On elliptic equations involving the 1-Laplacian operator

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This dissertation is submitted for the degree of
PhD in Mathematics

April 2018

I declare that this dissertation entitled *On elliptic equations involving the 1-Laplacian operator* and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a degree of PhD in Mathematics at Universitat de València.
- Where I have consulted the published works of others, this is always clearly attributed.
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- I have acknowledged all main sources of help.

Valencia, April 17th, 2018

Marta Latorre Balado

I declare that this dissertation presented by **Marta Latorre Balado** entitled *On elliptic equations involving the 1-Laplacian operator* has been done under my supervision at Universitat de València. I also state that this work corresponds to the thesis project approved by this institution and it satisfies all the requisites to obtain the degree of PhD in Mathematics.

Valencia, April 17th, 2018

Sergio Segura de León

Acknowledgements

I would like to express my gratitude for the help and support of my PhD advisor, Sergio. Without his efficient supervision, it would have been impossible to finish this work. I also want to thank J.M. Mazón for guiding me in my initial steps in the mathematical research.

I am also very thankful to Francesco Petitta and the students from Rome, specially to Francescantonio and Martina, who had made my stay in Italy a really pleasant time.

I give thanks to all the professors who taught me during my studies at university and also to all the members of the Departamento de Análisis Matemático.

Special thanks are given to the PhD students from Facultad de Matemáticas, whom I consider my friends, for the great moments in the last years.

I also want to express my sincere gratitude to Asun, who introduced me into the mathematics and inspired me to become a mathematician.

I would like to thank my family for their constant and unconditional support and encouragement during all my life.

I am thankful to Ministerio de Economía y Competitividad for funding my PhD during the last four years (grant BES-2013-066655) and for supporting my stay in Italy (grant EEBB-I-17-12226).

According to the normative of the Universitat de València, this PhD dissertation starts with three abstracts. The first two are in the official languages of Valencia Community, Valencian and Spanish, and the last one is written in English. The thesis really begins at page 1, after the three abstracts.

We would like also to point out that in each chapter we have described the methodologies, objectives, results and conclusions of this work.

Resum

Aquesta tesis preten donar a conèixer els resultats obtinguts en l'estudi de l'existència, unicitat i regularitat de les solucions de diferents equacions el·líptiques regides per l'operador 1-Laplacià.

La tesi comença amb una breu introducció i amb distintes explicacions relatives a la notació, definicions bàsiques i propietats elementals de les ferramentes utilitzades al llarg d'aquest treball.

El primer capítol està dedicat a l'estudi del següent problema de Dirichlet:

$$\begin{cases} -\mathrm{div}\left(\dfrac{Du}{|Du|}\right) + |Du| = f(x) & \text{a } \ \Omega\,, \\ u = 0 & \text{a } \ \partial\Omega\,, \end{cases} \tag{1}$$

on Ω és un obert fitat de $\mathbb{R}^N$ amb frontera $\partial\Omega$ Lipschitz i la dada f és una funció de l'espai de Marcinkiewicz $L^{N,\infty}(\Omega)$.

La motivació de l'estudi d'aquest problema ve donada per un article de J.M. Mazón i S. Segura de León (veure [55]), en el que proven l'existència de solucions fitades quan es trien les dades en l'espai de Lebesgue $L^q(\Omega)$ amb $q > N$.

En problemes semblants en els que en lloc del 1-Laplacià tenim l'operador p-Laplacià amb $1 < p < \infty$, l'espai natural en què es troben les solucions (seguinet un punt de vista variacional) és l'espai de Sobolev $W_0^{1,p}(\Omega)$, i les dades pertanyen al seu dual, és a dir, $W^{-1,p'}(\Omega)$ amb $p' = \frac{p}{p-1}$. En particular, quan $p = 1$, deuríem trobar solucions en l'espai de Sobolev $W_0^{1,1}(\Omega)$ quan utilitzem dades en l'espai $W^{-1,\infty}(\Omega)$.No obstant això, gràcies al teorema d'inmersió de Sobolev i utilitzant arguments de dualitat, s'observa que, per obtindre solucions febles, el millor espai en què podem triar les dades entre els espais de Lebesgue és $L^N(\Omega)$ i entre els espais de Lorentz és $L^{N,\infty}(\Omega)$. El nostre objectiu és millorar els resultats d'existència i unicitat obtinguts en [55] prenent les dades en l'espai òptim.

Hem indicat que l'espai d'energia deuria ser l'espai de Sobolev $W_0^{1,1}(\Omega)$, no obstant això, i contràriament al que ocorre amb els espais $W_0^{1,p}(\Omega)$ aquest espai no és reflexiu. És per això que en els problemes regits per l'operador 1-Laplacià treballem en un espai major i amb millors propietats: l'espai de les funcions de variació fitada, que denotarem per $BV(\Omega)$ i està format pel conjunt de totes les funcions integrables on la seua derivada en el sentit de les distribucions és una mesura de Radon amb variació total finita.

La primera dificultat que trobem a l'enfrontar-nos a una equació on apareix l'operador 1-Laplacià és definir la solució del problema. En particular, hem de donar-li sentit al quocient $\frac{Du}{|Du|}$, tenint en compter que $|Du|$ és una mesura. En [8], F. Andreu, C. Ballester, V. Caselles i J.M. Mazón van resoldre aquest problema utilitzant un camp vectorial $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^N)$ que exercix el paper del quocient $\frac{Du}{|Du|}$. En particular, necessitem que $\|\mathbf{z}\|_{L^\infty(\Omega)} \leq 1$ i que el producte $(\mathbf{z}, Du)$ estiga ben definit i complisca $(\mathbf{z}, Du) = |Du|$.

El producte $(\mathbf{z}, Du)$ va ser definit per G. Anzellotti en [13] i per G.- Q. Chen i H. Frid en [26] i és una generalització del producte escalar entre el camp $\mathbf{z}$ i Du. Recordem que per a tota funció $\varphi \in C_0^\infty(\Omega)$, Anzellotti definix la distribució

$$\langle (\mathbf{z}, Du), \varphi \rangle = -\int_\Omega u\, \varphi \operatorname{div} \mathbf{z} - \int_\Omega \mathbf{z} \cdot \nabla \varphi \, dx \,.$$

En [13] va provar que si prenem un camp vectorial $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^N)$ tal que $\operatorname{div} \mathbf{z}$ és una mesura fitada a Ω i la funció $u \in BV(\Omega)$ és fitada i contínua, aleshores $(\mathbf{z}, Du)$ és una mesura de Radon amb variació total finita.

Quan prenem dades en l'esapi $L^q(\Omega)$ amb $q > N$, les solucions del problema (1) són funcions de l'espai $BV(\Omega) \cap L^\infty(\Omega)$ (veure [55]), però no són, necessàriament, contínues, per tant $(\mathbf{z}, Du)$ no està ben definit. La generalització d'aquesta definició es deguda a G.-Q. Chen i H. Frid, des de una perspectiva diferent, i a A. Mercaldo, S. Segura de León i C. Trombetti, seguint la teoría d'Anzellotti. En els dos casos treballen amb un camp vectorial $\mathbf{z}$ en l'espai $\mathcal{DM}^\infty(\Omega)$, és a dir, $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^N)$ i a més $\operatorname{div} \mathbf{z}$ és una mesura de Radon amb variació total finita, i una funció u pertany a l'espai $BV(\Omega) \cap L^\infty(\Omega)$. Utilitzant el representant precís de u (denotat per u^*), van provar que $(\mathbf{z}, Du)$ és una mesura de Radon de variació total finita. No obstant això, quan prenem la dada $f \in L^{N,\infty}(\Omega)$ en el problema (1), les solucions que obtenim són no fitades i no podem utilitzar aquesta definició del producte $(\mathbf{z}, Du)$. És per això que necessitem el resultat que s'enuncia a continuació i que es prova al final de la Secció 1.4.

Teorema. *Siga u una funció de l'espai $BV(\Omega)$ i siga $\mathbf{z}$ un camp vectorial de $\mathcal{DM}^\infty(\Omega)$ amb $\operatorname{div} \mathbf{z} = \xi + f$ on ξ és una mesura de Radon tal que $\xi \geq 0$ o $\xi \leq 0$ i la funció f és de l'espai de Marcinkiewicz $L^{N,\infty}(\Omega)$. Aleshores, donat $\varphi \in C_0^\infty(\Omega)$, la distribució definida per*

$$\langle (\mathbf{z}, Du), \varphi \rangle = -\int_\Omega u^* \varphi \operatorname{div} \mathbf{z} - \int_\Omega u \, \mathbf{z} \cdot \nabla \varphi \, dx$$

és una mesura de Radon amb variació total finita i verifica

$$|(\mathbf{z}, Du)| \leq \|\mathbf{z}\|_{L^\infty(\Omega)} |Du| \, .$$

A més, assumint les hipòtesis abans mencionades també hem provat la següent generalització de la fórmula de Green:

$$\int_\Omega u^* \operatorname{div} \mathbf{z} + \int_\Omega (\mathbf{z}, Du) = \int_{\partial\Omega} u\,[\mathbf{z},\nu]\, d\mathcal{H}^{N-1}\,,$$

on $[\mathbf{z}, u]$ denota la traça feble a la frontera $\partial\Omega$ de la component normal de $\mathbf{z}$, definida per Anzellotti en [13].

Finalment, en la Secció 1.5 hem provat el resultat d'existència d'una única funció u no negativa que és solució del problema (1) quan prenem com a dada una funció $f \geq 0$ de l'espai de Marcinkiewicz $L^{N,\infty}(\Omega)$. Encara que les solucions del problema de Dirichlet (1) no són necessàriament fitades, sí tenen una certa regularitat ja que són funcions de variació fitada sense part de salt, és a dir, $D^j u = 0$. A més, necessitem un camp vectorial $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ que actuarà com el quocient $\frac{Du}{|Du|}$ en l'equació. La definició de solució es la següent:

Definició. *Siga $f \in L^{N,\infty}(\Omega)$ amb $f \geq 0$. Diem que $u \in BV(\Omega)$ amb $D^j u = 0$ és una solució feble del problema* (1) *si existeix un camp vectorial $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ amb $\|\mathbf{z}\|_{L^\infty(\Omega)} \leq 1$ i tal que*

$$\begin{aligned}
&\textbf{(i)} \quad -\operatorname{div} \mathbf{z} + |Du| = f \quad \textit{en} \;\; \mathcal{D}'(\Omega)\,,\\
&\textbf{(ii)} \quad (\mathbf{z}, Du) = |Du| \quad \textit{com a mesures a} \;\; \Omega\,,\\
&\textbf{(iii)} \quad u|_{\partial\Omega} = 0\,.
\end{aligned}$$

Presentem a continuació el resultat principal d'aquesta secció.

Teorema. *Siga Ω un obert fitat de $\mathbb{R}^N$ amb frontera Lipschitz i donada una funció $f \in L^{N,\infty}(\Omega)$ amb $f \geq 0$, hi ha una única solució feble no negativa del problema* (1).

A més, quan prenem una funció f de norma menuda sempre obtenim la solució nul·la, com mostra el següent resultat:

Proposició. *Siga* $u \in BV(\Omega)$ *la solució no negativa del problema* (1) *amb la dada* $0 \leq f \in L^{N,\infty}(\Omega)$. *Aleshores, la solució es nul·la si i només si* $\|f\|_{W^{-1,\infty}(\Omega)} \leq 1$.

És habitual trobar resultats semblants quan es treballa amb l'operador 1-Laplacià. En particular, en [57] es va provar que la solució del problema de Dirichlet amb l'equació $-\mathrm{div}\left(\frac{Du}{|Du|}\right) = f$ és nul·la si $\|f\|_{W^{-1,\infty}(\Omega)} < 1$. Quan es dóna la igualtat, la funció $u = 0$ sempre és solució encara que també pot haver-hi una solució no nul·la per a certes dades.

Finalitzem la secció amb un resultat sobre la regularitat de les solucions. Si prenem com a dada una funció $f \in L^q(\Omega)$ amb $q > N$, la solució del problema (1) sempre és fitada (veure [55]) . No obstant això, en el cas límit, quan triem la dada f en l'espai de Marcinkiewicz $L^{N,\infty}(\Omega)$, la solució u és de l'espai de Lebesgue $L^q(\Omega)$ per a tot $1 \leq q < \infty$. Per provar que, en efecte, les solucions del problema (1) amb dades no negatius de l'espai $L^{N,\infty}(\Omega)$ no són, necessàriament, fitats, mostrem el següent exemple.

Exemple. *Siguen* $0 < \rho < R$ *i* $0 < \lambda < N-1$. *Si considerem* $\Omega = B_R(0)$, *la solució del problema* (1) *amb dada* $\frac{\lambda}{|x|}\chi_{B_\rho(0)}(x) \in L^{N,\infty}(\Omega) \setminus L^N(\Omega)$ *ve donada per*

$$u(x) = (N-1-\lambda)\log\left(\frac{|x|}{\rho}\right), \quad |x| < \rho\,,$$

que pertany als espais de Lebesgue $L^q(\Omega)$ *per a* $1 \leq q < \infty$ *però és una funció no fitada.*

Les últimes seccions del Capítol 1 estan dedicades a l'estudi d'una generalització del problema (1) al què hem afegit una funció g en el terme del gradient. És a dir, estudiem el següent problema de Dirichlet:

$$\begin{cases} -\mathrm{div}\left(\dfrac{Dv}{|Dv|}\right) + g(v)|Dv| = f(x) & \text{a } \Omega\,, \\ v = 0 & \text{a } \partial\Omega\,, \end{cases} \tag{2}$$

on Ω és un obert fitat de $\mathbb{R}^N$ amb frontera $\partial\Omega$ Lipschitz, f és una funció no negativa de l'espai de Marcinkiewicz $L^{N,\infty}(\Omega)$ i la funció $g : [0,\infty[\to [0,\infty[$ és contínua i no negativa.

El nostre objectiu és veure com afecta la funció g als resultats d'existència, unicitat i regularitat de solucions. El terme amb la variació total és essencial per a la unicitat de solució ja que si considerem el mateix problema però sense la variació total $-\mathrm{div}\left(\frac{Dv}{|Dv|}\right) = f$, i denotem per v a la seua solució, la funció $h(v)$ també ha de ser solució per a tota funció h suau i creixent.

A més, en [9] van provar que una equació semblant al problema (1) però en la que tampoc apareix el terme amb la variació total, no complix la mateixa regularitat que les solucions de (1). En particular van provar que l'equació $u - \mathrm{div}\left(\frac{Du}{|Du|}\right) = f(x)$ té una única solució encara que la dita solució pot tindre part de salt.

Al llarg de les Seccions 1.7, 1.8 i 1.9 del Capítol 1 veiem que depenent de les característiques de la funció g, la solució del problema (2) satisfà diferents propietats. A més, en els casos més extrems, quan la funció g s'anul·la, també hem de modificar el concepte de solució ja que la solució no és, necessàriament, una funció de variació fitada.

Independentment de les propietats de la funció g, necessitem definir la funció auxiliar

$$G(s) = \int_0^s g(\sigma)\, d\sigma\,.$$

En la Secció 1.7 estudiem les condicions de la funció g baix les quals la solució v satisfà millors propietats. En particular, introduïm la definició de solució del problema (2) quan tenim una funció $g : [0, \infty[\to [0, \infty[$ contínua i tal que $g(s) \geq m > 0$ per a tot $s \geq 0$.

Definició. *Diem que v és una solució feble del problema (2) si $v \in BV(\Omega)$ amb $D^j v = 0$ i existeix un camp vectorial $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ amb $\|\mathbf{z}\|_{L^\infty(\Omega)} \leq 1$ i tal que*

$$
\begin{aligned}
&\textbf{(i)} \quad && -\operatorname{div}\mathbf{z} + g(v)^*|Dv| = f \quad \textit{en} \ \ \mathcal{D}'(\Omega)\,, \\
&\textbf{(ii)} \quad && (\mathbf{z}, Dv) = |Dv| \quad \textit{com a mesures a} \ \ \Omega\,, \\
&\textbf{(iii)} \quad && v|_{\partial\Omega} = 0\,.
\end{aligned}
$$

És important destacar que en aquest cas, la solució v complix les mateixes propietats de regularitat que les solucions del problema (1). Només afecta la funció g en la igualtat distribucional.

Quan triem una funció g contínua i tal que $g(s) \geq m > 0$, hem de distingir dos casos per provar l'existència de solució. Si la funció g és fitada, utilitzem els resultats enunciats a continuació.

Teorema.

(i) *Siga u la solució no negativa del problema (1). Aleshores la funció v tal que $u = G(v)$ és solució del problema (2).*

(ii) *Siga v una solució no negativa del problema (2). Aleshores la funció $u = G(v)$ és la solució del problema (1).*

En general, per provar l'existència de solució es necessita utilitzar la regla de la cadena per a funcions de variació fitada. En [7] van provar que si $v \in BV(\Omega)$ és tal que $D^j v = 0$ i $\psi : \mathbb{R} \to \mathbb{R}$ és una funció Lipschitz, aleshores $\psi(v) \in BV(\Omega)$ i a més $D\psi(v) = \psi'(v)Dv$. No obstant això, no sempre podem utilitzar aquest resultat ja que, amb les nostres hipòtesis,

ψ' no és, necessàriament, fitada. És per això que també hem provat una lleugera generalització de la regla de la cadena:

Proposició. *Siga v una funció de variació fitada sense part de salt i siga g una funció real, contínua i no negativa. Si $u = G(v) \in L^1(\Omega)$, aleshores $u \in BV(\Omega)$ si i només si $g(v)|Dv|$ és una mesura finita. A més, $|Du| = g(v)|Dv|$ com a mesures a Ω.*

Quan la funció $g(s)$ està separada de l'eix s però és no fitada, les proves dels resultats anteriors no funcionen. En aquest cas, per provar l'existència de solució utilitzem una successió de problemes aproximants. Per a cada $n \in \mathbb{N}$, considerem el problema (2) amb una funció $g_n(s)$ tal que hi ha una solució v_n del problema i a més $g_n(s)$ convergix a $g(s)$. Cal provar que la successió de solucions $\{v_n\}$ és convergent i el límit és la solució de (2) amb la funció $g(s)$.

El resultat principal de la Secció 1.7 s'enuncia a continuació:

Teorema. *Siga Ω un obert fitat de $\mathbb{R}^N$ amb frontera Lipschitz i siga $f \in L^{N,\infty}(\Omega)$ amb $f \geq 0$. Si g és una funció real i contínua i a més complix $g(s) \geq m > 0$ per a tot $s \geq 0$, aleshores hi ha una única solució no negativa del problema (2). A més, la dita solució pertany als espais de Lebesgue $L^q(\Omega)$ per a tot $1 \leq q < \infty$.*

La Secció 1.8 està dedicada a l'estudi del problema (2) quan la funció $g : [0, \infty[\to [0, \infty[$ pot ser nul·la en algun punt. En particular, treballem amb una funció g contínua, fitada, no integrable i tal que $g(s) > 0$ per a quasi tot $s \geq 0$.

El primer resultat que provem en aquesta secció mostra que hi ha una única solució del problema (2) quan la funció g complix, a més, la restricció següent:

(C) *Existeixen $m, \sigma > 0$ tals que $g(s) \geq m > 0$ per a tot $s \geq \sigma$.*

Si prenem una funció $g(s)$ amb les condicions indicades, la prova de l'existència de solució es basa a prendre una aproximació amb funcions $g_n(s)$ que estan separades de l'eix d'abcisses. Quan no es complix la condició (C), hem de continuar utilitzant una aproximació amb funcions $g_n(s)$ separades de l'eix s, no obstant això, el límit de les solucions d'aquestos problemes ja no és, necessàriament, una funció de variació fitada. Utilitzem el següent exemple per mostrar que la solució del problema (2) amb una funció g tal que $\lim_{s\to\infty} g(s) = 0$ no és de variació fitada.

Exemple. *Siguen $R > 0$ i $\lambda > 2N - 2$. Si considerem $\Omega = B_R(0)$, la dada $f(x) = \frac{\lambda}{|x|}$ i la funció $g(s) = \frac{1}{1+s}$, la solució del problema (2) ve donada per*

$$v(x) = \left(\frac{|x|}{R}\right)^{N-1-\lambda} - 1\,,$$

que no és de variació fitada ja que $|Dv| = \frac{\lambda-N+1}{R^{N-1-\lambda}}|x|^{N-2-\lambda}$ no és una funció integrable.

Per tant, les propietats de la solució del problema (2) canvien quan prenem una funció $g : [0,\infty[\to [0,\infty[$ contínua, fitada, no integrable i tal que $g(s) > 0$ per a quasi tot $s \geq 0$. La nueva definició de solució es la següent:

Definició. *Diem que una funció v és solució feble del problema (1.43) si $G(v) \in BV(\Omega)$ amb $D^j G(v) = 0$ i existeis un camp vectorial $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ amb $\|\mathbf{z}\|_{L^\infty(\Omega)} \leq 1$ i tal que*

$$\textbf{(i)} \quad -\operatorname{div}\mathbf{z} + g(v)^*|Dv| = f \quad \textit{en} \;\; \mathcal{D}'(\Omega)\,,$$

$$\textbf{(ii)} \quad (\mathbf{z}, DG(v)) = |DG(v)| \quad \textit{com a mesures a} \;\; \Omega\,,$$

$$\textbf{(iii)} \quad G(v)|_{\partial\Omega} = 0\,.$$

Encara que la solució v no és de variació fitada, sí ho és la funció $G(v)$, que a més, no ha de tindre part de salt. A més, com la funció v no és de variació fitada, el producte $(\mathbf{z}, Dv)$ no està ben definit. És per això que utilitzem $(\mathbf{z}, G(v))$ en el seu lloc.

El següent resultat prova que el problema (2) té una única solució que complix la nova definició.

Teorema. *Siga Ω un obert fitat de $\mathbb{R}^N$ amb frontera Lipschitz i siga $f \in L^{N,\infty}(\Omega)$ una funció no negativa. Aleshores hi ha una única solució del problema (2) en el sentit de la definició anterior quan triem una funció $g : [0, \infty[\to [0, \infty[$ contínua, fitada, no integrable i tal que $g(s) > 0$ per a quasi tot $s > 0$.*

L'última secció del primer capítol està dedicada a l'estudi d'alguns casos particulars en què, segons les propietats de la funció g, no tenim necessàriament existència o unicitat o bé les solutions poden tindre salt o no verificar la condició a la frontera.

Suposant que la funció g és integrable, l'existència de solució ve determinada per la dada f. En particular, quan la norma de f en l'espai dual de $W_0^{1,1}(\Omega)$ és menor que 1, la solució del problema (2) sempre és nul·la. No obstant això, quan aquesta norma és major que certa constant, el problema no té solució. Enunciem el resultat a continuació.

Proposició. *Siga $f \in L^{N,\infty}(\Omega$ amb $f \geq 0$ i siga $g \in L^1([0, \infty[)$. Aleshores,*

(i) *si $\|f\|_{W^{-1,\infty}(\Omega)} < 1$, la solució del problema (2) és trivial;*

(ii) *si $\|f\|_{W^{-1,\infty}(\Omega)} > e^{G\infty)}$, el problema (2) no té solució;*

sent $G(\infty) = \sup\{G(s) \, : \, s \in]0, \infty[\}$.

Finalment, també hem estudiat les propietats de la solució del problema (2) quan la funció g s'anul·la en un interval. En particular, hem provat que mai tenim unicitat de solució i a més, encara que hi haja solució, aquesta pot tindre part de salt o fins i tot no complir la condició a la frontera, tal com mostren el següent exemple.

Exemple. *Donats* $R > 0$ *y la dada* $f(x) = \frac{N}{|x|}$*, si prenem* $\Omega = B_R(0)$ *i la funció* $g : [0, \infty[\to [0, \infty[$ *definida per*

(a) $g(s) = 0$ *si* $0 \leq s \leq a$ *y* $g(s) = s - a$ *si* $s > a$*, aleshores la solució* u *no s'anul·la a la frontera, encara que sí es complix la condició frontera en un sentit feble, és a dir,* $[\mathbf{z}, \nu] = -\operatorname{sign}(u)$*.*

(b) $g(s) = a - s$ *si* $0 \leq s < a$*,* $g(s) = 0$ *si* $a \leq s \leq b$ *y* $g(s) = s - b$ *si* $s > b$*, aleshores la solució del problema* (2) *té part de salt.*

Els resultats que apareixen en el primer capítol estan publicats en el següent article.

M. Latorre and S. Segura de León, Elliptic equations involving the 1-Laplacian and a total variation term with $L^{N,\infty}$-data, *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* **28** (2017), no. 4, 817–859. DOI: 10.4171/RLM/787

El segon capítol d'aquesta tesi està dedicat a generalitzar els resultats obtinguts en l'estudi del problema de Dirichlet (1) quan prenem com a dada una funció no negativa de l'espai de les funcions integrables $L^1(\Omega)$.

Quan triem una dada $f \in L^1(\Omega)$, i tal com succeïx quan les dades són funcions de l'espai de Marcinkiewicz, necessitem generalitzar la teoria d'Anzellotti definint el producte $(\mathbf{z}, Du)$, ja que si u és solució del problema s'ha de complir la igualtat $(\mathbf{z}, Du) = |Du|$ com a mesures.

És per això que necessitem que $(\mathbf{z}, Du)$ siga una mesura de Radon amb variació total finita. No obstant això, al no ser les solucions a aquest problema necessàriament fitades, no podem utilitzar els resultats de [13], i com a més f és una funció de l'espai de Lebesgue $L^1(\Omega)$, tampoc podem provar que $(\mathbf{z}, Du)$ siga una mesura de Radon amb variació total finita utilitzant els arguments del Capítol 1.

Per tant, hem de modificar el concepte de solució utilitzant truncaments, tal com van fer F. Andreu, C. Ballester, V. Caselles i J.M. Mazón en [8] quan van definir la solució del problema parabòlic $u_t = \operatorname{div}\left(\frac{Du}{|Du|}\right)$.

L'ús dels truncaments en la definició de solució és degut a que considerem dades de l'espai $L^1(\Omega)$. En alguns problemes semblants en què, en lloc del 1-Laplacià tenim l'operador p-Laplacià amb $1 < p \le N$, ja s'han usat els truncaments, tant en solucions renormalitzades com en solucions d'entropia (veure [30] i [16] respectivament).

Si denotem per $T_k(s) = \min\{|s|, k\} \operatorname{sign}(s)$ a la funció truncament, la definició de solució del problema (1) es la següent:

Definició. *Diem $u \in BV(\Omega)$ és una solució del problema (1) si $D^j u = 0$ i existeis un camp vectorial $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ amb $\|\mathbf{z}\|_{L^\infty(\Omega)} \le 1$ i tal que*

(i) $-\operatorname{div}\mathbf{z} + |Du| = f \quad en \quad \mathcal{D}'(\Omega)\,,$

(ii) $(\mathbf{z}, DT_k(u)) = |DT_k(u)| \quad com\ a\ mesures\ a \quad \Omega\ (per\ a\ tot\ k > 0)\,,$

(iii) $u|_{\partial\Omega} = 0\,.$

Cal remarcar que la funció $T_k(u)$ és de l'espai $BV(\Omega) \cap L^\infty(\Omega)$ i per tant $(\mathbf{z}, DT_k(u))$ està ben definit i és una mesura de Radon amb variació total finita.

D'altra banda, és important destacar que en [9] estudien una equació semblant amb dades en l'espai $L^1(\Omega)$ però sense el terme de la variació total. Tal com nosaltres, definixen la solució utilitzant els truncaments però al no tindre la variació total, la solució perd regularitat. En particular, la condició a la frontera es complix en un sentit feble, no sent així en el nostre cas.

Mostrem a continuació el resultat principal d'aquesta secció que prova l'existència de solució en el sentit de la definició anterior.

Teorema. *Si f és una funció integrable no negativa, aleshores hi ha almenys una solució del problema (1) en el sentit de la definició anterior.*

D'altra banda, quant a la unicitat de solució, en la Secció 2.2 del Capítol 2 hem provat un principi de comparació que no sols millora el resultat d'unicitat de solució, sinó que també simplifica la seua demostració quan utilitzem dades de l'espai de Lebesgue $L^q(\Omega)$ amb $q > N$. El resultat és el següent:

Teorema. *Siga Ω un obert fitat de $\mathbb{R}^N$ amb frontera Lipschitz i siguen f_1 i f_2 dos funcions de l'espai $L^1(\Omega)$ tals que $0 \leq f_1 \leq f_2$. Si u_1 i u_2 són solucions del problema (1) amb dades f_1 i f_2 respectivament, aleshores $u_1 \leq u_2$.*

Cal destacar que encara que $(\mathbf{z}, Du)$ no estiga ben definit, provem que el producte $(e^{-u}\mathbf{z}, Du)$, i també $(\mathbf{z}, De^{-u})$, és una mesura de Radon amb variació total finita. Ambdós expressions són imprescindibles tant en la prova de l'existència de solució com en la demostració del principi de comparació.

Finalment, i gràcies al principi de comparació, queda provada la unicitat de solució.

Teorema. *Siga Ω un obert fitat de $\mathbb{R}^N$ amb frontera Lipschitz i siga $f \in L^1(\Omega)$ una funció no negativa. Aleshores hi ha una única solució del problema (1).*

La Secció 2.3 d'aquest capítol està destinada a l'estudi de la regularitat de les solucions del problema (1) quan prenem les dades en l'espai de Lebesgue $L^q(\Omega)$ amb $1 < q < N$. Com ja s'ha indicat en el Capítol 1, si la dada pertany a $L^N(\Omega)$, la solució és de l'espai $L^q(\Omega)$ per a tot

$1 \leq q < \infty$ i si la dada és de $L^q(\Omega)$ amb $q > N$, per [55] sabem que la solució és fitada.

El resultat principal d'aquesta secció és el següent:

Teorema. *Siga Ω un obert fitat de $\mathbb{R}^N$ amb frontera Lipschitz. Si prenem una funció $f \geq 0$ de l'espai de Lebesgue $L^q(\Omega)$ amb $1 < q < N$, aleshores l'única solució u del problema (1) satisfà $u \in BV(\Omega) \cap L^{\frac{Nq}{N-q}}(\Omega)$.*

En particular, hem provat que la màxima regularitat que aconsegueixen les solucions del problema (1) s'ajusta amb continuïtat en relació a la regularitat de la dada f.

Per acabar el capítol hi ha una secció en què mostrem que la regularitat de la solució és, en efecte, òptima.

Exemple. *Siguen $R > 0$, $1 < q < N$ i $\lambda > 0$. Si considerem $\Omega = B_R(0)$ i la dada $f(x) = \frac{\lambda}{|x|^q}$ de l'espai de Lebesgue $L^s(\Omega)$ amb $1 < s < \frac{N}{q}$, aleshores la solució del problema (1) ve donada per*

$$u(x) = \begin{cases} (N-1)\log\left(\frac{|x|}{\rho_\lambda}\right) + \frac{\lambda}{1-q}(\rho_\lambda^{1-q} - |x|^{1-q}) & si \quad 0 \leq |x| < \rho_\lambda \\ 0 & si \quad \rho_\lambda < |x| \leq R\,, \end{cases}$$

per a un cert valor $0 < \rho_\lambda < R$. És a dir, la solució u pertany a l'espai de Lebesgue $L^r(\Omega)$ per a tot $1 \leq r < \frac{N}{q-1}$.

Els resultats que apareixen en aquest capítol estan publicats en el següent article:

M. Latorre and S. Segura de León, Existence and comparison results for an elliptic equation involving the 1-Laplacian and L^1-data, *J. Evol. Equ.* **18** (2018), no. 1, 1–28. DOI: 10.1007/s00028-017-0388-0.

Finalment, en el Capítol 3 provem resultats d'existència i unicitat de solució d'un problema d'evolució. El dit problema consistix en una equació de tipus el·líptic en què apareix l'operador 1-Laplacià i una condició dinàmica de frontera. El problema és el següent:

$$\begin{cases} \lambda u - \operatorname{div}\left(\frac{Du}{|Du|}\right) = 0 & \text{a } (0,+\infty)\times\Omega\,, \\ \omega_t + \left[\frac{Du}{|Du|},\nu\right] = g(t,x) & \text{a } (0,+\infty)\times\partial\Omega\,, \\ u = \omega & \text{a } (0,+\infty)\times\partial\Omega\,, \\ \omega(0,x) = \omega_0(x) & \text{a } \partial\Omega\,, \end{cases} \tag{3}$$

on Ω és un obert fitat de $\mathbb{R}^N$ amb frontera $\partial\Omega$ suau, el paràmetre λ és un número real positiu, ν representa el vector exterior normal de norma 1, la funció g està en l'espai $L^1_{loc}(0,+\infty,L^2(\partial\Omega))$ i la dada inicial ω_0 és una funció de quadrat integrable a la frontera $\partial\Omega$. Denotem per ω_t a la derivada distribucional de ω respecte de la variable t.

Aquest tipus de problemes amb condicions dinàmiques de frontera apareixen quan es modeliza un problema en què la solució es pertorba, no a l'interior del domini, sinó a la frontera. En l'actualitat, l'estudi d'equacions el·líptiques o parabòliques amb aquest tipus de condició a la frontera és una àrea d'estudi molt activa, ja que s'ajusta a diferents processos com poden ser la transferència de calor d'un fluid en moviment a un sòlid o problemes de termoelasticitat o biologia.

Quant a problemes amb condicions dinàmiques de frontera i l'operador 1-Laplacià, pel que nosaltres sabem, aquesta és la primera vegada que s'estudia un problema amb aquestes condicions. F. Andreu, N. Igbida, J.M. Mazón i J. Toledo van estudiar en [11] un problema semblant amb una equació el · líptica en què apareix l'operador p-Laplacià, amb $1 < p < \infty$. Per provar l'existència de solució, els autors definixen un operador completament acretiu i utilitzant la teoria de semigrups no lineals obtenen una *mild solution* o solució en el sentit de la teoría de semigrups, que finalment proven que és, realment, una solució distribucional del problema estudiat.

Per obtindre una solució del problema (3), com no podem provar l'existència utilitzant problemes aproximants, hem seguit el mètode de [11], però adaptant-ho a les peculiaritats del 1-Laplacià.

Primer considerem el problema (3) restringint-nos al domini $(0,T)\times\Omega$, sent $T>0$. És a dir, estudiem el problema següent:

$$\begin{cases} \lambda u - \operatorname{div}\left(\frac{Du}{|Du|}\right) = 0 & \text{a } (0,T)\times\Omega\,, \\ \omega_t + \left[\frac{Du}{|Du|},\nu\right] = g(t,x) & \text{a } (0,T)\times\partial\Omega\,, \\ u = \omega & \text{a } (0,T)\times\partial\Omega\,, \\ \omega(0,x) = \omega_0(x) & \text{a } \partial\Omega\,, \end{cases} \tag{4}$$

on g denota una funció de l'espai $L^1(0,T;L^2(\partial\Omega))$.

En la Secció 3.2 d'aquest capítol presentem la notació específica i resultats previs necessaris a l'hora de provar l'existència de solució dels problemes (3) i (4). En particular, enunciem els resultats que emprem per provar l'existència d'una solució de semimgrups del problema de Cauchy

$$\begin{cases} \omega_t + \mathcal{B}(\omega) \ni g\,, \\ \omega(0) = \omega_0\,, \end{cases} \tag{5}$$

on g i ω_0 són funcions de $L^1(0,T;L^2(\partial\Omega))$ i $L^2(\partial\Omega)$ respectivament i $\mathcal{B}$ denota a un operador en $L^2(\partial\Omega)$. Aquestes solucions de semigrups s'obtenen com el límit de les solucions de les discretizaciones del problema (5).

Ja en la Secció 3.3 provem que la dita solució de semigrups existeix i és única i per això, definim un operador $\mathcal{B}$ en l'espai $L^2(\partial\Omega)$.

Definició. *Siga* $\omega \in L^2(\partial\Omega)$. *Diem que* $v \in \mathcal{B}(\omega)$ *si* $v \in L^\infty(\partial\Omega)$ *amb* $\|v\|_{L^\infty(\partial\Omega)} \leq 1$ *i existeix una funció* $u \in BV(\Omega) \cap L^2(\Omega)$ *i un camp vectorial* $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^N)$ *amb* $\|\mathbf{z}\|_{L^\infty(\Omega)} \leq 1$ *tals que*

$$\begin{aligned}
&\textbf{(i)} \quad \lambda u - \operatorname{div} \mathbf{z} = 0 \quad a \ \ \mathcal{D}'(\Omega)\,, \\
&\textbf{(ii)} \quad (\mathbf{z}, Du) = |Du| \quad \textit{com a mesures a} \ \ \Omega\,, \\
&\textbf{(iii)} \quad [\mathbf{z}, \nu] = v \quad \textit{per a quasi tot} \ \ x \in \partial\Omega\,, \\
&\textbf{(iv)} \quad [\mathbf{z}, \nu] \in \operatorname{sign}(\omega - u) \quad \textit{per a quasi tot} \ \ x \in \partial\Omega\,.
\end{aligned}$$

Cal destacar que l'equació $u - \operatorname{div}\left(\frac{Du}{|Du|}\right) = 0$ amb la condició de Dirichlet $u = \omega$ té una única solució i per tant, la funció u és única. No obstant això, el camp vectorial $\mathbf{z}$ no està unívocament determinat i tampoc ho està la funció v.

Amb aquesta definició, al llarg de la Secció 3.3 provem que l'operador $\mathcal{B}$ és m-acretiu en l'espai $L^2(\partial\Omega)$ i el seu domini és dens en $L^2(\partial\Omega)$, és a dir, $L^2(\partial\Omega) = \overline{D(\mathcal{B})}$ i per tant, hi ha una única solució de semigrups $\omega \in C([0,T]; L^2(\partial\Omega))$ del problema de Cauchy (5), per a dades $g \in L^1(0,T; L^2(\partial\Omega))$ i $\omega_0 \in L^2(\partial\Omega)$.

A més, en aquesta secció també hem provat un principi de comparació entre solucions de semigrups enunciat a continuació:

Teorema. *Si denotem per* ω^1 *i* $\omega^2 \in C([0,T]; L^2(\partial\Omega))$ *a les solucions de semigrups del problema de Cauchy* (5) *amb les dades* g_1 *i* $g_2 \in L^1(0,T; L^2(\partial\Omega))$ *i* ω_0^1 *i* $\omega_0^2 \in L^2(\partial\Omega)$ *respectivament, i a més*

$$\begin{aligned}
&g^1(t,x) \leq g^2(t,x)\,, \qquad \textit{per a quasi tot} \ \ (t,x) \in (0,T) \times \partial\Omega\,, \\
&i \ \ \omega_0^1(x) \leq \omega_0^2(x)\,, \qquad \textit{per a quasi tot} \ \ x \in \partial\Omega\,,
\end{aligned}$$

aleshores la desigualtat també és certa per a les solucions de semigrups:

$$\omega^1(t,x) \leq \omega^2(t,x)\,, \qquad \textit{per a quasi tot} \ \ (t,x) \in (0,T) \times \partial\Omega\,.$$

En la quarta secció provem els dos resultats principals d'aquest capítol: l'existència d'una única solució global forta del problema (3) i un principi de comparació.

Respecte al problema (4), la solució forta està formada per un parell de funcions (u, ω) i la definició és la següent:

Definició. *Diem que el parell* (u, ω) *és una solució forta del problema* (4) *si* $u \in L^\infty(0,T;BV(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ *i* $\omega \in C([0,T];L^2(\partial\Omega)) \cap W^{1,1}(0,T;L^2(\partial\Omega))$ *i és tal que* $\omega(0) = \omega_0$. *A més existeix un camp vectorial* $\mathbf{z} \in L^\infty((0,T) \times \Omega;\mathbb{R}^N)$ *amb* $\|\mathbf{z}\|_{L^\infty((0,T)\times\Omega)} \leq 1$ *que satisfà les següents condicions:*

$$\begin{aligned}
&\textbf{(i)} \quad \lambda u(t) - \operatorname{div}(\mathbf{z}(t)) = 0 \quad \textit{en} \quad \mathcal{D}'(\Omega)\,, \\
&\textbf{(ii)} \quad (\mathbf{z}(t), Du(t)) = |Du(t)| \quad \textit{com a mesures a} \quad \Omega\,, \\
&\textbf{(iii)} \quad [\mathbf{z}(t), \nu] = g(t) - \omega_t(t) \quad \textit{per a quasi tot} \quad x \in \partial\Omega\,, \\
&\textbf{(iv)} \quad [\mathbf{z}(t), \nu] \in \operatorname{sign}(\omega(t) - u(t)) \quad \textit{per a quasi tot} \quad x \in \partial\Omega\,,
\end{aligned}$$

per a quasi tot $t \in (0,T)$.

Diem que la solució del problema (4) és forta ja que totes aquestes condicions es complixen puntualment per a quasi tot $t \in (0,T)$.

S'enuncia a continuació el teorema que prova l'existència d'una única solució del problema (4).

Teorema. *Donat* $\lambda > 0$ *i donades les funcions* $g \in L^1(0,T;L^2(\partial\Omega))$ *i* $\omega_0 \in L^2(\partial\Omega)$, *hi ha una única solució forta del problema* (4). *A més, es complixen les estimacions següents:*

$$\|\omega\|_{L^\infty(0,T;L^2(\partial\Omega))} \leq \|\omega_0\|_{L^2(\partial\Omega)} + \|g\|_{L^1(0,T;L^2(\partial\Omega))}\,, \qquad \textit{per a tot} \;\; T > 0\,,$$

$$\lambda \|u(t)\|^2_{L^2(\Omega)} + \|u(t)\|_{BV(\Omega)} \leq \|\omega(t)\|_{L^1(\partial\Omega)}\,, \qquad \textit{per a quasi tot} \;\; t > 0\,.$$

D'altra banda, si considerem el problema (3) i les dades $\omega_0 \in L^2(\partial\Omega)$ i $g \in L^1_{loc}(0,+\infty; L^2(\partial\Omega))$, la solució (u,ω) és global si és solució forta del problema (4) per a tot $T > 0$. Per tant, com el problema (4) té una única solució, també hi ha una única solució global forta del problema (3).

És important remarcar que la regularitat de les funcions que conformen la solució millora si triem les dades g i ω_0 en espais més regulars. En particular, si triem una funció g de l'espai $L^\infty(0,+\infty; L^2(\partial\Omega))$, deduïm que $\omega_t \in L^\infty(0,+\infty; L^2(\partial\Omega))$ ja que es complix la igualtat $\omega_t(t) = [\mathbf{z}(t), u] + g(t)$ en $\partial\Omega$. Per tant, la solució ω és Lipschitz respecte a la variable t.

Utilitzant el principi de comparació per a les solucions de semigrups i certs resultats de convergències que provem en la demostració d'existència de solució del problema (4), també demostrem un principi de comparació per a les solucions (u,ω). El resultat és el següent:

Teorema. *Siguen les funcions g^1 i $g^2 \in L^1(0,T; L^2(\partial\Omega))$ i ω_0^1 i $\omega_0^2 \in L^2(\partial\Omega)$ tals que*

$$g^1(t,x) \le g^2(t,x)\,, \qquad \textit{per a quasi tot } \ (t,x) \in (0,T)\times\partial\Omega\,,$$
$$\textit{i} \ \ \omega_0^1(t,x) \le \omega_0^2(t,x)\,, \qquad \textit{per a quasi tot } \ x \in \partial\Omega\,.$$

Si denotem per (u^1,ω^1) i (u^2,ω^2) a les respectives solucions del problema (4), aleshores es complix

$$\omega^1(t,x) \le \omega^2(t,x)\,, \qquad \textit{per a quasi tot } \ (t,x) \in (0,T)\times\partial\Omega\,,$$
$$\textit{i} \ \ u^1(t,x) \le u^2(t,x)\,, \qquad \textit{per a quasi tot } \ (t,x) \in (0,T)\times\Omega\,.$$

Acabem la secció provant un resultat sobre el comportament a llarg termini de la solució (u,ω) del problema (3). En particular, veiem que

ω convergeix feblement a una funció de quadrat integrable sobre $\partial\Omega$ i la funció u convergeix feblement en $L^2(\Omega)$ i fortament en $L^1(\Omega)$ a una certa funció v de l'espai $BV(\Omega) \cap L^2(\Omega)$. A més, Du convergix a Dv feble-$*$ com a mesures en Ω.

Finalment, en la Secció 3.5 del Capítol 3 mostrem un resultat que compara les solucions del problema (3) amb diferents dades. En particular, el resultat obtingut ens permet estimar la distància entre solucions en relació a la distància entre les dades.

Teorema. *Si* (u_1, ω_1) *i* (u_2, ω_2) *són solucions del problema* (3) *amb les dades* g_1*,* $g_2 \in L^1(0,T; L^2(\partial\Omega))$ *i* ω_{01}*,* $\omega_{02} \in L^2(\partial\Omega)$ *respectivament, aleshores es complixen les desigualtats següents:*

$$\|\omega_1 - \omega_2\|_{L^\infty(0,T;L^2(\partial\Omega))} \leq \|\omega_{01} - \omega_{02}\|_{L^2(\partial\Omega)} + \|g_1 - g_2\|_{L^1(0,T;L^2(\partial\Omega))} \,,$$

$$\lambda\|u_1 - u_2\|^2_{L^2(0,T;L^2(\Omega))} \leq \frac{1}{2}\|\omega_{01} - \omega_{02}\|^2_{L^2(\partial\Omega)} + \frac{1}{2}\|g_1 - g_2\|^2_{L^1(0,T;L^2(\partial\Omega))} \,.$$

Els resultats que apareixen en aquest capítol estan publicats en el següent article:

M. Latorre and S. Segura de León, Elliptic 1-Laplacian equations with dynamical boundary conditions, *J. Math. Anal. Appl.* **464** (2018), no. 2, 1051–1081. DOI: 10.1016/j.jmaa.2018.02.006

Resumen

Esta tesis pretende dar a conocer los resultados obtenidos en el estudio de la existencia, unicidad y regularidad de las soluciones de diferentes ecuaciones elípticas regidas por el operador 1-Laplaciano.

La tesis empieza con una breve introducción y con distintas explicaciones relativas a la notación, definiciones básicas y propiedades elementales de las herramientas utilizadas a lo largo de este trabajo.

El primer capítulo está dedicado al estudio del siguiente problema de Dirichlet:

$$\begin{cases} -\operatorname{div}\left(\dfrac{Du}{|Du|}\right) + |Du| = f(x) & \text{en } \Omega\,, \\ u = 0 & \text{sobre } \partial\Omega\,, \end{cases} \tag{1}$$

donde Ω es un abierto acotado de $\mathbb{R}^N$ con frontera $\partial\Omega$ Lipschitz y el dato f es una función del espacio de Marcinkiewicz $L^{N,\infty}(\Omega)$.

La motivación del estudio de este problema viene dada por un artículo de J.M. Mazón y S. Segura de León (ver [55]), en el que prueban la existencia de soluciones acotadas cuando se eligen los datos en el espacio de Lebesgue $L^q(\Omega)$ con $q > N$.

En problemas similares en los que en lugar del 1-Laplaciano tenemos el operador p-Laplaciano con $1 < p < \infty$, el espacio natural en el que se encuentran las soluciones (siguiendo un punto de vista variacional) es el espacio de Sobolev $W_0^{1,p}(\Omega)$, y los datos pertenecen a su dual,

es decir, $W^{-1,p'}(\Omega)$ siendo $p' = \frac{p}{p-1}$. En nuestro caso particular en el que $p = 1$, deberíamos encontrar soluciones en el espacio de Sobolev $W_0^{1,1}(\Omega)$ cuando tomamos como dato una función del espacio $W^{-1,\infty}(\Omega)$. No obstante, gracias al teorema de inmersión de Sobolev y utilizando argumentos de dualidad, se observa que, para obtener soluciones débiles, el mejor espacio en el que podemos escoger los datos entre los espacios de Lebesgue es $L^N(\Omega)$ y entre los espacios de Lorentz es $L^{N,\infty}(\Omega)$. Nuestro objetivo es mejorar los resultados de existencia y unicidad obtenidos en [55] tomando los datos en el espacio óptimo.

Hemos indicado que el espacio de energía debería ser el espacio de Sobolev $W_0^{1,1}(\Omega)$, sin embargo, y contrariamente a lo que ocurre con los espacios $W_0^{1,p}(\Omega)$ con $1 < p < \infty$, este espacio no es reflexivo. Es por ello que en los problemas regidos por el operador 1-Laplaciano trabajamos en un espacio mayor y con mejores propiedades: el espacio de las funciones de variación acotada, que denotaremos por $BV(\Omega)$ y está formado por el conjunto de todas las funciones integrables cuya derivada en el sentido de las distribuciones es una medida de Radon con variación total finita.

La primera dificultad que encontramos al enfrentarnos a una ecuación donde aparece el operador 1-Laplaciano es definir la solución del problema. En particular, hemos de darle sentido al cociente $\frac{Du}{|Du|}$, siendo $|Du|$ una medida. En [8], F. Andreu, C. Ballester, V. Caselles y J.M. Mazón solventaron este problema utizando un campo vectorial $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^N)$ que desempeña el papel del cociente $\frac{Du}{|Du|}$. En particular, necesitamos que $\|\mathbf{z}\|_{L^\infty(\Omega)} \leq 1$ y que el par $(\mathbf{z}, Du)$ esté bien definido y cumpla $(\mathbf{z}, Du) = |Du|$.

El par $(\mathbf{z}, Du)$ fue definido por G. Anzellotti en [13] y por G.-Q. Chen y H. Frid en [26] y es una generalización del producto escalar entre el campo $\mathbf{z}$ y Du. Recordamos que para toda función $\varphi \in C_0^\infty(\Omega)$, Anzellotti define la distribución

$$\langle (\mathbf{z}, Du), \varphi \rangle = -\int_\Omega u\,\varphi \operatorname{div} \mathbf{z} - \int_\Omega \mathbf{z} \cdot \nabla\varphi\, dx\,.$$

En [13] probó que si tomamos un campo vectorial $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^N)$ tal que $\operatorname{div} \mathbf{z}$ es una medida acotada en Ω y la función $u \in BV(\Omega)$ es acotada y continua, entonces $(\mathbf{z}, Du)$ es una medida de Radon con variación total finita.

Cuando tomamos datos en el espacio $L^q(\Omega)$ con $q > N$, las soluciones del problema (1) son funciones del espacio $BV(\Omega) \cap L^\infty(\Omega)$ (ver [55]), pero no son, necesariamente, continuas, luego $(\mathbf{z}, Du)$ no está bien definido. La generalización de esta definición es debida a G.-Q. Chen y H. Frid, visto desde un punto de vista diferente, y a A. Mercaldo, S. Segura de León y C. Trombetti, siguiendo la teoría de Anzellotti. En ambos casos toman un campo vectorial $\mathbf{z}$ en el espacio $\mathcal{DM}^\infty(\Omega)$, es decir, $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^N)$ y además $\operatorname{div} \mathbf{z}$ es una medida de Radon con variación total finita, y una función u del espacio $BV(\Omega) \cap L^\infty(\Omega)$. Utilizando el representante preciso de u (denotado por u^*), probaron que $(\mathbf{z}, Du)$ es una medida de Radon de variación total finita. No obstante, como tomamos el dato $f \in L^{N,\infty}(\Omega)$ en el problema (1), las soluciones que obtenemos son no acotadas y no podemos utilizar esta definición del par $(\mathbf{z}, Du)$. Es por ello que necesitamos el resultado que se enuncia a continuación y que se prueba al final de la Sección 1.4.

Teorema. *Sea u una función del espacio $BV(\Omega)$ y $\mathbf{z}$ un campo vectorial de $\mathcal{DM}^\infty(\Omega)$ con $\operatorname{div} \mathbf{z} = \xi + f$ donde ξ es una medida de Radon tal que $\xi \geq 0$ o $\xi \leq 0$ y la función f es del espacio de Marcinkiewicz $L^{N,\infty}(\Omega)$. Entonces, dado $\varphi \in C_0^\infty(\Omega)$, la distribución definida por*

$$\langle (\mathbf{z}, Du), \varphi \rangle = -\int_\Omega u^*\,\varphi \operatorname{div} \mathbf{z} - \int_\Omega u\,\mathbf{z} \cdot \nabla\varphi\, dx$$

es una medida de Radon con variación total finita y verifica

$$|(\mathbf{z}, Du)| \leq \|\mathbf{z}\|_{L^\infty(\Omega)}|Du| \,.$$

Además, asumiendo las hipótesis antes mencionadas también hemos probado la siguiente generalización de la fórmula de Green:

$$\int_\Omega u^* \operatorname{div} \mathbf{z} + \int_\Omega (\mathbf{z}, Du) = \int_{\partial\Omega} u \, [\mathbf{z}, \nu] \, d\mathcal{H}^{N-1} \,,$$

donde $[\mathbf{z}, \nu]$ denota la traza débil sobre la frontera $\partial\Omega$ de la componente normal de $\mathbf{z}$, definida por Anzellotti en [13].

Finalmente, en la Sección 1.5 hemos probado el resultado de existencia de una única función u no negativa que es solución del problema (1) cuando tomamos como dato una función $f \geq 0$ del espacio de Marcinkiewicz $L^{N,\infty}(\Omega)$. Aunque las soluciones del problema de Dirichlet (1) no son necesariamente acotadas, sí tienen cierta regularidad ya que son funciones de variación acotada sin parte de salto, es decir, $D^j u = 0$. Además, necesitamos un campo vectorial $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ que $\mathbf{z}$ actuará como el cociente $\frac{Du}{|Du|}$ en la ecuación. La definición de solución es la siguiente:

Definición. *Dada $f \in L^{N,\infty}(\Omega)$ con $f \geq 0$. Decimos que una función $u \in BV(\Omega)$ sin parte de salto, es decir, si $D^j u = 0$ es una solución débil del problema (1) si existe un campo vectorial $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ con $\|\mathbf{z}\|_{L^\infty(\Omega)} \leq 1$ y tal que*

(i) $-\operatorname{div} \mathbf{z} + |Du| = f \quad \textit{en} \quad \mathcal{D}'(\Omega) \,,$

(ii) $(\mathbf{z}, Du) = |Du| \quad \textit{como medidas en} \quad \Omega \,,$

(iii) $u|_{\partial\Omega} = 0 \,.$

Presentamos a continuación el resultado principal de esta sección.

Teorema. *Dado Ω un abierto acotado de $\mathbb{R}^N$ con frontera Lipschitz y dada una función $f \in L^{N,\infty}(\Omega)$ con $f \geq 0$, existe una única solución débil no negativa del problema* (1)*.*

Además, cuando tomamos una función f de norma pequeña siempre obtenemos la solución nula, como muestra el siguiente resultado:

Proposición. *Sea $u \in BV(\Omega)$ la solución no negativa del problema* (1) *con el dato $0 \leq f \in L^{N,\infty}(\Omega)$. Entonces, $u \equiv 0$ si y solo si $\|f\|_{W^{-1,\infty}(\Omega)} \leq 1$.*

Es habitual encontrar resultados similares cuando se trabaja con el operador 1-Laplaciano. En particular, en [57], se probó que la solución del problema de Dirichlet con la ecuación $-\mathrm{div}\left(\frac{Du}{|Du|}\right) = f$ es nula si $\|f\|_{W^{-1,\infty}(\Omega)} < 1$ y no existe solución del problema cuando $\|f\|_{W^{-1,\infty}(\Omega)} > 1$. Cuando se da la igualdad, $u = 0$ siempre es solución aunque también puede existir una solución no nula para ciertos datos.

Finalizamos la sección con un resultado sobre la regularidad de las soluciones. Si tomamos como dato una función $f \in L^q(\Omega)$ con $q > N$, la solución del problema (1) siempre es acotada (ver [55]). Sin embargo, en el caso límite, cuando elegimos el dato f en el espacio de Marcinkiewicz $L^{N,\infty}(\Omega)$, la solución u es del espacio de Lebesgue $L^q(\Omega)$ para todo $1 \leq q < \infty$. Para probar que, en efecto, las soluciones del problema (1) con datos no negativos del espacio $L^{N,\infty}(\Omega)$ no son, necesariamente, acotados, mostramos el siguiente ejemplo.

Ejemplo. *Sean $0 < \rho < R$ y $0 < \lambda < N-1$. Si consideramos $\Omega = B_R(0)$, la solución del problema* (1) *con dato $\frac{\lambda}{|x|}\chi_{B_\rho(0)}(x) \in L^{N,\infty}(\Omega) \setminus L^N(\Omega)$ viene dada por*

$$u(x) = (N-1-\lambda)\log\left(\frac{|x|}{\rho}\right), \quad |x| < \rho,$$

que pertenece a los espacios de Lebesgue $L^q(\Omega)$ para $1 \le q < \infty$ pero es una función no acotada.

