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PhD Dissertation

Higher-dimensional Reidemeister torsion invariants for cusped hyperbolic 3-manifolds

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Agraïments

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Summary

The aim of this dissertation is to study a class of Reidemeister torsion invariants for a complete, hyperbolic three-manifold of finite volume.

Let M be an oriented, complete, hyperbolic three-manifold of finite volume. The hyperbolic structure of M yields the holonomy representation:

$$\mathrm{Hol}_M \colon \pi_1(M,p) \to \mathrm{Isom}^+ \mathbf{H}^3,$$

where $\mathrm{Isom}^+ \mathbf{H}^3$ denotes the isometry group of hyperbolic 3-space $\mathbf{H}^3$. Using the upper half-space model, $\mathrm{Isom}^+ \mathbf{H}^3$ is naturally identified with $\mathrm{PSL}(2,\mathbf{C}) = \mathrm{SL}(2,\mathbf{C})/\{\pm 1\}$. It is known that Hol_M can be lifted to $\mathrm{SL}(2,\mathbf{C})$; moreover, such lifts are in canonical one-to-one correspondence with spin structures on M. Thus, attached to a fixed spin structure η on M, we get a representation

$$\mathrm{Hol}_{(M,\eta)} \colon \pi_1(M,p) \to \mathrm{SL}(2,\mathbf{C}).$$

On the other hand, for all $n > 0$ there exists a unique (up to isomorphism) n-dimensional, complex, irreducible representation of the Lie group $\mathrm{SL}(2,\mathbf{C})$, say:

$$\varsigma_n \colon \mathrm{SL}(2,\mathbf{C}) \to \mathrm{SL}(n,\mathbf{C}).$$

Hence, composing $\mathrm{Hol}_{(M,\eta)}$ with ς_n we get the following representation:

$$\rho_n \colon \pi_1(M,p) \to \mathrm{SL}(n,\mathbf{C}).$$

This representation will be called the *canonical n-dimensional representation* of the spin-hyperbolic manifold (M,η).

Roughly speaking, the Reidemeister torsion invariants that we want to study are those coming from ρ_n. The first issue that arises in trying to define the Reidemeister torsion concerns the cohomology groups $\mathrm{H}^*(M;\rho_n)$ (i.e. the cohomology groups of M in the local system defined by ρ_n). If all these groups vanish, then it makes sense to consider the Reidemeister torsion $\tau(M;\rho_n)$; however, if some of them are not trivial, then a choice of bases for $\mathrm{H}^*(M;\rho_n)$ is required.

An important case for which ρ_n is acyclic (i.e. $\mathrm{H}^*(M;\rho_n) = 0$) is when M is closed. This is a particular case of Raghunathan's vanishing theorem. For M closed, the invariant $\tau(M;\rho_n)$ has been considered by W. Müller in [Mül], and for $n = 3$ by J. Porti in [Por97].

In general, the representation ρ_n need not be acyclic for a cusped manifold M. Therefore, we need to choose bases in (co)homology to define $\tau(M;\rho_n)$. Obviously, if we want an invariant of the manifold, these bases must be chosen in a somehow canonical way. Unfortunately,

we do not know if this is possible. J. Porti proved in [Por97] that for $n = 3$ a natural choice of bases can be done once a basis for $\mathrm{H}^1(\partial\overline{M}; \mathbf{Z})$ is chosen. Using the same approach, we will prove the following result: given non-trivial cycles $\{\theta_i\}$ in $\mathrm{H}_1(\partial\overline{M}; \mathbf{Z})$, one for each connected component of $\partial\overline{M}$, there is a canonical family of bases of $\mathrm{H}^*(M; \rho_n)$ such that any member of this family yields the same Reidemeister torsion, say $\tau(M; \rho_n; \{\theta_i\})$. Moreover, we will show that for $k > 1$ the following quotients are independent of the choices $\{\theta_i\}$:

$$\begin{aligned} \mathcal{T}_{2k+1}(M,\eta) &:= \frac{\tau(M;\rho_{2k+1};\{\theta_i\})}{\tau(M;\rho_3;\{\theta_i\})} \in \mathbf{C}^*/\{\pm 1\}, \\ \mathcal{T}_{2k}(M,\eta) &:= \frac{\tau(M;\rho_{2k};\{\theta_i\})}{\tau(M;\rho_2;\{\theta_i\})} \in \mathbf{C}^*/\{\pm 1\}. \end{aligned}$$

Thus, for all $n \geq 4$, $\mathcal{T}_n(M,\eta)$ is an invariant of the spin-hyperbolic manifold (M,η). Notice that if n is odd the quantity $\mathcal{T}_n(M,\eta)$ is independent of the spin structure (this is an immediate consequence of the fact that an odd dimensional irreducible complex representation of $\mathrm{SL}(2,\mathbf{C})$ factors through $\mathrm{PSL}(2,\mathbf{C})$), and hence we will denote it simply by $\mathcal{T}_{2k+1}(M)$. The invariant $\mathcal{T}_n(M,\eta)$ will be called the *normalized n-dimensional Reidemeister torsion* of the cusped spin-hyperbolic manifold M. We will also refer to these invariants as the higher-dimensional Reidemeister torsion invariants. These invariants are the focus of study of the present dissertation.

Remark. It is possible to assign a well defined sign to $\mathcal{T}_n(M,\eta)$: if n is even, this can be done for $\tau(M;\rho_n)$ (see Turaev's book [Tur01]); if n is odd, this can be done because, roughly speaking, the sign indeterminacy of $\tau(M;\rho_n)$ is the same for $\tau_3(M;\rho_n)$. In spite of this, we will work up to sign in general, as our main results concern just the modulus of $\mathcal{T}_n(M,\eta)$.

This dissertation is organized into two parts, which we briefly summarize separately.

Twisted cohomology

Most of the material presented in the two chapters of this part is not new, and some of their contents will be probably well known to the reader. However, we think it is worthwhile to outline it in an elementary, self-contained way to make it more accessible to the non-expert.

The first chapter concerns the definition of a spin-hyperbolic 3-manifold and some general properties about them. The definition of the canonical n-dimensional representation and some other related notions are also given in that chapter.

The second chapter deals with the cohomology groups $\mathrm{H}^*(M;\rho_n)$ of a complete spin-hyperbolic 3-manifold (M,η) in the local system of coefficients given by ρ_n. Our main result is the following.

Theorem. *Let (M,η) be a complete, spin-hyperbolic 3-manifold of finite volume, or, more generally, non-elementary and topologically finite, and let $n \geq 2$. Then the inclusion $\partial\overline{M} \subset \overline{M}$ induces an injection,*

$$\mathrm{H}^1(\overline{M};\rho_n) \hookrightarrow \mathrm{H}^1(\partial\overline{M};\rho_n),$$

with $\dim \mathrm{H}^1(\overline{M};\rho_n) = \frac{1}{2}\dim \mathrm{H}^1(\partial\overline{M};\rho_n)$*, and an isomorphism*

$$\mathrm{H}^2(\overline{M};\rho_n) \cong \mathrm{H}^2(\partial\overline{M};\rho_n).$$

The chapter then continues with the analysis of the cohomology groups of the ends of the manifold, namely the groups $\mathrm{H}^*(\partial\overline{M};\rho_n)$. As a result of this, we prove the following result.

Theorem. *Let* (M,η) *be a complete spin-hyperbolic* 3*-manifold of* finite volume *with a* single cusp. *If* $k \geq 1$*, then* $\mathrm{H}^*(M;\rho_{2k}) = 0$.

Finally, it is worth noting that these two theorems are infinitesimal rigidity results, and that they have applications to local rigidity. This is discussed at the end of the second chapter.

Higher-dimensional Reidemeister torsion invariants

To simplify the exposition in this introduction, we will restrict ourselves to odd-dimensional representations. Thus we do not need any spin structure on M. Our main result concerns the asymptotic behaviour of $\{\mathcal{T}_{2k+1}(M)\}$.

Theorem. *Let* M *be an oriented, complete, finite-volume, hyperbolic* 3*-manifold. Then*

$$\lim_{k\to\infty} \frac{\log|\mathcal{T}_{2k+1}(M)|}{(2k+1)^2} = -\frac{\mathrm{Vol}(M)}{4\pi}.$$

For M compact, this result was established by W. Müller in [Mül]. To explain our approach to the above theorem, we need to discuss Müller's result.

Let us assume that M is closed. According to Müller's Theorem on the equivalence between Reidemeister torsion and Ray-Singer torsion for *unimodular* representations (see [Mül93]), we have

$$|\tau(M;\rho_n)| = \mathrm{Tor}(M;\rho_n),$$

where $\mathrm{Tor}(M;\rho_n)$ is the Ray-Singer torsion of M with respect to ρ_n. For a manifold of negative curvature and a unitary representation ρ, D. Fried established in [Fri86] and [Fri95] a deep relationship between $\mathrm{Tor}(M,\rho_n)$ and the twisted Ruelle zeta function. The twisted Ruelle zeta function of M and ρ is formally defined by

$$R_\rho(s) = \prod_{\varphi\in\mathcal{PC}(M)} \det\left(\mathrm{Id} - \rho_n(\varphi)e^{-sl(\varphi)}\right), \tag{1}$$

where $\mathcal{PC}(M)$ denotes the set of oriented, prime, closed geodesics in M, and $l(\varphi)$ is the length of φ (we are using the identification between $\mathcal{PC}(M)$ and the set of hyperbolic conjugacy classes of $\pi_1 M$, so the expression appearing inside the above product makes sense). D. Fried proved that, for any representation ρ, the function $R_\rho(s)$ admits a meromorphic extension to the whole plane; moreover, if ρ is assumed to be acyclic and unitary, then $|R_\rho(0)| = \mathrm{Tor}(M,\rho_n)^2$. The work of U. Bröcker [Brö98] and A. Wotzke [Wot08] shows that a similar result also holds for a compact hyperbolic manifold and representations of its fundamental group arising from representations of $\mathrm{Isom}^+\mathbf{H}^n$. In our particular case, the result is the following.

Theorem (A. Wotzke, [Wot08]). *Let (M, η) be a compact spin-hyperbolic 3-manifold. Then, for $n > 1$, $R_{\rho_n}(s)$ admits a meromorphic extension to the whole complex plane and*

$$|R_{\rho_n}(0)| = \mathrm{Tor}(M; \rho_n)^2.$$

O. Bröcker established in [Brö98] a functional equation for $R_{\rho_n}(s)$ involving the volume of the manifold. Using this equation and other related material, Müller has recently established in [Mül] the following formula for $|\tau(M; \rho_n)|$, which involves the volume of the closed manifold M and some related Ruelle zeta functions $R_k(s)$,

$$\log\left|\frac{\tau(M, \rho_{2k+1})}{\tau(M, \rho_5)}\right| = \sum_{j=3}^{k} \log|R_{2j}(j)| - \frac{1}{\pi}\,\mathrm{Vol}\,M\,(k(k+1) - 6)\,. \tag{2}$$

One of the advantages of this formula is that the Ruelle zeta functions $R_k(s)$ are evaluated inside the corresponding region of convergence, and hence they have an expression similar to that of Equation (1). The result about the asymptotics of the torsion is then deduced by showing that the sum appearing in the right hand side of Equation (2) is uniformly bounded in k.

In trying to adapt Müller's proof to the non-compact case, some difficulties arise, the main one being the fact that the Ray-Singer torsion is *a priori* not defined for non-compact manifolds. Nevertheless, the terms appearing in Equation (2) still make sense for cusped manifolds. Thus this equation is meaningful for such manifolds also; we prove that this true in Chapter 6. Roughly speaking, our proof will consist in approximating the manifold M by the compact manifolds $\{M_{p/q}\}$ obtained by performing Dehn fillings on M. Then we will get a formula relating $\mathcal{T}_{2k+1}(M)$ and $\mathcal{T}_{2k+1}(M_{p/q})$ in Chapter 4. This will be done using a Mayer-Vietoris argument. As a by-product of this formula, the behaviour of the higher-dimensional Reidemeister torsion invariants under Dehn filling will be established as well.

The other thing we must take into account concerns the limit of the Ruelle zeta functions of the manifolds $M_{p/q}$ as (p, q) goes to infinity. Our main tool to deal with this will be the continuity of the complex-length spectrum, which we briefly discuss now.

Definition. The *prime complex-length spectrum* of M, denoted $\mu_{\mathrm{sp}}\,M$, is the measure on $\mathbf{C}$ defined by

$$\mu_{\mathrm{sp}}\,M = \sum_{\varphi \in \mathcal{PC}(M)} \delta_{e^{\lambda(\varphi)}},$$

where λ is the complex-length function of M, and δ_x denotes the Dirac measure centered at x. In other words, $\mu_{\mathrm{sp}}\,M$ is the image measure of the counting measure in $\mathcal{PC}(M)$ under the exponential of the complex-length function.

Remark. The complex-length spectrum is usually regarded as a collection of complex numbers and multiplicities. This is of course equivalent to our definition; however, we think that regarding it as a measure puts some questions in a natural context.

We can consider the prime complex-length spectrum as a map from $\mathcal{M}$, the set of complete oriented hyperbolic 3-manifolds of finite volume, to $M(\mathbf{C}\setminus\overline{D})$, the set of measures on the exterior of the unit disc $\overline{D}$. Both spaces are endowed with natural topologies: the former with the geometric topology, and the latter with the topology of weak convergence. Using standard techniques from hyperbolic geometry we will prove the continuity of this map in Chapter 5.

Theorem. *The map $\mu_{\mathrm{sp}}\colon\mathcal{M}\to M(\mathbf{C}\setminus\overline{D})$ which assigns to every finite volume complete oriented hyperbolic 3-manifold its complex-length spectrum is continuous.*

With this formalism, Equation (2) can be expressed in terms of the complex-length spectrum measure. Using some complex analysis, we will prove that if we know all the values $\{|\mathcal{T}_{2k+1}(M)|\}_{k\geq N}$, for some $N\geq 4$, then we also know the values of the following integrals

$$M_k=\int_{|z|>1}(z^{-k}+\bar{z}^{-k})\,\mathrm{d}\,\mu_{\mathrm{sp}}\,M(z),\quad k\geq N.$$

Using the Cauchy transform we will prove that for this kind of measure this information is enough to recover the measure up to complex conjugation, that is, we do not know $\mu_{\mathrm{sp}}\,M$ but

$$\mu_{\mathrm{sp}}\,M+\overline{\mu_{\mathrm{sp}}\,M},$$

where $\overline{\mu_{\mathrm{sp}}\,M}$ denotes the image measure of $\mu_{\mathrm{sp}}\,M$ under complex conjugation. As a consequence, we will obtain the following result.

Theorem. *Let M be an oriented complete hyperbolic 3-manifold of finite volume. For all $N\geq 4$, the sequence of values $\{|\mathcal{T}_{2k+1}(M)|\}_{k\geq N}$ determines the complex-length spectrum of M up to complex conjugation.*

Remark. This theorem may be regarded as a geometric interpretation of the information encoded in the higher-dimensional Reidemeister torsion invariants.

As a particular case, if M admits an orientation-reversing isometry (this is for instance the case of the complement of the figure eight knot), then $\overline{\mu_{\mathrm{sp}}\,M}=\mu_{\mathrm{sp}}\,M$, and hence the sequence $\{|\mathcal{T}_{2k+1}(M)|\}_{k\geq N}$ determines the complex-length spectrum completely.

Using Wotzke's Theorem we obtain the following corollary of the above theorem.

Corollary. *Let M be an oriented* compact *hyperbolic 3-manifold. Knowing the invariants $|\mathcal{T}_{2k+1}(M)|$ for all $k\geq N\geq 4$ is equivalent to knowing the complex-length spectrum of M up to complex conjugation.*

Our last result concerns the behaviour of $\tau(M;\rho_n)$ under mutation. More concretely, if K is a hyperbolic knot in S^3 and K^τ is a mutant of K, then we have the following theorem.

Theorem. *Let μ be a meridian of K, and consider ρ and ρ^τ lifts of the holonomy representation of $\pi_1(S^3\setminus K)$ and $\pi_1(S^3\setminus K^\tau)$ respectively such that* $\operatorname{trace}\rho(\mu)=\operatorname{trace}\rho^\tau(\mu)$. *Then the following equality of signed refined holds:*

$$\tau(S^3\setminus K;\rho)=\tau(S^3\setminus K^\tau;\rho^\tau)\in\mathbf{C}.$$

Remark. It is worth noting that the above theorem is no longer true for $n > 2$, that is $\tau(S^3 \setminus K; \rho_n) \neq \tau(S^3 \setminus K^\tau; \rho_n^\tau)$ in general. For $n = 4, 6$ the Kinoshita-Terasaka knot and the Conway knot provide an example of this fact, see Chapter 8. Thus for $n > 2$ the higher-dimensional Reidemeister torsion invariants may be used to distinguish mutant knots.

Part I

Twisted cohomology for hyperbolic three-manifolds

Chapter 1

Spin-hyperbolic three-manifolds

The aim of this chapter is to review and establish some facts and constructions concerning a spin-hyperbolic 3-manifold. The definition of the object under consideration is quite obvious.

Definition. A *spin-hyperbolic* 3-manifold is a pair (M, η) where M is an oriented hyperbolic 3-manifold and η a spin structure on M.

The first section reviews the relation between spin structures and lifts of the holonomy representation; although this material is well known (see for instance [Cul86]), we think it is worth to expose it in an elementary and self-contained way. In the second section we give the definition of the n-dimensional canonical representation of a spin-hyperbolic 3-manifold; some basic results about irreducible finite-dimensional complex representations of $\mathrm{SL}(2, \mathbf{C})$ are also recalled.

1.1 Lifts of the holonomy representation

Let M be a connected, oriented, hyperbolic 3-manifold which is not necessarily complete. We will use the following definition of a spin structure, see [Kir89]. The SO(3)–principal bundle of orthonormal positively oriented frames on M is denoted by $P_{\mathrm{SO}(3)}M$.

Definition. A *spin structure* on M is a (double) cover of $P_{\mathrm{SO}(3)}M$ by a Spin(3)–principal bundle over M.

The above definition is equivalent to say that a spin structure on M is a double cover of $P_{\mathrm{SO}(3)}M$ such that the preimage of any fiber of $P_{\mathrm{SO}(3)}M$ is connected. One can deduce from this observation that there is a natural identification between the set of spin structures on M and the following set:

$$\left\{\alpha \in \mathrm{H}^1(P_{\mathrm{SO}(3)}M; \mathbf{Z}/2\mathbf{Z}) \mid i^*(\alpha) = 1 \in \mathrm{H}^1(\mathrm{SO}(3); \mathbf{Z}/2\mathbf{Z})\right\}.$$

On the other hand, the hyperbolic structure of M defines a canonical flat $\mathrm{Isom}^+\,\mathbf{H}^3$–principal bundle over M, see [Thu97]. Let us recall how it is defined. Let $\mathbf{H}^3$ be hyperbolic space of dimension three with a fixed orientation. Consider an $(\mathrm{Isom}^+\,\mathbf{H}^3, \mathbf{H}^3)$–atlas on M

defining the hyperbolic structure. Thus we have local charts $\phi_i \colon U_i \to \mathbf{H}^3$ covering M such that the changes of coordinates are restrictions of orientation-preserving isometries of $\mathbf{H}^3$. We can assume that the local charts preserve the fixed orientations on both M and $\mathbf{H}^3$. Let ψ_{ij} be the change of coordinates from (ϕ_j, U_j) to (ϕ_i, U_i), that is,

$$\psi_{ij} \colon U_i \cap U_j \to \operatorname{Isom}^+ \mathbf{H}^3, \quad \psi_{ij} \circ \phi_j = \phi_i.$$

The analyticity of the elements of $\operatorname{Isom}^+ \mathbf{H}^3$ implies that ψ_{ij} is a locally constant map. Since these maps also satisfy the cocycle condition $\psi_{ij} \circ \psi_{jk} = \psi_{ik}$, they define a flat $\operatorname{Isom}^+ \mathbf{H}^3$–principal bundle over M,

$$\operatorname{Isom}^+ \mathbf{H}^3 \to P_{\operatorname{Isom}^+ \mathbf{H}^3} M \xrightarrow{\pi} M.$$

Let us fix a base point $p \in M$. Given $u \in P_{\operatorname{Isom}^+ \mathbf{H}^3} M$ with $\pi(u) = p$, it makes sense to consider the holonomy representation of this principal bundle,

$$\operatorname{Hol}_u \colon \pi_1(M, p) \to \operatorname{Isom}^+ \mathbf{H}^3.$$

By definition, if $\sigma \colon [0,1] \to M$ is a loop based at p, $\operatorname{Hol}_u(\sigma)$ is the unique element of $\operatorname{Isom}^+ \mathbf{H}^3$ such that

$$\tilde{\sigma}(1) \cdot \operatorname{Hol}_u(\sigma) = \tilde{\sigma}(0),$$

where $\tilde{\sigma}(t)$ is the horizontal lift of $\sigma(t)$ starting at u. It can be checked that this holonomy is, up to a conjugation, the same as the one given in terms of the developing map. In other words, for some suitable initial choices, we have $\operatorname{Hol}_u = \operatorname{Hol}_M$.

In what follows, we will identify $\operatorname{Isom}^+ \mathbf{H}^3$ with $\operatorname{PSL}(2, \mathbf{C})$.

Proposition 1.1.1. *There is a canonical one-to-one correspondence between the following sets:*

1. *The set of covers of* $P_{\operatorname{PSL}(2,\mathbf{C})} M$ *by* $\operatorname{SL}(2, \mathbf{C})$*–principal bundles over* M*.*

2. *The set of lifts of* Hol_M *to* $\operatorname{SL}(2, \mathbf{C})$*.*

Proof. Let us assume that we have chosen a base point $u \in P_{\operatorname{PSL}(2,\mathbf{C})} M$ with $\pi(u) = p \in M$, such that $\operatorname{Hol}_u = \operatorname{Hol}_M$. Let $P_{\operatorname{SL}(2,\mathbf{C})} M$ be an $\operatorname{SL}(2, \mathbf{C})$–principal bundle over M covering $P_{\operatorname{PSL}(2,\mathbf{C})} M$. Take one of the two points $\tilde{u} \in P_{\operatorname{SL}(2,\mathbf{C})} M$ that projects to u, and consider the corresponding holonomy representation $\operatorname{Hol}_{\tilde{u}}$. It is clear that $\operatorname{Hol}_{\tilde{u}}$ is a lift of Hol_u; moreover, it is independent of the choice of the base point $\tilde{u}$, since the other choice is obtained by conjugating it by $-\operatorname{Id} \in \operatorname{SL}(2, \mathbf{C})$. This gives a well defined correspondence between the set of covers of $P_{\operatorname{PSL}(2,\mathbf{C})} M$ by $\operatorname{SL}(2, \mathbf{C})$–principal bundles over M and the set of lifts of Hol_M to $\operatorname{SL}(2, \mathbf{C})$. Finally, this correspondence is one-to-one, as we can recover the flat bundle from its holonomy representation. $\square$

Next, we want to embed the frame bundle $P_{\operatorname{SO}(3)} M$ into $P_{\operatorname{PSL}(2,\mathbf{C})} M$. To that end, identify $\operatorname{PSL}(2, \mathbf{C})$ with $P_{\operatorname{SO}(3)} \mathbf{H}^3$ by fixing a positively oriented frame $R_O \in P_{\operatorname{SO}(3)} \mathbf{H}^3$ based at $O \in \mathbf{H}^3$. Notice that this gives a concrete embedding of $\operatorname{SO}(3)$ into $\operatorname{PSL}(2, \mathbf{C})$ as the isometry group of the tangent space at O with fixed basis R_O. Now let $u \in P_{\operatorname{PSL}(2,\mathbf{C})} M$

and $p = \pi(u)$. A local chart (ϕ_j, U_j) of the hyperbolic structure containing p gives a local trivialization $U_j \times \mathrm{PSL}(2, \mathbf{C})$ of $P_{\mathrm{PSL}(2,\mathbf{C})}M$, with respect to which the point u is written as a pair $(p, g) \in U_j \times \mathrm{PSL}(2, \mathbf{C})$. We will say that u *is based at* $p \in M$, if $g \in \mathrm{PSL}(2, \mathbf{C}) \cong P_{\mathrm{SO}(3)}\mathbf{H}^3$ is a frame based at $\phi_j(p)$. It can be checked that this definition does not depend on the choice of the local chart (ϕ_j, U_j), and that we have the following identification:

$$\left\{u \in P_{\mathrm{PSL}(2,\mathbf{C})}M \mid u \text{ is a frame based at } \pi(u)\right\} \cong P_{\mathrm{SO}(3)}M.$$

Thus we have obtained a concrete embedding $P_{\mathrm{SO}(3)}M \hookrightarrow P_{\mathrm{PSL}(2,\mathbf{C})}M$, which is easily seen to be compatible with the actions of the respective structural groups $\mathrm{SO}(3)$ and $\mathrm{PSL}(2, \mathbf{C})$. In other words, we have an explicit reduction of the structural group with respect to the fixed embedding $\mathrm{SO}(3) \subset \mathrm{PSL}(2, \mathbf{C})$. Although this embedding depends on the choices that we have done, it must be pointed out that its homotopy class does not.

Proposition 1.1.2. *There is a canonical one-to-one correspondence between the following sets:*

1. *The set of covers of* $P_{\mathrm{PSL}(2,\mathbf{C})}M$ *by* $\mathrm{SL}(2, \mathbf{C})$*–principal bundles over* M*.*
2. *The set of spin structures on* M*.*

Proof. The set of spin structures on M is canonically identified with the following set:

$$\left\{\alpha \in \mathrm{H}^1(P_{\mathrm{SO}(3)}M; \mathbf{Z}/2\mathbf{Z}) \mid i^*(\alpha) = 1 \in \mathrm{H}^1(\mathrm{SO}(n); \mathbf{Z}/2\mathbf{Z})\right\}.$$

The same argument shows that the set of covers of $P_{\mathrm{PSL}(2,\mathbf{C})}M$ by $\mathrm{SL}(2, \mathbf{C})$–principal bundles over M is identified with

$$\left\{\alpha \in \mathrm{H}^1(P_{\mathrm{PSL}(2,\mathbf{C})}M; \mathbf{Z}/2\mathbf{Z}) \mid i^*(\alpha) = 1 \in \mathrm{H}^1(\mathrm{SL}(2, \mathbf{C}); \mathbf{Z}/2\mathbf{Z})\right\}.$$

The result then follows from the fact that the map $P_{\mathrm{SO}(3)}M \hookrightarrow P_{\mathrm{Isom}^+\mathbf{H}^3}M$ defined above, whose homotopy class is canonical, is a homotopy equivalence, for $\mathrm{SO}(3) \simeq \mathrm{PSL}(2, \mathbf{C})$. □

Corollary 1.1.3. *The holonomy representation of a hyperbolic* 3*-manifold can be lifted to* $\mathrm{SL}(2, \mathbf{C})$*. The number of such lifts is* $|\,\mathrm{H}^1(M; \mathbf{Z}/2\mathbf{Z})|$.

Proof. An oriented 3-manifold admits $|\,\mathrm{H}^1(M; \mathbf{Z}/2\mathbf{Z})|$ different spin structures. □

1.2 Positive spin structures

Let M be a complete, oriented, hyperbolic 3-manifold of finite volume. Thus M is the interior of a compact manifold whose boundary consists of tori $T_1, \ldots, T_k$.

Definition. We will say that a spin structure η on M is *positive* on T_i if for all $g \in \pi_1 T_i$ we have:

$$\operatorname{trace} \mathrm{Hol}_{(M,\eta)}(g) = +2.$$

Otherwise, we will say that η is nonpositive on T_i.

The aim of this section is to prove the existence of spin structures that are nonpositive on each torus T_i. Let T^2 be one of these tori. We can assume that T^2 is a horospheric cross-section, and that

$$\mathrm{Hol}_M(\pi_1 T^2) = \left\langle \left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right], \left[\begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}\right] \right\rangle < \mathrm{PSL}(2, \mathbf{C}), \qquad \text{with } \operatorname{Im}\tau > 0.$$

Let $P_{\mathrm{SO}(3)}T^2 \subset P_{\mathrm{PSL}(2,\mathbf{C})}T^2$ be the restriction of $P_{\mathrm{SO}(3)}M \subset P_{\mathrm{PSL}(2,\mathbf{C})}M$ to T^2, and let Hol_{T^2} be the restriction of Hol_M to $\pi_1 T^2$. Using the Euclidean structure of T^2 and the outward normal vector of T^2, we can construct a section s of the bundle $P_{\mathrm{SO}(3)}T^2$ that is canonical up homotopy as follows: fix $p \in T^2$ and define $s(p) \in P_{\mathrm{SO}(3)}T^2$ as any frame based at p whose third component is equal to the outward normal vector at p; for all $q \in T^2$, define $s(q)$ as the parallel transport (with respect to the Euclidean structure) of $s(p)$ along a curve joining p and q on T^2. This yields a well defined section which is canonical up to homotopy. Thus we have a canonical trivialization $P_{\mathrm{SO}(3)}T^2 \cong T^2 \times \mathrm{SO}(3)$, and hence a distinguished spin structure $T^2 \times \mathrm{Spin}(3)$. All other spin structures arise as quotients of the form

$$\eta_\alpha = \left(\widetilde{T^2} \times \mathrm{Spin}(3)\right) / \pi_1 T^2,$$

where $\alpha \in \mathrm{H}^1(T^2; \mathbf{Z}/2\mathbf{Z}) = \mathrm{Hom}(\pi_1 T^2; \{\pm 1\})$, and the action of $\sigma \in \pi_1 T^2$ on $\mathrm{Spin}(3)$ is by multiplication by $\alpha(\sigma)\,\mathrm{Id}$. Therefore, spin structures of $P_{\mathrm{SO}(3)}T^2$ are in canonical one-to-one correspondence with $\mathrm{H}^1(T^2; \mathbf{Z}/2\mathbf{Z})$.

A similar argument proves the following result.

Lemma 1.2.1. *Let $\alpha \in \mathrm{H}^1(T^2; \mathbf{Z}/2\mathbf{Z}) = \mathrm{Hom}(\pi_1 T^2; \{\pm 1\})$, η_α be the associated spin structure on T^2, and $\mathrm{Hol}_{(T^2,\eta_\alpha)}$ be the corresponding lift of the holonomy representation. Then we have*

$$\alpha(\sigma) = \operatorname{sgn} \operatorname{trace} \mathrm{Hol}_{(T^2,\eta_\alpha)}(\sigma), \quad \textit{for all } \sigma \in \pi_1 T^2.$$

Now we can prove the existence of spin structures that are nonpositive on each torus T_i.

Proposition 1.2.2. *Let M be an oriented, complete, hyperbolic 3-manifold of finite volume. For each boundary component T_i take a closed simple curve γ_i. Then there exists a spin structure η on M such that*

$$\operatorname{trace} \mathrm{Hol}_{(M,\eta)}([\gamma_i]) = -2,$$

where $[\gamma_i]$ denotes the conjugacy class of $\pi_1(M, p)$ defined by γ_i.

Proof. Let N be the manifold obtained by performing a Dehn filling along each of the curves $\{\gamma_i\}$. Fix a spin structure η on N. We claim that the restriction of η to M gives the required spin structure.

Assume that γ is one of the curves γ_i, and that it is contained in a horospheric cross-section T^2. We can assume also that γ is a closed geodesic with respect to the Euclidean structure of T^2. Let $P_{\mathrm{Spin}(3)}T^2 \to P_{\mathrm{SO}(3)}T^2$ be the corresponding Spin(3)–bundle over T^2 defined by η, and $\alpha \in \mathrm{H}^1(T^2; \mathbf{Z}/2\mathbf{Z})$ the associated cohomology class. Consider the canonical section $s\colon T^2 \to P_{\mathrm{SO}(3)}T^2$ constructed above using as starting frame one whose first vector is tangent to γ. Then the closed curve $s \circ \gamma$ can be lifted to $P_{\mathrm{Spin}(3)}T^2$ if and only if $\alpha(\gamma) = 1$. On the other hand, if $s \circ \gamma$ could be lifted to $P_{\mathrm{Spin}(3)}T^2$, then such a lift could be extended to the added disk bounding γ (there is no obstruction in doing it because $\pi_1 \mathrm{Spin}(3) = \{1\}$), and hence $s \circ \gamma$ could be extended to that disk, which is not possible by construction. Thus $\alpha(\gamma) = -1$, and the preceding lemma implies the result. □

As a corollary of the proof of the Proposition 1.2.2, we obtain the following result.

Corollary 1.2.3. *Let $\gamma \subset \partial M$ be a simple closed curve non-homotopically trivial in ∂M, and M_γ be the manifold obtained by performing a Dehn filling along γ. A spin structure η on M extends to a spin structure on M_γ if and only if*

$$\operatorname{trace} \mathrm{Hol}_{(M,\eta)}(\gamma) = -2.$$

The following corollary of Proposition 1.2.2 gives a sufficient condition to guarantee the existence of nonpositive spin structures.

Corollary 1.2.4. *Assume that for each boundary component T_i of M, the map induced by the inclusion,*

$$\mathrm{H}_1(T_i; \mathbf{Z}/2\mathbf{Z}) \to \mathrm{H}_1(M; \mathbf{Z}/2\mathbf{Z}),$$

has non-trivial kernel. Then all spin structures on M are nonpositive on each T_i. In particular, if M has only one cusp, all spin structures on M are nonpositive on each T_i.

Proof. If the hypothesis holds, then for each T_i there exists a closed simple curve $\gamma_i \in T_i$ that is zero in $\mathrm{H}_1(M; \mathbf{Z}/2\mathbf{Z})$. Take a spin structure on M such that $\operatorname{trace} \mathrm{Hol}_{(M,\eta)}([\gamma_i]) = -2$, for all γ_i. Now let η' be another spin structure on M, and $\alpha \in \mathrm{H}^1(M; \mathbf{Z}/2\mathbf{Z})$ be the cohomology class relating η and η'. Then, using multiplicative notation, we have

$$\mathrm{Hol}_{(M,\eta')}(\gamma_i) = \alpha(\gamma_i)\, \mathrm{Hol}_{(M,\eta)}(\gamma_i).$$

Since $[\gamma] \in \mathrm{H}_1(M; \mathbf{Z}/2\mathbf{Z})$ is zero, we have $\alpha(\gamma_i) = 1$, and hence $\mathrm{Hol}_{(M,\eta')}(\gamma_i)$ has trace -2, as we wanted to prove. The rest of the result follows from the fact that in any compact 3-manifold M the map

$$i_* \colon \mathrm{H}_1(\partial M; \mathbf{Z}/2\mathbf{Z}) \to \mathrm{H}_1(M; \mathbf{Z}/2\mathbf{Z})$$

induced by the inclusion $i \colon \partial M \to M$ has a non-trivial kernel. □

1.3 The canonical n-dimensional representation

Irreducible, complex, finite-dimensional representations of $\mathrm{SL}(2, \mathbf{C})$ are well known: for all n there is exactly one irreducible representation of dimension n which is given by $V_n = \mathrm{Sym}^{n-1} V_2$, the $(n-1)$-th symmetric power of the standard representation $V_2 = \mathbf{C}^2$. We use the convention that $\mathrm{Sym}^0 V_2$ is the base field.

Definition. Let (M, η) be a spin-hyperbolic 3-manifold with spin holonomy representation $\mathrm{Hol}_{(M,\eta)}$. We define the *canonical n-dimensional representation* of M as the composition of $\mathrm{Hol}_{(M,\eta)}$ with V_n.

The decomposition into irreducible factors of the tensor product of two representations of $\mathrm{SL}(2,\mathbf{C})$ is given by the Clebsch-Gordan formula (see [FH91, §11.2]).

