretical predictions that had been made. Härterich [Här98] studies bifocal homoclinic orbits arising in reversible systems, hence again with $-\rho_1 = \rho_2$. He proves that for any $N \geq 2$ there exists infinitely many N-homoclinic orbits in a neighborhood of the primary homoclinic orbit. Each of them is accumulated by one or more families of N-periodic orbits.

As for Shil'nikov homoclinic orbits, it has been proved (see [OS91]) that homoclinic tangencies to hyperbolic periodic orbits are dense in the space of vector fields with a bifocal homoclinic orbit. Nevertheless, despite the abundant literature regarding bifocal homoclinic orbit, as far as we know, no result has been established relating the existence of these homoclinic bifurcations with the existence of persistent strange attractors. This, in spite of a bifocal homoclinic orbit seems to be a scenario for more complicated dynamics than those inherent to Shil'nikov homoclinic orbits, where the existence of such strange attractors has been proved. In fact it seems natural to think that the dynamical complexity associated with homoclinic cycles increases with dimension. For instance, strange attractors with more than one positive Lyapunov exponent could appear.

4.1.1 Generic unfoldings

Let X be a C^∞ vector field in $\mathbb{R}^n$ with $X(0)=0$ and 1-jet at the origin linearly conjugated to

$$\sum_{k=1}^{n-1} x_{k+1}\frac{\partial}{\partial x_k}.$$

Introducing appropriate C^∞ coordinates, X can be written as:

$$\sum_{k=1}^{n-1} x_{k+1}\frac{\partial}{\partial x_k} + f(x_1,\ldots,x_n)\frac{\partial}{\partial x_n}, \tag{4.1}$$

with $f(x)=O(\|x\|^2)$ where $x=(x_1,\ldots,x_n)$. It is said that X has a nilpotent singularity of codimension n at 0 if the generic condition $\partial^2 f/\partial x_1^2(0)\neq 0$ is fulfilled. The vector field X itself will be often referred to as a nilpotent singularity of codimension n.

Nilpotent singularities of codimension n are generic in families depending on at least n parameters and according to [DIR07, Lemma 2.1] we can state the following result:

Lemma 4.1. *Any n-parametric generic unfolding of a nilpotent singularity of codimension n in $\mathbb{R}^n$ can be written as*

$$\sum_{k=1}^{n-1} x_{k+1}\frac{\partial}{\partial x_k} + \big(\mu_1 + \sum_{k=2}^{n}\mu_k x_k + x_1^2 + h(x,\mu)\big)\frac{\partial}{\partial x_n}, \tag{4.2}$$

where $\mu=(\mu_1,\ldots,\mu_n)\in\mathbb{R}^n$, $h(0,\mu)=0$, $\partial h/\partial x_i(0,\mu)=0$ for $i=1,\ldots,n$, $\partial^2 h/\partial x_1^2(0,\mu)=0$, $h(x,\mu)=O(\|(x,\mu)\|^2)$ and $h(x,\mu)=O(\|(x_2,\ldots,x_n)\|)$.

Remark 4.2. *Besides the condition $\partial^2 f/\partial x_1^2(0)\neq 0$ in (4.1), genericity assumptions in Lemma 4.1 include a transversality condition involving derivatives of the family with respect to parameters.*

The classical techniques of reduction to normal forms could be used to remove terms in the Taylor expansion of h but we do not need to work with simpler expressions. To obtain the results provided in the next sections we will have to impose

$$\kappa = \frac{\partial^2 h}{\partial x_1 \partial x_2}(0,0)\neq 0, \tag{4.3}$$

as an additional generic assumption.

4.1.2 Rescalings and limit families

Generalizing the techniques used in [DI96] for dimension three, we rescale variables and parameters by means of

$$\begin{aligned} \mu_1 &= \varepsilon^{2n}\nu_1, \\ \mu_k &= \varepsilon^{n-k+1}\nu_k \quad \text{for } k=2,\ldots,n, \\ x_k &= \varepsilon^{n+k-1}y_k \quad \text{for } k=1,\ldots,n, \end{aligned} \tag{4.4}$$

with $\varepsilon>0$ and $\nu_1^2+\ldots+\nu_n^2=1$, and also multiply the whole family by a factor $1/\varepsilon$.

In new coordinates and parameters (4.2) can be written as

$$\sum_{k=1}^{n-1} y_{k+1}\frac{\partial}{\partial y_k} + \big(\nu_1 + \sum_{k=2}^{n} \nu_k y_k + y_1^2 + \varepsilon\kappa y_1 y_2 + O(\varepsilon^2)\big)\frac{\partial}{\partial y_n}, \tag{4.5}$$

with κ as introduced in (4.3) and where $y = (y_1, \ldots y_n)$ belongs to an arbitrarily big compact in $\mathbb{R}^n$.

The first step to understand the dynamics arising in generic unfoldings of n-dimensional nilpotent singularities of codimension n is the study of the bifurcation diagram of the *limit family*

$$\sum_{k=1}^{n-1} y_{k+1}\frac{\partial}{\partial y_k} + \big(\nu_1 + \sum_{k=2}^{n} \nu_k y_k + y_1^2\big)\frac{\partial}{\partial y_n}, \tag{4.6}$$

obtained by taking $\varepsilon = 0$ in (4.5). Structurally stable behaviours and generic bifurcations in (4.6) should persist in (4.5) for $\varepsilon > 0$ small enough.

If $\nu_1 > 0$ then (4.6) has no equilibrium points. Moreover the function

$$L(y_1, \ldots, y_n) = y_n - \nu_2 y_1 - \nu_3 y_2 - \ldots - \nu_n y_{n-1}$$

is strictly increasing along the orbits and therefore the maximal compact invariant set is empty. Hence we only need to pay attention to the case $\nu_1 \leq 0$.

On the other hand, up to a change of sign, family (4.6) is invariant under the transformation

$$\begin{aligned}(\nu, y) \mapsto \big(&\nu_1, (-1)^{n-1}\nu_2, (-1)^{n-2}\nu_3, \ldots, \nu_{n-1}, -\nu_n,\\ &(-1)^n y_1, (-1)^{n-1} y_2, (-1)^{n-2} y_3, \ldots, y_{n-1}, -y_n\big),\end{aligned} \tag{4.7}$$

with $\nu = (\nu_1, \ldots, \nu_n)$. As a first consequence, the study of bifurcations can be reduced to the region $\mathcal{R} = \{(\nu_1, \ldots, \nu_n) \in \mathbb{S}^{n-1} : \ \nu_1 \leq 0, \ \nu_n \leq 0\}$. Moreover, since the limit family is invariant under (4.7) up to a change of sign, for parameter values on the set

$$\mathcal{T} = \{(\nu_1, \ldots, \nu_n) \in \mathbb{S}^{n-1} : \nu_{n-2i} = 0 \text{ with } i = 0, \ldots, \lfloor (n-2)/2 \rfloor\},$$

where $\lfloor \cdot \rfloor$ denotes the floor function, the correspondent vector fields in the limit family (4.6) are time-reversible with respect to the involution

$$R : (y_1, y_2, y_3, \ldots, y_n) \mapsto ((-1)^n y_1, (-1)^{n-1} y_2, \ldots, y_{n-1}, -y_n).$$

We said that the manifold $\mathcal{T}$ of dimension $\lfloor n/2 \rfloor - 1$ is the *reversibility set* of the n-dimensional nilpotent limit family.

Note that the divergence of the limit family (4.6) takes the constant value ν_n. Therefore the condition $\nu_n = 0$ characterizes a subfamily of volume-preserving vector fields. Assuming that n is even and defining $m = n/2$, for parameter values in $\mathcal{T}$ the limit family (4.6) can be written as

$$\sum_{k=1}^{n-1} y_{k+1}\frac{\partial}{\partial y_k} + \big(\nu_1 + \sum_{k=1}^{m-1} \nu_{2k+1} y_{2k+1} + y_1^2\big)\frac{\partial}{\partial y_n}. \tag{4.8}$$

Theorem 4.3. *Introducing the new variables $q = S\cdot(y_1, y_3, \dots, y_{n-1})^t$ and $p = (y_2, y_4, \dots, y_n)^t$,*

$$S = \begin{pmatrix} -\nu_3 & -\nu_5 & \dots & -\nu_{n-1} & 1 \\ -\nu_5 & & \iddots & \iddots & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ -\nu_{n-1} & \iddots & \iddots & & \vdots \\ 1 & 0 & \dots & \dots & 0 \end{pmatrix},$$

the family (4.8) transforms into a Hamiltonian vector field X_H, $H(q,p) = \frac{1}{2} < Sp, p > +V(q)$. The potential V is defined as

$$\begin{aligned} V(q) = &-\frac{1}{3}q_m^3 - \frac{1}{2}\sum_{k=1}^{m-1}\nu_{2k+1}b_{k+1}q_m^2 - \frac{1}{2}\sum_{j=1}^{\lfloor m/2\rfloor} b_{m-2j+1}q_{m-j}^2 \\ &- \sum_{k=1}^{m-1}\sum_{i=m-k}^{m-1}\nu_{2k+1}b_{i-m+k+1}q_iq_m - \sum_{j=1}^{\lfloor m/2\rfloor}\sum_{i=j}^{m-j-1} b_iq_iq_{m-j} - \nu_1 q_m, \end{aligned}$$

where, given $b_1 = 1$,

$$b_i = \sum_{\ell=1}^{i-1}\nu_{2(m-i+\ell)+1}b_\ell \qquad \text{for } i = 2, \dots, m.$$

In order to prove this theorem we will use the following technical result:

Lemma 4.4. *Given a symmetric upper anti-triangular matrix*

$$A = \begin{pmatrix} a_m & a_{m-1} & \dots & a_2 & 1 \\ a_{m-1} & & \iddots & \iddots & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ a_2 & \iddots & \iddots & & \vdots \\ 1 & 0 & \dots & \dots & 0 \end{pmatrix},$$

A^{-1} is a lower anti-triangular symmetric matrix

$$A^{-1} = \begin{pmatrix} 0 & \dots & \dots & 0 & 1 \\ \vdots & & \iddots & \iddots & b_2 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ 0 & \iddots & \iddots & & b_{m-1} \\ 1 & b_2 & \dots & b_{m-1} & b_m \end{pmatrix}$$

where, given $b_1 = 1$,

$$b_i = -\sum_{\ell=1}^{i-1} a_{i-\ell+1}b_\ell, \quad \text{for } i = 2, \dots, m.$$

Proof. Let P be an anti-diagonal matrix with all entries equal to 1. Hence

$$L = PA = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ a_2 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{m-1} & & \ddots & \ddots & 0 \\ a_m & a_{m-1} & \dots & a_2 & 1 \end{pmatrix}.$$

is a lower triangular matrix. Therefore, $L^{-1} = (b_{i,j})$ is also a lower triangular matrix and hence $A^{-1} = L^{-1}P^{-1} = L^{-1}P$ is a lower anti-triangular matrix. In fact, using the well know formulas for the calculation of the inverse of a triangular matrix, it follows that, for all $j = 1, \dots, m$

$$\begin{aligned} b_{j,j} &= 1, \\ b_{i,j} &= 0 \quad \text{for all } i = 1, \dots, j-1, \\ b_{i,j} &= -\sum_{\ell=j}^{i-1} a_{i-\ell+1} b_{\ell,j} \quad \text{for all } i = j+1, \dots, m. \end{aligned}$$

On the other hand, $b_{i,j} = b_{i+1,j+1}$ for all $i = j+1, \dots, m-1$. Indeed it is clear for $i = j+1$. For $i = j+2, \dots, m-1$ we can argue by induction. Finally, by defining $b_i = b_{i,1}$ for all $i = 1, \dots, m$, and calculating $A^{-1} = L^{-1}P$ the proof is finished. □

Proof of Theorem 4.3. It follows from Lemma 4.4 that

$$S^{-1} = \begin{pmatrix} 0 & \dots & \dots & 0 & 1 \\ \vdots & & \iddots & \iddots & b_2 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ 0 & \iddots & \iddots & & b_{m-1} \\ 1 & b_2 & \dots & b_{m-1} & b_m \end{pmatrix},$$

where, defining $b_1 = 1$,

$$b_i = \sum_{\ell=1}^{i-1} \nu_{2(m-i+\ell)+1} b_\ell \quad \text{for } i = 2, \dots, m.$$

or equivalently

$$b_{m-j+1} = \sum_{\ell=1}^{m-j} \nu_{2(j+\ell-1)+1} b_\ell = \sum_{k=j}^{m-1} \nu_{2k+1} b_{k-j+1} \quad \text{for } j = 1, \dots, m-1. \tag{4.9}$$

Writing family (4.8) in the new variables we get

$$Sp \frac{\partial}{\partial q} + \sum_{k=1}^{m-1} \Big(\sum_{i=m-k}^{m} b_{i-m+k+1} q_i \Big) \frac{\partial}{\partial p_k} + \big(\nu_1 + \sum_{k=1}^{m-1} \nu_{2k+1} \dot{p}_k + q_m^2\big) \frac{\partial}{\partial p_m}.$$

To obtain a function $V(q)$ such that $\dot{p} = -\nabla V(q)$ we need

$$-\frac{\partial V}{\partial q_i} = \dot{p}_i \qquad \text{for all } i = 1, \dots, m.$$

In particular

$$-\frac{\partial V}{\partial q_m} = \nu_1 + \sum_{k=1}^{m-1} \nu_{2k+1}\dot{p}_k + q_m^2,$$

and therefore

$$-V(q) = \nu_1 q_m + \sum_{k=1}^{m-1} \nu_{2k+1}\big(\frac{1}{2}b_{k+1}q_m^2 + \sum_{i=m-k}^{m-1} b_{i-m+k+1}q_i q_m\big) + \frac{1}{3}q_m^3 + \varphi_{m-1}(q_1,\dots,q_{m-1}).$$

From the identity $-\partial V/\partial q_{m-1} = \dot{p}_{m-1}$ and taking into account the equation (4.9) we get

$$\frac{\partial \varphi_{m-1}}{\partial q_{m-1}} = \sum_{i=1}^{m} b_i q_i - \sum_{k=1}^{m-1} \nu_{2k+1} b_k q_m = \sum_{i=1}^{m-1} b_i q_i$$

and therefore

$$\varphi_{m-1}(q_1,\dots,q_{m-1}) = \frac{1}{2}b_{m-1}q_{m-1}^2 + \sum_{i=1}^{m-2} b_i q_i q_{m-1} + \varphi_{m-2}(q_1,\dots,q_{m-2}).$$

Since $-\partial V/\partial q_{m-2} = \dot{p}_{m-2}$, a similar computation leads to

$$\frac{\partial \varphi_{m-2}}{\partial q_{m-2}} = \sum_{i=2}^{m} b_{i-1}q_i - \sum_{k=2}^{m-1} \nu_{2k+1} b_{k-1} q_m - b_{m-2}q_{m-1} = \sum_{i=2}^{m-2} b_{i-1}q_i.$$

and hence

$$\varphi_{m-2}(q_1,\dots,q_{m-2}) = \frac{1}{2}b_{m-3}q_{m-2}^2 + \sum_{i=2}^{m-3} b_{i-1}q_i q_{m-2} + \varphi_{m-3}(q_1,\dots,q_{m-3}).$$

A recursive argument provides

$$\frac{\partial \varphi_{m-j}}{\partial q_{m-j}} = \sum_{i=j}^{m} b_{i-j+1}q_i - \sum_{k=j}^{m-1} \nu_{2k+1} b_{k-j+1} - \sum_{i=m-j+1}^{m-1} b_{i-j+1}q_i = \sum_{i=j}^{m-j} b_{i-j+1}q_i,$$

for all $j = 1,\dots,\lfloor m/2 \rfloor$ and consequently,

$$\varphi_{m-j}(q_1,\dots,q_{m-j}) = \frac{1}{2}b_{m-2j+1}q_{m-j}^2 + \sum_{i=j}^{m-j-1} b_i q_i q_{m-j} + \varphi_{m-j-1}(q_1,\dots,q_{m-j-1})$$

where for $j = \lfloor m/2 \rfloor$ the function $\varphi_{m-\lfloor m/2 \rfloor -1}$ is constant. Therefore we get a function $V(q)$ with

$$\begin{aligned}-V(q) = \nu_1 q_m &+ \sum_{k=1}^{m-1} \nu_{2k+1}\left(\frac{1}{2}b_{k+1}q_m^2 + \sum_{i=m-k}^{m-1} b_{i-m+k+1}q_i q_m\right) + \frac{1}{3}q_m^3 \\ &+ \sum_{j=1}^{\lfloor m/2 \rfloor} \big(\frac{1}{2}b_{m-2j+1}q_{m-j}^2 + \sum_{i=j}^{m-j-1} b_i q_i q_{m-j}\big) + \varphi_{m-\lfloor m/2 \rfloor -1},\end{aligned}$$

such that $\dot{p} = -\nabla V(q)$. This concludes the proof of Theorem 4.3. □

4.2 Nilpotent singularity of codimension four in $\mathbb{R}^4$

Along this section we will take $n = 4$ in all the general expressions introduced in §4.1.2 and we will prove Theorem G. That is, we will prove that in any generic unfolding of a nilpotent singularity of codimension four in $\mathbb{R}^4$ there exists a bifurcation hypersurface of homoclinic connections to bifocus equilibria.

It follows from Lemma 4.1 that any generic unfolding of the nilpotent singularity of codimension four in $\mathbb{R}^4$ can be written as in (4.2). After applying the rescaling (4.4) we get

$$y_2\frac{\partial}{\partial y_1} + y_3\frac{\partial}{\partial y_2} + y_4\frac{\partial}{\partial y_3} + \left(\nu_1 + \nu_2 y_2 + \nu_3 y_3 + \nu_4 y_4 + y_1^2 + \varepsilon\kappa y_1 y_2 + O(\varepsilon^2)\right)\frac{\partial}{\partial y_4}, \tag{4.10}$$

with $\nu = (\nu_1, \nu_2, \nu_3, \nu_4) \in \mathbb{S}^3$ and $\varepsilon > 0$. As mentioned in §4.1.2 the first step to understand the dynamics arising in (4.10) is the study of the limit family

$$y_2\frac{\partial}{\partial y_1} + y_3\frac{\partial}{\partial y_2} + y_4\frac{\partial}{\partial y_3} + \left(\nu_1 + \nu_2 y_2 + \nu_3 y_3 + \nu_4 y_4 + y_1^2\right)\frac{\partial}{\partial y_4}, \tag{4.11}$$

obtained from (4.10) taking $\varepsilon = 0$. As argued in §4.1.2 one only need to pay attention to parameters in the region $\mathcal{R} = \{(\nu_1, \nu_2, \nu_3, \nu_4) \in \mathbb{S}^3 : \nu_1 \le 0,\ \nu_4 \le 0\}$. When $\nu \in \mathcal{R}$, vector fields in the limit family (4.11) have equilibrium points $p_\pm = (\pm\sqrt{-\nu_1}, 0, 0, 0)$ with characteristic equations

$$r^4 - \nu_4 r^3 - \nu_3 r^2 - \nu_2 r \mp 2\sqrt{-\nu_1} = 0. \tag{4.12}$$

Local bifurcations arising in the family were discussed in [Dru09].