Las últimas secciones del Capítulo 1 están dedicadas al estudio de una generalización del problema (1), al que hemos añadido una función g en el término del gradiente. Es decir, estudiamos el siguiente problema de Dirichlet:

$$\begin{cases} -\mathrm{div}\left(\dfrac{Dv}{|Dv|}\right) + g(v)|Dv| = f(x) & \text{en } \Omega\,, \\ v = 0 & \text{sobre } \partial\Omega\,, \end{cases} \tag{2}$$

donde Ω es un abierto acotado de $\mathbb{R}^N$ con frontera $\partial\Omega$ Lipschitz, f es una función no negativa del espacio de Marcinkiewicz $L^{N,\infty}(\Omega)$ y la función $g : [0,\infty[\to [0,\infty[$ es continua y no negativa.

Nuestro objetivo es ver cómo afecta la función g a los resultados de existencia, unicidad y regularidad de soluciones. El término con la variación total es esencial para la unicidad de solución ya que si consideramos el mismo problema pero sin la variación total $-\mathrm{div}\left(\frac{Dv}{|Dv|}\right) = f$, y denotamos por v a su solución, la función $h(v)$ también debe ser solución para toda función h suave y creciente.

Además, en [9] probaron que una ecuación similar al problema (1) pero en la que tampoco aparece el término con la variación total, no cumple la misma regularidad que las soluciones de (1). En particular probaron que la ecuación $u - \mathrm{div}\left(\frac{Du}{|Du|}\right) = f(x)$ tiene una única solución aunque dicha solución puede tener parte de salto.

A lo largo de las Secciones 1.7, 1.8 y 1.9 del Capítulo 1 vemos que dependiendo de las características de la función g, la solución del problema (2) satisface diferentes propiedades. Además, en los casos más extremos, cuando la función g se anula, también tenemos que modificar el concepto de solución ya que la solución no es, necesariamente, una función de variación acotada.

Independientemente de las propiedades de la función g, necesitamos definir la función auxiliar

$$G(s) = \int_0^s g(\sigma)\, d\sigma\,.$$

En la Sección 1.7 estudiamos las condiciones de la función g bajo las cuales la solución v satisface mejores propiedades. En particular, introducimos la definición de solución del problema (2) cuando tenemos una función $g : [0,\infty[\to [0,\infty[$ continua y tal que $g(s) \geq m > 0$ para todo $s \geq 0$.

Definición. *Decimos que v es una solución débil del problema (2) si $v \in BV(\Omega)$ con $D^j v = 0$ y existe un campo vectorial $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ con $\|\mathbf{z}\|_{L^\infty(\Omega)} \leq 1$ y tal que*

$$\begin{aligned}
&\textbf{(i)} \quad -\operatorname{div}\mathbf{z} + g(v)^*|Dv| = f \quad \textit{en} \;\; \mathcal{D}'(\Omega)\,,\\
&\textbf{(ii)} \quad (\mathbf{z}, Dv) = |Dv| \;\; \textit{como medidas en} \;\; \Omega\,,\\
&\textbf{(iii)} \quad v|_{\partial\Omega} = 0\,.
\end{aligned}$$

Es importante destacar que en este caso, la solución v cumple las mismas propiedades de regularidad que las soluciones del problema (1). Sólo afecta la función g en la igualdad distribucional.

Cuando escogemos una función g continua y tal que $g(s) \geq m > 0$, hemos de distinguir dos casos para probar la existencia de solución. Si la función g es acotada, utilizamos los resultados enunciados a continuación.

Teorema.

(i) *Sea u la solución no negativa del problema (1). Entonces la función v tal que $u = G(v)$ es solución del problema (2).*

(ii) *Sea v una solución no negativa del problema (2). Entonces la función $u = G(v)$ es la solución del problema (1).*

En general, para probar la existencia de solución se necesita utilizar la regla de la cadena para funciones de variación acotada. En [7] probaron que si $v \in BV(\Omega)$ tal que $D^j v = 0$ y $\psi : \mathbb{R} \to \mathbb{R}$ es una función Lipschitz, entonces $\psi(v) \in BV(\Omega)$ y además $D\psi(v) = \psi'(v)Dv$. Sin embargo, no siempre podemos utilizar este resultado ya que, con nuestras hipótesis, ψ' no es, necesariamente, acotada. Es por ello que también hemos probado una ligera generalización de la regla de la cadena:

Proposición. *Sea v una función de variación acotada sin parte de salto y sea g una función real continua y no negativa. Si $u = G(v) \in L^1(\Omega)$, entonces $u \in BV(\Omega)$ si y solo si $g(v)|Dv|$ es una medida finita. Además, $|Du| = g(v)|Dv|$ como medidas en Ω.*

Cuando la función $g(s)$ está separada del eje s pero es no acotada, las pruebas de los resultados anteriores no funcionan. En este caso, para probar la existencia de solución utilizamos una sucesión de problemas aproximantes. Para cada $n \in \mathbb{N}$, consideramos el problema (2) con una función $g_n(s)$ tal que existe una solución v_n del problema y además $g_n(s)$ converge a $g(s)$. Hay que probar que la sucesión de soluciones $\{v_n\}$ es convergente y el límite es la solución de (2) con la función $g(s)$.

El resultado principal de la Sección 1.7 se enuncia a continuación:

Teorema. *Sea Ω un abierto acotado de $\mathbb{R}^N$ con frontera Lipschitz y sea $f \in L^{N,\infty}(\Omega)$ con $f \geq 0$. Si g es una función real y continua y además cumple $g(s) \geq m > 0$ para todo $s \geq 0$, entonces existe una única solución no negativa del problema (2). Además, dicha solución pertenece a los espacios de Lebesgue $L^q(\Omega)$ para todo $1 \leq q < \infty$.*

La Sección 1.8 está dedicada al estudio del problema (2) cuando la función $g : [0, \infty[\to [0, \infty[$ puede ser nula en algún punto. En particular, trabajamos con una función g continua, acotada, no integrable y tal que $g(s) > 0$ para casi todo $s \geq 0$.

El primer resultado que probamos en esta sección muestra que existe una única solución del problema (2) cuando la función g cumple, además, la siguiente restricción:

(C) *Existen* $m, \sigma > 0$ *tales que* $g(s) \geq m > 0$ *para todo* $s \geq \sigma$.

Tomando una función $g(s)$ con las condiciones indicadas, la prueba de la existencia de solución se basa en tomar una aproximación con funciones $g_n(s)$ que están separadas del eje de abcisas. Cuando no se cumple la condición (C), hemos de seguir utilizando una aproximación con funciones $g_n(s)$ separadas del eje s, sin embargo, el límite de las soluciones de estos problemas ya no es, necesariamente, una función de variación acotada. Utilizamos el siguiente ejemplo para mostrar que la solución del problema (2) con una función g tal que $\lim_{s\to\infty} g(s) = 0$ no es de variación acotada.

Ejemplo. *Sean* $R > 0$ *y* $\lambda > 2N - 2$. *Si consideramos* $\Omega = B_R(0)$, *el dato* $f(x) = \frac{\lambda}{|x|}$ *y la función* $g(s) = \frac{1}{1+s}$, *la solución del problema* (2) *viene dada por*

$$v(x) = \left(\frac{|x|}{R}\right)^{N-1-\lambda} - 1\,,$$

que no es de variación acotada ya que $|Dv| = \frac{\lambda-N+1}{R^{N-1-\lambda}}|x|^{N-2-\lambda}$ *no es una función integrable.*

Por lo tanto, las propiedades de la solución del problema (2) cambian cuando tomamos una función $g : [0,\infty[\to [0,\infty[$ continua, acotada, no integrable y tal que $g(s) > 0$ para casi todo $s \geq 0$. La nueva definición de solución es la siguiente:

Definición. *Decimos que una función* v *es solución débil del problema* (1.43) *si* $G(v) \in BV(\Omega)$ *con* $D^j G(v) = 0$ *y existe un campo vectorial*

$\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ *con* $\|\mathbf{z}\|_{L^\infty(\Omega)} \leq 1$ *y tal que*

$$\begin{aligned} &\textbf{(i)} \quad -\operatorname{div} \mathbf{z} + g(v)^*|Dv| = f \quad \text{en} \quad \mathcal{D}'(\Omega)\,, \\ &\textbf{(ii)} \quad (\mathbf{z}, DG(v)) = |DG(v)| \quad \text{como medidas en} \quad \Omega\,, \\ &\textbf{(iii)} \quad G(v)|_{\partial\Omega} = 0\,. \end{aligned}$$

Aunque la solución v no es de variación acotada, sí lo es la función $G(v)$, que además no debe tener parte de salto. Además, como la función u no es de variación acotada, el par $(\mathbf{z}, Dv)$ no está bien definido. Es por ello que utilizamos $(\mathbf{z}, G(v))$ en su lugar.

El siguiente resultado prueba que el problema (2) tiene una única solución que cumple la nueva definición.

Teorema. *Sea* Ω *un abierto acotado de* $\mathbb{R}^N$ *con frontera Lipschitz y sea* $f \in L^{N,\infty}(\Omega)$ *una función no negativa. Entonces existe una única solución del problema* (2) *en el sentido de la definición anterior cuando elegimos una función* $g : [0, \infty[\to [0, \infty[$ *continua, acotada, no integrable y tal que* $g(s) > 0$ *para casi todo* $s > 0$.

La última sección del primer capítulo está dedicada al estudio de algunos casos particulares en los que, según las propiedades de la función g, no tenemos necesariamente existencia o unicidad de solución, o estas pueden tener parte de salto o no cumplir la condición sobre la frontera.

Suponiendo que la función g es integrable, la existencia de solución viene determinada por el dato f. En particular, cuando la norma de f en el espacio dual de $W_0^{1,1}(\Omega)$ es menor que 1, la solución del problema (2) siempre es nula. Sin embargo, cuando dicha norma es mayor que cierta constante, el problema no tiene solución. Enunciamos el resultado a continuación.

Proposición. *Sea* $f \in L^{N,\infty}(\Omega)$ *con* $f \geq 0$ *y sea* $g \in L^1([0, \infty[)$. *Entonces,*

(i) *si* $\|f\|_{W^{-1,\infty}(\Omega)} < 1$, *la solución del problema* (2) *es trivial;*

(ii) *si* $\|f\|_{W^{-1,\infty}(\Omega)} > e^{G(\infty)}$, *el problema* (2) *no tiene solución;*

siendo $G(\infty) = \sup\{G(s) \, : \, s \in]0, \infty[\}$.

Por último, también hemos estudiado las propiedades de la solución del problema (2) cuando la función g se anula en un intervalo. En particular, hemos probado que nunca tenemos unicidad de solución y además, aunque existe solución, esta puede tener parte de salto o incluso no cumplir la condición en la frontera, tal y como muestra el siguiente ejemplo:

Ejemplo. *Dados* $R > 0$ *y el dato* $f(x) = \frac{N}{|x|}$, *si tomamos* $\Omega = B_R(0)$ *y la función* $g : [0, \infty \to [0, \infty[$ *definida por*

(a) $g(s) = 0$ *si* $0 \le s \le a$ *y* $g(s) = s - a$ *si* $s > a$, *entonces la solución* u *no se anula en la frontera, aunque sí se cumple la condición frontera en un sentido débil, es decir,* $[\mathbf{z}, \nu] = -\operatorname{sign}(u)$.

(b) $g(s) = a - s$ *si* $0 \le s < a$, $g(s) = 0$ *si* $a \le s \le b$ *y* $g(s) = s - b$ *si* $s > b$, *entonces la solución del problema* (2) *tiene parte de salto.*

Los resultados que aparecen en el primer capítulo están publicados en el siguiente artículo.

M. Latorre and S. Segura de León, Elliptic equations involving the 1-Laplacian and a total variation term with $L^{N,\infty}$-data, *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* **28** (2017), no. 4, 817–859. DOI: 10.4171/RLM/787

El segundo capítulo de esta tesis está dedicado a generalizar los resultados obtenidos en el estudio del problema de Dirichlet (1) cuando tomamos como dato una función no negativa del espacio de las funciones integrables $L^1(\Omega)$.

Cuando escogemos un dato $f \in L^1(\Omega)$, y al igual que sucede cuando los datos son funciones del espacio de Marcinkiewicz, necesitamos generalizar la teoría de Anzellotti definiendo el par $(\mathbf{z}, Du)$, ya que si u es solución del problema se debe cumplir la igualdad $(\mathbf{z}, Du) = |Du|$ como medidas.

Es por ello que necesitamos que $(\mathbf{z}, Du)$ sea una medida de Radon con variación total finita. No obstante, al no ser las soluciones a este problema necesariamente acotadas, no podemos utilizar los resultados de [13], y como además f es una función del espacio de Lebesgue $L^1(\Omega)$, tampoco podemos probar que $(\mathbf{z}, Du)$ sea una medida de Radon con variación total finita utilizando los argumentos del Capítulo 1.

Por lo tanto, hemos de modificar el concepto de solución utilizando truncamientos, tal y como hicieron F. Andreu, C. Ballester, V. Caselles y J.M. Mazón en [8] cuando definieron la solución del problema parabólico $u_t = \operatorname{div}\left(\frac{Du}{|Du|}\right)$.

El uso de los truncamientos en la definición de solución es debido a que consideramos datos en el espacio $L^1(\Omega)$. En algunos problemas similares en los que, en lugar del 1-Laplaciano tenemos el operador p-laplaciano con $1 < p \leq N$, ya se han usado los truncamientos, tanto en soluciones renormalizadas como en soluciones de entropía (ver [30] y [16] respectivamente).

Si denotamos por $T_k(s) = \min\{|s|, k\} \operatorname{sign}(s)$ a la función truncamiento, la definición de solución del problema (1) es la siguiente:

Definición. *Decimos que* $u \in BV(\Omega)$ *es una solución del problema* (1) *si* $D^j u = 0$ *y existe un campo vectorial* $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ *con* $\|\mathbf{z}\|_{L^\infty(\Omega)} \leq 1$ *y tal que*

(i) $-\operatorname{div} \mathbf{z} + |Du| = f \quad \textit{en} \quad \mathcal{D}'(\Omega)\,,$

(ii) $(\mathbf{z}, DT_k(u)) = |DT_k(u)| \quad \textit{como medidas en} \quad \Omega \ \textit{(para todo } k > 0)\,,$

(iii) $u|_{\partial\Omega} = 0$.

Hay que remarcar que la función $T_k(u)$ es del espacio $BV(\Omega) \cap L^\infty(\Omega)$ y por lo tanto $(\mathbf{z}, DT_k(u))$ está bien definido y es una medida de Radon con variación total finita.

Por otro lado, es importante destacar que en [9] estudian una ecuación similar con datos en el espacio $L^1(\Omega)$ pero sin el término de la variación total. Al igual que nosotros, definen la solución utilizando los truncamientos pero al no tener la variación total, la solución pierde regularidad. En particular, la condición sobre la frontera se cumple en un sentido débil, no siendo así en nuestro caso.

Enunciamos a continuación el resultado principal de esta sección que prueba la existencia de solución en el sentido de la definición anterior.

Teorema. *Si f es una función integrable no negativa, entonces existe al menos una solución del problema* (1) *en el sentido de la definición anterior.*

Por otro lado, en cuanto a la unicidad de solución, en la Sección 2.2 del Capítulo 2 hemos probado un principio de comparación que no solo mejora el resultado de unicidad de solución, sino que también simplifica su demostración cuando utilizamos datos del espacio de Lebesgue $L^q(\Omega)$ con $q > N$. El resultado es el siguiente:

Teorema. *Sea Ω un abierto acotado de $\mathbb{R}^N$ con frontera Lipschitz y sean f_1 y f_2 dos funciones del espacio $L^1(\Omega)$ tales que $0 \le f_1 \le f_2$. Si u_1 y u_2 son soluciones del problema* (1) *con datos f_1 y f_2 respectivamente, entonces $u_1 \le u_2$.*

Hay que destacar que aunque $(\mathbf{z}, Du)$ no esté bien definido, probamos que el par $(e^{-u}\mathbf{z}, Du)$, y también $(\mathbf{z}, De^{-u})$, es una medida de Radon con variación total finita. Ambas expresiones son imprescindibles tanto en la prueba de la existencia de solución como en la demostración del principio de comparación.

Finalmente, y gracias al principio de comparación, queda probada la unicidad de solución:

Teorema. *Sea Ω un abierto acotado de $\mathbb{R}^N$ con frontera Lipschitz y sea $f \in L^1(\Omega)$ una función no negativa. Entonces existe una única solución del problema* (1).

La Sección 2.3 de este capítulo está destinada al estudio de la regularidad de las soluciones del problema (1) cuando tomamos los datos en el espacio de Lebesgue $L^q(\Omega)$ con $1 < q < N$. Como ya se ha indicado en el Capítulo 1, si el dato pertenece a $L^N(\Omega)$, la solución es del espacio $L^q(\Omega)$ para todo $1 \leq q < \infty$ y si el dato es de $L^q(\Omega)$ con $q > N$, por [55] sabemos que la solución es acotada.

El resultado principal de esta sección es el siguiente:

Teorema. *Sea Ω un abierto acotado de $\mathbb{R}^N$ con frontera Lipschitz. Si tomamos una función $f \geq 0$ del espacio de Lebesgue $L^q(\Omega)$ con $1 < q < N$, entonces la única solución u del problema* (1) *satisface $u \in BV(\Omega) \cap L^{\frac{Nq}{N-q}}(\Omega)$.*

En particular, hemos probado que la máxima regularidad que alcanzan las soluciones del problema (1) se ajusta con continuidad en relación a la regularidad del dato f.

Acabamos el capítulo con una sección en la que mostramos que la regularidad de la solución es, en efecto, óptima.

Ejemplo. *Sean $R > 0$, $1 < q < N$ y $\lambda > 0$. Si consideramos $\Omega = B_R(0)$ y el dato $f(x) = \frac{\lambda}{|x|^q}$ del espacio de Lebesgue $L^s(\Omega)$ con $1 \leq s < \frac{N}{q}$,* *entonces la solución del problema* (1) *viene dada por*

$$u(x) = \begin{cases} (N-1)\log\left(\frac{|x|}{\rho_\lambda}\right) + \frac{\lambda}{1-q}(\rho_\lambda^{1-q} - |x|^{1-q}) & \textit{si} \quad 0 \le |x| < \rho_\lambda \\ 0 & \textit{si} \quad \rho_\lambda < |x| \le R\,, \end{cases}$$

para cierto valor $0 < \rho_\lambda < R$. *Es decir, la solución* u *pertenece al espacio de Lebesgue* $L^r(\Omega)$ *para todo* $1 \le r < \frac{N}{q-1}$.

Los resultados que aparecen en este capítulo están publicados en el siguiente artículo.

M. Latorre and S. Segura de León, Existence and comparison results for an elliptic equation involving the 1-Laplacian and L^1-data, *J. Evol. Equ.* **18** (2018), no. 1, 1–28. DOI: 10.1007/s00028-017-0388-0.

Finalmente, en el Capítulo 3 probamos resultados de existencia y unicidad de solución de un problema de evolución. Dicho problema consiste en una ecuación de tipo elíptico en la que aparece el operador 1–laplaciano y una condición dinámica de frontera. El problema es el siguiente:

$$\begin{cases} \lambda u - \operatorname{div}\left(\frac{Du}{|Du|}\right) = 0 & \text{en} \;\; (0,+\infty)\times\Omega\,, \\ \omega_t + \left[\frac{Du}{|Du|},\nu\right] = g(t,x) & \text{sobre} \;\; (0,+\infty)\times\partial\Omega\,, \\ u = \omega & \text{sobre} \;\; (0,+\infty)\times\partial\Omega\,, \\ \omega(0,x) = \omega_0(x) & \text{sobre} \;\; \partial\Omega\,, \end{cases} \tag{3}$$

donde Ω es un abierto acotado de $\mathbb{R}^N$ con frontera $\partial\Omega$ suave, el parámetro λ es un número real positivo, ν representa el vector exterior normal de norma 1, la función g está en el espacio $L^1_{loc}(0,+\infty,L^2(\partial\Omega))$ y el dato inicial ω_0 es una función de cuadrado integrable en la frontera

$\partial\Omega$. Denotamos por ω_t a la derivada distribucional de ω respecto de la variable t.

Este tipo de problemas con condiciones dinámicas de frontera aparecen cuando se modeliza un problema en el que la solución se perturba, no en el interior del dominio, sino en la frontera. En la actualidad, el estudio de ecuaciones elípticas o parabólicas con este tipo de condición sobre la frontera es un área de estudio muy activa, ya que se ajusta a diferentes procesos como pueden ser la transferencia de calor de un fluido en movimiento a un sólido o problemas de termoelasticidad o biología.

En cuanto a problemas con condiciones dinámicas de frontera y el operador 1-Laplaciano, por lo que sabemos, esta es la primera vez que se estudia un problema con estas condiciones. F. Andreu, N. Igbida, J.M. Mazón y J. Toledo estudiaron en [11] un problema similar con una ecuación elíptica en la que aparece el operador p-laplaciano, con $1 < p < \infty$. Para probar la existencia de solución, los autores definen un operador completamente acretivo y utilizando la teoría de semigrupos no lineales obtienen una *mild solution* o solución en el sentido de la teorá de semigrupos, que finalmente prueban que es, realmente, una solución distribucional del problema estudiado.

Para obtener una solución del problema (3) como no podemos probar la existencia utilizando problemas aproximantes, hemos seguido el método de [11], pero adaptándolo a las peculiaridades del 1-Laplaciano.

Primero consideramos el problema (3) restringiéndonos al domino $(0,T)\times\Omega$, siendo $T>0$. Es decir, estudiamos el siguiente problema:

$$\begin{cases} \lambda u - \operatorname{div}\left(\frac{Du}{|Du|}\right) = 0 & \text{en } \ (0,T)\times\Omega\,, \\ \omega_t + \left[\frac{Du}{|Du|},\nu\right] = g(t,x) & \text{sobre } \ (0,T)\times\partial\Omega\,, \\ u = \omega & \text{sobre } \ (0,T)\times\partial\Omega\,, \\ \omega(0,x) = \omega_0(x) & \text{sobre } \ \partial\Omega\,, \end{cases} \tag{4}$$

donde g denota a una función del espacio $L^1(0,T;L^2(\partial\Omega))$.

En la Sección 3.2 de este capítulo presentamos la notación específica y resultados previos necesarios a la hora de probar la existencia de solución de los problemas (3) y (4). En particular, enunciamos los resultados que empleamos para probar la existencia de una solución de semigrupos del problema de Cauchy

$$\begin{cases} \omega_t + \mathcal{B}(\omega) \ni g\,, \\ \omega(0) = \omega_0\,, \end{cases} \tag{5}$$

donde g y ω_0 son funciones de $L^1(0,T;L^2(\partial\Omega))$ y $L^2(\partial\Omega)$ respectivamente y $\mathcal{B}$ denota a un operador en $L^2(\partial\Omega)$. Estas soluciones de semigrupos se obtienen como el límite de las soluciones de las discretizaciones del problema (5).

Ya en la Sección 3.3 probamos que dicha solución de semigrupos existe y es única y para ello, definimos un operador $\mathcal{B}$ en el espacio $L^2(\partial\Omega)$.

Definición. *Sea $\omega \in L^2(\partial\Omega)$. Decimos que $v \in \mathcal{B}(\omega)$ si $v \in L^\infty(\partial\Omega)$ con $\|v\|_{L^\infty(\partial\Omega)} \le 1$ y existe una función $u \in BV(\Omega) \cap L^2(\Omega)$ y un campo vectorial $\mathbf{z} \in L^\infty(\Omega;\mathbb{R}^N)$ con $\|\mathbf{z}\|_{L^\infty(\Omega)} \le 1$ tales que*

$$\begin{aligned}
&\textbf{(i)} \quad \lambda u - \operatorname{div}\mathbf{z} = 0 \quad \textit{en} \quad \mathcal{D}'(\Omega)\,,\\
&\textbf{(ii)} \quad (\mathbf{z}, Du) = |Du| \quad \textit{como medidas en} \quad \Omega\,,\\
&\textbf{(iii)} \quad [\mathbf{z},\nu] = v \; paracasitodo \; x \in \partial\Omega\,,\\
&\textbf{(iv)} \quad [\mathbf{z},\nu] \in \operatorname{sign}(\omega - u) \quad \textit{para casi todo} \quad x \in \partial\Omega\,.
\end{aligned}$$

Hay que destacar que la ecuación $u - \operatorname{div}\left(\frac{Du}{|Du|}\right) = 0$ con la condición de Dirichlet $u = \omega$ tiene una única solución, luego la función u es única. No obstante, el campo vectorial $\mathbf{z}$ no está unívocamente determinado y por tanto, tampoco lo está la función v.

Con esta definición, a lo largo de la Sección 3.3 probamos que el operador $\mathcal{B}$ es m-acretivo en el espacio $L^2(\partial\Omega)$ y su dominio es denso en $L^2(\partial\Omega)$, es decir, $L^2(\partial\Omega) = \overline{D(\mathcal{B})}$ y por lo tanto, existe una única solución de semigrupos $\omega \in C([0,T]; L^2(\partial\Omega))$ del problema de Cauchy (5), para datos $g \in L^1(0,T; L^2(\partial\Omega))$ y $\omega_0 \in L^2(\partial\Omega)$.

Además, en esta sección también hemos probado un principio de comparación entre soluciones de semigrupos enunciado a continuación.

Teorema. *Si denotamos por ω^1 y $\omega^2 \in C([0,T]; L^2(\partial\Omega))$ a las soluciones de semigrupos del problema de Cauchy (5) con los datos g_1 y $g_2 \in L^1(0,T; L^2(\partial\Omega))$ y ω_0^1 y $\omega_0^2 \in L^2(\partial\Omega)$ respectivamente, y además*

$$g^1(t,x) \le g^2(t,x)\,, \qquad \textit{para casi todo } \ (t,x) \in (0,T) \times \partial\Omega\,,$$

$$y \ \ \omega_0^1(x) \le \omega_0^2(x)\,, \qquad \textit{para casi todo } \ x \in \partial\Omega\,,$$

entonces la desigualdad también es cierta para las soluciones de semigrupos:

$$\omega^1(t,x) \le \omega^2(t,x)\,, \qquad \textit{para casi todo } \ (t,x) \in (0,T) \times \partial\Omega\,.$$

En la cuarta sección probamos los dos resultados principales de este capítulo: la existencia de una única solución global fuerte del problema (3) y un principio de comparación.

Con respecto al problema (4), la solución fuerte está formada por un par de funciones (u,ω) y la definición es la siguiente:

Definición. *Decimos que el par (u,ω) es una solución fuerte del problema (4) si u es del espacio $L^\infty(0,T; BV(\Omega)) \cap L^\infty(0,T; L^2(\Omega))$ y $\omega \in C([0,T]; L^2(\partial\Omega)) \cap W^{1,1}(0,T; L^2(\partial\Omega))$ es tal que $\omega(0) = \omega_0$. Además, existe un campo vectorial $\mathbf{z} \in L^\infty((0,T) \times \Omega; \mathbb{R}^N)$ con $\|\mathbf{z}\|_{L^\infty((0,T)\times\Omega)} \le 1$ que satisface las siguientes condiciones:*

(i) $\lambda u(t) - \operatorname{div}(\mathbf{z}(t)) = 0$ *en* $\mathcal{D}'(\Omega)\,,$

(ii) $(\mathbf{z}(t), Du(t)) = |Du(t)|$ *como medidas en* $\Omega\,,$

(iii) $[\mathbf{z}(t), \nu] = g(t) - \omega_t(t)$ *para casi todo* $x \in \partial\Omega\,,$

(iv) $[\mathbf{z}(t), \nu] \in \operatorname{sign}(\omega(t) - u(t))$ *para casi todo* $x \in \partial\Omega\,,$

para casi todo $t \in (0, T)$*.*

Decimos que la solución del problema (4) es fuerte ya que todas estas condiciones se cumplen puntualmente para casi todo $t \in (0, T)$.

Se enuncia a continuación el teorema que prueba la existencia de una única solución del problema (4).

Teorema. *Dado* $\lambda > 0$ *y dadas las funciones* $g \in L^1(0, T; L^2(\partial\Omega))$ *y* $\omega_0 \in L^2(\partial\Omega)$*, existe una única solución fuerte del problema* (4)*. Además, se cumplen las siguientes estimaciones:*

$$\|\omega\|_{L^\infty(0,T;L^2(\partial\Omega))} \leq \|\omega_0\|_{L^2(\partial\Omega)} + \|g\|_{L^1(0,T;L^2(\partial\Omega))}\,, \qquad \textit{para todo } \; T > 0\,,$$

$$\lambda\|u(t)\|^2_{L^2(\Omega)} + \|u(t)\|_{BV(\Omega)} \leq \|\omega(t)\|_{L^1(\partial\Omega)}\,, \qquad \textit{para casi todo } \; t > 0\,.$$

Por otro lado, si consideramos el problema (3) y los datos $g \in L^1_{loc}(0, +\infty; L^2(\partial\Omega))$ y $\omega_0 \in L^2(\partial\Omega)$, la solución (u, ω) es global si es solución fuerte del problema (4) para todo $T > 0$. Consecuentemente, como el problema (4) tiene una única solución, también existe una única solución global fuerte del problema (3).

Es importante remarcar que la regularidad de las funciones que conforman la solución mejora si escogemos los datos g y ω_0 en espacios más regulares. En particular, si escogemos una función g del espacio $L^\infty(0, +\infty; L^2(\partial\Omega))$, deducimos que $\omega_t \in L^\infty(0, +\infty; L^2(\partial\Omega))$ ya que se cumple la igualdad $\omega_t(t) = [\mathbf{z}(t), \nu] + g(t)$ en $\partial\Omega$. Por lo tanto, la solución ω es Lipschitz con respecto a la variable t.

Utilizando el principio de comparación para las soluciones de semigrupos y ciertos resultados de convergencias que probamos en la demostración de existencia de solución del problema (4), también probamos un principio de comparación para las soluciones (u, ω). El resultado es el siguiente:

Teorema. *Sean las funciones* g^1 *y* $g^2 \in L^1(0,T;L^2(\partial\Omega))$ *y* ω_0^1 *y* $\omega_0^2 \in L^2(\partial\Omega)$ *tales que*

$$\begin{aligned} &g^1(t,x) \leq g^2(t,x)\,, \qquad \textit{para casi todo } (t,x) \in (0,T) \times \partial\Omega\,, \\ &y\ \ \omega_0^1(t,x) \leq \omega_0^2(t,x)\,, \qquad \textit{para casi todo } x \in \partial\Omega\,. \end{aligned}$$

Si denotamos por (u^1, ω^1) *y* (u^2, ω^2) *a las respectivas soluciones del problema* (4), *entonces se cumple*

$$\begin{aligned} &\omega^1(t,x) \leq \omega^2(t,x)\,, \qquad \textit{para casi todo } (t,x) \in (0,T) \times \partial\Omega\,, \\ &y\ \ u^1(t,x) \leq u^2(t,x)\,, \qquad \textit{para casi todo } (t,x) \in (0,T) \times \Omega\,. \end{aligned}$$

Acabamos la sección probando un resultado sobre el comportamiento a largo plazo de la solución (u, ω) del problema (3). En particular, vemos que ω converge débilmente a una función de cuadrado integrable sobre $\partial\Omega$ y la función u converge débilmente en $L^2(\Omega)$ y fuertemente en $L^1(\Omega)$ a cierta función v del espacio $BV(\Omega) \cap L^2(\Omega)$. Además, Du converge a Dv débil-$*$ como medidas en Ω.

Por último, en la Sección 3.5 del Capítulo 3 mostramos un resultado que compara las soluciones del problema (3) con diferentes datos. En particular, el resultado obtenido nos permite estimar la distancia entre soluciones en relación a la distancia entre los datos.

Teorema. *Si* (u_1, ω_1) *y* (u_2, ω_2) *son soluciones del problema* (3) *con los datos* g_1, $g_2 \in L^1(0,T;L^2(\partial\Omega))$ *y* ω_{01}, $\omega_{02} \in L^2(\partial\Omega)$ *respectivamente,* *entonces se cumplen las siguientes desigualdades:*

$$\|\omega_1 - \omega_2\|_{L^\infty(0,T;L^2(\partial\Omega))} \leq \|\omega_{01} - \omega_{02}\|_{L^2(\partial\Omega)} + \|g_1 - g_2\|_{L^1(0,T;L^2(\partial\Omega))} \,,$$

$$\lambda\|u_1 - u_2\|^2_{L^2(0,T;L^2(\Omega))} \leq \frac{1}{2}\|\omega_{01} - \omega_{02}\|^2_{L^2(\partial\Omega)} + \frac{1}{2}\|g_1 - g_2\|^2_{L^1(0,T;L^2(\partial\Omega))} \,.$$

Los resultados que aparecen en este capítulo están publicados en el siguiente artículo.

M. Latorre and S. Segura de León, Elliptic 1-Laplacian equations with dynamical boundary conditions, *J. Math. Anal. Appl.* **464** (2018), no. 2, 1051–1081. DOI: 10.1016/j.jmaa.2018.02.006

Abstract

This dissertation is devoted to the study of the existence, uniqueness and regularity of solutions of different elliptic equations involving the 1-Laplacian operator.

We start with a brief introduction to the topic as well as notations, basic definitions and elementary properties of the tools that we are using in this work.

In the first chapter we study the following Dirichlet problem:

$$\begin{cases} -\mathrm{div}\left(\dfrac{Du}{|Du|}\right) + |Du| = f(x) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1}$$

where Ω is a bounded open set of $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$ and the datum f is a function in the Marcinkiewicz space $L^{N,\infty}(\Omega)$.

The motivation to study this kind of problem comes from a work due to J.M. Mazón and S. Segura de León (see [55]) where it is proved that there exit bounded solutions when we take data in the Lebesgue space $L^q(\Omega)$ with $q > N$.

In similar problems driven by the p-Laplacian with $1 < p < \infty$, the natural space in which we find solutions, from a variational point of view, should be the Sobolev space $W_0^{1,p}(\Omega)$ and data in its dual, that is, $W^{-1,p'}(\Omega)$ where $p' = \frac{p}{p-1}$. In our case, when $p = 1$, we should find solutions in the Sobolev space $W_0^{1,1}(\Omega)$ when we take data in $W^{-1,\infty}(\Omega)$.

Nevertheless, thanks to the embedding Sobolev's theorem and using duality arguments, we observe that the best space for taking data is the Lebesgue $L^N(\Omega)$ and Lorentz space $L^{N,\infty}(\Omega)$ in order to get weak solutions. Our main aim is to improve the existence and uniqueness results obtained in [55] taking data in the optimal space.

We have pointed out that the energy space should be the Sobolev space $W_0^{1,1}(\Omega)$. However, and contrary to what happens with the spaces $W_0^{1,p}(\Omega)$ with $1 < p < \infty$, this space is not reflexive. This is the reason to work, in problems with the 1-Laplacian operator, with a larger space with better properties: the space of functions of bounded variation, denoted by $BV(\Omega)$. We say that $u \in BV(\Omega)$ if $u \in L^1(\Omega)$ and its derivative, in the sense of distributions, is a Radon measure with finite total variation.

The first obstacle that we find when we work with an equation with the 1-Laplacian operator is to give the appropriated definition of solution. In particular, we have to give sense to the quotient $\frac{Du}{|Du|}$, where $|Du|$ is a measure. In [8], F. Andreu, C. Ballester, V. Caselles and J.M. Mazón solved this problem using a vector field $\mathbf{z} \in L^\infty(\Omega;\mathbb{R}^N)$ which plays role of the quotient $\frac{Du}{|Du|}$. In particular, we need that $\|\mathbf{z}\|_{L^\infty(\Omega)} \le 1$ and also that the pairing $(\mathbf{z}, Du)$ is well-defined and the equality $(\mathbf{z}, Du) = |Du|$ holds as measures.

The pairing $(\mathbf{z}, Du)$ was defined by G. Anzellotti in [13] and also by G.-Q. Chen and H. Frid in [26], and it is a generalization of the scalar product between the field $\mathbf{z}$ and Du. We recall that for every function $\varphi \in C_0^\infty(\Omega)$, Anzellotti defines the distribution

$$\langle (\mathbf{z}, Du), \varphi \rangle = -\int_\Omega u\,\varphi \operatorname{div} \mathbf{z} - \int_\Omega \mathbf{z} \cdot \nabla\varphi \, dx\,.$$

He proved in [13] that if we take a vector field $\mathbf{z} \in L^\infty(\Omega;\mathbb{R}^N)$ such that $\operatorname{div} \mathbf{z}$ is a bounded measure in Ω and a continuous bounded function

$u \in BV(\Omega)$, then the pairing $(\mathbf{z}, Du)$ is a Radon measure with finite total variation.

When we take data in the space $L^q(\Omega)$ with $q > N$, the solutions of problem (1) are functions of the space $BV(\Omega) \cap L^\infty(\Omega)$ (see [55]) but they are not necessarily continuous and then $(\mathbf{z}, Du)$ is not well-defined. The generalization of this definition is due to G.-Q. Chen and H. Frid, from a different point of view, and to A. Mercaldo, S. Segura de León and C. Trombetti, following Anzellotti's theory. In both cases they use a vector field $\mathbf{z}$ in the space $\mathcal{DM}^\infty(\Omega)$, that is, $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^N)$, and also $\operatorname{div} \mathbf{z}$ is a Radon measure with finite total variation and a function u in the space $BV(\Omega) \cap L^\infty(\Omega)$. Using the precise representative of u (denoted by u^*), they proved that $(\mathbf{z}, Du)$ is a Radon measure with finite total variation. Nevertheless, since we take data $f \in L^{N,\infty}(\Omega)$ in problem (1), the solutions that we obtained are not bounded and we cannot use this definition for the pairing $(\mathbf{z}, Du)$. For that reason we need the following result which we prove at the end of Section 1.4.

Theorem. *Let $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ and $u \in BV(\Omega)$ and assume that $\operatorname{div} \mathbf{z} = \xi + f$ where ξ is a Radon measure satisfying either $\xi \geq 0$ or $\xi \leq 0$ and $f \in L^{N,\infty}(\Omega)$. Then, let $\varphi \in C_0^\infty(\Omega)$, the distribution defined by*

$$\langle (\mathbf{z}, Du), \varphi \rangle = -\int_\Omega u^* \varphi \operatorname{div} \mathbf{z} - \int_\Omega u\, \mathbf{z} \cdot \nabla \varphi \, dx$$

is a Radon measure with finite tota variation and satisfies

$$|(\mathbf{z}, Du)| \leq \|\mathbf{z}\|_{L^\infty(\Omega)} |Du| \,.$$

Moreover, assuming the same hypothesis as before, we prove the following generalization of Green's formula:

$$\int_\Omega u^* \operatorname{div} \mathbf{z} + \int_\Omega (\mathbf{z}, Du) = \int_{\partial\Omega} u \, [\mathbf{z}, \nu] \, d\mathcal{H}^{N-1} \,,$$

where $[\mathbf{z}, \nu]$ denotes the weak trace on the boundary $\partial\Omega$ of the normal component of $\mathbf{z}$ defined by Anzellotti in [13].

Finally, in Section 1.5, taking a function $f \geq 0$ in the Marcinkiewicz space $L^{N,\infty}(\Omega)$, we prove the existence of a unique nonnegative function u which is solution to problem (1). Although the solutions of the Dirichlet problem (1) are not necessarily bounded, we get some regularity since the solutions are in the space of functions of bounded variation without jump part, that is, $D^j u = 0$. In addition, we need a vector field which acts as the quotient $\frac{Du}{|Du|}$ in the equation. The definition of solution is the following:

Definition. *Let $f \in L^{N,\infty}(\Omega)$ with $f \geq 0$. We say that a function $u \in BV(\Omega)$ with $D^j u = 0$ is a weak solution of problem (1) if there exists a vector field $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ with $\|\mathbf{z}\|_{L^\infty(\Omega)} \leq 1$ and such that*

$$\begin{array}{rl} \textbf{(i)} & -\operatorname{div}\mathbf{z} + |Du| = f \quad \textit{in} \;\; \mathcal{D}'(\Omega)\,, \\ \textbf{(ii)} & (\mathbf{z}, Du) = |Du| \quad \textit{as measures in} \;\; \Omega\,, \\ \textbf{(iii)} & u|_{\partial\Omega} = 0\,. \end{array}$$

We state now the main result of this section.

Theorem. *Let Ω be a bounded open set in $\mathbb{R}^N$ with Lipschitz boundary and let $f \in L^{N,\infty}(\Omega)$ with $f \geq 0$. Then, there is a unique nonnegative weak solution of problem (1).*

Moreover, when we take a function f with small norm, we always get the null solution as we can see in the next result.

Proposition. *Let $u \in BV(\Omega)$ be the nonnegative solution of problem (1) with $0 \leq f \in L^{N,\infty}(\Omega)$. Then, $u \equiv 0$ if and only if $\|f\|_{W^{-1,\infty}(\Omega)} \leq 1$.*

Is usual to find similar results when we work with the 1-Laplacian operator. In particular, in [57], it was proved that the solution to the Dirichlet problem with equation $-\text{div}\left(\frac{Du}{|Du|}\right) = f$ vanishes if $\|f\|_{W^{-1,\infty}(\Omega)} < 1$ and there is no solution when $\|f\|_{W^{-1,\infty}(\Omega)} > 1$. When we have the equality, $u = 0$ is always a solution although it may exist a nonzero solution for certain data.

We finish this section with a result concerning the regularity of the solutions. If we take as a data function $f \in L^q(\Omega)$ with $q > N$, the solution of problem (1) is always bounded (see [55]). Nevertheless, in the limit case, when we choice the data f in the Marcinkiewicz space $L^{N,\infty}(\Omega)$, the solution u belongs to the Lebesgue space $L^q(\Omega)$ for all $1 \le q < \infty$. Indeed, in order to prove that the solutions of problem (1) with nonnegative data of the space $L^{N,\infty}(\Omega)$ are not necessarily bounded, we present the following example.

Example. *Let $0 < \rho < R$ and $0 < \lambda < N-1$. If we consider $\Omega = B_R(0)$, then the solution to problem (1) with data $\frac{\lambda}{|x|}\chi_{B_\rho(0)}(x) \in L^{N,\infty}(\Omega)\backslash L^N(\Omega)$ is given by*

$$u(x) = (N-1-\lambda)\log\left(\frac{|x|}{\rho}\right), \quad |x| < \rho,$$

which belongs to the Lebesgue space $L^q(\Omega)$ with $1 \le q < \infty$ but is not bounded.

The last sections of Chapter 1 are devoted to study a generalization of problem (1) in which we add a function g in the gradient term. That is, we study the following Dirichlet problem:

$$\begin{cases} -\text{div}\left(\dfrac{Dv}{|Dv|}\right) + g(v)|Dv| = f(x) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where Ω is a bounded open set of $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$, f is a nonnegative function which belongs to the Marcinkiewicz space $L^{N,\infty}(\Omega)$ and $g : [0,\infty[\to [0,\infty[$ is a continuous function.

Our aim here is to see how the function g affects the existence, uniqueness and regularity results of the solutions. The total variation term is essential to have uniqueness of solutions. If we consider the same problem without the total variation, i.e., $-\mathrm{div}\left(\frac{Dv}{|Dv|}\right) = f$ and v denotes its solution, the function $h(v)$ should be also a solution for all smooth increasing function h.

Moreover, in [9] it was showed that a similar equation to problem (1) without the total variation term, does not have the regularity of the solutions of (1). In particular, they proved that the equation $u - \mathrm{div}\left(\frac{Du}{|Du|}\right) = f(x)$ has a unique solution, although it may has jump part.

In Sections 1.7, 1.8 and 1.9 of Chapter 1, we see that, depending on the characteristics of function g, the solution of problem (2) satisfies different properties. Moreover, in the extreme cases, when the function g vanishes, we also have to modify the definition of solution since it is not, necessarily, a function of bounded variation.

In any case, we need to define the following auxiliary function

$$G(s) = \int_0^s g(\sigma)\, d\sigma\,.$$

In Section 1.7 we study the conditions of function g such that the solution v satisfies better properties. In particular, we introduce the notion of solution to problem (2) when we take a continuous function $g : [0,\infty[\to [0,\infty[$ such that $g(s) \geq m > 0$ for all $s \geq 0$.

Definition. *We say that a function v is a weak solution to problem* (2) *if $v \in BV(\Omega)$ with $D^j v = 0$ and there exists a field $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ with $\|\mathbf{z}\|_{L^\infty(\Omega)} \leq 1$ such that*

$$
\begin{aligned}
&\textbf{(i)} && -\operatorname{div}\mathbf{z} + g(v)^*|Dv| = f \quad \textit{in} \;\; \mathcal{D}'(\Omega)\,,\\
&\textbf{(ii)} && (\mathbf{z}, Dv) = |Dv| \quad \textit{as measures in} \;\; \Omega\,,\\
&\textbf{(iii)} && v|_{\partial\Omega} = 0\,.
\end{aligned}
$$

It is important to highlight that, in this case, the solution v satisfies the same regularity properties as the solution of problem (1).

When we take a continuous function g such that $g(s) \geq m > 0$, we have to distinguish two cases in order to prove the existence of solution. If g is bounded, then we use the following results.

Theorem.

(i) *Let u be the nonnegative solution of problem (1). Then, function v such that $u = G(v)$ is a solution to problem (2).*

(ii) *Let v be a nonnegative solution of problem (2). Then, function $u = G(v)$ is the solution of problem (1).*

In general, to prove the existence of solution we need to apply the chain rule for functions of bounded variation. In [7] it was showed that if $v \in BV(\Omega)$ with $D^j v = 0$ and $\psi : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function, then $\psi(v) \in BV(\Omega)$ and moreover $D\psi(v) = \psi'(v)Dv$. Nevertheless, we cannot use this result under our hypothesis since function ψ' is not, necessarily, bounded. For that reason we prove the following slight generalization of the chain rule:

Proposition. *Let v a function of bounded variation without jump part and let g be a continuous nonnegative real function. If $u = G(v) \in L^1(\Omega)$, then $u \in BV(\Omega)$ if and only if $g(v)|Dv|$ is a finite measure. Moreover, $|Du| = g(v)|Dv|$ as measures in Ω.*

When the function $g(s)$ does not touch the s-axis and it is not bounded, then the proofs of the previous results do not work. In this case, in order to show the existence of solution we use approximating problems. For each $n \in \mathbb{N}$, we consider the problem (2) with a function $g_n(s)$ such that there exists a solution v_n to the problem and also $g_n(s)$ converges to $g(s)$. We have to prove that the sequence of solutions $\{v_n\}$ is convergent and the limit is the solution of (2) with function $g(s)$.

The main result of Section 1.7 is the following:

Theorem. *Let Ω be a bounded open set in $\mathbb{R}^N$ with Lipschitz boundary and let $f \in L^{N,\infty}(\Omega)$ with $f \geq 0$. If g is a continuous real function such that $g(s) \geq m > 0$ for every $s \geq 0$, then there exists a unique nonnegative solution of problem (2). Moreover, that solution belongs to the Lebesgue space $L^q(\Omega)$ for every $1 \leq q < \infty$.*

Section 1.8 in devoted to the study of problem (2) when function $g : [0, \infty[\to [0, \infty[$ may vanish at some points. In particular, we take a non integrable bounded continuous function g such that $g(s) > 0$ for almost every $s \geq 0$.

The first result we prove in this section shows the existence of a unique solution to (2) when g also satisfies the following restriction:

(C) *There exist $m, \sigma > 0$ such that $g(s) \geq m > 0$ for every $s \geq \sigma$.*

Taking a function $g(s)$ with the above conditions, the existence proof is based on the approximating argument. We approximate the function $g(s)$ by a sequence $g_n(s)$ which does not touch the s-axis. When condition (C) does not hold, we follow the approximating argument but, in this case, the limit of the sequence $\{g_n(s)\}$ is not necessarily a function of bounded variation. We use the following example to show that the solution of problem (2) with a function g such that $\lim_{s\to\infty} g(s) = 0$ does not belong to the BV space.

Example. *Let $R > 0$ and $\lambda > 2N - 2$. If we consider $\Omega = B_R(0)$, the datum $f(x) = \frac{\lambda}{|x|}$ and the function $g(s) = \frac{1}{1+s}$, then the solution of the* *problem* (2) *is given by*

$$v(x) = \left(\frac{|x|}{R}\right)^{N-1-\lambda} - 1\,,$$

which is not of bounded variation since $|Dv| = \frac{\lambda-N+1}{R^{N-1-\lambda}}|x|^{N-2-\lambda}$ *is not a integrable function.*

Therefore, the properties of solution to problem (2) change when we take a non integrable bounded continuous function $g : [0,\infty[\to [0,\infty[$ such that $g(s) > 0$ for almost every $s \geq 0$. The new notion of solution is the following:

Definition. *We say that a function* v *is a weak solution to problem* (1.43) *if* $G(v) \in BV(\Omega)$ *with* $D^j G(v) = 0$ *and there exists a field* $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ *with* $\|\mathbf{z}\|_{L^\infty(\Omega)} \leq 1$ *such that*

$$\begin{aligned} &\textbf{(i)} \quad -\operatorname{div}\mathbf{z} + g(v)^*|Dv| = f \quad \text{in} \quad \mathcal{D}'(\Omega)\,, \\ &\textbf{(ii)} \quad (\mathbf{z}, DG(v)) = |DG(v)| \quad \text{as measures in} \quad \Omega\,, \\ &\textbf{(iii)} \quad G(v)|_{\partial\Omega} = 0\,. \end{aligned}$$

Although the solution v does not belong to the BV space, $G(v)$ does and it does not have jump part. Moreover, since u is not a function of bounded variation, then the pairing $(\mathbf{z}, Dv)$ is not well-defined. Because of this, we use $(\mathbf{z}, G(v))$ instead of it.

The next result shows that problem (2) has a unique solution satisfying the previous definition.

Theorem. *Let* Ω *be a bounded open set in* $\mathbb{R}^N$ *with Lipschitz boundary and let* $f \in L^{N,\infty}(\Omega)$ *with* $f \geq 0$*. Then, there exists a unique nonnegative solution to problem* (2) *in the sense of the previous definition when we* *take a non integrable bounded continuous function* $g : [0, \infty[\to [0, \infty[$ *such that* $g(s) > 0$ *for almost every* $s > 0$.

The last section of the first chapter is devoted to the study of some particular cases in which the properties of function g does not allow us to have existence or uniqueness of solution. Moreover, solutions may have jump part or do not satisfy the boundary condition.

Assuming g is an integrable function, the existence of solution depends on the datum f. In particular, when the norm of f in the dual space of $W_0^{1,1}(\Omega)$ is smaller than 1, then the solution of problem (2) is always trivial. Nevertheless, when this norm is bigger than some constant, the problem has no solution as we can see in the next result.

Proposition. *Let* $f \in L^{N,\infty}(\Omega)$ *with* $f \geq 0$ *and let* $g \in L^1([0, \infty[)$. *Then,*

- **(i)** *if* $\|f\|_{W^{-1,\infty}(\Omega)} < 1$, *the solution to problem* (2) *is trivial;*
- **(ii)** *if* $\|f\|_{W^{-1,\infty}(\Omega)} > e^{G(\infty)}$, *problem* (2) *has no solution;*

where $G(\infty) = \sup\{G(s) : s \in]0, \infty[\}$.

We finish the chapter by studying the properties of solutions to problem (2) when g vanishes on an interval. In particular, we show that there is no uniqueness of solution at all and although problem (2) has a solution u, this may have jump part or does not satisfy the boundary condition. This is shown in the following example.

Example. *Given* $R > 0$ *and the datum* $f(x) = \frac{N}{|x|}$, *if we take* $\Omega = B_R(0)$ *and a function* $g : [0, \infty \to [0, \infty[$ *defined by*

- **(a)** $g(s) = 0$ *if* $0 \leq s \leq a$ *and* $g(s) = s - a$ *if* $s > a$, *then solution* u *does not vanish on the boundary. Nevertheless, it satisfies the boundary condition in a weak sense, that is,* $[\mathbf{z}, \nu] = -\operatorname{sign}(u)$.

(b) *$g(s) = a - s$ if $0 \leq s < a$, $g(s) = 0$ if $a \leq s \leq b$ and $g(s) = s - b$ if $s > b$, then solution to problem (2) has jump part.*

The contents of this chapter are published in the following paper:

M. Latorre and S. Segura de León, Elliptic equations involving the 1-Laplacian and a total variation term with $L^{N,\infty}$-data, *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* **28** (2017), no. 4, 817–859. DOI: 10.4171/RLM/787

The second chapter of this dissertation is devoted to the generalization of the results that we have got in the study of Dirichlet problem (1), taking now as a data f a nonnegative function in $L^1(\Omega)$.

When we choice a datum $f \in L^1(\Omega)$, and in the same way that we have done with data in the Marcinkiewicz space, we have to generalize the Anzellotti's theory by defining the pairing $(\mathbf{z}, Du)$.

This pairing $(\mathbf{z}, Du)$ must be is a Radon measure with finite total variation. Nevertheless, since the solution to this problem is not necessarily bounded, we cannot use the results from [13]. Moreover, since f is a function in the Lebesgue space $L^1(\Omega)$, we cannot prove that $(\mathbf{z}, Du)$ is a Radon measure with finite total variation using the arguments from Chapter 1.

For this reason, we need to modify the definition of solution using truncations as F. Andreu, C. Ballester, V. Caselles and J.M. Mazón did in [8], where they defined the solution to the parabolic problem $u_t = \operatorname{div}\left(\frac{Du}{|Du|}\right)$.

Every time we take data in the space $L^1(\Omega)$, we have to use the truncations, even when the equation is not driven by a 1-Laplacian operator. For equations involving the p-Laplacian with $1 < p \leq N$ we refer to [30] and [16] in order to see the use of truncation in the renormalized and entropy solutions, respectively.

If we denote by $T_k(s) = \min\{|s|, k\} \operatorname{sign}(s)$ the truncation function, the definition of solution to problem (1) is the following:

Definition. *We say that $u \in BV(\Omega)$ is a solution to problem* (1) *if $D^j u = 0$ and there exists a vector field $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ with $\|\mathbf{z}\|_{L^\infty(\Omega)} \leq 1$ such that*

$$
\begin{aligned}
&\textbf{(i)} \quad -\operatorname{div}\mathbf{z} + |Du| = f \quad \textit{in} \;\; \mathcal{D}'(\Omega)\,,\\
&\textbf{(ii)} \quad (\mathbf{z}, DT_k(u)) = |DT_k(u)| \;\; \textit{as measures in} \;\; \Omega \;\textit{(for all } k > 0)\,,\\
&\textbf{(iii)} \quad u|_{\partial\Omega} = 0\,.
\end{aligned}
$$

We highlight that function $T_k(u)$ belongs to $BV(\Omega) \cap L^\infty(\Omega)$ and then $(\mathbf{z}, DT_k(u))$ is well-defined and it is also a Radon measure with finite total variation.

On the other hand, it should be pointed out that a similar equation with L^1-data was studied in [9], but without the gradient term. Since they have data in $L^1(\Omega)$, they have to use truncations in order to have a proper definition. However, their solution satisfies the boundary condition in a weak sense. So, it loses regularity because the boundary condition in our problem holds in the trace sense.

The main result of this section shows the existence of solutions to problem (1) in the sense of the above definition.

Theorem. *If f is a nonnegative integrable function, then there is, at least, one solution to problem* (1) *in the sense of above definition.*

On the other hand, concerning the uniqueness of solution, in Section 2.2 of Chapter 2 we prove a comparison principle which improves the uniqueness results and also simplify the proof when we take data in the Lebesgue space $L^q(\Omega)$ with $q > N$. The result is the following:

Theorem. *Let Ω be a bounded open set in $\mathbb{R}^N$ with Lipschitz boundary and let f_1 and f_2 be two functions in the space $L^1(\Omega)$ such that $0 \leq f_1 \leq f_2$. If u_1 and u_2 are solutions to problem (1) with data f_1 and f_2, respectively, then $u_1 \leq u_2$.*

We highlight that although $(\mathbf{z}, Du)$ is not well-defined, we show that the pairing $(e^{-u}\mathbf{z}, Du)$, and also $(\mathbf{z}, De^{-u})$, are Radon measures with finite total variation. Both expressions are essential for the proof of existence of solution as well as the comparison principle.

Finally, thanks to the comparison principle, we have proved the uniqueness of solution.

Theorem. *Let Ω be a bounded open set in $\mathbb{R}^N$ with Lipschitz boundary and let $f \in L^1(\Omega)$ be a nonnegative function. Then, there is a unique solution to problem (1).*

Section 2.3 of this chapter is devoted to the study of regularity of solution to problem (1) when we take data in the Lebesgue space $L^q(\Omega)$ with $1 < q < N$. As we have observed in Chapter 1, if datum f belongs to $L^N(\Omega)$, then the solution belongs to $L^q(\Omega)$ with $1 \leq q < \infty$; and if we have data in $L^q(\Omega)$ with $q > N$, we know by [55] that the solution is bounded.

The main result of this section is the following.