Theorem 1.3.1 (Clebsch-Gordan formula). *For nonnegative integers n and k we have:*

$$V_n \otimes V_{n+k} = \bigoplus_{i=0}^{n-1} V_{2(n-i)+k-1}.$$

Lemma 1.3.2. *Let V be a finite-dimensional complex representation of $\mathrm{SL}(2,\mathbf{C})$. Then there exists a nondegenerate $\mathbf{C}$–bilinear invariant pairing*

$$\phi\colon V \times V \to \mathbf{C}.$$

Moreover, if V is irreducible, then there exists, up to multiplication by nonzero scalars, a unique $\mathbf{C}$–bilinear invariant pairing, which a fortiori *is non-degenerate.*

Proof. On one hand, the natural pairing between V^* and V always yields a nondegenerate $\mathbf{C}$–bilinear invariant map. From the classification of irreducible representations of $\mathrm{SL}(2,\mathbf{C})$, we deduce that V^* is isomorphic to V, and hence the first part of the lemma is proved. On the other hand, invariant bilinear maps are in one-to-one correspondence with fixed vectors of $V^* \otimes V^*$. Thus the second assertion follows from the Clebsch-Gordan formula, which shows that $(V_n \otimes V_n)^* \cong V_n \otimes V_n$ has a unique irreducible factor of dimension 1, on which $\mathrm{SL}(2,\mathbf{C})$ acts trivially. $\square$

Remark. Roughly speaking, the $\mathbf{C}$–bilinear invariant pairing on $V_n = \mathrm{Sym}^{n-1} V_2$ is the $(n-1)$-th symmetric power of the determinant. To be precise, let $\mathrm{S}(V_2)$ be the symmetric algebra on V_2, that is,

$$\mathrm{S}(V_2) = \bigoplus_{i \geq 0} \mathrm{Sym}^i V_2.$$

With respect to a fixed basis (e_1, e_2) of V_2, the determinant is given by:

$$\det = e_1^* \otimes e_2^* - e_2^* \otimes e_1^*,$$

where (e_1^*, e_2^*) is the dual basis of (e_1, e_2). The determinant thus can be regarded as an element of $\mathrm{S}(V^*) \otimes \mathrm{S}(V^*)$. This latter vector space is an algebra in a natural way, and hence it makes sense to consider the power $\det^n$. Notice that $\det^n \in \mathrm{Sym}^n(V^*) \otimes \mathrm{Sym}^n(V^*)$, so $\det^n$ defines a bilinear pairing on V_{n+1}. On the other hand, it can be checked that we have:

$$g \cdot \det^n = (g \cdot \det)^n, \quad \text{for all } g \in \mathrm{SL}(2,\mathbf{C}).$$

Hence, $\det^n$ is $\mathrm{SL}(2,\mathbf{C})$–invariant, for so is det, and Lemma 1.3.2 implies that this pairing is nondegenerate. Notice also that $\det^n$ is alternating for n odd and symmetric for n even.

From Lemma 1.3.2 we get the following result (see [Gol86, Sec. 2.2]), which will be used very often in the sequel.

Corollary 1.3.3. *Poincaré duality with coefficients in ρ_n holds.*

Let $\mathrm{Ad}\colon \mathrm{SL}(n,\mathbf{C}) \to \mathrm{Aut}(\mathfrak{sl}(n,\mathbf{C}))$ denote the adjoint representation of $\mathrm{SL}(n,\mathbf{C})$. Composing it with the representation V_n we get a representation

$$\mathrm{SL}(2,\mathbf{C}) \to \mathrm{Aut}(\mathfrak{sl}(n,\mathbf{C})),$$

which makes $\mathfrak{sl}(n,\mathbf{C})$ into an $\mathrm{SL}(2,\mathbf{C})$–module. Next we want to decompose $\mathfrak{sl}(n,\mathbf{C})$ into irreducible factors.

Lemma 1.3.4. *As* $\mathrm{SL}(2,\mathbf{C})$*–modules, we have:*

$$\mathfrak{sl}(n,\mathbf{C}) \cong V_{2n-1} \oplus V_{2n-3} \cdots \oplus V_3.$$

Proof. Consider the action of $\mathrm{SL}(2,\mathbf{C})$ on $\mathfrak{gl}(n,\mathbf{C})$ obtained by composing the n-dimensional representation V_n with the adjoint representation. We have the following isomorphisms of $\mathrm{SL}(2,\mathbf{C})$–modules:

$$V_n \otimes V_n^* \cong \mathfrak{gl}(n,\mathbf{C}) \cong \mathfrak{sl}(n,\mathbf{C}) \oplus \mathbf{C},$$

where the factor $\mathbf{C}$ corresponds to diagonal matrices. The result now follows from the Clebsh-Gordan formula applied to $V_n \otimes V_n^* \cong V_n \otimes V_n$. $\square$

Chapter 2

Vanishing cohomology

Let (M, η) be a spin complete hyperbolic 3-manifold. For $n > 0$, consider its n-dimensional canonical representation ρ_n. The associated flat vector bundle will be denoted by E_{ρ_n}. Let us recall the following definition.

Definition. A hyperbolic 3-manifold M is said to be *topologically finite* if it is the interior of a compact manifold $\overline{M}$.

Remark. By the proof of Marden's conjecture [Ago, CG06], to be topologically finite is equivalent to say that $\pi_1(M)$ is finitely generated.

In what follows, we will assume that M is non-elementary, which means, in the context of three-manifolds, that its holonomy is an irreducible representation in $\mathrm{PSL}(2, \mathbf{C})$. The following results are still true for elementary manifolds with a straightforward proof because of the simplicity of these manifolds (cf. Lemma 2.2.3).

Theorem 2.0.5. *Let (M, η) be a complete, non-elementary, spin-hyperbolic* 3*-manifold that is topologically finite, and let $n \geq 2$. Then the inclusion $\partial\overline{M} \subset \overline{M}$ induces an injection,*

$$\mathrm{H}^1(M; E_{\rho_n}) \hookrightarrow \mathrm{H}^1(\partial\overline{M}; E_{\rho_n}),$$

with $\dim \mathrm{H}^1(M; E_{\rho_n}) = \frac{1}{2} \dim \mathrm{H}^1(\partial\overline{M}; E_{\rho_n})$*, and an isomorphism*

$$\mathrm{H}^2(M; E_{\rho_n}) \cong \mathrm{H}^2(\partial\overline{M}; E_{\rho_n}).$$

The above theorem implies that (M, η) is ρ_n-acyclic if and only if so is $\partial\overline{M}$. Therefore, by Poincaré duality and an Euler characteristic argument, this can only happen if $\partial\overline{M}$ is a union of tori $T_1, \ldots, T_k$, and $\mathrm{H}^0(T_i; E_{\rho_n}) = 0$ for each torus T_i. An easy computation shows that for $n \geq 3$ odd $\mathrm{H}^0(T_i; E_{\rho_n})$ is never trivial (see Section 2.2), whereas for $n \geq 2$ even $\mathrm{H}^0(T_i; E_{\rho_n})$ is zero if and only if η is nonpositive on T_i.

Definition. Let M be an oriented, complete, hyperbolic 3-manifold of finite volume. A spin structure η on M will be called *acyclic* if the cohomology groups $\mathrm{H}^*(M; \mathrm{Hol}_{(M,\eta)})$ are all trivial, or, equivalently, if η is nonpositive on all connected components of $\partial\overline{M}$.

As we have said, the existence of spin acyclic structures is equivalent to the existence of spin structures that are nonpositive on each T_i. Thus, by Corollary 1.2.4, all spin structures are acyclic on one-cusped manifolds; hence, Theorem 2.0.5 gives the following result.

Theorem 2.0.6. *Let (M, η) be a complete spin-hyperbolic 3-manifold of* finite volume *with a* single cusp. *Then for $k \geq 1$ we have*

$$\mathrm{H}^*(M; E_{\rho_{2k}}) = 0.$$

Theorem 2.0.5 has applications to infinitesimal rigidity. The space of infinitesimal deformations of ρ_n is isomorphic to $\mathrm{H}^1(M; E_{\mathrm{Ad}\circ\rho_n})$, where

$$\mathrm{Ad}\colon \mathrm{SL}(n, \mathbf{C}) \to \mathrm{Aut}(\mathfrak{sl}(n, \mathbf{C}))$$

is the adjoint representation.

The following theorem is an infinitesimal rigidity result for ρ_n in $\mathrm{SL}(n, \mathbf{C})$ relative to the boundary. Its proof, which uses the decomposition of the representation $\mathfrak{sl}(n, \mathbf{C})$ into irreducible factors given in Lemma 1.3.4, will be given in Section 2.3.

Theorem 2.0.7. *Let (M, η) be a complete, hyperbolic, non-elementary spin 3-manifold that is topologically finite. If $\partial\overline{M}$ is the union of k tori and l surfaces of genus $g_1, \ldots, g_l \geq 2$, and $n \geq 2$, then*

$$\dim_{\mathbf{C}} \mathrm{H}^1(M; E_{\mathrm{Ad}\circ\rho_n}) = k(n-1) + \sum (g_i - 1)(n^2 - 1).$$

In particular, if M is closed then $\mathrm{H}^1(M; E_{\mathrm{Ad}\circ\rho_n}) = 0$. In addition, all nontrivial elements in $\mathrm{H}^1(M; E_{\mathrm{Ad}\circ\rho_n})$ are nontrivial in $\mathrm{H}^1(\partial\overline{M}; E_{\mathrm{Ad}\circ\rho_n})$ and have no L^2-representative.

When $n = 2$, this is Weil's infinitesimal rigidity in the compact case, and Garland's L^2-infinitesimal rigidity in the noncompact case. This has been generalized to cone three-manifolds by Hodgson-Kerckhoff [HK98], Weiss [Wei05] and Bromberg [Bro04].

Let $X(M, \mathrm{SL}(n, \mathbf{C}))$ be the variety of characters of $\pi_1(M)$ in $\mathrm{SL}(n, \mathbf{C})$. The character of ρ_n is denoted by χ_{ρ_n}. From the previous theorem and standard results on the variety of characters, we deduce the following result (for $n = 2$, this is Theorem 8.44 of Kapovich [Kap01]).

Theorem 2.0.8. *Let M be a topologically finite, hyperbolic, non-elementary and orientable 3-manifold as in Theorem 2.0.7. If $n \geq 2$, then the character χ_{ρ_n} is a smooth point of $X(M, \mathrm{SL}(n, \mathbf{C}))$ with tangent space $\mathrm{H}^1(M; E_{\mathrm{Ad}\circ\rho_n})$.*

This chapter is organized as follows. Section 2.1 is devoted to Raghunathan's vanishing theorem and to Theorem 2.0.5. Theorem 2.0.6 is proved in Section 2.2, where we compute the cohomology of the ends, and discuss the existence of acyclic spin structures. Section 2.3 deals with applications to infinitesimal and local rigidity, in particular, we prove Theorems 2.0.7 and 2.0.8. Appendix A reviews some results about principal bundles that are required in Section 2.1.

2.1 Raghunathan's vanishing theorem

The aim of this section is to prove Theorem 2.1.1 stated below. This theorem is a particular case of a theorem due to Raghunathan [Rag65]. Before stating it, let us recall some well known facts.

The homogeneous manifold $\mathrm{SL}(2,\mathbf{C})/\mathrm{SU}(2)$ is endowed with a Riemannian structure using the Killing form on $\mathfrak{sl}(2,\mathbf{C})$ (see section 2.1.1 for details), which makes this space isometric to hyperbolic 3-dimensional space $\mathbf{H}^3$.

Let Γ be a discrete torsion-free subgroup of $\mathrm{SL}(2,\mathbf{C})$, and $M = \Gamma\backslash\mathbf{H}^3$ be the corresponding complete hyperbolic manifold. Let V be a finite-dimensional representation of $\mathrm{SL}(2,\mathbf{C})$, $\rho\colon \Gamma \to \mathrm{SL}(V)$ be the induced representation and E_ρ be the associated flat vector bundle over M. The space of E_ρ–valued differential forms on M will be denoted by $\Omega^*(M;E_\rho)$. An $\mathrm{SU}(2)$–invariant inner product on V yields a well defined inner metric on E_ρ, and hence on $\Omega^*(M;E_\rho)$ as well. In particular, it makes sense to talk about L^2-forms in $\Omega^*(M;E_\rho)$ as those which are square integrable.

Theorem 2.1.1 ([Rag65])**.** *Let Γ be a discrete torsion-free subgroup of* $\mathrm{SL}(2,\mathbf{C})$*. Let V be an irreducible, finite-dimensional, complex representation of* $\mathrm{SL}(2,\mathbf{C})$*, and consider the induced representation,*

$$\rho\colon \Gamma \to \mathrm{SL}(V).$$

Then, for $p = 1,2$, every closed L^2-form in $\Omega^p(\Gamma\backslash\mathbf{H}^3;E_\rho)$ is exact.

As an immediate corollary of Theorem 2.1.1 we get a particular case of Raghunathan's vanishing theorem.

Corollary 2.1.2 ([Rag65])**.** *Let M be a* closed *hyperbolic three-manifold. If V is an irreducible, finite-dimensional, complex representation of* $\mathrm{SL}(2,\mathbf{C})$*, then*

$$\mathrm{H}^1(M;E_\rho) = 0.$$

Remark. Raghunathan's theorem applies to lattices of a semisimple Lie group G and a broader family of representations, see [Rag65].

From Theorem 2.1.1 we can easily deduce Theorem 2.0.5.

Proof of Theorem 2.0.5. For some discrete torsion-free subgroup Γ of $\mathrm{SL}(2,\mathbf{C})$, we have

$$(M,\eta) = \Gamma\backslash\mathrm{SL}(2,\mathbf{C})/\mathrm{SU}(2).$$

If M is compact then the result is clear from Theorem 2.1.1, so we can assume M is noncompact. The space $\mathrm{H}^p(\overline{M},\partial\overline{M};E_\rho)$ can be identified with the cohomology group of compactly supported E_ρ–valued p-forms on M; hence, an element $[\alpha] \in \mathrm{H}^p(\overline{M},\partial\overline{M};E_\rho)$ is represented by a closed form α on M with compact support. Therefore, Theorem 2.1.1 implies that for $p = 1,2$ the image of $[\alpha]$ under the map $\mathrm{H}^p(\overline{M},\partial\overline{M};E_\rho) \to \mathrm{H}^p(M;E_\rho)$ induced by the inclusion is zero. The theorem now follows from the long exact sequence of the pair, and

Poincaré duality. Indeed the long exact sequence of the pair $(\overline{M}, \partial\overline{M})$ splits into two short exact sequences

$$0 \to \mathrm{H}^1(M; E_{\rho_n}) \to \mathrm{H}^1(\partial\overline{M}; E_{\rho_n}) \to \mathrm{H}^2(\overline{M}, \partial\overline{M}; E_{\rho_n}) \to 0, \tag{2.1}$$

$$0 \to \mathrm{H}^2(M; E_{\rho_n}) \to \mathrm{H}^2(\partial\overline{M}; E_{\rho_n}) \to \mathrm{H}^3(\overline{M}, \partial\overline{M}; E_{\rho_n}) \to 0. \tag{2.2}$$

Poincaré duality yields:

$$\dim \mathrm{H}^1(M; E_{\rho_n}) = \dim \mathrm{H}^2(\overline{M}, \partial\overline{M}; E_{\rho_n}),$$

and from sequence (2.1) we deduce the first assertion of Theorem 2.0.5. On the other hand, Lemma 2.2.5, to be proved later, states that $\dim_{\mathbf{C}} \mathrm{H}^0(M; E_{\rho_n}) = 0$. Poinacré duality then shows that sequence (2.2) yields an isomorphism,

$$0 \to \mathrm{H}^2(M; E_{\rho_n}) \to \mathrm{H}^2(\partial\overline{M}; E_{\rho_n}) \to 0.$$

□

The proof given by Raghunathan in [Rag65] of its vanishing theorem is based in two facts, which we briefly discuss now.

The first one is the following theorem due to Andreotti and Vesentini [AV65]. Although the original theorem is for complex manifolds, there is an adaptation by Garland [Gar67, Thm. 3.22] to the real case.

Theorem 2.1.3 (Andreotti-Vesentini [AV65], Garland [Gar67])**.** *Let E be a flat vector bundle over a complete Riemannian manifold M. Assume that E is endowed with an inner product, and there exists $c > 0$ such that for every $\alpha \in \Omega^p(M; E)$ with compact support we have:*

$$(\Delta\alpha, \alpha) \geq c(\alpha, \alpha),$$

where $(\,,)$ denotes the inner product on the space of E–valued forms. Then every square-integrable closed p-form is exact.

The second point in Raghunathan's proof is the work by Matsushima and Murakami concerning the theory of harmonic forms on a locally symmetric manifold [MM63]. One of the goals of that work is the proof of a Weitzenböck formula for the Laplacian. This allows to prove the strong-positivity hypothesis of the Laplacian required in Theorem 2.1.3 by establishing the positivity of a certain linear operator H, which is defined on a finite-dimensional space, see Subsection 2.1.1. Although this is an important technical reduction, it remains to prove the positivity of the operator H. Raghunathan was able to prove it for a large family of locally symmetric manifolds and representations (see [Rag65]), which also include the ones we are dealing with.

The rest of this section is divided into two parts. The first one is a review of the work of Matsushima and Murakami concerning the Laplacian of a locally symmetric manifold. The material presented here is almost entirely based on Matsushima-Murakami [MM63], and Raghunathan's book [Rag72]. Although it does not bring in a new conceptual approach, we think that it is worth to review it here for both completeness and to make it more accessible to the non-expert. The proof of Theorem 2.1.1 will be given in Subsection 2.1.2.

2.1.1 Harmonic forms on a locally symmetric manifold

Let G be a connected semi-simple Lie group and $K < G$ be a maximal compact subgroup of G. The respective Lie algebras are denoted by $\mathfrak{g}$ and $\mathfrak{k}$, with the convention that they are Lie algebras of left invariant vector fields.

Let B denote the Killing form of $\mathfrak{g}$. We recall that it is defined by

$$B(V, W) = \mathrm{trace}(\mathrm{ad}_V \circ \mathrm{ad}_W), \quad \text{for } V, W \in \mathfrak{g}.$$

Cartan's criterion implies that B is nondegenerate if, and only if, $\mathfrak{g}$ is semisimple. In that case, we have a canonical decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$, where $\mathfrak{m}$ is the orthogonal complement to $\mathfrak{k}$ with respect to B. This decomposition satisfies the following well known properties: B is negatively-definite on $\mathfrak{k}$, B is positively-definite on $\mathfrak{m}$, $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$ and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$.

The Killing form defines a pseudo-Riemannian metric on G which is invariant under the action of G by right translations and positively-definite (resp. negatively-definite) on $\mathfrak{m}$ (resp. $\mathfrak{k}$). Therefore, the Killing form defines a Riemannian metric on the homogeneous space

$$X = G/K.$$

Notice that G acts on the left on X by orientation preserving isometries. Let Γ be a discrete torsion-free subgroup of G. The quotient $M = \Gamma\backslash X$ is then a Riemannian manifold, which is called a *locally symmetric manifold*.

The quotient map $X \to M$ identifies X with the universal cover of M, and Γ with $\pi_1 M$. Just for notational convenience, we will regard X as a principal bundle over M with structure group Γ.

Remark. In what follows we will use the convention that the action of the structure group on a principal bundle is on the right.

Therefore we must turn the left action of Γ on X into a right action, which is done as usual: if $g \in \Gamma$ and $x \in X$, then $x \cdot g := g^{-1} \cdot x$. We will also regard X as a flat bundle.

Consider the G–principal bundle $P = X \times_\Gamma G$ over M endowed with the flat connection induced from the trivial connection on the product $X \times G$ (see Appendix A for notation). We can embed X on P using the section $X \to X \times G$ whose second coordinate is the identity element, and then projecting it to P. Thus we can think of X as a reduction of the structure group. Obviously, the connection on P is reducible to X, as the horizontal leaves of X are also horizontal leaves of P.

On the other hand, the principal bundle P has a canonical reduction of its structure group from G to K. To get such a reduction, consider the embedding

$$i \colon \Gamma\backslash G \to P \tag{2.3}$$
$$[g] \mapsto [(gK, g)]. \tag{2.4}$$

Denote by Q the image of $\Gamma\backslash G$ under this embedding. It is easily checked that Q is invariant under the bundle action of $K < G$ on P. Hence, Q is the wanted canonical reduction. However, it is worth noticing that the connection defined on P is not reducible to Q; this follows from the fact that the horizontal distribution on P is not tangent to Q (to see this, take a curve on $X \times G$ whose second component is constant; this curve is horizontal on P, and if the horizontal distribution were tangent to Q, then this curve would be contained in Q, which does not happen). Nevertheless, the action of K on $\mathfrak{g}$ preserves the decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$ because $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$. This allows us to apply the following standard result, see [KN96].

Remark. We may identify $\mathfrak{g}$ with the space of vector fields on $\Gamma\backslash G$ that are projections of left invariant vector fields on G. In what follows we will tacitly do this identification.

Proposition 2.1.4. *Let $\eta \in \Omega^1(P; \mathfrak{g})$ be the connection form of the connection defined on P above. Put $\eta = \eta_{\mathfrak{m}} + \eta_{\mathfrak{k}}$, where $\eta_{\mathfrak{m}}$ and $\eta_{\mathfrak{k}}$ are the $\mathfrak{m}$ and $\mathfrak{k}$ components of η respectively. Then we have:*

1. *The restriction of $\eta_{\mathfrak{k}}$ to Q is a connection form on Q.*
2. *Let $\omega \in \Omega^1(\Gamma\backslash G; \mathfrak{g})$ be the left Maurer-Cartan form of G. Then $i^*(\eta) = \omega$, where i is the embedding defined in* (2.3)*.*
3. *The horizontal distribution on $\Gamma\backslash G$ defined by $i^*(\eta_{\mathfrak{k}})$ agrees with the one defined by $\mathfrak{m}$.*

Consider a finite-dimensional linear representation $\rho\colon G \to \mathrm{Aut}(V)$, and the associated vector bundle $E_\rho = X \times_\Gamma V$. This bundle is canonically identified with both $P \times_G V$ and $Q \times_K V$. The flat connection on P defines an exterior covariant differential

$$d_\rho\colon \Omega^r(M; E_\rho) \to \Omega^{r+1}(M; E_\rho).$$

The space $\Omega^r(M; E_\rho)$ consists of sections of $\bigwedge^r T^*M \otimes E_\rho$, and the covariant differential d_ρ is defined using the exterior differential to differentiate forms and the connection on E_ρ to differentiate sections of E_ρ. We can however differentiate forms on M by using the connection that we have on Q, which should be regarded as a "Levi-Civita" connection. Formally, this can be expressed as follows.

Consider the space $\Omega^*_{\mathrm{Hor}}(\Gamma\backslash G; V)^K$ of K–equivariant horizontal V–valued differential forms on $\Gamma\backslash G$ (see Appendix A). The connection defined on Q defines a covariant differential

$$D\colon \Omega^r_{\mathrm{Hor}}(\Gamma\backslash G; V)^K \to \Omega^{r+1}_{\mathrm{Hor}}(\Gamma\backslash G; V)^K.$$

Using the canonical isomorphism between $\Omega^*_{\mathrm{Hor}}(\Gamma\backslash G; V)^K$ and $\Omega^*(M; E_\rho)$ (see Appendix A) we can then define another covariant differential on $\Omega^*(M; E_\rho)$. We want to compare these two differentials. It is easier to perform this comparison in $\Omega^*_{\mathrm{Hor}}(\Gamma\backslash G; V)^K$ rather than in $\Omega^*(M; E_\rho)$. To that end, we transfer the operator d_ρ to an operator D_ρ on $\Omega^*_{\mathrm{Hor}}(\Gamma\backslash G; V)^K$ through the canonical isomorphism. The following proposition relates the operators D and D_ρ. Before stating it, let us introduce the following operator:

$$T\colon \Omega^r_{\mathrm{Hor}}(\Gamma\backslash G; V)^K \to \Omega^{r+1}_{\mathrm{Hor}}(\Gamma\backslash G; V)^K$$
$$\alpha \mapsto \rho(\omega_{\mathfrak{m}}) \wedge \alpha,$$

where $\omega_{\mathfrak{m}}$ is the $\mathfrak{m}$–component of the left Maurer-Cartan form on $\Gamma\backslash G$.

Proposition 2.1.5. *Let α be a form in $\Omega^r_{\mathrm{Hor}}(\Gamma\backslash G;V)^K$. We have the following decomposition*

$$D_\rho\alpha = D\alpha + T\alpha.$$

Proof. The differential covariant on P is given by $d\alpha+\rho(\eta)\wedge\alpha$, see Proposition A.0.4. Hence, if we transfer it to Q via i, we get $D_\rho\alpha = d\alpha + \rho(i^*\eta)\wedge\alpha$, and the proposition follows from the fact that $i^*\eta = \omega$. □

Notation. We will use the following conventions. Let V be a finite-dimensional vector space. If $(e_1,\dots,e_n)$ is a basis for V, then its dual basis will be denoted by $(e^1,\dots,e^n)$. If $A \in \bigotimes^r V^*$ is an r-times covariant tensor, then its components relative to a fixed basis will be denoted as usual by $A_{i_1,\dots,i_r}$. Concerning the exterior product $\bigwedge^* V^*$, we will follow the convention that $e^1\wedge\cdots\wedge e^n$ is the determinant with respect to the basis $(e_1,\dots,e_n)$. We will use Einstein notation. Thus an alternate form $\alpha\in\bigwedge^r V^*$ is written as

$$\alpha = \alpha_{i_1,\dots,i_r} e^{i_1}\otimes\cdots\otimes e^{i_r},$$

where $\alpha_{i_1,\dots,i_r}$ are scalars satisfying $\alpha_{i_{\sigma(1)},\dots,i_{\sigma(r)}} = \mathrm{sgn}(\sigma)\alpha_{i_1,\dots,i_r}$, for any permutation σ on the set $\{1,\dots,r\}$.

Let us fix an orientation on $\mathfrak{k}$ and $\mathfrak{m}$, and an orthonormal basis for $\mathfrak{g}$,

$$(X_1,\dots,X_n,Y_1,\dots,Y_m),$$

such that $(X_1,\dots,X_n)$ and $(Y_1,\dots,Y_m)$ are positively-oriented orthonormal bases for $\mathfrak{k}$ and $\mathfrak{m}$, respectively. Here, orthonormality means that

$$B(X_i,X_j) = -\delta_{ij} \quad B(Y_i,Y_j)=\delta_{ij}, \quad B(X_i,Y_j)=0.$$

From now on, all the tensors will be written in the fixed basis of $\mathfrak{g}$. The following proposition gives the expression of D and T with respect to the fixed basis.

Proposition 2.1.6. *For $\alpha\in\Omega^r_{\mathrm{Hor}}(\Gamma\backslash G;V)^K$, the operators D and T are given by the following equations:*

$$(D\alpha)_{i_1,\dots,i_{r+1}} = \sum_{k=1}^{r+1}(-1)^{k+1} Y_{i_k}\alpha_{i_1,\dots,\widehat{i_k},\dots,i_{r+1}} \tag{2.5}$$

$$(T\alpha)_{i_1,\dots,i_{r+1}} = \sum_{k=1}^{r+1}(-1)^{k+1} \rho(Y_{i_k})\alpha_{i_1,\dots,\widehat{i_k},\dots,i_{r+1}}. \tag{2.6}$$

Proof. Put $\alpha = \alpha_{i_1,\dots,i_r}\otimes Y^{i_1}\wedge\cdots\wedge Y^{i_r}$, with $\alpha_{i_1,\dots,i_r}\in V$. By definition, $D\alpha$ is the horizontal component of $d\alpha$. On one hand, it is immediate to check that dY^k has no horizontal component, indeed we have:

$$(dY^k)(Y_i,Y_j) = Y^k([Y_i,Y_j]) = 0,$$

because $[Y_i, Y_j] \in \mathfrak{k}$. Thus we get,

$$D\alpha = (d\alpha_{i_1,\dots,i_r}) \wedge Y^{i_1} \wedge \cdots \wedge Y^{i_r}.$$

And hence:

$$D\alpha = (Y_j \alpha_{i_1,\dots,i_r}) \otimes Y^j \wedge Y^{i_1} \wedge \cdots \wedge Y^{i_r}.$$

Rearranging indices we get Equation (2.5). The other equation follows easily from the definition of T. $\square$

Let us define the following two forms $\Omega_K = X^1 \wedge \cdots \wedge X^n$ and $\Omega_M = Y^1 \wedge \cdots \wedge Y^m$. These forms are independent of the orthonormal bases chosen, and hence they yield well-defined forms on $\Gamma\backslash G$. Note that Ω_K is vertical whereas Ω_M is horizontal, and that they both are right K–invariant (this is a consequence of the fact that the right action of K on $\mathfrak{g}$ leaves both the Killing form and the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ invariant). Observe also that Ω_M defines a volume form on M which is compatible with the metric structure on M.

Next we want to define an inner product on the fibers of E_ρ. To that end, fix a K–invariant inner product $\langle\,,\rangle_V$ on V, and use it to define a metric on the fibers of $E_\rho = Q \times_K V$. Then define an inner product on $\Omega^*(M; E_\rho)$ as usual:

$$(\alpha, \beta) = \int_M \langle \alpha(x), \beta(x) \rangle_x \Omega_M, \quad \text{for all } \alpha, \beta \in \Omega^*(M; E_\rho),$$

where $\langle\,,\rangle_x$ denotes the extension of the inner product defined on the fiber of E_ρ at x to its exterior powers. Here Ω_M is interpreted as a form on M. On the other hand, we can define an inner product on $\Omega^*_{\mathrm{Hor}}(\Gamma\backslash G; V)^K$ by

$$(\tilde\alpha, \tilde\beta) = \frac{1}{\mu(K)} \int_{\Gamma\backslash G} \langle \tilde\alpha(u), \tilde\beta(u) \rangle_u \Omega_K \wedge \Omega_M, \quad \text{for all } \tilde\alpha, \tilde\beta \in \Omega^*_{\mathrm{Hor}}(\Gamma\backslash G; V)^K,$$

where $\langle\,,\rangle_u$ is the inner product on $\bigwedge^r \mathrm{H}^* \otimes V$ induced by the Killing form and the inner product on V, and $\mu(K) = \int_K \Omega_K$ is the volume of K. Proposition A.0.6 gives the relationship between these two products.

Proposition 2.1.7. *The canonical isomorphism between the spaces $\Omega^*_{\mathrm{Hor}}(\Gamma\backslash G; V)^K$ and $\Omega^*(M; E_\rho)$ is an isometry.*

Next we want to get an expression of the formal adjoint of D and T. We consider the Hodge star operator defined on the horizontal bundle

$$*\colon \Omega^r_{\mathrm{Hor}}(\Gamma\backslash G; V)^K \to \Omega^{m-r}_{\mathrm{Hor}}(\Gamma\backslash G; V)^K.$$

Recall that if $\alpha = \alpha_{i_1,\dots,i_r} \otimes Y^{i_1} \wedge \cdots \wedge Y^{i_r}$, with $\alpha_{i_1,\dots,i_r} \in V$, then

$$*\alpha = \alpha_{i_1,\dots,i_r} \otimes Y^{j_{r+1}} \wedge \cdots \wedge Y^{j_m},$$

where $Y^{i_1} \wedge \cdots \wedge Y^{i_r} \wedge Y^{j_{r+1}} \wedge \cdots \wedge Y^{j_m} = Y^1 \wedge \cdots \wedge Y^m$.

Proposition 2.1.8. *Let $\alpha \in \Omega^r_{\mathrm{Hor}}(\Gamma\backslash G; V)^K$ with compact support. Then,*

$$\begin{aligned} D^*\alpha &= (-1)^r *^{-1} D * \alpha && (2.7)\\ T^*\alpha &= (-1)^{r-1} *^{-1} \rho(\omega)^* \wedge (*\alpha), && (2.8)\end{aligned}$$

where $\rho(\omega)^*$ denotes the adjoint of $\rho(\omega)$ with respect to the fixed inner product on V, that is, for all vector field X on $\Gamma\backslash G$, $\rho(\omega)^*(X)$ is the adjoint of $\rho(\omega)(X) \in \mathrm{End}(V)$.

Proof. We want to use Proposition A.0.7. We claim that

$$\int_P D\alpha \wedge \beta \wedge \Omega_K = (-1)^r \int_P \alpha \wedge D\beta \wedge \Omega_K, \tag{2.9}$$

for α and β forms of $\Omega^*_{\mathrm{Hor}}(\Gamma\backslash G; V)$ with compact support of degree $r-1$ and $m-r$ respectively. Indeed, since $D\alpha$ is the horizontal component of $d\alpha$, we have $D\alpha \wedge \Omega_K = d\alpha \wedge \Omega_K$. Then,

$$d(\alpha \wedge \beta \wedge \Omega_K) = d\alpha \wedge \beta \wedge \Omega_K + (-1)^{r-1}\alpha \wedge d\beta \wedge \Omega_K,$$

since Ω_K is closed. Therefore, equation (2.9) follows by Stokes' Theorem. Proposition A.0.7 then gives formula (2.7).

Let us prove now Equation (2.8). By Proposition A.0.7, it suffices to prove that

$$(\rho(\omega) \wedge \alpha) \wedge \beta = (-1)^{r-1}\alpha \wedge (\rho(\omega)^* \wedge \beta).$$

Let us fix an orthonormal basis for V so that we can write α and β as column vectors of forms and $\rho(\omega)$ as a matrix of 1-forms. The expression of $(\rho(\omega) \wedge \alpha) \wedge \beta$ in this basis is $(\rho(\omega) \wedge \alpha)^t \wedge \bar{\beta}$, where the bar denotes complex conjugation and the superscript t denotes matrix transposition. We have:

$$(\rho(\omega) \wedge \alpha)^t \wedge \bar{\beta} = (-1)^{r-1}\alpha^t \wedge \rho(\omega)^t \wedge \bar{\beta}.$$

The result then follows by taking into account that $\overline{\rho(\omega)}^t = \rho(\omega)^*$. □

A similar proof of Proposition 2.1.6 and the formulae found in the previous proposition give the following result.

Proposition 2.1.9. *For $\alpha \in \Omega^r_{\mathrm{Hor}}(\Gamma\backslash G; V)^K$ with compact support, the operators D^* and T^* are given by the following equations:*

$$\begin{aligned} (D^*\alpha)_{i_1,\dots,i_{r-1}} &= \sum_{k=1}^m -Y_k\alpha_{k,i_1,\dots,i_{r-1}} && (2.10)\\ (T^*\alpha)_{i_1,\dots,i_{r-1}} &= \sum_{k=1}^m \rho(Y_k)\alpha_{k,i_1,\dots,i_{r-1}}. && (2.11)\end{aligned}$$

Lemma 2.1.10. *If the inner product on V is symmetric with respect to the action of $\mathfrak{m}$, then the operator $S = TD^* + T^*D + DT^* + D^*T$ is zero for every form with compact support.*

Before proving the lemma we need the following result.