For parameters on the reversibility curve $\mathcal{T} = \left\{(\nu_1, \nu_2, \nu_3, \nu_4) \in \mathbb{S}^3 : \nu_2 = \nu_4 = 0\right\}$ with $\nu_1 \le 0$, the characteristic equations reduces to $r^4 - \nu_3 r^2 \mp 2\sqrt{-\nu_1} = 0$. It follows that the linear part at p_+ always have a pair of real eigenvalues and a pair of complex eigenvalues with non-zero real part. Local behaviour at p_- is richer and it is depicted in Figure A. Note that we only have to pay attention to $\nu_1^2 + \nu_3^2 = 1$ with $\nu_1 \le 0$. It easily follows that the linear part at p_- has

- a double zero eigenvalue and eigenvalues ± 1 at $\mathrm{BT} = (0, 0, 1, 0)$;
- a double zero eigenvalue and a pair of pure imaginary eigenvalues at $\mathrm{HDZ} = (0, 0, -1, 0)$;
- two double real eigenvalues $\pm(\nu_3/2)^{1/2}$ at $\mathrm{BD} = (\nu_1, 0, \nu_3, 0)$ with $\nu_3^2 - 8\sqrt{-\nu_1} = 0$ and $\nu_3 > 0$;
- two double pure imaginary eigenvalues $\pm i(-\nu_3/2)^{1/2}$ at $\mathrm{HH} = (\nu_1, 0, \nu_3, 0)$ with $\nu_3^2 = 8\sqrt{-\nu_1}$ and $\nu_3 < 0$;
- four non-zero real eigenvalues $\pm\lambda_k$, with $k = 1, 2$ for parameters along the open arc $\mathcal{SR}$ between BD and BT;
- four complex eigenvalues with non-zero real part $\rho \pm \omega i$ and $-\rho \pm \omega i$ for parameters along the open arc $\mathcal{DF}$ between BD and HH; and
- four pure imaginary eigenvalues $\pm\omega_k i$, with $k = 1, 2$, for parameters along the open arc $\mathcal{HH}$ between HH and HDZ.

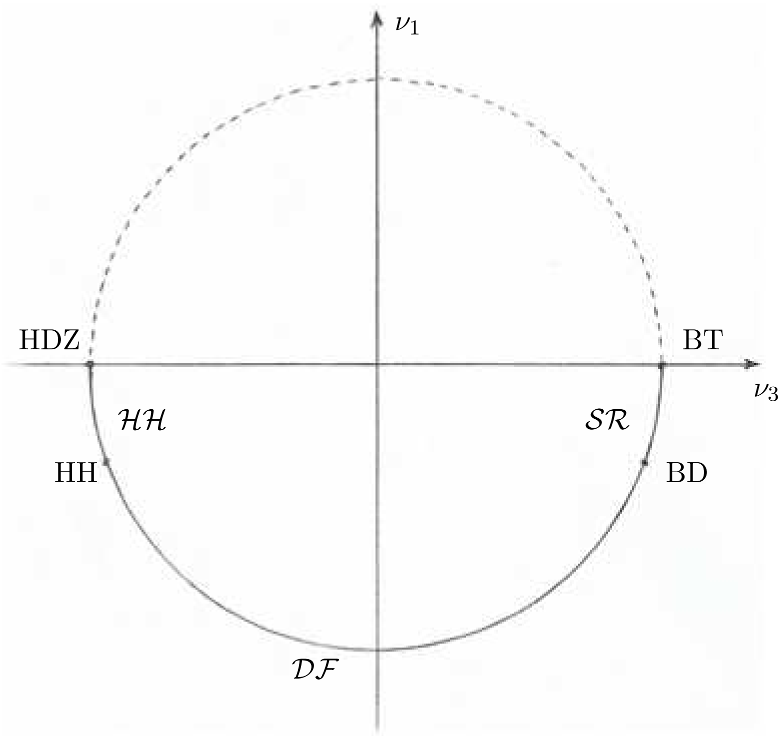

Fig. A: The reversibility curve $\mathcal{T}$ attending to the type of eigenvalues at p_-.

From the analysis of the linear part at the equilibrium points it follows that a bifocal homoclinic orbit is only possible at p_-. Then, we will study the existence of homoclinic orbits to p_- for parameter values along $\mathcal{T}$ and in this case, since the limit family (4.11) for this parameter values a Hamiltonian vector field, it said to be *conservative bifocal homoclinic orbit.*

4.2.1 Conservative bifocal homoclinic orbits in the limit family

In order to study the family (4.10) close to the reversibility curve $\mathcal{T}$ with $\nu_1 < 0$ it is more convenient to use a directional version of the rescaling (4.4) taking $\nu_1 = -1$ and $(\nu_2, \nu_3, \nu_4) = (\bar{\nu}_2, \bar{\nu}_3, \bar{\nu}_4) \in \mathbb{R}^3$ to get

$$y_2\frac{\partial}{\partial y_1} + y_3\frac{\partial}{\partial y_2} + y_4\frac{\partial}{\partial y_3} + \big(-1 + \bar{\nu}_2 y_2 + \bar{\nu}_3 y_3 + \bar{\nu}_4 y_4 + y_1^2 + \varepsilon\kappa y_1 y_2 + O(\varepsilon^2)\big)\frac{\partial}{\partial y_4}. \tag{4.13}$$

The equilibrium points when $\varepsilon = 0$ are given by $q_\pm = (\pm 1, 0, 0, 0)$. Note that in fact $q_\pm$ are the only equilibrium points even for $\varepsilon > 0$ because in (4.2) $h(x,\mu) = O(\|(x_2,\ldots,x_n)\|)$ and this property is preserved by the rescaling. In order to compare with equations already considered in the literature we translate q_- to the origin applying the change of coordinates

$$x_1 = (y_1+1)/2, \qquad x_2 = y_2/2^{5/4}, \qquad x_3 = y_3/2^{6/4}, \qquad x_4 = y_4/2^{7/4},$$

to (4.13) and multiplying by the factor $2^{1/4}$ to obtain

$$x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} + x_4\frac{\partial}{\partial x_3} + \big(-x_1 + \eta_2 x_2 + \eta_3 x_3 + \eta_4 x_4 + x_1^2 + \bar{\varepsilon}\kappa x_1 x_2 + O(\bar{\varepsilon}^2)\big)\frac{\partial}{\partial x_4} \tag{4.14}$$

with $\eta_2 = 2^{-3/4}(\bar{\nu}_2 - \varepsilon\kappa)$, $\eta_3 = 2^{-1/2}\bar{\nu}_3$, $\eta_4 = 2^{-1/4}\bar{\nu}_4$ and $\bar{\varepsilon} = 2^{1/4}\varepsilon$.

The equilibrium point q_- in (4.13) corresponds to the equilibrium point of (4.14) at the origin. The limit subfamily for $\eta_2 = \eta_4 = \bar{\varepsilon} = 0$ is now given as

$$x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3} + \big(- x_1 + \eta_3 x_3 + x_1^2 \big) \frac{\partial}{\partial x_4}. \tag{4.15}$$

Writing $u = x_1$, the vector field (4.15) is equivalent to the fourth order differential equation

$$u^{(iv)}(t) + Pu''(t) + u(t) - u(t)^2 = 0, \tag{4.16}$$

with $P = -\eta_3$. This equation has been widely studied [AT92, CT93, BCT96, Buf96] due to its role in some applications as the study of travelling waves of the Korteweg- de Vries equation

$$u_t = u_{xxxx} - bu_{xxx} + 2uu_x,$$

or the description of the displacement of a compressed strut with bending softness resting on a nonlinear elastic foundation [Cha98]. In particular, according to [AT92], when $\eta_3 = 2$ the vector field (4.15) has a homoclinic orbit to a hyperbolic equilibrium point at which the linear part has a pair of double real eigenvalues ± 1. An essential fact used in [AT92] to prove the existence of homoclinic orbits in (4.15) is that it is a family of hamiltonian vector fields as we have stated in Theorem 4.3 for a more general case. This permits to apply the general theory developed in [HT84, Theorem 2] to conclude that, for each $P \leq -2$, there exists an even solution u with $u(t) \to 0$ when $t \to \pm\infty$ satisfying that $u > 0$, $u' < 0$ and $(P/2)u' + u''' > 0$ on $(0, \infty)$. They also prove that for all $P \leq -2$ any such even solution is unique. From [BCT96] it follows that this unique homoclinic orbit γ is transversal for the restriction to the level surface of the hamiltonian function which contains it and, consequently, it is non-degenerate in the following sense:

Definition 4.1. *A homoclinic orbit γ to a hyperbolic equilibrium point p of a vector field X is said* non-degenerate *if*

$$\dim T_x W^s(p) \cap T_x W^u(p) = 1,$$

with $x \in \gamma$. Otherwise γ is said degenerate.

Moreover, again in [AT92], the persistence of such homoclinic solutions is argued for $P > -2$ but close enough to -2. Variational methods used in [Buf96] allow to prove that at least one homoclinic solution exists for $P < 2$. On the other hand, in [BCT96, Section 2] authors check all hypothesis required in [CT93, Theorem 4.4] to conclude that a Belyakov-Devaney bifurcation takes place at $P = -2$. It consists in the emerging from the primary homoclinic solution and for each $n \in \mathbb{N}$ of a finite number of n-modal secondary homoclinics (or n-pulses) which cut n times a section transversal to the primary homoclinic orbit [Dev76, Bel84, BS90]. Heuristic arguments in [BCT96], supported by numerical results, show that the non-degenerate n-modal homoclinic orbits arising at $P = -2$ become in degenerate orbits and disappear gradually when P varies from $P = -2$ to $P = 2$ through a cascade of coalescences and bifurcations. In particular, it is known from [IP93] that for P close to $P = 2$ there exist at least two even homoclinic solutions and from the numerical results it seems that no other homoclinic orbits reaches $P = 2$.

All the above results about the existence of homoclinic solutions of (4.16) can be directly translated to family (4.15) and also to the reversible subfamily of (4.11) obtained restricting to parameter values along the previously defined reversibility curve $\mathcal{T}$ (or, in this even dimensional case, also called *conservative curve* because Theorem 4.3). For the later case we can conclude that (see Figure A):

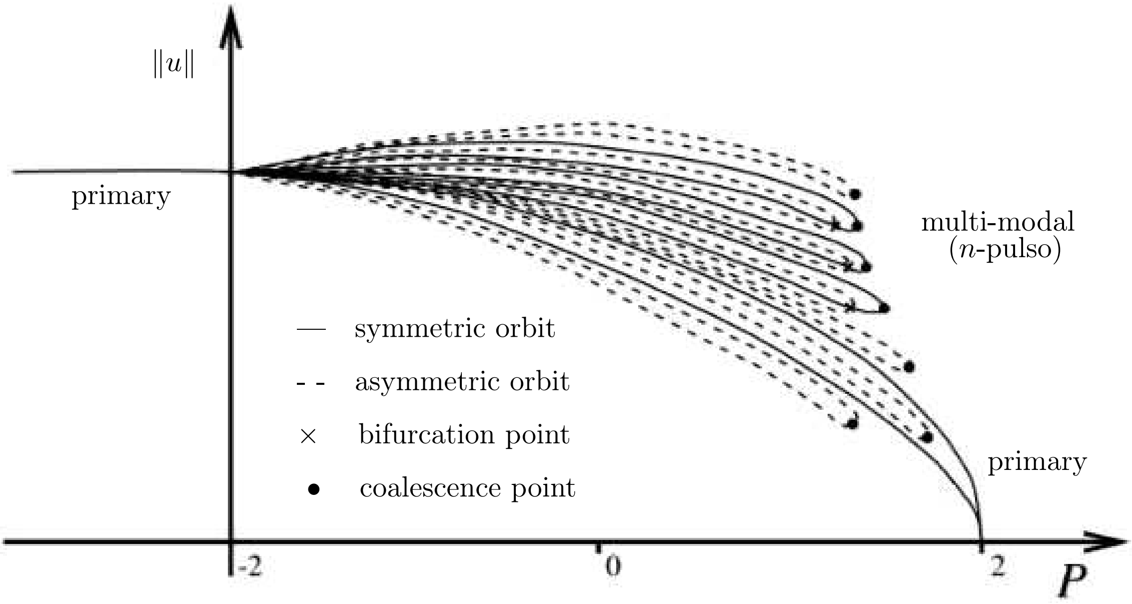

Fig. B: Schematic bifurcation diagram of homoclinic solutions of (4.16).

- for parameter values along $\mathcal{DF} \cup \{\mathrm{BD}\} \cup \mathcal{SR}$ there exists a symmetric homoclinic orbit at p_- which is unique and non-degenerate along $\{\mathrm{BD}\} \cup \mathcal{SR}$,
- BD is a Belyakov-Devaney bifurcation point and thus non-degenerate bifocal homoclinic orbits arising at this point,
- numerical continuation shows that non-degenerate n-modal homoclinic orbits arising at BD become in degenerate orbits and disappear gradually when parameters move along $\mathcal{DF}$ in the direction of HH. Close to that point only two symmetric homoclinic orbits persist.

4.2.2 Bifocal homoclinic orbits in generic unfoldings

In order to study the persistence of homoclinic orbits we will consider (4.10) as an unfolding of the Belyakov-Devaney bifurcation point BD. As already mentioned it is better to work with expression (4.14) for the rescaled unfolding. With respect to parameters $(\eta_2, \eta_3, \eta_4, \bar{\varepsilon})$ the point BD corresponds to $(0, 2, 0, 0)$. Note that (4.14) can be written as

$$x' = f(x) + g(\lambda, x), \tag{4.17}$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (\eta_2, \eta_3 - 2, \eta_4, \bar{\varepsilon})$,

$$f(x) = (x_2, x_3, x_4, -x_1 + 2x_3 + x_1^2)$$

and

$$g(\lambda, x) = (0, 0, 0, \lambda_1 x_2 + \lambda_2 x_3 + \lambda_3 x_4 + \lambda_4 \kappa x_1 x_2 + O(\lambda_4^2)).$$

As already mentioned, $q_\pm$ are the only equilibrium points of (4.13) for all $\varepsilon \geq 0$ and hence $g(\lambda, 0) = 0$ for all λ. Observe that only bifurcations occurring inside the region of parameters with $\lambda_4 > 0$ will be observed in the unfolding of the singularity.

Theorem 4.5. *In a neighbourhood of $\lambda = 0$ there exists a bifurcation hypersurface $\mathcal{H}om$ corresponding to parameter values for which (4.17) has homoclinic orbits to the origin. Moreover:*

There exist two bifurcation surfaces $\mathcal{H}om^-$ and $\mathcal{H}om^+$ contained in $\mathcal{H}om$ corresponding to parameter values for which the origin has a double negative and positive, respectively, real eigenvalue. The surfaces $\mathcal{H}om^+$ and $\mathcal{H}om^-$ intersect transversely along a curve $\mathcal{H}om^\pm$ corresponding to parameter values for which the origin has a pair of double real eigenvalues $\{r_1, r_2\}$ with $r_1 < 0 < r_2$. $\mathcal{H}om^- \cup \mathcal{H}om^+$ splits $\mathcal{H}om$ into four regions:

- *$\mathcal{H}om_{FF}$: homoclinic orbits to a focus-focus equilibrium (bifocus case),*
- *$\mathcal{H}om_{N^+F^-}$: homoclinic orbits to a (repelling) node-(attracting) focus equilibrium,*
- *$\mathcal{H}om_{F^+N^-}$: homoclinic orbits to a (repelling) focus-(attracting) node equilibrium,*
- *$\mathcal{H}om_{NN}$: homoclinic orbits to a node-node equilibrium.*

All bifurcations are transverse to $\lambda_4 = 0$.

Since all bifurcations are transverse to $\lambda_4 = 0$ they are also present in the unfolding of the nilpotent singularity of codimension four. Particularly, Theorem G follows as a corollary of this theorem.

Remark 4.6. *Recall that the bifurcation point $\lambda = 0$ in* (4.17) *corresponds to the Belyakov-Devaney bifurcation point* BD *in* (4.10)*. In particular, the hypersurface of parameters corresponding to homoclinic orbits to a node-node equilibrium point in family* (4.10) *cuts $\varepsilon = 0$ along the curve $\mathcal{SR}$. Similarly, $\mathcal{H}om_{FF}$ is unfolded from $\mathcal{DF}$.*

The proof of Theorem 4.5 requires some background on exponential dichotomies. In Appendix A we include a brief summary of results about dichotomies in order to get a precise formulation of the bifurcation equation (see Theorem A.12) which is required in the following proof of the above result.

Proof of Theorem 4.5. Family (4.17) fulfills all the hypothesis imposed to Equation (A.1) in Appendix A. In particular, $x' = f(x)$ satisfies the following:

(BD1) It has a first integral

$$H(x_1, x_2, x_3, x_4) = \frac{1}{2}x_1^2 - \frac{1}{3}x_1^3 - x_2^2 + x_2x_4 - \frac{1}{2}x_3^2$$

(BD2) It is time reversible with respect to

$$R : (x_1, x_2, x_3, x_4) \mapsto (x_1, -x_2, x_3, -x_4).$$

(BD3) The origin is a hyperbolic equilibrium point at which the linear part has a pair of double real eigenvalues ± 1.

(BD4) According to [AT92], there exists a non-degenerate homoclinic orbit

$$\gamma = \{p(t) = (p_1(t), p_2(t), p_3(t), p_4(t)) \,:\, t \in \mathbb{R}\}$$

to the origin such that $p_1(t)$ and $p_3(t)$ are even functions and $p_2(t)$ and $p_4(t)$ are odd functions and, moreover, $p_1 > 0$, $p_2 < 0$ and $p_4 - p_2 > 0$ on $(0, \infty)$.