Theorem. *Let Ω be a bounded open set in $\mathbb{R}^N$ with Lipschitz boundary. If we take a function $f \geq 0$ in the Lebesgue space $L^q(\Omega)$ with $1 < q < N$, then the unique solution u to problem (1) satisfy $u \in BV(\Omega) \cap L^{\frac{Nq}{N-q}}(\Omega)$.*

In particular, we prove the best regularity that the solutions of problem (1) achieve continuously adjust with the regularity of data f.

We finish the chapter with a section which shows that the regularity of solution is in fact optimal.

Example. *Let* $R > 0$, $1 < q < N$ *and* $\lambda > 0$. *If we consider* $\Omega = B_R(0)$ *and the datum* $f(x) = \frac{\lambda}{|x|^q}$ *from the Lebesgue space* $L^s(\Omega)$ *with* $1 \leq s < \frac{N}{q}$, *then the solution to problem* (1) *is given by*

$$u(x) = \begin{cases} (N-1)\log\left(\frac{|x|}{\rho_\lambda}\right) + \frac{\lambda}{1-q}(\rho_\lambda^{1-q} - |x|^{1-q}) & \text{if } \ 0 \leq |x| < \rho_\lambda \\ 0 & \text{if } \ \rho_\lambda < |x| \leq R\,, \end{cases}$$

for certain value $0 < \rho_\lambda < R$. *That is, the solution* u *belongs to the Lebesgue space* $L^r(\Omega)$ *for every* $1 \leq r < \frac{N}{q-1}$.

The contents of this chapter are published in the following paper:

M. Latorre and S. Segura de León, Existence and comparison results for an elliptic equation involving the 1-Laplacian and L^1-data, *J. Evol. Equ.* **18** (2018), no. 1, 1–28. DOI: 10.1007/s00028-017-0388-0.

Finally, in Chapter 3 we prove an existence and uniqueness result for an evolution problem. It consists in an elliptic equation involving the 1-Laplacian operator and a dynamical boundary condition. The problem is the following:

$$\begin{cases} \lambda u - \operatorname{div}\left(\frac{Du}{|Du|}\right) = 0 & \text{in } \ (0,+\infty)\times\Omega\,, \\ \omega_t + \left[\frac{Du}{|Du|}, \nu\right] = g(t,x) & \text{on } \ (0,+\infty)\times\partial\Omega\,, \\ u = \omega & \text{on } \ (0,+\infty)\times\partial\Omega\,, \\ \omega(0,x) = \omega_0(x) & \text{on } \ \partial\Omega\,, \end{cases} \tag{3}$$

where Ω is a bounded open set in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, λ is a positive real number, ν stands for the unit outward normal vector on $\partial\Omega$, function g belongs to $L^1_{loc}(0,+\infty, L^2(\partial\Omega))$ and the initial datum ω_0

is a square-integrable function in the boundary $\partial\Omega$. We denote by ω_t the distributional derivative of ω with respect to t.

This type of problems with dynamical boundary conditions appear in applications where there is a reaction term in the problem that concentrates in a small strip around the boundary of the domain, while in the interior there is no reaction and only diffusion matters. Currently, the study of elliptic or parabolic equations with this type of boundary condition is a very active field since it adjusts to many mathematical models including heat transfer in a solid in contact with a moving fluid, in thermoelasticity, in biology, etc.

Concerning problems with dynamical boundary conditions and an equation driven by the 1-Laplacian operator, as far as we know, this is the first time that this problem is tackled. F. Andreu, N. Igbida, J.M. Mazón and J. Toledo studied a similar problem with an elliptic equation with the p-Laplacian operator with $1 < p < \infty$ (see [11]). In order to prove the existence of solution, the authors defined a completely accretive operator and using nonlinear semigroups theory, they obtained a mild solution. Finally, they showed that this mild solution is in fact a distributional solution.

To obtain a solution to problem (3), since we cannot prove the existence using approximating problems, we adapt the method used in [11] to the context of the 1-Laplacian operator.

First, we consider problem (3) restricted to the domain $(0,T) \times \Omega$ with $T > 0$. That is, we study the following problem:

$$\begin{cases} \lambda u - \operatorname{div}\left(\frac{Du}{|Du|}\right) = 0 & \text{in } \ (0,T) \times \Omega\,, \\ \omega_t + \left[\frac{Du}{|Du|}, \nu\right] = g(t,x) & \text{on } \ (0,T) \times \partial\Omega\,, \\ u = \omega & \text{on } \ (0,T) \times \partial\Omega\,, \\ \omega(0,x) = \omega_0(x) & \text{on } \ \partial\Omega\,, \end{cases} \tag{4}$$

where g denotes a function in the space $L^1(0,T;L^2(\partial\Omega))$.

In Section 3.2 we present the notation and previous results that we need to prove the existence of solution to problems (3) and (4). In particular, we state the results that we will use to prove the existence of a mild solution to the Cauchy problem

$$\begin{cases} \omega_t + \mathcal{B}(\omega) \ni g\,, \\ \omega(0) = \omega_0\,, \end{cases} \tag{5}$$

where g and ω_0 are functions in $L^1(0,T;L^2(\partial\Omega))$ and $L^2(\partial\Omega)$, respectively, and $\mathcal{B}$ denotes an operator in $L^2(\partial\Omega)$. These mild solutions are obtained as the limit of solutions of the discretizations of problem (5).

In Section 3.3 we prove that this mild solution exists and it is unique. To do this, we define an operator $\mathcal{B}$ in the space $L^2(\partial\Omega)$.

Definition. *Let $\omega \in L^2(\partial\Omega)$. We say that $v \in \mathcal{B}(\omega)$ if $v \in L^\infty(\partial\Omega)$ with $\|v\|_{L^\infty(\partial\Omega)} \leq 1$ and there exist a function $u \in BV(\Omega)\cap L^2(\Omega)$ and a vector field $\mathbf{z} \in L^\infty(\Omega;\mathbb{R}^N)$ with $\|\mathbf{z}\|_{L^\infty(\Omega)} \leq 1$ such that*

$$\begin{aligned} &\textbf{(i)} && \lambda u - \operatorname{div}\mathbf{z} = 0 \quad \textit{in} \ \ \mathcal{D}'(\Omega)\,, \\ &\textbf{(ii)} && (\mathbf{z}, Du) = |Du| \quad \textit{as measures in} \ \ \Omega\,, \\ &\textbf{(iii)} && [\mathbf{z},\nu] = v \quad \mathcal{H}^{N-1}\textit{-a.e. on} \ \ \partial\Omega\,, \\ &\textbf{(iv)} && [\mathbf{z},\nu] \in \operatorname{sign}(\omega - u) \quad \mathcal{H}^{N-1}\textit{-a.e. on} \ \ \partial\Omega\,. \end{aligned}$$

We highlight that since the equation $u - \operatorname{div}\left(\frac{Du}{|Du|}\right) = 0$ with the Dirichlet condition $u = \omega$ has a unique solution, function u is unique. Nevertheless, the vector field $\mathbf{z}$ is not unequivocally determined and so does not v.

With this definition, in Section 3.3 we prove that the operator $\mathcal{B}$ is m-accretive in the space $L^2(\partial\Omega)$ and its domain is dense in $L^2(\partial\Omega)$, that is, $L^2(\partial\Omega) = \overline{D(\mathcal{B})}$. Therefore, there exists a unique mild solution $\omega \in C([0,T];L^2(\partial\Omega))$ to the Cauchy problem (5) for the data $g \in L^1(0,T;L^2(\partial\Omega))$ and $\omega_0 \in L^2(\partial\Omega)$.

Moreover, in this section we also prove a comparison principle between mild solutions as we show in the next result.

Theorem. *Let ω^1 and $\omega^2 \in C([0,T];L^2(\partial\Omega))$ be the mild solutions to the Cauchy problem (5) with data g_1 and $g_2 \in L^1(0,T;L^2(\partial\Omega))$ and ω_0^1 and $\omega_0^2 \in L^2(\partial\Omega)$, respectively. If*

$$g^1(t,x) \le g^2(t,x)\,, \qquad \text{for almost every } (t,x) \in (0,T)\times\partial\Omega\,,$$
$$\text{and } \omega_0^1(x) \le \omega_0^2(x)\,, \qquad \text{for almost every } x \in \partial\Omega\,,$$

then, the corresponding mild solutions also satisfy

$$\omega^1(t,x) \le \omega^2(t,x)\,, \qquad \text{for almost every } (t,x) \in (0,T)\times\partial\Omega\,.$$

In the forth section we prove the main results of this chapter: the existence of a unique global strong solution to problem (3) and a comparison principle.

With respect to problem (4), the strong solution is given by a pairing (u,ω) and the definition is the following:

Definition. *We say that the pairing (u,ω) is a strong solution to problem (4) if $u \in L^\infty(0,T;BV(\Omega))\cap L^\infty(0,T;L^2(\Omega))$ and $\omega \in C([0,T];L^2(\partial\Omega))\cap W^{1,1}(0,T;L^2(\partial\Omega))$ such that $\omega(0)=\omega_0$ and there exists a vector field $\mathbf{z} \in L^\infty((0,T)\times\Omega;\mathbb{R}^N)$ with $\|\mathbf{z}\|_{L^\infty((0,T)\times\Omega)} \le 1$ satisfying the following conditions:*

$$\textbf{(i)} \quad \lambda u(t) - \operatorname{div}(\mathbf{z}(t)) = 0 \ \text{ in } \mathcal{D}'(\Omega)\,,$$

$$\textbf{(ii)} \quad (\mathbf{z}(t), Du(t)) = |Du(t)| \ \text{ as measures in } \Omega\,,$$

$$\textbf{(iii)} \quad [\mathbf{z}(t),\nu] = g(t) - \omega_t(t) \ \text{ for almost every } x \in \partial\Omega\,,$$

$$\textbf{(iv)} \quad [\mathbf{z}(t), \nu] \in \operatorname{sign}(\omega(t) - u(t)) \ \textit{ for almost every } \ x \in \partial\Omega\,,$$

for almost every $t \in (0, T)$.

We say that the solution of problem (4) is strong since all these conditions are pointwisely satisfied for almost every $t \in (0, T)$.

Next we highlight the theorem that proves the existence of a unique solution to problem (4).

Theorem. *Let* $\lambda > 0$ *and let* $g \in L^1(0, T; L^2(\partial\Omega))$ *and* $\omega_0 \in L^2(\partial\Omega)$*. Then, there exists a unique strong solution to problem* (4)*. Furthermore, the following estimates hold:*

$$\|\omega\|_{L^\infty(0,T;L^2(\partial\Omega))} \leq \|\omega_0\|_{L^2(\partial\Omega)} + \|g\|_{L^1(0,T;L^2(\partial\Omega))}\,, \qquad \textit{for every } \ T > 0\,,$$

$$\lambda \|u(t)\|^2_{L^2(\Omega)} + \|u(t)\|_{BV(\Omega)} \leq \|\omega(t)\|_{L^1(\partial\Omega)}\,, \qquad \textit{for almost every } \ t > 0\,.$$

On the other hand, if we consider problem (3) with data $g \in L^1_{loc}(0, +\infty; L^2(\partial\Omega))$ and $\omega_0 \in L^2(\partial\Omega)$, the solution (u, ω) is global if it is a strong solution to problem (4) for all $T > 0$. Therefore, since problem (4) has a unique solution, there also exists a unique global strong solution to problem (3).

It is important to remark that the regularity of the solutions get better if we choice the data g and ω_0 in more regular spaces. In particular, if we take a function g in the space $L^\infty(0, +\infty; L^2(\partial\Omega))$, we deduce that $\omega_t \in L^\infty(0, +\infty; L^2(\partial\Omega))$ since the equality $\omega_t(t) = [\mathbf{z}(t), \nu] + g(t)$ holds in $\partial\Omega$. Therefore, the solution ω is Lipschitz with respect to the variable t.

Using the comparison principle for the mild solution and certain convergence results showed in the proof of the existence of solution to problem (4), we may also prove a comparison principle to the solutions (u, ω). This results is enunciated below.

Theorem. *Let* g^1 *and* $g^2 \in L^1(0,T;L^2(\partial\Omega))$ *and let* ω_0^1 *and* $\omega_0^2 \in L^2(\partial\Omega)$ *such that*

$$g^1(t,x) \le g^2(t,x)\,, \qquad \textit{for almost every } \ (t,x) \in (0,T)\times\partial\Omega\,,$$
$$\textit{and } \ \omega_0^1(t,x) \le \omega_0^2(t,x)\,, \qquad \textit{for almost every } \ x \in \partial\Omega\,.$$

If we denote by (u^1, ω^1) *and* (u^2, ω^2) *the corresponding solutions to problem* (4)*, then*

$$\omega^1(t,x) \le \omega^2(t,x)\,, \qquad \textit{for almost every } \ (t,x) \in (0,T)\times\partial\Omega\,,$$
$$\textit{and } \ u^1(t,x) \le u^2(t,x)\,, \qquad \textit{for almost every } \ (t,x) \in (0,T)\times\Omega\,.$$

We finish this section by proving a result about the long term behaviour of the solution (u, ω) to problem (3). In particular we see that ω converges weakly to a square-integrable function on $\partial\Omega$ and u converges weakly in $L^2(\Omega)$ and strongly in $L^1(\Omega)$ to some function v in the space $BV(\Omega)\cap L^2(\Omega)$. Moreover, Du converges weakly-* to Dv as measures in Ω.

Finally, in Section 3.5 of Chapter 3, we show a result which compares the solutions of problem (3) with different data. In particular, the obtained result allow us to estimate the distance between solutions in relation to the distance between data.

Let (u_1, ω_1) *and* (u_2, ω_2) *be the solutions to problem* (3) *with data* g_1*,* $g_2 \in L^1(0,T;L^2(\partial\Omega))$ *and* ω_{01}*,* $\omega_{02} \in L^2(\partial\Omega)$*, respectively. Then, it holds*

$$\|\omega_1 - \omega_2\|_{L^\infty(0,T;L^2(\partial\Omega))} \le \|\omega_{01} - \omega_{02}\|_{L^2(\partial\Omega)} + \|g_1 - g_2\|_{L^1(0,T;L^2(\partial\Omega))}\,,$$

$$\lambda\|u_1 - u_2\|^2_{L^2(0,T;L^2(\Omega))} \leq \frac{1}{2}\|\omega_{01} - \omega_{02}\|^2_{L^2(\partial\Omega)} + \frac{1}{2}\|g_1 - g_2\|^2_{L^1(0,T;L^2(\partial\Omega))}\,.$$

The contents of this chapter are published in the following paper:

M. Latorre and S. Segura de León, Elliptic 1-Laplacian equations with dynamical boundary conditions, *J. Math. Anal. Appl.* **464** (2018), no. 2, 1051–1081. DOI: 10.1016/j.jmaa.2018.02.006

Contents

Introduction

This dissertation is devoted to the study of some elliptic equations involving the 1-Laplacian operator. In Chapter 1 we focus on the study of the equation

$$- \operatorname{div}\left(\frac{Du}{|Du|}\right) + g(u)|Du| = f(x)\,, \tag{I.1}$$

in a bounded open set Ω in $\mathbb{R}^N$ with Lipschitz boundary with the Dirichlet condition $u = 0$ on the boundary, where the datum f denotes a nonnegative function and $g : [0, \infty[\to [0, \infty[$ is a continuous real function.

Our aim is twofold. On the one hand, we deal with unbounded solutions when datum f belongs to the Marcinkiewicz space $L^{N,\infty}(\Omega)$, so we have to introduce the suitable concept of this kind of solutions. On the other hand, this equation allows us to deal with many related problems having a different gradient term, depending on the function g. We show that the total variation term induces a regularizing effect, that is, the bigger g, the better the properties of the solution.

Chapter 2 is devoted to study equation (I.1) when $g \equiv 1$ and nonnegative data in the Lebesgue space $L^1(\Omega)$. We prove an existence result and a comparison principle. Moreover, we seek the optimal summability of the solution when L^q-data, with $1 < q < N$, are considered: If f belongs to the Lebesgue space $L^q(\Omega)$ with $1 < q < N$, the solutions are in the space $L^{\frac{Nq}{N-q}}(\Omega)$. Moreover, we show with an example that this regularity is optimal.

Finally, in Chapter 3 we deal with an evolution problem. Let λ be a a nonnegative parameter. We study the equation

$$\lambda u - \operatorname{div}\left(\frac{Du}{|Du|}\right) = 0\,,$$

with dynamical boundary conditions. Applying nonlinear semigroup theory we obtain a mild solution to the problem and we prove that this is, in fact, a strong solution. We also prove a comparison principle and a result which shows that the distance between the solutions depends on the distance between the data.

Notation

Let us now introduce some notation and basic results which will be used throughout this dissertation. In what follows, we consider $N \geq 2$ and $\mathcal{H}^{N-1}(E)$ denotes the $(N-1)$-dimensional Hausdorff measure of a set E and $|E|$ its Lebesgue measure.

The set Ω will always stands for a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary. Thus, an outward normal unit vector $\nu(x)$ is defined for $\mathcal{H}^{N-1}$-almost every $x \in \partial\Omega$. We are working with the usual Lebesgue and Sobolev spaces denoted by $L^q(\Omega)$ and $W_0^{1,p}(\Omega)$, respectively. For sake of completeness, we recall the Sobolev embedding

$$\left[\int_\Omega |u|^{p^*}\right]^{1/p^*} \leq C\left[\int_\Omega |\nabla u|^p\right]^{1/p}, \qquad \text{for all } \ u \in W_0^{1,p}(\Omega)\,. \tag{I.2}$$

We also recall that this constant C just depend on N and p, and this dependence is continuous on p.

We refer the reader to [24] and [34] for more information about these spaces.

On the other hand, $C(\Omega)$ stands for the space of all continuous functions on Ω and given $k > 0$, we denote by $C_0^k(\Omega)$ the set of all functions with compact support which are k-times continuously differentiable on Ω, and $C_0^\infty(\Omega) = \cap_{k\geq 0} C^k(\Omega)$.

We recall that for a Radon measure μ in Ω and a Borel set $A \subset \Omega$, the measure $\mu \llcorner A$ is defined by $(\mu \llcorner A)(B) = \mu(A \cap B)$ for any Borel set $B \subset \Omega$.

We also use truncation functions which are defined, for a given $k > 0$, by

$$T_k(s) = \min\{|s|, k\} \operatorname{sign}(s)\,, \tag{I.3}$$

for all $s \in \mathbb{R}$. Moreover, throughout this work, we also use another auxiliary real function defined by

$$G_k(s) = s - T_k(s)\,. \tag{I.4}$$

Functions of bounded variation

In an equation driven by the p-Laplacian, that is, the operator defined by

$$\Delta_p u = |\nabla u|^{p-2} \nabla u\,, \qquad \text{with } \; p > 1\,,$$

the natural energy space to look for solutions is the Sobolev space $W_0^{1,p}(\Omega)$. Nevertheless, in problems involving the 1-Laplacian we cannot work with the Sobolev space $W_0^{1,1}(\Omega)$ since it is not reflexive. We seek solutions in a bigger space with better properties, the space of all functions of bounded variation denoted by $BV(\Omega)$. We say that a function $u : \Omega \to \mathbb{R}$ belongs to $BV(\Omega)$ if $u \in L^1(\Omega)$ and its derivative in the sense of distributions Du is a Radon measure with finite total variation. This space is endowed with the norm defined by

$$\|u\|_{BV(\Omega)} = \int_\Omega |u|\,dx + \int_\Omega |Du|\,,$$

for any $u \in BV(\Omega)$. We recall that the notion of trace can be extended to every function of bounded variation and this fact allow us to interpret it as the boundary values of u and we may write $u|_{\partial\Omega}$. Moreover, the trace is a bounded linear operator $BV(\Omega) \hookrightarrow L^1(\partial\Omega)$ which is also onto. As a consequence of this, an equivalent norm on $BV(\Omega)$ can be defined as

$$\|u\|_{BV(\Omega)} = \int_{\partial\Omega} |u|\,d\mathcal{H}^{N-1} + \int_\Omega |Du|\,.$$

Let $u \in BV(\Omega)$. We can decompose the Radon measure Du into its absolutely continuous and its singular parts with respect to the Lebesgue measure: $Du = D^a u + D^s u$. We denote by S_u the set of all $x \in \Omega$ such that the approximate limit of u does not exist at x, that is, $x \notin S_u$ if there exists $\tilde{u}(x)$ such that

$$\lim_{\rho\downarrow 0} \frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} |u(y) - \tilde{u}(x)|\,dy = 0\,.$$

We say that $x \in \Omega$ is an approximate jump point of u, denoted by $x \in J_u$, if there exist two real numbers $u^+(x) > u^-(x)$ and $\nu_u(x)$ with $|\nu_u(x)| = 1$ such that

$$\lim_{\rho\downarrow 0} \frac{1}{|B_\rho^+(x,\nu_u(x))|} \int_{B_\rho^+(x,\nu_u(x))} |u(y) - u^+(x)|\,dy = 0\,,$$

$$\lim_{\rho\downarrow 0} \frac{1}{|B_\rho^-(x,\nu_u(x))|} \int_{B_\rho^-(x,\nu_u(x))} |u(y) - u^-(x)|\,dy = 0\,,$$

where

$$B_\rho^+(x,\nu_u(x)) = \{y \in B_\rho(x) \mid \langle y - x, \nu_u(x)\,, \rangle > 0\}$$

and

$$B_\rho^-(x,\nu_u(x)) = \{y \in B_\rho(x) \mid \langle y-x, \nu_u(x)\rangle < 0\}\,.$$

We know that S_u is countably $\mathcal{H}^{N-1}$-rectifiable and $\mathcal{H}^{N-1}(S_u\backslash J_u) = 0$ by the Federer–Vol'pert Theorem (see [7, Theorem 3.78]). Moreover, we also know that

$$Du \llcorner J_u = (u^+ - u^-)\nu_u \mathcal{H}^{N-1} \llcorner J_u\,.$$

Using S_u and J_u, we can split $D^s u$ in two parts: the jump part $D^j u$ and the Cantor part $D^c u$, defined, respectively, by

$$D^j u = D^s u \llcorner J_u \quad \text{ and } \quad D^c u = D^s u \llcorner (\Omega\backslash S_u)\,.$$

Therefore, we have

$$D^j u = (u^+ - u^-)\nu_u \mathcal{H}^{N-1} \llcorner J_u\,.$$

In addition, if $x \in J_u$, then $\nu_u(x) = \frac{Du}{|Du|}(x)$ where $\frac{Du}{|Du|}$ is the Radon–Nikodým derivative of Du with respect to its total variation $|Du|$.

We will use the precise representative of u, denoted by u^*, in the definition of the pairing $(\mathbf{z}, Du)$ when u is a merely BV-function (see (I.9) below). We say that $u^* : \Omega\backslash(S_u\backslash J_u) \to \mathbb{R}$ is the precise representative of u if it is equal to $\tilde{u}$ (the approximate limit of u) on $\Omega\backslash S_u$ and equal to $\frac{u^- + u^+}{2}$ on J_u. It is well known (see, for instance, [7, Corollary 3.80]) that if ρ is a symmetric mollifier, then the mollified functions $u \star \rho_\varepsilon$ converge pointwise to u^* in its domain. In order to simplify the notation, most of the time we denote both function and its precise representative by u.

A compactness result in $BV(\Omega)$ will be used several times in this work. It states that every bounded sequence in $BV(\Omega)$ has a subsequence which strongly converges in $L^1(\Omega)$ to a certain $u \in BV(\Omega)$ and the subsequence of gradients $*$-weakly converges to Du in the sense of measures.

To pass to the limit we often use that some functionals defined on $BV(\Omega)$ are lower semicontinuous with respect to the convergence in $L^1(\Omega)$. The most important are the functionals defined by

$$u \mapsto \int_\Omega |Du| \,, \tag{I.5}$$

and

$$u \mapsto \int_\Omega |Du| + \int_{\partial\Omega} |u-\omega| \, d\mathcal{H}^{N-1} \,, \tag{I.6}$$

for any $\omega \in L^1(\partial\Omega)$. In the same way, it yields that each $\varphi \in C_0^1(\Omega)$ with $\varphi \geq 0$ defines a functional

$$u \mapsto \int_\Omega \varphi \, |Du| \,, \tag{I.7}$$

which is lower semicontinuous in $L^1(\Omega)$.

Moreover, in Chapter 1 we use the chain rule, but only when u is a function of bounded variation without jump part. That is, if $u \in BV(\Omega)$ with $D^j u = 0$ and f is a Lipschitz function in Ω, then $v = f \circ u$ belongs to $BV(\Omega)$ and $Dv = f'(u)Du$, so that $D^j v = 0$. It is worth noting that f is only differentiable a.e., so that $f'(u)$ could be undefined in a non-empty set. Nevertheless, the above formula $f'(u)Du$ is well defined since $f'(u)$ is not defined in a $|Du|$-null set due to the assumption $D^j u = 0$ (see [7, Proposition 3.92]).

For further information about functions of bounded variation, we refer the reader to [7], [35] and [66].

L^∞-divergence-measure fields

In every problem involving the 1-Laplacian operator we have to give sense to the quotient $\frac{Du}{|Du|}$ even if Du is a Radon measure and if, besides that, it vanishes in a zone of the domain. Following [9], we use a vector field $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^N)$ which plays the role of $\frac{Du}{|Du|}$ in our equation and such that it satisfies two conditions: $\|\mathbf{z}\|_{L^\infty(\Omega)} \leq 1$ and the dot product of $\mathbf{z}$ and Du is equal to $|Du|$.

In each chapter, since the solution $u \in BV(\Omega)$ satisfies different conditions, we will look for vector fields $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^N)$ which satisfy different properties. In Chapters 1 and 2 we will need a vector field $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$, that is, a vector field in $L^\infty(\Omega; \mathbb{R}^N)$ such that $\operatorname{div} \mathbf{z}$ is a Radon measure in Ω with finite total variation. On the other hand, in Chapter 3 we just ask for $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^N)$ with $\operatorname{div} \mathbf{z} \in L^2(\Omega)$.

The validity of this dot product between gradients of BV-functions and L^∞-divergence-measure vector fields is due to G. Anzellotti [13] and, independently, to G.-Q. Chen and H. Frid [26]. In spite of their different points of view, both approaches introduce the normal trace of a vector field through the boundary and establish the same generalized Gauss–Green's formula. It should be pointed out that the theory of divergence-measure fields has been extended later (see, for instance, [27] and [67]).

In [13], Anzellotti proved, in particular, that if one of the following conditions holds

(i) a vector field $\mathbf{z}$ belongs to $\mathcal{DM}^\infty(\Omega)$ and a function $u \in BV(\Omega) \cap C(\Omega) \cap L^\infty(\Omega)$;

(ii) a vector field $\mathbf{z}$ belongs to $L^\infty(\Omega; \mathbb{R}^N)$ with $\operatorname{div} \mathbf{z} \in L^2(\Omega)$ and a function $u \in BV(\Omega) \cap L^2(\Omega)$;

then, the functional defined by

$$\langle (\mathbf{z}, Du), \varphi \rangle = -\int_\Omega u\,\varphi \operatorname{div} \mathbf{z} - \int_\Omega u\,\mathbf{z} \cdot \nabla\varphi\, dx\,,$$

for every function $\varphi \in C_0^\infty(\Omega)$, is a distribution of order 0 since it satisfies the inequality

$$|\langle (\mathbf{z}, Du), \varphi \rangle| \leq \|\varphi\|_{L^\infty(\Omega)} \|\mathbf{z}\|_{L^\infty(\Omega)} \int_\Omega |Du|\,.$$

Hence, it is actually a Radon measure with finite total variation and the following inequality holds

$$|(\mathbf{z}, Du)| \leq \|\mathbf{z}\|_{L^\infty(\Omega)}|Du|\,, \tag{I.8}$$

as measures in Ω. In particular, the Radon measure $(\mathbf{z}, Du)$ is absolutely continuous with respect to $|Du|$.

However, in [26] G.–Q. Chen and H. Frid consider general functions $u \in BV(\Omega)\cap L^\infty(\Omega)$ but it is only shown that the Radon measure $(\mathbf{z}, Du)$ is absolutely continuous with respect to $|Du|$. In this dissertation we need that the inequality (I.8) holds for every $u \in BV(\Omega)$ and every $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ satisfying a certain condition.

As we had already commented, Anzellotti's theory also provides the definition of a weak trace on the boundary $\partial\Omega$ of the normal component of the vector field $\mathbf{z}$. It is denoted by $[\mathbf{z}, \nu]$ and it satisfies $\|[\mathbf{z}, \nu]\|_{L^\infty(\partial\Omega)} \leq \|\mathbf{z}\|_{L^\infty(\Omega)}$.

In addition, a Green's formula involving all these elements holds: if $\mathbf{z}$ and u satisfy either **(i)** or **(ii)**, then

$$\int_\Omega u \operatorname{div}\mathbf{z} + \int_\Omega (\mathbf{z}, Du) = \int_{\partial\Omega} u\,[\mathbf{z}, \nu]\; d\mathcal{H}^{N-1}\,.$$

Let us stress that $(\mathbf{z}, Du)$ can be defined for other pairings; for instance, $\operatorname{div}\mathbf{z} \in L^N(\Omega)$ and $u \in BV(\Omega)$ or $\operatorname{div}\mathbf{z} \in L^p(\Omega)$ and $u \in BV(\Omega) \cap L^{p'}(\Omega)$, where p' is the conjugate of p, that is, $1 = \frac{1}{p} + \frac{1}{p'}$. In any case, the above results are also true.

Moreover, with a slight modification in the definition of the pairing $(\mathbf{z}, Du)$, all these results also hold for a general $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ and $u \in BV(\Omega) \cap L^\infty(\Omega)$ (see [55]). Using the precise representative of u, the pairing $(\mathbf{z}, Du)$ is defined by

$$\langle(\mathbf{z}, Du), \varphi\rangle = -\int_\Omega u^*\,\varphi \operatorname{div}\mathbf{z} - \int_\Omega u\,\mathbf{z}\cdot\nabla\varphi\, dx\,, \tag{I.9}$$

for all $\varphi \in C_0^\infty(\Omega)$, and a Green's formula remains true:

$$\int_\Omega u^* \operatorname{div} \mathbf{z} + \int_\Omega (\mathbf{z}, Du) = \int_{\partial\Omega} u \, [\mathbf{z}, \nu] \; d\mathcal{H}^{N-1} \, .$$

Chapter 1

Problem with a general gradient term and $L^{N,\infty}$-data

1.1 Introduction

This chapter deal with the Dirichlet problem for equations involving the 1-Laplacian operator and a total variation term, namely

$$\begin{cases} -\operatorname{div}\left(\frac{Du}{|Du|}\right) + g(u)|Du| = f(x) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set with Lipschitz boundary $\partial\Omega$, $g(s)$ is a continuous nonnegative function defined for $s \geq 0$ and f is a nonnegative function which belongs to the Marcinkiewicz space $L^{N,\infty}(\Omega)$ (see Section 1.3 for the definition of this space).

A related class of elliptic problems involving the p-Laplacian operator with a gradient term has been widely studied. We recall the seminal paper [52] for a gradient term of exponent $p-1$ and the systematic study of equations having a gradient term with natural growth initiated by L. Boccardo, F. Murat and J.-P. Puel (see [20–22]). The variational approach seeks for solutions in the Sobolev space $W_0^{1,p}(\Omega)$ and considers data belonging to its dual $W^{-1,p'}(\Omega)$. Notice that in the setting of Lebesgue spaces, data are naturally taken in $L^{\frac{Np}{Np-N+p}}(\Omega)$ as a consequence of the Sobolev's embedding (see I.2).

It should be pointed out that the natural space to look for a solution to problem (1.1) is the Sobolev space $W_0^{1,1}(\Omega)$ although we extend it to the larger space of BV-functions, and the space of data, from a variational point of view, is the dual space of $W_0^{1,1}(\Omega)$, that is, $W^{-1,\infty}(\Omega)$. The Sobolev embedding theorem and duality arguments lead to consider the spaces $L^N(\Omega)$ and $L^{N,\infty}(\Omega)$ as *the right* function spaces of data among all the Lebesgue spaces and the Lorentz spaces, respectively. Evidences that the norm of $L^{N,\infty}(\Omega)$ is suitable enough to deal with this kind of problems can be found in [28, 57]. Therefore, our framework is the following: given a nonnegative $f \in L^{N,\infty}(\Omega)$, find $u \in BV(\Omega)$ that solves problem (1.1) in an appropriate sense (see Definition 1.5.1).

Two important cases of problem (1.1) have already been studied. When $g(s) \equiv 0$ we obtain just the 1-Laplacian operator: $-\operatorname{div}\left(\frac{Du}{|Du|}\right)$. There is a big amount of literature on this equation in recent years, started in [46]. Other papers dealing with this equation are [8, 15, 28, 31, 47, 57]. The interest in studying this equation came from an optimal design problem in the theory of torsion and also related geometrical problems (see [46]). The equation is also important from the variational approach to image restoration (see [8] and also [12] for a review on the development of variational models in image processing).

On the other hand, when $g(s) \equiv 1$, we get $-\operatorname{div}\left(\frac{Du}{|Du|}\right) + |Du|$ which occurs in the level set formulation of the inverse mean curvature flow (see [44]; related developments can be found in [45, 61, 62]). Nevertheless, the framework of these papers is different from ours since Ω is unbounded. Furthermore, the concept of solution is based on the minimization of certain functional and does not coincide with the one considered when

$g \equiv 0$. This operator has also been studied in a bounded domain in [55], where it was proved the existence and uniqueness of a bounded solution for a datum regular enough.

It is worth noting that, contrary to the p-Laplacian setting with $p > 1$, where even the gradient term does not appear in equation, we always have uniqueness of solution (see, for instance, [22] when $g \equiv 1$ and [52] for the equation without the gradient term and $W^{-1,p'}(\Omega)$-data), the properties of solutions to problem (1.1) with $g(s) \equiv 0$ are very different from those with $g(s) \equiv 1$. Indeed, the presence of the gradient term has a strong regularizing effect because when $g \equiv 0$ the following facts hold:

(i) Existence of BV-solutions is only guaranteed for data small enough; for large data solutions become infinity in a set of positive measure.

(ii) There is no uniqueness at all: given a solution u, we also obtain that $h(u)$ is a solution for every smooth increasing function h.

Whereas, when $g \equiv 1$, the properties are:

(i) There is always a solution, even in the case where the datum is large.

(ii) An uniqueness result holds.

Regarding regularity of solutions, even an equation related to the case $g(s) \equiv 0$ like $u - \operatorname{div}\left(\frac{Du}{|Du|}\right) = f(x)$ (for which existence and uniqueness hold) has solutions with jump part (see, for instance, [8]). On the other hand, in [55] proved that solutions to problem (1.1) with $g(s) \equiv 1$ have no jump part. Moreover, solutions to $u - \operatorname{div}\left(\frac{Du}{|Du|}\right) = f(x)$ satisfy the boundary condition only in a weak sense (and in general, $u|_{\partial\Omega} \neq 0$), while if $g(s) \equiv 1$, then the boundary condition holds in the trace sense, that is, the value is attained "pointwise" on the boundary.

We stress that the situation concerning existence of the solutions of problem (1.1) is very similar to the problem

$$\begin{cases} -\Delta u + |\nabla u|^2 = \lambda \dfrac{u}{|x|^2} & \text{in } \ \Omega\,, \\ u = 0 & \text{on } \ \partial\Omega\,, \end{cases} \tag{1.2}$$

in domains satisfying $0 \in \Omega$, since the presence of the quadratic gradient term induces a regularizing effect (see [1, 3] and Remark 1.6.4 below). Indeed, the existence of a positive solution to (1.2) is proved for all $\lambda > 0$. Nevertheless, if the gradient term does not appear, solutions can be found just for small values of λ due to Hardy's inequality.

Our purpose here is to study how the function g affects the properties of the solutions of (1.1). Roughly speaking, we see that the bigger g, the better the properties of the solution. The standard case occurs when $g(s) \geq m > 0$ for all $s \geq 0$ and the situation degenerates as soon as $g(s)$ touch the s-axis.

We begin by considering the case $g(s) = 1$ for all $s \geq 0$. To get an idea of the difficulties one may finds, let us recall previous works on this subject. As we have mentioned, this problem was already handled in [55] for data $f \in L^q(\Omega)$ with $q > N$. This condition is somewhat artificial and was taken in this way due to the necessity of obtaining bounded solutions. This requirement derives from the use of the theory of L^∞-divergence-measure fields. Since we must expect unbounded solutions starting from the most natural space of data $L^{N,\infty}(\Omega)$, the first result we need is to find an appropriated definition for the dot product $(\mathbf{z}, Du)$ when $u \in BV(\Omega)$ can be unbounded. This was achieved in [2], but we include it for the sake of completeness.

Endowed with this tool, in the first part of this chapter, we prove an existence and uniqueness result for problem (1.1) in the particular case $g(s) \equiv 1$. The second part is fully devoted to our main concern, that is, to investigate which properties satisfy solutions to problem (1.1) depending on the function g.

Table 1.1

Function $g(s)$	Existence	Uniqueness	Regularity
$0 < m \leq g(s)$	For every datum$^{(1)}$	Yes$^{(1)}$	No jump part$^{(1)}$ Better summability$^{(2)}$
g vanishes at some points $g \notin L^1([0, \infty[)$	For every datum$^{(3)}$	Yes$^{(3)}$	No jump part$^{(3)}$
g vanishes at infinity $g \notin L^1([0, \infty[)$	For every datum$^{(4)}$, with another concept of solution$^{(5)}$	Yes$^{(4)}$	No jump part$^{(4)}$
$g \in L^1([0, \infty[)$	For data small enough$^{(6,7)}$	Yes$^{(7)}$	No jump part$^{(7)}$
g vanishes on an interval	For data small enough$^{(8)}$	No$^{(8)}$	With jump part$^{(10)}$ No boundary condition$^{(11)}$

Notes: (1) Theorem 1.7.4 and Theorem 1.7.5, (2) Proposition 1.7.6, (3) Theorem 1.8.1, (4) Theorem 1.8.3, (5) Definition 1.8.2 and Example 1.8.4, (6) Example 1.9.4, (7) Theorem 1.9.1, (8) Remark 1.9.5, (9) Remark 1.9.5 and Remark 1.9.7, (10) Example 1.9.8, (11) Example 1.9.6.

Let us describe the contents of this chapter. In Section 1.2 we give the necessary background and we would like to highlight Proposition 1.2.1, a generalization of [55, Proposition 2.2] for which we give a new proof. This result and Proposition 1.2.4 are essential in the definition of pairings involving functions like $\varphi(u)$, where u denotes the solution to problem (2.1) and φ is a function with certain properties, as, for example, truncations (see (I.3)). Section 1.3 is devoted to define and give some properties of the space $L^{N,\infty}(\Omega)$. In Section 1.4 we generalize the theory of L^∞-divergence-measure fields in order to take pairings $(\mathbf{z}, Du)$ of a certain vector field $\mathbf{z}$ and any $u \in BV(\Omega)$. This theory is applied in Section 1.5 to extend the result of existence and uniqueness of [55] to $L^{N,\infty}(\Omega)$-data. In Section 1.6 we show explicit radial examples of solutions. Section 1.7 is devoted to study the standard cases of problem (1.1), those where $g(s)$ is bounded from below by a positive constant. A non standard case is shown in Section 1.8 when $g(s)$ touches the s-axis; in this case we need to change our definition of solution since solutions no longer belong to $BV(\Omega)$. Finally, in Section 1.9 we deal with really odd cases for which the considered properties are not necessarily satisfied.

For better understanding, we have summarized the results of this chapter in Table 1.1.

1.2 Some properties of pairings $(\mathbf{z}, Du)$

We start by taking $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ and $u \in BV(\Omega) \cap C(\Omega) \cap L^\infty(\Omega)$. If we denote by $\theta(\mathbf{z}, Dw, \cdot) : \Omega \to \mathbb{R}$ the Radon–Nikodým derivative of $(\mathbf{z}, Dw)$ with respect to $|Dw|$, due to [13], it follows that

$$\int_B (\mathbf{z}, Dw) = \int_B \theta(\mathbf{z}, Dw, x)\, |Dw| \,, \qquad \text{for all Borel sets } \ B \subset \Omega \,,$$

and

$$\|\theta(\mathbf{z}, Dw, \cdot)\|_{L^\infty(\Omega, |Dw|)} \le \|\mathbf{z}\|_{L^\infty(\Omega)} \,.$$

Moreover, if $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous increasing function, then

$$\theta(\mathbf{z}, D(f \circ w), x) = \theta(\mathbf{z}, Dw, x) \quad |Dw|\text{-a.e. in } \ \Omega \,. \tag{1.3}$$

As we have mentioned, for a general $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$, Anzellotti's theory assumes $w \in BV(\Omega) \cap C(\Omega) \cap L^\infty(\Omega)$ in order to define $(\mathbf{z}, Dw)$ and to prove a Green's formula. Indeed, in [58] the authors generalize it to any $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ and $w \in BV(\Omega) \cap L^\infty(\Omega)$. Then, for every

$\varphi \in C_0^\infty(\Omega)$ we may define the functional

$$\langle (\mathbf{z}, Du), \varphi \rangle = -\int_\Omega u^* \, \varphi \operatorname{div} \mathbf{z} - \int_\Omega u \, \mathbf{z} \cdot \nabla \varphi \, dx \, .$$

We explicitly mention that we have shown in Section 1.4 that the precise representative u^* is summable with respect to the measure $\operatorname{div} \mathbf{z}$ and that this definition depends on the chosen representative of the function.

At first sight, it is not clear that (1.3) holds in the case that $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ and $u \in BV(\Omega) \cap L^\infty(\Omega)$. However, we will see in Proposition 1.2.4 that (1.3) holds if we assume that the jump part $D^j u$ vanishes. This result was proved in [55, Proposition 2.2] but an extra hypothesis is needed in the proof, namely the set of discontinuities of u is $\mathcal{H}^{N-1}$-null. We next prove this result under the general assumption $D^j u = 0$. Following Anzellotti's theory, the main ingredient to prove the above formula is a "slicing" result which connects the measure $(\mathbf{z}, Du)$ with the measures $(\mathbf{z}, D\chi_{E_{u,t}})$ where $E_{u,t} = \{x \in \Omega : u(x) > t\}$.

Proposition 1.2.1. *Let $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ and consider $u \in BV(\Omega) \cap L^\infty(\Omega)$ with $D^j u = 0$. Let $E_{u,t} = \{x \in \Omega : u(x) > t\}$. Then, for all $\varphi \in C_0^\infty(\Omega)$, the function $t \mapsto \langle (\mathbf{z}, D\chi_{E_{u,t}}), \varphi \rangle$ is Lebesgue measurable and*

$$\langle (\mathbf{z}, Du), \varphi \rangle = \int_{-\infty}^{+\infty} \langle (\mathbf{z}, D\chi_{E_{u,t}}), \varphi \rangle \, dt \, .$$

Proof. First we observe that we may assume $u \geq 0$; otherwise, we consider the function $u + \|u\|_{L^\infty(\Omega)}$.

For every set E (measurable with respect to Lebesgue measure), we denote by $\partial^* E$ its essential boundary (see [7, Definition 3.60]). Note that for every measurable set $E \subset \Omega$ having finite perimeter, the condition $|\operatorname{div} \mathbf{z}|(\partial^* E) = 0$ implies

$$\chi_E \operatorname{div} \mathbf{z} = \chi_E^* \operatorname{div} \mathbf{z} \, .$$

As a consequence, we obtain the following claim:
If E is a measurable set in Ω with finite perimeter such that $|\mathrm{div}\,\mathbf{z}|(\partial^* E) = 0$, then

$$\langle (\mathbf{z}, D\chi_E), \varphi\rangle = -\int_E \varphi\,\mathrm{div}\,\mathbf{z} - \int_E \mathbf{z}\cdot\nabla\varphi\,dx\,,$$

for all $\varphi \in C_0^\infty(\Omega)$.

In what follows, recall that u stands for the precise representative of the BV-function. Observe that, thanks to the coarea formula (see, for instance, [7, Theorem 3.40]), the level sets $E_{u,t}$ have finite perimeter for $\mathcal{L}^1$-almost all $t \in \mathbb{R}$. Moreover, since $D^j u = 0$, it follows that

$$\mathcal{H}^{N-1}\left(\partial^* E_{u,t} \cap \partial^* E_{u,s}\right) = 0\,, \qquad \text{for} \quad s \neq t\,.$$

Moreover, since the measure $\mathrm{div}\,\mathbf{z}$ is absolutely continuous with respect to $\mathcal{H}^{N-1}$, i.e., $|\mathrm{div}\,\mathbf{z}| \ll \mathcal{H}^{N-1}$ (by Proposition 1.4.1), we have

$$|\mathrm{div}\,(\mathbf{z})|\left(\partial^* E_{u,t} \cap \partial^* E_{u,s}\right) = 0 \quad \text{ if } \ s \neq t\,.$$

Therefore, there exists $A \subset \mathbb{R}$ countable such that

$$|\mathrm{div}\,(\mathbf{z})|\left(\partial^* E_{u,t}\right) = 0 \quad \text{ if } \ t \in \mathbb{R}\backslash A\,.$$

In other words, we have seen that $|\mathrm{div}\,(\mathbf{z})|\left(\partial^* E_{u,t}\right) = 0$ for $\mathcal{L}^1$-almost all $t > 0$. Thus, our claim implies that if $\varphi \in C_0^\infty(\Omega)$, then

$$\langle (\mathbf{z}, D\chi_{E_{u,t}}), \varphi\rangle = -\int_{E_{u,t}} \varphi\,\mathrm{div}\,\mathbf{z} - \int_{E_{u,t}} \mathbf{z}\cdot\nabla\varphi\,dx\,, \tag{1.4}$$

for $\mathcal{L}^1$-almost all $t > 0$.

Considering $\varphi \in C_0^\infty(\Omega)$, we apply the slicing formula for integrable functions (see, for instance, [66, Lemma 1.5.1]) and (1.4) to get that the function

$$t \mapsto -\int_{E_{u,t}} \varphi\,\mathrm{div}\,\mathbf{z}\,dt - \int_{E_{u,t}} \mathbf{z}\cdot\nabla\varphi\,dx$$

is Lebesgue measurable and

$$\begin{aligned}
\langle(\mathbf{z}, Du), \varphi\rangle &= -\int_\Omega u^* \varphi \operatorname{div} \mathbf{z} - \int_\Omega u\, \mathbf{z} \cdot \nabla \varphi \, dx \\
&= \int_0^\infty \left[-\int_{E_{u,t}} \varphi \operatorname{div} \mathbf{z} - \int_{E_{u,t}} \mathbf{z} \cdot \nabla \varphi \, dx \right] dt \\
&= \int_0^\infty \langle(\mathbf{z}, D\chi_{E_{u,t}}), \varphi\rangle \, dt \, ,
\end{aligned}$$

as desired. □

Proposition 1.2.2. *Let $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ and consider $u \in BV(\Omega) \cap L^\infty(\Omega)$ with $D^j u = 0$. Let $E_{u,t} = \{x \in \Omega : u(x) > t\}$. Then, for all Borel set $B \subset \Omega$, the function $t \mapsto \int_B (\mathbf{z}, D\chi_{E_{u,t}})$ is Lebesgue measurable and*

$$\int_B (\mathbf{z}, Du) = \int_{-\infty}^{+\infty} \left[\int_B (\mathbf{z}, D\chi_{E_{u,t}}) \right] dt. \tag{1.5}$$

Proof. Let S be a countable set in $C_0^\infty(\Omega)$ which is dense with respect to the uniform convergence. Then, for every $t \in \mathbb{R}$ such that $E_{u,t}$ has finite perimeter and for every $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geq 0$, it yields

$$\langle(\mathbf{z}, D\chi_{E_{u,t}})^+, \varphi\rangle = \sup \left\{ \langle(\mathbf{z}, D\chi_{E_{u,t}}), \psi\rangle \ : \ \psi \in S \, , \ 0 \leq \psi \leq \varphi \right\} .$$

Thus, taking the positive part of the measure, $t \mapsto \langle(\mathbf{z}, D\chi_{E_{u,t}})^+, \varphi\rangle$ defines a Lebesgue measurable function since it is the supremum of a countable quantity of Lebesgue measurable functions. Recalling the Riesz representation theorem, we may go further considering an open set $B \subset \Omega$. It follows from

$$\int_B (\mathbf{z}, D\chi_{E_{u,t}})^+ = \sup \left\{ \langle(\mathbf{z}, D\chi_{E_{u,t}})^+, \psi\rangle \ : \ \psi \in S \, , \ 0 \leq \psi \leq \chi_B \right\} ,$$

that $t \mapsto \int_B (\mathbf{z}, D\chi_{E_{u,t}})^+$ defines a Lebesgue measurable function. The regularity of the measure leads to the same conclusion for an arbitrary Borel set. This function is $\mathcal{L}^1$-summable since

$$\int_B (\mathbf{z}, D\chi_{E_{u,t}})^+ \leq \int_B |(\mathbf{z}, D\chi_{E_{u,t}})| \leq \|\mathbf{z}\|_{L^\infty(\Omega)} \int_B |D\chi_{E_{u,t}}| \,,$$

for $\mathcal{L}^1$-almost all $t \in \mathbb{R}$, and $t \mapsto \int_B |D\chi_{E_{u,t}}|$ defines an $\mathcal{L}^1$-summable function due to the coarea formula (see, for instance, [7, Theorem 3.40]).

On the other hand, a similar argument can be used for the negative part of the measures $(\mathbf{z}, D\chi_{E_{u,t}})$, so that $t \mapsto \int_B (\mathbf{z}, D\chi_{E_{u,t}})^-$ defines an $\mathcal{L}^1$-summable function for every Borel set $B \subset \Omega$. As a consequence, $t \mapsto \int_B (\mathbf{z}, D\chi_{E_{u,t}})$ defines an $\mathcal{L}^1$-summable function for every Borel set $B \subset \Omega$ too.

Finally, consider a distribution μ defined by

$$\langle \mu, \varphi \rangle = \langle (\mathbf{z}, Du), \varphi \rangle - \int_{-\infty}^{+\infty} \langle (\mathbf{z}, D\chi_{E_{u,t}}), \varphi \rangle \, dt \,.$$

Proposition 1.2.1 gives $\langle \mu, \varphi \rangle = 0$ for all $\varphi \in C_0^\infty(\Omega)$, so that μ is a Radon measure which vanishes identically. Therefore, (1.5) holds. $\square$

Corollary 1.2.3. *Let* $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ *and consider* $u \in BV(\Omega) \cap L^\infty(\Omega)$ *with* $D^j u = 0$*. Then,*

$$\theta(\mathbf{z}, Du, x) = \theta(\mathbf{z}, D\chi_{E_{u,t}}, x) \quad |D\chi_{E_{u,t}}|\text{-a.e. in } \Omega \,, \tag{1.6}$$

for $\mathcal{L}^1$*-almost all* $t \in \mathbb{R}$*.*

Proof. Let $a, b \in \mathbb{R}$, with $a < b$ and let $B \subset \Omega$ be a Borel set. Applying (1.5) to the set $\{x \in \Omega : a \leq u(x) \leq b\} \cap B$, we obtain

$$\int_{\{a \leq u \leq b\} \cap B} (\mathbf{z}, Du) = \int_a^b \left[\int_B (\mathbf{z}, D\chi_{E_{u,t}}) \right] dt \,. \tag{1.7}$$

Now we are analyzing both sides of (1.7). On the one hand, the coarea formula (see, for instance, [7, Theorem 3.40]) implies

$$\int_{\{a\le u\le b\}\cap B}(\mathbf{z}, Du) = \int_{\{a\le u\le b\}\cap B}\theta(\mathbf{z}, Du, x)|Du|$$
$$= \int_a^b \left[\int_B \theta(\mathbf{z}, Du, x)|D\chi_{E_{u,t}}|\right] dt\,.$$

On the other hand,

$$\int_a^b \left[\int_B (\mathbf{z}, D\chi_{E_{u,t}})\right] dt = \int_a^b \left[\int_B \theta(\mathbf{z}, D\chi_{E_{u,t}}, x)|D\chi_{E_{u,t}}|\right] dt\,.$$

Hence, (1.7) becomes

$$\int_a^b \left[\int_B \theta(\mathbf{z}, Du, x)|D\chi_{E_{u,t}}|\right] dt = \int_a^b \left[\int_B \theta(\mathbf{z}, D\chi_{E_{u,t}}, x)|D\chi_{E_{u,t}}|\right] dt\,.$$

It follows that, for $\mathcal{L}^1$-almost all $t\in\mathbb{R}$,

$$\int_B \theta(\mathbf{z}, Du, x)|D\chi_{E_{u,t}}| = \int_B \theta(\mathbf{z}, D\chi_{E_{u,t}}, x)|D\chi_{E_{u,t}}|$$

holds for every Borel set B and the desired equality (1.6) is proved. □

Proposition 1.2.4. *Let $\mathbf{z}\in\mathcal{DM}^\infty(\Omega)$ and consider $u\in BV(\Omega)\cap L^\infty(\Omega)$ with $D^j u = 0$. If $f:\mathbb{R}\to\mathbb{R}$ is a Lipschitz continuous nondecreasing function, then*

$$\theta(\mathbf{z}, D(f\circ u), x) = \theta(\mathbf{z}, Du, x) \quad |D(f\circ u)|\text{-}a.e. \text{ in } \Omega\,. \tag{1.8}$$

Proof. We may follow Anzellotti (see [13, Proposition 2.8]) for the case of an increasing function. For the general case, consider f nondecreasing and let $\varepsilon>0$. Since the function given by $t\mapsto f(t)+\varepsilon\, t$ is increasing, it follows that

$$
\begin{aligned}
(\mathbf{z}, D(f\circ u)) + \varepsilon(\mathbf{z}, Du) &= (\mathbf{z}, D((f\circ u) + \varepsilon\, u)) \\
&= \theta(\mathbf{z}, Du, x)|D((f\circ u) + \varepsilon\, u)| \\
&= \theta(\mathbf{z}, Du, x)(f'(u) + \varepsilon)|Du|
\end{aligned}
$$

as measures in Ω. Letting $\varepsilon \to 0^+$, we deduce

$$
(\mathbf{z}, D(f\circ u)) = \theta(\mathbf{z}, Du, x)|D(f\circ u)| \quad \text{as measures in} \quad \Omega\,.
$$

Therefore, we have seen that (1.8) holds. $\square$

Now, we present an auxiliary result that we will need in the proof of the existence of solution to problem (1.27).

Proposition 1.2.5. *Let $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ such that $\|\mathbf{z}\|_{L^\infty(\Omega)} \le 1$. Let $u \in BV(\Omega)$ and assume that $\operatorname{div}\mathbf{z} = \xi + f$ where ξ is a Radon measure satisfying either $\xi \ge 0$ or $\xi \le 0$ and $f \in L^{N,\infty}(\Omega)$. Then, $(\mathbf{z}, Du) = |Du|$ as measures if and only if $(\mathbf{z}, DT_k(u)) = |DT_k(u)|$ as measures for all $k > 0$.*

Proof. First assume $(\mathbf{z}, Du) = |Du|$ and let $k > 0$. Using the auxiliary real functions $T_k(s)$ and $G_k(s)$ defined in (I.3) and (I.4) respectively, we get

$$
\begin{aligned}
|Du| = (\mathbf{z}, Du) &= (\mathbf{z}, DT_k(u)) + (\mathbf{z}, DG_k(u)) \\
&\le |DT_k(u)| + |DG_k(u)| = |Du|\,.
\end{aligned}
$$

Then, $(\mathbf{z}, DT_k(u)) = |DT_k(u)|$ as measures in Ω.

Conversely, we assume $(\mathbf{z}, DT_k(u)) = |DT_k(u)|$ for all $k > 0$. For each $\varphi \in C_0^\infty(\Omega)$, we argue as in the proof of Proposition 1.4.6 to obtain:

$$\lim_{k\to\infty} \langle(\mathbf{z}, DT_k(u)), \varphi\rangle = \langle(\mathbf{z}, Du), \varphi\rangle \,,$$

and

$$\lim_{k\to\infty} \int_\Omega \varphi\, |DT_k(u)| = \int_\Omega \varphi\, |Du| \,.$$

So, using the hypothesis, we conclude that $\langle(\mathbf{z}, Du), \varphi\rangle = \int_\Omega \varphi\, |Du|$ for every $\varphi \in C_0^\infty(\Omega)$. That is, $(\mathbf{z}, Du) = |Du|$ as measures in Ω. □

On the other hand, we prove a slight generalization of [55, Proposition 2.3] for an unbounded function $u \in BV(\Omega)$, which will be very useful in the proofs of Chapters 1 and 2.