Lemma 2.1.11. *For every function f with compact support on $\Gamma\backslash G$ and for all $Y \in \mathfrak{g}$ we have:*

$$\int_{\Gamma\backslash G} (Yf)\Omega_M \wedge \Omega_K = 0.$$

Proof. On one hand, Cartan's formula $L_Y = \iota_Y \circ d + d \circ \iota_Y$ yields

$$L_Y(f\Omega_M \wedge \Omega_K) = d\left(\iota_Y(f\Omega_M \wedge \Omega_K)\right).$$

Since f has compact support, Stokes' Theorem then shows that we have:

$$0 = \int_{\Gamma\backslash G} L_Y(f\Omega_M \wedge \Omega_K),$$

On the other hand, since Y is an infinitesimal isometry, we have $L_Y\Omega_K = L_Y\Omega_M = 0$, and thus

$$L_Y(f\Omega_M \wedge \Omega_K) = (Yf)\Omega_M \wedge \Omega_K.$$

The result then follows immediately. □

Proof of Lemma 2.1.10. Since S is a self-adjoint operator, $S = 0$ if and only if $(S\alpha, \alpha) = 0$ for every α with compact support. Let us take $\alpha \in \Omega^*_{\mathrm{Hor}}(\Gamma\backslash G; V)^K$ with compact support. We must show that

$$(S\alpha, \alpha) = (D\alpha, T\alpha) + (T\alpha, D\alpha) + (D^*\alpha, T^*\alpha) + (T^*\alpha, D^*\alpha) = 0.$$

It suffices to prove that $(D\alpha, T\alpha) + (D^*\alpha, T^*\alpha) = 0$. Moreover, using the fact that the Hodge $*$ operator is an isometry, we must prove that

$$(D\alpha, T\alpha) + (D(*\alpha), T(*\alpha)) = 0.$$

Let us compute first $(D\alpha, T\alpha)$. Put $\alpha = \alpha_{i_1,\dots,i_r} \otimes Y^{i_1} \wedge \cdots \wedge Y^{i_r}$. Using the expression of D and T given in Proposition 2.1.6, we see that $(D\alpha, T\alpha)$ is a sum of terms of the form:

$$(-1)^{i+j} \int_{\Gamma\backslash G} \left\langle Y_{i_j}\alpha_{i_1,\dots,\widehat{i_j},\dots,i_{r+1}}, \rho(Y_{i_k})\alpha_{i_1,\dots,\widehat{i_k},\dots,i_{r+1}} \right\rangle_V d\mu_G.$$

Let us write the summands according to whether the avoided sub-indices $\widehat{i_j}$ and $\widehat{i_k}$ are equal or not. Therefore, one term is a sum of terms of the form

$$\int_{\Gamma\backslash G} \langle Y_j\alpha_{i_1,\dots,i_r}, \rho(Y_j)\alpha_{i_1,\dots,i_r}\rangle_V \, d\mu_G, \quad j \notin \{i_1, \dots, i_r\},$$

and the rest is a sum of terms of the form

$$(-1)^{j+k} \int_{\Gamma\backslash G} \left\langle Y_{i_j}\alpha_{i_1,\dots,\widehat{i_j},\dots,i_k,\dots,i_r}, \rho(Y_{i_k})\alpha_{i_1,\dots,i_j,\dots,\widehat{i_k},\dots,i_r} \right\rangle_V d\mu_G, \tag{2.12}$$

with $i_j \neq i_k$. We can apply this formula to $*\alpha$ to compute $(D(*\alpha), T(*\alpha))$. The formula we get is just the above formula with the range of the indices changed by their complementary; that is, on one hand we get terms of the form

$$\int_{\Gamma\backslash G} \langle Y_j \alpha_{i_1,\ldots,i_r}, \rho(Y_j)\alpha_{i_1,\ldots,i_r}\rangle_V \, d\mu_G, \quad j \in \{i_1, \ldots, i_r\},$$

and on the other hand terms of the form

$$(-1)^{j+k} \int_{\Gamma\backslash G} \left\langle Y_{i_k} \alpha_{i_1,\ldots,\widehat{i_j},\ldots,i_k,\ldots,i_r}, \rho(Y_{i_j})\alpha_{i_1,\ldots,i_j,\ldots,\widehat{i_k},\ldots,i_r} \right\rangle_V d\mu_G, \tag{2.13}$$

for $i_j \neq i_k$. By Lemma 2.1.11, the term (2.13) is minus the term (2.12). Hence, it suffices to prove that for every $Y \in \mathfrak{m}$, and $f \in \mathcal{C}(\Gamma\backslash G; V)$, we have

$$\int_{\Gamma\backslash G} \langle Yf, \rho(Y)f\rangle_V \, d\mu_G = 0.$$

This is a consequence of Lemma 2.1.11 and the symmetry of $\rho(Y)$. The lemma now follows immediately. □

Corollary 2.1.12 (Matsushima-Murakami formula). *Assume that the inner product on V is symmetric with respect to the action of $\mathfrak{m}$. Then*

$$\Delta_\rho = \Delta + H,$$

where $\Delta = DD^* + D^*D$, and $H = TT^* + T^*T$.

Proof. We have $\Delta_\rho = D_\rho D_\rho^* + D_\rho^* D_\rho = \Delta + H + S$, and Lemma 2.1.10 yields the result. □

The operator T can be extended in an obvious way to operators on $\Omega^*_{\mathrm{Hor}}(\Gamma\backslash G; V)$, namely its definition does not use the K–equivariance. Moreover, T is $\mathcal{C}^\infty(\Gamma\backslash G)$–linear. On the other hand, since the horizontal bundle on $\Gamma\backslash G$ is trivial, we have:

$$\Omega^r_{\mathrm{Hor}}(\Gamma\backslash G; V) = \mathcal{C}^\infty(\Gamma\backslash G) \otimes \left(V \otimes \bigwedge^r \mathfrak{m}^* \right).$$

Let us define $\mathbf{T}$ as the restriction of T to $V \otimes \bigwedge^r \mathfrak{m}^*$. The $\mathcal{C}^\infty(\Gamma\backslash G)$–linearity of T then implies that we can recover T from $\mathbf{T}$; hence, all properties of $\mathbf{T}$ are encoded somehow in T. Analogous considerations applied to T^* and H yield operators $\mathbf{T}^*$ and $\mathbf{H}$.

Notice that H is positive definite if and only so is $\mathbf{H}$. Therefore, by Theorem 2.1.3 we get the following result.

Corollary 2.1.13. *If the operator $\mathbf{H}$ is positively-definite in degree r, then every closed square integrable r-form in $\Omega^r_{\mathrm{Hor}}(\Gamma\backslash G; V)^K$ is exact.*

The next result gives an explicit expression of the operator $\mathbf{H}$ in terms of the fixed basis of $\mathfrak{m}$.

Proposition 2.1.14. *Let $\alpha \in V \otimes \bigwedge^p \mathfrak{m}^*$. Then we have,*

$$(\mathbf{H}\alpha)_{i_1,\dots,i_r} = \sum_{j=1}^{m} \rho(Y_j)^2 \alpha_{i_1,\dots,i_r} + \sum_{k=1}^{r}\sum_{j=1}^{m} (-1)^{k+1}\rho([Y_{i_k}, Y_j])\alpha_{j,i_1,\dots,\widehat{i_k},\dots,i_r}$$

Proof. Put $\beta_{i_1,\dots,i_{r+1}} = (\mathbf{T}\alpha)_{i_1,\dots,i_{r+1}}$ and $\gamma_{i_1,\dots,i_{r-1}} = (\mathbf{T}^*\alpha)_{i_1,\dots,i_{r-1}}$. On one hand we have

$$\begin{aligned}(\mathbf{TT}^*\alpha)_{i_1,\dots,i_r} &= \sum_{k=1}^{r} (-1)^{k+1}\rho(Y_{i_k})\gamma_{i_1,\dots,\widehat{i_k},\dots,i_r} \\ &= \sum_{k=1}^{r} (-1)^{k+1}\rho(Y_{i_k}) \sum_{j=1}^{m} \rho(Y_j)\alpha_{j,i_1,\dots,\widehat{i_k},\dots,i_r}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(\mathbf{T}^*\mathbf{T}\alpha)_{i_1,\dots,i_r} &= \sum_{j=1}^{m} \rho(Y_j)\beta_{j,i_1,\dots,i_r} \\ &= \sum_{j=1}^{m} \rho(Y_j)\left(\rho(Y_j)\alpha_{i_1,\dots,i_r} + \sum_{k=1}^{r}(-1)^k \rho(Y_{i_k})\alpha_{j,i_1,\dots,\widehat{i_k},\dots,i_r}\right).\end{aligned}$$

The proposition then follows by rearranging terms. $\square$

2.1.2 Proof of the vanishing cohomology theorem

We want to apply Corollary 2.1.13 to our particular case. First we need to choose an orthonormal basis for $\mathfrak{su}(2)$ with respect to the Killing form (in fact, it will be a constant multiple of it). Let us define

$$X_1 = \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}.$$

Then (X_1, X_2, X_3) is an orthonormal basis for $\mathfrak{su}(2)$. The orthogonal complement to $\mathfrak{su}(2)$ with respect to the Killing form is given by $Y_k = \mathbf{i}X_k$, for $k = 1, 2, 3$. On the other hand, we have $[X_i, X_{i+1}] = 2X_{i+2}$, for $i = 1, 2, 3$, where the indices are taken modulo 3.

Lemma 2.1.15. *Let $\rho\colon \mathfrak{sl}(2, \mathbf{C}) \to \mathrm{End}(V)$ be a* complex*, finite-dimensional, irreducible representation with* $\dim(V) \geq 2$. *Then the operator* $\mathbf{H}$ *is positively definite on degree* 1 *and* 2.

Proof. Since $\mathbf{H} = \mathbf{TT}^* + \mathbf{T}^*\mathbf{T}$, to show that $\mathbf{H}$ is positively definite is equivalent to show that its kernel is trivial. Let $\alpha \in V \otimes \mathfrak{m}^*$. We have $\alpha = \sum_{i=1}^{3} \alpha_i \otimes Y^i$, with $\alpha_i \in V$. Assume $\mathbf{H}\alpha = 0$. Then $\mathbf{T}\alpha = 0$ must vanish too, and from Proposition 2.1.6 Equation (2.6) we obtain

$$0 = (\mathbf{T}\alpha)(Y_i, Y_j) = \rho(Y_i)\alpha_j - \rho(Y_j)\alpha_i, \quad i, j = 1, 2, 3. \tag{2.14}$$

Proposition 2.1.14 yields

$$(\mathbf{H}\alpha)(Y_j) = \sum_{k=1}^{3} \left(\rho(Y_k)^2\alpha_j + \rho([Y_j, Y_k])\alpha_k\right).$$

Taking the indices modulo 3, and using the Lie algebra relations, we get

$$\begin{aligned}
\sum_{k=1}^{3} \rho([Y_j, Y_k])\alpha_k &= \rho([Y_j, Y_{j+1}])\alpha_{j+1} + \rho([Y_j, Y_{j+2}])\alpha_{j+2} \\
&= 2\big(\rho(-X_{j+2})\alpha_{j+1} + \rho(X_{j+1})\alpha_{j+2}\big) \\
&= 2\mathbf{i}\big(\rho(Y_{j+2})\alpha_{j+1} - \rho(Y_{j+1})\alpha_{j+2}\big).
\end{aligned}$$

Notice that in the last equality we have used the complex structure. Hence, using Equation (2.14), we get $(\mathbf{H}\alpha)(Y_j) = \sum_{k=1}^{3} \rho(Y_k)^2\alpha_j$, and then

$$\begin{aligned}
0 = \langle \mathbf{H}\alpha, \alpha\rangle &= \sum_{j=1}^{3}\sum_{k=1}^{3} \left\langle \rho(Y_k)^2\alpha_j, \alpha_j\right\rangle \\
&= \sum_{j,k=1}^{3} \langle \rho(Y_k)\alpha_j, \rho(Y_k)\alpha_j\rangle,
\end{aligned}$$

which implies $\rho(Y_j)\alpha_k = 0$ for $j, k = 1, 2, 3$. Hence, for a fixed k, we have $\rho(Z)\alpha_k = 0$ for every $Z \in \mathfrak{sl}(2, \mathbf{C})$. Since we are assuming that ρ is irreducible and nontrivial, we get $\alpha_k = 0$ for all k. It proves the lemma in degree 1. Since $\mathfrak{m}^* \cong \bigwedge^2 \mathfrak{m}^*$, the same proof holds in degree 2. $\square$

2.2 Cohomology of the ends

Assume that (M, η) is a complete, noncompact, non-elementary, spin-hyperbolic manifold with finite topology; in particular, M is the interior of a compact manifold $\overline{M}$ with boundary $\partial\overline{M}$. The aim of this section is to analyse the cohomology groups of $\mathrm{H}^*(\partial\overline{M}; E_{\rho_n})$. When the ends of the manifold are cusps, these cohomology groups depend on the chosen lift of the holonomy.

Definition. Let G be a group acting on a vector space V. The *subspace of invariants* of V, denoted by V^G, is the subspace consisting of elements of V that are fixed by G. That is,

$$V^G = \{v \in V \mid g \cdot v = v, \text{ for all } g \in G\}.$$

Lemma 2.2.1. *Let F be a connected component of $\partial\overline{M}$. For every $n > 1$ we have,*

$$\begin{aligned}
\dim_{\mathbf{C}} \mathrm{H}^0(F; E_{\rho_n}) &= \dim_{\mathbf{C}} V_n^{\pi_1(F)} \\
\dim_{\mathbf{C}} \mathrm{H}^1(F; E_{\rho_n}) &= 2\dim_{\mathbf{C}} V_n^{\pi_1(F)} - n\chi(F), \\
\dim_{\mathbf{C}} \mathrm{H}^2(F; E_{\rho_n}) &= \dim_{\mathbf{C}} V_n^{\pi_1(F)}.
\end{aligned}$$

Proof. Since F is a $K(\pi_1(F), 1)$ space, $\mathrm{H}^0(F; E_{\rho_n}) = \mathrm{H}^0(\pi_1(F); E_{\rho_n})$, and this is identified with $V_n^{\pi_1(F)}$. It proves the first equality. The third one follows from Poincaré duality, and the second one from an Euler characteristic argument. $\square$

Therefore, all the cohomological information comes from the subspace of invariants $V_n^{\pi_1(F)}$. We distinguish two cases according to whether F has genus $g \geq 2$ or F is a torus.

Proposition 2.2.2. *Let F be a connected component of $\partial\overline{M}$ and $n > 1$. If F has genus $g \geq 2$, then $V_n^{\pi_1(F_g)} = 0$. If F is a torus T^2, then we have the following cases,*

$$V_n^{\pi_1(T^2)} = \begin{cases} 0 & \text{for } n \text{ even and } \eta \text{ is nonpositive on } T^2, \\ \mathbf{C} & \text{for } n \text{ even and } \eta \text{ positive on } T^2, \\ \mathbf{C} & \text{for } n \text{ odd.} \end{cases}$$

Before proving it, we need the following lemmas. The first one can be found in standard references about Kleinian groups (see [Kap01]).

Lemma 2.2.3. *Let M be a hyperbolic three-manifold. Then the following are equivalent:*

1. *M is elementary (its holonomy is reducible in* $\mathrm{PSL}(2, \mathbf{C})$*).*
2. *$\pi_1(M)$ is abelian.*
3. *M is homeomorphic to either the product of the plane with a circle, $\mathbf{R}^2 \times S^1$, or to the product of a 2-torus with a line, $S^1 \times S^1 \times \mathbf{R}$.*

Lemma 2.2.4. *Let F be a connected component of $\partial\overline{M}$. If F has genus $g \geq 2$, then* $\mathrm{Hol}_{(M,\eta)}(\pi_1(F))$ *is an* irreducible *subgroup of* $\mathrm{SL}(2, \mathbf{C})$.

Proof. When F is π_1–injective (i.e. when $\pi_1(F)$ injects into $\pi_1(M)$) then the holonomy restricts to a discrete faithful representation of $\pi_1(F)$, and irreducibility follows because $\pi_1(F)$ is non-abelian. Otherwise, when F is not π_1–injective, according to Bonahon [Bon83] and McCullough-Miller [MM86] there are two possibilities: either M is a *handlebody* or F is a boundary component of a *characteristic compression body* $C \subseteq M$. A handlebody is the result of attaching one handles to a 3-ball; in particular when M is a handlebody then $\pi_1(F)$ surjects onto $\pi_1(M)$, thus $\mathrm{Hol}(\pi_1(F)) = \mathrm{Hol}(\pi_1(M))$ and irreducibility comes from the hypothesis that M is non-elementary. Next, assume that F is the positive boundary of a characteristic compression body C, namely $C \subseteq M$ is a codimension 0 closed submanifold, whose boundary splits as a union $\partial C = \partial_- C \cup \partial_+ C$, so that $\partial_+ C = F$, the components of $\partial_- C$ are π_1–injective in M, and C is the result of gluing 1-handles to $\partial_- C \times [0,1]$ along $\partial_- C \times \{1\}$. In particular $\pi_1(F)$ surjects onto $\pi_1(C)$ and $\mathrm{Hol}(\pi_1(F)) = \mathrm{Hol}(\pi_1(C))$. Thus, if $F = \partial_+ C$ and one of the components of $\partial_- C$ has genus ≥ 2, then we are done by the π_1–injective case. Finally if $F = \partial_+ C$ and all components of $\partial_- C$ are tori, since incompressible tori in M are boundary parallel, then the inclusion $C \subseteq M$ is a homotopy equivalence. Thus $\pi_1(F)$ surjects onto $\pi_1(M)$ and irreducibility follows again because M is non-elementary. $\square$

Lemma 2.2.5. *Let (M,η) be a non-elementary spin-hyperbolic three-manifold. Then, for $n \geq 2$ the subspace of invariants of V_n is trivial, that is,*

$$V_n^{\pi_1(M)} = 0.$$

Proof. Let us fix a basis for V_n. Let $e_1 = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $e_2 = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$, so that (e_1, e_2) is the standard basis for $V_2 = \mathbf{C}^2$. Thus

$$\left(e_1^{n-1}, e_1^{n-2}e_2, \ldots, e_2^{n-1}\right)$$

is a basis for $V_n = \mathrm{Sym}^{n-1}(V_2)$. Since M is non-elementary, there exists at least one element $\gamma \in \pi_1(M)$ whose holonomy is non-parabolic (see [Kap01, Corollary 3.25]). Up to conjugation, we have

$$\mathrm{Hol}_{(M,\eta)}(\gamma) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

for some $\lambda \in \mathbf{C}$, with $|\lambda| > 1$. This means that the vectors e_1 and e_2 of the standard basis for $\mathbf{C}^2$ are eigenvectors. Since V_n is the $(n-1)$-symmetric power of $\mathbf{C}^2$, for n even the only element of V_n that is γ-invariant is zero. For n odd, the subspace of γ-invariants of V_n is the line generated by $e_1^{\frac{n-1}{2}} e_2^{\frac{n-1}{2}}$. Any other matrix of $\mathrm{SL}(2,\mathbf{C})$ that fixes $e_1^{\frac{n-1}{2}} e_2^{\frac{n-1}{2}}$ is either diagonal or antidiagonal (zero entries in the diagonal). Antidiagonal matrices have trace zero, hence they have order four, so they cannot occur because the holonomy of M has no torsion elements. Also, any element $\gamma' \in \pi_1(M)$ that does not commute with γ has non-diagonal holonomy, thus 0 is the only element of V_n invariant by both γ and γ'. $\square$

Proof of Proposition 2.2.2. When F has genus $g \geq 2$, then by Lemma 2.2.4 the following quotient is a non-elementary spin-hyperbolic 3-manifold.

$$\mathrm{Hol}_{(M,\eta)}(\pi_1(F)) \backslash \mathbf{H}^3.$$

We apply Lemma 2.2.5 to conclude that $V_n^{\pi_1(F)} = 0$. The case of the torus follows easily. $\square$

Applying Lemma 2.2.1, Proposition 2.2.2, Theorem 2.0.5 and Lemma 2.2.5, we get the following corollaries.

Corollary 2.2.6. *Let (M,η) be a spin-hyperbolic 3-manifold with k cusps and l ends of infinite volume of genus $g_1, \ldots, g_l$, and let $n \geq 2$. Then we have:*

$$\begin{aligned} \dim_{\mathbf{C}} \mathrm{H}^0(\partial \overline{M}; E_{\rho_n}) &= a, \\ \dim_{\mathbf{C}} \mathrm{H}^1(\partial \overline{M}; E_{\rho_n}) &= \sum_{i=1}^{l} 2n(g_i - 1) + 2a, \\ \dim_{\mathbf{C}} \mathrm{H}^2(\partial \overline{M}; E_{\rho_n}) &= a, \end{aligned}$$

where a is equal to k if n is odd, and is equals to the number of cusps for which η is nonpositive if n is even.

Corollary 2.2.7. *Let (M,η) be as in Corollary 2.2.6. Then we have*

$$\begin{aligned}\dim_{\mathbf{C}} \mathrm{H}^0(M;E_{\rho_n}) &= 0,\\ \dim_{\mathbf{C}} \mathrm{H}^1(M;E_{\rho_n}) &= \sum_{i=1}^{l} n(g_i-1)+a,\\ \dim_{\mathbf{C}} \mathrm{H}^2(M;E_{\rho_n}) &= a.\end{aligned}$$

2.3 Infinitesimal Rigidity

In this section we prove Theorem 2.0.7 which we restate now.

Theorem 2.3.1. *Let M be a complete hyperbolic 3-manifold that is topologically finite. If $\partial\overline{M}$ is the union of k tori and l surfaces of genus $g_1,\dots,g_l \geq 2$, and $n\geq 2$, then*

$$\dim_{\mathbf{C}} \mathrm{H}^1(M;E_{\mathrm{Ad}\circ\rho_n}) = k(n-1)+\sum(g_i-1)(n^2-1).$$

In particular, if M is closed then $\mathrm{H}^1(M;E_{\mathrm{Ad}\circ\rho_n})=0$. In addition, all nontrivial elements in $\mathrm{H}^1(M;E_{\mathrm{Ad}\circ\rho_n})$ are nontrivial in $\mathrm{H}^1(\partial\overline{M},E_{\mathrm{Ad}\circ\rho_n})$ and have no L^2-representative.

Proof. By Lemma 1.3.4 we have $\mathfrak{sl}(n,\mathbf{C})\cong V_{2n-1}\oplus V_{2n-3}\cdots\oplus V_3$. Hence,

$$\mathrm{H}^1(M;E_{\mathrm{Ad}\circ\rho_n})\cong \mathrm{H}^1(M;E_{\rho_{2n-1}})\oplus \mathrm{H}^1(M;E_{\rho_{2n-3}})\oplus\cdots\oplus \mathrm{H}^1(M;E_{\rho_3}). \tag{2.15}$$

The theorem now follows from this isomorphism, Corollary 2.2.6 and Theorem 2.0.5. □

Next we want to prove Theorem 2.0.8. See [LM85] for basic results about representation and character varieties. The variety of representations of $\pi_1(M)$ in $\mathrm{SL}(n,\mathbf{C})$ is

$$R(M,\mathrm{SL}(n,\mathbf{C}))=\hom(\pi_1(M),\mathrm{SL}(n,\mathbf{C})).$$

Since $\pi_1(M)$ is finitely generated, this is an algebraic affine set. The group $\mathrm{SL}(n,\mathbf{C})$ acts by conjugation on $R(M,\mathrm{SL}(n,\mathbf{C}))$ algebraically, and the quotient in the algebraic category is the variety of characters:

$$X(M,\mathrm{SL}(n,\mathbf{C}))=R(M,\mathrm{SL}(n,\mathbf{C}))/\!/\,\mathrm{SL}(n,\mathbf{C}).$$

For a representation $\rho\in R(M,\mathrm{SL}(n,\mathbf{C}))$ its character is the map

$$\begin{aligned}\chi_\rho\colon \pi_1(M) &\to \mathbf{C}\\ \gamma &\mapsto \mathrm{trace}(\rho(\gamma)).\end{aligned}$$

The projection $R(M,\mathrm{SL}(n,\mathbf{C}))\to X(M,\mathrm{SL}(n,\mathbf{C}))$ maps each representation ρ to its character χ_ρ.

Weil's construction gives a natural isomorphism between the Zariski tangent space to a representation $T^{Zar}_\rho R(M,\mathrm{SL}(n,\mathbf{C}))$ and $Z^1(\pi_1(M),V_{\mathrm{Ad}\circ\rho})$, the space of group cocycles valued in the Lie algebra $\mathfrak{sl}(n, \mathbf{C})$, which as $\pi_1(M)$–module is also written as $V_{\mathrm{Ad}\circ\rho}$. Namely, $Z^1(\pi_1(M), V_{\mathrm{Ad}\circ\rho})$ is the set of maps $d\colon \pi_1(M) \to V_{\mathrm{Ad}\circ\rho}$ that satisfy the cocycle relation

$$d(\gamma_1\gamma_2) = d(\gamma_1) + \mathrm{Ad}_{\rho(\gamma_1)}\, d(\gamma_2), \qquad \forall \gamma_1, \gamma_2 \in \pi_1(M).$$

Notice that $R(M, \mathrm{SL}(n, \mathbf{C}))$ may be a non reduced algebraic set, so the Zariski tangent space may be larger than the Zariski tangent space of the underlying algebraic variety.

The space of coboundaries $B^1(\pi_1(M), V_{\mathrm{Ad}\circ\rho})$ is the set of cocycles that satisfy $d(\gamma) = \mathrm{Ad}_{\rho(\gamma)}\, m - m$ for all $\gamma \in \pi_1(M)$ and for some fixed $m \in V_{\mathrm{Ad}\circ\rho}$. The space of coboundaries is the tangent space to the orbit by conjugation, so under some hypothesis the cohomology may be identified with the tangent space of the variety of characters (Proposition 2.3.2). Since M is aspherical, the group cohomology of $\pi_1(M)$

$$\mathrm{H}^1(\pi_1(M); V_{\mathrm{Ad}\circ\rho}) = Z^1(\pi_1(M), V_{\mathrm{Ad}\circ\rho})/B^1(\pi_1(M), V_{\mathrm{Ad}\circ\rho})$$

is naturally isomorphic to $\mathrm{H}^1(M; E_{\mathrm{Ad}\circ\rho})$.

Definition. A representation $\rho\colon \pi_1(M) \to \mathrm{SL}(n, \mathbf{C})$ is *semi-simple* if every subspace of $\mathbf{C}^n$ invariant by $\rho(\pi_1(M))$ has an invariant complement.

Thus a semi-simple representation decomposes as a direct sum of simple representations, where simple means without proper invariant subspaces.

The following summarizes the relation between tangent spaces and cohomology. See [LM85] for a proof.

Proposition 2.3.2. *Let $\rho \in R(M, \mathrm{SL}(n, \mathbf{C}))$.*

1. *There is a natural isomorphism*

$$Z^1(\pi_1(M), V_{\mathrm{Ad}\circ\rho}) \cong T_\rho^{Zar} R(M, \mathrm{SL}(n, \mathbf{C})).$$

2. *If ρ is semisimple, then it induces an isomorphism*

$$\mathrm{H}^1(\pi_1(M); V_{\mathrm{Ad}\circ\rho}) \cong T_\rho^{Zar} X(M, \mathrm{SL}(n, \mathbf{C})).$$

3. *If ρ is semisimple and a smooth point of $R(M, \mathrm{SL}(n, \mathbf{C}))$, then its character χ_ρ is a smooth point of $X(M, \mathrm{SL}(n, \mathbf{C})$.*

A point in an algebraic affine set is smooth if and only if it has the same dimension that its Zariski tangent space. So to prove smoothness we need to compute these dimensions.

Lemma 2.3.3. *Let ρ_n be as in Theorem 2.0.8, and T^2 a component of $\partial\overline{M}$ corresponding to a cusp. Then the restriction of ρ_n to $\pi_1(T^2)$ is a smooth point of $R(T^2, \mathrm{SL}(n, \mathbf{C}))$.*

Proof. Knowing that $\dim R(T^2, \mathrm{SL}(n, \mathbf{C})) \leq \dim Z^1(T^2, V_{\mathrm{Ad}\circ\rho_n})$, we want to show that equality of dimensions holds. Before the cocycle space, we first compute the dimension of the cohomology group. By Equation (2.15) in the proof of Theorem 2.3.1:

$$\dim \mathrm{H}^1(T^2; E_{\mathrm{Ad}\circ\rho_n}) = \sum_{i=2}^{n} \dim \mathrm{H}^1(T^2; E_{\rho_{2i-1}}).$$

Hence, by Corollary 2.2.6,

$$\dim \mathrm{H}^1(T^2; E_{\mathrm{Ad}\circ\rho_n}) = 2(n-1).$$

We apply the same splitting for computing the dimension of the coboundary space. It is the sum of terms $\dim B^1(T^2, E_{\rho_k})$, for k odd from 3 to $2n-1$. Since we have an exact sequence

$$0 \to V_k^{\pi_1(T^2)} \to V_k \to B^1(T^2, E_{\rho_k}) \to 0,$$

$\dim B^1(T^2, E_{\rho_k}) = k - \dim V_k^{\pi_1(T^2)} = k-1$, by Lemma 2.2.2. Thus

$$\dim B^1(T^2, E_{\mathrm{Ad}\circ\rho_n}) = (2n-2) + (2n-4) + \cdots + 2 = n^2 - n.$$

Hence as $\mathrm{H}^1(T^2; E_{\mathrm{Ad}\circ\rho_n}) = Z^1(T^2, E_{\mathrm{Ad}\circ\rho_n})/B^1(T^2, E_{\mathrm{Ad}\circ\rho_n})$, we have:

$$\begin{aligned} \dim Z^1(T^2, E_{\mathrm{Ad}\circ\rho_n}) &= \dim \mathrm{H}^1(T^2, E_{\mathrm{Ad}\circ\rho_n}) + \dim B^1(T^2, E_{\mathrm{Ad}\circ\rho_n}) \\ &= n^2 + n - 2. \end{aligned}$$

Now we look for a lower bound of $\dim R(T^2, \mathrm{SL}(n, \mathbf{C}))$. Fix $\{\gamma_1, \gamma_2\}$ a generating set of $\pi_1(T^2)$. The representation ρ_n restricted to $\pi_1(T^2)$ has eigenvalues equal to ± 1. By deforming the representation of $\pi_1(T^2)$ to $\mathrm{SL}(2, \mathbf{C})$, and by composing it with the representation of $\mathrm{SL}(2, \mathbf{C})$ to $\mathrm{SL}(n, \mathbf{C})$, there exists a representation $\rho' \in R(T^2, \mathrm{SL}(n, \mathbf{C}))$ arbitrarily close to ρ_n such that all eigenvalues of $\rho'(\gamma_1)$ are different, in particular $\rho'(\gamma_1)$ diagonalises. Now, to find deformations of ρ', notice that $\rho'(\gamma_1)$ can be deformed with $n^2 - 1 = \dim(\mathrm{SL}(n, \mathbf{C}))$ parameters, and having all eigenvalues different is an open condition. As $\rho'(\gamma_2)$ has to commute with $\rho'(\gamma_1)$, it has the same eigenspaces, but one can still choose $n-1$ eigenvalues for $\rho'(\gamma_2)$. This proves that the dimension of some irreducible component of $R(T^2, \mathrm{SL}(n, \mathbf{C}))$ that contains ρ_n is at least

$$n^2 - 1 + n - 1 = n^2 + n - 2.$$

As this is $\dim Z^1(T^2, E_{\mathrm{Ad}\circ\rho_n})$, it must be a smooth point. □

Proof of Theorem 2.0.8. Using Proposition 2.3.2, we just need to prove that ρ_n is a smooth point of the variety of representations.

Given a Zariski tangent vector $v \in Z^1(M, V_{\mathrm{Ad}\circ\rho_n})$, we have to show that it is integrable, i.e. that there is a path in the variety of representations whose tangent vector is v. For this, we use the algebraic obstruction theory, see [Gol86, HP05]. There exists an infinite sequence of obstructions that are cohomology classes in $\mathrm{H}^2(M, V_{\mathrm{Ad}\circ\rho_n})$, each obstruction being defined only if the previous one vanishes. These are related to the analytic expansion in power series of a deformation of a representation, and to Kodaira's theory of infinitesimal deformations. Our aim is to show that this infinite sequence vanishes. This gives a formal power series, that does not need to converge, but this is sufficient for v to be a tangent vector by a theorem of Artin [Art68] (see [HP05] for details). We do not give the explicit construction of these obstructions; we just use that they are natural and that they live in the second cohomology group.

By Theorem 2.0.5 we have an isomorphism:

$$\mathrm{H}^2(M; E_{\mathrm{Ad}\circ\rho_n}) \cong \mathrm{H}^2(\partial\overline{M}; E_{\mathrm{Ad}\circ\rho_n}). \tag{2.16}$$

Now, $\mathrm{H}^2(\partial\overline{M}; E_{\mathrm{Ad}\circ\rho_n})$ decomposes as the sum of the connected components of $\partial\overline{M}$. If F_g has genus $g \geq 2$ then $\mathrm{H}^2(F_g; E_{\mathrm{Ad}\circ\rho_n}) = 0$. Thus, only the components of $\partial\overline{M}$ that are tori appear in $\mathrm{H}^2(\partial\overline{M}; E_{\mathrm{Ad}\circ\rho_n})$. By Lemma 2.3.3 and naturality, the obstructions vanish when restricted to $\mathrm{H}^2(T^2; E_{\mathrm{Ad}\circ\rho_n})$, hence they vanish in $\mathrm{H}^2(M; E_{\mathrm{Ad}\circ\rho_n})$ by the isomorphism (2.16). $\square$

Appendix A

Some results on principal bundles

Throughout this appendix G will be a Lie group and P a G-principal bundle over a manifold M. The bundle projection will be denoted by

$$\pi_P \colon P \to M.$$

We will follow the convention that G acts on P on the right. For $g \in G$, we will denote the action of g on P as

$$R_g \colon P \to P.$$

Let us recall the following common construction. Let F be a differentiable manifold on which G acts on the left. The associated bundle $P \times_G F$ is the quotient of $P \times F$ by the diagonal right action of G, that is

$$(u, x) \cdot g = (ug, g^{-1}x), \quad \text{with } g \in G \text{ and } (u, x) \in P \times F.$$

The space $P \times_G F$ has in a natural way a structure of fiber bundle over M with typical fiber F.

Remark. The definition of $P \times_G F$ shows that a point u in P can be interpreted as an isomorphism between F and the fiber of $P \times_G F$ at $\pi_P(u)$: if π denotes the quotient map $P \times F \to P \times_G F$, then $\pi(u, \cdot)$ is an isomorphism from F to $F_{\pi_P(u)}$. Note that $\pi(ug, x) = \pi(u, gx)$.

Assume that we have a connection on P defined by a $\mathfrak{g}$-valued form $\omega \in \Omega^1(P; \mathfrak{g})$. Recall that ω then satisfies the following two conditions (see [KN96]):

$$\begin{aligned} R_g^*\omega &= \mathrm{Ad}(g^{-1})\omega, && \text{for all } g \in G, \\ \omega(X^*) &= X, && \text{for all } X \in \mathfrak{g}, \end{aligned}$$

where Ad is the adjoint action of G on $\mathfrak{g}$, and X^* is the fundamental vector field on P generated by $X \in \mathfrak{g}$, see [KN96]. This connection defines a horizontal vector bundle H over P whose fibers are given by

$$H_u = \operatorname{Ker} \omega_u, \quad u \in P.$$

A vector field on P is called *horizontal* if it is tangent to H. The differential of the projection map π_P is an isomorphism when restricted to H. Thus given $V_p \in T_pM$ and $u \in \pi_P^{-1}(p)$, there exists a unique $\tilde{V}_u \in H_u$ that is projected to V_p. The vector $\tilde{V}_u$ is called the *horizontal lift* of V_p at u.

The above definitions are extended in a natural way to the cotangent bundle and its exterior powers. Thus it makes sense to consider the space of V-valued horizontal r-forms on P, which we will denote as $\Omega^r_{\mathrm{Hor}}(P;V)$.

Remark. A form $\alpha \in \Omega^r(P;V)$ is horizontal if and only if it vanishes on vertical directions, that is

$$\iota_{X^*}\alpha = 0, \quad \text{for all } X \in \mathfrak{g}.$$

Let us fix a linear representation $\rho\colon G \to \mathrm{GL}(V)$. Then it makes sense to consider the space $\Omega^r(P;V)^G$ of V-valued G-equivariant forms on P, that is

$$\Omega^r(P;V)^G = \left\{\alpha \in \Omega^r(P;V) \mid R_g^*\alpha = \rho(g^{-1})\alpha \quad \text{for all } g \in G\right\}.$$

The space of horizontal V-valued differential forms over P that are G-equivariant will be denoted by

$$\Omega^*_{\mathrm{Hor}}(P;V)^G.$$

Consider now $E = P \times_K V$ the vector bundle over M defined by ρ, and $\Omega^r(M;E)$ the space of E-valued r-forms on M.

We want to recall the definition of the canonical isomorphism between $\Omega^r(M;E)$ and $\Omega^*_{\mathrm{Hor}}(P;V)^G$. To that end, consider the bundle $Q = \bigwedge^r H^* \otimes V$, where H is the horizontal vector bundle of P. We let G act on Q on the right by

$$(\alpha_p \otimes w_p)\cdot g = \left(R^*_{g^{-1}}\alpha_p\right) \otimes \rho(g)^{-1}w_p \in Q_{pg}, \qquad \text{for all } \alpha_p \otimes w_p \in Q_p.$$

The quotient space Q/G defines a vector bundle over M whose sections are identified with $\Omega^*_{\mathrm{Hor}}(P;V)^G$.

We define an isomorphism between Q/G and $\bigwedge^r T^*M \otimes E$ as follows. Let $p \in M$ and $u \in \pi^{-1}(p)$. Use the horizontal lift process to define an isomorphism

$$\psi_u\colon \bigwedge^r H^* \to \bigwedge^r T_p^*M.$$

Interpreting u as an isomorphism between V and E_p, we get an isomorphism

$$\varphi_u = \psi_u \otimes u\colon Q_u = \bigwedge^r H^* \otimes V \to \bigwedge^r T_p^*M \otimes E_p.$$

Notice that we have $\varphi_u(v) = \varphi_{ug}(vg)$ for all $v \in Q$. Therefore, it induces an isomorphism between Q/G and $\bigwedge^r T^*M \otimes E$, and hence an isomorphism,

$$\varphi\colon \Omega^r_{\mathrm{Hor}}(P;V)^G \to \Omega^r(M;E). \tag{A.1}$$

The connection on P defines an exterior covariant differential on $\Omega^r_{\mathrm{Hor}}(P;V)^G$ by

$$D\alpha = (d\alpha)\circ \pi_h, \quad \text{for} \quad \alpha \in \Omega^r_{\mathrm{Hor}}(P;V)^G,$$

where π_h is the projection on the horizontal distribution H. On the other hand, a connection on P induces a connection on the vector bundle E, and hence an exterior covariant differential d_ρ on $\Omega^r(M;E)$. It can be verified that the canonical isomorphism (A.1) commutes with the exterior covariant differentiation, see [KN96, p. 76].