(BD5) According to Proposition A.9, since γ is a non-degenerate homoclinic orbit, both the variational equation $z' = Df(p(t))z$ and its adjoint $z' = -Df(p(t))^* z$ has a unique non trivial linearly independent bounded solution. The function $\varphi(t) = f(p(t))$ is a bounded solution of the variational equation and

$$\psi(t) = \nabla H(p(t)) = (p_1(t) - p_1(t)^2, p_4(t) - 2p_2(t), -p_3(t), p_2(t))$$

is a bounded solution of adjoint equation.

Finally, let us consider the bifurcation equation for homoclinic solutions $\xi^\infty(\lambda) = 0$, with $\xi^\infty : \Lambda \to \mathbb{R}$ and $\Lambda \subset \mathbb{R}^4$ a neighbourhood of the origin, as introduced in Lemma A.11. It follows from Theorem A.12 that under the generic condition $\nabla\xi^\infty(0) = (\xi_{\lambda_1}, \xi_{\lambda_2}, \xi_{\lambda_3}, \xi_{\lambda_4}) \neq 0$, where

$$\xi_{\lambda_i} = \int_{-\infty}^{\infty} \langle \psi(t), \frac{\partial g}{\partial \lambda_i}(0, p(t)) \rangle \, dt,$$

then (4.17) has homoclinic orbits (continuation of γ) for parameters on a hypersurface $\mathcal{H}om$ with tangent subspace at $\lambda = 0$ given by

$$\xi_{\lambda_1}\lambda_1 + \xi_{\lambda_2}\lambda_2 + \xi_{\lambda_3}\lambda_3 + \xi_{\lambda_4}\lambda_4 = 0. \tag{4.18}$$

Note that

$$\xi_{\lambda_1} = \int_{-\infty}^{\infty} p_2^2(t)\, dt, \qquad \xi_{\lambda_2} = \int_{-\infty}^{\infty} p_2(t)p_3(t)\, dt,$$
$$\xi_{\lambda_3} = \int_{-\infty}^{\infty} p_2(t)p_4(t)\, dt, \qquad \xi_{\lambda_4} = \int_{-\infty}^{\infty} \kappa p_1(t)p_2^2(t)\, dt.$$

Clearly $\xi_{\lambda_1} \neq 0$. Since p_2p_3 is an odd function $\xi_{\lambda_2} = 0$. Integrating by parts one gets

$$\xi_{\lambda_3} = -\int_{-\infty}^{\infty} p_3(t)^2\, dt \neq 0.$$

Finally, since p_1 is a positive function, we also get that $\xi_{\lambda_4} \neq 0$. Therefore the tangent subspace (4.18) intersects $\lambda_4 = 0$ transversely. Consequently $\mathcal{H}om$ also meets $\lambda_4 = 0$ transversely.

Now we have to study the eigenvalues at the equilibrium point in order to determine which types of homoclinic orbits can be unfolded by the singularity. Since for $\lambda = 0$ the linear part at $x = 0$ has a pair of double real eigenvalues ± 1 and $\dim W^s(0) = \dim W^u(0) = 2$, for all λ small enough, then we can expect three different types of equilibrium: a focus-focus (bifocus), a node-node or a focus-node. It easily follows that the characteristic polynomial at $x = 0$ is given by

$$Q(r, \lambda) = r^4 - D(\lambda)r^3 - C(\lambda)r^2 - B(\lambda)r - A(\lambda),$$

with

$$A(\lambda) = -1 + O(\lambda_4^2), \quad B(\lambda) = \lambda_1 + O(\lambda_4^2), \quad C(\lambda) = 2 + \lambda_2 + O(\lambda_4^2), \quad D(\lambda) = \lambda_3 + O(\lambda_4^2).$$

The condition for an improper node is given by the discriminant equations

$$Q(r,\lambda)=0, \qquad \frac{\partial Q}{\partial r}(r,\lambda)=0.$$

Note that $(r,\lambda)=(\pm 1,0)$ are both solutions of the discriminant equations. Now it follows from a straightforward application of the Implicit Function Theorem that there exist two hypersurfaces $\mathcal{D}^-$ and $\mathcal{D}^+$ through the origin in the parameter space such that, for parameter values on $\mathcal{D}^-$ (resp. $\mathcal{D}^+$) the equilibrium point at the origin has a double negative (resp. positive) real eigenvalue. Moreover the respective tangent subspaces at $\lambda=0$ are $\lambda_1-\lambda_2+\lambda_3=0$ and $\lambda_1+\lambda_2+\lambda_3=0$. Let $N_{\mathcal{H}om}=(\xi_{\lambda_1},\xi_{\lambda_2},\xi_{\lambda_3},\xi_{\lambda_4})$, $N_{\mathcal{D}^-}=(1,-1,1,0)$ and $N_{\mathcal{D}^+}=(1,1,1,0)$ be the normal vectors to the tangent spaces of $\mathcal{H}om$, $\mathcal{D}^-$ and $\mathcal{D}^+$ at $\lambda=0$, respectively. Moreover denote $N_{\lambda_4=0}=(0,0,0,1)$. Since $\operatorname{rank}(N_{\mathcal{H}om},N_{\mathcal{D}^-},N_{\lambda_4=0})=3$, there exists a surface $\mathcal{H}om^-=\mathcal{H}om\cap\mathcal{D}^-$ transverse to $\lambda_4=0$ of homoclinic orbits to an equilibrium point with a double negative real eigenvalue. Moreover, since $\operatorname{rank}(N_{\mathcal{H}om},N_{\mathcal{D}^+},N_{\lambda_4=0})=3$, there exists a surface $\mathcal{H}om^+=\mathcal{H}om\cap\mathcal{D}^+$ transverse to $\lambda_4=0$ of homoclinic orbits to an equilibrium point with a double positive real eigenvalue. On the other hand $\operatorname{rank}(N_{\mathcal{H}om},N_{\mathcal{D}^-},N_{\mathcal{D}^+},N_{\lambda_4=0})=4$ if and only if $\xi_{\lambda_1}-\xi_{\lambda_3}\neq 0$. But, taking into account that p_2 and p_4 are odd functions and also that $p_2<0$ and $p_4-p_2>0$ on $(0,\infty)$ it follows that

$$\xi_{\lambda_1}-\xi_{\lambda_3}=\int_{-\infty}^{\infty}p_2(t)(p_2(t)-p_4(t))\,dt>0.$$

Hence we can conclude that $\operatorname{rank}(N_{\mathcal{H}om},N_{\mathcal{D}^-},N_{\mathcal{D}^+},N_{\lambda_4=0})=4$. Therefore there exists a curve $\mathcal{H}om^{\pm}=\mathcal{H}om\cap\mathcal{D}^-\cap\mathcal{D}^+$ transverse to $\lambda_4=0$ of homoclinic orbits to an equilibrium point with a pair of double real eigenvalues one positive and the other negative. □

The eigenvalues of the linear part in the equilibrium point p_- of (4.10) is $\pm(\rho\pm iw)$ for parameter values in $\mathcal{DF}$. Hence, for parameter values close to this curve los autovalores en p_- are also $\rho_1\pm iw_1$ and $\rho_2\pm iw_2$ with $\rho_1<0<\rho_2$. The divergence of this limit family (4.10) is equal to ν_4. Then it follows that $\rho_1+\rho_2=\nu_4/2$. Well then, since the homoclinic orbits are continued for the parameter values $\lambda_3\neq 0$ where $\lambda_3=\eta_4=2^{-1/4}\bar{\nu}_4$, according to the equation $\xi_{\lambda_2}\lambda_2+\xi_{\lambda_3}\lambda_3+\xi_{\lambda_4}\lambda_4=0$, it is concluded that $-\rho_1\neq\rho_2$ for some parameter values in $\mathcal{H}om_{FF}$.

Remark 4.7. *Bifocal homoclinic orbits under the generic condition (no-resonant case) $-\rho_1\neq\rho_2$ are unfolded from $\mathcal{DF}$ for parameter values in $\mathcal{H}om_{FF}$.*

4.3 Return map for a conservative bifocal homoclinic orbits

Let us consider a smooth Hamiltonian vector field X_H on $\mathbb{R}^4$ under the following assumptions:

(H1) p is an equilibrium point of X_H with eigenvalues $\pm\lambda\pm iw$, $\lambda\omega\neq 0$,

(H2) there exists $\gamma\subset W^s(p)\cap W^u(p)$ non-degenerate homoclinic orbit.

A *bifocus equilibrium*, as (H1), possesses two local smooth two-dimensional submanifolds, stable $W^s_{loc}(p)$ and unstable $W^u_{loc}(p)$, lying both in the singular level $H(p)$. This set is a smooth three-dimensional submanifold near every point, except for the point p, where it has a singularity

We will study the dynamical behavior in a neighborhood of the non-degenerate bifocal homoclinic orbit γ of the conservative vector field X_H. First result in this direction was obtained by

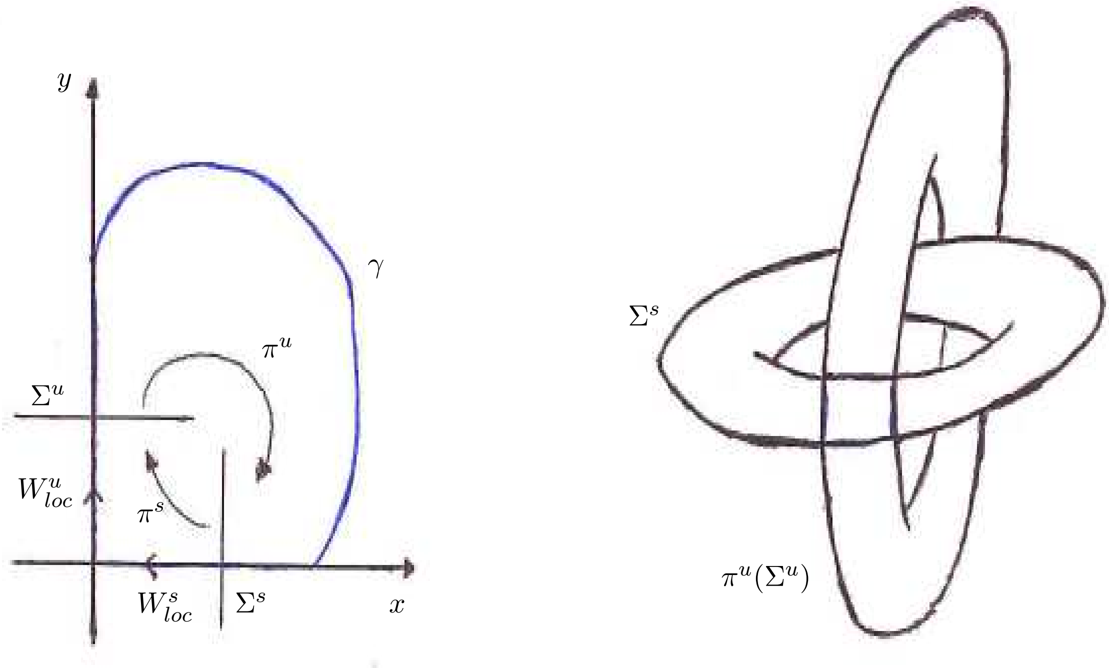

Fig. C: Cross-sections and Poincaré return map

Devaney [Dev76] who carried over the results by Shil'nikov [Shi67] from general systems to Hamiltonian ones which required of a special (sympletic) tool. In [Dev76], the existence of infinitely many hyperbolic subsets in a neighborhood of γ accumulating onto the bifocal homoclinic orbit were showed. More precisely, for any positive integer N, Devaney found an invariant subset of the flow in a critical level which was described as a suspension over the Bernoulli shift with N symbols. The presence of subsidiaries bifocal homoclinic orbits was described by Belyakov in [Bel84, BS90]. A complete symbolic description of the set of all orbits lying wholly on the critical level in a neighborhood of γ was given by Lerman in [Ler91, Ler97, Ler00].

4.3.1 Local coordinates and cross-sections

Next, we will describe the orbits of X_H lying entirely in some neighborhood U of γ. Their study will be carried out by means of a Poincaré map π associated with γ. This map will be constructed, see Figure C, as a composition $\pi = \pi^u \circ \pi^s$ of two maps. The map π^s is defined in a neighborhood of p between two cross-section Σ^s and Σ^u of γ. The map $\pi^s : \Sigma^s \to \Sigma^u$ will be called the local map. The other π^u to be defined from Σ^u to Σ^s will be called global map.

Local map

Assume that $p = 0$ and $H(p) = 0$. As X_H is a smooth Hamiltonian vector field, in order to describe the local behavior of flow orbits near a bifocus equilibrium point we can use the Moser's normal (see [Mos58] in the analytic case, [Lyč77] for C^∞ vector fields or [BLW96, BK96] in some sufficiently smooth cases). Theses results guarantees the existence in some neighborhood V of p of local symplectic coordinates (x_1, x_2, y_1, y_2) such that Hamiltonian takes the form

$$H(x_1, x_2, y_1, y_2) = h(\xi, \eta) = \lambda\xi + \omega\eta + \ldots, \qquad \xi = x_1y_1 + x_2y_2, \quad \eta = x_1y_2 - x_2y_1$$

where h is a smooth function and dots means higher order terms in ξ, η. We will work locally in these coordinates. Then, we have the following differential equations in the neighborhood V:

$$\begin{cases} \dot{x}_1 = -H_{y_1} = -h_\xi x_1 + h_\eta x_2, \\ \dot{x}_2 = -H_{y_2} = -h_\eta x_1 - h_\xi x_2, \\ \dot{y}_1 = \quad H_{x_1} = \quad h_\xi y_1 + h_\eta y_2, \\ \dot{y}_2 = \quad H_{x_2} = -h_\eta y_1 + h_\xi y_2, \end{cases} \tag{4.19}$$

where the lower indices with respect to variable denote related partial derivatives.

By means of the angular coordinates

$$x_1 = r_s \cos\theta_s, \quad x_2 = r_s \sin\theta_s \quad \text{and} \quad y_1 = r_u \cos\theta_u, \quad y_2 = r_u \sin\theta_u.$$

(4.19) can be written as

$$\dot{r}_s = -h_\xi r_s, \quad \dot{\theta}_s = -h_\eta, \quad \dot{r}_u = h_\xi r_u, \quad \dot{\theta}_u = -h_\eta. \tag{4.20}$$

Remark 4.8. *Notice that*

i) In this coordinates in the neighborhood V, the invariant manifolds are linear. The stable (unstable) manifold coincides with te stable (unstable) subspace $r_u = 0$ ($r_s = 0$).

ii) The functions ξ na η are first integral of the Hamiltonian vector field in (4.19).

According to (i) in the above remark, the following three-dimensional tori

$$\Sigma^s = \{(r_s, \theta_s, r_u, \theta_u) : \ r_s = \varepsilon, \ r_u \le \delta, \ \theta_s, \theta_u \in S^1\},$$
$$\Sigma^u = \{(r_s, \theta_s, r_u, \theta_u) : \ r_s \le \delta, \ r_u = \varepsilon, \ \theta_s, \theta_u \in S^1\}.$$

are cross-section for the flow of (4.20). Let us denote $q^s = \gamma \cap \Sigma^s$ and $q^u = \gamma \cap \Sigma^u$. With an appropriate choice of $\varepsilon > 0$, we can assume that $q^s = (\varepsilon, 0, 0, 0)$ and $q^u = (0, 0, \varepsilon, 0)$.

From (ii) in Remark 4.8 it follows that h_ξ and h_η only depend on ξ and η and both, ξ and η, remain constant along to orbits of (4.20) in V. Thus, the equation in (4.20) are immediately integrated providing the solutions

$$r_s(t) = r_{s0} e^{-\lambda_0 t}, \quad \theta_s(t) = \theta_{s0} - \omega_0 t, \quad r_u(t) = r_{u0} e^{\lambda_0 t}, \quad \theta_u(t) = \theta_{u0} - \omega_0 t, \tag{4.21}$$

where $x_{10} = r_{s0}\cos\theta_{s0}$, $x_{20} = r_{s0}\sin\theta_{s0}$, $y_{10} = r_{u0}\cos\theta_{u0}$ and $y_{20} = r_{u0}\sin\theta_{u0}$ are the inicial conditions and λ_0, ω_0 are the values of h_ξ, h_η, in $\xi_0 = x_{10}y_{10} + x_{20}y_{20}$ and $\eta_0 = x_{10}y_{20} - x_{20}y_{10}$. So, we obtain that the time of pass time of passage for any orbit from Σ^s to Σ^u is $T = \lambda_0^{-1}\log(\varepsilon/r_{u0})$. Thus, the local map $\pi^s : \Sigma^s \to \Sigma^u$ is given by

$$r_s^* = r_{u0}, \qquad \theta_s^* = \theta_{s0} - \frac{\omega_0}{\lambda_0}\log\frac{\varepsilon}{r_{u0}}, \qquad \theta_u^* = \theta_{u0} - \frac{\omega_0}{\lambda_0}\log\frac{\varepsilon}{r_{u0}}. \tag{4.22}$$

where starts are used to distinguis the coordinates on Σ^u.