Proposition 1.2.6. *If $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ and $u \in BV(\Omega)$ and $v \in BV(\Omega) \cap L^\infty(\Omega)$ are functions with $D^j u = D^j v = 0$ then,*

$$(v\,\mathbf{z}, Du) = v^*(\mathbf{z}, Du) \quad \textit{as Radon measures in} \;\; \Omega\,. \tag{1.9}$$

Proof. Let $k > 0$ and consider $T_k(u) \in BV(\Omega) \cap L^\infty(\Omega)$. By [55, Proposition 2.3], it yields

$$(v\,\mathbf{z}, DT_k(u)) = v^*(\mathbf{z}, DT_k(u)) \;\; \text{as Radon measures in} \;\; \Omega\,.$$

Then, for every $\varphi \in C_0^\infty(\Omega)$ it holds

$$\begin{aligned} &-\int_\Omega T_k(u)\,\varphi \,\mathrm{div}\,(v\,\mathbf{z}) - \int_\Omega T_k(u)\, v\,\mathbf{z}\cdot\nabla\varphi\, dx \\ &\qquad = -\int_\Omega v^*\, T_k(u)\,\varphi\,\mathrm{div}\,\mathbf{z} - \int_\Omega v^*\, T_k(u)\,\mathbf{z}\cdot\nabla\varphi\, dx\,, \end{aligned}$$

where we can take limits (thanks to the dominated convergence theorem) to get

$$
\begin{aligned}
&-\int_\Omega u\,\varphi\,\mathrm{div}\,(v\,\mathbf{z})-\int_\Omega u\,v\,\mathbf{z}\cdot\nabla\varphi\,dx\\
&\qquad=-\int_\Omega v^*\,u\,\varphi\,\mathrm{div}\,\mathbf{z}-\int_\Omega v^*\,u\,\mathbf{z}\cdot\nabla\varphi\,dx\,,
\end{aligned}
$$

and the proof is done. □

1.3 The data space

In this section we present the Marcinkiewicz and Lorentz spaces and give some properties related to those spaces which will used throughout this chapter.

Given a measurable function $u:\Omega\to\mathbb{R}$, we denote by μ_u the distribution function of u: the function $\mu_u:[0,+\infty[\to[0,+\infty[$ defined by

$$\mu_u(t)=|\{x\in\Omega:|u(x)|>t\}|\,,\quad t\geq 0\,.$$

For $1<q<\infty$, the space $L^{q,\infty}(\Omega)$, known as Marcinkiewicz or weak-Lebesgue space, is the space of Lebesgue measurable functions $u:\Omega\to\mathbb{R}$ such that

$$[u]_q=\sup_{t>0}\{t\,\mu_u(t)^{1/q}\}<+\infty\,. \tag{1.10}$$

The relation between Marcinkiewicz and Lebesgue spaces is given by the following inclusions

$$L^q(\Omega)\hookrightarrow L^{q,\infty}(\Omega)\hookrightarrow L^{q-\varepsilon}(\Omega)\,,$$

for a suitable $\varepsilon > 0$. Note that expression (1.10) defines a quasi-norm which is not a norm in $L^{q,\infty}(\Omega)$. We refer to (1.12), (1.13) and (1.14) below for a suitable norm in this space.

Some properties of Lorentz spaces $L^{q,1}(\Omega)$ for $1 < q < \infty$ must be applied throughout this chapter and we define them as follows. Consider the decreasing rearrangement of u as the function $u^\star :]0, |\Omega|] \to [0, \infty[$ given by

$$u^\star(s) = \sup\{t > 0 : \mu_u(t) > s\}, \qquad s \in]0, |\Omega|].$$

We refer the reader to [18, 43, 66] for the main properties of rearrangements. Then, in terms of $u^\star$, the quasi-norm (1.10) becomes

$$[u]_q = \sup_{s>0} \{s^{1/q} u^\star(s)\}.$$

We say that a measurable function $u : \Omega \to \mathbb{R}$ belongs to the Lorentz space $L^{q,1}(\Omega)$ if

$$\|u\|_{L^{q,1}(\Omega)} = \frac{1}{q} \int_0^\infty s^{1/q} u^\star(s) \frac{ds}{s} < \infty, \tag{1.11}$$

where (1.11) defines a norm on $L^{q,1}(\Omega)$ (see, for instance, [18, Theorem 5.13]). The classical reference where these spaces are systematically studied is [43] (see also [18, 66]). Now, let us describe some important properties of Lorentz spaces:

1. $L^{q,1}(\Omega)$ is a Banach space endowed with the norm (1.11).
2. Simple functions are dense in $L^{q,1}(\Omega)$.
3. The norm (1.11) is absolutely continuous.

Concerning duality, the Marcinkiewicz space $L^{q',\infty}(\Omega)$ is the dual space of $L^{q,1}(\Omega)$ where q' is the conjugated of q, that is, $\frac{1}{q} + \frac{1}{q'} = 1$. Indeed, it follows from a Hardy–Littlewood inequality that if $f \in L^{q',\infty}(\Omega)$ and

$u \in L^{q,1}(\Omega)$, then $fu \in L^1(\Omega)$ and a Hölder's type inequality holds as we can see below:

$$\begin{aligned}\left|\int_\Omega fu\,dx\right| &\leq \int_0^\infty f^\star(s)u^\star(s)\,ds = \int_0^\infty s^{1/q'}f^\star(s)s^{1/q}u^\star(s)\,\frac{ds}{s}\\ &\leq q[f]_{q'}\|u\|_{L^{q,1}(\Omega)}\,.\end{aligned}$$

Thus,

$$\|f\|_{L^{q',\infty}(\Omega)} = \sup\left\{\frac{\left|\int_\Omega fu\,dx\right|}{\|u\|_{L^{q,1}(\Omega)}}\ :\ u \in L^{q,1}(\Omega)\backslash\{0\}\right\} \tag{1.12}$$

defines a norm in the Marcinkiewicz space and $\|f\|_{L^{q',\infty}(\Omega)} \leq q\,[f]_{q'}$ holds. Moreover, applying the density of simple functions, we deduce that

$$\begin{aligned}\|f\|_{L^{q',\infty}(\Omega)} &= \sup\left\{\left|\int_\Omega fu\,dx\right|\ :\ u = |E|^{-\frac{1}{q}}\chi_E\,,\ \text{with } |E|>0\right\} \qquad (1.13)\\ &= \sup\left\{|E|^{-1/q}\int_E |f|\,dx\ :\ |E|>0\right\}\,.\end{aligned}$$

This implies $[f]_{q'} \leq \|f\|_{L^{q',\infty}(\Omega)}$, so that, the quasi-norm $[\,\cdot\,]_{q'}$ is equivalent to the norm $\|\cdot\|_{L^{q',\infty}(\Omega)}$. It also yields

$$\|f\|_{L^{q',\infty}(\Omega)} = \sup_{s>0}\left\{s^{1/q'}f^{\star\star}(s)\right\}, \tag{1.14}$$

where $f^{\star\star}(s) = \dfrac{1}{s}\displaystyle\int_0^s f^\star(\sigma)\,d\sigma\,.$

On the other hand, we recall that Sobolev's inequality can be improved in the context of Lorentz spaces (see [4]): the continuous embedding

$$W_0^{1,1}(\Omega) \hookrightarrow L^{\frac{N}{N-1},1}(\Omega)$$

holds. The best constant of this embedding is denoted by

$$S_N = \sup\left\{ \frac{\|u\|_{L^{\frac{N}{N-1},1}(\Omega)}}{\int_\Omega |\nabla u|\, dx} \;:\; u \in W_0^{1,1}(\Omega)\backslash\{0\} \right\}, \tag{1.15}$$

and its value is

$$S_N = \frac{\Gamma\left(\frac{N}{2}+1\right)^{1/N}}{N\sqrt{\pi}} = \frac{1}{NC_N^{1/N}},$$

where C_N is the measure of the unit ball in $\mathbb{R}^N$. It should be pointed out that this is the value for the best constant taking into account the norm of the Lorentz space defined in (1.11).

Furthermore, by an approximation argument, this inclusion may be extended to BV-functions with the same best constant S_N (see, for instance, [66]):

$$BV(\Omega) \hookrightarrow L^{\frac{N}{N-1},1}(\Omega).$$

It is worth remarking that the supremum in (1.15) is attained in $BV(\Omega)$. As a consequence of this embedding, given $f \in L^{N,\infty}(\Omega)$ and $u \in BV(\Omega)$, it yields $fu \in L^1(\Omega)$. This fact will be essential in what follows.

Another feature concerning Lorentz spaces and duality is in order. We denote by $W^{-1,q'}(\Omega)$ the dual space of $W_0^{1,q}(\Omega)$ for $1 \le q < \infty$. Here we recall just that the norm in $W^{-1,\infty}(\Omega)$ is given by

$$\|\mu\|_{W^{-1,\infty}(\Omega)} = \sup\left\{ \left| < \mu, u >_{W^{-1,\infty}(\Omega), W_0^{1,1}(\Omega)} \right| \;:\; \int_\Omega |\nabla u|\, dx \le 1 \right\}.$$

Since the norm in $L^{\frac{N}{N-1},1}(\Omega)$ is absolutely continuous, it follows that $C_0^\infty(\Omega)$ is dense in $L^{\frac{N}{N-1},1}(\Omega)$ and a duality argument shows that $L^{N,\infty}(\Omega) \hookrightarrow W^{-1,\infty}(\Omega)$. Moreover, if $f \in L^{N,\infty}(\Omega)$ then, keeping in mind (1.12) and (1.15), it yields

$$\|f\|_{L^{N,\infty}(\Omega)} = \sup\left\{ \frac{\left|\int_\Omega f u \, dx\right|}{\|u\|_{L^{\frac{N}{N-1},1}(\Omega)}} \, : \, u \in W_0^{1,1}(\Omega)\backslash\{0\} \right\}$$
$$= \sup\left\{ \frac{\left|\int_\Omega f u \, dx\right|}{\int_\Omega |\nabla u| \, dx} \cdot \frac{\int_\Omega |\nabla u| \, dx}{\|u\|_{L^{\frac{N}{N-1},1}(\Omega)}} \, : \, u \in W_0^{1,1}(\Omega)\backslash\{0\} \right\}$$
$$\geq S_N^{-1} \|f\|_{W^{-1,\infty}(\Omega)} \, .$$

Therefore,

$$\|f\|_{W^{-1,\infty}(\Omega)} \leq \frac{1}{N C_N^{1/N}} \|f\|_{L^{N,\infty}(\Omega)} \, , \tag{1.16}$$

for every $f \in L^{N,\infty}(\Omega)$. We refer to [57, Remark 3.3] for a related inequality in a ball.

1.4 Extending Anzellotti's theory

In this section we study some properties involving L^∞-divergence-measure vector fields and functions of bounded variation. Our aim is to extend Anzellotti's theory (see Introduction for more details about this theory).

We begin by recalling a result proved in [26].

Proposition 1.4.1. *[26, Proposition 3.1] For every $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$, the measure $\mu = \operatorname{div} \mathbf{z}$ is absolutely continuous with respect to $\mathcal{H}^{N-1}$, that is, $|\mu| \ll \mathcal{H}^{N-1}$.*

Consider now $\mu = \operatorname{div} \mathbf{z}$ with $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ and let $u \in BV(\Omega)$. As we have mentioned before, the precise representative u^* is equal to $\lim_{\varepsilon\to 0^+} \rho_\varepsilon \star u$ in its domain , where $\{\rho_\varepsilon\}$ is a symmetric mollifier. Then,

u^* is equal $\mathcal{H}^{N-1}$-a.e. to a Borel function. Therefore, it is deduced from Proposition 1.4.1 that the precise representative u^* is also equal μ-a.e. to a Borel function and then, given $u \in BV(\Omega)$, its precise representative u^* is always μ-measurable. Moreover, $u \in BV(\Omega) \cap L^\infty(\Omega)$ implies $u \in L^\infty(\Omega, \mu) \subset L^1(\Omega, \mu)$.

1.4.1 Preservation of the norm

The aim of this subsection is to prove that we are able to define the distribution $(\mathbf{z}, Du)$ for a general vector field $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ and a unbounded $u \in BV(\Omega)$. Indeed, we will show in Section 1.4.2 that with these results, the pairing $(\mathbf{z}, Du)$ is, in fact, a Radon measure with finite total variation.

We start noticing that every $\operatorname{div}\mathbf{z}$ with $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ defines a functional on $W_0^{1,1}(\Omega)$ by

$$\langle \operatorname{div}\mathbf{z}, u\rangle_{W^{-1,\infty}(\Omega),W_0^{1,1}(\Omega)} = -\int_\Omega \mathbf{z}\cdot\nabla u\,dx\,. \tag{1.17}$$

To express this functional in terms of an integral with respect to the measure $\mu = \operatorname{div}\mathbf{z}$, we need the following Meyers–Serrin type theorem (see [7, Theorem 3.9] for its extension to BV-functions).

Proposition 1.4.2. *Let $\mu = \operatorname{div}\mathbf{z}$, with $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$. For every $u \in BV(\Omega)\cap L^\infty(\Omega)$ there exists a sequence $\{u_n\}$ in $W^{1,1}(\Omega)\cap C^\infty(\Omega)\cap L^\infty(\Omega)$ such that*

$$\begin{aligned}
&(1)\quad u_n \to u^* \quad \textit{in} \quad L^1(\Omega,\mu)\,,\\
&(2)\quad \int_\Omega |\nabla u_n|\,dx \to |Du|(\Omega)\,,\\
&(3)\quad u_n|_{\partial\Omega} = u|_{\partial\Omega} \quad \textit{for all} \quad n\in\mathbb{N}\,,\\
&(4)\quad |u_n(x)| \le \|u\|_{L^\infty(\Omega)} \quad |\mu|\textit{-a.e. for all} \quad n\in\mathbb{N}\,.
\end{aligned}$$

Moreover, if $u \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$*, then one may find a sequence* $\{u_n\}$ *satisfying, instead of (2), condition*

$$(2') \quad u_n \to u \ \ in \ \ W^{1,1}(\Omega)\,.$$

Since

$$-\int_\Omega \mathbf{z} \cdot \nabla\varphi\, dx = \int_\Omega \varphi\, d\mu$$

holds for every $\varphi \in C_0^\infty(\Omega)$, we can also obtain this equality for every $W_0^{1,1}(\Omega) \cap C^\infty(\Omega)$. Given $u \in W_0^{1,1}(\Omega) \cap L^\infty(\Omega)$ and applying Proposition 1.4.2, we may find a sequence $\{u_n\}$ in $W_0^{1,1}(\Omega) \cap C^\infty(\Omega) \cap L^\infty(\Omega)$ satisfying (1) and $(2')$. Letting n go to infinity, it follows from

$$-\int_\Omega \mathbf{z} \cdot \nabla u_n\, dx = \int_\Omega u_n\, d\mu\,,$$

for every $n \in \mathbb{N}$, that

$$-\int_\Omega \mathbf{z} \cdot \nabla u\, dx = \int_\Omega u^*\, d\mu\,,$$

and so

$$\langle \operatorname{div} \mathbf{z}, u\rangle_{W^{-1,\infty}(\Omega), W_0^{1,1}(\Omega)} = \int_\Omega u^*\, d\mu$$

holds for every $u \in W_0^{1,1}(\Omega) \cap L^\infty(\Omega)$. Then the norm of this functional is given by

$$\|\mu\|_{W^{-1,\infty}(\Omega)} = \sup\left\{\left|\int_\Omega u^*\, d\mu\right| : u \in W_0^{1,1}(\Omega) \cap L^\infty(\Omega),\ \|u\|_{W_0^{1,1}(\Omega)} \le 1\right\},$$

where $\|u\|_{W_0^{1,1}(\Omega)} = \int_\Omega |\nabla u|\, dx$. We have seen that $\mu = \operatorname{div} \mathbf{z}$ can be extended from $W_0^{1,1}(\Omega)$ to $BV(\Omega) \cap L^\infty(\Omega)$. In fact, this extension can be given as an integral with respect to μ and it preserves the norm (see Theorem 1.4.4). To prove this result, we will use the following lemma which was showed in [13].

Lemma 1.4.3. *[13, Lemma 5.5] For every $u \in BV(\Omega)$, which implies $u|_{\partial\Omega} \in L^1(\partial\Omega)$, there exists a sequence $\{w_n\}$ in $W^{1,1}(\Omega) \cap C(\Omega)$ such that*

$$\begin{aligned}
&(1) \quad w_n|_{\partial\Omega} = u|_{\partial\Omega}\,,\\
&(2) \quad \int_\Omega |\nabla w_n|\,dx \le \int_{\partial\Omega} |u|\,d\mathcal{H}^{N-1} + \frac{1}{n}\,,\\
&(3) \quad \int_\Omega |w_n|\,dx \le \frac{1}{n}\,,\\
&(4) \quad w_n(x) = 0 \ \ \textit{if} \ \ \operatorname{dist}(x, \partial\Omega) > \frac{1}{n}\,,\\
&(5) \quad w_n(x) \to 0 \ \ \textit{for all} \ \ x \in \Omega\,.
\end{aligned}$$

Moreover, if $u \in BV(\Omega) \cap L^\infty(\Omega)$, then $w_n \in L^\infty(\Omega)$ and $\|w_n\|_{L^\infty(\Omega)} \le \|u|_{\partial\Omega}\|_{L^\infty(\partial\Omega)}$ for all $n \in \mathbb{N}$.

Theorem 1.4.4. *Let $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ and denote $\mu = \operatorname{div}\mathbf{z}$. Then, the functional given by (1.17) can be extended to $BV(\Omega) \cap L^\infty(\Omega)$ as an integral with respect to μ and its norm satisfies*

$$\|\mu\|_{W^{-1,\infty}(\Omega)} = \sup\left\{\left|\int_\Omega u^*\,d\mu\right| \,:\, u \in BV(\Omega) \cap L^\infty(\Omega),\ \|u\|_{BV(\Omega)} \le 1\right\},$$

where $\|u\|_{BV(\Omega)} = \int_{\partial\Omega} |u|\,d\mathcal{H}^{N-1} + \int_\Omega |Du|$.

Proof. Since we already know that $BV(\Omega) \cap L^\infty(\Omega)$ is a subset of $L^1(\Omega, \mu)$, all we have to show is

$$\left|\int_\Omega u^*\,d\mu\right| \le \|\mu\|_{W^{-1,\infty}(\Omega)}\Big(|Du|(\Omega) + \int_{\partial\Omega} |u|\,d\mathcal{H}^{N-1}\Big)\,, \tag{1.18}$$

for all $u \in BV(\Omega) \cap L^\infty(\Omega)$. We prove this inequality in two steps.

Step 1: Assuming that $u \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$, we consider the sequence $\{w_n\}$ in $W^{1,1}(\Omega) \cap C(\Omega)$ of the Lemma 1.4.3. Hence, $w_n \in L^\infty(\Omega)$ and

$\|w_n\|_{L^\infty(\Omega)} \le \|u|_{\partial\Omega}\|_{L^\infty(\partial\Omega)}$ for all $n \in \mathbb{N}$. Then,

$$\begin{aligned}\left|\int_\Omega (u^* - w_n^*)\, d\mu\right| &= \left|\langle \mu, (u - w_n)\rangle_{W^{-1,\infty}(\Omega), W_0^{1,1}(\Omega)}\right| \\ &\le \|\mu\|_{W^{-1,\infty}(\Omega)} \int_\Omega |\nabla u - \nabla w_n|\, dx \\ &\le \|\mu\|_{W^{-1,\infty}(\Omega)} \left(\int_\Omega |\nabla u|\, dx + \int_{\partial\Omega} |u|\, d\mathcal{H}^{N-1} + \frac{1}{n}\right),\end{aligned}$$

and it follows that

$$\begin{aligned}\left|\int_\Omega u^*\, d\mu\right| \le \left|\int_\Omega (u^* - w_n^*)\, d\mu\right| + \left|\int_\Omega w_n^*\, d\mu\right| \qquad (1.19) \\ \le \|\mu\|_{W^{-1,\infty}(\Omega)} \left(\int_\Omega |\nabla u|\, dx + \int_{\partial\Omega} |u|\, d\mathcal{H}^{N-1} + \frac{1}{n}\right) + \left|\int_\Omega w_n^*\, d\mu\right|.\end{aligned}$$

Since the sequence $\{w_n\}$ converges pointwise to 0 and it is uniformly bounded in $L^\infty(\Omega)$, by Lebesgue's theorem,

$$\lim_{n\to\infty} \int_\Omega w_n^*\, d\mu = 0\,.$$

Now, taking the limit in (1.19), we obtain (1.18).

Step 2: In the general case, we apply Proposition 1.4.2 and find a sequence $\{u_n\}$ in $W^{1,1}(\Omega) \cap C^\infty(\Omega) \cap L^\infty(\Omega)$ such that

(1) $u_n \to u^*$ in $L^1(\Omega, \mu)\,,$

(2) $\displaystyle\int_\Omega |\nabla u_n|\, dx \to |Du|(\Omega)\,,$

(3) $u_n|_{\partial\Omega} = u|_{\partial\Omega}$ for all $n \in \mathbb{N}\,,$

(4) $|u_n(x)| \le \|u\|_{L^\infty(\Omega)}$ $|\mu|$-a.e. for all $n \in \mathbb{N}\,.$

Then, it follows from

$$\left| \int_\Omega u_n \, d\mu \right| \le \|\mu\|_{W^{-1,\infty}(\Omega)} \left(\int_\Omega |\nabla u_n| \, dx + \int_{\partial\Omega} |u_n| \, d\mathcal{H}^{N-1} \right) ,$$

for all $n \in N$, that (1.18) holds. □

Thanks to Theorem 1.4.4 we are able to prove the following result, which will be essential in Subsection 1.4.2.

Corollary 1.4.5. *Let* $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ *be such that* $\operatorname{div} \mathbf{z} = \xi + f$ *for a certain Radon measure* ξ *and a certain* $f \in L^{N,\infty}(\Omega)$*. If either* $\xi \ge 0$ *or* $\xi \le 0$*, then* $\mu = \operatorname{div} \mathbf{z}$ *can be extended to* $BV(\Omega)$ *and*

$$\|\mu\|_{W^{-1,\infty}(\Omega)} = \sup \left\{ \left| \int_\Omega u^* \, d\mu \right| \, : \, u \in BV(\Omega) \ \ \textit{with} \ \ \|u\|_{BV(\Omega)} \le 1 \right\} ,$$

where $\|u\|_{BV(\Omega)} = |Du|(\Omega) + \int_{\partial\Omega} |u| \, d\mathcal{H}^{N-1}$.

Moreover, $BV(\Omega) \hookrightarrow L^1(\Omega, \mu)$*.*

Proof. Consider $u \in BV(\Omega)$. Denote $u_+ = \max\{u, 0\}$ and apply Theorem 1.4.4 to $T_k(u_+)$ for every $k > 0$ (see (I.3) for definition of truncations). Then,

$$\begin{aligned} \left| \int_\Omega T_k(u_+)^* \, d\mu \right| &\le \|\mu\|_{W^{-1,\infty}(\Omega)} \left(|DT_k(u_+)|(\Omega) + \int_{\partial\Omega} T_k(u_+) \, d\mathcal{H}^{N-1} \right) \\ &\le \|\mu\|_{W^{-1,\infty}(\Omega)} \left(|Du_+|(\Omega) + \int_{\partial\Omega} u_+ \, d\mathcal{H}^{N-1} \right) . \end{aligned} \tag{1.20}$$

On the other hand, observing that u^* is a ξ-measurable function we obtain

$$\int_\Omega T_k(u_+)^* \, d\mu = \int_\Omega T_k(u_+)^* \, d\xi + \int_\Omega T_k(u_+(x)) f(x) \, dx \, ,$$

for every $k > 0$. We may now apply Levi's theorem and Lebesgue's theorem to deduce

$$\lim_{k\to+\infty}\int_\Omega T_k(u_+)^*\,d\xi = \int_\Omega (u_+)^*\,d\xi\,,$$

and

$$\lim_{k\to+\infty}\int_\Omega T_k(u_+(x))f(x)\,dx = \int_\Omega u_+(x)f(x)\,dx\,.$$

Thus,

$$\lim_{k\to+\infty}\int_\Omega T_k(u_+)^*\,d\mu = \int_\Omega (u_+)^*\,d\xi + \int_\Omega u_+(x)f(x)\,dx = \int_\Omega (u_+)^*\,d\mu\,.$$

Now, taking the limit when $k\to\infty$ in (1.20), it yields

$$\left|\int_\Omega (u_+)^*\,d\mu\right| \le \|\mu\|_{W^{-1,\infty}(\Omega)}\left(|Du_+|(\Omega) + \int_{\partial\Omega} u_+\,d\mathcal{H}^{N-1}\right). \tag{1.21}$$

Assume, in order to be concrete, that $\xi \ge 0$ and let $\mu = \mu^+ - \mu^-$ be a decomposition such that μ^+ and μ^- are both nonnegative measures. Since

$$\int_\Omega (u_+)^*\,d\mu^- = \int_\Omega u_+(x)f_-(x)\,dx\,,$$

we already have that $(u_+)^*$ is μ^--integrable. Hence, as a consequence of (1.21), we deduce that $(u_+)^*$ is also μ^+-integrable and then, $(u_+)^*$ is μ-integrable too.

Since we may prove a similar inequality to $u_- = \max\{-u, 0\}$, adding both inequalities we deduce that u^* is μ-integrable and that

$$\left|\int_\Omega u^*\,d\mu\right| \le \|\mu\|_{W^{-1,\infty}(\Omega)}\left(|Du|(\Omega) + \int_{\partial\Omega} |u|\,d\mathcal{H}^{N-1}\right)$$

holds. $\square$

1.4.2 A Green's formula

As we have seen in Introduction, a Green's formula holds for a general vector field $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ and a function $u \in BV(\Omega) \cap L^\infty(\Omega)$ and a vector field $\mathbf{z}$ satisfying $\operatorname{div} \mathbf{z} \geq f \in L^{N,\infty}(\Omega)$. Now, we generalize that result in our context, that is, for a general function $u \in BV(\Omega)$. We have to begin with the appropriated definition of the pairing $(\mathbf{z}, Du)$.

Let $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ and $u \in BV(\Omega)$. Assume that $\operatorname{div} \mathbf{z} = \xi + f$, with ξ a Radon measure satisfying either $\xi \geq 0$ or $\xi \leq 0$, and $f \in L^{N,\infty}(\Omega)$. In the spirit of [13], we define the following distribution on Ω. For every $\varphi \in C_0^\infty(\Omega)$, we write

$$\langle (\mathbf{z}, Du), \varphi \rangle = -\int_\Omega u^* \, \varphi \, d\mu - \int_\Omega u \, \mathbf{z} \cdot \nabla \varphi \, dx \,, \tag{1.22}$$

where $\mu = \operatorname{div} \mathbf{z}$. Note that subsection 1.4.1 implies that every term in the above definition makes sense. In what follows, we prove that this distribution is actually a Radon measure with finite total variation.

Proposition 1.4.6. *Let $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ and $u \in BV(\Omega)$ and assume that $\operatorname{div} \mathbf{z} = \xi + f$ where ξ is a Radon measure satisfying either $\xi \geq 0$ or $\xi \leq 0$ and $f \in L^{N,\infty}(\Omega)$. Then, the distribution $(\mathbf{z}, Du)$ defined in (1.22) satisfies*

$$|\langle (\mathbf{z}, Du), \varphi \rangle| \leq \|\varphi\|_{L^\infty(U)} \|\mathbf{z}\|_{L^\infty(U)} \int_U |Du| \,, \tag{1.23}$$

for all open set $U \subset \Omega$ and for all $\varphi \in C_0^\infty(U)$.

Proof. If $U \subset \Omega$ is an open set and $\varphi \in C_0^\infty(U)$, then it was proved in [58, Proposition A.1] that

$$\begin{aligned} |\langle (\mathbf{z}, DT_k(u)), \varphi \rangle| &\leq \|\varphi\|_{L^\infty(U)} \|\mathbf{z}\|_{L^\infty(U)} \int_U |DT_k(u)| \\ &\leq \|\varphi\|_{L^\infty(U)} \|\mathbf{z}\|_{L^\infty(U)} \int_U |Du| \end{aligned} \tag{1.24}$$

holds for every $k > 0$. On the other hand,

$$\langle(\mathbf{z}, DT_k(u)), \varphi\rangle = -\int_\Omega T_k(u)^*\varphi\, d\mu - \int_\Omega T_k(u)\, \mathbf{z}\cdot\nabla\varphi\, dx\,.$$

We may let $k \to \infty$ in each term on the right-hand side, due to $u^* \in L^1(\Omega, \mu)$ and $u \in L^1(\Omega)$. Therefore, the dominated convergence theorem implies

$$\lim_{k\to\infty}\langle(\mathbf{z}, DT_k(u)), \varphi\rangle = -\int_\Omega u^*\varphi\, d\mu - \int_\Omega u\, \mathbf{z}\cdot\nabla\varphi\, dx = \langle(\mathbf{z}, Du), \varphi\rangle\,,$$

and so (1.24) implies (1.23). □

Due to Proposition 1.4.6, we get the following result.

Corollary 1.4.7. *The distribution $(\mathbf{z}, Du)$ is a Radon measure. It and its total variation $|(\mathbf{z}, Du)|$ are absolutely continuous with respect to the measure $|Du|$ and the following inequalities*

$$\left|\int_B (\mathbf{z}, Du)\right| \le \int_B |(\mathbf{z}, Du)| \le \|\mathbf{z}\|_{L^\infty(U)} \int_B |Du|$$

hold for all Borel sets B and for all open sets U such that $B \subset U \subset \Omega$.

We recall that for every $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$, a weak trace on $\partial\Omega$ of the normal component of $\mathbf{z}$ is defined in [13] and denoted by $[\mathbf{z}, \nu]$. Using this weak trace, we show in the following proposition that a Green's formula holds for $\mathbf{z}$ and u defined as above.

Proposition 1.4.8. *Let $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ and $u \in BV(\Omega)$ and assume that $\operatorname{div}\mathbf{z} = \xi + f$ where ξ is a Radon measure satisfying either $\xi \ge 0$ or $\xi \le 0$ and $f \in L^{N,\infty}(\Omega)$. With the above definitions, the following Green's formula holds:*

$$\int_\Omega u^*\, d\mu + \int_\Omega (\mathbf{z}, Du) = \int_{\partial\Omega} u\, [\mathbf{z}, \nu]\, d\mathcal{H}^{N-1}\,, \qquad (1.25)$$

where $\mu = \operatorname{div} \mathbf{z}$.

Proof. Applying the Green's formula proved in [58, Theorem A.1], we obtain

$$\int_\Omega T_k(u)^* \, d\mu + \int_\Omega (\mathbf{z}, DT_k(u)) = \int_{\partial\Omega} T_k(u) \, [\mathbf{z}, \nu] \; d\mathcal{H}^{N-1} \,, \tag{1.26}$$

for every $k > 0$. We can argue in the same way that in Proposition 1.4.6 to deduce that

$$\lim_{k\to\infty} \int_\Omega (\mathbf{z}, DT_k(u)) = \int_\Omega (\mathbf{z}, Du) \,.$$

Moreover, we may apply the dominated convergence theorem in the other terms since $u^* \in L^1(\Omega, \mu)$ and $u \in BV(\Omega) \hookrightarrow L^1(\partial\Omega)$. Hence, taking the limit $k \to \infty$ in equation (1.26), we get (1.25). □

1.4.3 The chain rule

We stress that there is a chain rule for BV-functions, the more general formula is due to L. Ambrosio and G. Dal Maso (see [7, Theorem 3.101]; see also [7, Theorem 3.96]). In our framework, it states that if $v \in BV(\Omega)$ satisfies $D^j v = 0$ and $u = G(v)$, where G is a Lipschitz continuous real function, then $u \in BV(\Omega)$ and

$$Du = G'(v) Dv \,.$$

We cannot apply directly this result in our context since G' need not to be bounded. Hence, the following slight generalization is needed.

Theorem 1.4.9. *Let* $v \in BV(\Omega)$ *such that* $D^j v = 0$ *and let* g *be a continuous, nonnegative and unbounded real function. We define*

$$G(s) = \int_0^s g(\sigma) \, d\sigma \,.$$

Assuming $u = G(v) \in L^1(\Omega)$, we have that $u \in BV(\Omega)$ if and only if $g(v)^*|Dv|$ is a finite measure and, in this case, $|Du| = g(v)^*|Dv|$ as measures in Ω.

Proof. Let $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geq 0$. We apply the chain rule (see [7, Theorem 3.96]) to get the next equality:

$$\int_{\{v<k\}} \varphi\,|Du| = \int_{\{v<k\}} \varphi\, g(T_k(v))^*\,|Dv| = \int_{\{v<k\}} \varphi\, g(v)^*\,|Dv|\,.$$

Now, using the monotone convergence theorem, we take limits when $k \to \infty$ and we get that

$$\int_\Omega \varphi\,|Du| = \int_\Omega \varphi\, g(v)^*\,|Dv|\,,$$

and if one integral is finite, so is the other. Finally, since $\varphi = \varphi^+ - \varphi^-$ with $\varphi^+ = \max\{\varphi, 0\}$ and $\varphi^- = \max\{-\varphi, 0\}$, and both are nonnegative, the result follows for every $\varphi \in C_0^\infty(\Omega)$. □

1.5 Solutions for $L^{N,\infty}$-data

This section is devoted to solve problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{Du}{|Du|}\right) + |Du| = f(x) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.27}$$

for a nonnegative datum $f \in L^{N,\infty}(\Omega)$. We begin by introducing the notion of solution to this problem.

Definition 1.5.1. *Let $f \in L^{N,\infty}(\Omega)$ with $f \geq 0$. We say that $u \in BV(\Omega)$ satisfying $D^j u = 0$ is a weak solution of problem (1.27) if there exists*

$\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ *with* $\|\mathbf{z}\|_{L^\infty(\Omega)} \le 1$ *such that*

$$\begin{aligned} &\textbf{(i)} \quad -\operatorname{div}\mathbf{z} + |Du| = f \quad \textit{in} \ \ \mathcal{D}'(\Omega)\,, \\ &\textbf{(ii)} \quad (\mathbf{z}, Du) = |Du| \quad \textit{as measures in} \ \ \Omega\,, \\ &\textbf{(iii)} \quad u|_{\partial\Omega} = 0\,. \end{aligned}$$

Remark 1.5.2. Let us notice that any solution $u \in BV(\Omega) \cap L^\infty(\Omega)$ to problem (1.27) with datum $0 \le f \in L^q(\Omega)$ for $q > N$ satisfies

$$-\text{div}\,\left(e^{-u}\mathbf{z}\right) = e^{-u} f$$

in the sense of distributions (see [55, Remark 3.4]).

Theorem 1.5.3. *Let* $f \in L^{N,\infty}(\Omega)$ *with* $f \ge 0$*. Then, there is a unique nonnegative weak solution of problem* (1.27)*.*

Proof. The proof is divided in several steps.

Step 1: Approximating problems.

The function f is in $L^{N,\infty}(\Omega)$ so, there exists a sequence $\{f_n\}$ in $L^\infty(\Omega)$ such that f_n converges to f in $L^1(\Omega)$ (we may take, for example, $f_n(x) = T_n(f(x))$, where the truncation function $T_n(s)$ is defined in (I.3)).

In [55, Theorem 3.5] it was proved that there exists $u_n \in BV(\Omega) \cap L^\infty(\Omega)$ with $D^j u_n = 0$ and $u_n \ge 0$ which is a solution to problem

$$\begin{cases} -\text{div}\left(\dfrac{Du_n}{|Du_n|}\right) + |Du_n| = f_n(x) & \text{in} \ \ \Omega\,, \\ u_n = 0 & \text{on} \ \ \partial\Omega\,. \end{cases} \tag{1.28}$$

That is, there exists a vector field $\mathbf{z}_n$ in $\mathcal{DM}^\infty(\Omega)$ such that

$$\textbf{(i)} \quad -\operatorname{div}\mathbf{z}_n + |Du_n| = f_n \quad \text{in} \ \ \mathcal{D}'(\Omega)\,,$$

$$\textbf{(ii)} \quad (\mathbf{z}_n, Du_n) = |Du_n| \ \text{ as measures in } \ \Omega\,,$$

$$\textbf{(iii)} \quad u_n|_{\partial\Omega} = 0\,.$$

Taking into account Remark 1.5.2, we have that

$$-\operatorname{div}(e^{-u_n}\mathbf{z}_n) = e^{-u_n} f_n \ \text{ in } \ \mathcal{D}'(\Omega)\,. \tag{1.29}$$

Step 2: BV-estimate.

Taking the function test $\frac{T_k(u_n)}{k}$ in problem (1.28), we get

$$\begin{aligned}\frac{1}{k}\int_\Omega (\mathbf{z}_n, DT_k(u_n)) + \frac{1}{k}\int_\Omega T_k(u_n)^*|Du_n| &= \int_\Omega f_n \frac{T_k(u_n)}{k}\,dx \\ &\le \int_\Omega f_n\,dx \le C\,,\end{aligned}$$

where C does not depend on n. Since $(\mathbf{z}_n, Du_n) = |Du_n|$, it follows from Proposition 1.2.5 that $(\mathbf{z}_n, DT_k(u_n)) = |DT_k(u_n)|$, which is nonnegative. Thus,

$$\frac{1}{k}\int_\Omega T_k(u_n)^*|Du_n| \le C\,.$$

Then, letting $k \to 0^+$ in the above inequality we get

$$\int_\Omega |Du_n| \le C\,.$$

Therefore, the sequence $\{u_n\}$ is bounded in $BV(\Omega)$ and there exist $u \in BV(\Omega)$ such that, up to a subsequence, $u_n \to u$ in $L^1(\Omega)$ and, moreover, $\{Du_n\}$ converges to Du $*$-weakly as measures when $n \to \infty$.

Step 3: Vector field.

Now, we would like to find a vector field $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ with $\|\mathbf{z}\|_{L^\infty(\Omega)} \le 1$ such that

$$-\operatorname{div}\mathbf{z} + |Du| \le f \ \text{ in } \ \mathcal{D}'(\Omega)\,.$$

Since the sequence $\{\mathbf{z}_n\}$ is bounded in $L^\infty(\Omega;\mathbb{R}^N)$, there exists $\mathbf{z} \in L^\infty(\Omega;\mathbb{R}^N)$ such that $\mathbf{z}_n \rightharpoonup \mathbf{z}$ $*$-weakly in $L^\infty(\Omega;\mathbb{R}^N)$. In addition, since $\|\mathbf{z}_n\|_{L^\infty(\Omega)} \leq 1$ we get $\|\mathbf{z}\|_{L^\infty(\Omega)} \leq 1$.

Using $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geq 0$ as a function test in (1.28), we obtain the following equality:

$$\int_\Omega \mathbf{z}_n \cdot \nabla\varphi\, dx + \int_\Omega \varphi\, |Du_n| = \int_\Omega f_n\, \varphi\, dx\,,$$

and when we take $n \to \infty$, using the lower semicontinuity of the functional (I.5), it becomes

$$\int_\Omega \mathbf{z} \cdot \nabla\varphi\, dx + \int_\Omega \varphi\, |Du| \leq \int_\Omega f\, \varphi\, dx\,.$$

Therefore,

$$-\operatorname{div}\mathbf{z} + |Du| \leq f \quad \text{in} \quad \mathcal{D}'(\Omega) \tag{1.30}$$

and $-\operatorname{div}\mathbf{z}$ is a Radon measure. Moreover, since $-\operatorname{div}\mathbf{z}_n = f_n - |Du_n|$ holds for every $n \in \mathbb{N}$, the sequence $\{-\operatorname{div}\mathbf{z}_n\}$ is bounded in the space of measures and, since $\{-\operatorname{div}\mathbf{z}_n\}$ converges to $-\operatorname{div}\mathbf{z}$ in a distributional sense, we deduce that $-\operatorname{div}\mathbf{z}$ is a Radon measure with finite total variation.

On the other hand, multiplying (1.29) by $\varphi \in C_0^\infty(\Omega)$, Green's formula provides us

$$\int_\Omega e^{-u_n}\mathbf{z}_n \cdot \nabla\varphi\, dx = \int_\Omega f_n e^{-u_n}\varphi\, dx\,,$$

and letting $n \to \infty$ we get

$$\int_\Omega e^{-u}\mathbf{z} \cdot \nabla\varphi\, dx = \int_\Omega f e^{-u}\varphi\, dx\,.$$

Therefore,

$$-\operatorname{div}(e^{-u}\mathbf{z}) = f e^{-u} \quad \text{in} \quad \mathcal{D}'(\Omega)\,. \tag{1.31}$$

Step 4: $D^j u = 0$.

In this step, we are adapting an argument used in [40], which relies on [6, Proposition 3.4] and [25, Lemma 5.6] (see also [2, Proposition 2]). We will make some manipulations on the restriction of the measure $\operatorname{div}(u\, e^{-u}\, \mathbf{z})$ over J_u, i.e., the set of all jump points of u. Nevertheless, we have to begin by proving that inequality

$$|De^{-u}| \le (e^{-u}\mathbf{z}, Du) \tag{1.32}$$

holds as measures in Ω. We begin by recalling

$$-\operatorname{div}(e^{-u_n}\mathbf{z}_n) = e^{-u_n} f_n \quad \text{in} \quad \mathcal{D}'(\Omega)\,,$$

since u_n is the solution to problem (1.28). Using that $u_n = G_k(u_n) + T_k(u_n)$ (see definitions (I.3) and (I.4)), we can write

$$\begin{aligned} -\operatorname{div}(e^{-u_n}\mathbf{z}_n) &= -e^{-G_k(u_n)}\operatorname{div}(e^{-T_k(u_n)}\mathbf{z}_n) - (e^{-T_k(u_n)}\mathbf{z}_n, De^{-G_k(u_n)}) \\ &= -e^{-G_k(u_n)}\operatorname{div}(e^{-T_k(u_n)}\mathbf{z}_n) - (e^{-T_k(u_n)})^*(\mathbf{z}_n, De^{-G_k(u_n)}) \\ &= -e^{-G_k(u_n)}\operatorname{div}(e^{-T_k(u_n)}\mathbf{z}_n) + (e^{-u_n})^*|DG_k(u_n)|\,, \end{aligned}$$

where we have used Proposition 1.2.6 and [55, Proposition 2.2], and so

$$\begin{aligned} e^{-T_k(u_n)} f_n &= -\operatorname{div}(e^{-T_k(u_n)}\mathbf{z}_n) + (e^{-T_k(u_n)})^*|DG_k(u_n)| \\ &= -\operatorname{div}(e^{-T_k(u_n)}\mathbf{z}_n) + e^{-k}|DG_k(u_n)|\,. \end{aligned} \tag{1.33}$$

Applying first the chain rule (see [7, Theorem 3.96]) and then Propositions 1.2.5 and 1.2.6, we have

$$\begin{aligned} |De^{-T_k(u_n)}| &= (e^{-T_k(u_n)})^*|DT_k(u_n)| \\ &= (e^{-T_k(u_n)})^*(\mathbf{z}_n, DT_k(u_n)) = (e^{-T_k(u_n)}\mathbf{z}_n, DT_k(u_n))\,. \end{aligned} \tag{1.34}$$

Let $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geq 0$, due to (1.34) and (1.33), we get

$$\begin{aligned}
&\int_\Omega \varphi\,|De^{-T_k(u_n)}| = \langle (e^{-T_k(u_n)}\mathbf{z}_n, DT_k(u_n)), \varphi\rangle \\
&= -\int_\Omega T_k(u_n)\,\varphi\,\mathrm{div}\,(e^{-T_k(u_n)}\mathbf{z}_n) - \int_\Omega T_k(u_n)\,e^{-T_k(u_n)}\,\mathbf{z}_n\cdot\nabla\varphi\,dx \\
&= \int_\Omega T_k(u_n)\,\varphi\,e^{-T_k(u_n)} f_n\,dx - \int_\Omega k\,e^{-k}\varphi\,|DG_k(u_n)| \\
&\quad - \int_\Omega T_k(u_n)\,e^{-T_k(u_n)}\,\mathbf{z}_n\cdot\nabla\varphi\,dx\,.
\end{aligned}$$

That is,

$$\begin{aligned}
&\int_\Omega \varphi\,|De^{-T_k(u_n)}| + \frac{k}{e^k}\int_\Omega \varphi\,|DG_k(u_n)| \\
&= \int_\Omega T_k(u_n)\,\varphi\,e^{-T_k(u_n)}\,f_n\,dx - \int_\Omega T_k(u_n)\,e^{-T_k(u_n)}\,\mathbf{z}_n\cdot\nabla\varphi\,dx\,.
\end{aligned}$$

Now, we can take limits for $n \to \infty$ and applying the lower semicontinuity of functional (I.7) we get

$$\begin{aligned}
&\int_\Omega \varphi\,|De^{-T_k(u)}| + \frac{k}{e^k}\int_\Omega \varphi\,|DG_k(u)| \\
&\leq \int_\Omega T_k(u)\,\varphi\,e^{-T_k(u)}\,f\,dx - \int_\Omega T_k(u)\,e^{-T_k(u)}\,\mathbf{z}\cdot\nabla\varphi\,dx\,.
\end{aligned}$$

Finally, letting $k \to \infty$ we have that

$$\int_\Omega \varphi\,|De^{-u}| \leq \int_\Omega u\,\varphi\,e^{-u}\,f\,dx - \int_\Omega u\,e^{-u}\,\mathbf{z}\cdot\nabla\varphi\,dx = \langle (e^{-u}\mathbf{z}, Du), \varphi\rangle\,.$$

Therefore, (1.32) holds:

$$|De^{-u}| \leq (e^{-u}\mathbf{z}, Du) \quad \text{as measures in} \quad \Omega\,.$$

On the other hand, we know that

$$\operatorname{div}(u\, e^{-u}\mathbf{z}) = u \operatorname{div}(e^{-u}\mathbf{z}) + (e^{-u}\mathbf{z}, Du)$$

as measures and now we will consider its restriction on the set J_u. Since, by using (1.31), we have

$$u \operatorname{div}(e^{-u}\mathbf{z}) = -u\, e^{-u} f \in L^1(\Omega)\,,$$

and $|J_u| = 0$, it follows that the measure $u \operatorname{div}(e^{-u}\mathbf{z})$ vanishes on J_u, so that

$$\operatorname{div}(u\, e^{-u}\mathbf{z}) \llcorner J_u = (e^{-u}\mathbf{z}, Du) \llcorner J_u \geq |De^{-u}| \llcorner J_u\,. \tag{1.35}$$

Now, we follow the notation used in [40] to denote the normal traces $[\mathbf{z}, \nu]^+$ and $[\mathbf{z}, \nu]^-$ in J_u. In addition, applying [40, Lemmas 2.4 and 2.5], the following manipulations can be performed on J_u:

$$\begin{aligned} \operatorname{div}(u\, e^{-u}\mathbf{z}) &= [u\, e^{-u}\mathbf{z}, \nu]^+ - [u\, e^{-u}\mathbf{z}, \nu]^- \\ &= u^+[e^{-u}\mathbf{z}, \nu]^+ - u^-[e^{-u}\mathbf{z}, \nu]^-\,. \end{aligned} \tag{1.36}$$

Moreover, we also deduce that, on J_u,

$$\operatorname{div}\left(e^{-u}\mathbf{z}\right) = [e^{-u}\mathbf{z}, \nu]^+ - [e^{-u}\mathbf{z}, \nu]^-\,,$$

and, since

$$\operatorname{div}\left(e^{-u}\mathbf{z}\right) = -e^{-u} f \in L^1(\Omega) \quad \text{and} \quad |J_u| = 0\,,$$

it follows that $[e^{-u}\mathbf{z},\nu]^+ = [e^{-u}\mathbf{z},\nu]^-$. We write this common value as $[e^{-u}\mathbf{z},\nu]$. With this notation, (1.36) becomes

$$\begin{aligned}\operatorname{div}(u\,e^{-u}\mathbf{z}) &= (u^+ - u^-)[e^{-u}\mathbf{z},\nu] = (u^+ - u^-)(e^{-u})^-[\mathbf{z},\nu]\\ &= (u^+ - u^-)e^{-u^+}[\mathbf{z},\nu] \le (u^+ - u^-)e^{-u^+}\,.\end{aligned}$$

Thus, (1.35) and the previous inequality give us

$$\begin{aligned}(u^+ - u^-)e^{-u^+}\mathcal{H}^{N-1}\llcorner J_u \ge \operatorname{div}(u\,e^{-u}\mathbf{z})\llcorner J_u &\ge |De^{-u}|\llcorner J_u\\ &= \left(e^{-u^-} - e^{-u^+}\right)\mathcal{H}^{N-1}\llcorner J_u\,.\end{aligned}$$

Hence, for $\mathcal{H}^{N-1}$-almost all $x \in J_u$, we may use the mean value theorem to get

$$(u(x)^+ - u(x)^-)e^{-u(x)^+} \ge e^{-u(x)^-} - e^{-u(x)^+} = (u(x)^+ - u(x)^-)e^{-w(x)}$$

with $u(x)^- < w(x) < u(x)^+$. Therefore, it yields $u(x)^+ = u(x)^-$. Since this argument holds for $\mathcal{H}^{N-1}$-almost every point $x \in J_u$, we get

$$D^j u = 0\,.$$

Step 5: u is a solution to problem (1.27).

To finish the proof, it remains to check that u satisfies the three conditions of Definition 1.5.1. The previous step will be essential in this checking. Indeed, it allows us to perform the following calculations:

$$\begin{aligned}fe^{-u} &= -\operatorname{div}(e^{-u}\mathbf{z}) = -(\mathbf{z}, De^{-u}) - (e^{-u})^*\operatorname{div}\mathbf{z}\\ &\le |De^{-u}| + fe^{-u} - (e^{-u})^*|Du|\\ &= fe^{-u}\,,\end{aligned}$$

where we have apply that $|(\mathbf{z}, De^{-u})| \leq |De^{-u}|$ and (1.30). Hence,

$$-\mathrm{div}\,\mathbf{z} + |Du| = f \quad \text{in} \quad \mathcal{D}'(\Omega)\,.$$

To prove that $(\mathbf{z}, Du) = |Du|$ as measures in Ω, we just take into account (1.32), (1.9) and the chain rule (see [7, Theorem 3.96]) to get

$$|De^{-u}| \leq (e^{-u}\mathbf{z}, Du) = (e^{-u})^*(\mathbf{z}, Du) \leq (e^{-u})^*|Du| = |De^{-u}|\,,$$

from where the equality $(e^{-u})^*(\mathbf{z}, Du) = (e^{-u})^*|Du|$ as measures follows. Hence, we conclude that $(\mathbf{z}, Du) = |Du|$ as measures in Ω.

Now, we prove that $u(x) = 0$ for $\mathcal{H}^{N-1}$-almost all $x \in \partial\Omega$. To do so, we use the test function $T_k(u_n)$ in problem (1.28), so that

$$\int_\Omega (\mathbf{z}_n, DT_k(u_n)) + \int_\Omega (T_k(u_n))^*|Du_n| = \int_\Omega f\, T_k(u_n)\, dx\,.$$

Defining the auxiliary function J_k by

$$J_k(s) = \int_0^s T_k(\sigma)\, d\sigma = \begin{cases} \frac{s^2}{2} & \text{if} \quad 0 \leq s \leq k\,, \\ ks - \frac{k^2}{2} & \text{if} \quad k > s\,, \end{cases}$$

the previous equality becomes

$$\int_\Omega |DT_k(u_n)| + \int_{\partial\Omega} |T_k(u_n)|\, d\mathcal{H}^{N-1} + \int_\Omega |DJ_k(u_n)| + \int_{\partial\Omega} |J_k(u_n)|\, d\mathcal{H}^{N-1}$$
$$= \int_\Omega f\, T_k(u_n)\, dx\,.$$

Taking into account that $J_k(u_n) \to J_k(u)$ in $L^1(\Omega)$, we let $n \to \infty$ and due to the lower semicontinuity of functional (I.6) we get

$$\int_\Omega |DT_k(u)| + \int_{\partial\Omega} |T_k(u)|\, d\mathcal{H}^{N-1} + \int_\Omega |DJ_k(u)| + \int_{\partial\Omega} |J_k(u)|\, d\mathcal{H}^{N-1}$$

$$\le \int_\Omega f\, T_k(u)\, dx \le \int_\Omega f u\, dx\,.$$

Letting now $k \to \infty$ we obtain

$$\int_\Omega |Du| + \int_{\partial\Omega} |u|\, d\mathcal{H}^{N-1} + \int_\Omega \left| D\Big(\frac{u^2}{2}\Big)\right| + \int_{\partial\Omega} \frac{u^2}{2}\, d\mathcal{H}^{N-1} \le \int_\Omega f u\, dx\,.$$

On the other hand, Green's formula implies

$$\begin{aligned}\int_\Omega f u\, dx &= -\int_\Omega u^* \mathrm{div}\, \mathbf{z} + \int_\Omega u^* |Du| \\ &= \int_\Omega |Du| - \int_{\partial\Omega} u\, [\mathbf{z}, \nu]\, d\mathcal{H}^{N-1} + \int_\Omega u^* |Du|\,.\end{aligned}$$

Then,

$$\int_{\partial\Omega} (|u| + u\, [\mathbf{z}, \nu])\, d\mathcal{H}^{N-1} + \int_{\partial\Omega} \frac{u^2}{2}\, d\mathcal{H}^{N-1} \le 0\,,$$

and because of that, $u = 0$ $\mathcal{H}^{N-1}$-a.e. in $\partial\Omega$.

Now, in order to prove that there is a unique solution to our problem, we may argue as in Theorem 2.11 of Chapter 2 (we may also argue as in [55, Theorem 3.8] to get the uniqueness). □

Proposition 1.5.4. *The nonnegative solution u to problem* (1.27) *is trivial if and only if the function f is such that $\|f\|_{W^{-1,\infty}(\Omega)} \le 1$.*

Proof. Assume first that $\|f\|_{W^{-1,\infty}(\Omega)} \le 1$ and let $u \in BV(\Omega)$ be the solution to problem (1.27). Using the test function $T_k(u)$ in that problem we obtain

$$\int_\Omega (\mathbf{z}, DT_k(u)) + \int_\Omega T_k(u)^* |Du| = \int_\Omega f\, T_k(u)\, dx \le \int_\Omega f u\, dx\,. \tag{1.37}$$

Now, taking into account that $\int_\Omega T_k(u)^*|Du| \ge 0$, it yields

$$\int_\Omega (\mathbf{z}, DT_k(u)) = \int_\Omega |DT_k(u)| \le \int_\Omega f u\, dx\,.$$

Finally, letting $k \to \infty$ in (1.37) and using duality and an approximating procedure we get

$$\int_\Omega |Du| + \int_\Omega u^*|Du| \le \int_\Omega f u\, dx \le \|f\|_{W^{-1,\infty}} \int_\Omega |Du| \le \int_\Omega |Du|\,.$$

Then, $\int_\Omega u^*|Du| = 0$ and thus, $u^* = 0$ in Ω. We conclude $u(x) = 0$ for almost every $x \in \Omega$.

Now, we suppose that

$$\|f\|_{W^{-1,\infty}(\Omega)} = \sup\left\{\int_\Omega \varphi f\, dx \;:\; \int_\Omega |\nabla\varphi|\, dx = 1,\ \varphi \in W_0^{1,1}(\Omega)\right\} > 1\,,$$

that is, there exists $\psi \in W_0^{1,1}(\Omega)$ such that

$$\int_\Omega |\nabla\psi|\, dx = 1 \qquad \text{and} \qquad \int_\Omega \psi f\, dx > 1\,.$$

Finally, we use ψ as a test function in (1.27), so we get

$$\begin{aligned}\int_\Omega \psi\,|Du| &= \int_\Omega \psi\, f\, dx - \int_\Omega \mathbf{z}\cdot\nabla\psi\, dx > \int_\Omega |\nabla\psi|\, dx - \int_\Omega \mathbf{z}\cdot\nabla\psi\, dx \\ &\ge \int_\Omega |\nabla\psi|\, dx - \int_\Omega \|\mathbf{z}\|_{L^\infty(\Omega)}|\nabla\psi|\, dx \ge 0\,.\end{aligned}$$

Therefore, $|Du| \ne 0$ and so $u \ne 0$ in Ω. $\square$

Remark 1.5.5. This phenomenon of trivial solutions for nontrivial data is usual in problems involving the 1-Laplacian operator. It is worth comparing the above result with [57, Theorem 4.1] (see also [58, Theorem 4.2]), where the Dirichlet problem for the equation $-\mathrm{div}\left(\frac{Du}{|Du|}\right) = f(x)$ is studied. Indeed, for such a problem it is seen that a datum satisfying $\|f\|_{W^{-1,\infty}(\Omega)} < 1$ implies a trivial solution, while no BV-solution can exist for $\|f\|_{W^{-1,\infty}(\Omega)} > 1$. Obviously, the most interesting case is when $\|f\|_{W^{-1,\infty}(\Omega)} = 1$; then nontrivial solutions can be found for some data but the trivial solution always exists. In our case, this dichotomy does not hold: for $\|f\|_{W^{-1,\infty}(\Omega)} = 1$, only trivial solutions exist.