Proposition A.0.4. *Let $\omega \in \Omega^1(P;\mathfrak{g})$ be a connection form defined on P. Then the following formula holds*

$$D\alpha = d\alpha + \rho(\omega)\wedge\alpha, \quad \textit{for all } \alpha \in \Omega^r_{\mathrm{Hor}}(P;V)^G.$$

Remark. If $V_1,\ldots,V_{p+1}$ are vector fields on P, by definition,

$$(\rho(\omega)\wedge\alpha)(V_1,\ldots,V_{p+1}) = \sum_{i=1}^{p+1}(-1)^{i+1}\rho(\omega(V_i))\big(\alpha(V_1,\ldots,\widehat{V_i},\ldots,V_{p+1})\big).$$

Taking a basis of V, $\rho(w)$ is just a matrix of 1-forms, α a column vector of p-forms, and the product $\rho(\omega)\wedge\alpha$ is just the wedge product of a matrix by a vector.

Proof. Let $\alpha \in \Omega^r_{\mathrm{Hor}}(P;V)^G$. We must show that the form $d\alpha + \rho(\omega)\wedge\alpha$ is horizontal, and that agrees with $D\alpha$ on horizontal vectors. The second assertion is obvious from the definition of D and the fact that ω vanishes on horizontal vectors. Thus it remains to prove that $d\alpha + \rho(\omega)\wedge\alpha$ is horizontal. It is enough to prove that

$$\iota_{X^*}\,(d\alpha + \rho(\omega)\wedge\alpha) = 0, \qquad \text{for all } X \in \mathfrak{g},$$

where X^* is the fundamental vector field associated to $X\in\mathfrak{g}$. On one hand we have:

$$\iota_{X^*}\,(d\alpha + \rho(\omega)\wedge\alpha) = \iota_{X^*}(d\alpha) + \rho(X)\alpha,$$

as $\iota_{X^*}\alpha = 0$ and $\iota_{X^*}\omega = X$. On the other hand, Cartan's identity $L = d\circ\iota + \iota\circ d$ yields:

$$\iota_{X^*}(d\alpha) = L_{X^*}\alpha - d(\iota_{X^*}\alpha) = L_{X^*}\alpha.$$

Finally, the infinitesimal version of the G-equivariance of α states that $L_{X^*}\alpha = -\rho(X)\alpha$. We conclude from this that $d\alpha + \rho(\omega)\wedge\alpha$ is horizontal. □

Now assume that M is an oriented Riemannian manifold with volume form ω_M. From now on we will assume that G is compact. To avoid confusions with the previous chapter, we will denote G by K to emphasize that K is compact. Let us fix a K-invariant inner product on V, say $\langle\cdot,\cdot\rangle$. This inner product induces an inner product on the vector bundle $E = P\times_K V$.

We can define an inner product on $\Omega^r(M;E_\rho)$ as usual:

$$(\alpha,\beta) = \int_M \langle\alpha(x),\beta(x)\rangle_x\;\omega_M, \quad \text{with } \alpha,\beta\in\Omega^p(M;E),$$

where ω_M is the volume form of M.

On the other hand, we can define an inner product on $\Omega^r_{\mathrm{Hor}}(P;V)^K$ as follows. First, pull back the metric of TM to the horizontal bundle H through the projection map π_P; next, take ω_K a right-invariant volume form on K, and consider the right-invariant form ω_K^* on P that it defines; finally, define an inner product on $\Omega^r_{\mathrm{Hor}}(P;V)^K$ by

$$(\alpha,\beta) = \int_P \langle \alpha(u), \beta(u) \rangle_u \, \pi_P^*(\omega_M) \wedge \omega_K^*, \quad \text{with } \alpha, \beta \in \Omega^r_{\mathrm{Hor}}(P;V)^G.$$

To get the relationship between these two inner products under the canonical isomorphism, we will use the following lemma.

Lemma A.0.5. *Let f be a function defined on P, and define its average on the fibers by*

$$\bar{f}(u) = \int_K f(ug)\omega_K.$$

The function $\bar{f}(u)$, being constant along the fibers, can be seen as a function on M. The following equality then holds:

$$\int_P f(u)\pi_P^*(\omega_M) \wedge \omega_K = \int_M \bar{f}(x)\omega_M.$$

Proof. Take an open set $U \subset M$ so that there exists a trivializing map $\psi \colon U \times K \to \pi_P^{-1}(U)$. Denote by π_U and π_K the projection of $U \times K$ on the first and the second factor respectively. The formula of the change of variables then gives

$$\int_{\pi_P^{-1}(U)} f(u)\pi_P^*(\omega_M) \wedge \omega_K^* = \int_{U \times K} f(\psi(x,g))\omega_M \wedge \omega_K.$$

The right hand side integral is

$$\int_U \left(\int_K f(\psi(x,g))\omega_K \right) \omega_M = \int_U \bar{f}(x)\omega_M.$$

The result then follows by taking a partition of the unity subordinated to a trivializing open cover. □

The above lemma then implies the following result.

Proposition A.0.6. *With the above notation, we have:*

$$(\alpha,\beta) = \mu(K)(\varphi(\alpha), \varphi(\beta)), \qquad \textit{for all } \alpha, \beta \in \Omega^r_{\mathrm{Hor}}(P;V)^G,$$

where μ denotes the measure defined by the volume form ω_K, and φ is the canonical isomorphism (A.1)*.*

Proof. The function $\langle \tilde{\alpha}(u), \tilde{\beta}(u) \rangle_V$ is constant along the fibers, and is equal to $\langle \alpha(x), \beta(x) \rangle_x$, where $x = \pi_P(u)$. Lemma A.0.5 then implies the result. □

Using the usual wedge product and the inner product on V, it makes sense to consider the wedge product of two V-valued forms, which yields a $\mathbf{C}$-valued form. In particular, we have:

$$\begin{array}{ccc} \Omega^r_{\mathrm{Hor}}(P;V)^K \times \Omega^q_{\mathrm{Hor}}(P;V)^K & \longrightarrow & \Omega^{r+q}_{\mathrm{Hor}}(P;\mathbf{C})^K \\ (\alpha,\beta) & \longmapsto & \alpha\wedge\beta. \end{array}$$

Let us consider also the pairing

$$\begin{array}{ccc} \phi\colon \Omega^r_{\mathrm{Hor}}(P;V)^K \times \Omega^{m-r}_{\mathrm{Hor}}(P;V)^K & \longrightarrow & \mathbf{C} \\ (\alpha,\beta) & \longmapsto & \int_P(\alpha\wedge\beta)\wedge\omega_K, \end{array}$$

The metric on the horizontal bundle and the orientation that we have fixed on it allow us to consider the Hodge star operator on the space of horizontal forms:

$$*\colon \Omega^r_{\mathrm{Hor}}(P;V)^K \longrightarrow \Omega^{m-r}_{\mathrm{Hor}}(P;V)^K.$$

Note that we have $(\alpha,\beta)=\phi(\alpha,*\beta)$

Proposition A.0.7. *Let $T\colon \Omega^r_{\mathrm{Hor}}(P;V)^K \to \Omega^{r+k}_{\mathrm{Hor}}(P;V)^K$ be a linear operator that decreases supports. Assume that we have a linear operator*

$$S\colon \Omega^{m-(r+k)}_{\mathrm{Hor}}(P;V)^K \to \Omega^{m-r}_{\mathrm{Hor}}(P;V)^K$$

such that $\phi(T\alpha,\beta)=\phi(\alpha,S\beta)$. Then, the formal adjoint of T is given by

$$T^* = (-1)^{r(m-r)} * S * .$$

Proof. Let Ω_r denote $\Omega^r_{\mathrm{Hor}}(P;V)^K$. We have the following commutative diagram,

$$\begin{tikzcd} \Omega^*_{r+k} \arrow[r, "T^t"] & \Omega^*_r \\ \Omega_{r+k} \arrow[u] \arrow[r, "T^*"] & \Omega_r \arrow[u] \end{tikzcd}$$

where the vertical arrows are the isomorphisms given by the metrics, T^t is the dual map of T, and T^* is its adjoint. We have the following commutative diagram:

$$\begin{tikzcd} \Omega^*_{r+k} \arrow[r, "T^t"] & \Omega^*_r \\ \Omega_{m-(r+k)} \arrow[u, "\phi(\cdot,)"] \arrow[r, "S"] & \Omega_{m-r} \arrow[u, "\phi(\cdot,)"'] \\ \Omega_{r+k} \arrow[u, "*"] \arrow[r, "T^*"] & \Omega_r \arrow[u, "*"'] \end{tikzcd} .$$

The proposition now follows from the fact that on degree r we have $*^{-1}=(-1)^{r(m-r)}*$. $\square$

Part II

Higher-dimensional Reidemeister torsion invariants

Chapter 3

Higher-dimensional Reidemeister torsion

In this chapter we define the n-dimensional normalized Reidemeister torsion for a complete spin-hyperbolic 3-manifold of finite volume and an integer $n \geq 4$. We will refer to these invariants as the higher-dimensional Reidemeister torsion invariants.

Let (M, η) be a spin-hyperbolic 3-manifold, and ρ_n be its canonical n-dimensional representation. We want to define the Reidemeister torsion of M with respect to the representation ρ_n. However, to do that we need either M to be ρ_n-acyclic (i.e. the groups $\mathrm{H}^*(M; \rho_n)$ are all trivial), or, if it does not happen, to fix bases on (co)homology.

If M is compact, then, as a particular case of Raghunathan's vanishing Theorem (Corollary 2.1.2), the cohomology groups $\mathrm{H}^*(M; \rho_n)$ are all trivial. Thus for M closed the Reidemeister torsion $\tau(M; \rho_n)$ is already defined.

On the other hand, if M is non-compact, these groups do not need to be trivial, as we have seen in Chapter 2 (Corollary 2.2.7). Thus we need to choose bases in (co)homology in that case. Of course, if we want to get an invariant of the manifold we must choose bases in a somehow canonical way. Unfortunately, we do not know how to do this. Nevertheless, we have at least the following result. Its proof will be given in Section 3.2.

Remark. In the whole present chapter we will restrict ourselves to finite-volume manifolds. Thus M is the interior of a compact manifold $\overline{M}$ such that

$$\partial \overline{M} = T_1 \cup \cdots \cup T_l,$$

where each connected component T_i is homeomorphic to a torus T^2.

Proposition 3.0.8. *Let $n > 0$. For each connected boundary component T_i of M such that $\mathrm{H}^0(T_i; \rho_n)$ is not trivial, fix a non-trivial cycle $\theta_i \in \mathrm{H}_1(T_i; \mathbf{Z})$. Then there exists a canonical family of bases for the homology groups $\mathrm{H}_*(M; \rho_n)$ such that any basis of this family determines the same Reidemeister torsion, say $\tau(M; \rho_n; \{\theta_i\})$. Moreover, for all $k > 0$ the* *following quantities are independent of* $\{\theta_i\}$

$$\begin{aligned}\mathcal{T}_{2k+1}(M,\eta) &:= \frac{\tau(M;\rho_{2k+1};\{\theta_i\})}{\tau(M;\rho_3;\{\theta_i\})} \in \mathbf{C}^*/\{\pm 1\},\\ \mathcal{T}_{2k}(M,\eta) &:= \frac{\tau(M;\rho_{2k};\{\theta_i\})}{\tau(M;\rho_2;\{\theta_i\})} \in \mathbf{C}^*/\{\pm 1\}.\end{aligned}$$

Definition. Let (M,η) be a complete spin-hyperbolic 3-manifold of finite volume. For $n \geq 4$, the invariant $\mathcal{T}_n(M,\eta)$ defined in the above proposition will be called the *normalized n-dimensional Reidemeister torsion* of the spin-hyperbolic manifold (M,η). If $n = 2k+1$ is odd, $\mathcal{T}_{2k+1}(M;\eta)$ is independent of η, and will be denoted by $\mathcal{T}_{2k+1}(M)$.

The rest of this chapter is devoted to the proof of Proposition 3.0.8. To that end, we will analyse the groups $\mathrm{H}_*(M;\rho_n)$.

3.1 Cohomology of the boundary

Let T_j be a connected component of $\partial\overline{M}$ (recall that we are assuming that M has finite volume), and $U_j \cong T_j \times [0,\infty)$ be the corresponding cusp. It is well known that T_j can be identified with the set of rays contained in U_j, and that this endows T_j with a canonical similarity structure; in particular, T_j has a canonical holomorphic structure. Let us consider the canonical projection from U_j to T_j which sends a point in U_j to the ray it belongs to; denote this projection as

$$\pi_j \colon U_j \to T_j.$$

Let E_n be the flat vector bundle over $\overline{M}$ defined by the representation ρ_n. To compute $\mathrm{H}^*(T_i;\rho_n)$ we will interpret it as $\mathrm{H}^*(T_i;E_n)$, that is the cohomology of the de Rham complex

$$(\Omega^*(T_i;E_n), d_\nabla),$$

where d_∇ denotes the covariant differential defined by the flat connection on E_n. This complex is isomorphic to the complex $(\Omega^*(\widetilde{T}_i;V_n)^{\pi_1 T_i}, d)$ of equivariant V_n–valued differential forms on $\widetilde{T}_i$ with the usual exterior differential.

On the other hand, E_n is a holomorphic vector bundle with respect to the holomorphic structure of T_i. This yields the following canonical decomposition:

$$\Omega^1(T_i;E_n) = \Omega^{1,0}(T_i;E_n) \oplus \Omega^{0,1}(T_i;E_n),$$

where $\Omega^{1,0}(T_i;E_n)$ and $\Omega^{0,1}(T_i;E_n)$ are the spaces of E_n-valued 1-forms of type $(1,0)$ and $(0,1)$ respectively. Let us denote as $\mathrm{H}^{r,s}(T_i;E_n)$ the projection of $\Omega^{r,s}(T_i;E_n) \cap \operatorname{Ker} d$ onto $\mathrm{H}^1(T_i;E_n)$, with $(r,s) = (0,1),(1,0)$.

Proposition 3.1.1. *Assume that* $\mathrm{H}^0(T_i;E_n) \neq 0$. *Then,*

$$\mathrm{H}^1(T_i;E_n) = \mathrm{H}^{0,1}(T_i;E_n) \oplus \mathrm{H}^{1,0}(T_i;E_n).$$

Proof. We can assume that for all $\gamma \in \pi_1 T_i$ we have:

$$\mathrm{Hol}_M(\gamma) = \left[\begin{pmatrix} 1 & a(\gamma) \\ 0 & 1 \end{pmatrix}\right] \in \mathrm{PSL}(2, \mathbf{C}).$$

This choice of the holonomy representation gives a complex coordinated z on $\widetilde{T}_i$. Identifying V_n with the space of $(n-1)$-th degree homogeneous polynomials in the variables X and Y, we define the following two forms on $\Omega^1(\widetilde{T}_i; V_n)$,

$$\alpha = d\bar{z} \otimes X^{n-1}, \quad \beta = dz \otimes (zX + Y)^{n-1}.$$

Let us check that these forms are equivariant. Let $\gamma \in \pi_1 T_i$, and denote by L_γ the action of γ on $\widetilde{T}_i$. Notice that $L_\gamma(z) = z + a(\gamma)$. Hence, on one hand, we have:

$$\begin{aligned} L_\gamma^*(\alpha) &= d(\bar{z} + \bar{a}(\gamma)) \otimes X^{n-1} = \alpha, \\ L_\gamma^*(\beta) &= dz \otimes ((z + a(\gamma))X + Y)^{n-1}, \end{aligned}$$

and on the other hand:

$$\begin{aligned} \rho(\gamma)\alpha &= dz \otimes (\gamma \cdot X)^{n-1} = dz \otimes (\epsilon X)^{n-1}, \\ \rho(\gamma)\beta &= dz \otimes (z\gamma \cdot X + \gamma \cdot Y)^{n-1} = dz \otimes (\epsilon(zX + a(\gamma)X + Y))^{n-1}, \end{aligned}$$

where $\epsilon = \pm 1$ is the sign of the trace of γ determined by the lift of the holonomy representation. If n is odd, these two forms are clearly equivariant. If n is even, then the condition that $\mathrm{H}^0(T_i; E_n)$ is not trivial is equivalent to say that $\mathrm{Hol}_{(M,\eta)}(\sigma)$ has trace 2 for all $\sigma \in \pi_1 T_i$; hence, $\epsilon = 1$, and the two forms are equivariant. Since α and β are closed forms, they define cohomology classes in $\mathrm{H}^1(T_i; E_n)$, and hence $[\alpha] \in \mathrm{H}^{0,1}(T_i; E_n)$ and $[\beta] \in \mathrm{H}^{1,0}(T_i; E_n)$. To conclude the proof, it remains to prove that $[\alpha]$ and $[\beta]$ are linearly independent, as $\dim_{\mathbf{C}} \mathrm{H}^1(T_i; \rho_m) = 2$. This is equivalent to say that $[\alpha] \wedge [\beta] \in \mathrm{H}^2(T^2; \mathbf{C})$ is not zero. A simple computation shows that

$$\alpha \wedge \beta = \phi\left(X^{n-1}, (zX + Y)^{n-1}\right) d\bar{z} \wedge dz = d\bar{z} \wedge dz,$$

where ϕ is the non-degenerate $\mathrm{SL}(2, \mathbf{C})$–invariant pairing of V_n, see Section 1.3. This shows that $[\alpha] \wedge [\beta]$ is not zero, and hence the two classes must be linearly independent. $\square$

Remark. It may seem somehow artificial to consider the induced holomorphic structure on the tori T_i. Nevertheless, Proposition 3.1.1 shows that it yields a *canonical* decomposition (i.e. depending only on the hyperbolic structure) of the cohomology group $\mathrm{H}^1(T_i; E_n)$, which is all we need.

Next we want to characterize the image of the map induced by the inclusion

$$i^* \colon \mathrm{H}^1(\overline{M}; E_n) \to \mathrm{H}^1(\partial \overline{M}; E_n).$$

Although this description will not be complete, it will be enough to give bases for the homology groups $\mathrm{H}_*(M;\rho_n)$. Before analysing the general case, let us discuss briefly the case $n=3$.

The representation V_3 is the adjoint representation of $\mathrm{SL}(2;\mathbf{C})$, and the cohomology group $\mathrm{H}^1(M;E_3)$ has a geometrical interpretation in terms of infinitesimal deformations of the complete hyperbolic structure. The vector bundle E_3 is identified with the bundle of germs of Killing vector fields on M, and, with the same notation as in the proof of the above proposition, it can be checked that the global section X^2 corresponds to the vector field $\frac{\partial}{\partial z_j}$. With this description, the 1-form $d\bar{z}_j \otimes \frac{\partial}{\partial z_j}$ is a $(0,1)$-form that takes values in the vector bundle of holomorphic fields. According to the theory of deformations of complex manifolds, this cohomology class describes the deformations of the holomorphic structure of T_j by deformations of the defining lattice; in particular, it gives a deformation of the euclidean structure through euclidean structures. On the other hand, a non-trivial deformation of the complete hyperbolic structure is encoded by a cohomology class $\omega \in \mathrm{H}^1(M;E_3)$, and $i^*(\omega)$ encodes the corresponding deformation of the similarity structure in each torus. Since this deformation cannot be through euclidean structures on all tori (otherwise it will yield a complete hyperbolic structure on M, contradicting thus the Mostow-Prasad rigidity), then, for some T_j, the restriction of $i^*(\omega)$ to T_j can not be contained in $\mathrm{H}^{0,1}(T_j;E_3)$. This shows that we have the following decomposition:

$$\mathrm{H}^1(\partial\overline{M};E_3) = \operatorname{Im} i^* \bigoplus_{j=1}^{k} \mathrm{H}^{0,1}(T_j^2;E_3). \tag{3.1}$$

We will prove that the above decomposition holds also for $n \geq 2$. Since we do not have an interpretation of the cohomology group $\mathrm{H}^1(M;E_n)$ in geometrical terms such as deformations, we proceed in a different way. Our key tool will be Theorem 2.1.1, which states that a class $\omega \in \mathrm{H}^1(M;E_n)$ cannot be represented by a square-integrable form, with respect to a suitable inner product on E_n. Let us recall the definition of the inner product on E_n. Choose any $\mathrm{SU}(2)$–invariant inner product $\langle\cdot,\cdot\rangle$ on V_n (we are considering $\mathrm{SU}(2)$ as a subgroup of $\mathrm{SL}(2,\mathbf{C})$). Identify $\mathbf{H}^3$ with $\mathrm{SL}(2,\mathbf{C})/\,\mathrm{SU}(2)$, and let $p \in \mathbf{H}^3$ be the class of the identity. Define an inner product on the trivial vector bundle $\mathbf{H}^3 \times V_n$ by

$$\langle (q,w_1),(q,w_2)\rangle_q = \langle g w_1, g w_2\rangle, \quad \text{where } g\cdot q = p.$$

Then it induces an inner product on the vector bundle $E_n = \mathbf{H}^3 \times_{\pi_1(M,p)} V_n$.

Lemma 3.1.2. *Assume that* $\mathrm{H}^0(T_j;E_n) \neq 0$. *Then there exists a form* $\alpha_j \in \Omega^{0,1}(T_j;E_n)$ *representing a non-trivial element in* $\mathrm{H}^{0,1}(T_j;E_n)$ *such that* $\pi_j^*(\alpha_j) \in \Omega^1(U_j;E_n)$ *is* L^2.

Proof. Let us work in the model of the half-space $\mathbf{H}^3 = \mathbf{C} \times (0,\infty)$. If $(z,t) = (x,y,t) \in \mathbf{H}^3$, the metric is given by

$$g = \frac{1}{t^2}(dx^2 + dy^2 + dt^2).$$

Proceeding as in the proof of Proposition 3.1.1, we obtain the form $\alpha = d\bar{z} \otimes X^{n-1}$. We will be done if we prove that $\pi_j^*(\alpha)$ is L^2. To compute the norm of $d\bar{z} \otimes X^{n-1}$, we may assume that the cusp U_j is isometric to $\mathbf{C}\times[1,\infty)/(\mathrm{Hol}_M\,\pi_1T^2)$. Thus we have:

$$|d\bar z\otimes X^{n-1}|_{(w,t)}=|d\bar z|_{(w,t)}|X^{n-1}|_{(w,t)}.$$

On one hand,

$$|d\bar z|^2_{(w,t)}=|dx|^2_{(w,t)}+|dy|^2_{(w,t)}=2t^2.$$

On the other hand, by definition of the metric of E_n, it can checked that

$$|X^{n-1}|^2_{(w,t)}=t^{1-n}|X^{n-1}|^2,$$

where $|X^{n-1}|$ is the norm of X^{n-1} in V_n with respect to the fixed hermitian metric. Therefore, if R is a fundamental domain for T^2, we get

$$\int_{U_j}|d\bar z\otimes X^{n-1}|^2d\,\mathrm{Vol}_{U_j}=2|X^{n-1}|^2\int_{R\times[1,\infty]}\frac{t^{3-n}}{t^3}dxdydt=C\int_1^\infty t^{-n}dt<\infty,$$

and the lemma is proved. □

Now we can prove that the decomposition (3.1) holds for all $n\geq 2$.

Proposition 3.1.3. *Assume that $T_1,\ldots,T_r$ are all the connected components of $\partial\overline{M}$ such that $\mathrm{H}^0(T_j;E_n)\neq 0$. Then we have the following decomposition:*

$$\bigoplus_{j=1}^r\mathrm{H}^1(T_j;E_n)=\mathrm{Im}\,i^*\bigoplus_{j=1}^r\mathrm{H}^{0,1}(T_j;E_n).$$

Proof. It is enough to prove that $\mathrm{Im}\,i^*\cap\bigoplus_{j=1}^r\mathrm{H}^{0,1}(T_j;E_n)=0$. Let $[\omega]\in\mathrm{H}^1(M;E_n)$ such that $i^*([\omega])\in\bigoplus_{j=1}^k\mathrm{H}^{0,1}(T_j^2;E_n)$. Let us work with the cusps $U_j\cong T_j\times(0,\infty)$, and assume that they are disjoint. Let α_j be the forms given by the above lemma. Then

$$\omega=\lambda_j\pi_i^*(\alpha_j)+df_j,\quad\text{on }U_j,$$

for some $\lambda_j\in\mathbf{C}$ and $f_j\in\Omega^0(U_j;E_n)$. Let $F\in\Omega^0(M;E_n)$ such that $F_{|T_j\times[1,\infty)}=f_j$ and vanishing outside the cusps. By the above lemma, $\omega-dF$ is L^2, and hence the class $[\omega]$ has an L^2 representative, which implies that $[\omega]=0$, as we wanted to prove. □

3.2 The homology groups $\mathrm{H}_*(M;\rho_n)$

The aim of this section is to prove Proposition 3.0.8 concerning the existence of a distinguished family of bases for the groups $\mathrm{H}_*(M;\rho_n)$.

We will use the following construction for the homology of a finite CW–complex X in the local system defined by a representation $\rho\colon\pi_1(X,p)\to\mathrm{GL}(V)$. Consider the right action of $\pi_1(X,p)$ on V, so that $\gamma\in\pi_1(X,p)$ maps $v\in V$ to $\rho(\gamma)^{-1}v$. We will write V_ρ to emphasize the fact that V is a $\pi_1(X,p)$–right module. Let $C_*(\widetilde{X};\mathbf{Z})$ denote the complex of singular chains on the universal covering, in which $\pi_1(X,p)$ acts on the left by deck transformations, and let

$$C_*(X;V_\rho) = V_\rho \otimes_{\mathbf{C}[\pi_1(X,p)]} C_*(\widetilde{X};\mathbf{Z}).$$

Then $\mathrm{H}_*(X;\rho)$ is the homology of the following complex of $\mathbf{C}$-vector spaces,

$$(C_*(X;V_\rho), \mathrm{Id} \otimes \partial_*)\,.$$

We will use the Kronecker pairing between homology and cohomology with twisted coefficients. To define it we need an invariant and non-degenerated bilinear map

$$\phi\colon V \times V \to \mathbf{C}.$$

If X is a differentiable manifold, then the Kronecker pairing can be defined at the level of smooth chains and forms as follows:

$$\begin{aligned} C_r(X;V_\rho) \times \Omega^r(\widetilde{X};V_\rho)^{\pi_1 X} &\longrightarrow \mathbf{C} \\ (v_\theta \otimes \theta, \omega \otimes v_\omega) &\longmapsto \int_\theta \phi(v_\theta, v_\omega)\omega \end{aligned}$$

The Kronecker pairing does not depend on the different choices, but on the respective classes in cohomology and homology, and it is natural and non-degenerate.

We want to prove the following result from which Proposition 3.0.8 is immediately deduced. Before stating it, let us recall the following definition.

Definition. Let $\theta_1, \theta_2 \in \mathrm{H}_1(T_j;\mathbf{Z})$ be two non-trivial cycles in a boundary component T_j of $\overline{M}$. Using the natural identification between $\mathrm{H}_1(T_j;\mathbf{Z}) \cong \pi_1 T_j$, let us assume that

$$\mathrm{Hol}_M(\theta_i) = \left[\begin{pmatrix} 1 & a(\theta_i) \\ 0 & 1 \end{pmatrix}\right] \in \mathrm{PSL}(2,\mathbf{C})$$

for some $a(\theta_i) \in \mathbf{C}^*$, $i = 1, 2$. Then define the *cusp shape* of the pair (θ_1, θ_2) as

$$\mathrm{cshape}(\theta_1, \theta_2) = \frac{a(\theta_1)}{a(\theta_2)}.$$

Notice that $\mathrm{cshape}(\theta_1,\theta_2)$ is well defined, because $a\colon \pi_1 T_j \to \mathbf{C}$ is unique up to homothety.

Proposition 3.2.1. *Let $T_1, \ldots, T_r$ be the boundary components of $\overline{M}$ that are* not *ρ_n-acyclic. Let $G_j < \pi_1(M,p)$ be some fixed realization of the fundamental group of T_j as a subgroup of $\pi_1(M,p)$. For each T_j choose a non-trivial cycle $\theta_j \in \mathrm{H}_1(M;\mathbf{Z})$, and a non-trivial vector $w_j \in V_n$ fixed by $\rho_n(G_j)$. If $i_j\colon T_j \to M$ denotes the inclusion, then we have:*

1. *A basis for $\mathrm{H}_1(M;\rho_n)$ is given by*

$$(i_{1*}([w_1 \otimes \theta_1]), \ldots, i_{r*}([w_r \otimes \theta_r]))\,.$$

2. *Let $[T_j] \in \mathrm{H}_2(T_j;\mathbf{Z})$ be a fundamental class of T_j. A basis for $\mathrm{H}_2(M;\rho_n)$ is given by*

$$(i_{1*}([w_1 \otimes T_1]), \ldots, i_{r*}([w_r \otimes T_r]))\,.$$

3. *If we choose different non-trivial cycles $\theta'_1, \ldots, \theta'_r$ then*

$$i_{j*}([w_j \otimes \theta_j]) = \mathrm{cshape}(\theta_j, \theta'_j) i_{j*}([w_j \otimes \theta'_j]).$$

Proof. Let $[\alpha_j]$ and $[\beta_j]$ be generators of $\mathrm{H}^{0,1}(T_j^2; E_n)$ and $\mathrm{H}^{1,0}(T_j^2; E_n)$ respectively. We claim that the Kronecker pairing $([w_j \otimes \theta_j], [\alpha_k])$ is zero for all j, k, and $([w_j \otimes \theta_j], [\beta_k])$ is zero if and only if $j \neq k$. We can assume that $k = j$. Let us fix T_j. Proceeding as in the proof of Proposition 3.1.1, we may assume that $w_j = X^{n-1}$, $\alpha_j = d\bar{z} \otimes X^{n-1}$ and $\beta_j = dz \otimes (zX+Y)^{n-1}$. We have

$$([w_j \otimes \theta_j], [\beta_j]) = \int_\theta \phi\left(X^{n-1}, (zX+Y)^{n-1}\right) dz = \int_\theta \phi\left(X^{n-1}, Y^{n-1}\right) dz = \int_\theta dz \neq 0. \quad (3.2)$$

On the other hand, since $\phi(X^{n-1}, X^{n-1}) = 0$, $([w_j \otimes \theta], [\alpha_j]) = 0$. This proves the claim.

Let us prove now the first assertion. Assume that we have:

$$\sum_{j=1}^{r} \lambda_j i_*[w_j \otimes \theta_j] = 0, \quad \text{with } \lambda_j \in \mathbf{C}.$$

The naturality and the non-degeneracy of the Kronecker pairing imply that this is equivalent to

$$\sum_{j=1}^{r} \lambda_j \left(w_j \otimes \theta_j, i^*(\omega)\right) = 0, \quad \text{for all } [\omega] \in \mathrm{H}^1(M; E_n),$$

where $(\cdot,\cdot)$ denotes the Kronecker pairing. By Proposition 3.1.3, each β_j is uniquely written as

$$\beta_j = \gamma_j + \sum_{k=1}^{r} \mu_j^k \alpha_k, \quad \text{with } \gamma_j \in \operatorname{Im} i^* \text{ and } \mu_j^k \in \mathbf{C}.$$

Moreover, $(\gamma_1, \ldots, \gamma_r)$ is a basis of $\operatorname{Im} i^*$. The preceding discussion then implies $\lambda_j = 0$ for all j. The first assertion is thus proved.

Let us prove Assertion 2. The long exact sequence in homology for the pair $(\overline{M}, \partial\overline{M})$ shows that the inclusion $\partial\overline{M} \subset \overline{M}$ yields an isomorphism

$$i_*\colon\ \mathrm{H}_2(\partial\overline{M}; E_n) = \bigoplus_{j=1}^{r} \mathrm{H}_2(T_j; E_n) \to \mathrm{H}_2(M; E_n).$$

Thus it is enough to prove that $[w_j \otimes T_j]$ is not zero. This can be proved using Poincaré duality PD. Indeed, if we identify $\mathrm{H}^0(T_j; E_n)$ with the subspace of V_n of invariant vectors, then it can be checked that

$$\mathrm{PD}(w_j) = [w_j \otimes T_j].$$

Assertion 3 follows easily from Equation (3.2). □

Chapter 4

Behaviour under hyperbolic Dehn filling

The aim of this chapter is to analyse the behaviour of the n-dimensional Reidemeister torsion under hyperbolic Dehn surgery. Before discussing it, we need to fix some notation.

Throughout this chapter M will denote an oriented complete hyperbolic 3-manifold of finite volume with l cusps. For each connected boundary component T_i of M we fix two closed simple oriented curves a_i, b_i in T_i generating $\mathrm{H}_1(T_i;\mathbf{Z})$. We define the following sets:

$$\begin{aligned}
\mathcal{A} &= \{(p,q) = (p_1,\dots,p_l,q_1,\dots,q_l) \in \mathbf{Z}^l \times \mathbf{Z}^l \mid \gcd(p_i,q_i) = 1\},\\
\mathcal{A}_M &= \{(p,q) \in \mathcal{A} \mid M_{p/q} := M_{p_1/q_1,\dots,p_l/q_l} \text{ is hyperbolic }\}.
\end{aligned}$$

Remark. We may regard $\mathcal{A}$ as a directed set with respect to the following preorder:

$$(p,q) \leq (p',q') \Leftrightarrow (p_i)^2 + (q_i)^2 \leq (p'_i)^2 + (q'_i)^2 \text{ for all } i = 1,\dots,l.$$

The hyperbolic Dehn surgery theorem by Thurston implies that $\mathcal{A}_M$ is also a directed subset of $\mathcal{A}$, namely any two elements of $\mathcal{A}_M$ have a common greater element. The limit of an $\mathcal{A}_M$-net $\{x_{p/q}\}$ in some topological space, when it exists, will be denoted by:

$$\lim_{(p,q)\to\infty} x_{p/q}.$$

In analysing the relation between the n-dimensional torsion invariants of M with those of $M_{p/q}$, some issues arise. In order to discuss them, we distinguish two cases according to the parity of n.

We consider first the case $n = 2k+1$, with $k > 0$. In that case we find two difficulties. The first one is that we need some extra data in order to define the torsion invariant for M (we must choose non-trivial cycles $\theta_i \in \mathrm{H}_i(T_i;\mathbf{Z})$), whereas for $M_{p/q}$ this is already defined. The second one is due to the following result proved in [Por97, p. 110] (notice that our torsion is the inverse of the one considered in [Por97]),

$$\lim_{(p,q)\to\infty} \left|\tau_3(M_{p/q})\right| = 0.$$

The proof of the above limit also works for any odd number $n \geq 3$. Moreover, the asymptotic growth of these sequences does not depend on the dimension n. These facts suggest that the above question should be formulated in terms of normalized torsions. In that case, we will prove the following result.

Proposition 4.0.2. *The set of cluster points of the following net in* $\mathbf{C}/\{\pm 1\}$*,*

$$\left\{\mathcal{T}_{2k+1}(M_{p/q})\right\}_{(p,q)\in\mathcal{A}_M},$$

is the segment joining the origin and the point $2^{2(k-1)l}\mathcal{T}_{2k+1}(M)$*.*

Let us analyse now the even dimensional case $n = 2k$, for $k > 0$. In this case, the main difficulty comes from the fact that we need a spin structure to define the n-dimensional torsion invariant. Hence, we somehow need a way to relate spin structures on M with those of $M_{p/q}$. To that end, for a fixed spin structure η on M, we define the following set

$$\mathcal{A}_{M,\eta} = \left\{(p,q) \in \mathcal{A}_M \mid \eta \text{ can be extended to } M_{p/q} \supset M\right\}.$$

Remark. Notice that if η can be extended to $M_{p/q}$ then the extension is unique (this follows from the fact if a spin structure on ∂D^2 can be extended to D^2, then the extension is unique). In such case the extension will be denoted by $\eta_{p/q}$.

Using Corollary 1.2.3, we get easily the following characterization of $\mathcal{A}_{M,\eta}$.