Next, we will use the local invariance of the functions ξ, η to introduce the new coordinates (θ_s, ξ, η) and $(\xi^*, \eta^*, \theta_u^*)$ on Σ^s, Σ^u, respectively. These coordinates are given in the following way:

$$\begin{aligned} \xi &= \varepsilon r_u(\cos\theta_s\cos\theta_u + \sin\theta_s\sin\theta_u) = \varepsilon r_u\cos(\theta_u - \theta_s), \qquad & \xi^* &= \varepsilon r_s^*\cos(\theta_u^* - \theta_s^*) \\ \eta &= \varepsilon r_u(\cos\theta_s\sin\theta_u - \sin\theta_s\cos\theta_u) = \varepsilon r_u\sin(\theta_u - \theta_s), \qquad & \eta^* &= \varepsilon r_s^*\sin(\theta_u^* - \theta_s^*). \end{aligned} \tag{4.23}$$

Thus, denoting $\Phi = \theta_u - \theta_s$ one holds that

$$\xi^2 + \eta^2 = (r_u \varepsilon)^2, \quad \cos\Phi = \frac{\xi}{\sqrt{\xi^2+\eta^2}} \quad \text{and} \quad \sin\Phi = \frac{\eta}{\sqrt{\xi^2+\eta^2}}.$$

Similar expressions are followed for the coordinates on Σ^u. So, we obtain that

$$\Sigma^s = \{(\theta_s, \xi, \eta) : \ \theta_s \in S^1, \ |\xi|, |\eta| \leq \delta\varepsilon\} \quad \text{and} \quad \Sigma^u = \{(\xi^*, \eta^*, \theta_u^*) : \ \theta_u^* \in S^1, \ |\xi^*|, |\eta^*| \leq \delta\varepsilon\}.$$

Submanifolds Σ^s, Σ^u are foliated by levels $H = c$ into two-dimensional annuli Σ_c^s, Σ_c^s, respectively. In the neighborhood V of p, one may regard equation $h(\xi, \eta) = c$ to be uniquely solved with respect to ξ,

$$\xi = a_c(\eta) = \lambda^{-1}c - \lambda^{-1}\omega\eta + \dots$$

This allows us to replace ξ by a new coordinate c in each cross-section Σ^s and Σ^u. Thus, for each $c \in \mathbb{R}$ with $|c|$ small enough,

$$\Sigma_c^s = \{(\theta_s, \eta) : \ \theta_s \in S^1, \ |\eta| \leq \delta\varepsilon\} \quad \text{and} \quad \Sigma_c^u = \{(\eta^*, \theta_u^*) : \ \theta_u^* \in S^1, \ |\eta^*| \leq \delta\varepsilon\}.$$

Remark 4.9. *The intersection of the estable (reps. unstable) manifold of p with Σ^s (resp. Σ^u) is given by $c = 0$ and $\eta = 0$ (resp. $\eta^* = 0$).*

Next, we will look for the expressions of π^s restricted to the annuli Σ_c^s. From (4.22) and (4.23) and we conclude that

$$\eta^* = \varepsilon r_s^* \sin(\theta_u^* - \theta_s^*) = \varepsilon r_{u0} \sin(\theta_{u0} - \theta_{s0}) = \eta_0. \tag{4.24}$$

Now, since $h(a_c(\eta), \eta) = c$ then we follow that $a_c'(\eta) = -h_\eta(a_c(\eta), \eta)/h_\xi(a_c(\eta), \eta)$. In particular, evaluating in the initial point $a_c'(\eta_0) = -\omega_0/\lambda_0$. Also,

$$r_{u0} = \varepsilon^{-1}\sqrt{a_c(\eta_0)^2 + \eta_0^2} \quad \text{and} \quad \theta_{u0} - \theta_{s0} = \arctan\frac{\eta_0}{a_c(\eta_0)} \stackrel{\text{def}}{=} \Phi_c(\eta_0).$$

Thus, substituting into (4.22) we obtain

$$\theta_u^* = \theta_{u0} + a_c'(\eta_0)\log\frac{\varepsilon^2}{\sqrt{a_c(\eta_0)^2 + \eta_0^2}} = \theta_{s0} + a_c'(\eta_0)\log\frac{\varepsilon^2}{\sqrt{a_c(\eta_0)^2 + \eta_0^2}} + \Phi_c(\eta_0). \tag{4.25}$$

Therefore, removing in (4.24) and (4.25) the subscript zero which indiques the evaluation in the initial point we obtain the following expression of the local map π^s restricted to the annuli Σ_c^s:

$$\pi_c^s : \Sigma_c^s \to \Sigma_c^u, \qquad \pi_c^s(\theta_s, \eta) = (\eta, \theta_s + b_c(\eta) \bmod 2\pi), \tag{4.26}$$

where $b_c(\eta) = a_c'(\eta)\log(\varepsilon^2/\sqrt{a_c(\eta)^2 + \eta^2}) + \Phi_c(\eta)$. Here the function $\Phi_c(\eta)$ is defined as the principal branch of arctangent function $\arctan(\eta/a_c(\eta))$ with $\Phi_0(+0) = \pi - \arctan(\lambda/\omega)$ and $\Phi(-0) = -\arctan(\lambda/\omega)$. The local mapping obtained is symplectic (area preserving), it is discontinuous along the circle $\eta = 0$ for $c = 0$, and smooth for $c \neq 0$.

In the following lemma one can find properties of the local map π_c^s.

Lemma 4.10 ([Ler91, Ler97, Ler00]). *There exist $\epsilon > 0$ and $c_0 > 0$ such that for all $|c| \leq c_0$, $|\eta| \leq \epsilon$ the following representations hold*

$$b_0'(\eta) = \frac{\omega/\lambda + O(\eta)}{\eta}, \quad |b_0'(\eta)| \leq \frac{3\omega}{2\lambda|\eta|} \quad \text{and} \quad b_c'(\eta) = \frac{L(c,\eta) + a_c''(\eta)R(c,\eta) + O(\|(c,\eta)\|^2)}{\eta^2 + a_c(\eta)^2}$$

with $L(c,\eta) = (\lambda^{-1} - \lambda^{-3}\omega^2)c + \lambda^{-3}\omega(\omega^2+\lambda^2)\eta$ and $R(c,\eta) = -\frac{1}{2}(\eta^2 + a_c(\eta)^2)\log(\eta^2 + a_c(\eta)^2)$. Moreover, $b_c' : [-\epsilon,\epsilon] \to \mathbb{R}$ is a monotone function with a unique zero at the point η_c

$$\eta_c = \frac{\omega^2 - \lambda^2}{\omega(\omega^2+\lambda^2)} + O(c^2), \quad 0 < |c| \leq c_0.$$

The following lemma allows one to distinguish the regions of hyperbolicity and critical dynamic where the creation of non-hyperbolic fixed points can occur.

Lemma 4.11 ([Ler97, Ler00]). *For a given $K > 0$ there exist $\epsilon > 0$ and $\kappa > 0$ such that*

$$\begin{aligned} &|b_0'(\eta)| \geq K \quad \text{for all} \quad 0 < |\eta| \leq \epsilon, \text{ and} \\ &|b_c'(\eta)| \geq K \quad \text{for all} \quad 0 < |c| \leq c_0 \text{ and for } |\eta - \eta_c| \geq \kappa c^2 \text{ with } |\eta| \leq \epsilon. \end{aligned}$$

If $\theta_s = u(\eta)$ is a function given for $|\eta|$ small enough, then the image of its graph with respect to π_c^s is a curve in the strip (η^*, θ_u^*), being graph of a function $\theta_u^* = u(\eta^*) + b_c(\eta^*)$ with $\eta^* = \eta$. Next lemma estates properties of this kind of function.

Lemma 4.12 ([Ler97, Ler00]). *There are positive ϵ, c_0, d_0, d_1, d_2 small enough such that given a C^2-smooth function $u : [-\epsilon,\epsilon] \to \mathbb{R}$ with $|u(\eta)| \leq d_0$, $|u'(\eta)| \leq d_1$ and $|u''(\eta)| \leq d_0$ for $|\eta| \leq \epsilon$ then it holds that*

$$\varphi : [-\epsilon,\epsilon] \to \mathbb{R}, \qquad \varphi(\eta) = u(\eta) + b_c(\eta) \tag{4.27}$$

- *i) for $c = 0$, it is a C^2-smooth function everywhere on $|\eta| \leq \epsilon$ except for the point $\eta = 0$ where it has a logarithmic singularity. Derivative of φ satisfies $|\varphi'(\eta)| \geq 3\omega/2\lambda|\eta|$.*
- *ii) for $0 < |c| \leq c_0$, it is a C^2-smooth unimodal function which reaches at the point*

$$\eta_c = \frac{\omega^2 - \lambda^2}{\omega(\omega^2+\lambda^2)} + O(c^2), \tag{4.28}$$

its minimum/maximum value given by the representation $\varphi(\eta_c) = (\omega/\lambda)\log|c| + E(c)$ with a bounded function $E(c)$, and $(d/dc)\,\varphi(\eta_c) = (\omega/\lambda + O(c))/c$.

Remark 4.13 ([Ler97]). *In fact, the function $O(c^2)$ in (4.28) depends on u, but for a given d_1 small enough, $|u'(\eta)| \leq d_1$, one has $O(c^2)/c \to 0$ uniformly in these u.*

Next lemma is used for proofs that tangency is quadratic if stable and unstable manifolds of some periodic orbit in a neighborhood of γ are tangent.

Lemma 4.14 ([Ler97, Ler00]). *Let $v : [-\epsilon,\epsilon] \to \mathbb{R}$ be a C^2-smooth function with C^2-norm bounded. Then, there is $c_1 > 0$ such that for all $|c| \leq c_1$ the graph of any function $\varphi(\eta)$ from (4.27) in Lemma 4.12 and the graph of $v(\eta)$ are quadratically tangent if they have a tangent point.*

Global map

As was said above, γ intersects Σ^s, Σ^u at points q^s and q^u respectively with corresponding coordinates $(0,0,0)$ on both cross-sections. Therefore, one may choose two neighborhoods V^s and V^u of points $(0,0,0)$ in Σ^s, Σ^u, respectively, such that in these neighborhoods map $\pi^u: V^u \to V^s$, generated by flow orbits near the global piece of γ will be a well defined diffeomorphism. This map is represented as a family of symplectic maps π^u_c defined for every $|c|$ small enough from $V^u_c = \Sigma^u_c \cap V^u$ to $V^s_c = \Sigma^s_c \cap V^s$. These symplectic diffeomorphisms have the form

$$\pi^u_c : V^u_c \to V^s_c, \qquad \pi^u_c(\eta^*, \theta^*_u) = (P_c(\eta^*, \theta^*_u),\, Q_c(\eta^*, \theta^*_u)), \qquad \det \frac{D(P_c, Q_c)}{D(\eta^*, \theta^*_u)} \equiv \pm 1, \tag{4.29}$$

with smooth functions P_c and Q_c. According to Remark 4.9 the traces of W^s_{loc} and W^u_{loc} on Σ^s_0 and Σ^u_0 are given as $\eta = 0$ and $\eta^* = 0$ respectively. The transversally condition in these coordinates means that the image of the segment $\eta^* = 0$ on Σ^u_0 is transversal at the point $(0,0) \in \Sigma^s_0$ with respect to the segment $\eta = 0$ on Σ^s_0. This is expressed as $(\partial Q_0/\partial\theta^*_u)(0,0) \neq 0$. Therefore, the existence of a non-degenerate homoclinic orbit γ means that

$$P_0(0,0) = Q_0(0,0) = 0 \quad \text{and} \quad \frac{\partial Q_0}{\partial \theta^*_u}(0,0) \neq 0. \tag{4.30}$$

Ignoring the high ordem term, one can assume that $\pi^u_c : V^u \to V^s$ is given by

$$(\theta_s, \eta) = \pi^u_c(\eta^*, \theta^*_u) = \begin{pmatrix} \beta_c & \alpha_c \\ \delta_c & \gamma_c \end{pmatrix} \begin{pmatrix} \eta^* \\ \theta^*_u \end{pmatrix} + \begin{pmatrix} A_c \\ B_c \end{pmatrix} \bmod 2\pi \tag{4.31}$$

where from (4.29) and (4.30) it holds

$$|\gamma_c\beta_c - \alpha_c\delta_c| = 1, \quad A_0 = B_0 = 0 \quad \text{and} \quad \gamma_0 = \frac{\partial Q_0}{\partial \theta^*_u}(0,0) \neq 0.$$

Note that for $|c|$ small enough, we infer that $\gamma_c \neq 0$.

4.3.2 A return map like standard map

Next we will provide a symbolic description of the hyperbolic sets lying in a neighborhood of γ. To this end, we will study the Poincaré return map $\pi = \pi^u \circ \pi^s$ in a neighborhood of $q^s = \gamma \cap \Sigma^s$. Namely, we will use the one-parametric familie $\pi_c = \pi^u_c \circ \pi^s_c$ for $|c|$ small enough. When the value c is varied, different bifurcations create parabolic and elliptic fixed points, period two elliptic points, etc. Although all these results were obtained in [Ler91, Ler97, Ler00], we will present here a slightly different proof.

Recall that the Poincaré return map π_c is defined by means of the composition of the diffeomorphisms $\pi^s_c : \Sigma^s_c \to \Sigma^u_c$ and $\pi^u_c : V^u_c \to V^s_c$ with $V^s_c = V^s \cap \Sigma^s_c$, $V^u_c = V^u \cap \Sigma^u_c$ where V^s and V^u are neighborhoods of $q^s = \gamma \cap \Sigma^s$ and $q^u = \gamma \cap \Sigma^u$ respectively. We have to study the dynamic of π_c in a neighborhood of $\eta = 0$ on the annulus Σ^s_c. Thus, we will consider this map on the strip $S^1 \times [-\epsilon, \epsilon]$ where $\epsilon > 0$ is given from Lemmas 4.10, 4.11 and 4.12. Note that π_c is only well defined on $R_{c,\epsilon} = (\pi^s_c)^{-1}(V^u_c) \cap (S^1 \times [-\epsilon, \epsilon])$ and, hence $\pi_c(R_{c,\epsilon}) \subset V^s_c$ but not necessarily it is a subset in $R_{c,\varepsilon}$. Namely, $\pi_c : R_{c,\epsilon} \subset \Sigma^s_c \to \Sigma^s_c$ is given by

$$\pi(\theta_s, \eta) = (\beta_c\eta + \alpha_c\theta_s + \alpha_c b_c(\eta) + A_c,\, \delta_c\eta + \gamma_c\theta_s + \gamma_c b_c(\eta) + B_c) \bmod 2\pi.$$

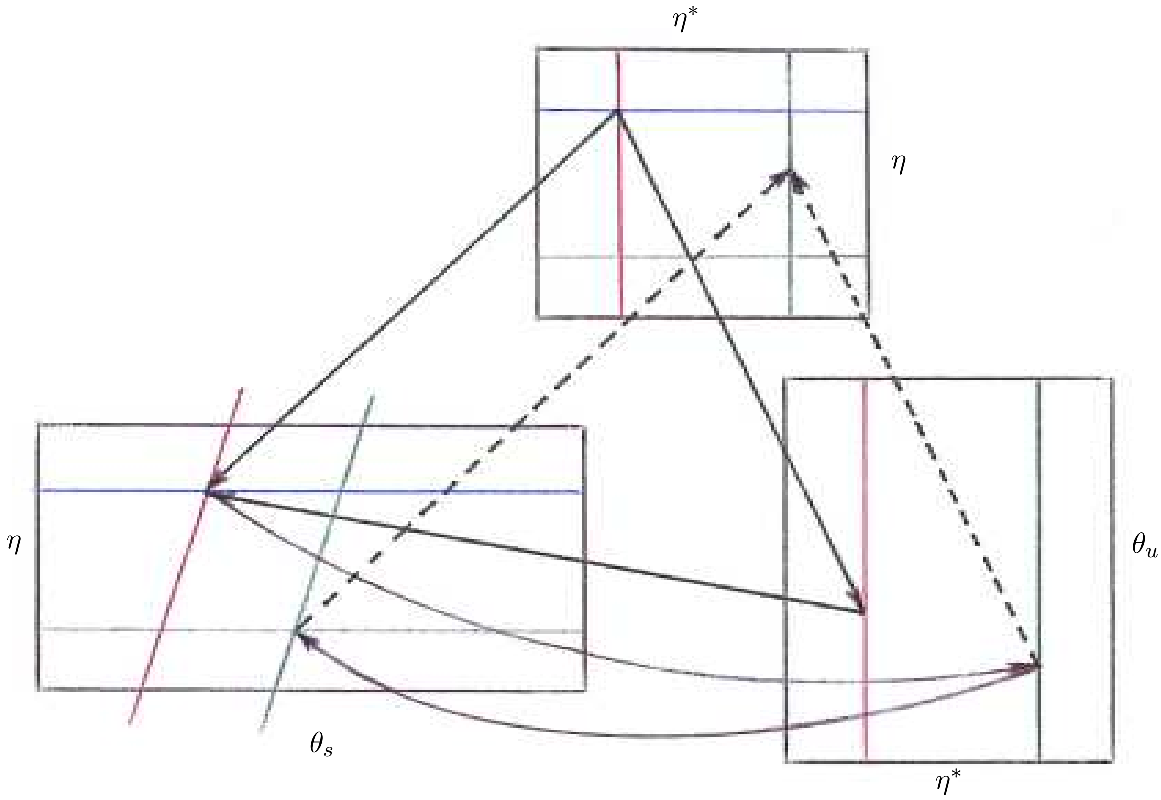

Fig. D: Construction of f_c

In fact, we will consider $\pi_c : \Delta_c \to \Delta_c$ where Δ_c is the maximal invariant set

$$\Delta_c = \bigcap_{n\in\mathbb{Z}} \pi_c^n(R_{c,\epsilon}).$$

In order to simplify the study of dynamics of $\pi_c|_{\Delta_c}$ we will introduce the family of maps

$$F_c : [-\epsilon,\epsilon]^2 \to [-\epsilon,\epsilon]\times S^1, \quad F_c(\eta^*,\eta) = (\eta,\, -\eta^* + \varphi_c(\eta) \bmod 2\pi), \tag{4.32}$$

where $\varphi_c : [-\epsilon,\epsilon] \to \mathbb{R}$ is the continuous unimodal function

$$\varphi_c(\eta) = (\alpha_c + \delta_c)\eta + \gamma_c b_c(\eta) + \gamma_c A_c + (1-\alpha_c)B_c. \tag{4.33}$$

Let Ω_c be the maximal invariant set of F_c in $[-\epsilon,\epsilon]^2$ then, we will prove $\pi_c|_{\Delta_c}$ is topologically conjugate to $F_c|_{\Omega_c}$.

Note that the map F_c evokes the so-called *standard map*: an area preserving map acting on the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and given by

$$G_c(x,y) = (y,\, -x + 2y + c\sin(2\pi y)), \quad (x,y)\in\mathbb{T}^2.$$

In the case of G_c, the corresponding function $\varphi_c : [0,1] \to \mathbb{R}$ is $\varphi_c(y) = 2y + c\sin(2\pi y)$ is a bimodal map. See [Dua94] for more details about the dynamic of this standard map family.