To study the summability of the solution to problem (1.27), we need the following technical result which is also useful in Section 1.7.

Lemma 1.5.6. *Let $u \in BV(\Omega)$ with $D^j u = 0$ and let $\mathbf{z}$ be a vector field in $\mathcal{DM}^\infty(\Omega)$ with $\|\mathbf{z}\|_{L^\infty(\Omega)} \le 1$ such that $\operatorname{div}\mathbf{z} = \xi + f$, where ξ is a Radon measure satisfying either $\xi \ge 0$ or $\xi \le 0$ and $f \in L^{N,\infty}(\Omega)$. If G is an increasing and C^1 function and $\lim_{s\to\infty} G(s) = \infty$, then, $(\mathbf{z}, Du) = |Du|$ implies $(\mathbf{z}, DG(u)) = |DG(u)|$.*

Proof. Since $(\mathbf{z}, Du) = |Du|$, we have $(\mathbf{z}, DT_k(u)) = |DT_k(u)|$ for all $k > 0$. Using [55, Proposition 2.2] we get $(\mathbf{z}, DG(T_k(u))) = |DG(T_k(u))|$ for all $k > 0$. Now, since $G(T_k(u)) = T_{G(k)}G(u)$ and $\lim_{s\to\infty} G(s) = \infty$ we apply Proposition 1.2.5 to get $(\mathbf{z}, DG(u)) = |DG(u)|$. □

Proposition 1.5.7. *If u is the nonnegative solution to problem (1.27), then $u^n \in L^1(\Omega)$ and $u^n \in BV(\Omega)$ for all $n \in \mathbb{N}$. Consequently, $u \in L^q(\Omega)$ for all $1 \le q < \infty$.*

Proof. We prove this result by induction. On the one hand, if u is the solution of problem (1.27), then choosing the truncation $T_k(u)$ with $k > 0$ as a test function in that problem, we get

$$\int_\Omega |DT_k(u)| + \int_\Omega T_k(u)^* \, |Du| = \int_\Omega f \, T_k(u) \, dx \le \int_\Omega f \, u \, dx \,. \tag{1.38}$$

On the other hand, since $BV(\Omega) \hookrightarrow L^1(\Omega)$ and $(T_k(u))^2 \in BV(\Omega) \cap L^\infty(\Omega)$, we get

$$\int_\Omega (T_k(u))^2 \, dx \le C \int_\Omega |D(T_k(u))^2| \le 2\, C \int_\Omega T_k(u)^* |Du| \,,$$

for some nonnegative constant C.

Now, since the first integral in expression (1.38) is nonnegative, we also obtain

$$\int_\Omega (T_k(u))^2\,dx \le 2\,C\int_\Omega T_k(u)^*|Du| \le 2\,C\int_\Omega f\,u\,dx < \infty\,.$$

Hence, taking limits when $k\to\infty$, it results:

$$\int_\Omega u^2\,dx \le 2\,C\int_\Omega f\,u\,dx < \infty\,,$$

and also

$$\int_\Omega u^*|Du| \le \int_\Omega f\,u\,dx < \infty\,.$$

That is, we have proved that $u^2\in L^1(\Omega)$ and $u^*|Du|$ is a finite measure. Thus, Theorem 1.4.9 provides us $u^2\in BV(\Omega)$ and $2u^*|Du| = |Du^2|$.

Now, let $n\in\mathbb{N}$ and assume $u^n\in BV(\Omega)$. Taking the test function $(T_k(u))^n$ in (1.27), it yields

$$\int_\Omega (\mathbf{z}, D(T_k(u))^n) + \int_\Omega ((T_k(u))^n)^*|Du| \le \int_\Omega f\,u^n\,dx\,.$$

Moreover, we also know that

$$\begin{aligned}\int_\Omega (T_k(u))^{n+1}\,dx \le C\int_\Omega |D(T_k(u))^{n+1}| &\le (n+1)\,C\int_\Omega ((T_k(u))^n)^*|Du|\\ &\le (n+1)\,C\int_\Omega f u^n\,dx\,,\end{aligned}$$

where we have used that $(\mathbf{z}, D(T_k(u))^n) = |D(T_k(u))^n| \ge 0$ (by Lemma 1.5.6). Finally, taking limits when $k\to\infty$ in the previous inequality, we get

$$\int_\Omega u^{n+1}\,dx \le (n+1)\,C\int_\Omega f\,u^n\,dx < \infty$$

and also

$$\int_\Omega (u^n)^*|Du| \le \int_\Omega f\,u^n\,dx < \infty\,.$$

Thus, $u^n \in L^1(\Omega)$ and the integral $\int_\Omega (u^n)^*|Du|$ is bounded. Consequently, $u^{n+1} \in BV(\Omega)$ by Theorem 1.4.9. □

Remark 1.5.8. Let f be a nonnegative function in $L^q(\Omega)$ for $q > N$. Then, the solution to problem (1.27) belongs to $L^\infty(\Omega)$ (see [55, Theorem 3.5]).

1.6 Radial solutions

In this section we show some radial solutions in $\Omega = B_R(0)$ with $R > 0$ for particular data in $L^{N,\infty}(\Omega)$. In [55, Section 4], some examples of bounded solutions for data $f \in L^q(\Omega)$ with $q > N$ can be found. In Example 1.6.1 we will show bounded solutions for $f \in L^{N,\infty}(\Omega)\backslash L^N(\Omega)$ while in Example 1.6.3 we will present unbounded solutions. Therefore, unbounded solutions really occur.

Throughout this section, we take $u(x) = h(|x|)$ with $h(r) \geq 0$, $h(R) = 0$ and $h'(r) \leq 0$. To deal with the examples, we consider two regions. If $h'(r) < 0$, we know that $\mathbf{z}(x) = \frac{Du}{|Du|} = -\frac{x}{|x|}$, so that $-\operatorname{div}\mathbf{z}(x) = \frac{N-1}{|x|}$. In the other case, $h'(r) = 0$ and then, the solution is constant and we only have to determine the radial vector field $\mathbf{z}(x) = \xi(|x|)\, x$, so that $\operatorname{div}\mathbf{z}(x) = \xi'(|x|)|x| + N\xi(|x|)$.

We would like to stress that we are interested in the continuity of the vector field. Otherwise, it would has a jump and, as a consequence, the measure $\operatorname{div}\mathbf{z}$ would have a singular part concentrated on a surface of the form $|x| = \varrho$, and measure $|Du|$ would also have that singular part. Hence, it would induce jumps on the solution which contradicts Definition 1.5.1.

Example 1.6.1. *Set* $\Omega = B_R(0)$ *and consider the following particular case of problem* (1.27)*:*

$$\begin{cases} -\mathrm{div}\left(\dfrac{Du}{|Du|}\right) + |Du| = \dfrac{N-1}{|x|} + \dfrac{\lambda}{|x|^q} & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

with $0 < q < 1$ *and* $\lambda > 0$*. Then, this problem has a bounded solution.*

Let $0 \le \rho_1 < \rho_2$. First, we assume that u is constant in a ring: $h'(r) = 0$ for all $\rho_1 < r < \rho_2$, and we consider the vector field $\mathbf{z}(x) = x\,\xi(|x|)$. Then, denoting $r = |x|$, the equation $-\mathrm{div}\,\mathbf{z} + |Du| = \frac{N-1}{|x|} + \frac{\lambda}{|x|^q}$ becomes

$$-(r\xi'(r) + N\xi(r)) = \frac{N-1}{r} + \frac{\lambda}{r^q}\,,$$

which is equivalent to

$$-(r^N\,\xi(r))' = (N-1)\,r^{N-2} + \lambda\,r^{N-1-q}\,.$$

Therefore, solving the equation we get the vector field

$$\mathbf{z}(x) = -x\,|x|^{-1} - \frac{\lambda}{N-q}x\,|x|^{-q} + Cx\,|x|^{-N}\,, \tag{1.39}$$

for all $\rho_1 < |x| < \rho_2$ and for some constant C. We next see under what conditions we can find a value for this constant satisfying $\|\mathbf{z}\|_{L^\infty(\Omega)} \le 1$. To this end, we distinguish three cases.

1. Assuming that $0 < \rho_1 < \rho_2 < R$ (and that $\mathbf{z}$ is continuous), if $|x| = \rho_1$, then

$$-x\,|x|^{-1} = -x\,|x|^{-1} - \frac{\lambda}{N-q}x\,|x|^{-q} + Cx\,|x|^{-N}\,,$$

and it implies $\frac{\lambda}{N-q} x\,|x|^{-q} = Cx\,|x|^{-N}$. Thus, we deduce that $C = \frac{\lambda}{N-q}\rho_1^{N-q}$. The same argument leads to $C = \frac{\lambda}{N-q}\rho_2^{N-q}$ when $|x| = \rho_2$. Therefore, $\rho_1 = \rho_2$ and we have got a contradiction.

2. If we assume $0 < \rho_1 < \rho_2 = R$, then we may argue as above and find $C = \frac{\lambda}{N-q}\rho_1^{N-q}$. Substituting this value in (1.39), we get

$$\mathbf{z}(x) = -x\,|x|^{-1} - \frac{\lambda}{N-q} x\,|x|^{-q} + \frac{\lambda}{N-q}\rho_1^{N-q} x\,|x|^{-N}.$$

Thus, condition $\|\mathbf{z}\|_{L^\infty(\Omega)} \le 1$ yields

$$\left|1 + \frac{\lambda}{N-q}\,|x|^{1-q} - \frac{\lambda}{N-q}\rho_1^{N-q}\,|x|^{1-N}\right| \le 1\,.$$

Nevertheless, this fact does not hold since

$$1 + \frac{\lambda}{N-q}\,r^{1-q} - \frac{\lambda}{N-q}\rho_1^{N-q}\,r^{1-N} > 1\,,$$

for $r > \rho_1$.

3. If we assume $0 = \rho_1 < \rho_2 < R$, then $\mathbf{z} \in L^\infty(\Omega;\mathbb{R}^N)$ implies $C = 0$. So (1.39) becomes

$$\mathbf{z}(x) = -x\,|x|^{-1} - \frac{\lambda}{N-q} x\,|x|^{-q}\,,$$

and it follows from $\|\mathbf{z}\|_{L^\infty(\Omega)} \le 1$ that $\frac{\lambda}{N-q} x\,|x|^{-q}$ vanishes. Hence, $\lambda = 0$ and a contradiction is obtained again.

In any case we get a contradiction, so that $h'(r) = 0$ cannot hold on $\rho_1 < r < \rho_2$. Therefore, we take $\mathbf{z}(x) = -\frac{x}{|x|}$. Then, the equation becomes

$$-h'(r) = \frac{\lambda}{r^q}\,,$$

and the solution satisfying the boundary condition is given by

$$u(x) = \frac{\lambda}{1-q}(R^{1-q} - |x|^{1-q})\,.$$

That is, the solution to problem (1.27) with datum $f(x) = \frac{N-1}{|x|} + \frac{\lambda}{|x|^q} \in L^{N,\infty}(\Omega) \setminus L^N(\Omega)$ is bounded.

Remark 1.6.2. We may perform similar computations to those of the previous example to study the problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{Du}{|Du|}\right) + |Du| = \dfrac{N-1}{|x|} + \lambda & \text{in } \ B_R(0)\,,\\ u = 0 & \text{on } \ \partial B_R(0)\,, \end{cases}$$

with $\lambda > 0$. Then, the solution is given by $u(x) = \lambda(R - |x|)$ with associated vector field $\mathbf{z}(x) = -\frac{x}{|x|}$.

Example 1.6.3. *Let $\Omega = B_R(0)$ and $0 < \rho \leq R$. Consider the following problem:*

$$\begin{cases} -\operatorname{div}\left(\dfrac{Du}{|Du|}\right) + |Du| = \dfrac{\lambda}{|x|}\chi_{B_\rho(0)}(x) & \textit{in } \ \Omega\,,\\ u = 0 & \textit{on } \ \partial\Omega\,, \end{cases}$$

with $\lambda > 0$. Then, the solution to this problem is not necessarily bounded.

Two cases according to the value of λ are distinguished:

- *Case 1:* $0 < \lambda \leq N-1$.

Assuming $h'(r) < 0$ for any $0 \leq r < R$, the vector field is given by $\mathbf{z}(x) = -\frac{x}{|x|}$ and the equation becomes

$$\frac{N-1}{r} - h'(r) = \frac{\lambda}{r}\chi_{]0,\rho[}(r)\,. \tag{1.40}$$

When $\rho < R$, we have to distinguish two regions: where $\rho \le r \le R$ in which equation (1.40) becomes $h'(r) = \frac{N-1}{r}$, and where $0 \le r < \rho$ in which we get $h'(r) = \frac{N-1-\lambda}{r}$. Both expressions are nonnegative and so they provide a contradiction with our hypothesis. We get the same contradiction when $\rho = R$. Therefore, $h'(r) = 0$ holds for all $0 \le r < R$ and then, $h(r) = 0$ for all $0 \le r < R$ because of the boundary condition.

To obtain the field $\mathbf{z}(x) = \xi(|x|)\, x$ we have to consider the equation

$$-(r^N \xi(r))' = \lambda\, r^{N-2} \chi_{]0,\rho[}(r)\,.$$

If $0 \le r < \rho$, then we obtain the field $\xi(r) = -\frac{\lambda}{N-1}\, r^{-1} + C r^{-N}$ but since we require $\|\mathbf{z}\|_{L^\infty(\Omega)} \le 1$, then $C = 0$. On the other hand, if $\rho \le r < R$, then we get $\xi(r) = -C r^{-N}$. In order to determine the value of C, we demand the continuity of ξ and then, the vector field becomes

$$\mathbf{z}(x) = \begin{cases} -\frac{\lambda}{N-1}\frac{x}{|x|} & \text{if} \quad 0 \le r < \rho\,, \\ -\frac{\lambda \rho^{N-1}}{N-1}\frac{x}{|x|^N} & \text{if} \quad \rho \le r < R\,. \end{cases}$$

- *Case 2:* $\lambda > N - 1$.

In the region $0 \le r < \rho$, we may argue as in Example 1.6.1 and get a contradiction when $h'(r) = 0$. So $h'(r) < 0$ and the solution is given, up to constants, by

$$u(x) = (N - 1 - \lambda) \log\left(\frac{|x|}{\rho}\right),$$

with the vector field $\mathbf{z}(x) = -\frac{x}{|x|}$. On the other hand, if $\rho < r < R$, we get a contradiction when $h'(r) < 0$, with which the solution is $u(x) = 0$ and the vector field is given by $\xi(r) = -C r^{-N}$. Since we have $\|\mathbf{z}\|_{L^\infty(\Omega)} = 1$ when $0 \le r < \rho$, in order to preserve the continuity, we require

$$1 = |\mathbf{z}\,(\rho)| = C \rho^{-N} \rho\,.$$

Therefore, the vector field becomes $\mathbf{z}(x) = -\rho^{N-1}\frac{x}{|x|^N}$ and the solution is given by

$$u(x) = \begin{cases} (N-1-\lambda)\log\left(\frac{|x|}{\rho}\right) & \text{if} \quad 0 \le |x| \le \rho\,, \\ 0 & \text{if} \quad \rho < |x| < R\,. \end{cases}$$

Remark 1.6.4. An important particular case of Example 1.6.3 is the problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{Du}{|Du|}\right) + |Du| = \dfrac{\lambda}{|x|} & \text{in} \quad B_R(0)\,, \\ u = 0 & \text{on} \quad \partial B_R(0)\,, \end{cases} \tag{1.41}$$

with $\lambda > 0$ which solution is given by

$$u(x) = \begin{cases} 0 & \text{when} \quad 0 < \lambda \le N-1\,, \\ (N-1-\lambda)\log\left(\frac{|x|}{R}\right) & \text{when} \quad \lambda > N-1\,. \end{cases}$$

Problem (1.41) can be seen as the limit case of problems with a Hardy-type potential, namely

$$\begin{cases} -\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) + |\nabla u|^p = \lambda\dfrac{u^{p-1}}{|x|^p} & \text{in} \quad B_R(0)\,, \\ u = 0 & \text{on} \quad \partial B_R(0)\,. \end{cases} \tag{1.42}$$

Problems with Hardy-type potential received much attention in recent years. We highlight that in [3] has been studied problem (1.42) with $p = 2$ showing the regularizing effect produced by the gradient term as absorption.

1.7 A more general gradient term

From now on, we generalize problem (1.27) adding a continuous function $g : [0, \infty[\to [0, \infty[$ in the gradient term:

$$\begin{cases} -\mathrm{div}\left(\dfrac{Dv}{|Dv|}\right) + g(v)\,|Dv| = f(x) & \text{in } \Omega\,, \\ v = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{1.43}$$

In this section, we study problem (1.43) for a function g bounded from below, which is the standard case.

The existence and uniqueness of solutions to problem (1.43) depend on the properties of the function g. Moreover, the definition of solution to this problem may also depend of the case we are studying. In any case, we have to give a precise meaning to $g(v)|Dv|$, since it depends on the representative of $g(v)$ which we are actually considering. First of all, we assume that a solution satisfies $D^j v = 0$ and then we take $g(v)$ as the precise representative $g(v)^* = g(v^*)$, which is integrable with respect to the measure $|Dv|$.

1.7.1 Bounded g

In this subsection, let g be a continuous and bounded function such that there exists $m > 0$ with $g(s) \geq m$ for all $s \geq 0$. We define the function

$$G(s) = \int_0^s g(\sigma)\, d\sigma\,.$$

With this notation, the term $g(v)|Dv|$ in the equation of problem (1.43) means $|DG(v)|$.

Let us now introduce the suitable concept of solution to this problem.

Definition 1.7.1. *Let $g : [0, \infty[\to [0, \infty[$ be a continuous function with $g(s) \geq m > 0$ for all $s \geq 0$. We say that a function v is a weak solution* *to problem* (1.43) *if* $v \in BV(\Omega)$ *with* $D^j v = 0$ *and there exists a field* $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ *with* $\|\mathbf{z}\|_{L^\infty(\Omega)} \le 1$ *such that*

$$\begin{aligned}
&\textbf{(i)} \quad -\operatorname{div}\mathbf{z} + g(v)^*|Dv| = f \quad \textit{in } \mathcal{D}'(\Omega)\,,\\
&\textbf{(ii)} \quad (\mathbf{z}, Dv) = |Dv| \quad \textit{as measures in } \Omega\,,\\
&\textbf{(iii)} \quad v|_{\partial\Omega} = 0\,.
\end{aligned}$$

The following results show that there exists a unique solution of problem (1.43) when we take a continuous and bounded function g such that $g(s) \ge m > 0$ for all $s \ge 0$. The proof of these results relies on the existence of solution to problem (1.27) and the chain rule.

Theorem 1.7.2. *Let* u *be the nonnegative solution to problem* (1.27). *Assume that* g *is a continuous function such that* $g(s) \ge m > 0$ *for all* $s \ge 0$ *and let* $u = G(v)$. *Then,* v *is a nonnegative solution to problem* (1.43).

Proof. Since $u \in BV(\Omega)$ is the nonnegative solution of problem (1.27), there exists a vector field $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ with $\|\mathbf{z}\|_{L^\infty(\Omega)} \le 1$ such that

$$\begin{aligned}
&\textbf{(i)} \quad -\operatorname{div}\mathbf{z} + |Du| = f \quad \text{in } \mathcal{D}'(\Omega)\,,\\
&\textbf{(ii)} \quad (\mathbf{z}, Du) = |Du| \quad \text{as measures in } \Omega\,,\\
&\textbf{(iii)} \quad u|_{\partial\Omega} = 0\,.
\end{aligned} \tag{1.44}$$

By the properties of g, function G is increasing, so there exists G^{-1} and its derivative is bounded. Then, we apply the chain rule (see [7, Theorem 3.96]) to get $v = G^{-1}(u) \in BV(\Omega)$ and $D^j v = 0$. We also deduce that the boundary condition holds:

$$v|_{\partial\Omega} = G^{-1}(u)|_{\partial\Omega} = 0\,.$$

Moreover, by Lemma 1.5.6 we have that

$$(\mathbf{z}, Dv) = |Dv| \ \text{ as measures in } \ \Omega\,.$$

Finally, making the substitution $u = G(v)$ in (1.44) and applying the chain rule (see Theorem 1.4.9) we get

$$-\operatorname{div} \mathbf{z} + g(v)^*|Dv| = f \ \text{ in } \ \mathcal{D}'(\Omega)\,.$$

□

In Theorem 1.7.2 we have used a continuous function g bounded from below but in the following result we need a function bounded both from below and above. That is because we cannot apply Theorem 1.4.9 instead of the chain rule from [7, Theorem 3.96], as we have done in Theorem 1.7.2.

Corollary 1.7.3. *If v is a nonnegative solution to problem (1.43) with g continuous, bounded and such that $g(s) \geq m > 0$ for all $s \geq 0$, then $u = G(v)$ is the nonnegative solution to problem (1.27).*

Proof. Applying the same argument which is used in Theorem 1.7.2 and keeping it in mind that g is bounded and G is increasing and unbounded, the result follows. □

We finally show the existence and uniqueness result.

Theorem 1.7.4. *There exists a unique nonnegative solution to problem (1.43) with g continuous, bounded and such that $g(s) \geq m > 0$ for all $s \geq 0$.*

Proof. Assuming there are two solutions v_1 and v_2 of problem (1.43), by the Corollary 1.7.3, $G(v_1)$ and $G(v_2)$ are solutions to problem (1.27). Thus, $G(v_1) = G(v_2)$ and since G is injective we get $v_1 = v_2$. □

1.7.2 Unbounded g

Now, we focus on the case when function $g : [0, \infty[\to [0, \infty[$ with $g(s) \geq m > 0$ may be unbounded.

Theorem 1.7.5. *There is a unique nonnegative solution to problem* (1.43) *with g continuous and such that $g(s) \geq m > 0$ for all $s \geq 0$.*

Proof. First of all, we consider the approximate problem

$$\begin{cases} -\text{div}\left(\dfrac{Dv_k}{|Dv_k|}\right) + T_k(g(v_k))|Dv_k| = f(x) & \text{in } \ \Omega\,, \\ v_k = 0 & \text{on } \ \partial\Omega\,. \end{cases} \tag{1.45}$$

By Theorem 1.7.4, it has a unique nonnegative solution. Then, there exists $v_k \in BV(\Omega)$ with $D^j v_k = 0$ and also a vector field $\mathbf{z}_k \in \mathcal{DM}^\infty(\Omega)$ with $\|\mathbf{z}_k\|_{L^\infty(\Omega)} \leq 1$ such that

$$\begin{aligned} &\textbf{(i)} \quad -\operatorname{div}\mathbf{z}_k + T_k(g(v_k))^*|Dv_k| = f \ \text{ in } \ \mathcal{D}'(\Omega)\,, \\ &\textbf{(ii)} \quad (\mathbf{z}_k, Dv_k) = |Dv_k| \ \text{ as measures in } \ \Omega\,, \\ &\textbf{(iii)} \quad v_k|_{\partial\Omega} = 0\,. \end{aligned}$$

First, we take the test function $\frac{T_h(v_k)}{h}$ in problem (1.45) and we get

$$\begin{aligned} \frac{1}{h}\int_\Omega (\mathbf{z}_k, DT_h(v_k)) + \int_\Omega T_k(g(v_k))^* \frac{T_h(v_k)^*}{h}|Dv_k| &= \int_\Omega f\,\frac{T_h(v_k)}{h}\,dx \\ &\leq \int_\Omega f\,dx\,. \end{aligned}$$

Keeping in mind that the first integral is nonnegative (by Lemma 1.5.6), we can take limits in the second integral when $h \to 0^+$ and so we obtain

$$\int_\Omega T_k(g(v_k))^*|Dv_k| \leq \int_\Omega f\,dx\,. \tag{1.46}$$

Since $T_k(g(v_k))$ is bigger than m, it yields

$$m\int_\Omega |Dv_k| \le \int_\Omega f\,dx\,.$$

Therefore, the sequence $\{v_k\}$ is bounded in $BV(\Omega)$ and there exists $v\in BV(\Omega)$ such that, up to subsequences, $v_k\to v$ in $L^1(\Omega)$ and a.e.. Moreover, $Dv_k\to Dv$ $*$-weak as measures when $k\to\infty$.

To prove $D^j v=0$, we argue as in the proof of Theorem 1.5.3. So we get $D^j G(v)=0$ and then, we deduce that $D^j v=0$. On the other hand, we define the function

$$F_k(s)=\int_0^s T_k(g(\sigma))\,d\sigma\,.$$

Using (1.46) and the chain rule (see [7, Theorem 3.96]) we have

$$\int_\Omega |DF_k(v_k)| \le \int_\Omega f\,dx\,,$$

which implies that the sequence $\{F_k(v_k)\}$ is bounded in $BV(\Omega)$ and converges in $L^1(\Omega)$ to w. Since $v_k\to v$ in $L^1(\Omega)$ and $F_k(s)\to G(s)$ when $k\to\infty$, then we deduce that $w=G(v)$.

Now, denoting $u_k=F_k(v_k)$ and $u=G(v)$ we get that $\{u_k\}$ converges to u in $L^1(\Omega)$ and

$$\int_\Omega |Du_k| \le \int_\Omega f\,dx\,.$$

Therefore, it is true that $u\in BV(\Omega)$. Moreover, due to Theorem 1.4.9 we get $|Du|=g(v)^*|Dv|$.

On the other hand, Corollary 1.7.3 implies that function u_k is the solution to problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{Du_k}{|Du_k|}\right)+|Du_k|=f(x) & \text{in }\ \Omega\,,\\ u_k=0 & \text{on }\ \partial\Omega\,,\end{cases}$$

and the same argument from the proof of Theorem 1.5.3 can be used in order to prove that u is the solution to problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{Du}{|Du|}\right) + |Du| = f(x) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Finally, since $g(s) \geq m > 0$ for all $s \geq 0$ and applying Theorem 1.7.2, we deduce that v is the solution to problem (1.43). $\square$

Concerning the summability of solutions to problem (1.43), we have the following result.

Proposition 1.7.6. *The solution v to problem* (1.43) *satisfies $v \in L^q(\Omega)$ for all $1 \leq q < \infty$.*

Proof. We can follow the proof of Proposition 1.5.7 to prove this result, taking into account that $g(s) \geq m > 0$ for all $s \geq 0$. $\square$

1.8 A nonstandard case: g touches the axis

In this section we assume that g is a continuous, bounded and non integrable function with $g(s) > 0$ for almost every $s \geq 0$. In this case, G is increasing but $(G^{-1})'$ may be unbounded.

First, we analyze the case when there are m, $\sigma > 0$ such that $g(s) \geq m > 0$ for all $s \geq \sigma$. Observe that this condition is similar to condition (1.7) in [1].

Theorem 1.8.1. *Let g be a continuous, bounded and non integrable function with $g(s) > 0$ for almost every $s \geq 0$ and such that $g(s) \geq m > 0$ for all $s \geq \sigma > 0$. Then, there exists a unique nonnegative solution to problem* (1.43).

Proof. By Theorem 1.7.4, there exist $v_n \in BV(\Omega)$ which is the nonnegative solution to the approximating problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{Dv_n}{|Dv_n|}\right) + \left(g(v_n) + \dfrac{1}{n}\right)|Dv_n| = f(x) & \text{in } \ \Omega\,, \\ v_n = 0 & \text{on } \ \partial\Omega\,, \end{cases}$$

with the associated vector field $\mathbf{z}_n$. We begin by using the test function $\frac{T_k(v_n - T_\sigma(v_n))}{k}$ in that problem. Taking into account that $DT_k(v_n - T_\sigma(v_n)) \neq 0$ if $0 \le v_n - \sigma < k$ and in that case $DT_k(v_n - T_\sigma(v_n)) = D(v_n - \sigma) = Dv_n$ and it holds $(\mathbf{z}_n, Dv_n) = |Dv_n| \ge 0$, then we get

$$\int_{\{v_n > \sigma\}} g(v_n)^* \frac{T_k(v_n - T_\sigma(v_n))^*}{k} |Dv_n| \le \int_{\{v_n > \sigma\}} f\, dx\,,$$

and taking limits when $k \to 0^+$ it yields

$$\int_{\{v_n > \sigma\}} g(v_n)^* |Dv_n| \le \int_{\{v_n > \sigma\}} f\, dx\,.$$

Since there exists $m > 0$ such that $g(s) \ge m$ for all $s \ge \sigma$, then the previous inequality becomes

$$\int_{\{v_n > \sigma\}} |Dv_n| \le \frac{1}{m} \int_\Omega f\, dx\,. \tag{1.47}$$

Now, we use the test function $T_\sigma(v_n)$ in the same problem and since $\int_\Omega g(v_n)^* T_\sigma(v_n)^* |Dv_n| \ge 0$, then we get

$$\int_{\{v_n \le \sigma\}} |Dv_n| \le \int_\Omega f\, T_\sigma(v_n)\, dx \le \sigma \int_\Omega f\, dx\,. \tag{1.48}$$

Finally, by (1.47) and (1.48) we have

$$\int_\Omega |Dv_n| \le \left(\sigma + \frac{1}{m}\right) \int_\Omega f\, dx\,, \qquad \text{for all } \ n \in \mathbb{N}\,,$$

that is, the sequence $\{v_n\}$ is bounded in $BV(\Omega)$ and this implies that, up to subsequences, there exists $v \in BV(\Omega)$ with $v_n \to v$ in $L^1(\Omega)$ and a.e. as well as $Dv_n \to Dv$ $*$-weak in the sense of measures. We conclude the proof arguing as in Theorem 1.5.3. $\square$

For a general function g we have to change the definition of solution. We show in Example 1.8.4 that Definition 1.7.1 does not really work in general.

Definition 1.8.2. *Let g be a continuous, bounded and non integrable function with $g(s) > 0$ for almost every $s \geq 0$. We say that a function v is a weak solution to problem (1.43) if $G(v) \in BV(\Omega)$ with $D^j G(v) = 0$ and there exists a field $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ with $\|\mathbf{z}\|_{L^\infty(\Omega)} \leq 1$ such that*

$$\begin{aligned} &\textbf{(i)} \quad -\operatorname{div}\mathbf{z} + g(v)^*|Dv| = f \quad \textit{in} \;\; \mathcal{D}'(\Omega)\,, \\ &\textbf{(ii)} \quad (\mathbf{z}, DG(v)) = |DG(v)| \quad \textit{as measures in} \;\; \Omega\,, \\ &\textbf{(iii)} \quad G(v)|_{\partial\Omega} = 0\,. \end{aligned}$$

where the function G is defined by

$$G(s) = \int_0^s g(\sigma)\, d\sigma\,.$$

Theorem 1.8.3. *Assume that the function g is continuous, bounded and non integrable with $g(s) > 0$ for almost every $s \geq 0$. Then, there exists a unique solution to problem (1.43) in the sense of Definition 1.8.2.*

Proof. The approximating problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{Dv_n}{|Dv_n|}\right) + \left(g(v_n) + \dfrac{1}{n}\right)|Dv_n| = f(x) & \text{in } \;\Omega\,, \\ v_n = 0 & \text{on } \;\partial\Omega\,, \end{cases} \tag{1.49}$$

has a unique solution for every $n \in \mathbb{N}$ by Theorem 1.7.4. That is, there exists a vector field $\mathbf{z}_n \in \mathcal{DM}^\infty(\Omega)$ with $\|\mathbf{z}_n\|_{L^\infty(\Omega)} \leq 1$ and a function $v_n \in BV(\Omega)$ with $D^j v_n = 0$ and such that

$$\begin{aligned} &\textbf{(i)} \quad -\operatorname{div}\mathbf{z}_n + \left(g(v_n) + \frac{1}{n}\right)^* |Dv_n| = f \ \text{ in } \ \mathcal{D}'(\Omega)\,, \\ &\textbf{(ii)} \quad (\mathbf{z}_n, DG_n(v_n)) = |DG_n(v_n)| \ \text{ as measures in } \ \Omega\,, \\ &\textbf{(iii)} \quad v_n|_{\partial\Omega} = 0\,. \end{aligned}$$

where we denote

$$G_n(s) = \int_0^s \left(g(\sigma) + \frac{1}{n}\right) d\sigma\,.$$

We show that the limit of the sequence $\{v_n\}$ is the solution to problem (1.43). First of all, we take the test function $\frac{T_k(v_n)}{k}$ in problem (1.49) and dropping the nonnegative term we get

$$\frac{1}{k}\int_\Omega T_k(v_n)^*|DG_n(v_n)| \leq \int_\Omega f\,dx$$

for every k. Now, letting $k \to 0^+$ and using Fatou's lemma we have that

$$\int_{\{v_n \neq 0\}} |DG_n(v_n)| \leq \int_\Omega f\,dx\,.$$

In addition, since $D^j v_n = 0$ it follows that $Dv_n = 0$ almost everywhere in $\{v_n = 0\}$. Thus,

$$\int_\Omega |DG_n(v_n)| \leq \int_\Omega f\,dx\,,$$

and so $G_n(v_n)$ is bounded in $BV(\Omega)$. This implies that, up to subsequences, there exist $w \in BV(\Omega)$ such that $G_n(v_n) \to w$ in $L^1(\Omega)$ and a.e., and also $DG_n(v_n) \to Dw$ $*$-weak in the sense of measures. We denote $v = G^{-1}(w)$.

In what follows, we argue as in Theorem 1.5.3 with minor modifications, hence we just sketch it. There exist a vector field $\mathbf{z} \in L^\infty(\Omega; R^N)$

which is the $*$-weakly limit of sequence $\{\mathbf{z}_n\}$ in $L^\infty(\Omega)$ with $\|\mathbf{z}\|_{L^\infty(\Omega)} \le 1$ and $-\mathrm{div}\,\mathbf{z}$ is a Radon measure with finite total variation. Moreover, using the test function $e^{-G_n(v_n)}\varphi$ with $\varphi \in C_0^\infty(\Omega)$ in problem (1.49) and letting $n \to \infty$, it leads $-\mathrm{div}\,(e^{-G(v)}\mathbf{z}) = e^{-G(v)}f$ in the sense of distributions. Next, we show that $D^jG(v) = 0$ and we also obtain

$$-\mathrm{div}\,\mathbf{z} + |DG(v)| = f \quad \text{in} \quad \mathcal{D}'(\Omega)\,,$$

and

$$(\mathbf{z}, DG(v)) = |DG(v)| \quad \text{as measures in} \quad \Omega\,.$$

Moreover, we use $T_k(G_n(v_n))$ as a test function in (1.49) to prove that the boundary condition $G(v)|_{\partial\Omega} = 0$ holds.

Finally, the uniqueness can be proved as in Theorem 2.11 of Chapter 2 (we may also argue as in [55, Theorem 3.8]). □

To remark the necessity to have a new definition to the concept of solution, we show in the next example that the solution to (1.43) when g is such that $\lim\limits_{s\to\infty} g(s) = 0$ is not in $BV(\Omega)$.

Example 1.8.4. *Set $\Omega = B_R(0)$. The solution to problem*

$$\begin{cases} -\mathrm{div}\left(\dfrac{Dv}{|Dv|}\right) + \dfrac{1}{1+v}|Dv| = \dfrac{\lambda}{|x|} & \textit{in} \ \ \Omega\,, \\ v = 0 & \textit{on} \ \ \partial\Omega\,, \end{cases} \tag{1.50}$$

is not in $BV(\Omega)$ for λ big enough.

First, we solve the related problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{Du}{|Du|}\right) + |Du| = \dfrac{\lambda}{|x|} & \text{in} \ \ \Omega\,, \\ u = 0 & \text{on} \ \ \partial\Omega\,, \end{cases} \tag{1.51}$$

and then, using the inverse function of

$$G(s) = \int_0^s \frac{1}{1+\sigma}\, d\sigma = \log(1+s)$$

we get the solution v. By Example 1.6.3 we know that for $\lambda > N-1$, the solution to problem (1.51) is given by $u(x) = (N-1-\lambda)\log\left(\frac{|x|}{R}\right)$ with the associated field $\mathbf{z}(x) = -\frac{x}{|x|}$. Moreover, the inverse of the function G is given by $G^{-1}(s) = e^s - 1$. Therefore, the solution to (1.50) is given by

$$v(x) = G^{-1}(u(x)) = \left(\frac{|x|}{R}\right)^{N-1-\lambda} - 1\,,$$

when $\lambda > N-1$. Nevertheless, v does not belong to $BV(\Omega)$ when λ is such that $N < \lambda/2 + 1$ because in this case, $|Dv| = \frac{\lambda-N+1}{R^{N-1-\lambda}}|x|^{N-2-\lambda}$ is not integrable.

1.9 Odd cases

In this last section we show some cases where the properties of the function g does not provide uniqueness, existence or regularity of solutions to problem (1.43).

1.9.1 First case

We start assuming that function g is integrable. In this case, the size of function f will determine the existence or absence of solution.

Theorem 1.9.1. *Let $f \in L^{N,\infty}(\Omega)$ with $f \geq 0$ and we consider problem (1.43) with $g \in L^1([0,\infty[)$. Then,*

(i) if $\|f\|_{W^{1,-\infty}(\Omega)} < 1$, the trivial solution holds;

(ii) if $\|f\|_{W^{1,-\infty}(\Omega)} > e^{G(\infty)}$, there is no solution;

with $G(\infty) = \sup\{G(s) : s \in]0, \infty[\}$.

Proof. Let u be the solution to problem (1.43). We use $T_k(u)$ as a test function in that problem to obtain

$$\int_\Omega |DT_k(u)| + \int_\Omega T_k(u)g(u)|Du| = \int_\Omega f\, T_k(u)\, dx \le \int_\Omega f\, u\, dx\,.$$

Taking limits when $k \to \infty$ and using duality arguments as in Proposition 1.5.4, we get

$$\int_\Omega |Du| + \int_\Omega u\, g(u)|Du| \le \int_\Omega f\, u\, dx \le \|f\|_{W^{-1,\infty}(\Omega)} \int_\Omega |Du| < \int_\Omega |Du|\,.$$

Thus, $\int_\Omega |Du| = 0$ and we deduce that $u = 0$.

On the other hand, let $\varphi \in W_0^{1,1}(\Omega)$. We use the distributional equality $-\mathrm{div}\,(e^{-G(v)}\mathbf{z}) = e^{-G(v)} f$ to get

$$\begin{aligned} e^{-G(\infty)} \int_\Omega f\, |\varphi|\, dx \le \int_\Omega e^{-G(u)} f\, |\varphi|\, dx &= \int_\Omega e^{-G(u)}\mathbf{z} \cdot \nabla|\varphi|\, dx \\ &\le \int_\Omega |\nabla\varphi|\, dx\,. \end{aligned}$$

Then, if $\|f\|_{W^{-1,\infty}(\Omega)} > e^{G(\infty)}$, there is no solution to problem (1.43). $\square$

Remark 1.9.2. Since we have shown in (1.16) that

$$\|f\|_{W^{-1,\infty}(\Omega)} \le S_N \|f\|_{L^{N,\infty}(\Omega)}\,,$$

Theorem 1.9.1 implies the following fact:

(i) If $\|f\|_{L^{N,\infty}(\Omega)} \le S_N^{-1}$, the trivial solution holds.

Remark 1.9.3. One may wonder what happens when datum f is such that $1 < \|f\|_{W^{-1,\infty}(\Omega)} \le e^{G(\infty)}$. Consider the approximate solutions v_n to problem (1.49) and let w satisfy $G(v_n) \to w$. Then $w \in [0, G(\infty)]$. In particular, if $w \in [0, G(\infty)[$, the function $v = G^{-1}(w)$ is finite a.e. in

Ω and it is the solution to problem (1.43). However, w can be equal to $G(\infty)$ in a set of positive measure and so v is infinite in the same set. We conclude that v, in this case, is not a solution.

Example 1.9.4. *Let $R > 0$ and $\lambda > 0$. Consider the following problem:*

$$\begin{cases} -\mathrm{div}\left(\dfrac{Dv}{|Dv|}\right) + \dfrac{1}{1+v^2}|Dv| = \dfrac{N-1}{|x|} + \lambda & \textit{in } B_R(0)\,, \\ v = 0 & \textit{on } \partial B_R(0)\,, \end{cases}$$

Then, there is no radial solution when λ is large enough.

Assuming there exists a radial solution $v(x) = h(|x|)$ with function $h : [0, R] \to \mathbb{R}$ satisfying $h(r) \geq 0$, $h(R) = 0$ and $h'(r) \leq 0$, we will get a contradiction.

First, let $0 \leq \rho_1 < \rho_2$. We suppose that $h'(r) = 0$ for $\rho_1 < r < \rho_2$ and, reasoning as in Example 1.6.1, we get a contradiction. Therefore, we only can have $h'(r) < 0$ for all $0 \leq r < R$. In this case, we know that the vector field is given by $\mathbf{z}(x) = -\frac{x}{|x|}$ and the equation becomes

$$-g(h(r))h'(r) = \lambda\,,$$

which is equivalent to $(G(h(r))' = -\lambda$. Then, the solution is given by $G(h(r)) = \lambda(R - r)$.

On the other hand, we know that $G(s) \in [0, \frac{\pi}{2}[$ because

$$G(s) = \int_0^s g(\sigma)\,d\sigma = \int_0^s \frac{1}{1+\sigma^2}\,d\sigma = \arctan(s)\,.$$

Thus, we have a radial solution if $\lambda < \frac{\pi}{2R}$ given by

$$v(x) = \tan\Big(\lambda(R - r)\Big)\,. \tag{1.52}$$

Moreover, since $G(\infty) = \frac{\pi}{2}$ is only attained in a null set, when $\lambda = \frac{\pi}{2R}$, we also obtain the radial solution 1.52.

1.9.2 Second case

Now we take the function $g : [0, \infty[\to [0, \infty[$ such that $g(s) = 0$ when $s \in [0, \ell]$ and $g(s) \geq m > 0$ for all $s > \ell$. We also assume $g \notin L^1([0, \infty[)$.

Remark 1.9.5. If g defined as above, then there is no uniqueness of solution. Indeed, if $\|f\|_{W^{1,-\infty}(\Omega)} \leq 1$ and $v \in BV(\Omega)$ is a nontrivial solution to problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{Dv}{|Dv|}\right) = f(x) & \text{in } \Omega\,, \\ v = 0 & \text{on } \partial\Omega\,, \end{cases}$$

(see [57, Proposition 4.1]), then function $T_\ell(v)$ is a solution to problem (1.43). Thus, there is no uniqueness.

Now, let $f \in L^{N,\infty}(\Omega)$. We define

$$h(s) = g(s + \ell)\,,$$

and let w be a solution to problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{Dw}{|Dw|}\right) + h(w)\,|Dw| = f(x) & \text{in } \Omega\,, \\ w = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.53}$$

with associated field $\mathbf{z}$. Therefore, $v(x) = w(x) + \ell$ is a solution to problem (1.43) with the same vector field $\mathbf{z}$.

Moreover, let $\psi : [0, \ell + 1] \to [\ell, \ell + 1]$ be an increasing and bijective C^1-function such that $\psi'(\ell + 1) = 1$. Then, we consider

$$h(s) = \begin{cases} \psi'(s) g(\psi(s)) & \text{if } \; 0 \leq s \leq \ell + 1\,, \\ g(s) & \text{if } \; \ell + 1 < s\,, \end{cases}$$

and let w be a solution to problem (1.53) with h defined as above. Therefore, the function

$$v(x) = \begin{cases} \psi(w(x)) & \text{if} \quad 0 \le w(x) \le \ell + 1\,, \\ w(x) & \text{if} \quad \ell + 1 < w(x)\,, \end{cases}$$

is a solution to (1.43), as we can see as follows. It is straightforward that the equation holds in the sense of distributions and since $w|_{\partial\Omega} = 0$, then $G(v)|_{\partial\Omega} = 0$. It only remains to be checked that $(\mathbf{z}, DG(v)) = |DG(v)|$ as measures in Ω. If $0 \le s \le \ell + 1$ we get

$$H(s) = \int_0^s h(\sigma)\,d\sigma = \int_0^s \psi'(\sigma) g(\psi(\sigma))\,d\sigma = \int_0^{\psi(s)} g(\sigma)\,d\sigma = G(\psi(s))\,,$$
$$H(\ell+1) = G(\psi(\ell+1)) = G(\ell+1)\,,$$

and for $s > \ell + 1$ we have

$$H(s) = H(\ell+1) + \int_{\ell+1}^s h(\sigma)\,d\sigma = G(\ell+1) + \int_{\ell+1}^s g(\sigma)\,d\sigma = G(s)\,.$$

Therefore, $DG(v(x)) = DH(w(x))$ and we conclude $(\mathbf{z}, DG(v)) = |DG(v)|$ as measures in Ω.

Example 1.9.6. *Set $\Omega = B_R(0)$. The solution to problem*

$$\begin{cases} -\mathrm{div}\left(\dfrac{Dv}{|Dv|}\right) + g(v)|Dv| = \dfrac{N}{|x|} & \text{in } \ \Omega\,, \\ v = 0 & \text{on } \ \partial\Omega\,, \end{cases}$$

with

$$g(s) = \begin{cases} 0 & \text{if} \quad s \le a\,, \\ s - a & \text{if} \quad a < s\,, \end{cases}$$

for $a > 0$ does not vanish on $\partial\Omega$.

We define

$$G(s)=\int_0^s g(\sigma)\,d\sigma=\begin{cases}0 & \text{if } \ 0\le s\le a\,,\\ \dfrac{a^2}{2}+\dfrac{s^2}{2}-a\,s & \text{if } \ a<s\,.\end{cases}$$

It can be checked that

$$v(x)=h(|x|)=h(r)=G^{-1}\left(-\log\left(\frac{r}{R}\right)\right)\,,$$

with $\mathbf{z}=-\frac{x}{|x|}$ is such that $(\mathbf{z},Dv)=|Dv|$ as measures in Ω and the distributional equation $-\operatorname{div}\mathbf{z}+g(v)^*|Dv|=\frac{N}{r}$ holds. However,

$$h(R)=G^{-1}(0)=a\,.$$

Although the boundary condition is not true, the solution achieves the boundary weakly (see [8]), that is,

$$[\mathbf{z},\nu]=-\frac{x}{|x|}\frac{x}{|x|}=-1=-\operatorname{sign}(v)\,.$$

1.9.3 Third case

Let $0<a<b$, and assume that $g:[0,\infty[\to[0,\infty[$ is a function with $g(s)=0$ when $s\in[a,b]$ and $g(s)\ge m>0$ for all $s<a$ and $s>b$. Moreover, we assume that $g\notin L^1([0,\infty[)$.

Remark 1.9.7. We argue in a similar way to Remark 1.9.5 to show that there is no uniqueness of solution to problem (1.43) with function g defined as above.

Let $\psi:[0,b]\to[0,a]$ be an increasing and bijective C^1-function. Now, we define

$$h(s)=\begin{cases}\psi'(s)g(\psi(s)) & \text{if } \ 0\le s\le b\,,\\ g(s) & \text{if } \ b<s\,.\end{cases}$$

If w is a solution to problem (1.53), then we have that

$$v(x) = \begin{cases} \psi(w(x)) & \text{if} \quad 0 \le w(x) \le b\,, \\ w(x) & \text{if} \quad b < w(x)\,, \end{cases}$$

is a solution to the original problem (1.43) because the equation holds in the sense of distributions and also $G(v)|_{\partial\Omega} = 0$ (because $w|_{\partial\Omega} = 0$). In addition, for $0 \le s \le b$ we have

$$H(s) = \int_0^s h(\sigma)\,d\sigma = \int_0^s \psi'(\sigma) g(\psi(\sigma))\,d\sigma = \int_0^{\psi(s)} g(\sigma)\,d\sigma = G(\psi(s))\,,$$
$$H(b) = G(\psi(b)) = G(a) = G(b)\,,$$

and for $s > b$ we get

$$H(s) = H(b) + \int_b^s h(\sigma)\,d\sigma = G(b) + \int_b^s g(\sigma)\,d\sigma = G(s)\,.$$

Therefore, we have proved that the remaining condition holds:

$$(\mathbf{z}, DG(v)) = |DG(v)| \quad \text{as measures in} \quad \Omega\,.$$

Example 1.9.8. *Set* $\Omega = B_R(0)$*. Problem*

$$\begin{cases} -\operatorname{div}\left(\dfrac{Dv}{|Dv|}\right) + g(v)|Dv| = \dfrac{N}{|x|} & \textit{in} \quad \Omega\,, \\ v = 0 & \textit{on} \quad \partial\Omega\,, \end{cases} \tag{1.54}$$

with

$$g(s) = \begin{cases} a - s & \textit{if} \quad s < a\,, \\ 0 & \textit{if} \quad a \le s \le b\,, \\ s - b & \textit{if} \quad b < s\,, \end{cases}$$

where $0 < a < b$*, has a discontinuous solution.*

Let us define

$$G(s) = \int_0^s g(\sigma)\,d\sigma = \begin{cases} \frac{-s^2}{2} + a\,s & \text{if} \quad 0 \le s \le a\,, \\ \frac{a^2}{2} & \text{if} \quad a \le s \le b\,, \\ \frac{a^2+b^2}{2} + \frac{s^2}{2} - b\,s & \text{if} \quad b < s\,. \end{cases}$$

We prove that the radial function

$$v(x) = h(|x|) = G^{-1}\left(-\log\left(\frac{|x|}{R}\right)\right)$$

is a solution to problem (1.54) and since G^{-1} is discontinuous, the solution v is discontinuous too. Having in mind that

$$h'(r) = \frac{-1}{g\left(G^{-1}\left(-\log\left(\frac{r}{R}\right)\right)\right)}\,,$$

and taking the vector field given by $\mathbf{z}(x) = -\frac{x}{|x|}$, then the distributional equality $-\operatorname{div}\mathbf{z} + g(v)^*|Dv| = \frac{N}{|x|}$ and the equality $(\mathbf{z}, Dv) = |Dv|$ as measures in Ω can be checked, as well as the boundary condition $h(R) = G^{-1}(0) = 0$.

Chapter 2

Non-variational data

2.1 Introduction

In this chapter, we study problem (1.1) but now considering $g \equiv 1$. More specifically, we are interested in investigate the following Dirichlet problem:

$$\begin{cases} -\mathrm{div}\left(\dfrac{Du}{|Du|}\right) + |Du| = f(x) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{2.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$ and f is a nonnegative integrable function in Ω.

Our aim now is to go a step further of problem (1.1) and study problem (2.1) when data are merely integrable functions. Hence, we cannot use the solution $u \in BV(\Omega)$ as a test function in the variational formulation of this problem since fu is not integrable.

This kind of non-variational problems with L^1-data has been extensively studied for equations involving the p-Laplacian with $1 < p \leq N$ for which there are two different formulations. For each one of them it was introduced different concepts of solutions. Indeed, in [16] it was used an entropy solution (see also [19]) and, on the other hand, renormalized solutions were adopted in [30]. We highlight that both approaches systematically use truncations of solutions.

In [8], in the framework of the 1-Laplacian, the authors also introduce a notion of solution by means of truncations. We follow the same concept, but adapted to our situation. It should be pointed out that although we follow the spirit of [8], due to the regularizing effect of the total variation (see Chapter 1), the solution to problem (2.1) has better properties that the one studied in [8]. Indeed, the boundary condition holds in the sense of traces, not in a weak sense.

Another feature derived from the regularizing effect is that the solution is a function of bounded variation without jump part. Nevertheless, this fact does not allow us to define the pairing of a general L^∞-divergence-measure vector field $\mathbf{z}$ and the solution u in the same way that we have done in Chapter 1 following Anzellotti's theory (see [13]). Then, truncations must remain in the definition of solution and instead of products of the form $(\mathbf{z}, Du)$ we have to handle with products such as $(\mathbf{z}, De^{-u})$ and $(e^{-u}\mathbf{z}, Du)$. Beyond these kind of technical complication, the existence theorem holds as it was expected and we will only make explicit those parts of the proof which are different from Theorem 1.5.3 with data $0 \le f \in L^{N,\infty}(\Omega)$.

It should be pointed out that the comparison principle is much more interesting since, even in the context of bounded solutions, its proof is new and simpler than the proof of the uniqueness result proved in [55].

On the other hand, we also investigate solutions when data belong to $L^p(\Omega)$ with $1 < p < N$, showing that solutions lie in $L^{\frac{Np}{N-p}}(\Omega)$. Notice that Lebesgue spaces continuously adjust with the known cases $p = 1$ (with solutions $u \in BV(\Omega) \subset L^{\frac{N}{N-1}}(\Omega)$) and $p = N$ (see Proposition 1.5.7).

This chapter is organized as follows. Section 2.2 is devoted to prove the main results, that is, the existence theorem and the comparison principle and in Section 2.3 we show the best summability that the solution can get when data belong to $L^p(\Omega)$ with $1 < p < N$. We finish the chapter by showing examples of radial solutions which give evidence that the obtained regularity is optimal.

2.2 Main results

In this section, we prove the existence theorem of problem (2.1) and a comparison principle. We begin by stating our concept of solution to this problem. The first obstacle we have to deal with is that, following the techniques from [13] or from Section 1.4, we are not able to define the distribution $(\mathbf{z}, Du)$ when u is a BV-function and the datum f is a merely integrable function. Following [8], we will solve this problem by using truncations in Definition 1.5.1.

Definition 2.2.1. *Let $f \in L^1(\Omega)$ with $f \geq 0$. We say that $u \in BV(\Omega)$ is a solution to problem (2.1) if $D^j u = 0$ and there exists a vector field $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ with $\|\mathbf{z}\|_{L^\infty(\Omega)} \leq 1$ such that*

$$\textbf{(i)} \quad -\operatorname{div}\mathbf{z} + |Du| = f \quad \textit{in} \;\; \mathcal{D}'(\Omega)\,, \tag{2.2}$$

$$\textbf{(ii)} \quad (\mathbf{z}, DT_k(u)) = |DT_k(u)| \quad \textit{as measures in} \;\; \Omega \; \textit{(for all } k > 0\textit{)}\,, \tag{2.3}$$

$$\textbf{(iii)} \quad u|_{\partial\Omega} = 0\,. \tag{2.4}$$

2.2.1 Existence Theorem

We next prove an existence theorem. In order to do this, we argue as in proof of Theorem 1.5.3. Nevertheless, we need to detail some important remarks. The result is the following.

Theorem 2.2.2. *Let Ω be an open and bounded subset of $\mathbb{R}^N$ with Lipschitz boundary and let f be a nonnegative function in $L^1(\Omega)$. Then, problem (2.1) has at least one solution.*

Proof. We begin by observing an important detail concerning the pairing $(e^{-u}\,\mathbf{z}, Du)$. If u is integrable with respect to the measure $\operatorname{div}(e^{-u}\mathbf{z})$ and $\varphi \in C_0^\infty(\Omega)$, then the integrals

$$\int_\Omega \varphi\, u \operatorname{div}(e^{-u}\mathbf{z}) \quad \text{and} \quad \int_\Omega u\, e^{-u}\mathbf{z}\cdot\nabla\varphi\, dx$$

are both finite; notice that the second integral is bounded due to the inequality $u\,e^{-u} \le e^{-1}$. Therefore,

$$\langle (e^{-u}\,\mathbf{z}, Du), \varphi\rangle = -\int_\Omega \varphi\, u \operatorname{div}(e^{-u}\mathbf{z}) - \int_\Omega u\, e^{-u}\mathbf{z}\cdot\nabla\varphi\, dx \tag{2.5}$$

is a well-defined distribution (although the distribution $(\mathbf{z}, Du)$ is not). Moreover, we may apply the Anzellotti's procedure and prove that (2.5) is a Radon measure.

Taking this fact in mind, we may argue as in the proof of Theorem 1.5.3. Starting from suitable approximating problems with data $f_n(x) = T_n(f(x))$, we get a limit of the approximate solutions $u \in BV(\Omega)$ such that $D^j u = 0$. In addition, we also get a vector field $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ such that $\|\mathbf{z}\|_{L^\infty(\Omega)} \le 1$. Moreover, equation (2.2) holds and so does

$$-\operatorname{div}(e^{-u}\mathbf{z}) = e^{-u} f \ \text{ in } \ \mathcal{D}'(\Omega)\,. \tag{2.6}$$

This last equality implies that u is integrable with respect to the measure $\operatorname{div}(e^{-u}\mathbf{z})$ and then $(e^{-u}\,\mathbf{z}, Du)$ is a Radon measure.

Two conditions of Definition 2.2.1 must still be proved, namely (2.3) and (2.4). We begin by seeing (2.3), that is,

$$(\mathbf{z}, DT_k(u)) = |DT_k(u)| \ \text{ as measures in } \ \Omega\,,$$

for every $k > 0$.