Proposition 4.0.3. *For each* T_i *let* $\epsilon_{a_i}, \epsilon_{b_i} = \pm 1$ *be the sign of the trace of* $\mathrm{Hol}_{(M,\eta)}(a_i)$ *and* $\mathrm{Hol}_{(M,\eta)}(b_i)$ *respectively. Then* $(p,q) \in \mathcal{A}_{M,\eta}$ *if and only if*

$$\epsilon_{a_i}^{p_i}\epsilon_{b_i}^{q_i} = -1, \quad \textit{for all } i = 1,\ldots,l.$$

Definition. We will say that a spin structure η on M is *compactly isolated* if $\mathcal{A}_{M,\eta}$ is empty; otherwise, we will say that η is *compactly approximable.*

As a corollary of the above proposition and the definition of an acyclic spin structure, we get the following result.

Corollary 4.0.4. *A spin structure* η *of* M *is compactly approximable if and only if it is acyclic.*

Remark. If η is compactly approximable, Proposition 4.0.3 implies that $\mathcal{A}_{M,\eta}$ is infinite; in particular, $\mathcal{A}_{M,\eta}$ is a directed set as well. The terminology introduced in the above definition is coherent with the geometric topology of the space $\mathcal{MS}$ of spin-hyperbolic 3-manifolds, see Chapter 5. For instance, if η is compactly approximable then the net of compact spin-hyperbolic manifolds $\{(M_{p/q}, \eta_{p/q})\}_{(p,q)\in\mathcal{A}_{M,\eta}}$ converges to (M,η) in $\mathcal{MS}$.

If η is compactly approximable, then $\mathrm{H}^*(M;\rho_{2k}) = 0$ for all $k > 0$, and hence it makes sense to consider the Reidemeister torsion $\tau(M;\rho_{2k})$. On the other hand, for all $(p,q) \in$

$\mathcal{A}_{M,\eta}$ we have the $2k$-dimensional canonical representation of the spin-hyperbolic manifold $(M_{p/q}, \eta_{p/q})$:

$$\rho_{2k}^{p/q} \colon \pi_1 M_{p/q} \to \mathrm{SL}(2k, \mathbf{C}).$$

The compactness of $M_{p/q}$ guarantees the acyclicity of this representation. Hence, it also makes sense to consider $\tau(M_{p/q}; \rho_{2k}^{p/q})$. We will prove the following result.

Proposition 4.0.5. *Let η be a compactly approximable (or acyclic) spin structure on M. The set of cluster points of the following net in $\mathbf{C}/\{\pm 1\}$,*

$$\left\{ \pm\, \tau(M_{p/q}; \rho_{2k}^{p/q}) \right\}_{(p,q)\in\mathcal{A}_{M,\eta}},$$

is the segment joining the origin and $\pm 2^{2kl}\, \tau(M; \rho_{2k})$.

The proof of both propositions will be based on surgery formulas for the torsion, which will be deduced from the Mayer-Vietoris formula. These formulae involve the spin complex lengths of the core geodesics added on the Dehn filling. The above results then will follow essentially from the fact that the cluster point set of the imaginary part of the spin complex lengths of the added geodesics in $M_{p/q}$, as (p, q) varies in $\mathcal{A}_{M,\eta}$, is $\mathbf{R}/\langle 4\pi \rangle$, see [Mey86].

The rest of this chapter is organized as follows. The first section is a brief account of the deformations of the holonomy representation of M. The second and third sections contain the proofs of Propositions 4.0.5 and 4.0.2 respectively.

4.1 Deformations

Consider a family of continuous local deformations of the complete hyperbolic structure of M given by:

$$\mathrm{Hol}_M \colon U \times \pi_1 M \to \mathrm{PSL}(2, \mathbf{C}), \quad U \subset \mathbf{C}^l,$$

with U an open ball containing the origin, and with

$$\mathrm{Hol}_M(0, \gamma) = \mathrm{Hol}_M(\gamma), \quad \text{for all } \gamma \in \pi_1(M).$$

The open set U is usually called Thurston's slice, and is a double branched covering of a neighborhood of the variety of characters of M around the complete hyperbolic structure. If we fix a boundary component T_i, then we can assume that

$$\mathrm{Hol}_M(u, a_i) = \left[\begin{pmatrix} e^{u_i/2} & 1 \\ 0 & e^{-u_i/2} \end{pmatrix} \right], \quad \mathrm{Hol}_M(u, b_i) = \left[\begin{pmatrix} e^{v_i(u)/2} & \tau_i(u) \\ 0 & e^{-v_i(u)/2} \end{pmatrix} \right],$$

where $v_i(u)$ and $\tau_i(u)$ are analytic functions on u which are related by

$$\sinh \frac{v_i(u)}{2} = \tau_i(u) \sinh \frac{u_i}{2}.$$

This last equation follows by imposing that the two matrices commute.

By Thurston's surgery theorem, for (p,q) large enough, the holonomy representation of the complete hyperbolic structure of $M_{p/q}$ is given at some value of u, say $u^{p/q}$. More concretely, we have the following commutative diagram,

$$\begin{tikzcd} \pi_1(M) \arrow[d, "i_*^{p/q}"'] \arrow[dr, "\mathrm{Hol}_M(u_{p/q},\cdot)"] & \\ \pi_1(M_{p/q}) \arrow[r, "\mathrm{Hol}_{M_{p/q}}"'] & \mathrm{PSL}(2;\mathbf{C}) \end{tikzcd}$$

where $i_*^{p/q}$ is the induced morphism on the fundamental groups by the inclusion

$$i^{p/q}\colon M \hookrightarrow M_{p/q}.$$

The map $i_*^{p/q}$ is surjective with kernel the normal subgroup generated by the curves $\{a_i^{p_i} b_i^{q_i}\}$ (here we are identifying $\mathrm{H}_1(T_i;\mathbf{Z})$ with $\pi_1 T_i$, and the latter group with a subgroup of $\pi_1 M$), we have the so-called Dehn filling equations

$$p_i u_i^{p/q} + q_i v_i(u^{p/q}) = 2\pi i, \quad \text{for all } i = 1, \ldots, l. \tag{4.1}$$

Moreover, we also have:

$$\lim_{(p,q)\to\infty} u^{p/q} = 0.$$

4.2 Even dimensional case

Let us retain the notation used in the previous section. Fix a spin structure η on M, and consider the lift of the whole family of representations $\mathrm{Hol}_M(u,\cdot)$ starting at $u=0$ with $\mathrm{Hol}_{(M,\eta)}$. By continuity, all these lifts are also group morphisms. Thus we obtain a family of representations

$$\mathrm{Hol}_{(M,\eta)}\colon U \times \pi_1 M \to \mathrm{SL}(2,\mathbf{C}).$$

The representation $\mathrm{Hol}_{(M,\eta)}(u_{p/q},\cdot)$ of $\pi_1 M$ needs no longer to yield a representation of $\pi_1 M_{p/q}$. We can characterize this condition in terms of spin structures.

Lemma 4.2.1. *The representation* $\mathrm{Hol}_{(M,\eta)}(u^{p/q},\cdot)$ *yields a representation of* $\pi_1 M_{p/q}$ *if and only if* $(p,q) \in \mathcal{A}_{M,\eta}$.

Proof. $\mathrm{Hol}_{(M,\eta)}(u^{p/q},\cdot)$ yields a representation of $\pi_1 M_{p/q}$ if and only if

$$\mathrm{Hol}_{(M,\eta)}(u_{p/q}, a_i^{p_i} b_i^{q_i}) = \mathrm{Id} \in \mathrm{SL}(2,\mathbf{C}), \quad \text{for all } i = 1,\ldots,l.$$

By Proposition 4.0.3, $(p,q) \in \mathcal{A}_{M,\eta}$ if and only if

$$\epsilon_{a_i}^{p_i} \epsilon_{b_i}^{q_i} = -1 \quad \text{for all } i = 1,\ldots,l,$$

where $\epsilon_{a_i}, \epsilon_{b_i} = \pm 1$ is the sign of the trace of $\mathrm{Hol}_{(M,\eta)}(0, a_i)$ and $\mathrm{Hol}_{(M,\eta)}(0, b_i)$ respectively. On the other hand, for a fixed i we can assume that

$$\mathrm{Hol}_{(M,\eta)}(u, a_i) = \epsilon_{a_i}(u) \begin{pmatrix} e^{u_i/2} & 1 \\ 0 & e^{-u_i/2} \end{pmatrix}, \quad \mathrm{Hol}_M(u, b_i) = \epsilon_{b_i}(u) \begin{pmatrix} e^{v_i(u)/2} & \tau_i(u) \\ 0 & e^{-v_i(u)/2} \end{pmatrix}.$$

By continuity, for u close to 0, $\epsilon_{a_i}(u) = \epsilon_{a_i}$ and $\epsilon_{b_i}(u) = \epsilon_{b_i}$. Thus, Equation (4.1) yields:

$$\left(\mathrm{Hol}_{(M,\eta)}(u^{p/q}, a_i)\right)^{p_i} \left(\mathrm{Hol}_{(M,\eta)}(u^{p/q}, b_i)\right)^{q_i} = -\epsilon_{a_i}^{p_i} \epsilon_{b_i}^{q_i} \,\mathrm{Id},$$

The result then follows immediately. □

Now let η be a *compactly approximable* (equivalently, acyclic) spin structure on M. Consider the composition of $\mathrm{Hol}_{(M,\eta)}(u, \cdot)$ with the $2k$-dimensional irreducible representation of $\mathrm{SL}(2, \mathbf{C})$

$$\rho_{2k} \colon U \times \pi_1 M \to \mathrm{SL}(2k, \mathbf{C}).$$

Since η is acyclic, for $u = 0$ the representation $\rho_{2k}(u, \cdot)$ is acyclic. The following more or less well-known result then implies that $\rho_{2k}(u, \cdot)$ is also acyclic for u close to 0.

Proposition 4.2.2. *Let X be a finite CW–complex, and consider a continuous family of representations*

$$\rho \colon U \times \pi_1(X, x_0) \to \mathrm{GL}(n, \mathbf{C}),$$

where U is some space of parameters. For a fixed $m \geq 0$, define the map $F \colon U \to \mathbf{Z}$ by $F(u) = \dim \mathrm{H}_m(X; \rho_u)$, where $\rho_u := \rho(u, \cdot)$. Then F is upper semicontinuous, that is,

$$\limsup_{u \to u_0} F(u) \leq F(u_0), \quad \textit{for all } u_0 \in U.$$

Proof. The idea is that the rank of a matrix, viewed as a map from the space of matrices to $\mathbf{Z}$, is lower a semicontinuous function. The details are as follows. The homology groups $\mathrm{H}_*(X; \rho_u)$ can be defined as the homology groups of the complex

$$\left(V \otimes_{\rho(u)} C_*(\widetilde{X}; \mathbf{Z}), \mathrm{Id} \otimes \partial_*\right).$$

Let us fix $(w_1, \ldots, w_n)$ a basis of V. Let $\{e^j_1, \ldots, e^j_{i_j}\}$ be the cells of X of dimension j, and let $\{\tilde{e}^j_1, \ldots, \tilde{e}^j_{i_j}\}$ be fixed lifts of these cells to $\widetilde{X}$. Then the set $\{w_i \otimes \tilde{e}^j_k\}$ gives a basis of $V \otimes_{\rho_u} C_j(\widetilde{X}; \mathbf{Z})$. With respect to these bases, the boundary map $\partial_j(u)$ is written as a matrix $A_j(u)$ whose entries depend continuously on u. Then we have

$$F(u) = \dim \mathrm{Ker}\, A_j(u) - \mathrm{rank}\, A_{j+1}(u).$$

Since the rank of a matrix is lower semicontinuous, the dimension of the kernel is upper-semicontinuous, and hence $F(u)$ is upper semicontinuous. □

Remark. The above result can be regarded as a special case of a further result stated in [Har77] as "The semicontinuity Theorem", which establishes the upper semicontinuity of the dimension function of some cohomology groups in a much more general context.

Let us put $\rho_{2k}(u) := \rho_{2k}(u,\cdot)$. The above proposition shows that it makes sense to consider $\tau(M;\rho_{2k}(u))$ for u close to 0. On the other hand, for $(p,q) \in \mathcal{A}_{M,\eta}$ large enough, Lemma 4.2.1 implies that we have the following commutative diagram:

$$\begin{tikzcd} \pi_1(M) \arrow[d, "i_*^{p/q}"'] \arrow[dr, "\rho_{2k}(u_{p/q})"] & \\ \pi_1(M_{p/q}) \arrow[r, "\rho_{2k}^{p/q}"] & \mathrm{SL}(2k;\mathbf{C}) \end{tikzcd}$$

Since $M_{p/q}$ is compact, the representation $\rho_{2k}^{p/q}$ is acyclic. Therefore, it also makes sense to consider $\tau(M_{p/q};\rho_{2k}^{p/q})$. The following lemma gives the relationship between these two quantities.

Lemma 4.2.3. *Let $\gamma_1,\ldots,\gamma_l$ be the core geodesics added on the (p,q)-Dehn filling $M_{p/q}$, and $\lambda_{p/q}$ be the spin-complex-length function with respect to the spin-hyperbolic structure $\eta_{p/q}$. Then we have*

$$\tau(M_{p/q};\rho_{2k}^{p/q}) = \pm\,\tau(M;\rho_{2k}(u_{p/q}))\prod_{j=0}^{k-1}\prod_{i=1}^{l}\left(e^{(\frac{1}{2}+j)\lambda_{p/q}(\gamma_i)}-1\right)\left(e^{-(\frac{1}{2}+j)\lambda_{p/q}(\gamma_i)}-1\right).$$

Proof. By induction, we can assume that M has only one cusp. We will apply the Mayer-Vietoris sequence to the decomposition $M_{p/q} = M \cup N(\gamma)$, where $N(\gamma)$ is a tubular neighbourhood of the core geodesic γ added on the Dehn filling. We must show first that all the involved spaces are $\rho_{2k}^{p/q}$-acyclic. We already know it for M. Since $\mathrm{Hol}_{M_{p/q}}(\gamma)$ has no fixed vector other than 0, $\mathrm{H}^0(\gamma;\rho_{2k}^{p/q})$ is trivial, and hence so is $\mathrm{H}^1(\gamma;\rho_{2k}^{p/q})$; this proves that $N(\gamma)\simeq\gamma$ is acyclic. The same argument shows that $\mathrm{H}^r(\partial N(\gamma);\rho_{2k}^{p/q})$ is trivial for $r=0,2$, which implies (Euler characteristic argument) that this holds for $r=1$ as well. The Mayer-Vietoris sequence then yields the formula,

$$\tau(M_{p/q};\rho_{2k}^{p/q})\,\tau(\partial N(\gamma);\rho_{2k}^{p/q}) = \tau(M;\rho_{2k}^{p/q})\,\tau(\gamma;\rho_{2k}^{p/q}).$$

The torsion of the torus $\partial N(\gamma)$ is ± 1, as it is the Reidemeister torsion of an even dimensional manifold, see [Mil62]. Finally, an easy computation shows that

$$\tau(\gamma;\rho_{2k}^{p/q}) = \prod_{j=0}^{k-1}\left(e^{(\frac{1}{2}+j)\lambda_{p/q}(\gamma)}-1\right)\left(e^{-(\frac{1}{2}+j)\lambda_{p/q}(\gamma)}-1\right).$$

$\square$

Now we can prove Proposition 4.0.5.

Proof of Propostion 4.0.5 . The formula of Lemma 4.2.3 can be written as

$$\frac{\tau(M_{p/q};\rho_{2k}^{p/q})}{\tau(M;\rho_{2k}(u_{p/q}))} = 2^{2kl}\prod_{j=0}^{k-1}\prod_{i=1}^{l}\frac{1-\cosh\left((\frac{1}{2}+j)\lambda_{p/q}(\gamma_i)\right)}{2}.$$

Since $\tau(M;\rho_{2k}(u_{p/q}))$ converges to $\tau(M;\rho_{2k})$ as (p,q) goes to infinity (this can be proved in the same way as Proposition 4.2.2), to prove the result we may restrict our attention to the product of the right hand side of the above equation. Consider the map defined by

$$\begin{aligned} F\colon [0,\infty)\times[0,4\pi] &\longrightarrow \mathbf{C} \\ (t,\theta) &\longmapsto \prod_{j=0}^{k-1}\frac{1-\cosh\left((\frac{1}{2}+j)(t+\theta\,\mathrm{i})\right)}{2}. \end{aligned}$$

The image of $\{0\}\times[0,4\pi]$ under F is $[0,1]$, since $F(\{0\}\times[0,4\pi])\subset[0,1]$, $F(0,0)=0$ and $F(0,2\pi)=1$. The result then follows from the fact that the real part of $\lambda_{p/q}(\gamma_i)$ goes to zero, and that the cluster point set of the imaginary parts of $(\lambda_{p/q}(\gamma_1),\ldots,\lambda_{p/q}(\gamma_l))_{(p,q)\in\mathcal{A}_{M,\eta}}$ is dense on $[0,4\pi]^l$, see [Mey86]. $\square$

4.3 Odd dimensional case

We will use the same notation as in the previous sections. Throughout this section we will assume that $n=2k+1$ and $k>0$.

Lemma 4.3.1. *Let T_j be a fixed boundary component of $\partial\overline{M}$. Assume that*

$$\mathrm{Hol}_M(u,a_j)=\left[\begin{pmatrix} e^{u_j/2} & 1 \\ 0 & e^{-u_j/2}\end{pmatrix}\right],\quad \mathrm{Hol}_M(u,b_j)=\left[\begin{pmatrix} e^{v_j(u)/2} & \tau_j(u) \\ 0 & e^{-v_j(u)/2}\end{pmatrix}\right],$$

where a_j,b_j are generators of the fundamental group of T_j. For $u=(u_1,\ldots,u_l)\in U\subset\mathbf{C}^l$ such that $u_j\neq 0$, consider the following vector

$$w_j(u):=X^k\left(X-2\sinh\frac{u_j}{2}Y\right)^k\in V_{2k+1}\cong S_{2k}[X,Y],$$

where $S_n[X,Y]$ is the space of homogeneous polynomials of degree n in the variables X,Y. Then, for u close to 0 with $u_j\neq 0$, the vector $w_j(u)$ is $\rho_{2k+1}(u)$-invariant. Moreover, the map

$$\begin{aligned} \Omega^*(T_j;\mathbf{C}) &\to \Omega^*(T_j;E_{\rho_{2k+1}(u)}) \\ \omega &\mapsto \omega\otimes w_j(u) \end{aligned}$$

induces isomorphisms in de Rham cohomology.

Proof. Let $\mathrm{Hol}_{(M,\eta)}$ be a lift of the holonomy representation. Since the matrices $\mathrm{Hol}_{(M,\eta)}(a_j)$ and $\mathrm{Hol}_{(M,\eta)}(b_j)$ diagonalize and commute, there exists a basis (e_1, e_2) of $\mathbf{C}^2$ that simultaneously diagonalize them. It can be checked that we can take

$$e_1 = X, \qquad e_2 = X - 2\sinh\frac{u_j}{2}Y.$$

The vector $e_1^k e_2^k \in V_{2k+1}$ is then independent of the chosen lift and invariant by both $\mathrm{Hol}_{(M,\eta)}(a_j)$ and $\mathrm{Hol}_{(M,\eta)}(b_j)$. This shows that $w_j(u)$ is $\rho_{2k+1}(u)$-invariant, and the first part of the lemma is proved.

For the second part, notice that the vector $w_j(u)$ gives a parallel nowhere-vanishing section of the flat vector bundle $E_{\rho_{2k+1}(u)}$. On the other hand, the $\mathrm{SL}(2,\mathbf{C})$-invariant pairing (see Section 1.3)

$$\phi\colon V_n \times V_n \to \mathbf{C},$$

defines a non-degenerate symmetric bilinear form on $E_{\rho_{2k+1}(u)}$. We have,

$$\phi\left(w_j(u), w_j(u)\right) = 2\left(-2\sinh\frac{u_j}{2}\right)^k.$$

Therefore, for $\sinh\frac{u_j}{2} \neq 0$, we have a decomposition $E_{\rho_{2k+1}(u)|T_j} = L \oplus L^\perp$, where L is the line bundle defined by $w_j(u)$, and $L^\perp$ is the orthogonal complement with respect to ϕ. Note that both sub-bundles are flat, so we have

$$\mathrm{H}^*(T_j; E_{\rho_{2k+1}(u)}) = \mathrm{H}^*(T_j; L) \oplus \mathrm{H}^*(T_j; L^\perp).$$

The line bundle L is trivialized using the section $w_j(u)$. Therefore, tensorization by $w_j(u)$ yields an isomorphism $\mathrm{H}^0(T_j; L) \cong \mathrm{H}^0(T_j; \mathbf{C})$. On the other hand, counting dimensions we deduce that $\mathrm{H}^0(T_j; L^\perp)$ is trivial. This proves the last assertion of the lemma for degree 0. The lemma then follows by Poincaré duality and an Euler characteristic argument. $\square$

Proposition 4.3.2. *There exists a neighbourhood of the origin $W \subset U$ such that for all $u \in W$,*

$$\dim_{\mathbf{C}} \mathrm{H}_1(M; \rho_{2k+1}(u)) = \dim_{\mathbf{C}} \mathrm{H}_2(M; \rho_{2k+1}(u)) = l,$$

where l is the number of connected components of $\partial\overline{M}$.

Proof. By Poincaré duality and an Euler characteristic argument, we deduce that

$$\dim_{\mathbf{C}} \mathrm{H}_1(\overline{M}; \rho_{2k+1}(u)) = \dim_{\mathbf{C}} \mathrm{H}_1(\overline{M}, \partial\overline{M}; \rho_{2k+1}(u)).$$

The long exact sequence of the pair $(\overline{M}, \partial\overline{M})$ yields the following short exact sequence,

$$\mathrm{H}_1(\overline{M}, \partial\overline{M}; \rho_{2k+1}(u)) \to \mathrm{H}_0(\partial\overline{M}; \rho_{2k+1}(u)) \to 0.$$

Therefore,

$$\dim_{\mathbf{C}} \mathrm{H}_1(\overline{M}; \rho_{2k+1}(u)) \geq \dim_{\mathbf{C}} \mathrm{H}_0(\partial\overline{M}; \rho_{2k+1}(u)) = \sum_{j=1}^{l} \dim_{\mathbf{C}} \mathrm{H}_0(T_j; \rho_{2k+1}(u)).$$

The vector space $\mathrm{H}_0(T_j;\rho_{2k+1}(u))$ has dimension 1. Indeed, if $u_j = 0$ this is clear by a direct inspection, and if $u_j \neq 0$, this follows from Lemma 4.3.1. Hence,

$$\dim_{\mathbf{C}} \mathrm{H}_1(M;\rho_{2k+1}(u)) \geq l, \quad \text{for } u \in U.$$

Since $\dim_{\mathbf{C}} \mathrm{H}_1(M;\rho_{2k+1}(0)) = l$, the upper semicontinuity of the dimension function (Proposition 4.2.2) implies the result. □

Proposition 4.3.3. *Let $\{\theta_j\}$ be a collection of nontrivial cycles with $\theta_j \in \mathrm{H}_1(T_j;\mathbf{Z})$. Then there exists a neighbourhood of the origin $W \subset U$ such that for all $u \in W$ the following assertions hold:*

1. *A basis of $\mathrm{H}_1(M;\rho_{2k+1}(u))$ is given by*

$$(i_*[w_1(u) \otimes \theta_1], \ldots, i_*[w_l(u) \otimes \theta_l])\,.$$

2. *A basis of $\mathrm{H}_2(M;\rho_{2k+1}(u))$ is given by*

$$(i_*[w_1(u) \otimes T_1], \ldots, i_*[w_l(u) \otimes T_l])\,.$$

In both cases, the vectors $w_j(u)$ are the ones given by Lemma 4.3.1, $[T_j] \in \mathrm{H}_2(T_j;\mathbf{Z})$ is a fundamental class of ∂M, and i_* is the map induced in homology by the inclusion $i\colon \partial\overline{M} \to \overline{M}$.

Proof. Proposition 3.2.1 shows that the two assertions are true for $u = 0$. The result then follows proceeding as in the proof of Proposition 4.2.2. □

It makes sense therefore to consider $\tau(M;\rho_{2k+1}(u_{p/q});\{\theta_j\})$, the Reidemeister torsion of M with respect to the representation $\rho_{2k+1}(u_{p/q})$ and the bases in homology associated to the family of non-trivial cycles $\{\theta_j\}$ given by the Proposition 4.3.3. We want to get a surgery formula for $\tau(M;\rho_{2k+1}(u_{p/q});\{\theta_j\})$. It turns out that it is easier to work with the bases given by the following lemma.

Lemma 4.3.4. *For sufficiently large (p,q), a basis of $\mathrm{H}_1(M;\rho_{2k+1}(u_{p/q}))$ is given by,*

$$\left(i_*^{p/q}[w_1(u_{p/q}) \otimes (p_1 a_1 + q_1 b_1)], \ldots, i_*^{p/q}[w_l(u_{p/q}) \otimes (p_l a_l + q_l b_l)]\right), \tag{4.2}$$

where $i_*^{p/q}$ is the map induced is the inclusion $i\colon M \to M_{p/q}$.

Proof. This is a Mayer-Vietoris argument as in Lemma 4.2.3. We have the decomposition

$$M_{p/q} = M \cup N, \quad \text{with } N = \bigcup_{j=1}^{l} N(\gamma_j),$$

where $\{N(\gamma_j)\}$ is a collection of disjoint tubular neighbourhoods of the core geodesics γ_j added in the Dehn filling. By compactness, $M_{p/q}$ is $\rho_n(u_{p/q})$-acyclic. The Mayer-Vietoris exact sequence then gives an isomorphism

$$\mathrm{H}_*(\partial M;\rho_n(u_{p/q})) \cong \mathrm{H}_*(M;\rho_n(u_{p/q})) \oplus \mathrm{H}_*(N;\rho_n(u_{p/q})).$$

The group $\mathrm{H}_*(T_j;\mathbf{C})$ is isomorphic to $\mathrm{H}_*(T_j;\rho_n(u_{p/q}))$ via tensorization $w_j(u_{p/q})\otimes -$ (this is the homological counterpart of Lemma 4.3.1). The same isomorphism also holds true for $N(\gamma_j)\simeq\gamma_j$. Since $[p_ja_j+q_jb_j]\in\mathrm{H}_1(N(\gamma_j);\mathbf{Z})$ is zero by construction, the vectors described in (4.2) must be linearly independent. □

The surgery formula is now easily obtained.

Lemma 4.3.5. *Let $\gamma_1,\ldots,\gamma_l$ be the core geodesics added on the (p,q)-Dehn filling $M_{p/q}$, and $\lambda_{p/q}$ be the complex-length function of $M_{p/q}$. Then we have*

$$\tau(M_{p/q};\rho^{p/q}_{2k+1})=\tau(M;\rho_{2k+1}(u_{p/q}),\{p_ja_j+q_jb_j\})\prod_{j=1}^{k}\prod_{i=1}^{l}(e^{j\lambda_{p/q}(\gamma_i)}-1)(e^{-j\lambda_{p/q}(\gamma_i)}-1).$$

Proof. This is again a Mayer-Vietoris argument. With the same notation as in the preceding proof, we have $M_{p/q}=\overline{M}\cup N$. The formula for the torsion is

$$\tau(M_{p/q};\rho^{p/q}_{2k+1})\tau(\partial\overline{M};\rho^{p/q}_{2k+1})=\tau(\overline{M};\rho^{p/q}_{2k+1},\{p_ja_j+q_jb_j\})\tau(N;\rho^{p/q}_{2k+1})\tau(\mathcal{H}_*),$$

where $\tau(\mathcal{H}_*)$ is the torsion of the Mayer-Vietoris complex computed using the bases that has been chosen to compute the involved torsions in the decomposition. To compute the torsions we choose bases in homology as follows. For $\mathrm{H}_*(T_j;\rho^{p/q}_{2k+1})$, we take in degree 0, $[P^j_n(u_{p/q})\otimes\sigma_j]$, where σ_j is a generator of $\mathrm{H}_0(T_j;\mathbf{Z})$, in degree 1, $[P^j_n(u_{p/q})\otimes(p_ja_j+q_jb_j)]$, and in degree 2, $[P^j_n(u_{p/q})\otimes T_j]$. For $\mathrm{H}_*(N(\gamma_j);\rho^{p/q}_{2k+1})$, we take in degree 0, $[P^j_n(u_{p/q})\otimes i_{2,*}(\sigma_j)]$, and in degree 1, $[P^j_n(u_{p/q})\otimes i_{2,*}(\gamma)]$, where $i_{2,*}$ is the map induced by the inclusion $i_2\colon\partial M=\partial N\to N$, and $\gamma\in\mathrm{H}_1(\partial\overline{M};\mathbf{Z})$ is such that $i_{1,*}(\gamma)\in\mathrm{H}_1(M;\rho_{2k+1}(u_{p/q}))$ is zero (notice that such a curve always exists and $i_{2,*}(\gamma)\in\mathrm{H}^1(D^2\times S^1;\mathbf{Z})$ is homologous to the core geodesic). Respect to these bases, we have $\tau(\mathcal{H}_*)=1$, since the isomorphism $i_{1,*}+i_{2,*}$ appearing in the Mayer-Vietoris sequence is represented by the identity matrix. On the other hand, the torsion of $\partial\overline{M}$ is ±1, as it is an even dimensional manifold. Thus we have

$$\tau(M_{p/q};\rho^{p/q}_{2k+1})=\tau(M;\rho^{p/q}_{2k+1},\{p_ja_j+q_jb_j\})\prod_{j=1}^{l}\tau(\gamma_j;\rho^{p/q}_{2k+1}).$$

Finally, a straightforward computation gives

$$\tau(\gamma_j;\rho^{p/q}_{2k+1})=\prod_{h=1}^{k}\left(e^{h\lambda_{p/q}(\gamma_j)}-1\right)\left(e^{-h\lambda_{p/q}(\gamma_j)}-1\right).$$

□

Let us normalize torsions in the formula of Lemma 4.3.5. Thus we get:

$$\mathcal{T}_{2k+1}(M_{p/q})=\frac{\tau(M;\rho_{2k+1}(u_{p/q}),\{p_ja_j+q_jb_j\})}{\tau(M;\rho_3(u_{p/q}),\{p_ja_j+q_jb_j\})}\prod_{j=2}^{k}\prod_{i=1}^{l}\left(e^{j\lambda_{p/q}(\gamma_i)}-1\right)\left(e^{-j\lambda_{p/q}(\gamma_i)}-1\right).$$

Let us focus on the quotient of torsions appearing in the right hand side of the above equation. We shall write down a formula relating the torsion of M with respect to the basis $\{a_j\}$ and $\{p_j a_j + q_j b_j\}$. To that end, let $A_{2k+1}(p,q)$ be the change of basis matrix from the basis $\{[w_j(u_{p/q}) \otimes a_j]\}$ to $\{[w_j(u_{p/q}) \otimes (p_j a_j + q_j b_j)]\}$. Then the change of basis formula for the torsion yields:

$$\tau(M; \rho_{2k+1}(u_{p/q}), \{p_j a_j + q_j b_j\}) \det A_{2k+1}(p,q) = \tau(M; \rho_{2k+1}(u_{p/q}), \{a_j\}).$$

This equation implies

$$\frac{\tau(M; \rho_{2k+1}(u_{p/q}), \{p_j a_j + q_j b_j\})}{\tau(M; \rho_3(u_{p/q}), \{p_j a_j + q_j b_j\})} = \frac{\tau(M; \rho_{2k+1}(u_{p/q}), \{a_j\})}{\tau(M; \rho_3(u_{p/q}), \{a_j\})} \frac{\det A_3(p,q)}{\det A_{2k+1}(p,q)}. \tag{4.3}$$

On one hand, working as in in Proposition 4.2.2, it can be checked that

$$\lim_{u \to 0} \tau(M; \rho_{2k+1}(u), \{a_j\}) = \tau(M; \rho_{2k+1}(0), \{a_j\}) = \tau(M; \rho_{2k+1}, \{a_j\}).$$

Hence,

$$\lim_{(p,q)\to\infty} \frac{\tau(M; \rho_{2k+1}(u_{p/q}), \{a_j\})}{\tau(M; \rho_3(u_{p/q}); \{a_j\})} = \mathcal{T}_{2k+1}(M). \tag{4.4}$$

On the other hand, we have the following result.

Lemma 4.3.6. *For any $k \geq 3$,*

$$\lim_{(p,q)\to\infty} \frac{\det A_{2k+1}(p,q)}{\det A_3(p,q)} = 1.$$

Proof. We have

$$A_{2k+1}(p,q) = \operatorname{diag}(p) + \operatorname{diag}(q) B_{2k+1}(u_{p/q}),$$

where $B_{2k+1}(u)$ is the change of basis matrix from the basis $\{[w_j(u) \otimes a_j]\}$ to the basis $\{[w_j(u) \otimes b_j]\}$. Working as in Proposition 4.2.2, it can be checked that $B_{2k+1}(u)$ depends analytically on u. Note that at $u = 0$ we have

$$B_{2k+1}(0) = \operatorname{diag}(\operatorname{cshape}(b_1, a_1), \ldots, \operatorname{cshape}(b_l, a_l)).$$

Let us write $P = \operatorname{diag}(p)$, $Q = \operatorname{diag}(q)$ and $C = B_{2k+1}(0)$. Notice that C is independent of k. The lemma will follow easily once we have proved the following equality:

$$\lim_{(p,q)\to\infty} \frac{\det(P + QC)}{\det(P + QB_{2k+1}(u_{p/q}))} = 1.$$

We have,

$$\frac{\det(P + QC)}{\det(P + QB_{2k+1}(u_{p/q}))} = \frac{\det(Q^{-1}P + C)}{\det(Q^{-1}P + B_{2k+1}(u_{p/q}))}$$

Let us put $D = Q^{-1}P + C$ and $E(u_{p/q}) = B_{2k+1}(u_{p/q}) - C$. Then we have

$$\begin{aligned}\frac{\det(P+QC)}{\det(P+QB_{2k+1}(u_{p/q}))} &= \frac{\det D}{\det(D+E_{2k+1}(u_{p/q}))}\\ &= \frac{1}{\det(\mathrm{Id}+D^{-1}E_{2k+1}(u_{p/q}))}.\end{aligned}$$

If $D = (d_{ij})$ then we have

$$|d_{jj}| = |p_j/q_j + \mathrm{cshape}(a_j, b_j)| > |\operatorname{Im}\mathrm{cshape}(a_j, b_j)| > 0.$$

Therefore, the entries of the diagonal matrix D^{-1} are bounded, and hence

$$\lim_{(p,q)\to\infty} D^{-1}E_{2k+1}(u_{p/q}) = \lim_{(p,q)\to\infty} D^{-1}(B_{2k+1}(u_{p/q}) - B_{2k+1}(0)) = 0.$$

□

Finally, taking limits in Equation (4.3), and using Equation (4.4) and Lemma 4.3.6, we get:

$$\lim_{(p,q)\to\infty} \frac{\tau(M;\rho_{2k+1}(u_{p/q}), \{p_ja_j + q_jb_j\})}{\tau(M;\rho_3(u_{p/q}); \{p_ja_j + q_jb_j\})} = \mathcal{T}_{2k+1}(M).$$

Just for future references, we summarize the preceding results in the following lemma.

Lemma 4.3.7. *With the above notation, for $k > 1$ we have*

$$\mathcal{T}_{2k+1}(M_{p/q}) = \frac{\det A_3(p,q)}{\det A_{2k+1}(p,q)} \frac{\tau(M;\rho_{2k+1}(u_{p/q}), \{a_j\})}{\tau(M;\rho_3(u_{p/q}), \{a_j\})} \prod_{j=2}^{k}\prod_{i=1}^{l}(e^{j\lambda_{p/q}(\gamma_i)} - 1)(e^{-j\lambda_{p/q}(\gamma_i)} - 1).$$

Moreover,

$$\begin{aligned}\lim_{(p,q)\to\infty} \frac{\det A_{2k+1}(p,q)}{\det A_3(p,q)} &= 1,\\ \lim_{(p,q)\to\infty} \frac{\tau(M;\rho_{2k+1}(u_{p/q}), \{a_j\})}{\tau(M;\rho_3(u_{p/q}), \{a_j\})} &= \mathcal{T}_{2k+1}(M).\end{aligned}$$

Proof of Proposition 4.0.2. By Lemma 4.3.7, the result is reduced to prove that the set of cluster points of the following net

$$\left\{\prod_{j=2}^{k}\prod_{i=1}^{l}(e^{j\lambda_{p/q}(\gamma_i)} - 1)(e^{-j\lambda_{p/q}(\gamma_i)} - 1)\right\}_{(p,q)\in\mathcal{A}_M}$$

is $[0, 4^{(k-1)l}]$, which may be proved in the same way as in the even dimensional case (see Proposition 4.0.5). □

Chapter 5

Complex-length spectrum

The aim of this chapter is to prove the continuity of the complex-length spectrum in a sense that we shall precise.