Conjugation map

In Figure D we represent graphically how to define the map $(\eta_1^*, \eta_1) = F_c(\eta^*,\eta)$ from the return map π_c. Next we will explain this construction in three step:

- For each $(\eta^*, \eta) \in [-\epsilon, \epsilon]^2$ and $\epsilon > 0$ small enough, using the Implicit Function Theorem, for $|c|$ small enough we infer that there exists two unique points $(\theta_s, \eta) \in \Sigma_c^s$ and $(\eta^*, \theta_u^*) \in \Sigma_c^u$ such that $(\theta_s, \eta) = \pi_c^u(\eta^*, \theta_u^*)$. Namely, solve (4.31) we get

$$\theta_s = S_c(\eta^*, \eta) = \frac{\alpha_c}{\gamma_c}\,\eta + (\beta_c - \frac{\alpha_c\delta_c}{\gamma_c})\,\eta^* + A_c - \frac{\alpha_c}{\gamma_c}B_c,$$
$$\theta_u^* = T_c(\eta^*, \eta) = \gamma_c^{-1}(\eta - \delta_c\,\eta^* - B_c).$$

- Using the local map π_c^s given in (4.26), the image of $(\theta_s, \eta) \in \Sigma_c^s$ by π_c^s is

$$(\eta_1^*, \theta_{u,1}^*) = (\eta,\ \theta_s + b_c(\eta) \bmod 2\pi) \in \Sigma_c^u.$$

- Finally, the coordinate η_1 is followed from $\pi_c^u(\theta_{u,1}^*, \eta_1^*)$. That is, substituting in (4.31)

$$\begin{aligned}\eta_1 &= \big(\gamma_c\theta_{u,1}^* + \delta_c\eta_1^* + B_c\big) \bmod 2\pi = \big(\gamma_c\theta_s + \gamma_c b_c(\eta) + \delta_c\eta + B_c\big) \bmod 2\pi \\ &= \big((\gamma_c\beta_c - \alpha_c\delta_c)\,\eta^* + (\alpha_c + \delta_c)\,\eta + \gamma_c b_c(\eta) + \gamma_c A_c + (1-\alpha_c)B_c\big) \bmod 2\pi\end{aligned}$$

  Observe that the coefficients of η^* is the determinante of $D\pi_c^u$.

Therefore

$$(\eta_1^*, \eta_1) = F_c^{\pm}(\eta^*, \eta) = (\eta,\ \pm\eta^* + \varphi_c(\eta) \bmod 2\pi), \quad (\eta, \eta^*) \in [-\epsilon, \epsilon]^2 \tag{4.34}$$

where φ_c is given in (4.33). Observe that, the coefficients the expression of φ_c are bounded functions of c for $|c|$ small enough. Notice that $\psi_c : [-\epsilon, \epsilon] \to \mathbb{R}$, $\psi_c(\eta) = \gamma_c^{-1}\varphi_c(\eta)$ can be written of the form (4.27) where, in this case, $u(\eta) = \gamma_c^{-1}(\alpha_c + \delta_c)\eta + A_c + \gamma_c^{-1}(1-\alpha_c)B_c$ depends on c. Since, as we have noted, its coefficients are bounded functions of c one has that u also is a bounded function on c. Hence, this observation and Remark 4.13 imply that the same statements of Lemma 4.12 are valid for ψ_c. In particular, it follows that φ_c for $|c| > 0$ is a unimodal function whose critical point is

$$\eta_c = \frac{\omega^2 - \lambda^2}{\omega(\omega^2 + \lambda^2)} + O(c^2),$$

and

$$\varphi(\eta_c) = \omega\lambda^{-1}\gamma_c \log|c| + \ldots = \omega\lambda^{-1}\gamma_0 \log|c| + E(c)$$

with a bounded function $E(c)$.

Next, we will study the family (4.34). The analysis of the dynamical behavior for both maps, F_c^+ and F_c^-, is analogous. We chose $F_c = F_c^-$ to develop the arguments below. Also, in what follows, since the dynamical behavior of (4.34) for both positive and negatives values of the parameter c is quite similar, for simplicity, we restrict ourselves to the family F_c with parameter $c \geq 0$.

An "increasing" family of hyperbolic basic sets

In order to find hyperbolic sets, we will use the following result:

Proposition 4.15. *Consider the invertible are-preserving map*

$$F(x, y) = (y,\ -x + \varphi(y)), \qquad (x, y) \in \mathbb{T}^2$$

and let Λ be a F-invariant compact set. Assume that there exists $\lambda > 2$ such that $|\varphi'(y)| \geq \lambda$ for all $(x, y) \in \Lambda$. Then, Λ is a hyperbolic set (of saddle type).

Proof. Note that

$$DF = \begin{pmatrix} 0 & 1 \\ -1 & \varphi'(y) \end{pmatrix},$$

In particular, the trace of DF verifies $|\mathrm{tr}\, DF| \geq \lambda > 2$ so that DF is uniformly hyperbolic. In fact, this follows from the fact $C^u_a = \{(v, w) \in \mathbb{R}^2 : |v| \leq a|w|\}$ is an unstable cone-field whenever $(\lambda - 1)^{-1} < a < 1$ (note that such a choice is possible since $\lambda > 2$). Indeed, if we write $DF(v, w)$ as (v', w'), we see that

$$|v'| = |w| \leq (\lambda - a)^{-1}|\varphi(y)w - v| = (\lambda - a)^{-1}|w'|$$

so that $DF(C^u_a) \subset C^u_{\theta a}$ where $\theta = (a(\lambda - a))^{-1} < 1$ by the choice of the parameter a. That is, C^u_a is DF-invariant. Furthermore, denoting by $\|(v, w)\| = \max\{|v|, |w|\}$, we get, for any $(v, w) \in C^u_a$,

$$\|Df(v, w)\| = |w'| \geq (\lambda - a)|w| = (\lambda - a)\|(v, w)\|$$

with $(\lambda - a) > 1$, i.e., DF (uniformly) expands any vector inside C^u_a. On the other hand, it is not hard to see that the same above argument can be applied to DF^{-1} in order to get a stable cone-field. Using the invariant cone-field criterion [KH95, Corolary 6.4.8], the proof is complete. ☐

Consider an alphabet $\{1, 2, \dots, \pm\infty\}$ consisting of all integers other than zero, supplemented with the two symbols $+\infty$ and $-\infty$. Let Σ_* be the set of bi-sequences in above alphabet of symbols satisfying the following condition: only the symbol $+\infty$ can follow $+\infty$, and only $-\infty$ can precede $-\infty$. Endowing Σ_* with the appropriate topology, one can make a compact space this set of bi-sequences.

For each parameter $c > 0$ let Λ_c be a basic set of a diffeomorphisms g_c. The family $\{\Lambda_c\}_{c>0}$ is said *dynamical "increasing"* if given $c > 0$, for any sufficiently small $\varepsilon > 0$, the set $\Lambda_{c-\varepsilon}$ contains the dynamical continuation of Λ_c. The following theorem shows the existence of dynamical increasing family of hyperbolic basic sets for the one-parametric family of maps F_c. The corresponding version of the this theorem for the Poincaré return map π_c was proved in [Ler91, Ler97, Theorem 1].

Theorem 4.16. *For the family F_c given in (4.32), there exist $\epsilon > 0$, $c_0 > 0$ and $\kappa > 0$ such that for every positive $c \leq c_0$, the maximal invariant set*

$$\Lambda_c = \bigcap_{n\in\mathbb{Z}} F^n_c(\{(\eta^*, \eta) \in [-\epsilon, \epsilon]^2 : \ |\eta - \eta_c| \geq \kappa c^2\})$$

is a hyperbolic set conjugated to Bernoulli shift of $n(c)$ symbols where

$$n(c) \sim -(\omega/\pi\lambda) \log c.$$

Moreover, these hyperbolic sets $\{\Lambda_c\}_{0<c\leq c_0}$ are a family of dynamical increasing basis sets and the restriction of F_0 to

$$\Omega_0 = \bigcap_{n\in\mathbb{Z}} F^n_0([-\epsilon, \epsilon]^2)$$

is conjugated with the Bernoulli shift map $\tau : \Sigma_* \to \Sigma_*$ where $\Sigma_* = \{1, 2, \dots, \pm\infty\}^{\mathbb{Z}}$.

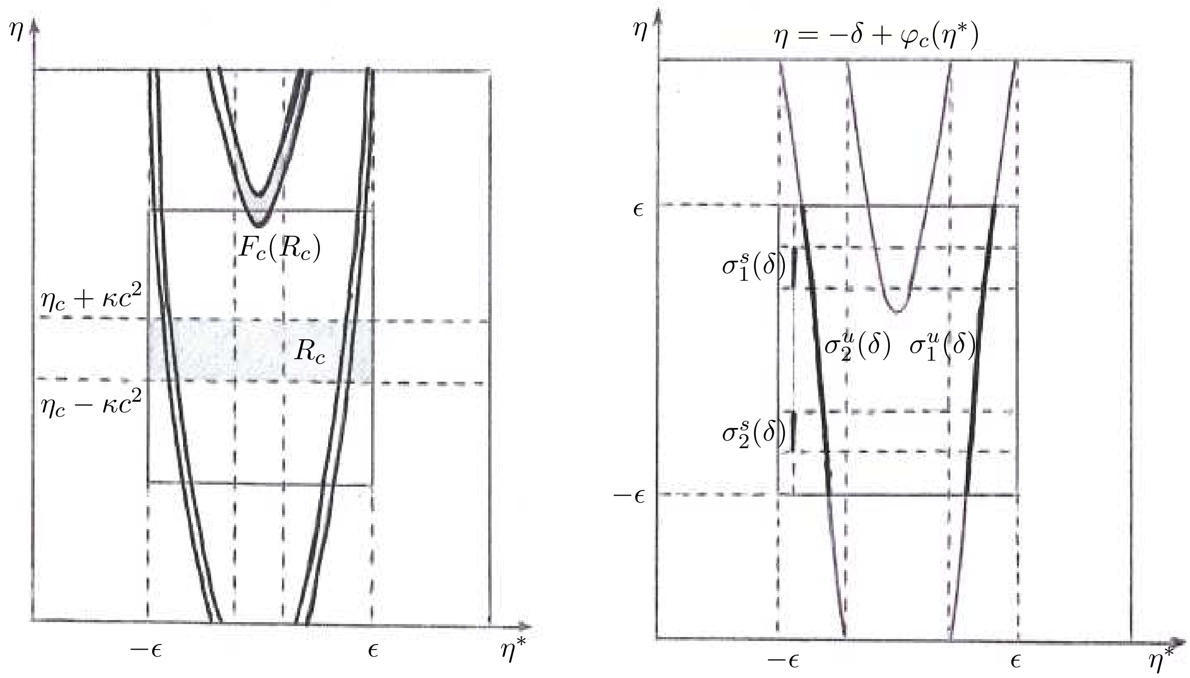

Fig. E: Scheme of the image of $[-\epsilon, \epsilon]^2$ by F_c

Proof. Since the coefficient α_c, δ_c and γ_c^{-1} in (4.33) are bounded function on c, then one can choose $K > 0$ greater than $(\lambda + |\alpha_c + \delta_c|)/|\gamma_c|$ with $\lambda > 2$. Hence, by Lemma 4.11 there exist $\epsilon > 0$ and $\kappa > 0$ such that

$$|\varphi_c'(\eta)| \geq |\gamma_c||b_c'(\eta)| - |\alpha_c + \delta_c| \geq |\gamma_c|K - |\alpha_c + \delta_c| \geq \lambda > 2$$

for all $c > 0$ small enough and $|\eta - \eta_c| \geq \kappa c^2$ with $|\eta| \leq \epsilon$. As immediate consequence of Proposition 4.15 is followed that Λ_c is a dynamically increasing family of hyperbolic sets.

For any $|\delta| \leq \epsilon$ and $c > 0$ small enough, the image by F_c of a vertical segment $\eta^* = \delta$, $|\eta| \leq \epsilon$ on the plane (η^*, η) is a curve

$$\eta_1 = -\delta + \varphi_c(\eta_1^*) \bmod 2\pi, \quad |\eta_1^*| \leq \epsilon \quad \text{(with parameter } \delta\text{)}$$

on the annulus $[-\epsilon, \epsilon] \times S^1$. This curve intersects $[-\epsilon, \epsilon]^2 \subset [-\epsilon, \epsilon] \times S^1$ into finitely many full (from the top to the bottom) branches $\sigma_i^u(\delta)$ as it is showed in Figure E. Each of these branches $\sigma_i^u(\delta)$ defines a sub-segment $\sigma_i^s(\delta)$ in the vertical segment $\eta^* = \delta$, $|\eta| \leq \epsilon$ such that $F_c(\sigma_i^s(\delta)) = \sigma_i^u(\delta)$. Let σ_i^s be the union all these vertical sub-segments $\sigma_i^s(\delta)$, $|\delta| \leq \epsilon$. Hence, σ_i^s is a horizontal strip on $[-\epsilon, \epsilon]^2$ such that $F_c(\sigma_i^s) = \sigma_i^u$ where σ_i^u is a vertical strip union of the branches $\sigma_i^u(\delta)$, $|\delta| \leq \epsilon$. This observation together with the hyperbolicity of Λ_c imply that F_c restricted to Λ_c is conjugated to a Bernoulli shift.

Let $n(c)$ be the number of symbols of the Bernoulli shift. Let us estimate $n(c)$. For this propose we find $n_+ = n_+(c)$ and $n_- = n_-(c)$ the numbers of subsegments $\sigma_i^s(\delta)$ lying in the segment $\eta^* = \delta$, $\eta_c + \kappa c^2 < \eta < \epsilon$ and in the segment $\eta^* = \delta$, $-\epsilon < \eta < \eta_c - \kappa c^2$ respectively. Hence $\min\{n_-, n_+\} \leq n(c)/2 \leq \max\{n_-, n_+\}$.

The number n_+ is determined by the last sub-segment $\sigma^s_{n+}(\delta)$ in $[-\epsilon, \epsilon] \times [\eta_c + \kappa c^2, \epsilon]$. Considering the affine coordinates of the standard covering the annulus $[-\epsilon, \epsilon] \times S^1$, one has that the image of a point $(\eta^*, \eta) \in \sigma^s_{n_+}$ by F_c is the point $\eta_1^* = \eta$, $\eta_1 - 2\pi n_+ = -\delta + \varphi_c(\eta)$ for some η_1 belongs to $[-\epsilon, \epsilon]$. Since $\varphi_c : [\eta_c + \kappa c^2, \epsilon] \to \mathbb{R}$ is monotone increasing and $\eta_c + \kappa c^2 < \eta \leq \epsilon$ then

$$\varphi_c(\eta_c + \kappa c^2) \leq \varphi_c(\eta) = \eta_1 + \delta - 2\pi n_+ \leq \varphi_c(\epsilon).$$

Thus, one obtains that

$$\frac{\delta - \epsilon}{2\pi} - \frac{1}{2\pi}\varphi_c(\epsilon) < n_+ < \frac{\delta + \epsilon}{2\pi} - \frac{1}{2\pi}\varphi_c(\eta_c + \kappa c^2).$$

Using the mean value theorem we have that $b_c(\eta_c+\kappa c^2) = b_c(\eta_c)+b_c'(\eta_c+\theta\kappa c^2)\kappa c^2$, with $0 < \theta < 1$, so that, from Lemma 4.12, it follows

$$b_c(\eta_c + \kappa c^2) = \frac{\omega}{\lambda}\log c + E(c) + K\kappa c^2.$$

In a similar way one may get the upper estimate for n_+, both estimates are asymptotically the same. The estimate for n_- is similar. Therefore, finally, we get

$$\frac{n(c)}{2} \sim -\frac{\omega}{2\pi\lambda}\log c + \text{const}.$$

Similarly in the case $c = 0$, one can defined infinitely many full horizontal and vertical strips, σ_i^s and σ_i^u in $[-\epsilon, \epsilon]^2$ such $F_0(\sigma_i^s) = \sigma_i^u$. For more details about the conjugation with $\tau : \Sigma_* \to \Sigma_*$ we refer the reader to [Ler91, Ler97] and this finishes the proof. □

By construction, periodic saddle orbits of the Hamiltonian vector field X_H correspond to fixed points of F_0. These orbits intersect the cross-section Σ_0^s of γ once. A periodic point of period N corresponds to a saddle orbit which intersects this cross-section N times before closing. Corresponding with a bi-sequence of the type $(\ldots - \infty, -\infty, \xi_1, \ldots, \xi_n \infty, \infty, \ldots)$ are homoclinic orbits of the point p that emerge from the trace $\eta^* = 0$ on Σ_0^u (the trace of W^u_{loc}) and then pass through a neighborhood of γ intersecting n times Σ_0^s and then reach $\eta = 0$ (the trace of W^s_{loc}). Consequently the following corollary holds:

Corollary 4.17 ([Bel84, Ler00]). *There exists a countable set of non-degenerate (bifocal) homoclinic orbits of any roundness in a neighborhood of γ.*

Hyperbolic windows and Newhouse intervals

We have show that in the limit $c \to H(p)$ the number of symbols of the Bernoulli shift increases approaching to ∞. Hence bifurcations have to occurs giving rise reconstructions in the orbit structure in the level set $H = c$. The following theorem show some of these bifurcations. The corresponding version of the this theorem for the Poincaré return map π_c was proved in [Ler00, Theorem 2]. Previously, we introduce some definitions.

Let p be a fixed point of a surface C^1-diffeomorphism g. It is said that p is an *elliptic fixed point* if the eigenvalues of $Dg(p)$ form a complex conjugate pair $\lambda_+ = \lambda$, $\lambda_- = \overline{\lambda}$, and a *parabolic fixed point* if $Dg(p)$ has a double eigenvalue $\lambda_\pm = 1$ but its Jordan form is not the identity matrix.