To prove this, we start with the following inequality as measures (proved in Theorem 1.5.3):

$$|De^{-u}| \leq (e^{-u}\mathbf{z}, Du)\,. \tag{2.7}$$

First, we will show that

$$|De^{-T_k(u)}| \leq (e^{-u}\mathbf{z}, DT_k(u))\,. \tag{2.8}$$

On the one hand, considering the restriction to the set $\{u \geq k\}$ and thanks to the chain rule (see [7, Theorem 3.96]) we have

$$|De^{-T_k(u)}| \llcorner \{u \geq k\} = e^{-T_k(u)}|DT_k(u)| \llcorner \{u \geq k\} = 0\,,$$

and on the other hand

$$|(e^{-u}\mathbf{z}, DT_k(u))| \llcorner \{u \geq k\} \leq |DT_k(u))| \llcorner \{u \geq k\} = 0\,.$$

Now, we just work with the restriction to the set $\{u < k\}$. For every $\varphi \in C_0^\infty(\Omega)$ such that $\varphi \geq 0$, using the definition of the distribution and applying (2.7) and the chain rule we get

$$\begin{aligned}
&\langle (e^{-u}\mathbf{z}, DT_k(u)) \llcorner \{u < k\}, \varphi\rangle \\
&\qquad = -\int_{\{u<k\}} \varphi\, u \operatorname{div}(e^{-u}\mathbf{z}) - \int_{\{u<k\}} u\, e^{-u}\mathbf{z}\cdot\nabla\varphi\, dx \\
&\qquad = \langle (e^{-u}\mathbf{z}, Du) \llcorner \{u < k\}, \varphi\rangle \geq \int_{\{u<k\}} \varphi\, |De^{-u}| \\
&\qquad = \int_\Omega \varphi\, e^{-u}|DT_k(u)| = \int_\Omega \varphi\, |De^{-T_k(u)}|\,,
\end{aligned}$$

and the proof of (2.8) is done.

Now, we have to prove that $(\mathbf{z}, DT_k(u)) = |DT_k(u)|$ as measures in Ω. We use Proposition 1.2.6 and the chain rule (see [7, Theorem 3.96]) to obtain

$$
\begin{aligned}
|De^{-T_k(u)}| &\le (e^{-u}\mathbf{z}, DT_k(u)) = e^{-u}(\mathbf{z}, DT_k(u)) \\
&\le e^{-u}|DT_k(u)| = |De^{-T_k(u)}|\,.
\end{aligned}
$$

Then, equality $e^{-u}(\mathbf{z}, DT_k(u)) = e^{-u}|DT_k(u)|$ holds as measures in Ω. Moreover, we deduce that

$$
(\mathbf{z}, DT_k(u)) = |DT_k(u)| \;\text{ as measures in }\; \Omega\,,
$$

since $e^{-u} = 0$ implies $T_k(u) = k$ for every $k > 0$.

In order to check the boundary condition (2.4), we consider the real function defined by

$$
J_1(s) = \int_0^s T_1(\sigma)\,d\sigma\,.
$$

Then, analogously to Theorem 1.5.3, we obtain

$$
\begin{aligned}
\int_\Omega |DT_1(u)| + \int_{\partial\Omega} |T_1(u)|\,d\mathcal{H}^{N-1} + \int_\Omega |DJ_1(u)| + \int_{\partial\Omega} |J_1(u)|\,d\mathcal{H}^{N-1} \\
\le \int_\Omega f\,T_1(u)\,dx\,. \qquad (2.9)
\end{aligned}
$$

Using now the distributional equation (2.2), Green's formula, definition of function $J_1(s)$ as well as condition (2.3), we get

$$
\begin{aligned}
\int_\Omega f\,T_1(u)\,dx &= -\int_\Omega T_1(u)\,\mathrm{div}\,\mathbf{z} + \int_\Omega T_1(u)|Du| \\
&= \int_\Omega (\mathbf{z}, DT_1(u)) - \int_{\partial\Omega} T_1(u)[\mathbf{z},\nu]\,d\mathcal{H}^{N-1} + \int_\Omega |DJ_1(u)| \\
&= \int_\Omega |DT_1(u)| - \int_{\partial\Omega} T_1(u)[\mathbf{z},\nu]\,d\mathcal{H}^{N-1} + \int_\Omega |DJ_1(u)|\,.
\end{aligned}
$$

Going back to (2.9) and simplifying it, we obtain

$$\int_{\partial\Omega} (|T_1(u)| + T_1(u)[\mathbf{z},\nu])\, d\mathcal{H}^{N-1} + \int_{\partial\Omega} |J_1(u)|\, d\mathcal{H}^{N-1} \le 0\,.$$

Observe that both integrals are nonnegative, so that both vanish. In particular, $J_1(u) = 0$ $\mathcal{H}^{N-1}$-a.e. on $\partial\Omega$. Therefore, the boundary condition holds. □

2.2.2 Comparison principle

Before proving the comparison principle, we need to present some preliminary results.

Proposition 2.2.3. *Let $\mathbf{z}$ be a vector field in $\mathcal{DM}^\infty(\Omega)$ and let u be a function of bounded variation with $D^j u = 0$ and such that $(\mathbf{z}, DT_k(u)) = |DT_k(u)|$ for every $k > 0$. If $g : \Omega \to \mathbb{R}$ is a bounded, increasing and Lipschitz function, then $(\mathbf{z}, Dg(u)) = |Dg(u)|$ holds as measures in Ω.*

Proof. Since the equality $(\mathbf{z}, DT_k(u)) = |DT_k(u)|$ holds, the Radon–Nikodým derivative of $(\mathbf{z}, DT_k(u))$ with respect to its total variation $|DT_k(u)|$ is $\theta(\mathbf{z}, DT_k(u), x) = 1$. Moreover, by Proposition 1.2.4 we get

$$\theta(\mathbf{z}, Dg(T_k(u)), x) = \theta(\mathbf{z}, DT_k(u), x) = 1\,,$$

that is, $(\mathbf{z}, Dg(T_k(u))) = |Dg(T_k(u))|$ for every $k > 0$. Now, we apply the chain rule (see, for instance, [7, Theorem 3.96]) to get $|Dg(T_k(u))| = g'(T_k(u))T_k'(u)|Du|$, taking into account that the set where $g'(T_k(u))T_k'(u)$ is undefined is $|Du|$-negligible. Thus, we get

$$(\mathbf{z}, Dg(T_k(u))) = g'(T_k(u))T_k'(u)|Du|\,,$$

for every $k > 0$. Next, the dominated convergence theorem leads to

$$(\mathbf{z}, Dg(u)) = g'(u)\,|Du|\,.$$

Finally, applying the chain rule again, we are done. □

We highlight that the distributional equality 2.6 holds for every solution to problem (2.1), not only for the limit of the approximating problems as we have shown.

Proposition 2.2.4. *Let $f \in L^1(\Omega)$ with $f \geq 0$. If $u \in BV(\Omega)$ is a solution to problem (2.1) and $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ is the associated vector field, then the following equality holds:*

$$-\operatorname{div}(e^{-u}\mathbf{z}) = e^{-u}f \quad \textit{in} \quad \mathcal{D}'(\Omega)\,.$$

Proof. Let $\varphi \in C_0^\infty(\Omega)$. Taking the test function $e^{-u}\varphi$ in problem (2.1) we obtain

$$-\int_\Omega e^{-u}\varphi \operatorname{div}\mathbf{z} + \int_\Omega e^{-u}\varphi\,|Du| = \int_\Omega e^{-u}\varphi f\,dx\,.$$

Now, since e^{-u} is bounded we can use the definition of pairing $(\mathbf{z}, De^{-u})$ and the former equality becomes

$$\int_\Omega e^{-u}\mathbf{z}\cdot\nabla\varphi\,dx + \int_\Omega \varphi\,(\mathbf{z}, De^{-u}) + \int_\Omega e^{-u}\varphi\,|Du| = \int_\Omega e^{-u}\varphi f\,dx\,.$$

Finally, using that $(\mathbf{z}, De^{-u}) = -e^{-u}|Du|$ holds (see Proposition 2.2.3) we deduce the desired distributional equality:

$$-\operatorname{div}(e^{-u}\mathbf{z}) = e^{-u}f \quad \text{in} \quad \mathcal{D}'(\Omega)\,.$$

□

We present now the main result of this chapter.

Theorem 2.2.5. *Let f_1 and f_2 be two nonnegative functions in $L^1(\Omega)$ with $f_1 \le f_2$, and consider problems*

$$\begin{cases} -\mathrm{div}\left(\dfrac{Du_1}{|Du_1|}\right) + |Du_1| = f_1(x) & \text{in } \Omega\,, \\ u_1 = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{2.10}$$

and

$$\begin{cases} -\mathrm{div}\left(\dfrac{Du_2}{|Du_2|}\right) + |Du_2| = f_2(x) & \text{in } \Omega\,, \\ u_2 = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{2.11}$$

If u_1 is a solution to problem (2.10) and u_2 is a solution to problem (2.11), then $u_1 \le u_2$.

Proof. For each $i = 1, 2$, we know that a solution $u_i \in BV(\Omega)$ satisfies $D^j u_i = 0$ and there exists a vector field $\mathbf{z}_i \in \mathcal{DM}^\infty(\Omega)$ such that $\|\mathbf{z}_i\|_{L^\infty(\Omega)} \le 1$. Moreover,

(i) $\quad -\operatorname{div} \mathbf{z}_i + |Du_i| = f_i \;$ in $\; \mathcal{D}'(\Omega)\,,$

(ii) $\quad (\mathbf{z}_i, DT_k(u_i)) = |DT_k(u_i)|\;$ as measures in $\;\Omega\;$ (for every $k > 0$),

(iii) $\quad u_i|_{\partial\Omega} = 0\,.$

We would like to show that $u_1 \le u_2$. To do so, we divide the proof in several steps.

Step 1: $(\mathbf{z}_1 - \mathbf{z}_2, D(T_k(u_1) - T_k(u_2))^+)$ is a positive Radon measure for all $k > 0$.

Let $\varphi \in C_0^\infty(\Omega)$ with $\varphi \ge 0$. We can perform the following manipulations over the measure $(\mathbf{z}_1 - \mathbf{z}_2, D(T_k(u_1) - T_k(u_2))^+)$ to obtain:

$$\int_\Omega \varphi\,(\mathbf{z}_1 - \mathbf{z}_2, D(T_k(u_1) - T_k(u_2))^+)$$

$$
\begin{aligned}
&= \int_{\{T_k(u_1)>T_k(u_2)\}} \varphi\,(\mathbf{z}_1 - \mathbf{z}_2, D(T_k(u_1) - T_k(u_2))) \\
&= \int_{\{T_k(u_1)>T_k(u_2)\}} \varphi \Big[(\mathbf{z}_1, DT_k(u_1)) - (\mathbf{z}_2, DT_k(u_1)) \\
&\qquad\qquad - (\mathbf{z}_1, DT_k(u_2)) + (\mathbf{z}_2, DT_k(u_2))\Big] \\
&= \int_{\{T_k(u_1)>T_k(u_2)\}} \varphi \Big[|DT_k(u_1)| - (\mathbf{z}_2, DT_k(u_1)) \\
&\qquad\qquad - (\mathbf{z}_1, DT_k(u_2)) + |DT_k(u_2)|\Big] \geq 0\,,
\end{aligned}
$$

because $(\mathbf{z}_i, Du_j) \leq |Du_j|$ for $i,j = 1,2$.

Therefore, we conclude that $(\mathbf{z}_1 - \mathbf{z}_2, D(T_k(u_1) - T_k(u_2))^+)$ is a positive Radon measure.

Step 2: $\displaystyle\int_{\{u_1>u_2\}} (e^{-u_2} - e^{-u_1})(|Du_2| - |Du_1|) \geq 0.$

First, we take the test function $(e^{-u_2} - e^{-u_1})^+$ in problem (2.10) and since $(e^{-u_2} - e^{-u_1})^+ \neq 0$ in the set $\{u_1 > u_2\}$, we get

$$
\begin{aligned}
&\int_{\{u_1>u_2\}} (\mathbf{z}_1, D(e^{-u_2} - e^{-u_1})) + \int_{\{u_1>u_2\}} (e^{-u_2} - e^{-u_1})\,|Du_1| \\
&= \int_\Omega (e^{-u_2} - e^{-u_1})^+ f_1\,dx\,.
\end{aligned}
$$

Moreover, using that $e^{-u_2} - e^{-u_1} = (1 - e^{-u_1}) - (1 - e^{-u_2})$ we also have

$$
\begin{aligned}
&\int_\Omega (e^{-u_2} - e^{-u_1})^+ f_1\,dx \qquad (2.12) \\
&= \int_{\{u_1>u_2\}} (\mathbf{z}_1, D(1 - e^{-u_1})) - \int_{\{u_1>u_2\}} (\mathbf{z}_1, D(1 - e^{-u_2})) \\
&\quad + \int_{\{u_1>u_2\}} e^{-u_2}\,|Du_1| - \int_{\{u_1>u_2\}} e^{-u_1}\,|Du_1|
\end{aligned}
$$

$$= \int_{\{u_1>u_2\}} |D(1-e^{-u_1})| - \int_{\{u_1>u_2\}} (\mathbf{z}_1, D(1-e^{-u_2}))$$

$$+ \int_{\{u_1>u_2\}} e^{-u_2}\,|Du_1| - \int_{\{u_1>u_2\}} e^{-u_1}\,|Du_1|$$

$$= - \int_{\{u_1>u_2\}} (\mathbf{z}_1, D(1-e^{-u_2})) + \int_{\{u_1>u_2\}} e^{-u_2}\,|Du_1|\,,$$

where we have used Proposition 1.2.4 and the chain rule (see [7, Theorem 3.96]).

Now, taking the same test function $(e^{-u_2}-e^{-u_1})^+$ but now in problem (2.11) and making similar computations we obtain

$$\int_\Omega (e^{-u_2}-e^{-u_1})^+ f_2\,dx \tag{2.13}$$

$$= \int_{\{u_1>u_2\}} (\mathbf{z}_2, D(1-e^{-u_1})) - \int_{\{u_1>u_2\}} e^{-u_1}\,|Du_2|\,.$$

Since $f_1 \le f_2$, we continue by joining expressions (2.12) and (2.13) to get the following inequality:

$$\int_{\{u_1>u_2\}} e^{-u_1}\,|Du_2| + \int_{\{u_1>u_2\}} e^{-u_2}\,|Du_1|$$

$$\le \int_{\{u_1>u_2\}} (\mathbf{z}_1, D(1-e^{-u_2})) + \int_{\{u_1>u_2\}} (\mathbf{z}_2, D(1-e^{-u_1}))$$

$$\le \int_{\{u_1>u_2\}} |(\mathbf{z}_1, D(1-e^{-u_2}))| + \int_{\{u_1>u_2\}} |(\mathbf{z}_2, D(1-e^{-u_1}))|$$

$$\le \int_{\{u_1>u_2\}} |D(1-e^{-u_2})| + \int_{\{u_1>u_2\}} |D(1-e^{-u_1})|$$

$$= \int_{\{u_1>u_2\}} e^{-u_2}|Du_2| + \int_{\{u_1>u_2\}} e^{-u_1}|Du_1|\,,$$

where we have used that $\|\mathbf{z}_i\|_{L^\infty(\Omega)} \le 1$ for $i = 1, 2$ and also the chain rule.

In conclusion, we have just proved

$$\int_{\{u_1>u_2\}} e^{-u_2}|Du_2| + \int_{\{u_1>u_2\}} e^{-u_1}|Du_1| \\ - \int_{\{u_1>u_2\}} e^{-u_1}\,|Du_2| - \int_{\{u_1>u_2\}} e^{-u_2}\,|Du_1| \geq 0\,,$$

and we are done.

Step 3: The Radon measure $(\mathbf{z}_1 - \mathbf{z}_2, D(T_k(u_1) - T_k(u_2))^+)$ *vanishes for all* $k > 0$.

Since u_1 is a solution to problem (2.10) and u_2 is a solution to problem (2.11), by Proposition 2.2.4 the following distributional equalities hold:

$$-\operatorname{div}(e^{-u_1}\mathbf{z}_1) = e^{-u_1} f_1 \tag{2.14}$$

and

$$-\operatorname{div}(e^{-u_2}\mathbf{z}_2) = e^{-u_2} f_2\,. \tag{2.15}$$

Let $k > 0$. We choose the test function $(T_k(u_1) - T_k(u_2))^+$ in equality (2.14). Applying Green's formula, we get

$$\int_\Omega (e^{-u_1}\mathbf{z}_1, D(T_k(u_1) - T_k(u_2))^+) = \int_\Omega (T_k(u_1) - T_k(u_2))^+ e^{-u_1} f_1\,dx\,,$$

and using the same test function but now in equality (2.15) we have

$$\int_\Omega (e^{-u_2}\mathbf{z}_2, D(T_k(u_1) - T_k(u_2))^+) = \int_\Omega (T_k(u_1) - T_k(u_2))^+ e^{-u_2} f_2\,dx\,.$$

Now, we put together the two previous equality to obtain

$$\int_\Omega (T_k(u_1) - T_k(u_2))^+ (e^{-u_1} f_1 - e^{-u_2} f_2)\,dx \tag{2.16}$$

$$= \int_\Omega (e^{-u_1}\mathbf{z}_1 - e^{-u_2}\mathbf{z}_2, D(T_k(u_1) - T_k(u_2))^+)\,.$$

Observe that the integral on the left-hand side is non-positive since $e^{-u_1}f_1 - e^{-u_2}f_2 \leq 0$ in the set $\{T_k(u_1) > T_k(u_2)\}$.

Our aim now is to prove the following limit:

$$\lim_{k\to\infty}\int_\Omega (e^{-u_1}\mathbf{z}_1 - e^{-u_2}\mathbf{z}_2, D(T_k(u_1) - T_k(u_2))^+) = 0\,, \tag{2.17}$$

which is non-positive (by (2.16)). To this end, we write

$$\begin{aligned}
&\int_\Omega (e^{-u_1}\mathbf{z}_1 - e^{-u_2}\mathbf{z}_2, D(T_k(u_1) - T_k(u_2))^+) \\
&= \int_{\{T_k(u_1)>T_k(u_2)\}} ((e^{-u_2} - e^{-u_1})\,\mathbf{z}_2, D(T_k(u_2) - T_k(u_1))) \\
&\quad + \int_{\{T_k(u_1)>T_k(u_2)\}} (e^{-u_1}(\mathbf{z}_2 - \mathbf{z}_1), D(T_k(u_2) - T_k(u_1))) = \mathrm{I.1} + \mathrm{I.2}\,,
\end{aligned}$$

and will see that the limit as $k \to \infty$ of (I.1) and of (I.2) are nonnegative and so (2.17) holds.

On the one hand, thanks to Proposition 1.2.6 we know that

$$\begin{aligned}
&\int_{\{T_k(u_1)>T_k(u_2)\}} ((e^{-u_2} - e^{-u_1})\,\mathbf{z}_2, D(T_k(u_2) - T_k(u_1))) \\
&= \int_{\{T_k(u_1)>T_k(u_2)\}} (e^{-u_2} - e^{-u_1})(\mathbf{z}_2, D(T_k(u_2) - T_k(u_1))) \\
&\geq \int_{\{T_k(u_1)>T_k(u_2)\}} (e^{-u_2} - e^{-u_1})\chi_{\{u_2<k\}}|Du_2| \\
&\quad - \int_{\{T_k(u_1)>T_k(u_2)\}} (e^{-u_2} - e^{-u_1})\chi_{\{u_1<k\}}|Du_1|\,.
\end{aligned}$$

And when we take limits when $k \to \infty$ we get

$$\lim_{k\to\infty}\int_{\{T_k(u_1)>T_k(u_2)\}} ((e^{-u_2} - e^{-u_1})\,\mathbf{z}_2, D(T_k(u_2) - T_k(u_1)))$$

$$\geq \int_{\{u_1>u_2\}}(e^{-u_2}-e^{-u_1})(|Du_2|-|Du_1|)\geq 0\,,$$

which is nonnegative due to Step 2.

On the other hand, we already know that integral (I.2) is nonnegative (because of Step 1); therefore, the limit when $k\to\infty$ is nonnegative too.

Furthermore, since (2.16)=(I.1)+(I.2) and the limit of integrals (I.1) and (I.2) are both nonnegative, it follows that both limits vanish.

In short, we have proved

$$\begin{aligned}&\lim_{k\to\infty}\int_\Omega (T_k(u_1)-T_k(u_2))^+(e^{-u_1}f_1-e^{-u_2}f_2)\,dx\\&=\lim_{k\to\infty}\int_\Omega (e^{-u_1}\mathbf{z}_1-e^{-u_2}\mathbf{z}_2, D(T_k(u_1)-T_k(u_2))^+)=0\,.\end{aligned}$$

Now, some remarks on Radon–Nikodým derivatives of these measures are in order. Let $\theta_k^1(\mathbf{z}_2, DT_k(u_1),x)$ be the Radon–Nikodým derivative of $(\mathbf{z}_2, DT_k(u_1))$ with respect to $|DT_k(u_1)|$, that is,

$$\theta_k^1(\mathbf{z}_2, DT_k(u_1),x)\,|DT_k(u_1)|=(\mathbf{z}_2, DT_k(u_1))\,.$$

Since $|(\mathbf{z}_2, DT_k(u_1))|\leq |DT_k(u_1)|$, it follows that $|\theta_k^1(\mathbf{z}_2, DT_k(u_1),x)|\leq 1$. We note that this function is $|DT_k(u_1)|$-measurable and, taking $\theta_k^1(\mathbf{z}_2, DT_k(u_1),x)=0$ in $\{u_1\geq k\}$, it is $|Du_1|$-measurable.

On the other hand, $(\mathbf{z}_2, DT_{k+1}(u_1))\llcorner\{u_1<k\}=(\mathbf{z}_2, DT_k(u_1))$ holds. Therefore,

$$\theta_{k+1}^1(\mathbf{z}_2, DT_{k+1}(u_1),x)\chi_{\{u_1<k\}}(x)=\theta_k^1(\mathbf{z}_2, DT_k(u_1),x)\,,$$

and then, $\theta_k^1(\mathbf{z}_2, DT_k(u_1),x)$ defines a nondecreasing sequence of $|Du_1|$-measurable functions.

Likewise, if we denote by $\theta_k^2(\mathbf{z}_1, DT_k(u_2),x)$ the Radon–Nikodým derivative of $(\mathbf{z}_1, DT_k(u_2))$ with respect to $|DT_k(u_2)|$, then we may deduce the inequality $|\theta_k^2(\mathbf{z}_1, DT_k(u_2), x)| \leq 1$. Moreover, $\theta_k^2(\mathbf{z}_1, DT_k(u_2), x)$ defines a nondecreasing sequence of $|Du_2|$-measurable functions.

Now, we define the functions $\theta^1(x)$ and $\theta^2(x)$ such that

$$\theta^1(x) = \theta_k^1(\mathbf{z}_2, DT_k(u_1), x) \quad \text{if } \ u_1(x) < k$$

and

$$\theta^2(x) = \theta_k^2(\mathbf{z}_1, DT_k(u_2), x) \quad \text{if } \ u_2(x) < k\,.$$

We know that θ^1 and θ^2 are $|Du_1|$ and $|Du_2|$-measurable, respectively, and they also satisfy $|\theta^1| \leq 1$ and $|\theta^2| \leq 1$.

So let us get back to expression (I.2). We know that

$$\begin{aligned}
&\int_{\{T_k(u_1)>T_k(u_2)\}} (e^{-u_1}(\mathbf{z}_1 - \mathbf{z}_2), D(T_k(u_1) - T_k(u_2))) \\
&= \int_{\{T_k(u_1)>T_k(u_2)\}} e^{-u_1} \Big[(\mathbf{z}_1, DT_k(u_1)) - (\mathbf{z}_2, DT_k(u_1)) \\
&\quad - (\mathbf{z}_1, DT_k(u_2)) + (\mathbf{z}_2, DT_k(u_2)) \Big] \\
&= \int_\Omega e^{-u_1} \chi_{\{T_k(u_1)>T_k(u_2)\} \cap \{u_1<k\}} (1 - \theta^1(x)) |Du_1| \\
&\quad + \int_\Omega e^{-u_1} \chi_{\{T_k(u_1)>T_k(u_2)\} \cap \{u_2<k\}} (1 - \theta^2(x)) |Du_2|\,.
\end{aligned}$$

Using the convergence dominated theorem we can take limits when $k \to \infty$ to get

$$0 = \int_{\{u_1>u_2\}} e^{-u_1}(1 - \theta^1(x))|Du_1| + \int_{\{u_1>u_2\}} e^{-u_1}(1 - \theta^2(x))|Du_2|\,,$$

and since both integrals are nonnegative, it yields

$$0 = \int_{\{u_1>u_2\}} e^{-u_1}(1 - \theta^1(x))|Du_1| = \int_{\{u_1>u_2\}} e^{-u_1}(1 - \theta^2(x))|Du_2|\,.$$

Therefore, we deduce that $1-\theta^i(x)=0$ $|Du_i|$-a.e. in $\{u_1>u_2\}$ for $i=1,2$ and then, the Radon–Nikodým derivative is

$$\theta^1_k(\mathbf{z}_2, DT_k(u_1), x) = 1 \quad |Du_1|\text{-a.e. in } \{u_1>u_2\}\cap\{u_1<k\}$$

and

$$\theta^2_k(\mathbf{z}_1, DT_k(u_2), x) = 1 \quad |Du_2|\text{-a.e. in } \{u_1>u_2\}\cap\{u_2<k\}\,,$$

for every $k>0$. That is, we have the following equalities as measures

$$|DT_k(u_1)| \llcorner \{u_1>u_2\} = (\mathbf{z}_2, DT_k(u_1)) \llcorner \{u_1>u_2\} \tag{2.18}$$

and

$$|DT_k(u_2)| \llcorner \{u_1>u_2\} = (\mathbf{z}_1, DT_k(u_2)) \llcorner \{u_1>u_2\}\,. \tag{2.19}$$

Finally, noting that $\{T_k(u_1)>T_k(u_2)\}\subset\{u_1>u_2\}$ and the measure $(\mathbf{z}_1-\mathbf{z}_2, D(T_k(u_1)-T_k(u_2))^+)$ is nonnegative, it follows:

$$\begin{aligned}
&(\mathbf{z}_1-\mathbf{z}_2, D(T_k(u_1)-T_k(u_2))^+)\\
&\quad= \Big[|DT_k(u_1)| - (\mathbf{z}_2, DT_k(u_1)) - (\mathbf{z}_1, DT_k(u_2)) + |DT_k(u_2)|\Big]\\
&\quad\llcorner \{T_k(u_1)>T_k(u_2)\}\\
&\quad\le \Big[|DT_k(u_1)| - (\mathbf{z}_2, DT_k(u_1)) - (\mathbf{z}_1, DT_k(u_2)) + |DT_k(u_2)|\Big]\\
&\quad\llcorner \{u_1>u_2\} = 0\,,
\end{aligned}$$

and Step 3 is proved.

Step 4: $(\mathbf{z}_i, DT_k(u_j)) \llcorner \{T_k(u_1) > T_k(u_2)\} = |DT_k(u_j)| \llcorner \{T_k(u_1) > T_k(u_2)\}$ *as measures for* $i, j = 1, 2$ *and for all* $k > 0$.

Since $\{T_k(u_1) > T_k(u_2)\} \subset \{u_1 > u_2\}$ and we have proved equalities (2.18) and (2.19), Step 4 is straightforward.

Step 5: $f_1 = f_2 = 0$ *in the set* $\{u_1 > u_2\}$.

In Step 3 we have proved that the limit of expression (2.16) when $k \to \infty$ is 0. Then, using the monotone convergence theorem, we obtain

$$0 = \int_\Omega (u_1 - u_2)^+ (e^{-u_1} f_1 - e^{-u_2} f_2)\, dx\,.$$

Notice that if $u_1 > u_2$, then $e^{-u_1} f_1 = e^{-u_2} f_2$ and $f_1 = e^{-(u_2 - u_1)} f_2 > f_2$ when $f_2 \neq 0$. We conclude that $u_1 > u_2$ implies $f_2 = f_1 = 0$.

Step 6: $\displaystyle\int_{\{u_1 > u_2\}} |Du_1| = \int_{\{u_1 > u_2\}} |Du_2|$.

We begin by picking $T_\varepsilon((T_k(u_1) - T_k(u_2))^+)$ as a test function in problems (2.10) and (2.11), and by Step 5, we get the following equalities:

$$0 = \int_{\{T_k(u_1) > T_k(u_2)\}} (\mathbf{z}_1, DT_\varepsilon(T_k(u_1) - T_k(u_2))) + \int_{\{T_k(u_1) > T_k(u_2)\}} T_\varepsilon(T_k(u_1) - T_k(u_2))\, |Du_1| \tag{2.20}$$

and

$$0 = \int_{\{T_k(u_1) > T_k(u_2)\}} (\mathbf{z}_2, DT_\varepsilon(T_k(u_1) - T_k(u_2))) + \int_{\{T_k(u_1) > T_k(u_2)\}} T_\varepsilon(T_k(u_1) - T_k(u_2))\, |Du_2|\,. \tag{2.21}$$

Now, since Step 3 holds, we have that

$$(\mathbf{z}_1 - \mathbf{z}_2, D(T_k(u_1) - T_k(u_2))) \llcorner \{T_k(u_1) > T_k(u_2)\} = 0\,,$$

for all $k > 0$. Furthermore, when we take the restriction to the set $\{0 < T_k(u_1) - T_k(u_2) < \varepsilon\}$, the measure $(\mathbf{z}_1 - \mathbf{z}_2, D(T_k(u_1) - T_k(u_2)))$ also vanishes for all $\varepsilon > 0$. Because of this, when we consider together equations (2.20) and (2.21), we obtain

$$\begin{aligned}&\int_{\{T_k(u_1)>T_k(u_2)\}} T_\varepsilon(T_k(u_1) - T_k(u_2))\, |Du_1| \\ &\quad = \int_{\{T_k(u_1)>T_k(u_2)\}} T_\varepsilon(T_k(u_1) - T_k(u_2))\, |Du_2|\,.\end{aligned}$$

Now, dividing both integrals by ε and using the dominated convergence theorem we can take limits as $\varepsilon \to 0^+$ and then get

$$\int_{\{T_k(u_1)>T_k(u_2)\}} |Du_1| = \int_{\{T_k(u_1)>T_k(u_2)\}} |Du_2|\,.$$

Finally, the dominated convergence theorem also allows us to take limits when $k \to \infty$ and provide the desired equality:

$$\int_{\{u_1>u_2\}} |Du_1| = \int_{\{u_1>u_2\}} |Du_2|\,. \tag{2.22}$$

Step 7: $Du_1 = Du_2 = 0$ in $\{u_1 > u_2\}$.

We begin by taking the test function $(T_k(u_1) - T_k(u_2))^+$ in problem (2.10) and, having in mind Steps 4 and 5, we get

$$\begin{aligned}0 = &\int_{\{T_k(u_1)>T_k(u_2)\}} |DT_k(u_1)| - \int_{\{T_k(u_1)>T_k(u_2)\}} |DT_k(u_2)| \\ &+ \int_{\{T_k(u_1)>T_k(u_2)\}} (T_k(u_1) - T_k(u_2))\, |Du_1|\,.\end{aligned}$$

Now, in the first two integrals we may apply the dominated convergence theorem and in the last one, the monotone convergence theorem.

Hence, when $k \to \infty$ we get

$$0 = \int_{\{u_1 > u_2\}} (|Du_1| - |Du_2|) + \int_{\{u_1 > u_2\}} (u_1 - u_2) \, |Du_1| \, ,$$

and since the first integral is finite, so is the last one.

On the other hand, we have proved in Step 6 that the first integral vanishes, then the above equality becomes

$$0 = \int_{\{u_1 > u_2\}} (u_1 - u_2) \, |Du_1| \, ,$$

and we deduce that $|Du_1| \llcorner \{u_1 > u_2\} = 0$ and also $Du_1 = 0$ in $\{u_1 > u_2\}$.

To prove that $Du_2 = 0$ in $\{u_1 > u_2\}$ we use (2.22), and since we already know that $Du_1 = 0$ in $\{u_1 > u_2\}$, it becomes

$$0 = \int_{\{u_1 > u_2\}} |Du_2| \, .$$

Therefore we have that $Du_2 = 0$ in $\{u_1 > u_2\}$.

Step 8: $u_1 \le u_2$ in Ω.

We have seen that $D(u_1 - u_2) = 0$ in $\{u_1 > u_2\}$ and since $D^j(u_1 - u_2) = 0$, it holds that $D(u_1 - u_2)^+ = 0$ in Ω. Moreover, we know that $(u_1 - u_2)^+ = 0$ in $\partial\Omega$, therefore we get that $0 = (u_1 - u_2)^+$ in Ω. $\square$

Corollary 2.2.6. *Let Ω be a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary. Let f be a nonnegative function in $L^1(\Omega)$. Then, problem*

$$\begin{cases} -\mathrm{div}\left(\dfrac{Du}{|Du|}\right) + |Du| = f(x) & \text{in } \Omega \, , \\ u = 0 & \text{on } \partial\Omega \, , \end{cases}$$

has a unique solution $u \in BV(\Omega)$.

Table 2.1 Optimal regularity of solutions

Data	Solution
$f \in L^p(\Omega)$ with $p > N$	$u \in L^\infty(\Omega)$
$f \in L^N(\Omega)$	$u \in L^q(\Omega)$ with $1 \le q < \infty$
$f \in L^p(\Omega)$ with $1 < p < N$	$u \in L^{\frac{Np}{N-p}}(\Omega)$
$f \in L^1(\Omega)$	$u \in L^{\frac{N}{N-1}}(\Omega)$

2.3 Better summability

In Section 2.2, we have seen that problem (2.1) has a solution for every nonnegative datum of $L^1(\Omega)$ and this solution belongs to $BV(\Omega) \subset L^{\frac{N}{N-1}}(\Omega)$. We expect that the solution satisfies a better summability if the datum belongs to $L^q(\Omega)$ with $q > 1$. In this regard, we recall that when the nonnegative datum f is in the space $L^p(\Omega)$ with $p > N$, it is proved in [55, Theorem 3.5] that the solution is always bounded. For a datum $f \in L^N(\Omega)$, we prove in Chapter 1 that the solution belongs to $L^q(\Omega)$ with $1 \le q < \infty$.

In this section, we show that solutions belong to $L^{\frac{Np}{N-p}}(\Omega)$ if data are in $L^p(\Omega)$ with $1 < p < N$. Observe that this result adjust continuously for $p = 1$ and $p = N$ with the known facts (see Table 2.1 for summarize).

The proof of our theorem relies on certain preliminary results. The first one enables us to take a power of our solution u^q as a test function in problem (2.1).

Proposition 2.3.1. *Let $1 < p < N$ and $0 \le f \in L^p(\Omega)$ with $1 < p < N$. If $u \in BV(\Omega)$ is a solution to problem (2.1) satisfying $u^q \in L^{p'}(\Omega)$ for certain $q > 1$ and where $p' = \frac{p}{p-1}$, then u^q and $u^{q+1} \in BV(\Omega)$. Moreover,*

$$\int_\Omega |Du^q| + \int_\Omega u^q\,|Du| = \int_\Omega u^q f\,dx\,.$$

Proof. Fixed $k > 0$, we recall (I.3) and (I.4) to take the test function $G_\delta(T_k(u)^q)$ with $\delta, k > 0$ in problem (2.1), obtaining the following equality

$$\int_\Omega (\mathbf{z}, DG_\delta(T_k(u)^q)) + \int_\Omega G_\delta(T_k(u)^q)\,|Du| = \int_\Omega G_\delta(T_k(u)^q) f\,dx\,.$$

Since we know that the Radon–Nikodým derivative of $(\mathbf{z}, DG_\delta(T_k(u)^q))$ and $(\mathbf{z}, DT_k(u))$ with respect their respective total variations are equal (see Proposition 1.2.4) and moreover $(\mathbf{z}, DT_k(u)) = |DT_k(u)|$ holds for all $k > 0$, we deduce that

$$(\mathbf{z}, G_\delta(T_k(u)^q)) = |DG_\delta((T_k(u))^q)|\,.$$

Then, we can write

$$\int_\Omega |DG_\delta(T_k(u)^q)| + \int_\Omega G_\delta(T_k(u)^q)\,|Du| = \int_\Omega G_\delta(T_k(u)^q) f\,dx\,. \tag{2.23}$$

Now, we use that $G_\delta(T_k(u)^q) \leq u^q$ and Hölder's inequality to get the following bound

$$\int_\Omega G_\delta(T_k(u)^q) f\,dx \leq \int_\Omega u^q f\,dx \leq \|u^q\|_{L^{p'}(\Omega)} \|f\|_{L^p(\Omega)} < \infty\,.$$

Therefore, each integral in left-hand side of (2.23) is also bounded, i.e.,

$$\int_\Omega |DG_\delta(T_k(u)^q)| \leq \|u^q\|_{L^{p'}(\Omega)} \|f\|_{L^p(\Omega)} < \infty \tag{2.24}$$

and

$$\int_\Omega G_\delta(T_k(u)^q)\,|Du| \leq \|u^q\|_{L^{p'}(\Omega)} \|f\|_{L^p(\Omega)} < \infty\,. \tag{2.25}$$

We will take advantage of these bounds taking limits in (2.23).

Now, we are able to prove that $u^q \in BV(\Omega)$. Using the chain rule (see [7, Theorem 3.96]) in (2.24) we can write

$$\int_\Omega \chi_{\{u<k\}\cap\{u^q>\delta\}}|Du^q| = \int_\Omega |DG_\delta(T_k(u)^q)| \le \|u^q\|_{L^{p'}(\Omega)}\|f\|_{L^p(\Omega)} < \infty\,,$$

and appealing to the monotone convergence theorem, we let $\delta \to 0^+$ to get

$$\int_\Omega \chi_{\{u<k\}}|Du^q| \le \|u^q\|_{L^{p'}(\Omega)}\|f\|_{L^p(\Omega)} < \infty\,.$$

We let k goes to ∞ and appealing to the monotone convergence theorem once more, it follows that u^q is a function of bounded variation.

Let $0<\delta<1$ and keeping in mind (2.25), we get the following bound

$$\int_\Omega \chi_{\{u<k\}\cap\{u^{q+1}>\delta\}}|Du^{q+1}| = (q+1)\int_\Omega u^q\chi_{\{u<k\}\cap\{u^{q+1}>\delta\}}|Du|$$

$$\le (q+1)\int_\Omega (G_\delta(T_k(u)^q)+\delta)\,|Du|$$

$$\le (q+1)(\|u^q\|_{L^{p'}(\Omega)}\|f\|_{L^p(\Omega)} + \delta\|u\|_{BV(\Omega)}) < \infty\,.$$

Taking limits when $\delta \to 0^+$ and also when $k\to\infty$ we get

$$\int_\Omega |Du^{q+1}| \le (q+1)\|u^q\|_{L^{p'}(\Omega)}\|f\|_{L^p(\Omega)} < \infty\,, \tag{2.26}$$

that is, $u^{q+1}\in BV(\Omega)$.

To conclude, we take limits in (2.23). First take $\delta\to 0^+$ and then $k\to\infty$ to obtain

$$\int_\Omega |Du^q| + \int_\Omega u^q|Du| = \int_\Omega u^q\, f\, dx\,.$$

□

Theorem 2.3.2. *Let $1 < p < N$ and let $f \in L^p(\Omega)$ be a nonnegative function. Then, the solution to problem (2.1) belongs to $BV(\Omega) \cap L^s(\Omega)$ for every $1 \leq s < \frac{Np}{N-p}$.*

Proof. Let $u \in BV(\Omega)$ denote the unique solution to problem (2.1). For every $j \in \mathbb{N}$, we will prove that $u \in L^{s_j}(\Omega)$, where

$$s_j = N' \sum_{k=0}^{j} \left(\frac{N'}{p'} \right)^k ,$$

and p' denotes the conjugate of p, given by $p' = \frac{p}{p-1}$.

It should be pointed out that $\lim_{j\to\infty} s_j = N' \sum_{k=0}^{\infty} \left(\frac{N'}{p'}\right)^k = \frac{Np}{N-p}$. Thus, proving $u \in L^{s_j}(\Omega)$ for all $j \in \mathbb{N}$, we are done.

We begin by choosing $q = \frac{N'}{p'}$ and since $u^q \in L^{p'}(\Omega)$, we may apply Proposition 2.3.1 to conclude that $u^{q+1} \in BV(\Omega) \subset L^{N'}(\Omega)$ and therefore $u \in L^{N'\left(\frac{N'}{p'}+1\right)}(\Omega)$, that is, $u \in L^{s_1}(\Omega)$.

Assuming now that $u \in L^{s_j}(\Omega)$, we take

$$q = \frac{N'}{p'} \sum_{k=0}^{j} \left(\frac{N'}{p'} \right)^k .$$

By hypothesis we already know that $u \in L^{qp'}(\Omega)$, and by Proposition 2.3.1 we get $u^{q+1} \in BV(\Omega) \subset L^{N'}(\Omega)$. Hence, $u \in L^{N'(q+1)}(\Omega)$ and since $s_{j+1} = N'(q+1)$, the proof is done. $\square$

Now we are ready to prove the main result of this section.

Theorem 2.3.3. *Let f be a nonnegative function which belongs to $L^p(\Omega)$ with $1 < p < N$. Then, the unique solution u to problem (2.1) satisfies $u \in BV(\Omega) \cap L^{\frac{Np}{N-p}}(\Omega)$.*

Proof. To show that $u \in L^{\frac{Np}{N-p}}(\Omega)$, we first prove that the following inequality

$$\left(\int_\Omega u^{(q+1)N'}\,dx\right)^{\frac{1}{N'}\left(1-\frac{q}{q+1}\right)} \le C(q+1)\|f\|_{L^p(\Omega)}|\Omega|^{\frac{1}{p'}-\frac{q}{(q+1)N'}} \tag{2.27}$$

holds for every $0 < q < \frac{N(p-1)}{N-p}$. Indeed, if q satisfies this condition, then we have that

$$qp' < \frac{N(p-1)}{N-p}\,\frac{p}{p-1} = \frac{Np}{N-p}\,.$$

Therefore, applying Theorem 2.3.2 and Proposition 2.3.1 we get $u^{q+1} \in BV(\Omega)$.

Now, we use Sobolev's inequality and inequality (2.26) to get

$$\left(\int_\Omega u^{(q+1)N'}\,dx\right)^{\frac{1}{N'}} \le C\int_\Omega |Du^{q+1}| \le C(q+1)\|f\|_{L^p(\Omega)}\left(\int_\Omega u^{qp'}\,dx\right)^{\frac{1}{p'}}\,.$$

where $C = C(p,N)$. Moreover, since $qp' < (q+1)N'$, we can apply Hölder's inequality and we also get

$$\int_\Omega u^{qp'}\,dx \le \left(\int_\Omega (u^{qp'})^{\frac{(q+1)N'}{qp'}}\,dx\right)^{\frac{qp'}{(q+1)N'}} |\Omega|^{1-\frac{qp'}{(q+1)N'}}\,.$$

Summing up, we have

$$\left(\int_\Omega u^{(q+1)N'}\,dx\right)^{\frac{1}{N'}} \le C(q+1)\|f\|_p\left(\int_\Omega u^{(q+1)N'}\,dx\right)^{\frac{q}{(q+1)N'}} |\Omega|^{\frac{1}{p'}-\frac{q}{(q+1)N'}}\,,$$

and then, inequality (2.27) holds.

Now, let $0 < q_n < \frac{N(p-1)}{N-p}$ define a nondecreasing sequence convergent to $\frac{N(p-1)}{N-p}$. Hence, for every $n \in \mathbb{N}$ it holds

$$\left(\int_\Omega u^{(q_n+1)N'}\,dx\right)^{\frac{1}{N'}\frac{q_n}{q_n+1}} \le C(q_n+1)\|f\|_{L^p(\Omega)}|\Omega|^{\frac{1}{p'}-\frac{q_n}{(q_n+1)N'}}\,.$$

Thanks to Fatou's lemma, letting $n \to \infty$, we get

$$\int_\Omega u^{\frac{p(N-1)}{N-p}N'}\,dx \le \liminf_{n\to\infty}\left[C(q_n+1)\|f\|_{L^p(\Omega)}|\Omega|^{\frac{1}{p'}-\frac{q_n}{(q_n+1)N'}}\right]^{\frac{N'(q_n+1)}{q_n}}$$
$$\le \left[C(p,N)\frac{p(N-1)}{N-p}\|f\|_{L^p(\Omega)}\right]^{\frac{p}{p-1}}.$$

Therefore, $u \in L^{\frac{Np}{N-p}}(\Omega)$. □

Remark 2.3.4. Going back to Proposition 2.3.1, it follows from $u^{\frac{N(p-1)}{N-p}} \in L^{p'}(\Omega)$ that $u^{\frac{N(p-1)}{N-p}}$ can be taken as a test function in problem (2.1). That is,

$$\int_\Omega |D(u^{\frac{N(p-1)}{N-p}})| + \int_\Omega u^{\frac{N(p-1)}{N-p}}|Du| = \int_\Omega f u^{\frac{N(p-1)}{N-p}}.$$

2.4 Explicit examples

This section is devoted to show radial examples of solutions in a ball. These examples allow us to provide evidence that our regularity result is sharp (see Remark 2.4.2 below).

Recall that $B_R(0)$ stands for the open ball of radius $R > 0$ centered at 0.

Example 2.4.1. *Set $\Omega = B_R(0)$. We consider a particular case of problem (2.1):*

$$\begin{cases} -\mathrm{div}\left(\dfrac{Du}{|Du|}\right) + |Du| = \dfrac{\lambda}{|x|^q} & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{2.28}$$

with $1 < q < N$ and $\lambda > 0$.

We know that a solution u to problem (2.28) must be a nonnegative function of bounded variation with no jump part and there also exists a vector field $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ with $\|\mathbf{z}\|_{L^\infty(\Omega)} \leq 1$ such that

$$\begin{aligned} &\textbf{(i)} \quad -\operatorname{div}\mathbf{z} + |Du| = \frac{\lambda}{|x|^q} \ \text{ in } \ \mathcal{D}'(\Omega)\,, \\ &\textbf{(ii)} \quad (\mathbf{z}, DT_k(u)) = |DT_k(u)| \ \text{ as measures in } \ \Omega \ \ (\text{for every } k > 0)\,, \\ &\textbf{(iii)} \quad u|_{\partial\Omega} = 0\,. \end{aligned} \tag{2.29}$$

We assume that the solution is radial, that is, $u(x) = h(|x|) = h(r)$. Moreover, in order to satisfy the Dirichlet condition, we need that $h(R) = 0$ holds. In addition, we also assume $h'(r) \leq 0$ for all $0 \leq r \leq R$.

If $h'(r) < 0$ in an interval, then the vector field is given by $\mathbf{z}(x) = \frac{Du(x)}{|Du(x)|} = -\frac{x}{|x|}$ and $\operatorname{div}\mathbf{z}(x) = -\frac{N-1}{|x|}$. Therefore, equation (2.29) becomes

$$\frac{N-1}{r} - h'(r) = \frac{\lambda}{r^q}\,. \tag{2.30}$$

Since we are assuming $h'(r) < 0$, then

$$\frac{N-1}{r} - \frac{\lambda}{r^q} < 0\,.$$

Now, we define

$$\rho_\lambda = \left(\frac{N-1}{\lambda}\right)^{\frac{1}{1-q}}\,.$$

Thus, if $r \leq \rho_\lambda$, then $h'(r) < 0$ may hold, and if $r > \rho_\lambda$ the solution must satisfy $h'(r) = 0$.

We assume $0 < \rho_\lambda < R$. Then, when $\rho_\lambda \leq r \leq R$ the solution to problem (2.28) is constant, and since we know that $h(R) = 0$, we deduce that $h(r) = 0$ for all $\rho_\lambda \leq r \leq R$.

Taking into account (2.30), if $0 \leq r < \rho_\lambda$, then solution is given by

$$h(\rho_\lambda) - h(r) = \int_r^{\rho_\lambda} h'(s)\,ds = \int_r^{\rho_\lambda} \left(\frac{N-1}{s} - \frac{\lambda}{s^q}\right) ds$$

$$= (N-1)\log\left(\frac{\rho_\lambda}{r}\right) + \frac{\lambda}{1-q}(r^{1-q} - \rho_\lambda^{1-q})\,.$$

Hence, the solution to problem (2.28) is

$$u(x) = \begin{cases} (N-1)\log\left(\frac{|x|}{\rho_\lambda}\right) + \frac{\lambda}{1-q}(\rho_\lambda^{1-q} - |x|^{1-q}) & \text{if} \quad 0 \le |x| < \rho_\lambda\,, \\ 0 & \text{if} \quad \rho_\lambda < |x| \le R\,, \end{cases}$$

and it only remains to be identify the vector field $\mathbf{z}$. When $0 \le r < \rho_\lambda$ we know that the vector field is $\mathbf{z}(x) = -x/|x|$, and when $\rho_\lambda \le r \le R$ we assume that the vector field is radial: $\mathbf{z}(x) = x\,\xi(|x|)$. Thus, $\operatorname{div}\mathbf{z}(x) = N\,\xi(|x|) + |x|\,\xi'(|x|)$, and equation (2.29) becomes

$$-(N\,\xi(r) + r\,\xi'(r)) = \frac{\lambda}{r^q}\,.$$

That is,

$$-r^N\,\xi(r) = -\int \left(r^N\,\xi(r)\right)'\,dr = \int \lambda\,r^{N-1-q}\,dr = \frac{\lambda}{N-q}\,r^{N-q} + C\,,$$

for some constant C to be determinate. Then,

$$\xi(r) = -\frac{\lambda}{N-q}\,r^{-q} - C\,r^{-N}\,.$$

Since we need a continuous vector field and we know that $\mathbf{z}(x) = -\frac{x}{\rho_\lambda}$ for all x such that $|x| = \rho_\lambda$, we get the following equation

$$\rho_\lambda^{-1} = \frac{\lambda}{N-q}\,\rho_\lambda^{-q} + C\,\rho_\lambda^{-N}\,.$$

Finally, using that $\lambda = (N-1)\,\rho_\lambda^{q-1}$ we deduce

$$C = \rho_\lambda^{N-1}\frac{1-q}{N-q}\,,$$

and therefore, the vector field is given by

$$\mathbf{z}(x) = \begin{cases} -\frac{x}{|x|} & \text{if } \ 0 \le |x| < \rho_\lambda\,, \\ -\frac{x}{N-q}\left((N-1)\frac{\rho_\lambda^{q-1}}{|x|^q} + (1-q)\frac{\rho_\lambda^{N-1}}{|x|^N}\right) & \text{if } \ \rho_\lambda < |x| \le R\,. \end{cases}$$

Remark 2.4.2. In Theorem 2.3.3 we have proved that if data f belong to the space $L^{\frac{N}{q}}(B_R(0))$, then $u \in L^{\frac{N}{q-1}}(B_R(0))$. Since $\frac{\lambda}{|x|^q} \in L^s(B_R(0))$ for all $s < \frac{N}{q}$, it follows that $u \in L^r(B_R(0))$ for all $r < \frac{N}{q-1}$. This is exactly what it is shown.

Remark 2.4.3. In Proposition 1.5.4 it was proved that for any "small" datum $f \in W^{-1,\infty}(B_R(0))$, the solution to problem (2.1) is always trivial. Nevertheless, in our examples we always get a nontrivial solution. This is due to the fact that the datum $f(x) = \lambda\,|x|^{-q}$ when $1 < q < N$ is not in the space $W^{-1,\infty}(B_R(0))$:

Let $s = N - q$, then function $v(x) = |x|^{-s} - R^{-s} \in W_0^{1,1}(B_R(0))$ since $s < N - 1$. However, the product $f(x)v(x) = \lambda|x|^{-N} - f(x)R^{-s} \notin L^1(B_R(0))$. We conclude that $f \notin W^{-1,\infty}(B_R(0))$.

It may be worth comparing our example with that occurring when the datum is $\frac{\lambda}{|x|^q}$ with $0 < q < 1$. In the same way as in Example 2.4.1, the solution to problem (2.28) depends on the value

$$r_\lambda = \left(\frac{N-q}{\lambda}\right)^{\frac{1}{1-q}}\,.$$

When $0 < q < 1$, the solution to problem (2.28) is given by

$$u(x) = \begin{cases} (N-1)\log\left(\frac{r_\lambda}{R}\right) + \frac{\lambda}{1-q}(R^{1-q} - r_\lambda^{1-q}) & \text{if } \ 0 \le |x| \le r_\lambda\,, \\ (N-1)\log\left(\frac{|x|}{R}\right) + \frac{\lambda}{1-q}(R^{1-q} - |x|^{1-q}) & \text{if } \ r_\lambda < |x| \le R\,, \end{cases}$$

with the associated vector field

$$\mathbf{z}(x) = \begin{cases} -\frac{\lambda}{N-q} x|x|^{-q} & \text{if} \quad 0 \le |x| \le r_\lambda\,, \\ -\frac{x}{|x|} & \text{if} \quad r_\lambda < |x| \le R\,. \end{cases}$$

We notice that since $0 < q < 1$, this solution is always bounded.

Chapter 3

Problem with dynamical boundary conditions

3.1 Introduction

In this chapter we deal with an existence and uniqueness result for an evolution problem. It consists in an elliptic equation involving the 1-Laplacian operator and a dynamical boundary condition, namely

$$\begin{cases} \lambda u - \operatorname{div}\left(\dfrac{Du}{|Du|}\right) = 0 & \text{in } (0,+\infty)\times\Omega\,, \\ \omega_t + \left[\dfrac{Du}{|Du|},\nu\right] = g(t,x) & \text{on } (0,+\infty)\times\partial\Omega\,, \\ u=\omega & \text{on } (0,+\infty)\times\partial\Omega\,, \\ \omega(0,x)=\omega_0(x) & \text{on } \partial\Omega\,; \end{cases} \tag{3.1}$$

where Ω is a bounded open set in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, λ is a nonnegative parameter, ν stands for the unit outward normal vector on $\partial\Omega$, $g\in L^1_{loc}(0,+\infty;L^2(\partial\Omega))$ and $\omega_0\in L^2(\partial\Omega)$. Here, we have denoted by ω_t the distributional derivative of ω with respect to t. As far as we know, this is the first time that dynamical boundary conditions for the 1-Laplacian are considered.

We point out that dynamical boundary conditions appear in applications where there is a reaction term in the problem that concentrates in a small strip around the boundary of the domain, while in the interior there is no reaction and only diffusion matters.

For that reason, it appears in many mathematical models including heat transfer in a solid in contact with a moving fluid, in thermoelasticity, in biology, etc. This fact has been attracted the attention of many authors to deal with problems having dynamical boundary conditions where mainly of those problems involve linear operators (see [5, 11, 10, 14, 29, 32, 33, 36, 41, 43, 48, 63]). The study of problems where an elliptic or parabolic equation occurs with this kind of boundary conditions is nowadays an active branch of research and we refer to [37, 39, 42, 59, 64] and references therein for recent papers.

The first study of an evolution problem having an elliptic equation driven by the p-Laplacian (with $p > 1$) and a dynamical boundary condition is due to [11] (see also [10]). To handle with that nonlinear problem, the authors define a completely accretive operator, apply the nonlinear semigroup theory to get a mild solution and finally, prove that this mild solution is actually a weak solution. Once their result is available, we may study problem (3.1) taking the solution corresponding to $p > 1$ and letting $p \to 1$. Nevertheless, we are not able to pass to the limit and this approach remains an open problem. Furthermore, once a solution to our problem is obtained, we cannot prove that it is the limit of mild solutions to problems involving the p-Laplacian because we should use a Modica type result on lower semicontinuity (see [60, Proposition 1.2]) for functionals depending on time.

Instead of trying this approach, we adapt the method used in [11] and apply the nonlinear semigroup theory (we refer to [17] for a good introduction to this theory). Obviously, the singular features of the 1-Laplacian do not allow us to follow every step.