5.1 Closed geodesics in a hyperbolic manifold

Although the material of this section is well known, we think it is worth to review it for the sake of completeness.

Let M be an oriented, complete, hyperbolic 3-manifold, and Hol_M be its holonomy representation. Let us consider $\mathcal{C}(M)$ the set of closed (constant-speed) geodesics in M up to orientation-preserving reparametrisation. We will describe $\mathcal{C}(M)$ as the following quotient set,

$$\mathcal{C}(M) = \left\{\varphi\colon S^1 \to M \mid \varphi \text{ is a geodesic}\right\} / S^1.$$

The action of S^1 on a closed geodesic is given by translation on the parameter. We are interpreting S^1 as $\mathbf{R}/\mathbf{Z}$. If $k \in \mathbf{Z}$ and $\varphi\colon S^1 \to M$ is a closed geodesic, $k\varphi$ will denote the closed geodesic $t \mapsto \varphi(kt)$.

Definition. A closed geodesic φ is said to be *prime* if $\varphi \neq k\psi$ for any $k > 1$ and any closed geodesic ψ (i.e. φ is prime if it traces its image exactly once). A class $[\varphi] \in \mathcal{C}(M)$ is said to be prime if φ is prime. The set of prime classes of $\mathcal{C}(M)$ will be denoted by $\mathcal{PC}(M)$.

We will also need the group theoretic definition of primality.

Definition. Let G be a group. An element $g \in G$ is said to be *prime* if $g \neq h^k$ for all $h \in G$ and $k > 1$ (note that we are excluding the identity from this definition). If $\mathrm{C}(G)$ denotes the set of conjugacy classes of G, then $[g] \in \mathrm{C}(G)$ is said to be prime if g is prime.

The identification between the set $\mathrm{C}\left(\pi_1(M,p)\right)$ of conjugacy classes of $\pi_1(M,p)$ and loops in M up to free homotopy yields a natural map

$$\psi\colon \mathcal{C}(M) \to \mathrm{C}\left(\pi_1(M,p)\right).$$

Let $\mathrm{HypC}(\pi_1(M,p))$ be the set of hyperbolic conjugacy classes of $\pi_1(M,p)$, that is,

$$\mathrm{HypC}(\pi_1(M,p)) = \{[\gamma] \in \mathrm{C}\,(\pi_1(M,p)) \mid \mathrm{Hol}_M(\gamma) \text{ is of hyperbolic type}\}\,.$$

The following result is well known, and is easily deduced from the fact that an isometry of $\mathbf{H}^3$ of hyperbolic type has exactly one axis.

Proposition 5.1.1. *The natural map $\psi\colon \mathcal{C}(M) \to \mathrm{C}(\pi_1(M,p))$ is a bijection onto the set $\mathrm{HypC}(\pi_1(M,p))$. Moreover, $[\varphi] \in \mathcal{C}(M)$ is prime if and only if so is $\psi([\varphi])$.*

Proof. Let $\pi\colon (\widetilde{M},\tilde{p}) \to (M,p)$ be the universal cover of M. Since M is complete $\widetilde{M}$ is isometric to hyperbolic 3-space. Let us take a closed geodesic $[\varphi] \in \mathcal{C}(M)$. By definition, $\psi([\varphi])$ is the conjugacy class of the loop $\tau = \sigma\varphi\sigma^{-1}$, where $\sigma\colon [0,1] \to M$ is any path such that $\sigma(0) = p$ and $\sigma(1) = \varphi(0)$. Now let $\tilde{\sigma}$ be the lift of σ with $\tilde{\sigma}(0) = \tilde{p}$, and $\widetilde{\varphi}\colon \mathbf{R} \to \widetilde{M}$ be the lift of $\varphi\colon S^1 \to M$ with $\widetilde{\varphi}(0) = \tilde{\sigma}(1)$. The deck transformation T_τ defined by τ sends $\widetilde{\varphi}(0)$ to $\widetilde{\varphi}(1)$. Since $\widetilde{\varphi}$ is a geodesic, it must be invariant by T_τ. Hence, T_τ is an isometry of hyperbolic type. This proves that the image of ψ is contained in $\mathrm{HypC}(\pi_1(M,p))$. Let us prove injectivity. If $[\varphi'] \in \mathcal{C}(M)$ is such that $\psi([\varphi]) = \psi([\varphi'])$, then there exists a path $\sigma'\colon [0,1] \to M$ such that $\sigma'(0) = p$, $\sigma'(1) = \varphi'(0)$, and $\sigma'\varphi'\sigma'^{-1} = \tau$. The same construction as before gives a lift $\widetilde{\varphi}'$ of φ' that is invariant under T_τ. Since T_τ is an isometry of hyperbolic type the two lines $\widetilde{\varphi}, \widetilde{\varphi}'$ must be equal, which implies that $[\varphi] = [\varphi']$. To prove surjectivity, let $[\gamma]$ be a conjugacy class in $\mathrm{HypC}(\pi_1(M,p))$. The isometry $\mathrm{Hol}_M(\gamma)$, being of hyperbolic type, has an axis which projects to a closed geodesic in M. As it is easily verified, the image of this closed geodesic under ψ is γ. The other assertion concerning primality is now quite obvious. $\square$

It is useful to endow the set $\mathcal{C}(M)$ with the (quotient) supremum metric. More explicitly, if $[\varphi_1], [\varphi_2] \in \mathcal{C}(M)$ then its distance is defined by

$$d([\varphi_1],[\varphi_2]) = \min_{s\in S^1} \max_{t\in S^1} \{d(\varphi_1(t+s), \varphi_2(t))\}\,.$$

The following observation is an immediate consequence of the previous proposition, and will be used quite often in the subsequent sections.

Proposition 5.1.2. *Let $[\varphi], [\varphi']$ be two distinct elements of $\mathcal{C}(M)$, and let m be the minimum of the injectivity radius at φ. Then $d([\varphi],[\varphi']) \geq m$.*

Proof. Assume that $d([\varphi],[\varphi']) = m' < m$. With suitable parametrisations, we have that for all $t \in S^1$, $d(\varphi(t), \varphi'(t)) \leq m'$. By the hypothesis on the injectivity radius, there exists a unique minimizing geodesic joining $\varphi(t)$ and $\varphi'(t)$. Therefore, we can define a free homotopy from φ to φ', which contradicts Proposition 5.1.1. $\square$

This section ends with an estimate on the growth of the number of closed geodesics in function of their length. The following estimate, though not the best possible (see for instance [Mar69], [CK02]), has the advantage of being explicit. Its proof is very close to the proof of Lemma 5.3 in [CK02].

Lemma 5.1.3. *Let M be a complete hyperbolic 3-manifold. For a compact domain $K \subset M$ define*

$$\mathcal{P}_K(t) = \#\left\{\varphi \in \mathcal{C}(M) \mid \varphi(S^1) \cap K \neq \emptyset, l(\varphi) \leq t\right\}.$$

Then, $\mathcal{P}_K(t) \leq Ce^{2t}$, with $C = \pi\frac{e^{8\operatorname{diam} K}}{\operatorname{Vol} K}$.

Proof. Let $M = \mathbf{H}^3/\Gamma$, with Γ a subgroup of $\operatorname{Isom}\mathbf{H}^3$, and let $\pi\colon \mathbf{H}^3 \to M$ denote the covering projection. Pick a point $p \in \mathbf{H}^3$ with $\pi(p) \in K$ and consider the Dirichlet domain centred at p:

$$D(p) = \{x \in \mathbf{H}^3 \mid d(x, \gamma(p)) \leq d(x,p),\ \forall \gamma \in \Gamma\}.$$

The intersection $\tilde{K} = D(p) \cap \pi^{-1}(K)$ is a fundamental domain for K, which means that

$$\pi^{-1}(K) = \bigcup_{\gamma\in\Gamma} \gamma(\tilde{K}),$$
$$\operatorname{Vol}(\gamma_1(\tilde{K}) \cap \gamma_2(\tilde{K})) = 0, \quad \text{for all } \gamma_1 \neq \gamma_2 \in \Gamma.$$

Moreover, $\operatorname{diam}\tilde{K} \leq 2\operatorname{diam} K$ and $\operatorname{Vol}\tilde{K} = \operatorname{Vol} K$. Now let $\varphi \in \mathcal{C}(M)$ intersecting K. Then there exists an isometry $\gamma \in \Gamma$ of hyperbolic type representing φ whose axis intersects $\tilde{K}$. We claim that

$$\gamma(\tilde{K}) \subset B(p, 4\operatorname{diam} K + l(\varphi)).$$

To prove this inclusion, we pick a point $q \in \tilde{K}$ that lies in the axis of γ. For any $q' \in \tilde{K}$,

$$d(p, \gamma(q')) \leq d(p,q) + d(q, \gamma(q)) + d(\gamma(q), \gamma(q')) \leq 4\operatorname{diam} K + d(q, \gamma(q))$$

and $d(q,\gamma(q)) = l(\varphi)$, which proves the claim. Hence, for any geodesic contributing to $\mathcal{P}_K(t)$, there is a hyperbolic isometry whose axis is a lift of this geodesic and such that $\gamma(\tilde{K}) \subset B(p, 4\operatorname{diam} K + t)$. In addition, $\operatorname{Vol}(\gamma_1(\tilde{K}) \cap \gamma_2(\tilde{K})) = 0$, for all $\gamma_1 \neq \gamma_2 \in \Gamma$. Thus we get the inequality:

$$\mathcal{P}_K(t)\operatorname{Vol} K = \mathcal{P}_K(t)\operatorname{Vol}\tilde{K} \leq \operatorname{Vol} B_p(4\operatorname{diam} K + t) \leq \pi e^{8\operatorname{diam} K + 2t}.$$

We have used that the volume of a ball of radius R in $\mathbf{H}^3$ is less than πe^{2R}. $\square$

5.2 Complex-length spectrum

Any closed geodesic $\varphi \in \mathcal{C}(M)$ has attached two geometric invariants: its length and its geometric torsion. Recall that the geometric torsion of φ is defined as the oriented angle between an orthogonal vector to φ and the parallel transport of it along φ. In terms of the holonomy representation, these two invariants are the translation distance and the rotational part of the corresponding hyperbolic isometry. More explicitly, if $[\gamma] \in \operatorname{HypC}(\pi_1(M,p))$, then

$$\operatorname{Hol}_M(\gamma) \sim \left[\begin{pmatrix} e^{\lambda/2} & 0 \\ 0 & e^{-\lambda/2} \end{pmatrix}\right] \in \operatorname{PSL}(2,\mathbf{C}), \quad \operatorname{Re}(\lambda) > 0,$$

$\mathrm{Re}(\lambda)$ is the length of the corresponding closed geodesic, and $\mathrm{Im}(\lambda)$ its geometric torsion. The parameter λ is called the *complex length* of γ, and it is only well defined up to $2\pi i$. We will regard this as a function

$$\begin{array}{rcl}\lambda\colon \mathcal{C}(M) & \to & \mathbf{C}/\langle 2\pi i\rangle \\ \varphi & \mapsto & \lambda(\varphi) = l(\varphi) + i\,\mathrm{torsion}(\varphi).\end{array}$$

To avoid the $2\pi i$ indeterminacy, we will work with the exponential of this map.

Definition. The *(prime) complex-length spectrum* of M, denoted as $\mu_{\mathrm{sp}}\, M$, is the measure on $\mathbf{C}$ defined by

$$\mu_{\mathrm{sp}}\, M = \sum_{\varphi \in \mathcal{PC}(M)} \delta_{e^{\lambda(\varphi)}},$$

where δ_x is the Dirac measure centered at x. In other words, $\mu_{\mathrm{sp}}\, M$ is the image measure of the counting measure in $\mathcal{PC}(M)$ under the exponential of the complex-length function. The *(prime) length spectrum* of M, denoted as $\mu_{\mathrm{lsp}}\, M$, is the measure on $\mathbf{R}$ defined by

$$\mu_{\mathrm{lsp}}\, M = \sum_{\varphi \in \mathcal{PC}(M)} \delta_{l(\varphi)}.$$

Thus we have:

$$\#\{\varphi \in \mathcal{PC}(M) \mid a < l(\varphi) < b\} = \mu_{\mathrm{sp}}\, M\{z \in \mathbf{C} \mid e^a < |z| < e^b\}.$$

Remark. The prime complex-length spectrum is usually regarded as a collection of numbers and multiplicities. This is of course equivalent to the definition made above; however, we think that some of the results that we will present in what follows are better expressed in these terms.

The following properties of $\mu_{\mathrm{sp}}\, M$ are immediately implied by Lemma 5.1.3 and the fact that a closed geodesic cannot be contained in a cusp.

Proposition 5.2.1. *Assume that M has finite volume. The following assertions then hold:*

1. *The measure $\mu_{\mathrm{sp}}\, M$ is locally finite with discrete support. In particular, it is a Radon measure on the complex plane.*

2. *Let $N_1, \ldots, N_j$ be cusps of M in such a way that $K = M \setminus \bigcup_{1\le j\le n} N_j$ is compact. Then for all $R > 1$,*
$$\mu_{\mathrm{sp}}\, M\big(\{|z| \le R\}\big) \le C_M R^2,$$
where $C_M = \pi \frac{e^{8\,\mathrm{diam}\, K}}{\mathrm{Vol}\, K}$.

Next we want to analyse the complex-length spectrum as a map

$$M \mapsto \mu_{\mathrm{sp}}\, M.$$

The domain of this map will be the set $\mathcal{M}$ of all (isometry classes of) oriented, complete, hyperbolic 3-manifolds of finite volume. This set is naturally endowed with the geometric topology, which is briefly discussed in next section. On the other hand, the target of this map will be $M(\mathbf{C}\setminus\overline{D})$, the vector space of $\mathbf{C}$-valued Radon measures defined on the complementary of the closed unit disk $\overline{D}$. We will endow $M(\mathbf{C}\setminus\overline{D})$ with the topology of the weak convergence. Thus a sequence $\{\mu_n\}$ converges weakly to μ in $M(\mathbf{C}\setminus\overline{D})$ if for every continuous function f with compact support contained in $\mathbf{C}\setminus\overline{D}$, we have:

$$\lim_{n\to\infty}\int_{|z|>1} f(z)d\mu_n(z) = \int_{|z|>1} f(z)d\mu(z).$$

The aim of the rest of this chapter is essentially to prove that this map is continuous.

Theorem 5.2.2. *The map* $\mu_{\mathrm{sp}}\colon \mathcal{M}\to M(\mathbf{C}\setminus\overline{D})$ *is continuous.*

Remark. If instead of the space $M(\mathbf{C}\setminus\overline{D})$ we consider $M(\mathbf{C})$, then the above theorem is no longer true. For instance, let $M\in\mathcal{M}$ be a one-cusped manifold, and $M_{p/q}$ be the manifold obtained by a hyperbolic (p,q)-Dehn filling. Then $\{M_{p/q}\}_{(p,q)}$ converges to M as (p,q) goes to infinity. However, the sequence of the corresponding measures do not even converge. To see this, let $\pm\varphi_{p/q}$ be the two (oriented) core prime geodesics added in the Dehn filling. Then the length of $\varphi_{p/q}$ goes to zero, and the geometric torsion is dense in $\mathbf{R}/2\pi\mathbf{Z}$, which implies that this sequence of measures does not converge. Restricting our attention to $M(\mathbf{C}\setminus\overline{D})$ we avoid these phenomena. Nevertheless, this bad behaviour is the worst that can happen; this is expressed in the following result.

Theorem 5.2.3. *Let* $M\in\mathcal{M}$ *with* $k>0$ *cusps, and* $\{M_n\}$ *be a sequence converging to* M *in* $\mathcal{M}$. *Assume that the number of cusps of* M_n *is eventually constant and is equal to* l. *Then the sequence of real-length spectrum measures* $\{\mu_{\mathrm{lsp}}\, M_n\}$ *converges weakly in* $M(\mathbf{R})$ *to the measure*

$$\mu_{\mathrm{lsp}}\, M + 2(k-l)\delta_0.$$

Theorem 5.2.2 and Theorem 5.2.3 will be proved in Section 5.4 after having discussed the geometric topology.

5.3 The geometric topology

Most of the material in this section is based on [CEM06].

Let $\mathcal{MF}$ be the set of (isometry classes of) oriented, complete, hyperbolic 3-manifolds of finite volume and with a baseframe. Thus an element of $\mathcal{MF}$ is a pair (M,E), where E is an orthonormal frame based at some point p in the oriented hyperbolic 3-manifold M of finite volume.

Remark. Our notation differs from [CEM06], where $\mathcal{MF}$ is defined without the finite volume restriction.

If we fix a base frame on hyperbolic space $\mathbf{H}^3$, then the holonomy representation of a member of $\mathcal{MF}$ is unambiguously defined (i.e. not only up to conjugation). Therefore, $\mathcal{MF}$ is in one-to-one correspondence with the set of discrete torsion-free subgroups of $\mathrm{PSL}(2, \mathbf{C})$ with *finite co-volume*. The latter set is endowed with the *geometric topology*. We recall its definition in the general context of Lie groups, see [Thu].

Definition. A sequence $\{\Gamma_n\}$ of closed subgroups of a Lie group G *converges geometrically* to a group Γ if the following conditions are satisfied:

1. Each $\gamma \in \Gamma$ is the limit of a sequence $\{\gamma_n\}$, with $\gamma_n \in \Gamma_n$.
2. The limit of every convergent sequence $\{\gamma_{n_j}\}$, with $\gamma_{n_j} \in \Gamma_{n_j}$, is in Γ (n_j is an increasing sequence of natural numbers).

Two related spaces are $\mathcal{MB}$ and $\mathcal{M}$. The former is obtained by forgetting the frame, but retaining the basepoint, and the latter by forgetting both the frame and the basepoint. Both sets are endowed with the quotient topology given by the corresponding forgetful maps.

The following results are well known, and will play an important role in the following sections. See [CEM06] for a proof.

Lemma 5.3.1. *Let* $\mathrm{inj}_R(M,p)$ *be the infimum of the injectivity radius on the ball* $B_R(p) \subset M$*. Then for any* $R > 0$ *the map* $\mathrm{inj}_R \colon \mathcal{MB} \to (0,\infty)$ *is continuous.*

Lemma 5.3.2. *Let* $\epsilon > 0$ *smaller than the Margulis constant. Let* $\{M_n\}$ *be a sequence converging to* M *in* $\mathcal{M}$*. Then there exists a uniform bound on the diameter of the thick parts* $\{M_{n,[\epsilon,\infty)}\}$*.*

Theorem 5.3.3 (Jorgensen)**.** *The map* $\mathrm{Vol} \colon \mathcal{M} \to \mathbf{R}$ *that assigns to each manifold its volume is continuous.*

The following theorem due to Thurston describes how a non-trivial convergence sequence in $\mathcal{M}$ is. We recall that we are assuming that all manifolds have finite volume.

Theorem 5.3.4 (Thurston)**.** *Let* $\{M_n\}$ *be a sequence converging to* M *in* $\mathcal{M}$*. Assume that* $\{M_n\}$ *is not eventually constant, and that* M *has* k *cusps. Then* M_n *is obtained by hyperbolic Dehn surgery* $M_{p_{1,n}/q_{1,n},\dots,p_{k,n}/q_{k,n}}$*, with* $p_{i,n}^2 + q_{i,n}^2 \to \infty$*, as* $n \to \infty$*.*

Corollary 5.3.5. *Let* $\{(M_n, E_n)\}$ *be a sequence converging to* (M,E) *in* $\mathcal{MF}$*. Then, for* n *large enough, we have a commutative diagram,*

$$\begin{tikzcd} \pi_1(M,p) \arrow[r, "\rho_n"] \arrow[d, "i_*^n"] & \mathrm{PSL}(2;\mathbf{C}) \\ \pi_1(M_n,p_n) \arrow[ur, "\mathrm{Hol}_{M_n}"'] & \end{tikzcd}$$

Moreover, the sequence of representations $\{\rho_n\}$ *converges to* Hol_M *both algebraically (that is, for all* $\sigma \in \pi_1(M,p)$ *the sequence* $\{\rho_n(\sigma)\}$ *converges to* $\mathrm{Hol}_M(\sigma)$*), and geometrically (that is, the sequence of discrete groups* $\{\rho_n(\pi_1(M,p))\}$ *converges geometrically to* $\mathrm{Hol}_M(\pi_1(M,p))$*).*

It can be proved that if a sequence $\{(M_n, p_n)\}$ converges to (M, p) in $\mathcal{MB}$, then it also converges to (M, p) in the pointed Hausdorff-Gromov sense, see [CEM06].

Next we want to give the following *ad hoc* definition concerning the convergence of geodesics.

Definition. With the above notation, we will say that a sequence of parametrised closed geodesics $\{\varphi_n : [0,1] \to M_n\}$ converges to $\varphi : [0,1] \to M$ if for all n there is a lift of φ_n (with respect to the covering map π_n)

$$\tilde{\varphi}_n : [0,1] \to \mathbf{H}^3,$$

such that the sequence of maps $\{\tilde{\varphi}_n\}$ converges pointwise to a lift of φ (with respect to the covering map π).

Remark. The above definition coincides with the more general (and natural) definition of convergence of maps $\{f_n : X_n \to Y_n\}$, where $\{X_n\}$ and $\{Y_n\}$ are sequences of compact metric space converging in the Hausdorff-Gromov sense to X and Y respectively, see [GP91].

With the above definition, it is quite obvious that the limit of parametrised closed geodesics is also a geodesic whose length is the limit of the lengths of the converging geodesics.

Definition. We will say that a sequence $\{\varphi_n\}$ of closed geodesics, with $\varphi_n \in \mathcal{C}(M_n)$, converges to $\varphi \in \mathcal{C}(M)$ if for all n we can choose parametrisations of φ_n converging to a parametrisation of φ (in the sense of the above definition).

Again the following result holds in a more general context, see [GP91]. Its proof in our case is quite obvious.

Theorem 5.3.6 (Ascoli-Arzela, Grove-Petersen.)**.** *Let $R > 0$ and $\{\varphi_n\}$ be a sequence of closed geodesics with $\varphi_n \subset B_R(p_n) \subset (M_n, p_n)$. If there exists a common upper bound on the lengths of $\{\varphi_n\}$, then $\{\varphi_n\}$ has a converging subsequence.*

5.4 Proof of the continuity

In this section we want to prove the continuity of the complex-length spectrum as a map from $\mathcal{M}$ to $M(\mathbf{C} \setminus \overline{D})$. An obvious observation is that we can assume that this map is defined from $\mathcal{MF}$ to $M(\mathbf{C} \setminus \overline{D})$, since the topology of $\mathcal{M}$ is the quotient topology coming from the forgetful map $\mathcal{MF} \to \mathcal{M}$.

Hereafter $\{(M_n, E_n)\}$ will denote a sequence converging to $\{(M, E)\}$ in $\mathcal{MF}$. In order to simplify notation, we will write μ_n and μ_∞ for $\mu_{\mathrm{sp}}\, M_n$ and $\mu_{\mathrm{sp}}\, M$, respectively. We want to prove that the sequence of measures $\{\mu_n\}$ converges to μ_∞ in $M(\mathbf{C} \setminus \overline{D})$. Our first task is to translate this into geometrical terms.

Recall from last section, Corollary 5.3.5, that we have a commutative diagram,

$$\begin{tikzcd}
\pi_1(M, p) \arrow[r, "\rho_n"] \arrow[d, "i_*^n"'] & \mathrm{PSL}(2; \mathbf{C}) \\
\pi_1(M_n, p_n) \arrow[ur, "\mathrm{Hol}_{M_n}"'] &
\end{tikzcd}$$

Furthermore, the sequence of representations $\{\rho_n\}$ converges both algebraically and geometrically to Hol_M.

Let $\sigma \in \pi_1(M,p)$ be a hyperbolic element. The algebraic convergence of $\{\rho_n\}$ implies that $\rho_n(\sigma)$ is also of hyperbolic type for large n (it follows for instance from the fact that the set of hyperbolic isometries is open in $\mathrm{PSL}(2,\mathbf{C})$). As a consequence, for large n, the conjugacy class of $i_*^n(\sigma)$ defines a closed geodesic in M_n; moreover, the complex length of $\rho_n(\sigma)$ is close to that of $\mathrm{Hol}_M(\sigma)$.

Let $0 < a < b$. Then, for large n, the map $i_*^n \colon \pi_1(M,p) \to \pi_1(M_n,p_n)$ gives a well defined map

$$\iota_{a,b,n} \colon \{\varphi \in \mathcal{C}(M) \mid a < l(\varphi) < b\} \to \{\varphi \in \mathcal{C}(M_n) \mid a < l(\varphi) < b\}\,.$$

Lemma 5.4.1. *Assume that for all* $0 < a < b$ not in the real-length spectrum *of* M *there exists* $N(a,b)$ *such that for all* $n > N(a,b)$ *the map* $\iota_{a,b,n}$ *is a bijection when restricted to prime geodesics. Then* $\{\mu_n\}$ *converges weakly to* μ_∞.

Proof. For two real numbers $a < b$ put $D_{a,b} = \{z \in \mathbf{C} \mid e^a < |z| < e^b\}$. Let f be a continuous function with compact support contained in the exterior of the unit disk. Take $1 < a < b$ such that $\operatorname{supp} f \subset D_{a,b}$, with both a and b not in the real length spectrum of M. Let $A = \{\varphi_1, \dots, \varphi_k\}$ be the set of prime closed geodesics in M with complex length in $D_{a,b}$. Therefore, we have

$$\int_{|z|>1} f(z)d\mu_\infty(z) = \int_{D_{a,b}} f(z)d\mu_\infty(z) = \sum_{i=1}^{k} f(\lambda(\varphi_i)).$$

By hypothesis, for $n > N(a,b)$, we have

$$\int_{|z|>1} f(z)d\mu_n(z) = \int_{D_{a,b}} f(z)d\mu_n(z) = \sum_{i=1}^{k} f(\lambda_n(\iota_{n,a,b}(\varphi_i))),$$

where λ_n is the complex-length function of M_n. The algebraic convergence implies

$$\lim_{n\to\infty} \lambda_n(\iota_{n,a,b}(\varphi_i)) = \lambda(\varphi_i),$$

and the continuity of f gives

$$\lim_{n\to\infty} \int_{|z|>1} f(z)d\mu_n(z) = \int_{|z|>1} f(z)d\mu_\infty(z).$$

Hence, μ_n converges weakly to μ_∞. $\square$

Next we want to prove that the hypothesis of the above lemma is satisfied. Hereafter, a and b will denote two fixed positive real numbers not in the length spectrum of M with $a < b$. We will write ι_n instead of $\iota_{a,b,n}$.

The following lemma is an immediate consequence of the convergence of $\{(M_n,E_n)\}$ to (M,E).

Lemma 5.4.2. *Let $\varphi \in \mathcal{C}(M)$. Then the sequence of closed geodesics $\{\iota_n(\varphi)\}$ converges to φ.*

Proposition 5.4.3. *Let $\varphi_1, \varphi_2 \in \mathcal{C}(M)$. If $\varphi_1 \neq \varphi_2$ then, for n large enough, $\iota_n(\varphi_1) \neq \iota_n(\varphi_2)$.*

Proof. We have $d(\varphi_1, \varphi_2) > 0$. The above lemma then implies that for large n also

$$d(\iota_n(\varphi_1), \iota_n(\varphi_2)) > 0.$$

□

Proposition 5.4.4. *If $\varphi \in \mathcal{C}(M)$ is prime, then, for n large enough, $\iota_n(\varphi)$ is also prime.*

Proof. Take $R > 0$ such that $\iota_n(\varphi) \subset B_R(p_n)$ for all n. If the lemma were false, then (up to a subsequence) for all n, $\iota_n(\varphi) = k_n\psi_n$ for some integer $k_n \geq 2$ and some $\psi_n \in \mathcal{PC}(M_n)$. By Lemma 5.3.1, the injectivity radius on $B_R(p_n)$ is uniformly bounded from below away from zero; hence, k_n must be bounded from above. Therefore, (up to a subsequence) for all n, $\iota_n(\varphi) = k\psi_n$, for some fixed $k \geq 2$. The geodesics $\{\psi_n\}$ have bounded length and are contained in $B_R(p_n)$; hence, by Ascoli-Arzela (up to a subsequence) they converge to a geodesic ψ which satisfies $\varphi = k\psi$, contradicting the primality of φ. □

These two preceding results imply that, for large n, ι_n gives an injective map

$$\{\varphi \in \mathcal{PC}(M) \mid a < l(\varphi) < b\} \to \{\varphi \in \mathcal{PC}(M_n) \mid a < l(\varphi) < b\},$$

Next we want to prove that, for a larger n, this map is surjective. We will proceed by contradiction using an Arzela-Ascoli argument. Before doing this, we need to prove that we have a control on the set of prime closed geodesics in M_n whose lengths are in (a, b). This is the content of the following result, which is just an application of the thick-thin decomposition of a complete finite-volume hyperbolic manifold.

Lemma 5.4.5. *There exists $R > 0$ such that for all n any* prime *closed geodesic in (M_n, p_n) of length in (a, b) is contained in $B_R(p_n)$.*

Proof. Let $\epsilon > 0$ be smaller than $a/2$ and the Margulis constant. If necessary, take a smaller $\epsilon > 0$ to guarantee that $p_n \in M_{n,[\epsilon,\infty)}$. If φ is a closed geodesic in M_n of length $l(\varphi) > a$, then φ must intersect the ϵ-thick part $M_{n,[\epsilon,\infty)}$ (otherwise φ would be the core of a Margulis tube in $M_{n,(0,\epsilon)}$, and the injectivity radius in that tube would be achieved by the curve φ, so $a/2 < l(\varphi)/2 < \epsilon$, which is absurd). The result then follows from the fact that the diameter of $M_{n,[\epsilon,\infty)}$ is uniformly bounded on n. □

Lemma 5.4.6. *There exists N such that for all $n > N$ the following holds: if φ_n is a prime closed geodesic in M_n of length $a < l(\varphi_n) < b$, then there exists a prime closed geodesic in M of length $a < l(\varphi) < b$ with $\varphi_n = \iota_n(\varphi)$.*

Proof. Assume that the lemma is false. Up to a subsequence, for all n there exists a prime closed geodesic φ_n on M_n with $l(\varphi_n) \in (a,b)$ such that $\varphi_n \neq \iota_n(\psi)$, for all $\psi \in \mathcal{PC}(M)$ of length in (a,b). Take the R given by Lemma 5.4.5. By the continuity of the injectivity radius, there exists a uniform lower bound $\epsilon > 0$ on the injectivity radius on $B_R(p_n)$. Therefore, by Proposition 5.1.2, for all $\psi \in \mathcal{PC}(M)$ of length in (a,b),

$$d(\iota_n(\psi), \varphi_n) > \epsilon.$$

Up to a subsequence, $\{\varphi_n\}$ converges to a closed geodesic φ in M. It is easily seen that φ must be prime. Since $l(\varphi) \in (a,b)$ (recall that a and b do not belong to the length spectrum of M), the above inequality gives

$$d(\iota_n(\varphi), \varphi_n) > \epsilon.$$

It contradicts the fact that both $\{\varphi_n\}$ and $\{\iota_n(\varphi)\}$ converge to φ. □

Proof of Theorem 5.2.2. Propositions 5.4.3 and 5.4.4 prove that ι_n is injective, and Lemma 5.4.6 states that ι_n is surjective. Then Lemma 5.4.1 proves that $\{\mu_n\}$ converges to μ_∞ weakly. □

It remains to prove Theorem 5.2.3. In order to do it, we can assume that M has k cusps, and that the sequence $\{M_n\}$ converging to M in $\mathcal{M}$ is obtained by performing Dehn fillings on l $(\leq k)$ fixed cusps of M. We must prove that the sequence of (real) length spectrum measures $\{\mu_{\mathrm{lsp}}\, M_n\}$ converges in $M(\mathbf{R})$ to

$$\mu_{\mathrm{lsp}}\, M + 2(k-l)\delta_0.$$

By Theorem 5.2.2, it is enough to prove that there exists $\delta > 0$ smaller than the length of the shortest geodesic in M such that

$$\lim_{n\to\infty} \mu_{\mathrm{lsp}}\, M_n([0,\delta)) = 2(k-l).$$

In geometrical terms, it is equivalent to the following well known result.

Lemma 5.4.7. *Let $\{\pm\varphi_n^1, \ldots, \pm\varphi_n^l\}$ be the core geodesics (oriented and prime) in M_n added on the Dehn filling. Let δ_s be the length of the shortest geodesic in M, and $\delta \in (0, \delta_s)$. Then, for large n, the only prime closed geodesics in M_n of length $< \delta$ are the core geodesics.*

Proof. Take $\epsilon > 0$ smaller than both the Margulis constant and $\delta/2$. Thus $M_{(0,\epsilon)}$ consists only of cusps. Since $l(\varphi_n^i)$ goes to zero as n goes to infinity, for large n, all the geodesics φ_n^i are in $M_{n,(0,\epsilon)}$. Let T_n^i be the Margulis tube corresponding to φ_n^i, and $\{C_n^{l+1}, \ldots, C_n^k\}$ be the cusp components of $M_{n,(0,\epsilon)}$ corresponding to the non-deformed cusps. Let

$$F_n = T_n^1 \cup \cdots \cup T_n^l \cup C_n^{l+1} \cup \cdots \cup C_n^k \subset M_{n,(0,\epsilon)}.$$

For large n, $M_{n,[\epsilon,\infty)}$ is homeomorphic to $M_{[\epsilon,\infty)}$; in particular, $M_{n,(0,\epsilon]}$ has k boundary components. It implies that, for large n, $F_n = M_{n,(0,\epsilon)}$, and the result follows. □

We will need the following improvement of Theorem 5.2.2 in the following chapter.

Proposition 5.4.8. *Let $f\colon \mathbf{C} \to \mathbf{C}$ be a continuous function with* $\operatorname{supp} f$ *not necessarily compact but contained in $\mathbf{C} \setminus \overline{D}$. Assume that there exists $\epsilon > 0$, and $K > 0$ such that*

$$|f(z)| \le \frac{K}{|z|^{2+\epsilon}},$$

for all $z \in \mathbf{C}$. Then we have:

1. *For any $M \in \mathcal{M}$,*
$$\int_{|z|>1} |f(z)| d\, \mu_{\mathrm{sp}}\, M(z) < \infty.$$

2. *If $\{M_n\}$ converges to M in $\mathcal{M}$, then*
$$\lim_{n\to\infty} \int_{|z|>1} f(z) d\, \mu_{\mathrm{sp}}\, M_n(z) = \int_{|z|>1} f(z) d\, \mu_{\mathrm{sp}}\, M(z).$$

Proof. Let δ be the Margulis constant. Then for all $M \in \mathcal{M}$ any prime closed geodesic in M of length $\ge 2\delta$ intersects the thick part $M_{[\delta,\infty)}$. Let $M \in \mathcal{M}$, and put $\mu = \mu_{\mathrm{sp}}\, M$. Fix $R \gg 1$. By Lemma 5.1.3, we have

$$\mu(\{e^{2\delta} \le |z| \le R\}) \le CR^2,$$

where $C = \pi \frac{e^{8 \operatorname{diam} M_{[\delta,\infty)}}}{\operatorname{Vol} M_{[\delta,\infty)}}$. Then we have,

$$\begin{aligned}
\int_{|z|\ge R} |f(z)| d\mu(z) &= \sum_{k=0}^{\infty} \int_{R2^k \le |z| < R2^{k+1}} |f(z)| d\mu(z) \\
&\le \sum_{k=0}^{\infty} \int_{R2^k \le |z| < R2^{k+1}} \frac{K}{|z|^{2+\epsilon}} d\mu(z) \\
&\le \sum_{k=0}^{\infty} \frac{K}{(R2^k)^{2+\epsilon}} \int_{R2^k \le |z| < R2^{k+1}} d\mu(z) \\
&\le \sum_{k=0}^{\infty} \frac{K}{(R2^k)^{2+\epsilon}} C(R2^{k+1})^2 \\
&= \frac{KC}{R^\epsilon} \sum_{k=0}^{\infty} \frac{2^{2k+2}}{2^{k(2+\epsilon)}} = \frac{4KC}{R^\epsilon} \frac{1}{1-\frac{1}{2^\epsilon}} = \frac{C'}{R^\epsilon},
\end{aligned}$$

where C' is a constant depending only on C, K, and ϵ. The first assertion is then proved. Now let $\{M_n\}$ be a sequence converging to M in $\mathcal{M}$. Let us put $\mu_n = \mu_{\mathrm{sp}}\, M_n$. Since both $\operatorname{diam} M_{n,[\delta,\infty)}$ and $\operatorname{Vol} M_n$ are uniformly bounded on n, the first assertion implies that there exists a constant C'' such that for all n

$$\int_{|z|\ge R} |f(z)| d\mu_n(z) \le \frac{C''}{R^\epsilon}.$$

Thus we have

$$\begin{aligned}\left|\int_{|z|>1} f(z)(d\mu_n(z) - d\mu(z))\right| &\leq \left|\int_{1<|z|<R} f(z)(d\mu_n(z) - d\mu(z))\right| \\ &+ \left|\int_{|z|\geq R} f(z)(d\mu_n(z) - d\mu(z))\right| \\ &\leq \left|\int_{1<|z|<R} f(z)(d\mu_n(z) - d\mu(z))\right| + \frac{C'' + C'}{R^\epsilon}.\end{aligned}$$

Theorem 5.2.2 shows that

$$\lim_{n\to\infty}\left|\int_{|z|>1} f(z)(d\mu_n(z) - d\mu(z))\right| \leq \frac{C'' + C'}{R^\epsilon}.$$

Since R is arbitrary and independent of both C and C'', the left hand side of the above equation must vanish. This proves the proposition. □

5.5 Spin-complex-length spectrum

Let (M, η) be an spin complete hyperbolic 3-manifold, and consider its holonomy representation,

$$\mathrm{Hol}_{(M,\eta)}\colon \pi_1(M, p) \to \mathrm{SL}(2, \mathbf{C}).$$

If $\gamma \in \pi_1(M, p)$ is of hyperbolic type then,

$$\mathrm{Hol}_{(M,\eta)}(\gamma) \sim \begin{pmatrix} e^{\lambda/2} & 0 \\ 0 & e^{-\lambda/2} \end{pmatrix} \in \mathrm{SL}(2, \mathbf{C}), \quad \mathrm{Re}(\lambda) > 0.$$

The *spin complex length* of γ is by definition the parameter $\lambda \in \mathbf{C}/\langle 4\pi i\rangle$. Hence, in contrast to the usual complex length, $e^{\lambda/2}$ is well defined (we have a well defined sign given by the lift of the holonomy). We propose the following definition.