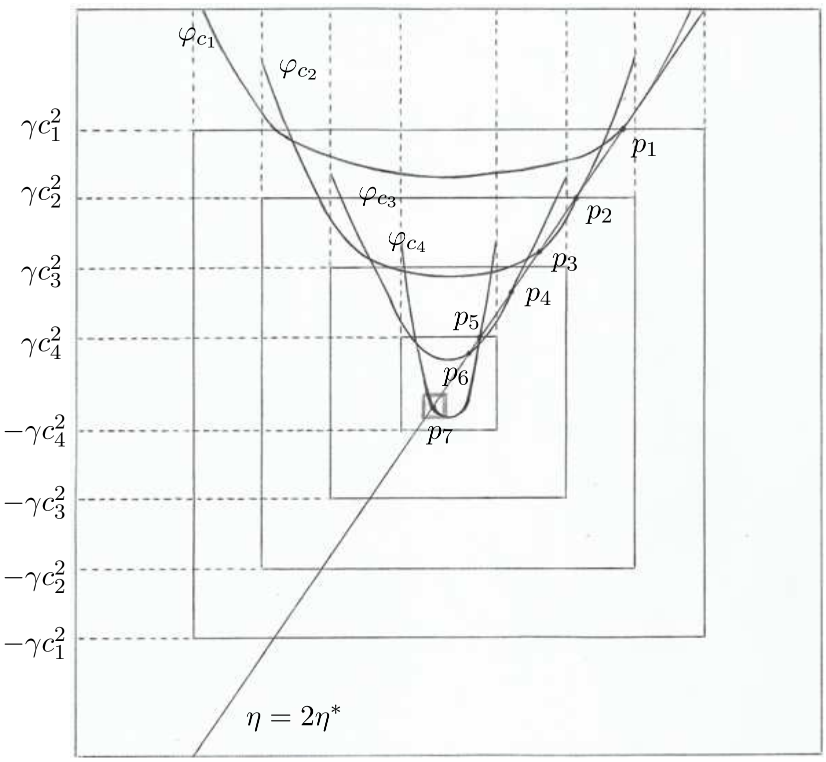

Fig. F: Map F_c on $[-\epsilon, \epsilon]^2$.

Theorem 4.18. *For the family F_c given in (4.32), in the interval of parameters $(0, c_0]$ there is a set accumulating zero disjoint intervals $I_n = (c_n, c'_n)$, $n \in \mathbb{N}$, such that*

i) for $c \in I_n$ the maximal invariant set Ω_c in $[-\epsilon, \epsilon]^2$ coincides with the hyperbolic basic set Λ_c,

ii) there are points $d_n^1, d_n^2 \in (c'_{n+1}, c_n)$ such that for

a) $c = d_n^1$ a parabolic fixed point of F_c appears inside of $[\eta_c - \kappa c^2, \eta_c + \kappa c^2]^2$,

b) $c \in (d_n^2, d_n^1)$ the parabolic point has bifurcated into a elliptic and a hyperbolic fixed points,

c) $c = d_n^2$ the elliptic point becomes a degenerate elliptic fixed point with eigenvalue $\pm i$, and

d) $c < d_n^2$ from the degenerate elliptic point appears a new hyperbolic fixed point and a cascade of period doubling bifurcation of elliptic periodic points.

The intervals I_n in the above theorem are called *hyperbolic windows*. When c varies in the interval between neighboring hyperbolic windows, the second part (ii) in Theorem A.12 shows the bifurcations associated with formation of a new, well-developed Smale horseshoe (see [YA83]).

Proof. Notice that the non-hyperbolic region where the creation of parabolic and elliptic fixed points can occur is located from Theorem 4.16 in $R_c = \{(\eta^*, \eta) : |\eta - \eta_c| \leq \kappa c^2,\ |\eta^*| \leq \epsilon\}$. The image by F_c of this critical region R_c is a solid piece in the annulus $[\eta_c - \kappa c^2, \eta_c + \kappa c^2] \times S^1$ as it is showed in Figure E. There is no generation of critical dynamic if $F_c(R_c) \cap R_c = \emptyset$ which occurs for parameters c in a set of disjoints intervals I_n, $n \in \mathbb{N}$ on $(0, c_0]$ accumulating at zero. In this situation, the maximal invariant set Ω_c in $[-\epsilon, \epsilon]^2$ coincides with the hyperbolic basic set Λ_c.

Now, let us find the fixed point of the map F_c which can appear in the critical region R_c. This is, $(\eta^*, \eta) \in R_c$ such that $(\eta^*, \eta) = F_c(\eta^*, \eta)$ or equivalently, $\eta^* = \eta$ and $2\eta^* = \varphi_c(\eta^*) \bmod 2\pi$. Thus, it comes to studying the intersection in the annulus $[\eta_c - \kappa c^2, \eta_c + \kappa c^2] \times S^1$ of the curves $\eta = 2\eta^*$ and $\eta = \varphi_c(\eta^*)$. Note that, the eigenvalues of linear part of F_c at a fixed point (η^*, η) are given by

$$\lambda_{\pm} = \frac{1}{2}\left(\varphi_c'(\eta^*) \pm \sqrt{\varphi_c'(\eta^*)^2 - 4}\right).$$

Hence, (η^*, η) is either hyperbolic or elliptic fixed point if $|\varphi_c'(\eta^*)| > 2$ or $|\varphi_c(\eta^*)| < 2$. If $\varphi_c'(\eta^*) = 2$ then $\lambda_{\pm} = 1$ and Jordan form of the linear part is not the identity matrix (it is the nilpotent one). Thus, in this case, the fixed point (η^*, η) is parabolic.

Figure F shows the different possibilities position of the graph $\eta = \varphi_c(\eta^*)$ when the parameter c is varies in an interval (c'_{n+1}, c_n) between the hyperbolic windows I_{n+1} and I_n. In this interval one can find a parameter $c_1 = d_n^1$ such that the curve $\eta = \varphi_{c_1}(\eta^*) \bmod 2\pi$ has a unique tangent point $\eta^* = p_1$ with the line $\eta = 2\eta^*$. This point stisfies that $\varphi_{c_1}(p_1) = 2p_1$ and $\varphi_{c_1}'(p_1) = 2$. Thus, $(p_1, 2p_1)$ is a parabolic fixed point of F_c. Moreover, it is non-degenerate fixed point since $\varphi_{c_1}''(p_1) \neq 0$. This point breaks up to $c_2 < c_1$ into hyperbolic fixed point $(p_2, 2p_2)$ and elliptic fixed point $(p_3, 2p_3)$, both of them persist till $c_3 = d_n^2$. For this parameter, the continuation $\eta^* = p_5$ of p_3 becames in the critical value the curve $\eta = \varphi_{c_3}(\eta^*)$. Then, $\varphi_{c_3}'(p_5) = 0$ and thus, the eigenvalue of linar part Df_{c_3} at $(p_5, 2p_5)$ are $\lambda_{\pm} = \pm i$. That is, $(p_5, 2p_5)$ is a degenerate elliptic fixed point of f_{c_3}. Finally, when $c_4 < c_3$ the degenerate elliptic fixed point becames in a hyperbolic fixed point and a new periodic 2 elliptic periodic point appears. This concludes the proof of the theorem. □

4.4 Blenders near conservative bifocal homoclinic orbits

Let $\pi = \pi^u \circ \pi^s$ be the Poincaré return map on a neighborhood of a non-degenerate bifocal homoclinic orbit γ of a smooth Hamiltonian vector field X_H on $\mathbb{R}^4$. This area preserving return map is defined on a solid tori Σ^s cross-section of γ. In adequate coordinates, it can be written as

$$\pi(\theta^s, \eta, c) = (\pi_c(\theta^s, \eta), c), \qquad \theta^s \in S^1,\ |\eta| \leq \varepsilon\delta,\ |c| \leq c_0$$

where π_c is a symplectic map defines on the annulus $\Sigma_c^s = \Sigma^s \cap H^{-1}(c)$. We introduce the notation $\pi = \pi_c \rtimes \mathrm{id}$ where $\mathrm{id} : I \to I$ is the identity function on the interval $I = [-c_0, c_0]$.

In §4.3.2 we have showed that $\pi_c|_{\Delta_c}$ is conjugated to the area preserving map $F_c|_{\Omega_c}$, where $F_c : [-\epsilon, \epsilon]^2 \to [-\epsilon, \epsilon] \times S^1$ is given in (4.32). Therefore, it follows that $\pi = \pi_c \rtimes \mathrm{id}$ is conjugated to $f = F_c \rtimes \mathrm{id}$ where

$$f(z, c) = (F_c(z), c), \qquad \text{with } c \in I \text{ and } z \in \Omega_c.$$

The following proposition show that we can conjugate the f with a direct product:

Proposition 4.19. *There is $\Delta \subset \Sigma^s$ such that $\pi|_\Delta$ is conjugated to $f_0 = F_0 \times \mathrm{id}$ from $\Lambda' \times I$ to itself where $\Lambda' \subset \Omega_0$ is a Smale horseshoe for F_0.*

Proof. According to Theorem 4.16, for each $c \in I$, $c \neq 0$, there is a hyperbolic basic set Λ_c of F_c in $[-\epsilon, \epsilon]^2$ such that $F|_{\Lambda_c}$ is conjugated with the Bernoulli shift of $n(c) \geq 2$ symbols. For $c = 0$, from the same theorem, we can take a Smale horseshoe $\Lambda' \subset \Omega_0$ for F_0 such that the continuation Λ'_c of Λ' for F_c, $0 < |c| \leq c_0$, is contained in Λ_c. We will show that $f = F_c \rtimes \mathrm{id}$ restricted to

$$\Lambda'_c \rtimes I \overset{\text{def}}{=} \{(z, c) : c \in I,\ z \in \Lambda'_c\}$$

is conjugated to $f_0 = F_0 \times \mathrm{id}$ restricted to $\Lambda' \times I$.

Since $\{\Lambda'_c\}_{|c| \leq c_0}$ is a dynamical "increasing" family of Smale horseshoes it follows that there exist homeomorphisms $H_c : \Lambda'_c \to \Lambda'_0$ such that $F_0 \circ H_c = H_c \circ F_c$ for all $|c| \leq c_0$. Consider

$$h : \Lambda'_c \rtimes I \to \Lambda' \times I, \qquad h(z, c) = (H_c(z), c)$$

shortly denoted by $h = H_c \rtimes \mathrm{id}$. Notice that this map is an homeomorphisms such that $f \circ h = h \circ f_0$. Therefore, we infer that $f|_{\Lambda'_c \rtimes I}$ is conjugated to $f_0|_{\Lambda' \times I}$. Finally, since π is conjugated to f then there is Δ in Σ^s such that $\pi|_\Delta$ is conjugated to $f_0|_{\Lambda' \times I}$. This concludes the proof. □

Since $F_0 : \Omega_c \to \Omega_c$ is conjugated to a Bernoulli shift in infinite many symbols, decreasing the size of the interval $I = [-c_0, c_0]$ and repeating the argue in the proof of the above result we obtain the following remak:

Remark 4.20. *For every $k \geq 2$ there exist*

$$0 < \bar{c} \leq c_0, \quad \Delta = \Delta(\bar{c}) \subset \Sigma^s \quad \textit{and} \quad \Lambda' = \Lambda'(\bar{c}) \subset \Omega_0$$

such that $\pi|_\Delta$ is conjugated to $f_0 = F_0 \times \mathrm{id}$ from $\Lambda' \times I$ to itself where $I = [-\bar{c}, \bar{c}]$ and $F_0|_{\Lambda'}$ is conjugated to $\tau : \Sigma_k \to \Sigma_k$.

One can study the bifurcation of the non-degenerate bifocal homoclinic orbit γ of X_H outside of the conservative vector field. This task can be carried out by studying the perturbations of the return map $f_0 = F_0 \times \mathrm{id}$. According to the theory developed in §2.1.2, perturbations of a dominated skew product diffeomorphism over a horseshoe are conjugated to locally Hölder symbolic skew products. Therefore, it suffices consider $\mathcal{S}^{1,\alpha}$-perturbations of $\Phi = \tau \times \mathrm{id}$ where $\tau : \Sigma_2 \to \Sigma_2$ is the shift the Bernoulli in two symbols. That is, perturbations of symbolic Hölder skew products with C^1-fiber maps in $\mathcal{S}_2(I)$, in the notation introduced in the second chapter, Definition 2.4. It is not difficult to construct an smooth arc of one-step skew products $\Phi_\mu = \tau \ltimes (\phi_{\mu,1}, \phi_{\mu,2})$, $\mu \in [0, \mu_0]$ such that $\Phi_0 = \Phi$ and Φ_μ has a symbolic blender-horseshoe in $\Sigma_2 \times I$ for all $0 < \mu \leq \mu_0$. Indeed, it suffices that the small perturbations $\phi_{\mu,1}$ and $\phi_{\mu,2}$ of the identity map on the interval I satisfy the covering property. This is possible with only two maps because the fiber space has dimension one. In this manner, via conjugation (see Proposition 2.1), we obtain an open set $\mathcal{V}$ of C^1-diffeomorphism with $f_0 \in \partial\mathcal{V}$ such that for every $g \in \mathcal{V}$ there exists a blender for g. This proves the following result:

Proposition 4.21. *Let X_H be a Hamiltonian vector field satisfying (H1) and (H2). Then there exists an open set $\mathcal{V}$ of C^1 vector fields with $X_H \in \partial\mathcal{V}$ such that each vector field $X \in \mathcal{V}$ has a suspended blender (contained in a neighborhood U of the bifocal homoclinic orbit γ of X_H).*

4.4.1 Possible blenders in generic unfoldings of nilpotent singularities

In order to conclude this chapter, we will study the possible existence of suspended blender-horseshoes in the generic unfoldings of the four-dimensional nilpotent singularities of codimension four. In order to establish the framework of this problem, we begin by summarizing our progress in this chapter until this point.

It follows from Equation (4.10) that any generic unfolding of the nilpotent singularity of codimension four in $\mathbb{R}^4$, denoted by $Y_{\nu,\varepsilon}$, can be written as

$$y_2\frac{\partial}{\partial y_1} + y_3\frac{\partial}{\partial y_2} + y_4\frac{\partial}{\partial y_3} + \left(\nu_1 + \nu_2 y_2 + \nu_3 y_3 + \nu_4 y_4 + y_1^2 + \varepsilon\kappa y_1 y_2 + O(\varepsilon^2)\right)\frac{\partial}{\partial y_4},$$

with $\nu = (\nu_1, \nu_2, \nu_3, \nu_4) \in \mathbb{S}^3$ and $\varepsilon > 0$. According to Theorem 4.3, the limit family, denoted by Y_ν,

$$y_2\frac{\partial}{\partial y_1} + y_3\frac{\partial}{\partial y_2} + y_4\frac{\partial}{\partial y_3} + \left(\nu_1 + \nu_2 y_2 + \nu_3 y_3 + \nu_4 y_4 + y_1^2\right)\frac{\partial}{\partial y_4},$$

for parameters on the reversibility curve

$$\mathcal{T} = \{(\nu_1, \nu_2, \nu_3, \nu_4) \in \mathbb{S}^3 : \nu_2 = \nu_4 = 0\}$$

is a Hamiltonian vector field. In §4.2.1 it is showed that the parameter value $\mathrm{BD} = (\nu_1, 0, \nu_3, 0)$ with $\nu_3^2 - 8\sqrt{-\nu_1} = 0$, $\nu_1 < 0$ and $\nu_3 > 0$, is a Belyakov-Devaney bifurcation point and, therefore, conservative non-degenerate bifocal homoclinic orbits arise for every vector field Y_ν with $\nu \in \mathcal{DF} \subset \mathcal{T}$ close enough to BD. The proof of Theorem 4.5 uses the exponential dichotomy theory to show that this bifocal homoclinic connections can be continued for the nilpotent singularity $Y_{\nu,\varepsilon}$ with parameter values in a codimension one manifold

$$\mathcal{H}om_{FF} \subset \mathcal{H}om = \{(\nu, \epsilon) \in \mathbb{S}^3 \times (0, \infty) : \ \xi^\infty(\nu, \varepsilon) = 0\}.$$

Fix a parameter ν^* on the double-focus arc $\mathcal{DF}$ close enough to BD. The Hamiltonian vector field Y_{ν^*} on $\mathbb{R}^4$ satisfies (H1) and (H2), i.e., there exists a non-degenerate bifocal homoclinic orbit γ. Thus, from Proposition 4.19, it follows that there is Δ contained in a cross-section Σ^s of the bifocal homoclinic orbit γ of Y_{ν^*} such that the Poincaré return map π_{ν^*} restricted to Δ is conjugated to $f_0 = F_0 \times \mathrm{id}$ from $\Lambda_0' \times I$ to itself. Here, $\Lambda_0' \subset [-\epsilon, \epsilon]^2$ is a Smale horseshoe of the map F_0 given in (4.32) and I is a close real interval $[-c_0, c_0]$, being ϵ and c_0 small enough positives constants. Since for parameter values $(\nu, \epsilon) \in \mathbb{S}^3 \times (0, \infty)$ close to $(\nu^*, 0)$ the vector field $Y_{\nu,\varepsilon}$ is a smooth perturbation of Y_{ν^*}, in order to understand its possible dynamics, we can study the perturbations of the return map $f_0 = F_0 \times \mathrm{id}$. As we argued in Proposition 4.21, from the theory developed in §2.1.2, every small C^1-perturbations of $f_0|_{\Lambda_0' \times I}$ is conjugated to a symbolic skew-product $\Phi = \tau \ltimes \phi_\xi$ in $\mathcal{S}_2(I)$ that is a small $\mathcal{S}^{1,\alpha}$-perturbations of $\Phi_0 = \tau \times \mathrm{id}$. Note that the fiber-maps are diffeomorphisms on the interval I and thus the endpoints are fixed points of these maps. Since we try to understand the dynamic of small perturbations near of homoclinic orbit γ of Y_{ν^*}, we only have to consider the invariant dynamic of Φ in $\Sigma_2 \times J$ with J an open interval in the interior of I containing $c = 0$ (level set of Φ_0 corresponded to the homoclinic connection).

Let us renormalice the close interval I to $I = [-1, 1]$. The generic expression of a fiber map diffeomorphism $\phi_\xi : I \to I$ of a $\mathcal{S}^{1,\alpha}$-perturbation $\Phi = \tau \ltimes \phi_\xi \in \mathcal{S}_2(I)$ of $\Phi_0 = \tau \ltimes \mathrm{id}$ is

$$\begin{aligned}\phi_\xi(c) &= -(c-1)(c+1)(\delta_0 c + \delta_1 c + \delta_2 c^2 + \dots)\\ &= \delta_0 + \delta_1 c + (\delta_0 - \delta_2)c^2 + \dots\end{aligned}$$

where the coefficients $\delta_i = \delta_i(\xi)$ depend locally Hölder continuously on ξ and satisfy that $\delta_1 \approx 1$ and $\delta_i \approx 0$ for all $i \neq 1$. Because of the robustness in the definition of blender we can assume that $\delta_i(\xi) = \delta_i(\xi_0)$, i.e., the coefficients only depend on the zero coordinate of the bisequence ξ. This implies that $\phi_\xi = \phi_i$ if $\xi_0 = i$ for $i = 1, 2$ and therefore we have a one-step symbolic skew-product $\Phi = \tau \ltimes (\phi_1, \phi_2)$ where ϕ_1 and ϕ_2 are smooth diffeomorphisms on I close enough to $\mathrm{id} : I \to I$. Then, two possibilites happen:

i) (non-generic generic): ϕ_1 and ϕ_2 have a fixed points in common in the interior of I,

ii) (generic case): ϕ_1 and ϕ_2 have no fixed points in common in the interior of I.