Among the special features verified by the 1-Laplacian, we highlight that boundary conditions do not hold, necessarily, in the sense of traces (we refer to [8] for the Dirichlet problem, to [56] for the Neumann problem as well as [9] for the homogeneous Neumann for a related equation, and to [54] for the Robin problem). This fact leads us to modify the procedure from the very beginning since it implies a change in the definition of the associated accretive operator. Indeed, the translation of the operator studied in [11] to our setting would be an operator $\mathfrak{B} : L^2(\partial\Omega) \to L^2(\partial\Omega)$ defined as follows:

Definition 3.1.1. *Let $v, \omega \in L^2(\partial\Omega)$. Then, $v \in \mathfrak{B}(\omega)$ if there exists $u \in BV(\Omega) \cap L^2(\Omega) \cap L^2(\partial\Omega)$ such that $u|_{\partial\Omega} = \omega$ and it is a solution to the Neumann problem*

$$\begin{cases} \lambda u - \operatorname{div}\left(\dfrac{Du}{|Du|}\right) = 0 & \text{in } \Omega\,, \\ \left[\dfrac{Du}{|Du|} \cdot \nu\right] = v & \text{on } \partial\Omega\,. \end{cases}$$

This is, in fact, a completely accretive operator but, unfortunately, we are not able to prove that it satisfies the range condition $R(I + \varepsilon\mathfrak{B}) = L^2(\partial\Omega)$ for all $\varepsilon > 0$. Thus, the nonlinear semigroup theory cannot be applied. We turn out to define our operator for $v, \omega \in L^2(\partial\Omega)$ as $v \in \mathcal{B}(\omega)$ if $v \in L^\infty(\partial\Omega)$, with $\|v\|_{L^\infty(\partial\Omega)} \le 1$, and there exists $u \in BV(\Omega) \cap L^2(\Omega)$ which is a solution to the Dirichlet problem with datum ω and it is also a solution of the Neumann problem with datum v (see Definition 3.3.2 below). Now, we do not know if this operator is completely accretive, we only prove that it is accretive in $L^2(\partial\Omega)$. Hence, we do not have to expect that our solution holds every feature satisfied by solutions to problems driven by the p-Laplacian (for instance, we just choose initial data belonging to $L^2(\partial\Omega)$). Moreover, even when our solution satisfies the same property, the proof of this fact can be different, as can be checked in the comparison principle. Despite these difficulties, we obtain global existence and uniqueness of solution for every datum $\omega_0 \in L^2(\partial\Omega)$ as well as a comparison principle. Furthermore, we prove that our solution is a strong solution in the sense that the problem holds for almost all $t > 0$. We also analyze some related properties as the continuous dependence on data. Our main result is the following.

Theorem 3.1.2. *Let $\lambda > 0$, and let $g \in L^1_{loc}(0, +\infty; L^2(\partial\Omega))$ and $\omega_0 \in L^2(\partial\Omega)$. Then, there exists a unique global strong solution (u, ω) to problem (3.1) in the sense of Definition 3.4.1. Indeed, this solution satisfies $u \in L^2_{loc}(0, +\infty; L^2(\Omega)) \cap L^\infty_{loc}(0, +\infty; BV(\Omega))$ and $\omega \in C([0, +\infty[; L^2(\partial\Omega)) \cap W^{1,1}_{loc}(0, +\infty; L^2(\partial\Omega))$.*

Furthermore, the following estimates hold:

$$\|\omega\|_{L^\infty(0,T;L^2(\partial\Omega))} \leq \|\omega_0\|_{L^2(\partial\Omega)} + \|g\|_{L^1(0,T;L^2(\partial\Omega))}\,, \qquad \textit{for every } \; T > 0\,,$$

$$\lambda\|u(t)\|^2_{L^2(\Omega)} + \|u(t)\|_{BV(\Omega)} \leq 2\|\omega(t)\|_{L^1(\partial\Omega)}\,, \qquad \textit{for almost all } \; t > 0\,.$$

This chapter is divided in five sections. In Section 3.2, we introduce our notation and state the main features of nonlinear semigroups theory. Section 3.3 is devoted to obtain the mild solution to the associated abstract Cauchy problem, while in Section 3.4 we check that this mild solution is actually a strong solution to problem (3.1). Finally, Section 3.5 deals with continuous dependence of data.

3.2 Preliminaries

In this section, we present some useful results and the notation used in this chapter.

As we have mentioned in Introduction, $L^q(\Omega)$ and $W_0^{1,p}(\Omega)$ denote the usual Lebesgue and Sobolev spaces, respectively. Then, if $T > 0$, the spaces $L^r(0,T;L^q(\Omega))$ are defined as follows: a Lebesgue measurable function $u : (0,T) \times \Omega \to \mathbb{R}$ belongs to $L^r(0,T;L^q(\Omega))$ if

$$\int_0^T \left(\int_\Omega |u(t,x)|^q dx \right)^{\frac{r}{q}} dt < \infty \,.$$

It is clear that for $q, r \geq 1$, the space $L^r(0,T;L^q(\Omega))$ is a Banach space equipped with the norm

$$||u||_{L^r(0,T;L^q(\Omega))} = \left(\int_0^T \left(\int_\Omega |u(t,x)|^q dx \right)^{\frac{r}{q}} dt \right)^{\frac{1}{r}} .$$

The spaces $L^r(0,T;W_0^{1,q}(\Omega))$ or $W^{1,r}(0,T;L^q(\Omega))$ are defined in a similar way. We refer to [34] for more details.

Given a Banach function space X, recall that $u \in L^r(0,T;X)$ implies that $u(t) \in X$ for almost all $t \in]0,T[$. Moreover, instead of writing "$u \in L^r(0,T;X)$ for every $T > 0$", we shall write $u \in L^r_{\text{loc}}(0,+\infty;X)$. Finally, if $\mathcal{I}$ is a real interval, then $C(\mathcal{I};X)$ stands for the space of all continuous functions from $\mathcal{I}$ into X.

3.2.1 Mild solutions

In this subsection we present some definitions and results concerning mild solutions.

Let X be a Banach space and let $\mathcal{P}(X)$ be the collection of all subsets of X. Every mapping $\mathcal{A} : X \to \mathcal{P}(X)$ will be called an operator in X

and we denote by

$$D(\mathcal{A}) = \{v \in X \,:\, \mathcal{A}(v) \neq \emptyset\}$$

the effective domain of $\mathcal{A}$ and by

$$R(\mathcal{A}) = \cup\{\mathcal{A}(v) \,:\, v \in D(\mathcal{A})\}$$

its range.

We say that operator $\mathcal{A}$ is accretive if

$$\|v - \widehat{v} + \alpha(\omega - \widehat{\omega})\|_X \geq \|v - \widehat{v}\|_X \,,$$

whenever $\alpha \geq 0$, and $v \in \mathcal{A}(\omega)$ and $\widehat{v} \in \mathcal{A}(\widehat{\omega})$. Moreover, when X is a Hilbert space, the operator $\mathcal{A}$ is accretive if and only if it is monotone, that is,

$$\langle v - \widehat{v}, \omega - \widehat{\omega}\rangle \geq 0 \,,$$

for every $v \in \mathcal{A}(\omega)$ and $\widehat{v} \in \mathcal{A}(\widehat{\omega})$.

On the other hand, we say that $\mathcal{A} : X \to \mathcal{P}(X)$ is m-accretive if it is accretive and $R(I + \varepsilon\mathcal{A}) = X$ for all $\varepsilon > 0$, where $I : X \to X$ denotes the identity operator.

Now, let us introduce the notion of mild solution to the following abstract Cauchy problem

$$\begin{cases} \omega_t + \mathcal{A}(\omega) \ni g \,, \\ \omega(0) = \omega_0 \,, \end{cases} \tag{3.2}$$

where $g \in L^1_{loc}(0, +\infty; X)$ and $\omega_0 \in X$.

Fix $T > 0$ and $\varepsilon > 0$. Let $0 \le t_0 < t_1 < \cdots < t_n \le T$ be a partition of the interval $[0,T]$ satisfying

$$\begin{aligned} &0 \le t_0 < \varepsilon\,, \\ &t_i - t_{i-1} < \varepsilon\,, \qquad \text{for} \;\; i = 1,2,\ldots,n\,, \\ &0 \le T - t_n < \varepsilon\,, \end{aligned}$$

and let $g_1, g_2, \ldots, g_n$ be a finite sequence in X such that

$$\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \|g(s) - g_i\|_X \, ds < \varepsilon\,.$$

Then, the system

$$\frac{\omega_i - \omega_{i-1}}{t_i - t_{i-1}} + \mathcal{A}(\omega_i) \ni g_i\,, \qquad \text{for} \;\; i = 1,2,\ldots,n\,, \tag{3.3}$$

is called an ε-discretization of (3.2) on $[0,T]$. Moreover, we say that $\omega_\varepsilon : [t_0, t_n] \to X$ is a solution to this ε-discretization if it is a piecewise constant function such that $\omega_\varepsilon(t_0) = \omega_0$, $\omega_\varepsilon(t) = \omega_i$ on $]t_{i-1}, t_i]$ for $i = 1,2,\ldots,n$, and system (3.3) holds.

Remark 3.2.1. The existence of an ε-discretization is based on the possibility of approximating any function $g \in L^1(0,T;X)$ by steps functions $\sum_{i=1}^n g_i \chi_{]t_{i-1},t_i]}$. We point out that this approximation can be taken in such way that $g_i = g(t_i)$, being t_i a Lebesgue point of g for $i = 1,\ldots,n$ (see [17, Proposition 1.5]).

Now, we are able to define mild solutions.

Definition 3.2.2. *Fixed $T > 0$, let $g \in L^1(0,T;X)$ and $\omega_0 \in X$. A mild solution of the abstract Cauchy problem (3.2) on $[0,T]$ is a function $\omega \in C([0,T];X)$ such that, for every $\varepsilon > 0$, there exists an ε-discretization* *of* (3.2) *on* $[0,T]$ *which has a solution* ω_ε *satisfying*

$$\|\omega(t)-\omega_\varepsilon(t)\|_X < \varepsilon\,, \qquad \textit{for all } \; t\in[0,T]\,.$$

Moreover, if $g\in L^1_{loc}(0,+\infty;X)$*, we say that* $\omega\in C([0,+\infty[;X)$ *is a mild solution of problem* (3.2) *on* $[0,+\infty[$ *if its restriction to each subinterval* $[0,T]$ *of* $[0,+\infty[$ *is a mild solution on* $[0,T]$*.*

Remark 3.2.3. From the definition of mild solution one deduces that solutions to discretizations satisfy the following convergence:

$$\omega_\varepsilon \to \omega \;\text{ in }\; L^\infty([0,T];X)\,,$$

for every $T>0$.

Finally, we present the result of [17] (see also [12, Theorem A.26]) that we will use to prove the existence of mild solutions in our context.

Theorem 3.2.4. *[17, Theorem 4.6] Let* $\mathcal{A}$ *be an* m*-accretive operator in* X*. Consider* $\omega_0\in\overline{D(\mathcal{A})}$ *and* $g\in L^1_{loc}([0,+\infty[;X)$*. Then, problem* (3.2) *has a unique mild solution* ω *on* $[0,+\infty[$*.*

We also need the following definition.

Definition 3.2.5. *Fixed* $T>0$*, let* $g\in L^1(0,T;X)$ *and* $\omega_0\in X$*. A strong solution of problem* (3.2) *on* $[0,T]$ *is an absolutely continuous function* $\omega:[0,T]\to X$ *which is differentiable almost everywhere on* $[0,T]$ *and satisfies* $\omega_t(t)+\mathcal{A}(\omega(t))\ni g(t)$ *for almost all* $t\in[0,T]$ *and* $\omega(0)=\omega_0$*.*

We point out that every strong solution is a mild solution (see [17, Theorem 1.4]), but the converse does not hold in general.

For further information about mild solutions and semigroups on Banach spaces we refer to [17] (and to [23] for semigroups on Hilbert spaces).

3.3 Existence of mild solutions

Let $T > 0$ and consider the problem

$$\begin{cases} \lambda u - \operatorname{div}\left(\dfrac{Du}{|Du|}\right) = 0 & \text{in } \;]0,T[\times\Omega\,, \\ \omega_t + \left[\dfrac{Du}{|Du|},\nu\right] = g(t,x) & \text{on } \;]0,T[\times\partial\Omega\,, \\ u = \omega & \text{on } \;]0,T[\times\partial\Omega\,, \\ \omega(0,x) = \omega_0(x) & \text{on } \; \partial\Omega\,. \end{cases} \tag{3.4}$$

As we have mentioned in the previous section, we would like to define an m-accretive operator in $L^2(\partial\Omega)$ in order to apply the semigroup theory and finally, using Theorem 3.2.4, get a mild solution. Afterwards, using this mild solution we will obtain a strong solution to problem (3.4).

Remark 3.3.1. We point out that our operator will be defined on the boundary, and so our mild solution is ω, while u appearing in problem (3.4) is just the corresponding auxiliary function. Nevertheless, this auxiliary function u is univocally determined by ω, since solutions to the Dirichlet problem for equation $\lambda u - \operatorname{div}\left(\frac{Du}{|Du|}\right) = 0$ are unique (see [8, Theorem 4]).

We start with the definition of the operator $\mathcal{B}$ in the space $L^2(\partial\Omega)$.

Definition 3.3.2. *Let $\omega \in L^2(\partial\Omega)$. We say that $v \in \mathcal{B}(\omega)$ if $v \in L^\infty(\partial\Omega)$ with $\|v\|_{L^\infty(\partial\Omega)} \leq 1$ and there exist a function $u \in BV(\Omega)\cap L^2(\Omega)$ and a vector field $\mathbf{z} \in L^\infty(\Omega;\mathbb{R}^N)$ with $\|\mathbf{z}\|_{L^\infty(\Omega)} \leq 1$ such that*

(i) $\lambda u - \operatorname{div}\mathbf{z} = 0$ *in* $\mathcal{D}'(\Omega)\,,$

(ii) $(\mathbf{z}, Du) = |Du|$ *as measures in* $\Omega\,,$

(iii) $[\mathbf{z},\nu] = v$ $\;\mathcal{H}^{N-1}$*-a.e. on* $\partial\Omega\,,$

$$\textbf{(iv)} \quad [\mathbf{z}, \nu] \in \operatorname{sign}(\omega - u) \quad \mathcal{H}^{N-1}\text{-}a.e. \;\; on \;\; \partial\Omega\,.$$

Using Green's formula and since conditions **(i)** and **(iii)** hold, we may deduce the following variational formulation:

$$\lambda \int_\Omega u\,\varphi\,dx + \int_\Omega (\mathbf{z}, D\varphi) = \int_{\partial\Omega} v\,\varphi\,d\mathcal{H}^{N-1}\,,$$

for every test function $\varphi \in BV(\Omega) \cap L^2(\Omega)$. Notice that function $v \in L^\infty(\partial\Omega)$ and $\varphi|_{\partial\Omega} \in L^1(\partial\Omega)$, so that the last integral is well defined.

In other words, we say that $v \in \mathcal{B}(\omega)$ if there exists $u \in BV(\Omega) \cap L^2(\Omega)$ such that u is a solution to equation

$$\lambda u - \operatorname{div}\left(\frac{Du}{|Du|}\right) = 0 \;\; \text{in} \;\; \Omega\,, \tag{3.5}$$

with the Dirichlet boundary condition

$$u = \omega \;\; \text{on} \;\; \partial\Omega\,, \tag{3.6}$$

and it is also a solution to equation (3.5) with Neumann boundary condition

$$\left[\frac{Du}{|Du|}, \nu\right] = v \;\; \text{on} \;\; \partial\Omega\,. \tag{3.7}$$

From another point of view, the operator $\mathcal{B}$ can be written as $v \in \mathcal{B}(\omega)$ if $v \in L^\infty(\partial\Omega)$ satisfies

(i) $\|v\|_{L^\infty(\partial\Omega)} \leq 1$,

(ii) $v \in \operatorname{sign}(\omega - u)$ where u is the solution to (3.5) with boundary condition (3.7).

3.3.1 Associated Robin problem

Now, we analyze the Robin problem for equation (3.5). To this end we follow [54]. For $\beta > 0$, we consider the Robin boundary condition given by

$$\beta u + \left[\frac{Du}{|Du|}, \nu\right] = g \quad \text{on} \quad \partial\Omega\,. \tag{3.8}$$

Definition 3.3.3. *Let $g \in L^2(\partial\Omega)$. We say that $u \in BV(\Omega) \cap L^2(\Omega)$ is a weak solution to Robin problem (3.8) for equation (3.5) if there exists a vector field $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^N)$ with $\|\mathbf{z}\|_{L^\infty(\Omega)} \le 1$ such that*

$$\begin{aligned}
&\textbf{(i)} && \lambda u - \operatorname{div} \mathbf{z} = 0 \quad \textit{in} \quad \mathcal{D}'(\Omega)\,,\\
&\textbf{(ii)} && (\mathbf{z}, Du) = |Du| \quad \textit{as measures in} \quad \Omega\,,\\
&\textbf{(iii)} && T_1(\beta u - g) = -[\mathbf{z}, \nu] \quad \mathcal{H}^{N-1}\textit{-a.e. on} \quad \partial\Omega\,.
\end{aligned}$$

As a consequence of Green's formula, we may write the following variational formulation:

$$\lambda \int_\Omega u\,\varphi\,dx + \int_\Omega (\mathbf{z}, D\varphi) + \int_{\partial\Omega} T_1(\beta u - g)\,\varphi\,d\mathcal{H}^{N-1} = 0\,, \tag{3.9}$$

for every $\varphi \in BV(\Omega) \cap L^2(\Omega)$.

Remark 3.3.4. Every solution to equation (3.5) with the Robin boundary condition (3.8) is also a solution to the same equation but with Dirichlet boundary condition (3.6) for ω satisfying $T_1(\beta u - g) = \beta\omega - g$ (see [54, Proposition 2.13]). Using this function, equality (3.9) becomes

$$\lambda \int_\Omega u\,\varphi\,dx + \int_\Omega (\mathbf{z}, D\varphi) + \int_{\partial\Omega} (\beta\omega - g)\,\varphi\,d\mathcal{H}^{N-1} = 0\,,$$

for every $\varphi \in BV(\Omega) \cap L^2(\Omega)$.

In the following observation we show that a comparison principle holds.

Remark 3.3.5. Consider data $g_1, g_2 \in L^2(\partial\Omega)$ and let $u_1, u_2 \in BV(\Omega) \cap L^2(\Omega)$ be the corresponding solutions to the Robin problem with data g_1 and g_2, respectively. Denote by $\mathbf{z}_i \in L^\infty(\Omega; \mathbb{R}^N)$ the associated vector fields and by ω_i the functions satisfying $T_1(\beta u_i - g_i) = \beta\omega_i - g_i$, for $i = 1, 2$.

Now, we prove that $g_1 \leq g_2$ on $\partial\Omega$ implies $u_1 \leq u_2$ in Ω and $\omega_1 \leq \omega_2$ on $\partial\Omega$. It is enough to take $\varphi = (u_1 - u_2)^+$ as test function in the respective variational formulations and perform straightforward manipulations to obtain

$$\lambda \int_\Omega \left[(u_1 - u_2)^+\right]^2 dx \leq \int_{\partial\Omega} (T_1(g_1 - \beta u_1) - T_1(g_2 - \beta u_2))(u_1 - u_2)^+ \, d\mathcal{H}^{N-1} . \tag{3.10}$$

Note that, on the set $\{u_1|_{\partial\Omega} \geq u_2|_{\partial\Omega}\}$, the assumption $g_1 \leq g_2$ implies

$$T_1(g_1 - \beta u_1) - T_1(g_2 - \beta u_2) \leq 0 .$$

Thus, the right-hand side of (3.10) is non-positive and so $(u_1 - u_2)^+$ vanishes in Ω. Moreover,

$$\beta\omega_1 = g_1 - T_1(g_1 - \beta u_1) \leq g_2 - T_1(g_2 - \beta u_1) \leq g_2 - T_1(g_2 - \beta u_2) = \beta\omega_2 ,$$

so that $\omega_1 \leq \omega_2$ on $\partial\Omega$.

3.3.2 Main properties of $\mathcal{B}$

In this subsection, we study the main properties of operator $\mathcal{B}$ that lead to a mild solution of problem (3.4). We begin showing that our operator is accretive.

Theorem 3.3.6. *The operator $\mathcal{B}$ given in Definition 3.3.2 is accretive in $L^2(\partial\Omega)$.*

Proof. Since $L^2(\partial\Omega)$ is a Hilbert space, we just have to prove that $\mathcal{B}$ is monotone. Let $v_i \in \mathcal{B}(\omega_i)$ for $i = 1, 2$. We will show that

$$\int_{\partial\Omega} (v_1 - v_2)(\omega_1 - \omega_2)\, d\mathcal{H}^{N-1} \geq 0\,.$$

Given $v_i \in \mathcal{B}(\omega_i)$, we may find functions $u_i \in BV(\Omega) \cap L^2(\Omega)$ and vector fields $\mathbf{z}_i \in L^\infty(\Omega; \mathbb{R}^N)$ with $\|\mathbf{z}_i\|_{L^\infty(\Omega)} \leq 1$ such that

$$\begin{aligned}
&\textbf{(i)} \quad \lambda u_i - \operatorname{div} \mathbf{z}_i = 0 \ \text{ in } \ \mathcal{D}'(\Omega)\,,\\
&\textbf{(ii)} \quad (\mathbf{z}_i, Du_i) = |Du_i| \ \text{ as measures in } \ \Omega\,,\\
&\textbf{(iii)} \quad [\mathbf{z}_i, \nu] = v_i \quad \mathcal{H}^{N-1}\text{-a.e. on } \ \partial\Omega\,,\\
&\textbf{(iv)} \quad [\mathbf{z}_i, \nu] \in \ \operatorname{sign}(\omega_i - u_i) \quad \mathcal{H}^{N-1}\text{-a.e. on } \ \partial\Omega\,,\\
&\textbf{(v)} \quad \lambda \int_\Omega u_i \,\varphi\, dx + \int_\Omega (\mathbf{z}_i, D\varphi) = \int_{\partial\Omega} v_i\, \varphi\, d\mathcal{H}^{N-1}\,,
\end{aligned}$$

for every $\varphi \in BV(\Omega) \cap L^2(\Omega)$ and for $i = 1, 2$. Taking $u_1 - u_2$ as a test function in condition **(v)** for both $i = 1, 2$ and subtracting one from the other, we get

$$\begin{aligned}
&\lambda \int_\Omega (u_1 - u_2)^2\, dx + \int_\Omega \left[|Du_1| - (\mathbf{z}_2, Du_1) + |Du_2| - (\mathbf{z}_1, Du_2) \right]\\
&= \int_{\partial\Omega} (v_1 - v_2)(u_1 - u_2)\, d\mathcal{H}^{N-1}\,.
\end{aligned}$$

Since the left-hand side is nonnegative (note that $(z_i, Du_j) \leq |Du_j|$ for $i, j = 1, 2$), we deduce

$$\begin{aligned}
0 &\leq \int_{\partial\Omega} (v_1 - v_2)(u_1 - u_2)\, d\mathcal{H}^{N-1} \qquad (3.11)\\
&= \int_{\partial\Omega} (v_1 - v_2)(\omega_1 - \omega_2)\, d\mathcal{H}^{N-1} + \int_{\partial\Omega} (v_1 - v_2)(u_1 - \omega_1)\, d\mathcal{H}^{N-1}\\
&\quad + \int_{\partial\Omega} (v_2 - v_1)(u_2 - \omega_2)\, d\mathcal{H}^{N-1}\,.
\end{aligned}$$

On the one hand, using conditions **(iii)** and **(iv)** and since $\|v_i\|_{L^\infty(\Omega)} \leq 1$, it holds

$$\begin{aligned}
&\int_{\partial\Omega} (v_1 - v_2)(u_1 - \omega_1)\, d\mathcal{H}^{N-1} \\
&= \int_{\partial\Omega} v_1(u_1 - \omega_1)\, d\mathcal{H}^{N-1} - \int_{\partial\Omega} v_2(u_1 - \omega_1)\, d\mathcal{H}^{N-1} \\
&= -\int_{\partial\Omega} \left(|u_1 - \omega_1| + v_2(u_1 - \omega_1) \right) d\mathcal{H}^{N-1} \leq 0\,,
\end{aligned}$$

and similarly

$$\begin{aligned}
&\int_{\partial\Omega} (v_2 - v_1)(u_2 - \omega_2)\, d\mathcal{H}^{N-1} \\
&= -\int_{\partial\Omega} \left(|u_2 - \omega_2| + v_1(u_2 - \omega_2) \right) d\mathcal{H}^{N-1} \leq 0\,.
\end{aligned}$$

Therefore, using (3.11) we conclude that

$$0 \leq \int_{\partial\Omega} (v_1 - v_2)(u_1 - u_2)\, d\mathcal{H}^{N-1} \leq \int_{\partial\Omega} (v_1 - v_2)(\omega_1 - \omega_2)\, d\mathcal{H}^{N-1}\,.$$

□

Proposition 3.3.7. *The operator $\mathcal{B}$ given in Definition 3.3.2 is m-accretive in $L^2(\partial\Omega)$.*

Proof. Denoting by I the identity operator in $L^2(\partial\Omega)$, we just have to prove that

$$R(I + \varepsilon\mathcal{B}) = L^2(\partial\Omega)\,, \qquad \text{for every } \ \varepsilon > 0\,.$$

Given $\varepsilon > 0$, it is enough to see that $L^2(\partial\Omega) \subset R(I + \varepsilon\mathcal{B})$.

Let $g \in L^2(\partial\Omega)$. We will show that there exists $\omega \in L^2(\partial\Omega)$ such that $g \in \omega + \varepsilon\mathcal{B}(\omega)$. That is, we will see that $\frac{1}{\varepsilon} g - \frac{1}{\varepsilon}\omega \in \mathcal{B}(\omega)$.

We begin considering the following Robin problem

$$\begin{cases} \lambda u - \operatorname{div}\left(\dfrac{Du}{|Du|}\right) = 0 & \text{in } \ \Omega\,, \\ \dfrac{1}{\varepsilon}u + \left[\dfrac{Du}{|Du|}, \nu\right] = \dfrac{1}{\varepsilon}g & \text{on } \ \partial\Omega\,. \end{cases}$$

Applying [54, Theorem 1.1], there exists a solution $u \in BV(\Omega) \cap L^2(\Omega)$, a vector field $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^N)$ with $\|\mathbf{z}\|_{L^\infty(\Omega)} \leq 1$ and a function $\omega \in L^2(\partial\Omega)$ such that $(\mathbf{z}, Du) = |Du|$ as measures in Ω and

$$[\mathbf{z}, \nu] = T_1\Big(\frac{1}{\varepsilon}g - \frac{1}{\varepsilon}u\Big) = \frac{1}{\varepsilon}g - \frac{1}{\varepsilon}\omega\,.$$

Then,

$$\frac{1}{\varepsilon}g - \frac{1}{\varepsilon}\omega \in L^\infty(\partial\Omega) \quad \text{ with } \quad \Big\|\frac{1}{\varepsilon}g - \frac{1}{\varepsilon}\omega\Big\|_{L^\infty(\partial\Omega)} \leq 1\,.$$

In addition, as we have explained in Remark 3.3.4, u is also a solution to the Dirichlet problem

$$\begin{cases} \lambda u - \operatorname{div}\left(\dfrac{Du}{|Du|}\right) = 0 & \text{in } \ \Omega\,, \\ u = \omega & \text{on } \ \partial\Omega\,. \end{cases}$$

Therefore, it also holds

$$[\mathbf{z}, \nu] \in \operatorname{sign}(\omega - u)\,.$$

Thus, $\frac{1}{\varepsilon}g - \frac{1}{\varepsilon}\omega \in \mathcal{B}(\omega)$ and the prove is done. □

Remark 3.3.8. Proposition 3.3.7 guarantees the existence of the resolvent

$$(I + \varepsilon\mathcal{B})^{-1} : L^2(\partial\Omega) \to L^2(\partial\Omega)$$

for every $\varepsilon > 0$. It should be pointed out that without this operator we cannot justify the existence of solution to the ε-discretization (3.3) of the Cauchy problem

$$\begin{cases} \omega_t + \mathcal{B}(\omega) \ni g\,, \\ \omega(0) = \omega_0\,, \end{cases}$$

with $g \in L^1(0,T; L^2(\partial\Omega))$ and $\omega_0 \in L^2(\partial\Omega)$.

On the other hand, taking into account Remark 3.3.5, we deduce that the resolvent is an order preserving operator.

Proposition 3.3.9. *Let $\mathcal{B}$ be the operator given in Definition 3.3.2. Then, it holds*

$$L^2(\partial\Omega) = \overline{D(\mathcal{B})}\,.$$

Proof. Since $\mathcal{B} : L^2(\partial\Omega) \to L^2(\partial\Omega)$, we just have to prove that $L^2(\partial\Omega) \subset \overline{D(\mathcal{B})}$. We begin by taking a function $g \in L^\infty(\partial\Omega)$. Given $n \in \mathbb{N}$, by Theorem 3.3.7, we know that $g \in R(I + \frac{1}{n}\mathcal{B})$. Then, there exists $\omega_n \in L^2(\partial\Omega)$ such that $g \in \omega_n + \frac{1}{n}\mathcal{B}(\omega_n)$. That is, $n(g - \omega_n) \in \mathcal{B}(\omega_n)$. Therefore, there exist $u_n \in BV(\Omega) \cap L^2(\Omega)$ and a vector field $\mathbf{z}_n \in L^\infty(\Omega; \mathbb{R}^N)$ with $\|\mathbf{z}_n\|_{L^\infty(\Omega)} \leq 1$ such that

$$\begin{aligned}
&\textbf{(i)} \quad \lambda u_n - \operatorname{div} \mathbf{z}_n = 0 \ \text{ in } \ \mathcal{D}'(\Omega)\,, \\
&\textbf{(ii)} \quad (\mathbf{z}_n, Du_n) = |Du_n| \ \text{ as measures in } \ \Omega\,, \\
&\textbf{(iii)} \quad [\mathbf{z}_n, \nu] = n(g - \omega_n) \quad \mathcal{H}^{N-1}\text{-a.e. on } \ \partial\Omega\,, \\
&\textbf{(iv)} \quad [\mathbf{z}_n, \nu] \in \operatorname{sign}(\omega_n - u_n) \quad \mathcal{H}^{N-1}\text{-a.e. on } \ \partial\Omega\,, \\
&\textbf{(v)} \quad \lambda \int_\Omega u_n\, \varphi\, dx + \int_\Omega (\mathbf{z}_n, D\varphi) = \int_{\partial\Omega} n(g - \omega_n)\, \varphi\, d\mathcal{H}^{N-1}\,,
\end{aligned} \tag{3.12}$$

for every $\varphi \in BV(\Omega) \cap L^2(\Omega)$.

Since $g \in L^\infty(\partial\Omega)$, we have $g = v|_{\partial\Omega}$ for some $v \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ (see [38, Teorema 1.II]), and we use $v - u_n$ as a test function in (3.12) to get

$$\lambda \int_\Omega u_n(v-u_n)\,dx + \int_\Omega (\mathbf{z}_n, D(v-u_n)) = \int_{\partial\Omega} n(g-\omega_n)(g-u_n)\,d\mathcal{H}^{N-1}\,.$$

Observe that, since $n(g-\omega_n) \in \operatorname{sign}(\omega_n - u_n)$, we also have

$$\begin{aligned}&\int_{\partial\Omega} n(g-\omega_n)(g-u_n)\,d\mathcal{H}^{N-1}\\ &\quad = n\int_{\partial\Omega}(g-\omega_n)^2\,d\mathcal{H}^{N-1} + \int_{\partial\Omega}|\omega_n - u_n|\,d\mathcal{H}^{N-1}\,.\end{aligned}$$

Putting together the two previous equations we get

$$\begin{aligned}&\lambda \int_\Omega u_n v\,dx + \int_\Omega (\mathbf{z}_n, Dv) - \int_\Omega |Du_n|\\ &\quad = n\int_{\partial\Omega}(g-\omega_n)^2\,d\mathcal{H}^{N-1} + \int_{\partial\Omega}|\omega_n - u_n|\,d\mathcal{H}^{N-1} + \lambda\int_\Omega u_n^2\,dx\,,\end{aligned}$$

and so it follows that

$$\lambda \int_\Omega u_n v\,dx + \int_\Omega (\mathbf{z}_n, Dv) \geq n\int_{\partial\Omega}(g-\omega_n)^2\,d\mathcal{H}^{N-1} + \lambda\int_\Omega u_n^2\,dx\,.$$

Then, using Young's inequality and the fact that $(\mathbf{z}_n, Dv) \leq |(\mathbf{z}_n, Dv)| \leq |Dv|$, we obtain

$$n\int_{\partial\Omega}(g-\omega_n)^2\,d\mathcal{H}^{N-1} + \lambda\int_\Omega u_n^2\,dx \leq \frac{\lambda}{2}\int_\Omega u_n^2\,dx + \frac{\lambda}{2}\int_\Omega v^2\,dx + \int_\Omega |Dv|\,.$$

Thus, simplifying,

$$n\int_{\partial\Omega}(g-\omega_n)^2\,d\mathcal{H}^{N-1} + \frac{\lambda}{2}\int_\Omega u_n^2\,dx \leq \frac{\lambda}{2}\int_\Omega v^2\,dx + \int_\Omega |Dv|\,,$$

and it yields

$$\int_{\partial\Omega}(g-\omega_n)^2\,d\mathcal{H}^{N-1} \leq \frac{1}{n}\left(\frac{\lambda}{2}\int_\Omega v^2\,dx + \int_\Omega |Dv|\right)\,.$$

Finally, since the right-hand side goes to 0 as $n \to \infty$, we deduce that $\omega_n \to g$ in $L^2(\partial\Omega)$ and therefore, $g \in \overline{D(\mathcal{B})}$.

Now, let $g \in L^2(\partial\Omega)$. We already know that each truncation $T_k(g) \in \overline{D(\mathcal{B})}$ and $T_k(g) \to g$ in $L^2(\partial\Omega)$ when $k \to +\infty$. Thus, $g \in \overline{D(\mathcal{B})}$. $\square$

Using the previous results, the main theorem of this subsection can be obtained applying Theorem 3.2.4.

Theorem 3.3.10. *Let $g \in L^1(0,T;L^2(\partial\Omega))$ and let $\omega_0 \in L^2(\partial\Omega)$. Then, there exists a unique mild solution to the abstract Cauchy problem*

$$\begin{cases} \omega_t + \mathcal{B}(\omega) \ni g\,, \\ \omega(0) = \omega_0\,, \end{cases}$$

on $[0,T]$.

Remark 3.3.11. Some remarks concerning the limiting case $\lambda = 0$ are in order. In this case, the definition of operator $\mathcal{B}$ must be modified. Now the auxiliary function u belongs to $BV(\Omega)$ (but, in general, does not belong to $L^2(\Omega)$). Furthermore, now the definition of $(\mathbf{z}, Du)$ depends on the duality $\operatorname{div}\mathbf{z} \in L^N(\Omega)$ and $u \in BV(\Omega) \subset L^{\frac{N}{N-1}}(\Omega)$. We point out that all the results proved in this section hold.

Nevertheless, this auxiliary function u is not longer determined by ω (see [53, Section 2.3] for examples of nonuniqueness of the Dirichlet problem for the 1-Laplacian) and, moreover, the arguments of the next section does not work. Hence, we may prove that a mild solution exists, but we are not able to see that it is actually a strong solution.

To finish this section, we present a result which compares two mild solutions when their data are ordered.

Theorem 3.3.12. *Let $g^1, g^2 \in L^1(0,T;L^2(\partial\Omega))$ and let $\omega_0^1, \omega_0^2 \in L^2(\partial\Omega)$. Denote by $\omega^k \in C([0,T];L^2(\partial\Omega))$ the mild solution corresponding to data g^k and ω_0^k, $k = 1, 2$.*

If $g^1(t,x) \le g^2(t,x)$ *for almost all* $(t,x) \in (0,T) \times \partial\Omega$ *and* $\omega_0^1(x) \le \omega_0^2(x)$ *for almost all* $x \in \partial\Omega$*, then the solutions to every* ε*-discretization satisfy* $\omega_\varepsilon^1(t,x) \le \omega_\varepsilon^2(t,x)$ *as well as the corresponding auxiliary functions* $u_\varepsilon^1(t,x) \le u_\varepsilon^2(t,x)$*. As a consequence,* $\omega^1(t,x) \le \omega^2(t,x)$ *for almost all* $(t,x) \in (0,T) \times \partial\Omega$*.*

Proof. Let $\varepsilon > 0$ and consider an ε-discretization of (3.2) for data g^k and ω_0^k. Observe that splitting the subintervals if necessary, we may take the same partition for both sets of data. In other words, there exist $t_0 < t_1 < \cdots < t_n$ satisfying

$$
\begin{aligned}
&0 \le t_0 < \varepsilon\,, \\
&t_i - t_{i-1} < \varepsilon\,, \qquad \text{for} \ \ i = 1,2,\ldots,n\,, \\
&0 \le T - t_n < \varepsilon\,,
\end{aligned}
$$

and $g_1^k, g_2^k, \ldots, g_n^k \in L^2(\partial\Omega)$ such that

$$
\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \|g^k(s) - g_i^k\|_{L^2(\partial\Omega)}\, ds < \varepsilon\,,
$$

for $k = 1,2$. Moreover, thanks to [17, Proposition 1.5] we may choose the corresponding $g_i^k = g^k(t_i)$, being each t_i a Lebesgue point of g^k. As a consequence, $g^1(t,x) \le g^2(t,x)$ for almost all $(t,x) \in (0,T) \times \partial\Omega$ implies $g_i^1(x) \le g_i^2(x)$ for almost all $x \in \partial\Omega$ and for $i = 1,\ldots,n$.

Consider now the systems

$$
\frac{\omega_i^k - \omega_{i-1}^k}{t_i - t_{i-1}} + \mathcal{B}(\omega_i^k) \ni g_i^k\,, \qquad \text{for} \ \ i = 1,2,\ldots,n\,, \ \ \text{and} \ \ k = 1,2\,,
$$

so that

$$
\omega_{i-1}^k + (t_i - t_{i-1}) g_i^k \in \Big(I + (t_i - t_{i-1})\mathcal{B}\Big)(\omega_i^k)\,,
$$

for $i = 1, 2, \ldots, n$ and $k = 1, 2$. Since $\omega_0^1(x) \leq \omega_0^2(x)$ and $g_i^1(x) \leq g_i^2(x)$ for almost all $x \in \partial\Omega$ and for $i = 1, \ldots, n$, and each resolvent $\big(I + (t_i - t_{i-1})\mathcal{B}\big)^{-1}$ is order preserving (see Remark 3.3.8), an appeal to induction leads to $\omega_i^1(x) \leq \omega_i^2(x)$ for almost all $x \in \partial\Omega$ as well as $u_i^1(x) \leq u_i^2(x)$ for almost every $x \in \Omega$ and for $i = 1, \ldots, n$.

Denoting by ω_ε^k the solution to the ϵ-discretization corresponding to data g_i^k and ω_0^k, it follows that $\omega_\varepsilon^1(t,x) \leq \omega_\varepsilon^2(t,x)$ for almost all $(t,x) \in (0,T) \times \partial\Omega$.

Finally, having in mind the following convergence:

$$\omega_\varepsilon^k \to \omega^k \ \text{ in } \ L^\infty([0,T]; L^2(\partial\Omega)),$$

for $k = 1, 2$ (see Remark 3.2.3), we deduce that $\omega^1(t,x) \leq \omega^2(t,x)$ for almost all $(t,x) \in (0,T) \times \partial\Omega$. $\square$

3.4 Existence of strong solutions

Now, we are able to prove that the mild solution we have obtained in the previous section is actually a strong solution to our problem. First, we introduce the concept of strong solution in our framework.

Definition 3.4.1. *Let $g \in L^1(0,T; L^2(\partial\Omega))$ and $\omega_0 \in L^2(\partial\Omega)$. We say that the pairing (u, ω) is a strong solution to problem (3.4) if $u \in L^\infty(0,T; BV(\Omega)) \cap L^\infty(0,T; L^2(\Omega))$ and $\omega \in C([0,T]; L^2(\partial\Omega)) \cap W^{1,1}(0,T; L^2(\partial\Omega))$ such that $\omega(0) = \omega_0$ and there exists a vector field $\mathbf{z} \in L^\infty((0,T) \times \Omega; \mathbb{R}^N)$ with $\|\mathbf{z}\|_{L^\infty((0,T)\times\Omega)} \leq 1$ satisfying the following conditions:*

(i) $\lambda u(t) - \operatorname{div}(\mathbf{z}(t)) = 0$ *in* $\mathcal{D}'(\Omega)$,

(ii) $(\mathbf{z}(t), Du(t)) = |Du(t)|$ *as measures in* Ω,

(iii) $[\mathbf{z}(t), \nu] = g(t) - \omega_t(t)$ *for almost every* $x \in \partial\Omega$,

(iv) $[\mathbf{z}(t), \nu] \in \operatorname{sign}(\omega(t) - u(t))$ *for almost every* $x \in \partial\Omega$,

for almost every $t \in (0, T)$.

Given $g \in L^1_{loc}(0, +\infty; L^2(\partial\Omega))$ *and* $\omega_0 \in L^2(\partial\Omega)$, *we say that* (u, ω) *is a global strong solution to problem* (3.1) *if it is a strong solution to* (3.4) *for every* $T > 0$.

As we have mentioned, functions u, ω, $\mathbf{z}$, g depend on two variables: t and x. For the sake of simplicity, most of the time we will write $u(t)$, $\omega(t)$, $\mathbf{z}(t)$ and $g(t)$ instead of $u(t, x)$, $\omega(t, x)$, $\mathbf{z}(t, x)$ and $g(t, x)$.

Theorem 3.4.2. *Let* $\lambda > 0$, *and let* $g \in L^1_{loc}(0, +\infty; L^2(\partial\Omega))$ *and* $\omega_0 \in L^2(\partial\Omega)$. *Then, there exists a global strong solution* (u, ω) *to problem* (3.1).

Furthermore, the following estimates hold:

$$\|\omega\|_{L^\infty(0,T;L^2(\partial\Omega))} \leq \|\omega_0\|_{L^2(\partial\Omega)} + \|g\|_{L^1(0,T;L^2(\partial\Omega))}\,, \tag{3.13}$$

for every $T > 0$ *and*

$$\lambda \|u(t)\|^2_{L^2(\Omega)} + \|u(t)\|_{BV(\Omega)} \leq 2\|\omega(t)\|_{L^1(\partial\Omega)}\,, \tag{3.14}$$

for almost all $t > 0$.

Proof. First fix $T > 0$. Applying Theorem 3.3.10, there exists a mild solution ω, with auxiliary function u, to the abstract Cauchy problem

$$\begin{cases} \omega_t + \mathcal{B}(\omega) \ni g\,, \\ \omega(0) = \omega_0\,, \end{cases} \tag{3.15}$$

on $[0, T]$. We will see that (u, ω) is actually a strong solution.

We will divide the proof in several steps.

Step 1: Solutions to ε-discretizations.

Since $\omega \in C([0,T]; L^2(\partial\Omega))$ is a mild solution, we may choose a family of ε-discretizations of (3.15), in such a way that their solutions ω_ε satisfy

$$\omega_\varepsilon \to \omega \ \text{ strongly in } \ L^\infty(0,T; L^2(\partial\Omega))\,. \tag{3.16}$$

Let us detail our notation. Fixed $0 < \varepsilon \leq 1$, there exists a partition $0 = t_0 < t_1 < \cdots < t_n < T$ such that $T - t_n < \varepsilon$ and $t_i - t_{i-1} < \varepsilon$ for every $i = 1, 2, \ldots, n$, and there exist functions $\widehat{g}_1, \ldots, \widehat{g}_n \in L^2(\partial\Omega)$ such that

$$\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(\int_{\partial\Omega} |g(t,x) - \widehat{g}_i(x)|^2 \, d\mathcal{H}^{N-1} \right)^{\frac{1}{2}} dt < \varepsilon\,, \tag{3.17}$$

and so the system

$$\frac{\omega_i - \omega_{i-1}}{t_i - t_{i-1}} + \mathcal{B}(\omega_i) \ni \widehat{g}_i\,, \qquad \text{for every } \ i = 1, 2, \ldots, n\,,$$

is an ε-discretization of Cauchy problem (3.15).

We denote $\varepsilon_i = t_i - t_{i-1}$. Observe that, splitting the intervals if necessary, there is no loss of generality in assuming $\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_{n-1}$. Hence, if $t \in]t_{i-1}, t_i]$, then $t - \varepsilon_i \in]t_{i-2}, t_{i-1}]$.

We also define $g_\varepsilon(t,x) = \widehat{g}_i(x)$ if $t \in]t_{i-1}, t_i]$ for $i = 1, 2, \ldots, n$. Therefore, the condition (3.17) becomes

$$\int_0^T \left(\int_{\partial\Omega} |g(t,x) - g_\varepsilon(t,x)|^2 \, d\mathcal{H}^{N-1} \right)^{\frac{1}{2}} dt < \varepsilon\,,$$

and we have the following convergence:

$$g_\varepsilon \to g \ \text{ strongly in } \ L^1(0,T; L^2(\partial\Omega))\,. \tag{3.18}$$

Now, the solution to the ε-discretization satisfies

$$\omega_\varepsilon(t,x) = \omega_i(x) \quad \text{if} \quad t \in]t_{i-1}, t_i]\,, \qquad \text{for } \ i = 1, 2, \ldots, n\,,$$

where

$$\widehat{g}_i + \frac{\omega_{i-1} - \omega_i}{\varepsilon_i} \in \mathcal{B}(\omega_i)\,, \qquad \text{for every} \quad i = 1, 2, \dots, n\,.$$

Due to the definition of the operator $\mathcal{B}$, for each $i = 1, 2, \dots, n$, it holds

(a) $\omega_i \in L^2(\partial\Omega)\,,$

(b) $\widehat{g}_i + \dfrac{\omega_{i-1} - \omega_i}{\varepsilon_i} \in L^\infty(\partial\Omega)$ with $\left\| \widehat{g}_i + \dfrac{\omega_{i-1} - \omega_i}{\varepsilon_i} \right\|_{L^\infty(\partial\Omega)} \le 1\,,$

(c) there exists $u_i \in BV(\Omega) \cap L^2(\Omega)\,,$

(d) there exists $\mathbf{z}_i \in L^\infty(\Omega; \mathbb{R}^N)$ with $\|\mathbf{z}_i\|_{L^\infty(\Omega)} \le 1\,,$

satisfying the following conditions

(i) $\lambda u_i - \operatorname{div} \mathbf{z}_i = 0$ in $\mathcal{D}'(\Omega)\,,$

(ii) $(\mathbf{z}_i, Du_i) = |Du_i|$ as measures in $\Omega\,,$ (3.19)

(iii) $[\mathbf{z}_i, \nu] = \widehat{g}_i + \dfrac{\omega_{i-1} - \omega_i}{\varepsilon_i}$ $\mathcal{H}^{N-1}$-a.e. on $\partial\Omega\,,$ (3.20)

(iv) $[\mathbf{z}_i, \nu] \in \operatorname{sign}(\omega_i - u_i)$ $\mathcal{H}^{N-1}$-a.e. on $\partial\Omega\,,$ (3.21)

(v) $\lambda \displaystyle\int_\Omega u_i\, \varphi\, dx + \int_\Omega (\mathbf{z}_i, D\varphi) = \int_{\partial\Omega} \Big(\widehat{g}_i + \frac{\omega_{i-1} - \omega_i}{\varepsilon_i} \Big) \varphi\, d\mathcal{H}^{N-1},$ (3.22)

for every $\varphi \in BV(\Omega) \cap L^2(\Omega)$.

Finally, given $\varepsilon > 0$, we define the following step functions:

$$u_\varepsilon(t, x) = u_i(x) \;\text{ if }\; t \in \,]t_{i-1}, t_i]\,, \qquad \text{for} \;\; i = 1, 2, \dots, n\,,$$

$$\mathbf{z}_\varepsilon(t, x) = \mathbf{z}_i(x) \;\text{ if }\; t \in \,]t_{i-1}, t_i]\,, \qquad \text{for} \;\; i = 1, 2, \dots, n\,.$$

We remark that all the above step functions are defined in $[0, t_n]$. To avoid lack of definiteness, we can extend them to $]t_n, T]$ giving their value at the point t_n.

Step 2: Existence of ω_t in the sense of distributions.

Due to Definition 3.3.2 we know that

$$\left\| \frac{\omega_i - \omega_{i-1}}{\varepsilon_i} - \hat{g}_i \right\|_{L^\infty(\partial\Omega)} \leq 1\,, \qquad \text{for every} \quad i = 1, 2, \ldots, n\,,$$

where $\varepsilon_i = t_i - t_{i-1}$. Denoting $\varepsilon(t) = \varepsilon_i$ for $t \in]t_{i-1}, t_i]$, the following equivalent bound holds:

$$\left\| \frac{\omega_\varepsilon(t) - \omega_\varepsilon(t - \varepsilon(t))}{\varepsilon(t)} - g_\varepsilon(t) \right\|_{L^\infty(\partial\Omega)} \leq 1\,, \tag{3.23}$$

for every $t \in]\varepsilon_1, T[\subset]\varepsilon, T[$.

Assuming $\eta > 0$, let $0 < \varepsilon < \eta$ and $t \in]\eta, T[$ be fixed. We may assume that this given t satisfies

$$g_\varepsilon(t) \to g(t) \text{ strongly in } \; L^2(\partial\Omega)\,, \tag{3.24}$$

which is a straightforward consequence of (3.18).

Due to (3.23), the sequence $\left\{ \frac{\omega_\varepsilon(t) - \omega_\varepsilon(t-\varepsilon(t))}{\varepsilon(t)} - g_\varepsilon(t) \right\}$ is bounded in $L^\infty(\partial\Omega)$. Then, there exists a subsequence and there exists a function $\rho(t) \in L^\infty(\partial\Omega)$ such that

$$\frac{\omega_\varepsilon(t) - \omega_\varepsilon(t - \varepsilon(t))}{\varepsilon(t)} - g_\varepsilon(t) \rightharpoonup \rho(t) \;\; *\text{-weakly in} \;\; L^\infty(\partial\Omega)\,. \tag{3.25}$$

Therefore, for every $\varphi \in L^2(\partial\Omega)$, we apply (3.25) and (3.24) to get

$$\begin{aligned}
&\int_{\partial\Omega} \rho(t)\varphi \, d\mathcal{H}^{N-1} \\
&= \lim_{\varepsilon \to 0^+} \left[\int_{\partial\Omega} \frac{\omega_\varepsilon(t) - \omega_\varepsilon(t - \varepsilon(t))}{\varepsilon(t)} \varphi \, d\mathcal{H}^{N-1} - \int_{\partial\Omega} g_\varepsilon(t)\varphi \, d\mathcal{H}^{N-1} \right] \\
&= \lim_{\varepsilon \to 0^+} \int_{\partial\Omega} \frac{\omega_\varepsilon(t) - \omega_\varepsilon(t - \varepsilon(t))}{\varepsilon(t)} \varphi \, d\mathcal{H}^{N-1} - \int_{\partial\Omega} g(t)\varphi \, d\mathcal{H}^{N-1}\,.
\end{aligned}$$

And so, we obtain

$$\lim_{\varepsilon\to 0^+}\int_{\partial\Omega}\frac{\omega_\varepsilon(t)-\omega_\varepsilon(t-\varepsilon(t))}{\varepsilon(t)}\varphi\, d\mathcal{H}^{N-1}=\int_{\partial\Omega}(g(t)+\rho(t))\varphi\, d\mathcal{H}^{N-1}\,,$$

that is,

$$\frac{\omega_\varepsilon(t)-\omega_\varepsilon(t-\varepsilon(t))}{\varepsilon(t)}\rightharpoonup g(t)+\rho(t)\quad\text{weakly in}\quad L^2(\partial\Omega)\,.$$

We take now the function $\psi\in C_0^1(0,T;L^2(\partial\Omega))$ such that $\operatorname{supp}\psi\subset\,]\varepsilon,T-\varepsilon[$, obtaining

$$\begin{aligned}
&\int_0^T\int_{\partial\Omega}\frac{\omega_\varepsilon(t)-\omega_\varepsilon(t-\varepsilon(t))}{\varepsilon(t)}\psi(t)\, d\mathcal{H}^{N-1}\, dt\\
&=\int_0^T\int_{\partial\Omega}\frac{\omega_\varepsilon(t)}{\varepsilon(t)}\psi(t)\, d\mathcal{H}^{N-1}\, dt-\int_0^T\int_{\partial\Omega}\frac{\omega_\varepsilon(t-\varepsilon(t))}{\varepsilon(t)}\psi(t)\, d\mathcal{H}^{N-1}\, dt\\
&=\int_0^T\int_{\partial\Omega}\frac{\omega_\varepsilon(t)}{\varepsilon(t)}\psi(t)\, d\mathcal{H}^{N-1}\, dt-\int_0^T\int_{\partial\Omega}\omega_\varepsilon(t)\frac{\psi(t+\varepsilon(t))}{\varepsilon(t)}\, d\mathcal{H}^{N-1}\, dt\\
&=-\int_0^T\int_{\partial\Omega}\omega_\varepsilon(t)\frac{\psi(t+\varepsilon(t))-\psi(t)}{\varepsilon(t)}\, d\mathcal{H}^{N-1}\, dt\,.
\end{aligned}$$

On the other hand, having in mind (3.18) and (3.25) (and also (3.23)), it follows that

$$\begin{aligned}
&\int_0^T\int_{\partial\Omega}(g(t)+\rho(t))\psi(t)\, d\mathcal{H}^{N-1}\, dt=\lim_{\varepsilon\to 0^+}\int_0^T\int_{\partial\Omega}g_\varepsilon(t)\psi(t)\, d\mathcal{H}^{N-1}\, dt\\
&\quad+\lim_{\varepsilon\to 0^+}\int_0^T\int_{\partial\Omega}\left(\frac{\omega_\varepsilon(t)-\omega_\varepsilon(t-\varepsilon(t))}{\varepsilon(t)}-g_\varepsilon(t)\right)\psi(t)\, d\mathcal{H}^{N-1}\, dt\\
&=\lim_{\varepsilon\to 0^+}\int_0^T\int_{\partial\Omega}\frac{\omega_\varepsilon(t)-\omega_\varepsilon(t-\varepsilon(t))}{\varepsilon(t)}\psi(t)\, d\mathcal{H}^{N-1}\, dt\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_0^T \int_{\partial\Omega} (g(t)+\rho(t))\psi(t)\, d\mathcal{H}^{N-1}\, dt \\
&\quad = \lim_{\varepsilon\to 0^+} \int_0^T \int_{\partial\Omega} \frac{\omega_\varepsilon(t)-\omega_\varepsilon(t-\varepsilon(t))}{\varepsilon(t)} \psi(t)\, d\mathcal{H}^{N-1}\, dt \\
&\quad = -\lim_{\varepsilon\to 0^+} \int_0^T \int_{\partial\Omega} \omega_\varepsilon(t) \frac{\psi(t+\varepsilon(t))-\psi(t)}{\varepsilon(t)}\, d\mathcal{H}^{N-1}\, dt \\
&\quad = -\int_0^T \int_{\partial\Omega} \omega(t)\psi_t(t)\, d\mathcal{H}^{N-1}\, dt\,,
\end{aligned}$$

due to (3.16). Then, the distributional derivative of ω is $\omega_t = g + \rho \in L^1(0,T;L^2(\partial\Omega))$ and it also holds $\|\omega_t(t) - g(t)\|_{L^\infty(\partial\Omega)} \leq 1$ for almost every $t \in (0,T)$.

Moreover, we have the following $L^2(\partial\Omega)$-weak-convergence:

$$\frac{\omega_\varepsilon(t)-\omega_\varepsilon(t-\varepsilon(t))}{\varepsilon(t)} \rightharpoonup g(t)+\rho(t) = \omega_t(t)\,. \tag{3.26}$$

We point out that, since the operator $\mathcal{B}$ is m-accretive, function ω is absolutely continuous and differentiable in almost every $t \in (0,T)$ and besides it is a mild solution to problem $\omega_t + \mathcal{B}(\omega) \ni g$ on $(0,T)$, it yields that function ω is also a strong solution (see [17, Theorem 7.1]). In other words, $g(t) - \omega_t(t) \in \mathcal{B}(\omega(t))$ holds for almost every $t \in (0,T)$. This concludes the proof in what the boundary concerns, which is where the semigroup is defined. Hence, for every fixed $t \in (0,T)$, there exist an auxiliary BV-function and a vector field satisfying Definition 3.3.2. Nevertheless, in the domain $(0,T)$ there may be a problem of measurability since the strong solution only provide us the functions pointwise in time. In the sequel, we will find $u \in L^\infty(0,T;BV(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ and $\mathbf{z} \in L^\infty((0,T)\times\Omega;\mathbb{R}^N)$ satisfying all the requirements of Definition 3.4.1.

Step 3: Existence of $\mathbf{z} \in L^\infty((0,T) \times \Omega; \mathbb{R}^N)$.

This fact is a directly consequence of $\|\mathbf{z}_\varepsilon\|_{L^\infty((0,T)\times\Omega)} \le 1$ for all $\varepsilon > 0$. Since the sequence is bounded, there exists a vector field $\mathbf{z} \in L^\infty((0,T) \times \Omega; \mathbb{R}^N)$ such that, up to subsequences,

$$\mathbf{z}_\varepsilon \rightharpoonup \mathbf{z} \quad * \text{-weakly in } \; L^\infty((0,T) \times \Omega)\,. \tag{3.27}$$

Step 4: $\{u_\varepsilon\}$ *is bounded in* $L^2(0,T; L^2(\Omega))$ *and* $L^1(0,T; BV(\Omega))$.