Definition. The (prime) spin-complex-length spectrum of (M, η) is defined by

$$\mu_{\mathrm{sp}}(M, \eta) = \sum_{\varphi\in\mathcal{PC}(M)} \delta_{e^{\lambda(\varphi)/2}},$$

where δ_x is the Dirac measure centered at x.

Remark. The image measure of $\mu_{\mathrm{sp}}(M, \eta)$ under the function $z \mapsto z^2$ is $\mu_{\mathrm{sp}}\, M$.

The results obtained for the length spectrum in the previous sections extend in a natural way for the spin-complex-length spectrum, and their proofs will be omitted. To do that we must consider the space $\mathcal{MSF}$ of spin-hyperbolic manifolds with a baseframe. In this case we have the identification between $\mathcal{MSF}$ and the space of discrete torsion-free subgroups of $\mathrm{SL}(2, \mathbf{C})$ with finite co-volume. We topologize $\mathcal{MSF}$ in such a way that this identification becomes a homeomorphism. The quotient spaces $\mathcal{MSB}$ and $\mathcal{MS}$ are then defined as in the non-spin case.

Theorem 5.5.1. *The map* $\mu_{\mathrm{sp}} \colon \mathcal{MS} \to M(\mathbf{C} \setminus \overline{D})$ *is continuous.*

As in the non-spin case, we can improve the continuity in the following sense. Notice that the condition on the decay at infinity must be replaced, since the measure of the ball $B_R(0) \subset \mathbf{C}$ under the measure $\mu_{\mathrm{sp}}(M,\eta)$ is equal to the measure of the ball $B_{R^2}(0)$ under the measure $\mu_{\mathrm{sp}}\, M$.

Proposition 5.5.2. *Let* $f \colon \mathbf{C} \to \mathbf{C}$ *be a continuous function with support contained in* $|z| > 1$. *Assume that there exists* $\epsilon > 0$, *and* $K > 0$ *such that*

$$|f(z)| \leq \frac{K}{|z|^{4+\epsilon}},$$

for all $|z| > 1$. *If* $\{(M_n, \eta_n)\}$ *converges to* (M,η) *in* $\mathcal{MS}$, *then*

$$\int_{|z|>1} |f(z)| d\mu(z), \int_{|z|>1} |f(z)| d\mu_n(z) < \infty,$$

and

$$\lim_{n\to\infty} \int_{|z|>1} f(z) d\mu_n(z) = \int_{|z|>1} f(z) d\mu(z).$$

Where $\mu = \mu_{\mathrm{sp}}(M,\eta)$ *and* $\mu_n = \mu_{\mathrm{sp}}(M_n,\eta_n)$.

Chapter 6

Asymptotic behavior

The aim of this chapter is to establish the asymptotic behavior of the n-dimensional hyperbolic Reidemeister torsion. More concretely, we will prove the following result.

Theorem 6.0.3. *Let M be a connected, complete, hyperbolic 3-manifold of finite volume. Then*

$$\lim_{k\to\infty}\frac{\log|\mathcal{T}_{2k+1}(M)|}{(2k+1)^2} = -\frac{\mathrm{Vol}(M)}{4\pi}.$$

In addition, if η is an acyclic spin structure on M, then

$$\lim_{k\to\infty}\frac{\log|\mathcal{T}_{2k}(M,\eta)|}{(2k)^2} = -\frac{\mathrm{Vol}(M)}{4\pi}.$$

For a compact manifold, the above result is due to Müller, see [Mül]. In this case, we can consider $\tau_n(M;\eta)$ for all n (i.e. there is no need to consider the normalized torsion $\mathcal{T}_n(M,\eta)$).

Theorem 6.0.4 (W. Müller, [Mül])**.** *Let (M,η) be a connected spin-hyperbolic 3-manifold. Then we have:*

$$\lim_{n\to\infty}\frac{\log|\,\tau_n(M;\eta)|}{n^2} = -\frac{\mathrm{Vol}(M)}{4\pi}.$$

The proof given by Müller is based on the fact that the Reidemeister torsion coincides with the Ray-Singer analytic torsion for a compact manifold. Since *a priori* the Ray-Singer torsion is not even defined for non-compact manifolds, it seems difficult to adapt Müller's proof to the non-compact case. Nevertheless, Müller's techniques are still powerful in the non-compact case, and will play a crucial role in our proof of Theorem 6.0.3. Roughly speaking, our approach will consist in approximating the cusp manifold M by compact manifolds obtained by hyperbolic Dehn filling; then we will apply Müller's theorem to these compact manifolds and the surgery formulas for the torsion stated in Chapter 4. The continuity of the (spin-)complex-length spectrum established in Chapter 5 will allow us to handle this limit process.

The distribution of this chapter is as follows. The first section is an exposition of the notions concerning the Ray-Singer analytic torsion and Ruelle zeta functions that will be needed in the subsequent sections; that section ends with Wotzke's theorem in dimension three, which gives the relationship between Ruelle zeta functions and the Reidemeister torsion invariants that we are studying. In the second section, we will state the theorem by Müller from which he deduces the asymptotic behaviour for the compact case. That theorem establishes a formula for the Ray-Singer analytic torsion, which will be the essential ingredient for the proof of Theorem 6.0.3 given in the last section.

6.1 Ruelle zeta functions

Let M be a differentiable closed n-manifold with a Riemannian metric g. Let us assume that we have an *acyclic* orthogonal (or unitary) representation of the fundamental group

$$\rho\colon \pi_1 M \to \mathrm{O}(n).$$

The analytic Ray-Singer torsion $T(M;\rho)$, introduced by Ray and Singer in the seminal paper [RS71], is a certain weighted alternating product of regularized determinants of the Laplacians

$$\Delta^q\colon \Omega^q(M;E_\rho) \to \Omega^q(M;E_\rho).$$

A theorem proved in [RS71] states that the Ray-Singer torsion is independent of the chosen metric. Hence, it is usually denoted simply as $T(M;\rho)$, without making reference to the metric g.

In the paper mentioned above, Ray and Singer conjectured that the Reidemeister torsion $\tau(M;\rho)$ agrees with the analytic torsion $T(M;\rho)$. This conjecture was proved independently by Cheeger and Müller in [Che79] and [Mül78] respectively. In [Mül93], Müller extended the definition of the analytic torsion to unimodular representations

$$\rho\colon \pi_1 M \to \mathrm{SL}(n,\mathbf{C}).$$

As in the orthogonal case, this definition requires a Riemannian metric, but, in contrast to the orthogonal case, this new analytic torsion is only metric independent for odd dimensions. In that paper, Müller also proved that both the analytic torsion and the Reidemeister torsion agree for an odd dimensional closed manifold.

An important part of this story concerns the relation between the Ray-Singer torsion and Ruelle zeta functions for a compact negatively curved manifold M. Since it will play a crucial role in the proof of our main theorem, we will spend the rest of this section to explain it.

Let Γ be a torsion free co-compact subgroup of $\mathrm{Isom}^+\,\mathbf{H}^n$, and let $M = \mathbf{H}^n/\Gamma$ be the corresponding hyperbolic manifold. The classical Ruelle zeta function associated to M is formally defined as

$$R(s) = \prod_{[\gamma]\in \mathrm{PC}(\Gamma)} \left(1 - e^{-sl(\gamma)}\right),$$

where $l(\gamma)$ is the length of the prime oriented closed geodesic defined by the prime conjugacy class $[\gamma]$ of Γ. The region of convergence of $R(s)$ can be determined using the asymptotic behaviour of the number of closed geodesics of length less or equal than a given value. To that end, define $P(t)$ as

$$P(t) = \#\Big\{[\gamma] \in \mathrm{PC}\Gamma \mid l(\gamma) \le t\Big\}.$$

Margulis studied the function $P(t)$ for a closed manifold of negative curvature in [Mar69]. Among other things, he proved that

$$\lim_{t\to\infty} \frac{P(t)}{e^{ht}/ht} = 1,$$

where h is the topological entropy of the geodesic flow. The topological entropy of a hyperbolic manifold of dimension n is $h = n-1$. Using Margulis' result, the region of convergence of $R(s)$ is easily seen to be

$$\Big\{ s \in \mathbf{C} \mid \operatorname{Re}(s) > n-1 \Big\}.$$

In [Fri86], Fried gave the following generalization on the definition of the Ruelle zeta function. Given an *orthogonal representation* $\rho\colon \pi_1(M) \to O(d)$, which need not to be acyclic, the twisted Ruelle zeta function associated to ρ is defined as

$$R_\rho(s) = \prod_{[\gamma]\in \mathrm{PC}\Gamma} \det\Big(\mathrm{Id} - \rho(\gamma)e^{-sl(\gamma)}\Big).$$

The region of convergence of $R_\rho(s)$ is the same as the one of the classical Ruelle zeta function (here we are using that ρ is an orthogonal representation). In that same paper, Fried proved that $R_\rho(s)$ has a meromorphic extension to the whole complex plane; moreover, if ρ is acyclic, then $R_\rho(s)$ is regular at $s=0$ and $|R_\rho(0)| = T(M;\rho)^2$ (if ρ is not acyclic, $R_\rho(s)$ can have a pole at $s=0$, and $T(M;\rho)^2$ is equal to the leading term of the Laurent expansion of $R_\rho(s)$ at the origin).

In a posterior paper [Fri95], Fried proved that for a general representation $\rho\colon \pi_1 M \to \mathrm{GL}(d;\mathbf{C})$ the twisted Ruelle zeta function $R_\rho(s)$ has also a meromorphic extension to the whole plane. However, he was not able to prove its relationship with the Ray-Singer analytic torsion. Nevertheless, three years later U. Bröcker proved in his thesis a similar result for representations of the fundamental group that are restrictions of finite-dimensional irreducible representations of $\mathrm{Isom}^+ \mathbf{H}^n \cong \mathrm{SO}_0(n,1)$, see [Brö98]. According to Müller [Mül], the methods used by Bröcker are based on elaborate computations which are difficult to verify. Nonetheless, this problem has been overcome by Wotzke in his thesis [Wot08]. The following section is dedicated to state Wotzke's Theorem in dimension 3.

6.1.1 Wotzke's Theorem

Let (M,η) be a connected, closed, spin-hyperbolic 3-manifold. If Γ is the image of $\pi_1(M,p)$ under the $\mathrm{Hol}_{(M,\eta)}$, then

$$(M,\eta) = \Gamma\backslash \mathrm{SL}(2;\mathbf{C})/\mathrm{SU}(2).$$

Let ρ be a *real* finite-dimensional representation of $\mathrm{SL}(2;\mathbf{C})$, regarded as a real Lie group. Denote by θ the Cartan involution of $\mathrm{SL}(2;\mathbf{C})$ with respect to $\mathrm{SU}(2)$, and put $\rho_\theta = \rho\circ\theta$. Let $E_\rho \to M$ be the flat vector bundle associated to ρ. Introduce some metric on E_ρ, and consider the Laplacians $\Delta^q\colon \Omega^r(M;E_\rho) \to \Omega^r(M;E_\rho)$.

Theorem 6.1.1 (Wotzke,[Wot08])**.** *With the above notation, the following assertions hold:*

1. *If ρ_θ is not isomorphic to ρ, then $R_\rho(s)$ is regular at $s = 0$ and*
$$|R_\rho(0)| = T(M;\rho)^2.$$

2. *Assume that $\rho \circ \theta$ is isomorphic to ρ. If ρ is not trivial, then the order h_ρ at $s = 0$ of $R_\rho(s)$ is given by*
$$h_\rho = 2\sum_{q=1}^{3}(-1)^q \dim\ker\Delta^q,$$
and for the trivial representation we have $h_\rho = 4 - 2\dim \mathrm{H}^1(M;\mathbf{R})$. The leading term of the Laurent expansion of $R_\rho(s)$ at s = 0 is given by
$$T(M;\rho)^2 s^{h_\rho}.$$

Remark. The Cartan involution of the real Lie algebra $\mathfrak{sl}(2;\mathbf{C})$ is given by $\theta(X) = -\overline{X}^t$. It can be checked that a *complex* representation ρ of $\mathrm{SL}(2;\mathbf{C})$ is not equivalent to $\rho\circ\theta$.

6.2 Müller's Theorem

Let us retain the same notation as in the previous section; in particular, M will be assumed to be closed. For $n > 0$, let ρ_n be the n-dimensional canonical representation of (M,η),
$$\rho_n\colon \pi_1(M,p) \cong \Gamma \to \mathrm{SL}(n;\mathbf{C}).$$

Müller's theorem on the equivalence of the Reidemeister torsion and the Ray-Singer analytic torsion implies that
$$T(M;\rho_{n,\eta}) = |\,\tau(M;\rho_n)|.$$
Let us denote by $R_n(s)$ the Ruelle zeta function associated to the representation ρ_n. Wotzke's Theorem gives
$$|R_{\rho_n}(0)| = |\,\tau(M;\rho_n)|^2.$$
Following [Mül], the Ruelle zeta function $R_{\rho_n}(s)$ can be expressed in terms of the following related Ruelle zeta functions,
$$R_k(s) = \prod_{[\gamma]\in \mathrm{PC}(\Gamma)} \left(1 - \sigma_k(\gamma)e^{-sl(\gamma)}\right),$$
where $\sigma_k(\gamma)$ is defined by
$$\sigma_k(\gamma) = e^{ki\,\mathrm{Im}\,\lambda(\gamma)/2} = e^{ki\theta(\gamma)/2},$$
with $\theta(\gamma)$ the geometric spin torsion of the closed geodesic defined by γ. A straightforward computation then shows that
$$R_{\rho_n}(s) = \prod_{k=0}^{n} R_{n-2k}(s-(n/2-k)).$$

The following theorem by Müller relates the Reidemeister torsion, Ruelle zeta functions and the volume of the manifold M.

Remark. Müller uses in [Mül] the notation τ_n to designate the representation coming from the nth symmetric power, so his τ_n is our ρ_{n+1}.

Theorem 6.2.1 (**Müller [Mül]**). *Let (M,η) be a closed spin-hyperbolic 3-manifold, and for $m \geq 3$ let ρ_m be its m-dimensional canonical representation. Then we have the following equations,*

$$\begin{aligned}
\log\left|\frac{\tau(M;\rho_{2m+1})}{\tau(M;\rho_5)}\right| &= \sum_{k=3}^{m}\log|R_{2k}(k)| - \frac{1}{\pi}\operatorname{Vol}M\,(m(m+1)-6)\,, \\
\log\left|\frac{\tau(M,\eta;\rho_{2m})}{\tau(M,\eta;\rho_4)}\right| &= \sum_{k=2}^{m-1}\log\left|R_{2k+1}\left(k+\frac{1}{2}\right)\right| - \frac{1}{\pi}\operatorname{Vol}M(m^2-4)
\end{aligned}$$

Müller then deduces Theorem 6.0.4 from the following lemma, [Mül].

Lemma 6.2.2. *For a closed spin-hyperbolic 3-dimensional manifold (M,η) there exists a constant $C > 0$, depending only on the manifold M, such that for all $m \geq 3$, we have*

$$\sum_{k=3}^{m}|\log|R_{2k}(k)|| < C, \quad \sum_{k=2}^{m-1}\left|\log\left|R_{2k+1}\left(k+\frac{1}{2}\right)\right|\right| < C.$$

6.3 The noncompact case

Let (M,η) be a compactly approximable spin-hyperbolic 3-manifolds of finite volume. In this section we want to prove that Theorem 6.0.3 holds for (M,η) as well. We will do this by proving that Theorem 6.2.1 holds also for (M,η).

The definition of the Ruelle zeta function R_{ρ_n} for (M,η) is obvious if we define it in terms of prime closed geodesics; more concretely, we define

$$R_{\rho_n}(s) = \prod_{\varphi\in\mathcal{PC}(M)} \det\left(\mathrm{Id} - \rho_n(\varphi)e^{-sl(\varphi)}\right).$$

Of course, it makes sense also to define

$$R_k(s) = \prod_{\varphi\in\mathcal{PC}(M)} \left(1 - \sigma_k(\varphi)e^{-sl(\gamma)}\right).$$

The function $R_{\rho_n}(s)$ is related to the functions $R(s,\sigma_k)$ in the same way as in the compact case. The estimations concerning the growth of closed geodesics in M imply that $R(s,\sigma_k)$ converges for $\mathrm{Re}(s) > 2$. More accurate estimations will probably allow to conclude that the region of convergence of $R(s,\sigma_k)$ is exactly that half-plane. Therefore, the region of convergence of $R_{\rho_n}(s)$ contains the half-plane $\mathrm{Re}(s) > 2 + n/2$.

It is worth noticing that the following equation holds.

Lemma 6.3.1. *For $k \geq 3$ we have:*

$$\log\left|R_k\left(\frac{k}{2}\right)\right| = \int_{|z|>1} \log|1-z^{-k}|d\,\mu_{\mathrm{sp}}(M,\eta)(z). \tag{6.1}$$

With the same notation as in Chapter 4, we have the following formula.

Lemma 6.3.2. *Let $(p,q) \in \mathcal{A}_{(M,\eta)}$, and $A = \{\pm\varphi_{p_1/q_1}, \ldots, \pm\varphi_{p_l/q_l}\}$ be the prime oriented core geodesics in $M_{p/q}$ added in the Dehn filling. For an integer $m \geq 3$, we have*

$$\log\left|\frac{\tau(M;\rho_{2m}^{p/q})}{\tau(M;\rho_4^{p/q})}\right| = -\frac{(m-2)(m+2)}{2}\sum_{i=1}^{l} l(\varphi^i_{p/q}) - \frac{1}{\pi}\operatorname{Vol}(M_{p/q})(m^2-4) + \sum_{k=2}^{m-1} B_{2k+1}^{p/q},$$

where

$$B_j^{p/q} = \sum_{\varphi\in\mathcal{PC}(M_{p/q})\setminus A} \log\left|1-e^{-j\lambda_{p/q}(\varphi)/2}\right|.$$

Proof. For the sake of simplicity we will prove it only for one-cusped manifolds. The surgery formula given by Lemma 4.2.3, yields

$$\log\left|\frac{\tau(M_{p/q};\rho_{2m}^{p/q})}{\tau(M;\rho_{2m}^{p/q})}\right| = \sum_{k=0}^{m-1}\log\left|\left(e^{\left(\frac{1}{2}+k\right)\lambda(\varphi_{p/q})}-1\right)\left(e^{-\left(\frac{1}{2}+k\right)\lambda(\varphi_{p/q})}-1\right)\right|.$$

It follows that,

$$\log\left|\frac{\tau(M_{p/q};\rho_{2m}^{p/q})\,\tau(M;\rho_4^{p/q})}{\tau(M_{p/q};\rho_4^{p/q})\,\tau(M;\rho_{2m}^{p/q})}\right| = \sum_{k=2}^{m-1}\log\left|\left(e^{\left(\frac{1}{2}+k\right)\lambda(\varphi_{p/q})}-1\right)\left(e^{-\left(\frac{1}{2}+k\right)\lambda(\varphi_{p/q})}-1\right)\right|.$$

Since $M_{p/q}$ is compact, Müller's Theorem 6.3.4 gives

$$\log\left|\frac{\tau(M_{p/q};\rho_{2m}^{p/q})}{\tau(M_{p/q};\rho_4^{p/q})}\right| = \sum_{k=2}^{m-1}\log\left|R_{2k+1}^{p/q}\left(k+\frac{1}{2}\right)\right| - \frac{1}{\pi}\operatorname{Vol}(M_{p/q})(m^2-4).$$

From these last two equations, we get

$$\begin{aligned}
-\log\left|\frac{\tau(M;\rho_4^{p/q})}{\tau(M;\rho_{2m}^{p/q})}\right| &= \sum_{k=2}^{m-1}\log\left|R_{2k+1}^{p/q}\left(k+\frac{1}{2}\right)\right| - \frac{1}{\pi}\operatorname{Vol}(M_{p/q})(m^2-4)\\
&- \sum_{k=2}^{m-1}\log\left|e^{(\frac{1}{2}+k)\lambda(\varphi_{p/q})}-1\right|\left|e^{-(\frac{1}{2}+k)\lambda(\varphi_{p/q})}-1\right|.
\end{aligned}$$

Using the expression

$$\log|R_{2k+1}^{p/q}(k+\frac{1}{2})| = \log|1-e^{-(k+\frac{1}{2})\overline{\lambda(\varphi_{p/q})}}|^2 + B_{2k+1}^{p/q},$$

the above equation is written as

$$\begin{aligned}\log\left|\frac{\tau(M;\rho_{2m}^{p/q})}{\tau(M;\rho_4^{p/q})}\right| &= \sum_{k=2}^{m-1}\log\frac{|1-e^{-(k+\frac{1}{2})\overline{\lambda(\varphi_{p/q})}}|^2}{|e^{(\frac{1}{2}+k)\lambda(\varphi_{p/q})}-1||e^{-(\frac{1}{2}+k)\lambda(\varphi_{p/q})}-1|}\\ &- \frac{1}{\pi}\mathrm{Vol}(M_{p/q})(m^2-4)+\sum_{k=2}^{m-1}B_{2k+1}^{p/q}.\end{aligned}$$

We have,

$$\frac{|1-e^{-(k+\frac{1}{2})\overline{\lambda(\varphi_{p/q})}}|^2}{|e^{(\frac{1}{2}+k)\lambda(\varphi_{p/q})}-1||e^{-(\frac{1}{2}+k)\lambda(\varphi_{p/q})}-1|}=e^{-(\frac{1}{2}+k)\operatorname{Re}\lambda(\varphi_{p/q})}.$$

Hence, summing up the terms, we get

$$\sum_{k=2}^{m-1}\log\left(\frac{|1-e^{-(k+\frac{1}{2})\overline{\lambda(\varphi_{p/q})}}|^2}{|e^{(\frac{1}{2}+k)\lambda(\varphi_{p/q})}-1||e^{-(\frac{1}{2}+k)\lambda(\varphi_{p/q})}-1|}\right)=-\frac{(m-2)(m+2)}{2}l(\varphi_{p/q}),$$

and the lemma follows. □

Lemma 6.3.3. *With the same notation as in the preceding lemma, for $k\geq 5$ we have*

$$\lim_{(p,q)\to\infty}B_k^{p/q}=\log\left|R_k\left(\frac{k}{2}\right)\right|.$$

Moreover, the following series is absolutely convergent

$$\sum_{k=5}^{\infty}\log\left|R_k\left(\frac{k}{2}\right)\right|.$$

Proof. Let δ be the length of the shortest closed geodesic in M. By Lemma 5.4.7, for (p,q) large enough, the only prime closed geodesics on $M_{p/q}$ whose lengths are less than $\delta/2$ are the core geodesics $A=\{\pm\varphi_{p_1/q_1},\dots,\pm\varphi_{p_l/q_l}\}$. In that case,

$$B_k^{p/q}=\sum_{\varphi\in\mathcal{PC}(M_{p/q})\setminus A}\log\left|1-e^{-k\lambda(\varphi)/2}\right|=\int_{|z|>e^{\delta/4}}\log|1-z^{-k}|d\mu_{p/q}(z).$$

where $\mu_{p/q}=\mu_{\mathrm{sp}}(M_{p/q},\eta_{p/q})$. Now we want to apply Proposition 5.5.2. We shall show that for large $|z|$ we have

$$|\log|1-z^{-k}||\leq\frac{C}{z^5},\qquad\text{for } k\geq 5,\tag{6.2}$$

where C is some constant. First notice that for $w\in\mathbf{C}$ with $|w|<1$ the following inequality holds

$$|\log|1-w||\leq-\log|1-|w||.$$

On the other hand, for $|w|$ small enough,

$$-\log|1-|w|| \sim |w|.$$

Inequality (6.2) then follows easily from the last two inequalities. Therefore, we can use Proposition 5.5.2 to conclude that

$$\lim_{(p,q)\to\infty} B_k^{p/q} = \log\left|R_k\left(\frac{k}{2}\right)\right|.$$

Finally, if $\mu = \mu_{\mathrm{sp}}(M,\eta)$, we have

$$\begin{aligned}
\sum_{k=5}^{\infty}\left|\log\left|R_k\left(\frac{k}{2}\right)\right|\right| &\leq \sum_{k=5}^{\infty}\int_{|z|>e^{\delta/2}} |\log|1-|z|^{-k}||d\mu(z) \\
&\leq \sum_{k=5}^{\infty}\int_{|z|>e^{\delta/2}} \frac{C}{|z|^k}d\mu(z) \\
&= \int_{|z|>e^{\delta/2}} \frac{C}{|z|^5}\frac{1}{1-\frac{1}{|z|}}d\mu(z) \\
&\leq \frac{C}{1-e^{\delta/2}}\int_{|z|>e^{\delta/2}} \frac{1}{|z|^5}d\mu(z) < \infty,
\end{aligned}$$

the last integral being finite by Proposition 5.5.2. □

Finally, letting (p,q) go to infinity in the equation of Lemma 6.3.2, using the continuity of the complex-length spectrum, the continuity of the volume, and the fact that the lengths of the core geodesics $\varphi^i_{p/q}$ go to zero, we deduce the following generalization of Theorem 6.2.1 for even dimensions n. In the following theorem we have also included the odd dimensional case, as its proof is handled in a similar way.

Theorem 6.3.4. *Let M be a complete hyperbolic 3-manifold of finite volume. Then for $m \geq 3$*

$$\log\left|\frac{\mathcal{T}_{2m+1}(M)}{\mathcal{T}_5(M)}\right| = \sum_{k=3}^{m}\log|R_{2k}(k)| - \frac{1}{\pi}\operatorname{Vol} M\,(m(m+1)-6)\,.$$

If in addition M is enriched with an acyclic spin structure, then for $m \geq 3$

$$\log\left|\frac{\mathcal{T}_{2m}(M,\eta)}{\mathcal{T}_4(M,\eta)}\right| = \sum_{k=2}^{m-1}\log\left|R_{2k+1}\left(k+\frac{1}{2}\right)\right| - \frac{1}{\pi}\operatorname{Vol} M(m^2-4).$$

The proof of Theorem 6.0.3 now follows easily.

Proof of Theorem 6.0.3. Theorem 6.3.4 and Lemma 6.3.3 imply that

$$\lim_{n\to\infty}\frac{\log|\mathcal{T}_n(M,\eta)|}{n^2} = -\frac{\operatorname{Vol} M}{4\pi}.$$

□

Chapter 7

Reidemeister torsion and length spectrum

The results of last chapter, especially Theorem 6.3.4, show that there is a close relationship between the spin-complex-length spectrum of a complete, acyclic, spin-hyperbolic 3-manifold of finite volume (M, η) and its higher-dimensional Reidemeister torsion invariants. In this chapter we want to focus on this question; more concretely, we want to study at what extent the sequence $\{\mathcal{T}_n(M, \eta)\}$ determines the spin-complex-length spectrum of the manifold. The equivalence between these two invariants should be regarded as a geometric interpretation of the information encoded in these invariants.

Definition. We will say that two (spin-)hyperbolic 3-manifolds are *(spin-)isospectral* if the have the same prime (spin-)complex-length spectrum.

The notion of isospectrality, as stated in the above definition, has been already considered by C. Maclachlan and A.W. Reid in [MR03]. They prove the following theorem.

Theorem 7.0.5 (C. Maclachlan and A.W. Reid, [MR03])**.** *For any integer $n \geq 2$, there are n isospectral non-isometric closed hyperbolic 3-manifolds.*

As an immediate consequence of Theorem 7.0.5 and Wotzke's Theorem 6.1.1, we get the following result.

Theorem 7.0.6. *For any integer $n \geq 2$, there are n non-isometric, closed, hyperbolic 3-manifolds $M_1, \ldots, M_n$ such that for all $k > 0$,*

$$|\tau_{2k+1}(M_i)| = |\tau_{2k+1}(M_j)|, \quad \textit{for all } i, j = 1, \ldots, n.$$

Unfortunately, we will need to weaken the notion of isospectrality, and rather consider isospectrality up to complex conjugation. Before giving its definition, let us make the following considerations. Let (M, η) be a spin-hyperbolic 3-manifold, and let $(\overline{M}, \eta)$ be the corresponding spin manifold with the orientation reversed (here we are using the canonical one-to-one correspondence between spin structures on M and $\overline{M}$). The relationship between the spin-complex-length spectra of these two manifolds is easily established. Indeed we have:

$$\begin{aligned} \mu_{\mathrm{sp}}(M,\eta) &= \sum_{\varphi\in\mathcal{PC}(M)} \delta_{e^{\lambda(\varphi)/2}}, \\ \mu_{\mathrm{sp}}(\overline{M},\eta) &= \sum_{\varphi\in\mathcal{PC}(M)} \delta_{e^{\overline{\lambda(\varphi)}/2}}, \end{aligned}$$

where $\lambda(\varphi)$ is the spin-complex-length function of (M,η). Notice that $\mu_{\mathrm{sp}}(\overline{M},\eta)$ is the image measure of $\mu_{\mathrm{sp}}(M,\eta)$ under the complex conjugation map.

Definition. We will say that two complete spin-hyperbolic 3-manifolds (M_1,η_1) and (M_2,η_2) are *spin isospectral up to complex conjugation* if they have the same spin-complex-length spectrum up to complex conjugation, that is,

$$\mu_{\mathrm{sp}}(M_1,\eta_1)+\mu_{\mathrm{sp}}(\overline{M_1},\eta_1)=\mu_{\mathrm{sp}}(M_2,\eta_2)+\mu_{\mathrm{sp}}(\overline{M_2},\eta_2).$$

The definition for "non-spin" manifolds is analogous.

Remark. The reason to consider isospectrality up to complex conjugation is essentially that Wotzke's Theorem 6.1.1 is an equality between the moduli of the Ruelle zeta function and Reidemeister torsion; if we had also equality between the arguments, then there should be no need to consider isospectrality *up to complex conjugation*.

Remark. If two complete spin-hyperbolic 3-manifolds are spin-isospectral up to complex conjugation, then they have the same real length spectrum. The same holds true for non-spin manifolds.

Theorem 7.0.7. *Let $(M_1,\eta_1),(M_2,\eta_2)$ be two complete spin acyclic hyperbolic 3-manifolds of finite volume. Assume that there exists $N\geq 4$ such that for all $n\geq N$ we have*

$$|\mathcal{T}_n(M_1,\eta_1)|=|\mathcal{T}_n(M_2,\eta_2)|.$$

Then the following assertions hold:

1. *The spin manifolds (M_1,η_1) and (M_2,η_2) are spin-isospectral up to complex conjugation. In particular, they have the same real length spectrum.*

2. *The equality $|\mathcal{T}_n(M_1,\eta_1)|=|\mathcal{T}_n(M_2,\eta_2)|$ holds for all $n\geq 4$.*

The proof of Theorem 7.0.7 will be given in Section 7.2. Before doing that, we need a result on complex analysis which we prove in the following section.

7.1 Some results on complex analysis

The aim of this section is to provide a proof of the following analytical result needed to prove Theorem 7.0.7. We are indebted to J. Ortega-Cerdà, N. Makarov, and A. Nicolau for the proof of Proposition 7.1.3.

Proposition 7.1.1. *Let μ be a Radon complex-valued measure with compact support* $\operatorname{supp}\mu$ *contained in the interior of the unit disk D. Assume that μ satisfies the following conditions:*

1. $\mathbf{C} \setminus \operatorname{supp}\mu$ *is connected.*
2. $\operatorname{supp}\mu$ *has zero Lebesgue measure.*
3. *There exists a positive integer N and a holomorphic function ψ on the open unit disk with $\psi(0) = 0$, $\psi'(0) = 1$ such that*
$$\int_D \frac{\psi(z^n)}{z^N} d\mu(z) = 0.$$
for all $n \geq N$.

Then $\mu = 0$.

We will prove first the special in which the holomorphic function ψ appearing in the third condition of the above proposition is the identity. Then we will show that if ψ is any holomorphic function on the open unit disk with $\psi(0) = 0$, and $\psi'(0) = 1$, for all $N > 0$ the linear span of $\{\frac{\psi(z^n)}{z^N}\}_{n \geq N}$ is dense in the space of holomorphic functions on the open unit disk endowed with the topology of the uniform convergence on compact sets.

A way to prove Proposition 7.1.1 is to use the Cauchy transform. If μ is a Radon complex-valued measure compactly supported in the complex plane, then its Cauchy transform is defined by
$$\widehat{\mu}(\zeta) = \int_{\mathbf{C}} \frac{d\mu(z)}{z - \zeta}.$$
We will need only the following properties of the Cauchy transform, see [Gam69].

Proposition 7.1.2. *Let $\widehat{\mathbf{C}}$ be the Riemann sphere. The Cauchy transform has the following properties:*

1. $\widehat{\mu}(\zeta)$ *is analytic on* $\widehat{\mathbf{C}} \setminus \operatorname{supp}\mu$ *and vanishes at infinity.*
2. *If $\widehat{\mu} = 0$ Lebesgue-almost everywhere, then $\mu = 0$.*

With this result we can prove the following particular case of Proposition 7.1.1 mentioned above.

Proposition 7.1.3. *Let μ be a Radon complex-valued measure compactly supported in the complex plane that satisfies the following conditions:*

1. $\mathbf{C} \setminus \operatorname{supp}\mu$ *is connected.*

2. $\operatorname{supp}\mu$ *has zero Lebesgue measure.*

3. For all $n \geq 0$, $\int_{\mathbf{C}} z^n d\mu(z) = 0$.

Then $\mu = 0$.

Proof. Let $\widehat{\mu}(\zeta)$ be the Cauchy transform of μ. We know that $\widehat{\mu}(\zeta)$ is analytic on $\widehat{\mathbf{C}} \setminus \operatorname{supp}\mu$ and vanishes at ∞. Take $|\zeta|$ large enough so that $|z/\zeta| < 1$ for all $z \in \operatorname{supp}\mu$. Then, we have

$$\widehat{\mu}(\zeta) = \int_{\mathbf{C}} \frac{d\mu(z)}{z-\zeta} = -\frac{1}{\zeta}\int_{\mathbf{C}} \frac{d\mu(z)}{1-\frac{z}{\zeta}} = -\frac{1}{\zeta}\sum_{n\geq 0}\int_{\mathbf{C}} \frac{z^n}{\zeta^n} d\mu(\zeta) = 0$$

The last term being zero by hypothesis. Thus $\widehat{\mu}$ is identically zero in a neighbourhood of ∞, and hence it must be identically zero in $\widehat{\mathbf{C}} \setminus \operatorname{supp}\mu$, as $\mathbf{C} \setminus \operatorname{supp}\mu$ is connected. Since $\operatorname{supp}\mu$ has zero Lebesgue measure, we have $\widehat{\mu} = 0$ Lebesgue-almost everywhere. Proposition 7.1.2 then implies that that $\mu = 0$, as we wanted to prove. □

Now, to prove Proposition 7.1.1, it remains to prove the following result.