Non-generic case

Let $c \in (-1, 1)$ be a fixed point of both, ϕ_1 and ϕ_2. Then the set $\Lambda = \Sigma_2 \times \{c\}$ is Φ-invariant and $\Phi|_\Lambda$ is conjugated to $\tau : \Sigma_2 \to \Sigma_2$. If $\phi_i'(c)$ are both less (resp. grater) than one, Λ is said to be a *symbolic normally hyperbolic horseshoe*. Notice that in this case any one-step smooth perturbation $\Psi = \tau \ltimes (\psi_1, \psi_2)$ of $\Phi = \tau \ltimes (\phi_1, \phi_2)$ satisfying the generic condition (ii) in a neighborhood of c has a symbolic blender-horseshoe. Indeed, it is enough to note that in a neighborhood of c the fiber maps ψ_i are both contractions (resp. expansions) and, from the proximity to the identity map, satisfy the covering property on the interval defined between its fixed points close to c. If $\phi_1'(c) \leq 1 \leq \phi_2'(c)$, Λ is said to be a *symbolic shear horseshoe*. In this case, only the continuation of the two fixed points in Λ survive under generic perturbations.

An interesting case is obtained when the fixed point in common is $c = 0$. In this case $\delta_0 = 0$. Shrinking the size of I, as follows of the Remark 4.20, we can increase the number of symbols k we are working with. Then, over a codimension one manifold where the non-generic condition $\delta_0 = 0$ is fulfilled, the dynamic of the restriction of Φ to $\Lambda = \Sigma_k \times \{0\}$ is conjugated to $\tau : \Sigma_k \to \Sigma_k$. In the limit, via conjugation, it is obtained that $Y_{\nu,\varepsilon}$ for a parameter value (ν, ϵ) in this codimension one manifold has infinitely many suspended Smale horseshoes. We expect that this situation occurs for parameter values in $\mathcal{H}om_{FF}$ where the results in [Shi67, FS91] imply the existence of infinitely many suspended Smale horseshoe in each neighborhood of the bifocal homoclinic orbit. Thus, we conjecture that this non-generic case occurs for values of the parameters in the codimension one manifold $\mathcal{H}om_{FF}$, where a bifocal homoclinic orbit takes place. The condition $\phi_i'(c) < 1$ (resp. $\phi_i'(c) > 1$)) for $i = 1, 2$ should correspond to the case $-\rho_1 > \rho_2 > 0$ (resp. $0 < -\rho_1 < \rho_2$) in Remark 4.7.

Generic case

Assume that ϕ_1 and ϕ_2 are Morse-Smale diffeomorphisms on the compact interval I with no periodic points in common in the interior of I. Recall that by a Morse-Smale diffeomorphisms on I we mean a diffeomorphism $f : I \to I$ with a non-empty finite set of fixed points in the interior of I and all of them hyperbolic. According to Spectral Decomposition Theorem of an IFS on the real line, Theorem 3.21 (see also Remark 3.22), we obtain that the limit set $L(\mathrm{IFS}(\phi_1, \phi_2))$ of the IFS generated by ϕ_1 and ϕ_2 is finite union of pairwise disjoint intervals. Moreover, this intervals are isolated and transitive set for $\mathrm{IFS}(\phi_1, \phi_2)$ (see this notions in Definition 3.1 and 3.3). Namely, each interval is either a $**$-interval for $\mathrm{IFS}(\phi_1, \phi_2)$ with $\{ss, su, uu, s, u\}$ or a single fixed point of ϕ_1 or ϕ_2. The isolated fixed points for IFS(ϕ_1, ϕ_2) correspond to an isolated periodic orbit $\Phi = \tau \ltimes (\phi_1, \phi_2)$ of period one. The $**$-intervals for IFS(ϕ_1, ϕ_2) correspond to the projection on the fiber space of non-trivial invariant set Γ^{**} for $\Phi = \tau \ltimes (\phi_1, \phi_2)$. Note that the s-intervals (resp. u-intervals) are always extremal intervals in the decomposition of the limit set of an IFS. Since we are interesting in the invariant dynamic for Φ in $\Sigma_k \times J$ with J an open interval in the interior of I, we can consider a s-interval (resp. the u-interval) as a subinterval of a ss-interval (resp. the uu-interval). In the contractive case, from the theory of symbolic blenders in the one-step setting (see in Section §2.3), the ss-intervals (resp. uu-intervals) for IFS(ϕ_1, ϕ_2) are the support of a symbolic cs-blender (resp. cu-blender). Notice that any non-empty open set B in an su-interval K^{su}_Φ is a blending region for IFS(ϕ_1, ϕ_2), and thus, from Proposition 2.21, for every one-step map Ψ close to Φ it holds that

$$W^{uu}\big((\vartheta, p_\Psi); \Psi\big) \cap \big(W^s_{loc}(\xi; \tau) \times U\big) \neq \emptyset,$$

for all $\xi \in \Sigma_k$ and non-empty open set U in B, where (ϑ, p_Ψ) is the continuation for Ψ of a fixed point $(\vartheta, p) \in \Sigma_k \times K^{su}_\Phi$ of Φ. That is, the intersection property in the definition of symbolic blender in the one-step setting (see Definition 2.11). On the other hand, noting that there is a fixed point ϑ of τ such that $\phi^n_\vartheta(x) \in K^{su}_\Phi$ for all $x \in K^{su}_\Phi$ for all $n \in \mathbb{Z}$, it follows that the Φ-invariant set Γ^{su} from the su-interval K^{su}_Φ for IFS(ϕ_1, ϕ_2) contains at least the spine $\{\vartheta\} \times K^{su}_\Phi$ (see definition of spine in the Section §2.2.2). This implies that $\Phi|_{\Gamma^{su}}$ is not conjugated to a shift the Bernoulli $\tau : \Sigma_2 \to \Sigma_2$. Moreover, it is not so hard to show the existence of heterodimensional cycles arbitrarily close to $\Phi = \tau \times (\phi_1, \phi_2)$. In any case, the $**$-intervals are the support blender-like dynamics: symbolic blenders or symbolic blender-like sets.

Anexo

Dichotomies and bifurcation equations

For a hyperbolic linear vector field, the decomposition of phase space as a direct sum of stable and unstable subspaces allows us to understand the behavior of their solutions. This property of the linear flow extends to the case of linear equations of non-autonomous systems under the name of exponential dichotomy. Such extension is useful to express the persistence of (homo)heteroclinic connections of a nonlinear autonomous vector field. For this it is essential to understand the dichotomy of the adjoint equation and the variational equation along these special solutions. The persistence of the connection is followed from the contact between the invariant manifolds of the hyperbolic equilibrium points. These contacts are formulated in terms of a bifurcation equation that allows us to know the set of parameter on which the (homo)heteroclinic connection persists.

A.1 Dichotamies

Let $x' = f(x)$ be a nonlinear equation, where $x \in \mathbb{R}^n$ and f is a regular enough vector field, and assume that it has a heteroclinic orbit $\gamma = \{p(t) : t \in \mathbb{R}\}$ connecting two hyperbolic equilibrium points p_+ and p_- (if $p_+ = p_-$, γ is said homoclinic). Consider a family

$$x' = f(x) + g(\lambda, x), \tag{A.1}$$

with $\lambda \in \mathbb{R}^k$ and g regular enough, such that $g(0, x) = 0$. For any λ small enough, family (A.1) has hyperbolic equilibrium points $p_+(\lambda)$ and $p_-(\lambda)$, continuation of p_+ and p_-, respectively, and the stability index is preserved. In order to study the persistence of the heteroclinic orbit for λ small enough we introduce the change of variables $x(t) = z(t) + p(t)$ in (A.1) to obtain

$$z'(t) = Df(p(t))z(t) + b(\lambda, t, z(t)), \tag{A.2}$$

where

$$b(\lambda, t, z(t)) = f(p(t) + z(t)) - f(p(t)) - Df(p(t))z(t) + g(\lambda, p(t) + z(t)).$$

Notice that $b(0, t, 0) = D_z b(0, t, 0) = 0$ for all $t \in \mathbb{R}$.

Persistence of heteroclinic orbits in (A.1) implies the existence of bounded solutions for (A.2) which, in turn, implies the existence of bounded solutions for a equation as

$$z'(t) = Df(p(t))z(t) + b(t), \tag{A.3}$$

where b belongs to the space $C_b^0(\mathbb{R}, \mathbb{R}^n)$. In the sequel $C_b^k(\mathbb{R}, \mathbb{R}^n)$ denotes the Banach space of bounded continuous $\mathbb{R}^n$-valued functions whose derivatives up to order k exist and are bounded

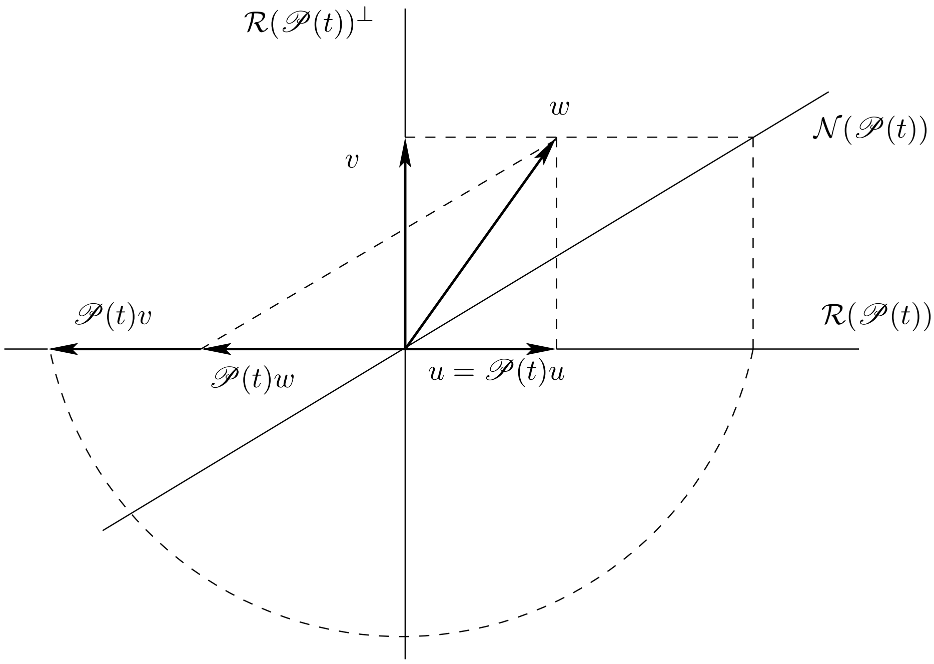

Fig. A: Geometric interpretation of projection $\mathscr{P}(t)$.

and continuous on $\mathbb{R}$. The existence of bounded solutions of a linear equation $x' = A(t)x + b(t)$, as that in (A.3), will be given in terms of exponential dichotomies of the homogeneous equation $x' = A(t)x$ and its adjoint $w' = -A(t)^*w$, where $A(t)^*$ stands for the conjugate transpose of $A(t)$. The classical references for the study of exponential dichotomies are [MS66, Cop78, Pal84, Pal00b]. Here, we will present a brief summary of results about dichotomies in order to get a precise formulation of the bifurcation equations. For an extended version of this introduction of dichotomies we recommend the reference [Bar09] where it is presented a complete exposition about dichotomies and bifurcations equations with the proof of the result.

A.1.1 Exponential dichotomy

Let $X(t)$ be a fundamental matrix of

$$x' = A(t)x, \qquad x \in \mathbb{R}^n, \tag{A.4}$$

where $A(t)$ is defined and continuous on an interval $J \subset \mathbb{R}$.

Definition A.1. *It is said that the equation (A.4) has an exponential dichotomy on J if there exists a projection $P : \mathbb{R}^n \to \mathbb{R}^n$, that is, an n by n matrix P with $P^2 = P$, and positive constants K, L, α and β such that*

$$\begin{aligned} \|X(t)PX^{-1}(s)\| &\le Ke^{-\alpha(t-s)} \quad \textit{for } t \ge s, \\ \|X(t)(I-P)X^{-1}(s)\| &\le Le^{-\beta(s-t)} \quad \textit{for } s \ge t. \end{aligned} \tag{A.5}$$

for all $s, t \in J$

Let us define $\mathscr{P}(s) = X(s)PX^{-1}(s)$ for each $s \in J$. Notice that, according with the above definition, $\mathscr{P}(s)$ is the projection corresponding to the fundamental matrix $Y(t) = X(t)X^{-1}(s)$ of (A.4) and we can give an alternative definition of exponential dichotomy.

Definition A.2. *It is said that the equation (A.4) has an exponential dichotomy on J if for all $s \in J$ there exists a projection $\mathscr{P}(s) : \mathbb{R}^n \to \mathbb{R}^n$ and positive constants K, L, α and β independents of s such that for all $t \in J$ the matrix $X^{-1}(t)\mathscr{P}(t)X(t)$ has constant coefficients and*

$$\begin{aligned} \|X(t)X^{-1}(s)\mathscr{P}(s)\| &\leq Ke^{-\alpha(t-s)} \quad \text{for all } t \geq s, \\ \|X(t)X^{-1}(s)(I - \mathscr{P}(s))\| &\leq Le^{-\beta(s-t)} \quad \text{for all } s \geq t. \end{aligned}$$

Although the notion of exponential dichotomy is stated for any $J \subset \mathbb{R}$, the most interesting cases are when J is not bounded. We are particularly interested in $J = [\tau, \infty)$ or $J = (-\infty, \tau]$. In such cases the notions of stable and unstable subspaces can be introduced in terms of the ranges of the projections of the exponential dichotomies.

Definition A.3. *Suppose that the matrix $A(t)$ in (A.4) is defined and continuous on $J = [\tau, \infty)$ (resp. $J = (-\infty, \tau]$). For each $t_0 \in J$ the stable (resp. unstable) subspace for initial time $t = t_0$ is defined as the set*

$$\begin{aligned} E^s_{t_0} &= \{\xi \in \mathbb{R}^n : \|X(t)X^{-1}(t_0)\xi\| \to 0 \text{ when } t \to \infty\} \\ (\textit{resp. } E^u_{t_0} &= \{\xi \in \mathbb{R}^n : \|X(t)X^{-1}(t_0)\xi\| \to 0 \text{ when } t \to -\infty\}). \end{aligned}$$

Below we give a collection of results which can be helpful to follow the paper. Their proofs are available in the literature.

Proposition A.1. *Suppose that the equation $x' = A(t)x$ has an exponential dichotomy on J.*

i) When $J = [\tau, \infty)$, $E^s_{t_0}$ coincides with the range $\mathcal{R}(\mathscr{P}(t_0))$ of $\mathscr{P}(t_0)$ for all $t_0 \in J$. Furthermore

$$\mathcal{R}(\mathscr{P}(t_0)) = \{\xi \in \mathbb{R}^n : \sup_{t \geq t_0} \|X(t)X^{-1}(t_0)\xi\| < \infty\},$$

and for all $t_0, t_1 \in J$ it follows that $E^s_{t_1} = X(t_1)X^{-1}(t_0)E^s_{t_0}$.

ii) When $J = (-\infty, \tau]$, $E^u_{t_0}$ coincides with the kernel $\mathcal{N}(\mathscr{P}(t_0))$ of $\mathscr{P}(t_0)$ for all $t_0 \in J$. Furthermore

$$\mathcal{N}(\mathscr{P}(t_0)) = \{\xi \in \mathbb{R}^n : \sup_{t \leq t_0} \|X(t)X^{-1}(t_0)\xi\| < \infty\},$$

and for all $t_0, t_1 \in J$ it follows that $E^u_{t_1} = X(t_1)X^{-1}(t_0)E^u_{t_0}$.

From the above proposition it follows that the linear flow sends $E^s_{t_0}$ and $E^u_{t_0}$ to $E^s_{t_1}$ and $E^u_{t_1}$, respectively. Accordingly, once $E^s_{t_0}$ and $E^u_{t_0}$ are fixed, the stable and unstable subspaces are determined for all t. Therefore, the projections are also determined for each $t \in J$ once they are defined for $t = t_0$. The same observation follows taking into account the uniqueness of solutions for the equation

$$\mathscr{P}'(s) = X'(s)PX^{-1}(s) + X(s)P(X^{-1}(s))' = A(s)\mathscr{P}(s) - \mathscr{P}(s)A(s).$$

Lemma A.2. *If the linear homogeneous equation $x' = A(t)x$, with $t \in (-\infty, \infty)$, has exponential dichotomy $[\tau, \infty)$ (resp. $(-\infty, \tau]$) for some $\tau \in \mathbb{R}$ then it has exponential dichotomy on $[t_0, \infty)$ (resp. $(-\infty, t_0]$) for all $t_0 \in \mathbb{R}$.*

The next result [Pal00b, Lemma 7.4] states that exponential dichotomy is a robust property with respect to small enough perturbations of $A(t)$.

Proposition A.3. *Suppose that $x' = A(t)x$ has an exponential dichotomy on $J = [a, b]$ (with $-\infty \leq a < b \leq \infty$) with projection matrix function $\mathscr{P}(t)$, with constants K_1, K_2 and exponents α_1, α_2. Let β_1 and β_2 be such that $0 < \beta_1 < \alpha_1$ and $0 < \beta_2 < \alpha_2$. Then*

there exists $\delta_0 = \delta_0(K_1, K_2, \alpha_1, \alpha_2, \beta_1, \beta_2) > 0$ such that if $B(t)$ is a continuous matrix function with $\|B(t)\| \leq \delta_t \leq \delta_0$ for all $t \in J$, the perturbed system

$$x' = [A(t) + B(t)]x$$

has an exponential dichotomy on J with constants L_1, L_2 exponents β_1, β_2 and projection matrix $\mathscr{Q}(t)$ satisfying that $\|\mathscr{Q}(t) - \mathscr{P}(t)\| \leq N\delta_t$, where L_1, L_2 and N are constants which only depend on K_1, K_2, α_1 and α_2.