We take $u_i \in BV(\Omega) \cap L^2(\Omega)$ as test function in (3.22) so we get

$$\begin{aligned}
\lambda \int_\Omega u_i^2\,dx + \int_\Omega (\mathbf{z}_i, Du_i) &= \int_{\partial\Omega} \left(\widehat{g}_i + \frac{\omega_{i-1} - \omega_i}{\varepsilon_i} \right) u_i\, d\mathcal{H}^{N-1} \\
&= \int_{\partial\Omega} \left(\widehat{g}_i + \frac{\omega_{i-1} - \omega_i}{\varepsilon_i} \right) (u_i - \omega_i)\, d\mathcal{H}^{N-1} \\
&\quad + \int_{\partial\Omega} \left(\widehat{g}_i + \frac{\omega_{i-1} - \omega_i}{\varepsilon_i} \right) \omega_i\, d\mathcal{H}^{N-1} \\
&= - \int_{\partial\Omega} |u_i - \omega_i|\, d\mathcal{H}^{N-1} + \int_{\partial\Omega} \widehat{g}_i \omega_i\, d\mathcal{H}^{N-1} + \int_{\partial\Omega} \frac{\omega_{i-1} - \omega_i}{\varepsilon_i} \omega_i\, d\mathcal{H}^{N-1}\,,
\end{aligned}$$

where we have used conditions (3.20) and (3.21). Then, condition (3.19) implies

$$\begin{aligned}
\lambda \int_\Omega u_i^2\,dx + \int_\Omega |Du_i| + \int_{\partial\Omega} |u_i - \omega_i|\, d\mathcal{H}^{N-1} + \int_{\partial\Omega} \frac{\omega_i - \omega_{i-1}}{\varepsilon_i} \omega_i\, d\mathcal{H}^{N-1} \\
= \int_{\partial\Omega} \widehat{g}_i \omega_i\, d\mathcal{H}^{N-1}
\end{aligned} \tag{3.28}$$

and dropping nonnegative terms, we get the following inequality:

$$\int_{\partial\Omega} \frac{\omega_i - \omega_{i-1}}{\varepsilon_i} \omega_i\, d\mathcal{H}^{N-1} \le \int_{\partial\Omega} \widehat{g}_i\, \omega_i\, d\mathcal{H}^{N-1}\,, \tag{3.29}$$

for every $i = 1, 2, \ldots, n$.

Next, we show that $\int_{\partial\Omega} \omega_i^2 \, d\mathcal{H}^{N-1}$ is bounded by a constant which does not depend on i. Using Hölder's inequality, condition (3.29) and then Hölder's inequality again, we get

$$\begin{aligned}
&\int_{\partial\Omega} \omega_i^2 \, d\mathcal{H}^{N-1} - \left(\int_{\partial\Omega} \omega_i^2 \, d\mathcal{H}^{N-1}\right)^{\frac{1}{2}} \left(\int_{\partial\Omega} \omega_{i-1}^2 \, d\mathcal{H}^{N-1}\right)^{\frac{1}{2}} \\
&\leq \int_{\partial\Omega} \omega_i^2 \, d\mathcal{H}^{N-1} - \int_{\partial\Omega} \omega_i \, \omega_{i-1} \, d\mathcal{H}^{N-1} = \int_{\partial\Omega} (\omega_i - \omega_{i-1}) \, \omega_i \, d\mathcal{H}^{N-1} \\
&\leq \varepsilon_i \int_{\partial\Omega} \widehat{g}_i \omega_i \, d\mathcal{H}^{N-1} \leq \varepsilon_i \left(\int_{\partial\Omega} \widehat{g}_i^2 \, d\mathcal{H}^{N-1}\right)^{\frac{1}{2}} \left(\int_{\partial\Omega} \omega_i^2 \, d\mathcal{H}^{N-1}\right)^{\frac{1}{2}} .
\end{aligned}$$

Now, if $\int_{\partial\Omega} \omega_i^2 \, d\mathcal{H}^{N-1} \neq 0$, we divide the previous inequality by integral $\left(\int_{\partial\Omega} \omega_i^2 \, d\mathcal{H}^{N-1}\right)^{\frac{1}{2}}$ and so we get

$$\left(\int_{\partial\Omega} \omega_i^2 \, d\mathcal{H}^{N-1}\right)^{\frac{1}{2}} - \left(\int_{\partial\Omega} \omega_{i-1}^2 \, d\mathcal{H}^{N-1}\right)^{\frac{1}{2}} \leq \varepsilon_i \left(\int_{\partial\Omega} \widehat{g}_i^2 \, d\mathcal{H}^{N-1}\right)^{\frac{1}{2}} ,$$

for every $i = 1, 2, \ldots, n$. We fix now $i \in \{1, 2, \ldots, n\}$ and sum the previous inequality for $k = 1, 2, \ldots, i-1, i$:

$$\begin{aligned}
&\left(\int_{\partial\Omega} \omega_i^2 \, d\mathcal{H}^{N-1}\right)^{\frac{1}{2}} - \left(\int_{\partial\Omega} \omega_0^2 \, d\mathcal{H}^{N-1}\right)^{\frac{1}{2}} \\
&= \sum_{k=1}^{i} \left[\left(\int_{\partial\Omega} \omega_k^2 \, d\mathcal{H}^{N-1}\right)^{\frac{1}{2}} - \left(\int_{\partial\Omega} \omega_{k-1}^2 \, d\mathcal{H}^{N-1}\right)^{\frac{1}{2}}\right] \\
&\leq \sum_{k=1}^{i} \varepsilon_k \left(\int_{\partial\Omega} \widehat{g}_k^2 \, d\mathcal{H}^{N-1}\right)^{\frac{1}{2}} = \sum_{k=1}^{i} \int_{t_{k-1}}^{t_k} \left(\int_{\partial\Omega} \widehat{g}_k^2 \, d\mathcal{H}^{N-1}\right)^{\frac{1}{2}} dt \, .
\end{aligned}$$

We can perform easy manipulations to get

$$\left(\int_{\partial\Omega} \omega_i^2 \, d\mathcal{H}^{N-1}\right)^{\frac{1}{2}} \leq \left(\int_{\partial\Omega} \omega_0^2 \, d\mathcal{H}^{N-1}\right)^{\frac{1}{2}} + \int_0^{t_i} \left(\int_{\partial\Omega} g_\varepsilon^2 \, d\mathcal{H}^{N-1}\right)^{\frac{1}{2}} dt$$

$$\leq \left(\int_{\partial\Omega} \omega_0^2 \, d\mathcal{H}^{N-1}\right)^{\frac{1}{2}} + \int_0^T \left(\int_{\partial\Omega} g_\varepsilon^2 \, d\mathcal{H}^{N-1}\right)^{\frac{1}{2}} dt \,. \tag{3.30}$$

Therefore, we deduce that

$$\begin{aligned} \|\omega_i\|_{L^2(\partial\Omega)} &\leq \|\omega_0\|_{L^2(\partial\Omega)} + \|g_\varepsilon\|_{L^1(0,T;L^2(\partial\Omega))} \\ &\leq \|\omega_0\|_{L^2(\partial\Omega)} + 1 + \|g\|_{L^1(0,T;L^2(\partial\Omega))} = M \,, \end{aligned} \tag{3.31}$$

for every $i = 1, 2, \ldots, n$ where M is a constant which does not depend on t. That is, the sequence $\{\omega_\varepsilon(t,x)\}$ is bounded in $L^\infty(0,T;L^2(\partial\Omega))$. In addition, since $g_\varepsilon(t)$, $\omega_\varepsilon(t) \in L^2(\partial\Omega)$, we can use Hölder's inequality to get

$$\begin{aligned} &\int_0^T \int_{\partial\Omega} g_\varepsilon \omega_\varepsilon \, d\mathcal{H}^{N-1} \, dt \leq \int_0^T \left(\int_{\partial\Omega} g_\varepsilon^2 \, d\mathcal{H}^{N-1}\right)^{\frac{1}{2}} \left(\int_{\partial\Omega} \omega_\varepsilon^2 \, d\mathcal{H}^{N-1}\right)^{\frac{1}{2}} dt \\ &\leq \int_0^T M \left(\int_{\partial\Omega} g_\varepsilon^2 \, d\mathcal{H}^{N-1}\right)^{\frac{1}{2}} dt = M \int_0^T \left(\int_{\partial\Omega} g_\varepsilon^2 \, d\mathcal{H}^{N-1}\right)^{\frac{1}{2}} dt \\ &\leq M^2 \,. \end{aligned} \tag{3.32}$$

On the other hand, we know that $\frac{\omega_i^2 - \omega_{i-1}^2}{2} \leq (\omega_i - \omega_{i-1})\,\omega_i$ and we deduce from (3.28) that

$$\begin{aligned} &\lambda \int_\Omega u_i^2 \, dx + \int_\Omega |Du_i| + \int_{\partial\Omega} |u_i - \omega_i| \, d\mathcal{H}^{N-1} \\ &\quad + \frac{1}{\varepsilon_i} \int_{\partial\Omega} \frac{\omega_i^2}{2} \, d\mathcal{H}^{N-1} - \frac{1}{\varepsilon_i} \int_{\partial\Omega} \frac{\omega_{i-1}^2}{2} \, dx \leq \int_{\partial\Omega} \widehat{g}_i \omega_i \, d\mathcal{H}^{N-1} \,. \end{aligned}$$

Now, we integrate the previous inequality between t_{i-1} and t_i to obtain

$$\begin{aligned} &\lambda \int_{t_{i-1}}^{t_i} \int_\Omega u_i^2 \, dx \, dt + \int_{t_{i-1}}^{t_i} \int_\Omega |Du_i| \, dt + \int_{t_{i-1}}^{t_i} \int_{\partial\Omega} |u_i - \omega_i| \, d\mathcal{H}^{N-1} \, dt \\ &+ \int_{\partial\Omega} \frac{\omega_i^2}{2} \, d\mathcal{H}^{N-1} - \int_{\partial\Omega} \frac{\omega_{i-1}^2}{2} \, dx \leq \int_{t_{i-1}}^{t_i} \int_{\partial\Omega} \widehat{g}_i \omega_i \, d\mathcal{H}^{N-1} \, dt \,, \end{aligned}$$

for every $i = 1, 2, \ldots, n$. The addition of all these terms from $i = 1$ to n provide us with

$$\lambda \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \int_\Omega u_i^2 \, dx \, dt + \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \int_\Omega |Du_i| \, dt$$

$$+ \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \int_{\partial\Omega} |u_i - \omega_i| \, d\mathcal{H}^{N-1} \, dt + \int_{\partial\Omega} \frac{\omega_n^2}{2} \, d\mathcal{H}^{N-1} - \int_{\partial\Omega} \frac{\omega_0^2}{2} \, dx$$

$$\leq \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \int_{\partial\Omega} \widehat{g}_i \omega_i \, d\mathcal{H}^{N-1} \, dt \,.$$

Then, due to (3.32) we get

$$\lambda \int_0^T \int_\Omega u_\varepsilon^2 \, dx \, dt + \int_0^T \int_\Omega |Du_\varepsilon| \, dt + \int_0^T \int_{\partial\Omega} |u_\varepsilon - \omega_\varepsilon| \, d\mathcal{H}^{N-1} \, dt$$

$$+ \int_{\partial\Omega} \frac{\omega_n^2}{2} \, d\mathcal{H}^{N-1} \leq \int_{\partial\Omega} \frac{\omega_0^2}{2} \, dx + \int_0^T \int_{\partial\Omega} g_\varepsilon \omega_\varepsilon \, d\mathcal{H}^{N-1} \, dt \leq 2M^2 \,.$$

Therefore, we have proved that

$$\lambda \int_0^T \int_\Omega u_\varepsilon^2 \, dx \, dt + \int_0^T \int_\Omega |Du_\varepsilon| \, dt \leq 2M^2 \,.$$

That is, the sequence $\{u_\varepsilon\}$ is bounded in $L^2(0, T; L^2(\Omega))$ and, by Hölder's inequality, it is also bounded in $L^1(0, T; BV(\Omega))$.

As a first consequence, there exists a measurable function $u \in L^2((0, T) \times \Omega)$ such that

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in} \quad L^2((0, T) \times \Omega) \,. \tag{3.33}$$

Step 5: u belongs to $L^\infty(0, T; BV(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$.

Let $t \in (0, T)$. Observe that (3.28), written in terms of the approximate solutions, becomes

$$\lambda \int_\Omega u_\varepsilon(t)^2 \, dx + \int_\Omega |Du_\varepsilon(t)| + \int_{\partial\Omega} |u_\varepsilon(t) - \omega_\varepsilon(t)| \, d\mathcal{H}^{N-1}$$

$$= \int_{\partial\Omega} \left(g_\varepsilon(t) - \frac{\omega_\varepsilon(t) - \omega_\varepsilon(t-\varepsilon_i)}{\varepsilon_i} \right) \omega_\varepsilon(t)\, d\mathcal{H}^{N-1} \le \int_{\partial\Omega} |\omega_\varepsilon(t)|\, d\mathcal{H}^{N-1},$$

because of (3.20). Dropping the nonnegative term and having in mind (3.31) and Hölder's inequality, we get

$$\lambda \int_\Omega u_\varepsilon(t)^2\, dx + \int_\Omega |Du_\varepsilon(t)| \le M\mathcal{H}^{N-1}(\partial\Omega)^{1/2} = M_1\,, \tag{3.34}$$

so that the sequence $\{u_\varepsilon\}$ is bounded in $L^\infty(0,T;L^2(\Omega))$ and also in $L^\infty(0,T;BV(\Omega))$ due to Hölder's inequality.

In order to see that $u \in L^\infty(0,T;BV(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, we need to let $\varepsilon \to 0^+$ in the above inequality. To this end, we first fix $\phi \in C_0^\infty((0,T))$ such that $\phi \ge 0$ and observe that, for each $v \in L^\infty(0,T;BV(\Omega))$, it holds

$$\int_0^T \int_\Omega |Dv(t,x)|\phi(t)\, dx\, dt = \sup\left\{ \int_0^T \int_\Omega v(t,x) \operatorname{div}\psi(x)\, \phi(t)\, dx\, dt \right\},$$

where the supremum is taken among all $\psi \in C_0^1(\Omega;\mathbb{R}^N)$ such that $|\psi(x)| \le 1$. Since for every fixed ϕ and ψ we have the continuity of operator

$$v \mapsto \int_0^T \int_\Omega v(t,x) \operatorname{div}\psi(x)\, \phi(t)\, dx\, dt$$

with respect to the weak convergence in $L^1((0,T)\times\Omega)$, we deduce that the functional

$$v \mapsto \int_0^T \int_\Omega |Dv(t,x)|\phi(t)\, dx\, dt$$

is lower semicontinuous with respect to the weak convergence in the space $L^1((0,T)\times\Omega)$. Thus, the convergence $u_\varepsilon \rightharpoonup u$ weakly in $L^2((0,T)\times\Omega)$ implies

$$\int_0^T \int_\Omega |Du(t,x)|\phi(t)\, dx\, dt \le \liminf_{\varepsilon\to 0^+} \int_0^T \int_\Omega |Du_\varepsilon(t,x)|\phi(t)\, dx\, dt\,.$$

It follows that $u \in L^\infty(0,T;BV(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ and also

$$\lambda \int_\Omega u(t)^2\, dx + \int_\Omega |Du(t)| \le M_1\,, \qquad \text{for almost all } \ t \in (0,T)\,.$$

Step 6: $\lambda u(t) - \operatorname{div} \mathbf{z}(t) = 0$ holds in $\mathcal{D}'(\Omega)$ for almost every $t \in (0,T)$.

Observe that, since (3.22) holds for every $i = 1, 2, \dots, n$, we have

$$\lambda \int_\Omega u_\varepsilon(t,x)\, \varphi(x)\, dx + \int_\Omega \mathbf{z}_\varepsilon(t,x) \cdot \nabla\varphi(x)\, dx = 0\,,$$

for every $\varphi \in C_0^\infty(\Omega)$ and for every $t \in (0,T)$. Considering $\phi \in C_0^\infty(0,T)$, from convergences (3.33) and (3.27) we deduce that

$$\lambda \int_0^T \int_\Omega u(t,x)\, \varphi(x)\, \phi(t)\, dx\, dt + \int_0^T \int_\Omega \mathbf{z}(t,x) \cdot \nabla\varphi(x)\, \phi(t)\, dx\, dt = 0\,.$$

Therefore, for almost every $t \in (0,T)$, we obtain

$$\lambda \int_\Omega u(t)\, \varphi\, dx + \int_\Omega \mathbf{z}(t) \cdot \nabla\varphi\, dx = 0\,,$$

and so Step 6 is proved and $\operatorname{div} \mathbf{z}(t) \in L^2(\Omega)$ for almost all $t \in (0,T)$.

Step 7: $\omega_t(t) + [\mathbf{z}(t), \nu] = g(t)$ for almost all $t \in (0,T)$.

As a consequence of $\operatorname{div} \mathbf{z}(t) \in L^2(\Omega)$, we may apply Green's formula to the vector field $\mathbf{z}(t)$. So Steps 3 and 6 imply

$$[\mathbf{z}_\varepsilon(t,x), \nu(x)] \rightharpoonup [\mathbf{z}(t,x), \nu(x)] \quad * \text{-weakly in } \ L^\infty((0,T) \times \partial\Omega)\,. \tag{3.35}$$

On the other hand, it also holds the following $*$-weakly convergence in $L^\infty(\partial\Omega)$:

$$-\,[\mathbf{z}_\varepsilon(t), \nu] = \frac{\omega_\varepsilon(t) - \omega_\varepsilon(t - \varepsilon(t))}{\varepsilon(t)} - g_\varepsilon(t) \rightharpoonup \rho(t)\,. \tag{3.36}$$

Taking $\varphi \in L^2(\partial\Omega)$ and $\phi \in L^2((0,T))$, we may compute the limit of

$$\int_0^T \int_{\partial\Omega} [\mathbf{z}_\varepsilon(t,x), \nu(x)]\varphi(x)\phi(t)\, d\mathcal{H}^{N-1}\, dt$$

using (3.35) and (3.36). Then,

$$\int_0^T \int_{\partial\Omega} [\mathbf{z}(t,x), \nu(x)]\varphi(x)\phi(t)\, d\mathcal{H}^{N-1}\, dt \\ = -\int_0^T \int_{\partial\Omega} \rho(t,x)\varphi(x)\phi(t)\, d\mathcal{H}^{N-1}\, dt\,.$$

Thus, $[\mathbf{z}(t), \nu] = -\rho(t)$ $\mathcal{H}^{N-1}$-a.e. on $\partial\Omega$ and

$$[\mathbf{z}_\varepsilon(t), \nu] \rightharpoonup [\mathbf{z}(t), \nu] \quad * \text{-weakly in } \; L^\infty(\partial\Omega)\,, \tag{3.37}$$

for almost all $t \in (0,T)$. Recalling (3.26), we also deduce that the identity

$$\omega_t(t) + [\mathbf{z}(t), \nu] = g(t)$$

holds on $\partial\Omega$ for almost all $t \in (0,T)$.

Step 8: For almost every $t \in (0,T)$, there exists a subsequence satisfying some useful convergences.

Let $t \in (0,T)$. From (3.34) it follows that

$$\lambda \int_\Omega u_\varepsilon(t)^2\, dx \le M_1\,.$$

Then, the sequence $\{u_\varepsilon(t)\}$ is bounded in $L^2(\Omega)$ and there exist $\widehat{u}(t) \in L^2(\Omega)$ and a subsequence $\{u_{\varepsilon^t}(t)\}$ (we remark that the subsequence we find depends on t) such that

$$u_{\varepsilon^t}(t) \rightharpoonup \widehat{u}(t) \;\text{ weakly in }\; L^2(\Omega)\,. \tag{3.38}$$

Now we go back to (3.34) which is an estimate of $\{u_{\varepsilon^t}(t)\}$ in $BV(\Omega)$ for a fixed $t \in (0,T)$. Thus, there exists a further subsequence (not relabelled) such that converges to a BV-function strongly in $L^1(\Omega)$. Since we have proved (3.38), we conclude that

$$u_{\varepsilon^t}(t) \to \widehat{u}(t) \ \text{ strongly in } \ L^1(\Omega)\,, \tag{3.39}$$

and $\widehat{u}(t) \in BV(\Omega)$.

On the other hand, fixed $t \in (0,T)$, the sequence $\{\mathbf{z}_{\varepsilon^t}(t)\}$ is bounded in $L^\infty(\Omega)$ since $\|\mathbf{z}_{\varepsilon^t}(t)\|_{L^\infty(\Omega)} \le 1$. Then, passing to a subsequence if necessary, there exists a vector field $\widehat{\mathbf{z}}(t) \in L^\infty(\Omega)$ such that

$$\mathbf{z}_{\varepsilon^t}(t) \rightharpoonup \widehat{\mathbf{z}}(t) \quad *\text{-weakly in } \ L^\infty(\Omega)\,. \tag{3.40}$$

Step 9: $\lambda\widehat{u}(t) - \operatorname{div}\widehat{\mathbf{z}}(t) = 0$ *for almost every* $t \in (0,T)$.

Observe that, since (3.22) holds for every $i = 1, 2, \ldots, n$, we have

$$\lambda \int_\Omega u_{\varepsilon^t}(t)\,\varphi\,dx + \int_\Omega \mathbf{z}_{\varepsilon^t}(t)\cdot\nabla\varphi\,dx = 0\,,$$

for every $\varphi \in C_0^\infty(\Omega)$ and for almost every $t \in (0,T)$. Then, it follows from (3.38) and (3.40) that

$$\lambda \int_\Omega \widehat{u}(t)\,\varphi\,dx + \int_\Omega \widehat{\mathbf{z}}(t)\cdot\nabla\varphi\,dx = 0\,,$$

and so $\operatorname{div}\widehat{\mathbf{z}}(t) \in L^2(\Omega)$ for almost all $t \in (0,T)$.

We point out that, as consequence of (3.40) and Green's formula, we also get

$$[\mathbf{z}_{\varepsilon^t}(t), \nu] \rightharpoonup [\widehat{\mathbf{z}}(t), \nu] \quad *\text{-weakly in } \ L^\infty(\partial\Omega)\,.$$

Having in mind (3.37), we conclude that $[\mathbf{z}(t), \nu] = [\widehat{\mathbf{z}}(t), \nu]$ on $\partial\Omega$.

Step 10: $(\widehat{\mathbf{z}}(t), D\widehat{u}(t)) = |D\widehat{u}(t)|$ *as measures in* Ω *for almost every* $t \in (0,T)$.

Fix $t \in (0,T)$ such that $u_{\varepsilon^t}(t) \to \widehat{u}(t)$ strongly in $L^1(\Omega)$, $\mathbf{z}_{\varepsilon^t}(t) \to \widehat{\mathbf{z}}(t)$ $*$-weakly in $L^\infty(\Omega)$ and the distributional equation $\lambda \widehat{u}(t) - \operatorname{div} \widehat{\mathbf{z}}(t) = 0$ holds. Given $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geq 0$, we take the test function $\varphi\, u_{\varepsilon^t}(t)$ in (3.22) and we obtain

$$\lambda \int_\Omega u_{\varepsilon^t}(t)^2 \varphi \, dx + \int_\Omega u_{\varepsilon^t}(t)\, \mathbf{z}_{\varepsilon^t}(t) \cdot \nabla\varphi \, dx + \int_\Omega \varphi \, |Du_{\varepsilon^t}(t)| = 0\,. \tag{3.41}$$

We want to take limits when ε^t goes to 0^+ in each term of (3.41).

On the one hand, the lower semicontinuity of the total variation (see (I.7)) implies

$$\int_\Omega \varphi \, |D\widehat{u}(t)| \leq \lim_{\varepsilon^t \to 0^+} \int_\Omega \varphi \, |Du_{\varepsilon^t}(t)|\,.$$

On the other hand, convergence (3.38) provide us

$$\lambda \int_\Omega \widehat{u}(t)^2 \varphi \, dx \leq \liminf_{\varepsilon^t \to 0^+} \lambda \int_\Omega u_{\varepsilon^t}(t)^2 \varphi \, dx\,.$$

Moreover,

$$\lim_{\varepsilon^t \to 0^+} \int_\Omega u_{\varepsilon^t}(t)\, \mathbf{z}_{\varepsilon^t}(t) \cdot \nabla\varphi \, dx = \int_\Omega \widehat{u}(t)\, \widehat{\mathbf{z}}(t) \cdot \nabla\varphi \, dx\,.$$

Therefore, letting $\varepsilon^t \to 0^+$ in (3.41) we get

$$\lambda \int_\Omega \widehat{u}(t)^2 \varphi \, dx + \int_\Omega \widehat{u}(t)\, \widehat{\mathbf{z}}(t) \cdot \nabla\varphi \, dx + \int_\Omega \varphi \, |D\widehat{u}(t)| \leq 0\,,$$

which, using the previous step, can be written as

$$\begin{aligned} \int_\Omega \varphi \, |D\widehat{u}(t)| &\leq -\lambda \int_\Omega \widehat{u}(t)^2 \varphi \, dx - \int_\Omega \widehat{u}(t)\, \widehat{\mathbf{z}}(t) \cdot \nabla\varphi \, dx \\ &= -\int_\Omega \widehat{u}(t)\, \varphi \operatorname{div} \widehat{\mathbf{z}}(t) - \int_\Omega \widehat{u}(t)\, \widehat{\mathbf{z}}(t) \cdot \nabla\varphi \, dx = \langle (\widehat{\mathbf{z}}(t), D\widehat{u}(t)), \varphi \rangle\,. \end{aligned}$$

Since this inequality holds for every $\varphi \geq 0$, we have that $|D\widehat{u}(t)| \leq (\widehat{\mathbf{z}}(t), D\widehat{u}(t))$ as measures in Ω. The reverse inequality is straightforward, so that the equality holds and Step 10 is proved.

Step 11: $[\mathbf{z}(t), \nu] \in \operatorname{sign}(\omega(t) - \widehat{u}(t))$ *on* $\partial\Omega$ *for almost every* $t \in (0, T)$.

As in Step 10, fix $t \in (0, T)$ such that the previous steps hold true and take $u_{\varepsilon^t}(t)$ as a test function in (3.22). Then,

$$\lambda \int_\Omega u_{\varepsilon^t}(t)^2\, dx + \int_\Omega (\mathbf{z}_{\varepsilon^t}(t), Du_{\varepsilon^t}(t)) = \int_{\partial\Omega} u_{\varepsilon^t}(t)[\mathbf{z}_{\varepsilon^t}(t), \nu]\, d\mathcal{H}^{N-1}.$$

Applying (3.19) and (3.21), we have

$$\lambda \int_\Omega u_{\varepsilon^t}(t)^2\, dx + \int_\Omega |Du_{\varepsilon^t}(t)| + \int_{\partial\Omega} |u_{\varepsilon^t}(t) - \omega_{\varepsilon^t}(t)|\, d\mathcal{H}^{N-1}$$
$$= \int_{\partial\Omega} \omega_{\varepsilon^t}(t)[\mathbf{z}_{\varepsilon^t}(t), \nu]\, d\mathcal{H}^{N-1},$$

which leads to

$$\lambda \int_\Omega u_{\varepsilon^t}(t)^2\, dx + \int_\Omega |Du_{\varepsilon^t}(t)| + \int_{\partial\Omega} |u_{\varepsilon^t}(t) - \omega(t)|\, d\mathcal{H}^{N-1}$$
$$\leq \int_{\partial\Omega} |\omega_{\varepsilon^t}(t) - \omega(t)|\, d\mathcal{H}^{N-1} + \int_{\partial\Omega} \omega_{\varepsilon^t}(t)[\mathbf{z}_{\varepsilon^t}(t), \nu]\, d\mathcal{H}^{N-1}.$$

To let $\varepsilon^t \to 0^+$, in the first term we use (3.38), while in the second and third terms we apply the lower semicontinuity of functional (I.6). The right-hand side is a consequence of convergences $\omega_{\varepsilon^t}(t) \to \omega(t)$ strongly in $L^2(\partial\Omega)$ and $[\mathbf{z}_{\varepsilon^t}(t), \nu] \rightharpoonup [\mathbf{z}(t), \nu]$ $*$-weakly in $L^\infty(\partial\Omega)$. Hence,

$$\lambda \int_\Omega \widehat{u}(t)^2\, dx + \int_\Omega |D\widehat{u}(t)| + \int_{\partial\Omega} |\widehat{u}(t) - \omega(t)|\, d\mathcal{H}^{N-1} \tag{3.42}$$
$$\leq \int_{\partial\Omega} \omega(t)[\mathbf{z}(t), \nu]\, d\mathcal{H}^{N-1}.$$

On the other hand, Steps 9, 10 and Green's formula imply

$$\lambda \int_\Omega \widehat{u}(t)^2\,dx + \int_\Omega |D\widehat{u}(t)| = \int_{\partial\Omega} \widehat{u}(t)[\mathbf{z}(t),\nu]\,d\mathcal{H}^{N-1}\,. \tag{3.43}$$

Combining (3.42) and (3.43), it yields

$$\int_{\partial\Omega} |\widehat{u}(t)-\omega(t)|\,d\mathcal{H}^{N-1} + \int_{\partial\Omega} (\widehat{u}(t)-\omega(t))[\mathbf{z}(t),\nu]\,d\mathcal{H}^{N-1} \leq 0\,,$$

from where Step 11 follows.

Step 12: $u_\varepsilon(t) \rightharpoonup \widehat{u}(t)$ in $L^2(\Omega)$ and $u_\varepsilon(t) \to \widehat{u}(t)$ in $L^1(\Omega)$ for almost every $t \in (0,T)$.

Fix $t \in (0,T)$ such that the previous steps hold. We have proved that there exist a subsequence $\{u_{\varepsilon^t}\}$ and a function $\widehat{u}(t) \in BV(\Omega)\cap L^2(\Omega)$ such that (3.38) and (3.39) hold, and $\widehat{u}(t)$ is a solution to the Dirichlet problem

$$\begin{cases} \lambda u - \operatorname{div}\left(\dfrac{Du}{|Du|}\right) = 0 & \text{in } \ \Omega\,,\\[2mm] u = \omega(t) & \text{on } \ \partial\Omega\,. \end{cases}$$

Following [8, Theorem 4], the previous Dirichlet problem has a unique solution which implies that the whole sequence $\{u_\varepsilon(t)\}$ converges to $\widehat{u}(t)$ weakly in $L^2(\Omega)$ and strongly in $L^1(\Omega)$. We remark that we may also assume that $\{u_\varepsilon(t)\}$ converges to $\widehat{u}(t)$ a.e. in Ω.

Step 13: $u(t) = \widehat{u}(t)$ for almost every $t \in (0,T)$.

Since u_ε are measurable functions in $(0,T)\times\Omega$, the pointwise limit function $\widehat{u}$ is also measurable in $(0,T)\times\Omega$.

Considering now $\varphi \in L^2(\Omega)$ and $\phi \in L^2((0,T))$, the following inequality holds

$$\left|\int_\Omega u_\varepsilon(t,x)\varphi(x)\phi(t)\,dx\right| \leq |\phi(t)| \left(\int_\Omega u_\varepsilon(t,x)^2\,dx\right)^{\frac12} \left(\int_\Omega \varphi(x)^2\,dx\right)^{\frac12}$$

$$\leq K|\phi(t)|\,,$$

for certain constant $K>0$, by (3.34). This inequality allows us to use the dominated convergence theorem and obtain

$$\lim_{\varepsilon\to 0^+}\int_0^T\int_\Omega u_\varepsilon(t,x)\varphi(x)\phi(t)\,dx\,dt=\int_0^T\left[\lim_{\varepsilon\to 0^+}\int_\Omega u_\varepsilon(t,x)\varphi(x)\phi(t)\,dx\right]dt\,,$$

so that

$$\int_0^T\int_\Omega u(t,x)\varphi(x)\phi(t)\,dx\,dt=\int_0^T\int_\Omega \widehat{u}(t,x)\varphi(x)\phi(t)\,dx\,dt\,.$$

Therefore, we get that $u(t,x)=\widehat{u}(t,x)$ for almost every $(t,x)\in(0,T)\times\Omega$.

Step 14: The pairing (u,ω) is a strong solution to problem (3.4).

Having in mind Steps 6, 7, 11 and 13, it only remains to check the equality $(\mathbf{z}(t),Du(t))=|Du(t)|$ for almost all $t\in(0,T)$. Now, a remark is in order. By Steps 10 and 13, we already know that $(\widehat{\mathbf{z}}(t),Du(t))=|Du(t)|$ holds for almost all $t\in(0,T)$. Nevertheless, the way we have obtained the vector field $\widehat{\mathbf{z}}$ does not imply that it is measurable in $(0,T)\times\Omega$. Hence, we cannot use this vector field to see that (u,ω) is a strong solution.

To prove $(\mathbf{z}(t),Du(t))=|Du(t)|$ for almost all $t\in(0,T)$, we first fix $t\in(0,T)$ satisfying the previous steps and observe that we have $\operatorname{div}\mathbf{z}(t)=\operatorname{div}\widehat{\mathbf{z}}(t)$ (by Steps 6, 9 and 13) and $[\mathbf{z}(t),\nu]=[\widehat{\mathbf{z}}(t),\nu]$ (see Step 9). Applying Green's formula, it yields

$$\int_\Omega u(t)\operatorname{div}\mathbf{z}(t)\,dx+\int_\Omega(\mathbf{z}(t),Du(t))=\int_{\partial\Omega}u(t)[\mathbf{z}(t),\nu]\,d\mathcal{H}^{N-1}$$

and

$$\int_\Omega u(t)\operatorname{div}\mathbf{z}(t)\,dx+\int_\Omega(\widehat{\mathbf{z}}(t),Du(t))=\int_{\partial\Omega}u(t)[\mathbf{z}(t),\nu]\,d\mathcal{H}^{N-1}\,,$$

which imply

$$\int_\Omega (\mathbf{z}(t), Du(t)) = \int_\Omega (\widehat{\mathbf{z}}(t), Du(t)) = \int_\Omega |Du(t)|\,,$$

thanks to Steps 10 and 13. Now, it follows from this identity and from $|(\mathbf{z}(t), Du(t))| \leq |Du(t)|$ that $(\mathbf{z}(t), Du(t)) = |Du(t)|$ as measures in Ω. Indeed, let $E \subset \Omega$ be a $|Du|$-measurable set, then

$$\begin{aligned}\int_\Omega |Du(t)| = \int_\Omega (\mathbf{z}(t), Du(t)) &= \int_E (\mathbf{z}(t), Du(t)) + \int_{\Omega\setminus E} (\mathbf{z}(t), Du(t)) \\ &\leq \int_E |Du(t)| + \int_{\Omega\setminus E} |Du(t)| = \int_\Omega |Du(t)|\,,\end{aligned}$$

and so the inequality becomes equality. Thus,

$$\int_E (\mathbf{z}(t), Du(t)) = \int_E |Du(t)|\,.$$

Step 15: Estimates (3.13) *and* (3.14).

To check (3.13), we only have to write (3.30) conveniently. Indeed, notice that

$$\begin{aligned}&\left(\int_{\partial\Omega} \omega_\varepsilon(t,x)^2\, d\mathcal{H}^{N-1}\right)^{\frac{1}{2}} \\ &\leq \left(\int_{\partial\Omega} \omega_0(x)^2\, d\mathcal{H}^{N-1}\right)^{\frac{1}{2}} + \int_0^T \left(\int_{\partial\Omega} g_\varepsilon(t,x)^2\, d\mathcal{H}^{N-1}\right)^{\frac{1}{2}} dt\,,\end{aligned}$$

for all $t \in (0,T)$ Then apply (3.18) and (3.16).

On the other hand, (3.14) is a consequence of (3.42). □

Remark 3.4.3. It is not difficult to obtain estimates (other than (3.13) and (3.14)) connecting data and solution, which may have some interest.

Indeed, we can easily deduce another estimate starting from (3.42):

$$\begin{aligned}
&\lambda \int_\Omega u(t)^2\, dx + \int_\Omega |Du(t)| + \int_{\partial\Omega} |u(t)|\, d\mathcal{H}^{N-1} \\
&\quad \le \int_{\partial\Omega} \big(g(t) - \omega_t(t)\big)\omega(t)\, d\mathcal{H}^{N-1} + \int_{\partial\Omega} |\omega(t)|\, d\mathcal{H}^{N-1} \\
&\quad \le \int_{\partial\Omega} g(t)\omega(t)\, d\mathcal{H}^{N-1} - \frac{1}{2}\frac{d}{dt}\int_{\partial\Omega} \omega(t)^2\, d\mathcal{H}^{N-1} + \int_{\partial\Omega} |\omega(t)|\, d\mathcal{H}^{N-1}\,.
\end{aligned}$$

Integrating in $[0,t]$ for $t \in (0,T]$, we get

$$\begin{aligned}
&\lambda \int_0^t \int_\Omega u(s)^2\, dx\, ds + \int_0^t \left[\int_\Omega |Du(s)| + \int_{\partial\Omega} |u(s)|\, d\mathcal{H}^{N-1}\right] ds \\
&\quad + \frac{1}{2}\int_{\partial\Omega} \omega(t)^2\, d\mathcal{H}^{N-1} \le \frac{1}{2}\int_{\partial\Omega} \omega_0^2\, d\mathcal{H}^{N-1} \\
&\quad + \int_0^t \int_{\partial\Omega} g(s)\omega(s)\, d\mathcal{H}^{N-1}\, ds + \int_0^t \int_{\partial\Omega} |\omega(s)|\, d\mathcal{H}^{N-1}\, ds\,,
\end{aligned}$$

and taking the supremum for $t \in (0,T]$, it yields

$$\begin{aligned}
&\lambda \|u\|_{L^2(0,T;L^2(\Omega))} + \|u\|_{L^1(0,T;BV(\Omega))} + \frac{1}{2}\|\omega\|^2_{L^\infty(0,T;L^2(\partial\Omega))} \\
&\le \frac{1}{2}\|\omega_0\|^2_{L^2(\partial\Omega)} + \|\omega\|_{L^1(0,T;L^1(\partial\Omega))} + \int_0^T \int_{\partial\Omega} g(s)\omega(s)\, d\mathcal{H}^{N-1}\, ds \\
&\le \frac{1}{2}\|\omega_0\|^2_{L^2(\partial\Omega)} + \|\omega\|_{L^1(0,T;L^1(\partial\Omega))} + \|\omega\|_{L^\infty(0,T;L^2(\partial\Omega))}\|g\|_{L^1(0,T;L^2(\partial\Omega))}\,.
\end{aligned}$$

Finally, Young's inequality implies

$$\begin{aligned}
&\lambda \|u\|_{L^2(0,T;L^2(\Omega))} + \|u\|_{L^1(0,T;BV(\Omega))} \\
&\quad \le \frac{1}{2}\|\omega_0\|^2_{L^2(\partial\Omega)} + \|\omega\|_{L^1(0,T;L^1(\partial\Omega))} + \frac{1}{2}\|g\|^2_{L^1(0,T;L^2(\partial\Omega))}\,.
\end{aligned}$$

Remark 3.4.4. We remark that choosing data in more regular spaces, we get better regularity of the solution. An easy instance is considered:

If $g \in L^\infty(0,+\infty; L^2(\partial\Omega))$, since the equality $\omega_t(t) = [\mathbf{z}(t), \nu] + g(t)$ holds on $\partial\Omega$, then $\omega_t \in L^\infty(0,+\infty; L^2(\partial\Omega))$ and thus, solution ω is Lipschitz continuous with respect to the time variable.

We finish this section with a comparison principle and a result on the long term behaviour.

Proposition 3.4.5. *Let $g^1, g^2 \in L^1(0,T; L^2(\partial\Omega))$ and let $\omega_0^1, \omega_0^2 \in L^2(\partial\Omega)$. Denote by (u^k, ω^k) the strong solution corresponding to data g^k and ω_0^k, $k = 1, 2$.*

If $g^1(t,x) \le g^2(t,x)$ for almost all $(t,x) \in (0,T) \times \partial\Omega$ and $\omega_0^1(x) \le \omega_0^2(x)$ for almost all $x \in \partial\Omega$, then $\omega^1(t,x) \le \omega^2(t,x)$ for almost all $(t,x) \in (0,T) \times \partial\Omega$ and $u^1(t,x) \le u^2(t,x)$ for almost all $(t,x) \in (0,T) \times \Omega$.

Proof. It is enough to apply Theorem 3.3.12 having in mind the L^1-convergence $u_\varepsilon^k(t) \to u^k(t)$ for almost every $t \in (0,T)$. □

Proposition 3.4.6. *If $g \in L^1(0,+\infty; L^2(\partial\Omega))$ and $\omega_0 \in L^2(\partial\Omega)$, then there exists a sequence $t_n \to +\infty$ and there exist $h \in L^2(\partial\Omega)$ and $v \in BV(\Omega) \cap L^2(\Omega)$ such that*

(i) $\omega(t_n) \rightharpoonup h$ *weakly in* $L^2(\partial\Omega)$,

(ii) $u(t_n) \rightharpoonup v$ *weakly in* $L^2(\Omega)$,

(iii) $u(t_n) \to v$ *strongly in* $L^1(\Omega)$,

(iv) $Du(t_n) \to Dv$ $*$*-weakly as measures in* Ω.

Proof. Since the datum $g \in L^1(0,+\infty; L^2(\partial\Omega))$, we deduce from estimate (3.13) that

$$\|\omega\|_{L^\infty(0,+\infty;L^2(\partial\Omega))} \le \|\omega_0\|_{L^2(\partial\Omega)} + \|g\|_{L^1(0,+\infty;L^2(\partial\Omega))} < +\infty\,.$$

Then, there exist a constant $M > 0$ such that $\|\omega(t)\|_{L^2(\partial\Omega)} \leq M$ for almost every $t > 0$. Therefore, there exist a sequence $t_n \to +\infty$ and a function $h \in L^2(\partial\Omega)$ such that $\omega(t_n) \rightharpoonup h$ weakly in $L^2(\partial\Omega)$.

On the other hand, from estimate (3.14), we also deduce

$$\lambda\|u(t)\|_{L^2(\Omega)} + \|u(t)\|_{BV(\Omega)} \leq M\,.$$

Thus, there exists another sequence $t_n \to +\infty$ and two functions v_1 and v_2 such that

$$u(t_n) \rightharpoonup v_1 \quad \text{in} \quad L^2(\Omega)$$

and

$$u(t_n) \to v_2 \quad \text{in} \quad L^1(\Omega)\,,$$

with

$$Du(t_n) \to Dv_2 \ \ast\text{-weakly as measures in in } \ \Omega\,.$$

Finally, due to the uniqueness of the limit, we denote $v = v_1 = v_2 \in BV(\Omega) \cap L^2(\Omega)$. $\square$

3.5 Continuous dependence on data

The present section is devoted to prove a result which compares solutions of problem (3.4) determined by different data. More precisely, the result allows us to estimate the distance of the solutions depending on the distance of the data.

Theorem 3.5.1. *Let (u_1, ω_1) and (u_2, ω_2) be the strong solution to problem (3.4) with initial data $g_1, g_2 \in L^1(0,T; L^2(\partial\Omega))$ and $\omega_{01}, \omega_{02} \in L^2(\partial\Omega)$, respectively. Then, it holds*

$$\|\omega_1 - \omega_2\|_{L^\infty(0,T;L^2(\partial\Omega))} \leq \|\omega_{01} - \omega_{02}\|_{L^2(\partial\Omega)} + \|g_1 - g_2\|_{L^1(0,T;L^2(\partial\Omega))}$$

and

$$\lambda\|u_1-u_2\|^2_{L^2(0,T;L^2(\Omega))} \leq \frac{1}{2}\|\omega_{01}-\omega_{02}\|^2_{L^2(\partial\Omega)} + \frac{1}{2}\|g_1-g_2\|^2_{L^1(0,T;L^2(\partial\Omega))}\,. \tag{3.44}$$

Proof. First, we fix $t \in (0,T)$ such that conditions **(i)** to **(iv)** of solution to problem (3.4) hold. Then, we take $u_1(t)-u_2(t)$ as a test function in the condition **(ii)** corresponding to (u_1,ω_1). Therefore, using Green's formula we get

$$\begin{aligned}0 =& \lambda\int_\Omega u_1(t)(u_1(t)-u_2(t))\,dx + \int_\Omega(\mathbf{z}_1(t), D(u_1(t)-u_2(t)))\\ &- \int_{\partial\Omega}[\mathbf{z}_1(t),\nu](u_1(t)-u_2(t))\,d\mathcal{H}^{N-1}\,.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}0 =& \lambda\int_\Omega u_2(t)(u_1(t)-u_2(t))\,dx + \int_\Omega(\mathbf{z}_2, D(u_1(t)-u_2(t)))\\ &- \int_{\partial\Omega}[\mathbf{z}_2(t),\nu](u_1(t)-u_2(t))\,d\mathcal{H}^{N-1}\,,\end{aligned}$$

and joining both equalities it holds

$$\begin{aligned}&\lambda\int_\Omega(u_1(t)-u_2(t))^2\,dx\\ &\quad+\int_\Omega\Big[|Du_1(t)|-(\mathbf{z}_1(t),Du_2(t))+|Du_2(t)|-(\mathbf{z}_2(t),Du_1(t))\Big]\\ &=\int_{\partial\Omega}[\mathbf{z}_1(t),\nu](u_1(t)-u_2(t))\,d\mathcal{H}^{N-1}\\ &\quad+\int_{\partial\Omega}[\mathbf{z}_2(t),\nu](u_2(t)-u_1(t))\,d\mathcal{H}^{N-1} = I_1+I_2\,.\end{aligned}$$

Now, since $(\mathbf{z}_1(t), Du_2(t)) \leq |Du_2(t)|$ and $(\mathbf{z}_2(t), Du_1(t)) \leq |Du_1(t)|$, we get the following inequality:

$$\lambda \int_\Omega (u_1(t) - u_2(t))^2 \, dx \leq I_1 + I_2 \, . \tag{3.45}$$

We are analyzing I_1 and I_2. First, we manipulate I_1 using conditions **(iii)** and **(iv)**:

$$\begin{aligned} I_1 &= \int_{\partial\Omega} [\mathbf{z}_1(t), \nu](u_1(t) - \omega_1(t) + \omega_1(t) - \omega_2(t) + \omega_2(t) - u_2(t)) \, d\mathcal{H}^{N-1} \\ &\leq - \int_{\partial\Omega} |u_1(t) - \omega_1(t)| \, d\mathcal{H}^{N-1} + \int_{\partial\Omega} |\omega_2(t) - u_2(t)| \, d\mathcal{H}^{N-1} \\ &\quad + \int_{\partial\Omega} (g_1(t) - \omega_{1t}(t))(\omega_1(t) - \omega_2(t)) \, d\mathcal{H}^{N-1} \, . \end{aligned}$$

In an analogous way we get

$$\begin{aligned} I_2 \leq &- \int_{\partial\Omega} |u_2(t) - \omega_2(t)| \, d\mathcal{H}^{N-1} + \int_{\partial\Omega} |\omega_1(t) - u_1(t)| \, d\mathcal{H}^{N-1} \\ &+ \int_{\partial\Omega} (g_2(t) - \omega_{2t}(t))(\omega_2(t) - \omega_1(t)) \, d\mathcal{H}^{N-1} \, , \end{aligned}$$

and adding both estimates it follows that

$$\begin{aligned} I_1 + I_2 \leq &\int_{\partial\Omega} (g_1(t) - \omega_{1t}(t))(\omega_1(t) - \omega_2(t)) \, d\mathcal{H}^{N-1} \\ &+ \int_{\partial\Omega} (g_2(t) - \omega_{2t}(t))(\omega_2(t) - \omega_1(t)) \, d\mathcal{H}^{N-1} \, . \end{aligned}$$

Therefore, (3.45) and Hölder's inequality imply

$$\begin{aligned} &\lambda \int_\Omega (u_1(t) - u_2(t))^2 \, dx \\ &\leq \int_{\partial\Omega} (g_1(t) - g_2(t))(\omega_1(t) - \omega_2(t)) \, d\mathcal{H}^{N-1} \\ &- \int_{\partial\Omega} (\omega_{1t}(t) - \omega_{2t}(t))(\omega_1(t) - \omega_2(t)) \, d\mathcal{H}^{N-1} \end{aligned}$$

$$\leq \left(\int_{\partial\Omega}(g_1(t)-g_2(t))^2\,d\mathcal{H}^{N-1}\right)^{\frac{1}{2}}\left(\int_{\partial\Omega}(\omega_1(t)-\omega_2(t))^2\,d\mathcal{H}^{N-1}\right)^{\frac{1}{2}}$$
$$-\int_{\partial\Omega}(\omega_{1t}(t)-\omega_{2t}(t))(\omega_1(t)-\omega_2(t))\,d\mathcal{H}^{N-1}\,.$$

Moreover, since $\omega_1,\omega_2\in W^{1,1}(0,T;L^2(\partial\Omega))$, we know that

$$\int_{\partial\Omega}(\omega_{1t}(t)-\omega_{2t}(t))(\omega_1(t)-\omega_2(t))\,d\mathcal{H}^{N-1}$$
$$=\frac{1}{2}\frac{d}{dt}\int_{\partial\Omega}\Big(\omega_1(t)-\omega_2(t)\Big)^2\,d\mathcal{H}^{N-1}\,.$$

Now, let $t\in(0,T)$ and integrate the previous equality with respect to the time to get

$$\lambda\int_0^t\int_\Omega(u_1(s)-u_2(s))^2\,dx\,ds\leq\int_0^t\left(\int_{\partial\Omega}(g_1(s)-g_2(s))^2\,d\mathcal{H}^{N-1}\right)^{\frac{1}{2}}\times$$
$$\times\left(\int_{\partial\Omega}(\omega_1(s)-\omega_2(s))^2\,d\mathcal{H}^{N-1}\right)^{\frac{1}{2}}\,ds$$
$$-\frac{1}{2}\int_{\partial\Omega}(\omega_1(t)-\omega_2(t))^2\,d\mathcal{H}^{N-1}+\frac{1}{2}\int_{\partial\Omega}(\omega_1(0)-\omega_2(0))^2\,d\mathcal{H}^{N-1}\,.$$

So, we have got the main estimate:

$$2\lambda\int_0^t\|u_1(s)-u_2(s)\|^2_{L^2(\Omega)}\,ds+\|\omega_1(t)-\omega_2(t)\|^2_{L^2(\partial\Omega)} \tag{3.46}$$
$$\leq 2\int_0^t\|g_1(s)-g_2(s)\|_{L^2(\partial\Omega)}\|\omega_1(s)-\omega_2(s)\|_{L^2(\partial\Omega)}\,ds$$
$$+\|\omega_1(0)-\omega_2(0)\|^2_{L^2(\partial\Omega)}\,,$$

for all $t\in(0,T)$. As a consequence we get the following the inequality:

$$\|\omega_1(t)-\omega_2(t)\|^2_{L^2(\partial\Omega)}\leq\|\omega_1(0)-\omega_2(0)\|^2_{L^2(\partial\Omega)}$$

$$+ 2\int_0^t \|g_1(s) - g_2(s)\|_{L^2(\partial\Omega)} \|\omega_1(s) - \omega_2(s)\|_{L^2(\partial\Omega)}\, ds\,,$$

which, due to an extension of Gronwall's inequality (see [65, Theorem 1.2]), allows us to have

$$\begin{aligned}
&\|\omega_1(t) - \omega_2(t)\|_{L^2(\partial\Omega)} \\
&\quad \le \|\omega_1(0) - \omega_2(0)\|_{L^2(\partial\Omega)} + \int_0^t \|g_1(s) - g_2(s)\|_{L^2(\partial\Omega)} ds \\
&\quad \le \|\omega_1(0) - \omega_2(0)\|_{L^2(\partial\Omega)} + \int_0^T \|g_1(s) - g_2(s)\|_{L^2(\partial\Omega)} ds\,,
\end{aligned}$$

for all $t \in (0, T)$. So that

$$\|\omega_1 - \omega_2\|_{L^\infty(0,T;L^2(\partial\Omega))} \le \|\omega_1(0) - \omega_2(0)\|_{L^2(\partial\Omega)} + \|g_1 - g_2\|_{L^1(0,T;L^2(\partial\Omega))}\,.$$

On the other hand, inequality (3.46) and Young's inequality imply

$$\begin{aligned}
&2\lambda \int_0^T \|u_1(s) - u_2(s)\|^2_{L^2(\Omega)}\, ds + \sup_{t\in[0,T]} \|\omega_1(t) - \omega_2(t)\|^2_{L^2(\partial\Omega)} \\
&\quad \le \|\omega_1(0) - \omega_2(0)\|^2_{L^2(\partial\Omega)} \\
&\qquad + 2\int_0^T \|g_1(s) - g_2(s)\|_{L^2(\partial\Omega)} \|\omega_1(s) - \omega_2(s)\|_{L^2(\partial\Omega)}\, ds \\
&\quad \le \|\omega_1(0) - \omega_2(0)\|^2_{L^2(\partial\Omega)} + \|g_1 - g_2\|^2_{L^1(0,T;L^2(\partial\Omega))} \\
&\qquad + \|\omega_1 - \omega_2\|^2_{L^\infty(0,T;L^2(\partial\Omega))}\,.
\end{aligned}$$

Simplifying, it leads to the desired inequality (3.44). $\square$

Conclusions

We would like to close this dissertation by summing up our main results, which were submitted and published in specifics journals during the PhD procedure.

In Chapter 1 we present the results from [49]. In particular, we show existence and uniqueness results of problem

$$\begin{cases} -\operatorname{div}\left(\frac{Du}{|Du|}\right) + g(u)|Du| = f(x) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where Ω is a bounded open set in $\mathbb{R}^N$ with Lipschitz boundary, f is a nonnegative datum in $L^{N,\infty}(\Omega)$ and a continuous function $g : [0,\infty[\to [0,\infty[$ is considered.

We start by showing a generalization of Anzellotti's theory. For the sake of completeness, we also add some results from [51].

The first existence theorem shows that if $g \equiv 1$, then there is a unique solution to this problem. Moreover, this solution belongs to the Lebesgue space $L^q(\Omega)$ with $1 \le q \le \infty$ but it is not necessarily bounded.

Furthermore, for a function $g(s)$ bounded from below we prove that there is a unique solution and it also belong to $L^q(\Omega)$ with $1 \le q \le \infty$.

Nevertheless, when we take g such that $g(s) > 0$ for almost every $s \ge 0$, then we have to change the notion of solution because u does not belong, necessarily, to the BV-space. Moreover, when g vanishes on an interval, we do not have uniqueness of solution, it can have jump discontinuities and also the boundary condition may not hold.

In Chapter 2 we deal with the problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{Du}{|Du|}\right) + |Du| = f(x) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where Ω is a bounded open set in $\mathbb{R}^N$ and datum f is a nonnegative integrable function in Ω.

Since we consider non-variation data and the pairing $(\mathbf{z}, Du)$ is not well-defined for a general $u \in BV(\Omega)$ and a vector field $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$, we have to use truncations in the definition of solution to this problem.

The main result of this chapter is the comparison principle, which not just improves the known results for this problem with less general data, but also its proof is simpler than the way the uniqueness is proved.

Finally, we also show some results concerning the regularity of solutions. In particular, we have proved that when we take L^q-data with $1 \le q < N$, the solution to this problem belongs to $BV(\Omega) \cap L^{\frac{Nq}{N-q}}(\Omega)$.

The contents of Section 1.2 of Chapter 1 and Chapter 2 are in [51].

Finally, in Chapter 3 we deal with an existence and uniqueness result for an evolution problem. It consists in an elliptic equation involving the 1-Laplacian operator and a dynamical boundary condition, namely

$$\begin{cases} \lambda u - \mathrm{div}\left(\dfrac{Du}{|Du|}\right) = 0 & \text{in } (0,+\infty)\times\Omega\,, \\ \omega_t + \left[\dfrac{Du}{|Du|}, \nu\right] = g(t,x) & \text{on } (0,+\infty)\times\partial\Omega\,, \\ u = \omega & \text{on } (0,+\infty)\times\partial\Omega\,, \\ \omega(0,x) = \omega_0(x) & \text{on } \partial\Omega\,; \end{cases}$$

where Ω is a bounded open set in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, λ is a nonnegative parameter, ν stands for the unit outward normal vector on $\partial\Omega$, $g \in L^1_{loc}(0, +\infty; L^2(\partial\Omega))$ and $\omega_0 \in L^2(\partial\Omega)$. Here, we have denoted by ω_t the distributional derivative of ω with respect to t.

Using the nonlinear semigroups theory, we show the existence of a mild solution and we prove that this solution is, in fact, a strong solution in the sense that every statement of the problem hold for almost every $t > 0$.

In addition, we also have proved a comparison principle and a result which shows that the distance between solutions depends on the distance between the data.

These results will appear in [50].

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