From the above result and Lemma A.2 it follows the existence of an exponential dichotomy for the homogeneous part $z' = Df(p(t))z$ of the equation (A.3). Since $\lim_{t\to\infty} p(t) = p_+$ and $\lim_{t\to-\infty} p(t) = p_-$ and according to Proposition A.3, the equation $x' = Df(p(t))x$ has the same exponential dichotomy than $x' = Df(p_+)x$ (resp. $x' = Df(p_-)x$) on $[t_0, \infty)$ (resp. $(-\infty, t_0]$). That is, if the stable (resp. unstable) subspace of $x' = Df(p_+)x$ (resp. $x' = Df(p_-)x$) has dimension k then $x' = Df(p(t))x$ has an exponential dichotomy on $[t_0, \infty)$ (resp. $(-\infty, t_0]$) with stable subspace $E^s_{t_0}$ (resp. unstable subspace $E^u_{t_0}$) with dimension k. In fact we have the following result:

Proposition A.4. *Let $p(t)$ be a solution of the equation $x' = f(x)$ parametrizing an orbit on the stable (resp. unstable) manifold of an equilibrium point p. Hence the variational equation $x' = Df(p(t))x$ has exponential dichotomy on $[t_0, \infty)$ (resp. $(-\infty, t_0]$). Moreover,*

$$\mathcal{R}(\mathscr{P}(t_0)) = T_{p(t_0)}W^s(p) \quad (\textit{resp. } \mathcal{N}(\mathscr{P}(t_0)) = T_{p(t_0)}W^u(p)).$$

Now we can apply to (A.3) the result below, which relates the existence of bounded solutions for a linear equation and for its adjoint.

Theorem A.5. [Pal84, Lemma 4.2] *Let $A(t)$ be a bounded and continuous matrix defined on $(-\infty, \infty)$. The linear equation $x' = A(t)x$ has exponential dichotomy on $[t_0, \infty)$ and on $(-\infty, t_0]$ if and only if the linear operator $L : x(t) \in C^1_b(\mathbb{R}, \mathbb{R}^n) \mapsto x'(t) - A(t)x(t) \in C^0_b(\mathbb{R}, \mathbb{R}^n)$ is Fredholm. The index of L is $\dim E^s_{t_0} + \dim E^u_{t_0} - n$. Moreover, $b \in \mathcal{R}(L)$ if and only if*

$$\int_{-\infty}^{\infty} < w(t), b(t) > \, dt = 0$$

for all bounded solutions $w(t)$ of the adjoint equation $w' = -A(t)^* w$.

To explore the existence of bounded solutions of the adjoint equation one has to study its properties of exponential dichotomy.

A.1.2 Exponential dichotomy for the adjoint equation

Let $X(t)$ be a fundamental matrix of the equation $x' = A(t)x$. It is well known that the conjugate transpose of its inverse $X^{-1}(t)^*$ is a fundamental matrix of the adjoint equation $w' = -A(t)^* w$. From this relationship between the fundamental matrices of both equations we can conclude the following result about the connection between their respective dichotomies.

Proposition A.6. *If the linear equation $x' = A(t)x$ has exponential dichotomy on J with projection matrix $\mathscr{P}(t)$ then the adjoint equation $w' = -A(t)^*w$ has exponential dichotomy on J with projection matrix $I - \mathscr{P}(t)^*$. Moreover, for each $t_0 \in J$*

$$\mathbb{R}^n = \mathcal{R}(\mathscr{P}(t_0)) \perp \mathcal{R}(I - \mathscr{P}(t_0)^*) = \mathcal{R}(\mathscr{P}(t_0)) \perp \mathcal{N}(\mathscr{P}(t_0)^*),$$
$$\mathbb{R}^n = \mathcal{R}(I - \mathscr{P}(t_0)) \perp \mathcal{R}(\mathscr{P}(t_0)^*) = \mathcal{N}(\mathscr{P}(t_0)) \perp \mathcal{R}(\mathscr{P}(t_0)^*).$$

As done in Definition A.3 we can define now the stable and unstable subspaces for adjoint equations.

Definition A.4. *Suppose that $J = [\tau, \infty)$ (resp. $J = (-\infty, \tau]$) is contained in the interval of definition of $x' = A(t)x$. For each $t_0 \in J$ the stable (resp. unstable) subspace for initial time $t = t_0$ of the adjoint equation $x' = -A(t)^*x$ is defined as*

$$E^{s*}_{t_0} = \{w \in \mathbb{R}^n : \|X^{-1}(t)^*X(t_0)^*w\| \to 0 \text{ when } t \to \infty\}$$
$$(\textit{resp. } E^{u*}_{t_0} = \{w \in \mathbb{R}^n : \|X^{-1}(t)^*X(t_0)^*w\| \to 0 \text{ when } t \to -\infty\}).$$

The following result about the relationship between the invariant subspaces of the equation $x' = A(t)x$ and its adjoint follows as a straight consequence of Proposition A.1 and Proposition A.6.

Proposition A.7. *Suppose that the equation $x' = A(t)x$ with $x \in \mathbb{R}^n$ and $t \in J$ has exponential dichotomy in J.*

i) If $J = [t_0, \infty)$ then

$$E^s_{t_0} = \mathcal{R}(\mathscr{P}(t_0)) = \{x \in \mathbb{R}^n : \sup_{t \geq t_0} \|X(t)X^{-1}(t_0)x\| < \infty\},$$
$$E^{s*}_{t_0} = \mathcal{N}(\mathscr{P}(t_0)^*) = \{w \in \mathbb{R}^n : \sup_{t \geq t_0} \|X^{-1}(t)^*X(t_0)^*w\| < \infty\},$$

and $\mathbb{R}^n = E^s_{t_0} \perp E^{s*}_{t_0}$.

ii) If $J = (-\infty, t_0]$ then

$$E^u_{t_0} = \mathcal{N}(\mathscr{P}(t_0)) = \{x \in \mathbb{R}^n : \sup_{t \leq t_0} \|X(t)X^{-1}(t_0)x\| < \infty\},$$
$$E^{u*}_{t_0} = \mathcal{R}(\mathscr{P}(t_0)^*) = \{w \in \mathbb{R}^n : \sup_{t \leq t_0} \|X^{-1}(t)^*X(t_0)^*w\| < \infty\},$$

and $\mathbb{R}^n = E^u_{t_0} \perp E^{u*}_{t_0}$.

In short, if the linear equation $x' = A(t)x$ has exponential dichotomy in $J = [t_0, \infty)$ (resp. $(-\infty, t_0]$) then the forward (resp. backward) bounded solutions of this equation and its adjoint are those which tend to zero exponentially when $t \to \infty$ (resp. $t \to -\infty$). On the other hand, from the decompositions of $\mathbb{R}^n$ given in Proposition A.7 it follows that, if $x' = A(t)x$ has m linearly independent forward (resp. backward) bounded solutions, then the adjoint equation $w' = -A(t)^*w$ has $n - m$ linearly independent forward (resp. backward) bounded solutions.

Proposition A.8. *If the linear equation $x' = A(t)x$ has exponential dichotomy in $[t_0, \infty)$ and in $(-\infty, t_0]$ then the number of linearly independent bounded solutions of the adjoint equation $w' = -A(t)^*w$ is*

$$\dim E^{s*}_{t_0} \cap E^{u*}_{t_0} = n - \dim E^s_{t_0} - \dim E^u_{t_0} + \dim E^s_{t_0} \cap E^u_{t_0}.$$

Now we apply the above result to determine the number of bounded solutions of the adjoint equation $z' = -Df(p(t))^* z$. As we have already noticed, the number of linearly independent forward (resp. backward) bounded solutions of the variational equation $x' = Df(p(t))x$ is given by the dimension of the stable (resp. unstable) subspace of the equation $x' = Df(p_+)x$ (resp. $x' = Df(p_-)x$). That is, such number coincides with the dimension of $W^s(p_+)$ (resp. $W^u(p_-)$). Therefore, taking into account that $E^s_{t_0} = T_{p(t_0)}W^s(p_+)$ and $E^u_{t_0} = T_{p(t_0)}W^u(p_-)$, we can conclude, from Proposition A.8, the following result.

Proposition A.9. *If $p(t)$ is a (homo)heteroclinic solution connecting two equilibrium points p_+ and p_- then the number of linearly independent bounded solutions of the adjoint variational equation $w' = -Df(p(t))^* w$ is the codimension of $T_{p(t_0)}W^s(p_+) + T_{p(t_0)}W^u(p_-)$, that is,*

$$n - \dim W^s(p_+) - \dim W^u(p_-) + \dim T_{p(t_0)}W^s(p_+) \cap T_{p(t_0)}W^u(p_-).$$

A (homo)heteroclinic orbit γ is said to be *non-degenerate* if $\dim T_pW^s(p_+) \cap T_pW^u(p_-) = 1$, with $p \in \gamma$. Otherwise γ is said to be *degenerate.*

Remark A.10. *If the (homo)heteroclinic orbit is non-degenerate, the number of linearly independent bounded solutions is obtained directly from the stability indexes of p_+ and p_-. Moreover, although* $\dim T_{p(t_0)}W^s(p_+) = \dim W^s(p_+)$ *and* $\dim T_{p(t_0)}W^u(p_-) = \dim W^u(p_-)$*, in general* $\dim T_{p(t_0)}W^s(p_+) \cap T_{p(t_0)}W^u(p_-)$ *does not coincide with* $\dim W^s(p_+) \cap W^u(p_-)$.

In the sequel the (homo)heteroclinic orbit $\gamma = \{p(t) \,:\, t \in \mathbb{R}\}$ will be non-degenerate.

A.2 Bifurcation equations

As already mentioned, the existence of (homo)heteroclinic orbits for (A.1) implies the existence of bounded solutions of (A.2) and, consequently, the existence of bounded solutions of (A.3) when $b(t) \in C^0_b(\mathbb{R}, \mathbb{R}^n)$. According to Proposition A.5, if the adjoint variational equation

$$w' = -Df(p(t))^* w$$

has d linearly independent bounded solutions w_i, then the persistence of the (homo)heteroclinic orbit requieres the fulfillment of the d conditions $\int_{-\infty}^{\infty} \langle w_i(t), b(t)\rangle \, dt = 0$ for $i = 1, \ldots, d$. The question now is the sufficiency of such conditions.

When $d = 1$ the sufficiency could be followed from [CHM80]. In general, for $d \geq 1$, the techniques to be used follow the first steps of the Lin's method [Lin90, San93]. For $\|\lambda\|$ small enough, one has to look for solutions $p^+_\lambda(\cdot)$ and $p^-_\lambda(\cdot)$ of (A.1), contained in the stable and unstable invariant manifolds of the equilibrium points $p_+(\lambda)$ and $p_-(\lambda)$, respectively. Initial values $p^\pm_\lambda(t_0)$ will belong to a section Σ_{t_0} transverse to the (homo)heteroclinic orbit γ. Namely

$$\Sigma_{t_0} = p(t_0) + \{f(p(t_0))\}^\perp = p(t_0) + \left(W^+_{t_0} \oplus W^-_{t_0} \oplus E^*_{t_0}\right)$$

where $E^*_{t_0} = E^{s*}_{t_0} \cap E^{u*}_{t_0}$ and $W^+_{t_0}$ (resp. $W^-_{t_0}$) is the orthogonal complement of $E^s_{t_0} \cap E^u_{t_0} = \operatorname{span}\{f(p(t_0))\}$ in $E^s_{t_0}$ (resp. $E^u_{t_0}$). Moreover the condition $\xi^\infty(\lambda) = p^-_\lambda(t_0) - p^+_\lambda(t_0) \in E^*_{t_0}$ will be required. Under these assumptions there will exist two unique solutions $p^\pm_\lambda(\cdot)$ for each λ. The jump $\xi^\infty(\lambda) = p^-_\lambda(t_0) - p^+_\lambda(t_0)$ measures the displacement between the stable and unstable invariant manifolds on the section Σ_{t_0} in the direction of the subspace $E^*_{t_0} = [E^s_{t_0} + E^u_{t_0}]^\perp$.

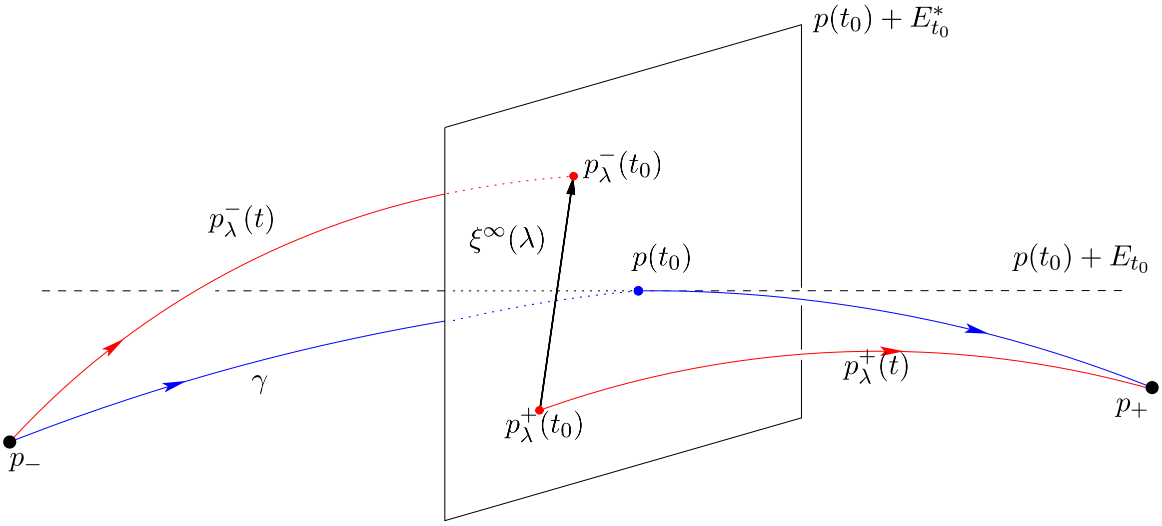

Fig. B: Heteroclinic connection in $\mathbb{R}^3$. In this case, $E^s_{t_0} = E^u_{t_0} = E_{t_0}$ (unidimensional), $E^{s*}_{t_0} = E^{u*}_{t_0} = E^*_{t_0}$ (bidimensional) and $\Sigma_{t_0} = p(t_0) + E^*_{t_0}$. For simplicity we have assumed that $f(p_\pm, \lambda) = 0$ for all λ.

The proof of the result below can be found in [San93, Lemma 3.3] and [Kno04, Lemma 2.1.2]. Namely, in [Kno04] only the first item is proved and, moreover, the proof is developed for the degenerate case although the non-degenerate one follows in a similar manner. The second item is proved in [San93] for the non-degenerate case. In [Bar09] is given a completed simpler slightly different proof of this result:

Lemma A.11. *There exists $\delta > 0$ such that for all $\lambda \in \mathbb{R}^k$, with $\|\lambda\| < \delta$,*

i) There exists a unique pair of solutions $p^+_\lambda(t)$ and $p^-_\lambda(t)$ of (A.1) parametrizing orbits on $W^s(p_+(\lambda))$ and $W^u(p_-(\lambda))$, respectively, such that $p^\pm_\lambda(t_0) \in \Sigma_{t_0}$ and

$$\xi^\infty(\lambda) = p^-_\lambda(t_0) - p^+_\lambda(t_0) \in E^*_{t_0}.$$

Writing the solutions as $p^\pm_\lambda(t) = p(t) + z^\pm_\lambda(t)$, then $z^\pm_\lambda(\cdot)$ are, respectively, forward and backward bounded solutions of the equation (A.2). They depend regularly on λ and the functions $z^\pm_0$ are identically zero.

ii) For $\varepsilon > 0$ small enough, there exists a (homo)heteroclinic solution $p_\lambda(t)$ such that $\|p_\lambda(t_0) - p(t_0)\| < \varepsilon$ if and only if $\xi^\infty(\lambda) = 0$, that is, the components $\xi^\infty_i(\lambda)$ of $\xi^\infty(\lambda)$ in the basis $\{w_i : i = 1 \dots d\}$ of $E^*_{t_0}$ satisfy

$$\begin{aligned}\xi^\infty_i(\lambda) \quad \equiv \quad & \int_{-\infty}^{t_0} < w_i(s), b(\lambda, s, z^-_\lambda(s)) > ds \\ & + \int_{t_0}^{\infty} < w_i(s), b(\lambda, s, z^+_\lambda(s)) > ds = 0.\end{aligned}$$

According with the above statement the persistence of (homo)heteroclinic orbits follows from the analysis of the bifurcation equation $\xi^\infty(\lambda) = 0$. The existence of non zero parameter values $\lambda \in \mathbb{R}^k$ such that $\xi^\infty(\lambda) = 0$ follows from the Implicit Function Theorem when $D_\lambda \xi^\infty(0)$ has rank $d < k$. Thus, the following result follows:

Theorem A.12. *Let $\xi^\infty(\lambda) = 0$, with $\lambda \in \mathbb{R}^k$, be the bifurcation equation of the differential equation (A.1). If $k > d$ and $\operatorname{rank} D_\lambda \xi^\infty(0) = d$, then (A.1) has a (homo)heteroclinic orbit for each parameter value λ on a regular manifold of dimension $k - d$ with tangent subspace at $\lambda = 0$ given by the solutions of the system*

$$\sum_{j=1}^{k} \xi_{ij}^\infty \lambda_j = 0 \qquad i = 1, \dots, d$$

where

$$\xi_{ij}^\infty \equiv \frac{\partial \xi_i^\infty}{\partial \lambda_j}(0) = \int_{-\infty}^{\infty} < w_i(s), D_{\lambda_j} g(0, p(s)) > ds$$

for $i = 1, \dots, d$ and $j = 1, \dots, k$.

Note that, when $k \leq d$, $\lambda = 0$ is the unique value of $\lambda \in \mathbb{R}^k$ for which there exists a (homo)heteroclinic orbit

$$\gamma_\lambda = \{p_\lambda(t) : \ p'_\lambda(t) = f(p_\lambda(t)) + g(\lambda, p_\lambda(t)) \ t \in \mathbb{R}\}$$

such that $\sup_{t \in \mathbb{R}} \|p_\lambda(t) - p(t)\|$ is small enough. If $k > d$ the homoclinic connection persists for parameter values on a manifold of codimension d where

$$d = n - \dim W^s(p_+) - \dim W^u(p_-) + 1.$$

In such a case we say that there is (homo)heteroclinic bifurcation of a non-degenerate orbit at $\lambda = 0$ which is of codimension d.

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