Proposition 7.1.4. *Let $H(D)$ be the space of holomorphic functions on the open unit disk endowed with the topology of the uniform convergence on compact sets, and let $\psi \in H(D)$ such that $\psi(0) = 0$ and $\psi'(0) = 1$. Then, for all $N \geq 1$, the linear span of*

$$\left\{ \frac{\psi(z^k)}{z^N} \right\}_{k \geq N}$$

is dense in $H(D)$.

Remark. We have not been able to find this result in the literature, we provide a proof of it in this section.

In what follows, $\psi(z)$ will denote a fixed holomorphic function in $H(D)$ such that $\psi(0) = 0$ and $\psi'(0) = 1$. Thus we have,

$$\psi(z) = z + \sum_{k \geq 1} \psi_k z^k, \quad \text{for all } z \in D.$$

The fact that the linear span of the monomials $\{z^n\}_{n\geq 0}$ is dense in $H(D)$ implies that Proposition 7.1.4 is equivalent to say that for all $n \geq 0$ there exists a sequence $\{a_k^n\}_{k\geq N}$ of complex numbers such that

$$z^n = \sum_{k\geq N} a_k^n \frac{\psi(z^k)}{z^N},$$

with the right hand side converging uniformly on every compact set of D. Using the power series expansion of $\psi(z)$, the above equality yields a linear system with $\{a_k^n\}_{k\geq N}$ as unknowns. Since $\psi(0) = 0$ and $\psi'(0) = 1$, this system is lower triangular with ones in the diagonal, and hence it has a unique solution. The difficult point is to prove the convergence of the corresponding sequence. Fortunately, we can proceed in a slightly different way.

Let us denote by $H(D_R)$ the space of holomorphic functions on the open disk of radius R,

$$D_R = \{z \in \mathbf{C} \mid |z| < R\}.$$

We will work with the Bergman space on D_R, which is defined by

$$A^2(D_R) = \left\{ f \in H(D_R) \mid \int_{D_R} |f(z)|^2 dA(z) < \infty \right\},$$

where $dA(z)$ is the usual area measure, see [HKZ00] for details. It is well known that $A^2(D_R)$ is a Hilbert space with respect to the following inner product (see [HKZ00]),

$$\langle f, g \rangle = \int_{D_R} f(z)\overline{g(z)} dA(z).$$

The reason to consider $A^2(D_R)$ instead of $H(D_R)$ is due to the fact that it is a Hilbert space (so it is easier to work with it), and to the fact that convergence in the former implies convergence in the latter, as expressed by the following result (see [HKZ00]).

Proposition 7.1.5. *If a sequence of functions $\{f_n\}$ in $A^2(D_R)$ converges to f in $A^2(D_R)$, then $\{f_n\}$ converges to f uniformly on each compact set of D_R.*

For $0 < R < 1$, consider the linear operator $A_\psi \colon A^2(D_R) \to A^2(D_R)$, with domain the linear space of monomials, defined as follows:

$$\begin{aligned} A_\psi(1) &= 1, \\ A_\psi(z^n) &= \psi(z^n) = z^n + \sum_{j \geq 2} \psi_j z^{nj} \quad \text{for } n \geq 1. \end{aligned}$$

The following result shows that A_ψ is a bounded operator.

Proposition 7.1.6. *For $R < 1$, let $A_\psi = I + B_\psi$. Then $B_\psi \colon A^2(D_R) \to A^2(D_R)$ is Hilbert-Schmidt. In particular, B_ψ is compact and A_ψ is bounded.*

Proof. A basis of $A^2(D_R)$ is given by the following functions, which are just normalizations of the monomials $\{z^k\}$,

$$\phi_n(z) = \sqrt{\frac{n+1}{\pi}} \frac{z^n}{R^{n+1}}.$$

To be Hilbert-Schmidt then means that

$$\sum_{n \geq 0} \langle B_\psi(\phi_n), B_\psi(\phi_n) \rangle < \infty.$$

In terms of the basis $\{\phi_n\}$, B_ψ is written as follows: $B_\psi(\phi_0) = 0$, and for $n \geq 1$,

$$\begin{aligned} B_\psi(\phi_n) &= \sqrt{\frac{n+1}{\pi}} \frac{1}{R^{n+1}} \sum_{j\geq 2} \psi_j z^{nj} = \sqrt{\frac{n+1}{\pi}} \frac{1}{R^{n+1}} \sum_{j\geq 2} \psi_j \sqrt{\frac{\pi}{nj+1}} R^{nj+1} \phi_{nj} \\ &= \sum_{j\geq 2} \psi_j \sqrt{\frac{n+1}{nj+1}} R^{n(j-1)} \phi_{nj}. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{n\geq 0} \langle B_\psi(\phi_n), B_\psi(\phi_n)\rangle &= \sum_{n\geq 1}\sum_{j\geq 2} |\psi_j|^2 \frac{n+1}{nj+1} R^{2n(j-1)} \leq \sum_{j\geq 2} \frac{2|\psi_j|^2}{j} \sum_{n\geq 1} R^{2n(j-1)} \\ &= \sum_{j\geq 2} \frac{2|\psi_j|^2}{j} \frac{R^{2(j-1)}}{1-R^{2(j-1)}} \leq \frac{2}{R^2(1-R^2)} \sum_{j\geq 2} \frac{|\psi_j|^2}{j} R^{2j}. \end{aligned}$$

The last series is finite because it is exactly π times the square of the norm in $A^2(D_R)$ of $(\psi(z) - z)/z$. Indeed,

$$\left\| \frac{\psi(z) - z}{z} \right\|^2_{A^2(D_R)} = \sum_{j\geq 1} |\psi_{j+1}|^2 \left\| z^j \right\|^2_{A^2(D_R)} = \pi \sum_{j\geq 1} |\psi_{j+1}|^2 \frac{R^{2(j+1)}}{j+1}.$$

□

Corollary 7.1.7. *For $R < 1$, the operator $A_\psi \colon A^2(D_R) \to A^2(D_R)$ is invertible.*

Proof. We have $A_\psi = I + B_\psi$, with B_ψ a compact operator. The matrix of the operator A_ψ in the basis $\{\phi_n\}$ is lower triangular, and has ones in the diagonal; hence, the kernel of A_ψ is trivial, and the Fredholm alternative implies that A_ψ is invertible. □

Corollary 7.1.8. *The linear span of $\{1, \psi(z), \psi(z^2), \dots\}$ is dense in $H(D)$.*

Proof. Let us fix $g(z) \in H(D)$. Let $0 < R < 1$. By the above corollary, there exists $f_R(z) \in A^2(D_R)$ such that $A_\psi(f_R) = g$. Then we have

$$f_R(z) = \sum_{n\geq 0} a_n(R) z^n,$$

and the series $a_0(R) + \sum_{n\geq 1} a_n(R)\psi(z^n)$ converges to g in $A^2(D_R)$, so it converges uniformly to g in every compact contained in D_R. Since $f_R(z)$ also belongs to $A^2(D_{R'})$ for all $0 < R' < R$, and is holomorphic, the coefficients $a_n(R)$ are independent of R, so $a_n(R) = a_n$. Hence $a_0 + \sum_{n\geq 0} a_n\psi(z^n)$ converges to g in every compact set contained in the unit disk, and this proves the result. □

The proof of Proposition 7.1.4 now follows easily.

Proof of Proposition 7.1.4. Consider the following linear subspace of $H(D)$

$$C_N = \{\phi \in H(D) \mid \phi^{(j)}(0) = 0,\ 0 \le j \le N-1\}.$$

Since the derivative is continuous in $H(D)$, C_N is closed in $H(D)$. By the preceding corollary, it follows that C_N is the closure of $\{\psi(z^N), \psi(z^{N+1}), \dots\}$. On the other hand, C_N is homeomorphic to $H(D)$ via the linear map

$$\begin{array}{rcl} H(D) & \to & C_N \\ \phi(z) & \mapsto & z^N \phi(z). \end{array}$$

Therefore, the closure of the linear span of $\{\frac{\psi(z^k)}{z^N}\}_{k \ge N}$ is the whole $H(D)$, as we wanted to prove. □

7.2 Isospectrality and torsion

We start this section with the proof of Theorem 7.0.7.

Proof of Theorem 7.0.7. We can assume that $N \ge 6$. Let us put $\mu_i = \mu_{\mathrm{sp}}(M_i, \eta_i)$ and $\overline{\mu}_i = \mu_{\mathrm{sp}}(\overline{M_i}, \eta_i)$, for $i = 1, 2$. From Theorem 6.3.4 we deduce that for $k \ge 3$,

$$\begin{array}{rcl} \log \left| \dfrac{\mathcal{T}_{2k+3}(M_i)}{\mathcal{T}_{2k+1}(M_i)} \right| & = & \log |R^{M_i}_{2k+2}(k+1)| - \dfrac{2(k+1)}{\pi} \operatorname{Vol} M_i \\ \log \left| \dfrac{\mathcal{T}_{2k+2}(M_i)}{\mathcal{T}_{2k}(M_i)} \right| & = & \log \left| R^{M_i}_{2k+1}\left(k + \dfrac{1}{2}\right) \right| - \dfrac{2k+1}{\pi} \operatorname{Vol} M_i. \end{array}$$

By hypothesis, for all $n \ge N$, $|\mathcal{T}_n(M_1, \eta_1)| = |\mathcal{T}_n(M_2, \eta_2)|$. Then, by Theorem 6.0.3, we have $\operatorname{Vol} M_1 = \operatorname{Vol} M_2$. On the other hand, by Lemma 6.3.1, we have:

$$\log \left| R^{M_i}_j \left(\frac{j}{2} \right) \right| = \int_{|z|>1} \log |1 - z^{-j}| d\mu_i(z).$$

Therefore, for all $n \ge N+1$, we have

$$\int_{|z|>1} \log |1 - z^{-n}| d\mu_1(z) = \int_{|z|>1} \log |1 - z^{-n}| d\mu_2(z). \tag{7.1}$$

On the other hand,

$$\int_{|z|>1} 2 \log |1 - z^{-n}| d\mu_i(z) = \int_{|z|>1} \log (1 - z^{-n}) d\mu_i(z) + \int_{|z|>1} \log (1 - z^{-n}) d\overline{\mu}_i(z).$$

Let ν_i be the image measure of $\mu_i + \overline{\mu}_i$ under the map $z \mapsto \frac{1}{z}$. Then Equation 7.1 is equivalent to,

$$\int_{|z|<1} \log(1 - z^n) d\nu_1(z) = \int_{|z|<1} \log(1 - z^n) d\nu_2(z),$$

for all $n \geq N+1$. The measure ν_i is not Radon since any neighbourhood of the origin has infinite measure. Nevertheless, by Proposition 5.5.2, $z^5\nu_i$ is finite. Hence, $\nu = z^{N+1}(\nu_1 - \nu_2)$ is a Radon measure that satisfies

$$\int_{|z|<1} \frac{\log(1-z^n)}{z^{N+1}} d\nu(z) = 0, \quad \text{for all } n \geq N+1.$$

Now we can apply Proposition 7.1.1 with $\psi(z) = -\log(1-z)$ to conclude that $\nu = 0$, which is equivalent to say that

$$\mu_1 + \overline{\mu}_1 = \mu_2 + \overline{\mu}_2.$$

The first part of the theorem is then proved. The second part is now easily deduced by using the first part and Theorem 7.0.7. □

A similar proof shows that the following result holds.

Theorem 7.2.1. *Let M_1 and M_2 be two complete hyperbolic 3-manifolds of finite volume. Assume that there exists $K \geq 2$ such that for all $k \geq K$ we have*

$$|\mathcal{T}_{2k+1}(M_1)| = |\mathcal{T}_{2k+1}(M_2)|.$$

Then the following assertions hold:

1. *The manifolds M_1 and M_2 are isospectral (as "non-spin" manifolds) up to complex conjugation. In particular, they have the same real length spectrum.*
2. *The equality $|\mathcal{T}_{2k+1}(M_1)| = |\mathcal{T}_{2k+1}(M_2)|$ holds for all $k \geq 2$.*

Proof of Theorem 7.2.1. The proof is the same as the proof of Theorem 7.0.7, but considering only odd dimensional representations. □

Wotzke's Theorem 6.1.1 and the above theorem yield the following result.

Theorem 7.2.2. *Let (M_1, η_1) and (M_2, η_2) be two* closed *spin-hyperbolic 3-manifolds. Then the following assertions are equivalent:*

1. *There exists $N \geq 2$ such that for all $n \geq N$,*

$$|\,\tau_n(M_1, \eta_1)| = |\,\tau_n(M_2, \eta_2)|.$$

2. *The manifolds (M_1, η_1) and (M_2, η_2) are isospectral up to conjugation.*

Proof. If the first assertion is true, then Theorem 7.0.7 implies that the two manifolds are isospectral up to complex conjugation. In order to prove the converse, let us make the following observation. Let (M, η) be a closed spin-hyperbolic 3-manifold. By definition, the spin-complex-length spectrum of $\mu_{\mathrm{sp}}(M, \eta)$ determines the Ruelle zeta function $R^M_{\rho_n}(s)$, and if we only know it up to conjugation, it determines the following function,

$$F_M(s) := R^M_{\rho_n}(s) R^{\overline{M}}_{\rho_n}(s).$$

By definition, for $\mathrm{Re}(s) > 2 + n/2$, we have:

$$\overline{R_{\rho_n}^{\overline{M}}(\bar{s})} = R_{\rho_n}^{M}(s).$$

Since both right and left hand side of the above equation are meromorphic functions, the above equality must hold for all $s \in \mathbf{C}$. By Wotzke's Theorem 6.1.1, we get then:

$$F_M(0) = R_{\rho_n}^{M}(0) R_{\rho_n}^{\overline{M}}(0) = |R_{\rho_n}^{M}(0)|^2 = |\tau_n(M, \eta)|^4.$$

The proof that Assertion 2 implies Assertion 1 is now clear. $\square$

Chapter 8

Mutation

Let K be a hyperbolic knot. The knot $K^\tau \subset S^3$ obtained by cutting along a Conway sphere C and gluing again after composing with an involution $\tau\colon C \to C$ is called the mutant knot. Ruberman [Rub87] showed that K^τ is also hyperbolic, and that $M^\tau = S^3 \setminus \mathcal{N}(K^\tau)$ has the same volume as $M = S^3 \setminus \mathcal{N}(K)$. See [DGST10, MR09] for a recent account on invariants that distinguish or not K from K^τ.

The aim of this chapter is to prove that the n-dimensional Reidemeister torsion invariants of M and M^τ agree for $n = 2$.

Theorem 8.0.3. *Let K, M, τ and M^τ be as above. Let ρ and ρ^τ be lifts of the holonomy representations of M and M^τ respectively, with $\operatorname{trace}(\rho(\mu)) = \operatorname{trace}(\rho^\tau(\mu'))$, where μ and μ' are two meridians of K and K^τ, respectively. Then*

$$\operatorname{tor}(M, \rho) = \operatorname{tor}(M^\tau, \rho^\tau).$$

This theorem is not true for any representation of $\pi_1(M)$. Wada proved in [Wad94] that the twisted Alexander polynomials could be used to distinguish mutant knots. N. Dunfield, S. Friedl and N. Jackson [DFJ] computed the torsion for the representation ρ twisted by the abelianization map (namely, the corresponding twisted Alexander polynomials), and proved that it distinguishes mutant knots. However, the evaluation at ± 1 of these polynomials gives numerical evidence of Theorem 8.0.3.

By computing the fourth and sixth dimensional torsion of the Conway and the Kinoshita-Terasaka mutants, we show that these invariants may be used to distinguish these two knots, see Section 8.6. In particular, Theorem 8.0.3 is not true in general for $n > 2$.

The chapter is organized as follows. In Section 8.1, we discuss the basic constructions for representations of mutants, and we give a sufficient criterion in Proposition 8.2.3 for invariance of the torsion under mutation. The sufficient criterion of Proposition 8.2.3 is stated in terms of the action of the involution τ on the cohomology of $S = M \cap C$ with twisted coefficients. This is applied to the proof of the case when the trace of the meridian is -2 in Section 8.3. The proof when the trace is $+2$ in Section 8.4 is different, because the involved cohomology groups are different. In Section 8.5 we compute an example, the Kinoshita-Terasaka and Conway mutants, and Section 8.6 is devoted to further discussion.

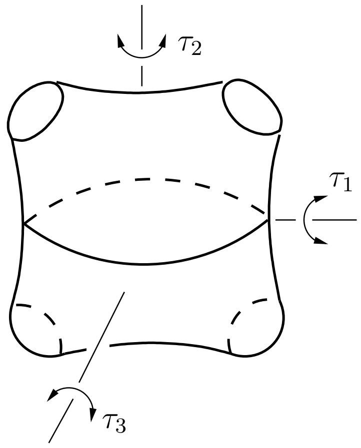

Figure 8.1: The involutions on the Conway sphere

8.1 Mutation

Let $K \subset S^3$ be a hyperbolic knot and $C \subset S^3$ be a Conway sphere; namely C intersects transversally K in four points. We write $\tau\colon (C, C \cap K) \to (C, C \cap K)$ to denote any of the three involutions in Figure 8.1.

Let B_1 and B_2 denote the components of $S^3 \setminus \mathcal{N}(C)$, so that the pairs $(B_i, B_i \cap K)$ are tangles with two strings. The exterior of the knot is denoted by

$$M = S^3 \setminus \mathcal{N}(K).$$

We also denote

$$S = C \cap M, \quad M_1 = M \cap B_1, \quad \text{and} \quad M_2 = M \cap B_2.$$

Write a commutative diagram for the inclusions:

$$\begin{tikzcd} S \arrow[r, "i_1"] \arrow[d, "i_2"'] & M_1 \arrow[d] \\ M_2 \arrow[r] & M \end{tikzcd}$$

so that $\pi_1(M)$ is an amalgamated product with respect the inclusions i_1 and i_2,

$$\pi_1(M) = \pi_1(M_1) *_{(\pi_1(S), i_{1*}, i_{2*})} \pi_1(M_2).$$

The knot $K^\tau \subset S^3$ obtained by cutting along a conway sphere C and gluing again after composing with $\tau\colon C \to C$ is called the mutant knot. The exterior of K^τ is denoted by

$$M^\tau = S^3 \setminus \mathcal{N}(K^\tau).$$

The fundamental group $\pi_1(M^\tau)$ is constructed also as the amalgamated product with the inclusions given by $j_1 = i_1$ and $j_2 = i_2 \circ \tau$,

$$\pi_1(M^\tau) = \pi_1(M_1) *_{(\pi_1(S), j_{1*}, j_{2*})} \pi_1(M_2).$$

Let $\rho\colon \pi_1(M) \to \mathrm{SL}(2, \mathbf{C})$ be a lift of the holonomy representation. A lift of the holonomy representation of M^τ is easily constructed thanks to the following result. We will use the following notation:

$$\rho^a(\cdot) = a\rho(\cdot)a^{-1}, \quad a \in \mathrm{SL}(2, \mathbf{C}).$$

Theorem 8.1.1 ([CL96, Rub87, Til04])**.** *There exists $a \in \mathrm{SL}(2, \mathbf{C})$ such that*

$$\rho^a(\tau_*(\gamma)) = \rho(\gamma), \quad \textit{for all } \gamma \in \pi_1(S) < \pi_1(M).$$

Moreover, such matrix a is unique up to sign.

Remark. Notice that the matrix a of the above theorem corresponds to a rotation of order two in hyperbolic space, therefore a is conjugate to

$$a \sim \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{8.1}$$

The representation $\rho^\tau\colon \pi_1(M^\tau) \to \mathrm{SL}(2, \mathbf{C})$ is then defined as follows:

$$\rho^\tau|_{\pi_1(M_1)} = \rho|_{\pi_1(M_1)} \qquad \text{and} \qquad \rho^\tau|_{\pi_1(M_2)} = \rho^a|_{\pi_1(M_2)}.$$

This is easily checked to be well defined by Theorem 8.1.1.

8.2 A sufficient condition for the invariance

We will use the Mayer-Vietoris exact sequence for the pair (M_1, M_2) to compute the torsion of M and M^τ. To this end, we need to compute some cohomology groups. We start with the planar surface with four boundary components $S = M_1 \cap M_2$.

Lemma 8.2.1. $\mathrm{H}^i(S; \rho) = 0$ *for* $i \neq 1$ *and* $\mathrm{H}^1(S; \rho) \cong \mathbf{C}^4$.

Proof. First, $\mathrm{H}^0(S, \rho) = 0$ because $\rho(\pi_1(S))$ contains hyperbolic elements, so there is no nontrivial fixed vector in $\mathbf{C}^2$. On the other hand, $\mathrm{H}^i(S, \rho) = 0$ for $i \geq 2$, because S has the homotopy type of a graph. Finally,

$$\dim_{\mathbf{C}} \mathrm{H}^1(S; \rho) = -\chi(S) \dim(\mathbf{C}^2) = 4.$$

□

Lemma 8.2.2. *For* $k = 1, 2$, $\mathrm{H}^j(M_k; \rho) = 0$ *for* $j \neq 1$ *and* $\mathrm{H}^1(M_k; \rho) = \mathbf{C}^2$.

Proof. By Mayer-Vietoris, and using the fact that $\mathrm{H}^*(M;\rho)=0$, we have

$$\mathrm{H}^j(M_1;\rho)\oplus\mathrm{H}^j(M_2;\rho)\cong\mathrm{H}^j(S;\rho).$$

The lemma follows from Lemma 8.2.1, and taking into account that

$$\chi(M_k)=\frac{1}{2}\chi(\partial M_k)=1,$$

as ∂M_k is an orientable closed surface of genus 2. $\square$

Mayer-Vietoris for M and M^τ gives the following isomorphisms:

$$i_1^*\oplus i_2^*\colon\ \mathrm{H}^1(M_1;\rho)\oplus\mathrm{H}^1(M_2;\rho)\to\mathrm{H}^1(S;\rho), \tag{8.2}$$
$$j_1^*\oplus j_2^*\colon\ \mathrm{H}^1(M_1;\rho)\oplus\mathrm{H}^1(M_2;\rho^a)\to\mathrm{H}^1(S;\rho). \tag{8.3}$$

In order to relate $\mathrm{H}^1(M_2;\rho)$ and $\mathrm{H}^1(M_2;\rho^a)$ we define a map at the level of the flat vector bundles $E_\rho=\widetilde{M_2}\times_\rho V$ and $E_{\rho^a}=\widetilde{M_2}\times_{\rho^a}V$ as follows:

$$\begin{array}{rccc}\varphi_a\colon & E_{\rho^a} & \to & E_\rho\\ & [(x,v)] & \mapsto & [(x,a^{-1}v)].\end{array}$$

It is straightforward to check that φ_a is an isomorphism of flat vector bundles. Thus it induces an isomorphism in cohomology $\varphi_a^*\colon\ \mathrm{H}^1(M_2;\rho)\to\mathrm{H}^1(M_2;\rho^a)$. On the other hand, τ induces an isomorphism of $E_\rho|_S$, since $\rho(\tau_*(\gamma))=\rho(\gamma)$ for all $\gamma\in\pi_1(S)$. We have the following commutative diagram:

$$\begin{tikzcd} E_\rho|_S \arrow[r, "i_2"] & E_\rho \\ E_\rho|_S \arrow[r, "j_2"] \arrow[u, "\varphi_a\circ\tau"'] & E_{\rho^\tau}. \arrow[u, "\varphi_a"] \end{tikzcd}$$

Let us write $\tau_a=\varphi_a\circ\tau$. Notice that $\tau_a^2(v_x)=-v_x$ for all $v_x\in E_\rho|_S$, as $a^2=-\mathrm{Id}$ (see Equation (8.1)) and $\tau^2=\mathrm{Id}$. The above commutative diagram yields the following commutative diagram in cohomology:

$$\begin{tikzcd} \mathrm{H}^1(M_2;\rho) \arrow[r, "i_2^*"] \arrow[d, "\varphi_a^*"'] & \mathrm{H}^1(S;\rho) \arrow[d, "(\varphi_a\circ\tau)^*=\tau_a^*"] \\ \mathrm{H}^1(M_2;\rho^a) \arrow[r, "j_2^*"] & \mathrm{H}^1(S;\rho) \end{tikzcd}$$

Notice that

$$(\tau_a^*)^2=-\mathrm{Id}\,. \tag{8.4}$$

Choose b_i a basis for $\mathrm{H}^1(M_i;\rho_i)$ as $\mathbf{C}$-vector space. In particular $\varphi_a^*(b_2)$ is a basis for $\mathrm{H}^1(M_2;\rho^a)$. By Milnor's formula [Mil66] for the torsion of a long exact sequence applied to (8.2) and (8.3):

$$\tau(M,\rho) = \pm\,\frac{\tau(M_1,\rho,b_1)\,\tau(M_2,\rho,b_2)}{\tau(S,\rho,i_1^*(b_1)\sqcup i_2^*(b_2))}$$
$$\tau(M^\tau,\rho^\tau) = \pm\,\frac{\tau(M_1,\rho,b_1)\,\tau(M_2,\rho^a,\varphi_a^*(b_2))}{\tau(S,\rho,j_1^*(b_1)\sqcup j_2^*(\varphi_a^*(b_2)))}.$$

Here $\sqcup$ denotes the disjoint union of bases. Notice that Milnor works with torsions up to sign in [Mil66], but his formalism applies even with sign. Since $j_2^*\circ\varphi_a^* = \tau_a^*\circ i_2^*$, we get

$$\frac{\tau(M,\rho)}{\tau(M^\tau,\rho^\tau)} = \det(i_1^*(b_1)\sqcup \tau_a^*(i_2^*(b_2)), i_1^*(b_1)\sqcup i_2^*(b_2)). \tag{8.5}$$

Namely, the determinant of the matrix whose entries are the coefficients of the basis

$$i_1^*(b_1)\sqcup \tau_a^*(i_2^*(b_2))$$

with respect to $i_1^*(b_1)\sqcup i_2^*(b_2)$.

The following is a sufficient criterion for invariance of torsion with respect to mutation.

Proposition 8.2.3. *Assume that $\tau_a^*\colon \mathrm{H}^1(S;\rho)\to \mathrm{H}^1(S;\rho)$ leaves invariant the image of $i_2^*\colon \mathrm{H}^1(M_2;\rho)\to \mathrm{H}^1(S,\rho)$. Then we have*

$$\tau(M,\rho) = \pm\,\tau(M^\tau,\rho^\tau).$$

Proof. Since $(\tau_a^*)^2 = -\,\mathrm{Id}$ by Formula (8.4), τ_a^* diagonalizes with eigenvalues $\pm i$. Hence, assuming that τ_a^* leaves invariant the image of i_2^*, the matrix in Equation (8.5) is conjugate to

$$\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \pm i & 0\\ 0 & 0 & 0 & \pm i\end{pmatrix},$$

hence it has determinant ± 1. □

8.3 Invariance when trace = −2.

We discuss first the case when the trace of a meridian μ of K is -2, that is

$$\mathrm{trace}(\rho(\mu)) = -2.$$

The proof has three parts. In Subsection 8.3.1 we consider a non-degenerate symmetric bilinear form B on $\mathrm{H}^1(S;\rho)$. We show that, for $k=1,2$, the images of $i_k^*\colon \mathrm{H}^1(M_k;\rho)\to \mathrm{H}^1(S;\rho)$ are isotropic subspaces. Then in Subsection 8.3.2 we analyze properties of isotropic planes of $\mathrm{H}^1(S;\rho)\cong \mathbf{C}^4$, which are viewed as lines in a ruled quadric in the projective complex space $\mathbf{P}^3$. The properties of this ruled quadric are used in a deformation argument in Subsection 8.3.3 to conclude the proof when $\mathrm{trace}(\mu)=-2$.

8.3.1 A perfect pairing

The cup product in its relative form gives the following non-degenerate bilinear map,

$$\begin{array}{rccccl} \cup\colon & \mathrm{H}^1(S,\partial S;E_\rho) & \times & \mathrm{H}^1(S;E_\rho) & \to & \mathrm{H}^2(S,\partial S;\mathbf{C})\cong\mathbf{C} \\ & ([\alpha] & , & [\beta]) & \mapsto & [\alpha\wedge\beta], \end{array}$$

where the exterior product is computed using the determinant on $\mathbf{C}^2$. The isomorphism $\mathrm{H}^2(S,\partial S;\mathbf{C})\cong\mathbf{C}$ is given by integration on the relative fundamental class $[S]$, with respect to a fixed orientation on S. Thus

$$[\alpha]\cup[\beta]=\int_S \alpha\wedge\beta.$$

Remark. The following lemma assumes that $\mathrm{trace}(\rho(\mu))=-2$, as the whole section, though it only requires $\mathrm{trace}(\rho(\mu))\neq 2$.

Lemma 8.3.1. *If* $\mathrm{trace}(\rho(\mu))=-2$ *then:*

1. $\mathrm{H}^*(\partial S;E_\rho)=0$.
2. $\mathrm{H}^*(S,\partial S;E_\rho)=\mathrm{H}^*(S;E_\rho)$.
3. *For* $k=1,2$, *the inclusion map induces an isomorphism*

$$i_k^*\colon\ \mathrm{H}^1(\partial M_k;\rho)\to\mathrm{H}^1(S;\rho).$$

Proof. Since $\mathrm{trace}(\rho(\mu))=-2$, $\rho(\mu)$ has no nontrivial fixed vectors in $\mathbf{C}^2$. Thus $\mathrm{H}^0(\partial S;E_\rho)$ is trivial, and by duality $\mathrm{H}^1(\partial S;E_\rho)$ is trivial as well. Since $\mathrm{H}^*(\partial S;E_\rho)=0$, the exact sequence for the pair $(S,\partial S)$ shows that the map induced by restriction $\mathrm{H}^1(S,\partial S;E_\rho)\to\mathrm{H}^1(S;E_\rho)$ is an isomorphism. The last assertion follows from the Mayer-Vietoris sequence applied to S and $\overline{\partial M_k\setminus S}$, which is the union of the two annuli around the arcs of $K\cap B_k$ that have the homotopy type of a component of ∂S. Hence by the first assertion $\mathrm{H}^*(\partial M_k\setminus S;E_\rho)=0$. □

Next we want to consider a bilinear form

$$B\colon\ \mathrm{H}^1(S;E_\rho)\times\mathrm{H}^1(S;E_\rho)\to\mathbf{C}, \tag{8.6}$$

defined as follows. Let $[\alpha],[\beta]\in\mathrm{H}^1(S;E_\rho)$. By the second assertion of Lemma 8.3.1, there exist 1-forms α_c,β_c compactly supported in the interior of S such that $[\alpha]=[\alpha_c]$ and $[\beta]=[\beta_c]$. Then we define:

$$B([\alpha],[\beta]):=\int_S \alpha_c\wedge\beta_c.$$

The following result is easily proved using Lemma 8.3.1 and the fact that Poincaré duality defines a non-degenerate pairing.

Proposition 8.3.2. *The bilinear form B is well defined, symmetric and nondegenerate.*

Lemma 8.3.3. *For $k = 1, 2$, the image of $i_k^*\colon \mathrm{H}^1(M_k; E_\rho) \to \mathrm{H}^1(S; E_\rho)$ is an isotropic plane for B.*

Proof. The fact that the image is a plane follows from Lemma 8.2.2 and its proof. To prove that it is an isotropic subspace, let $\alpha_1, \alpha_2 \in \Omega^1(M_k; E_\rho)$ be closed forms. We must show that $B(i_k^*([\alpha_1]), i_k^*([\alpha_2])) = 0$. Take $f_1, f_2 \in \Omega^0(S; E_\rho)$ such that $i_k^*(\alpha_1) + df_1, i_k^*(\alpha_2) + df_2$ have support contained in the interior of S. Then we have:

$$B(i_k^*([\alpha_1]), i_k^*([\alpha_2])) = \int_S (i_k^*(\alpha_1) + df_1) \wedge (i_k^*(\alpha_2) + df_2)$$

The forms $i_k^*([\alpha_j]) + df_j$ can be extended trivially to ∂M_k, as their support is contained in the interior of S; hence, the 1-forms df_i can also be extended to ∂M_k. Therefore,

$$\int_S (i_k^*(\alpha_1) + df_1) \wedge (i_k^*(\alpha_2) + df_2) = \int_{\partial M_k} (i_k^*(\alpha_1) + df_1) \wedge (i_k^*(\alpha_2) + df_2).$$

Now, since the inclusion induces an isomorphism $\mathrm{H}^1(\partial M_k; E_\rho) \cong \mathrm{H}^1(S; E_\rho)$, the 1-forms df_i are also exact on ∂M_k, and hence can be removed from the above integral as ∂M_k is a closed manifold. Finally, Stokes' theorem yields

$$\int_{\partial M_k} i_k^*(\alpha_1) \wedge i_k^*(\alpha_2) = \int_{M_k} d\,(\alpha_1 \wedge \alpha_2) = 0,$$

as α_1, α_2 are closed. □

8.3.2 Finding isotropic planes with the ruled quadric

Let $\mathbf{P}^3$ denote the projective space on $\mathrm{H}^1(S; \rho) \cong \mathbf{C}^4$. Isotropic planes of $\mathrm{H}^1(S; \rho)$ (with respect to B) are in bijection with projective lines in the quadric

$$Q = \{x \in \mathbf{P}^3 \mid B(x, x) = 0\}.$$

Since B is a non-degenrate paring, Q is the standard quadric, which is a ruled surface with two rulings. We recall next its basic properties.

Proposition 8.3.4. *There are two disjoint families of projective lines $\mathcal{L}_+$ and $\mathcal{L}_-$ in Q such that:*

(i) Every line in Q belongs to either $\mathcal{L}_+$ or $\mathcal{L}_-$.

(ii) Every point in Q belongs to precisely one line in $\mathcal{L}_+$ and one in $\mathcal{L}_-$. Thus there is a bijection between Q and $\mathcal{L}_+ \times \mathcal{L}_-$.

(iii) Two lines in Q intersect if, and only if, one is in $\mathcal{L}_+$ and the other one is in $\mathcal{L}_-$.

(iv) Each space of lines $\mathcal{L}_+$ and $\mathcal{L}_-$ is isomorphic (as projective varieties) to $\mathbf{P}^1$, in such a way that the bijection between Q and $\mathcal{L}_+ \times \mathcal{L}_-$ is also an isomorphism.

We shall also use the action of $\mathrm{SO}(4,\mathbf{C})$, the isometry group of $\mathrm{H}^1(S;\rho)$. Since the quadric Q is isomorphic to the product $\mathbf{P}^1\times\mathbf{P}^1\cong\mathcal{L}_+\times\mathcal{L}_-$, the automorphism groups of Q and $\mathbf{P}^1\times\mathbf{P}^1$ are isomorphic. Hence, we have an isomorphism:

$$\psi\colon\ \mathrm{PSL}(2,\mathbf{C})\times\mathrm{PSL}(2,\mathbf{C})\to\mathrm{PSO}(4,\mathbf{C}).$$

This isomorphism lifts to a unique isomorphism of Lie groups,

$$\widetilde{\psi}\colon\ \mathrm{SL}(2,\mathbf{C})\times\mathrm{SL}(2,\mathbf{C})/\pm(\mathrm{Id},\mathrm{Id})\to\mathrm{SO}(4,\mathbf{C}).$$

Therefore, $\mathrm{PSL}(2,\mathbf{C})\times\{\mathrm{Id}\}$ acts trivially on $\mathcal{L}_-$, and $\{\mathrm{Id}\}\times\mathrm{PSL}(2,\mathbf{C})$ acts trivially on $\mathcal{L}_+$.

For $i=1,2,3$, consider the involutions τ_i of the Conway sphere C, as in Figure 8.1, and the associated maps τ_{a_i} defined in the preceding sections.

Lemma 8.3.5. *The induced maps $\tau_{a_i}^*$ on $\mathbf{P}^3$, regarded as elements of $\mathrm{PSL}(2,\mathbf{C})\times\mathrm{PSL}(2,\mathbf{C})$, lie in one of the two factors $\mathrm{PSL}(2,\mathbf{C})\times\{\mathrm{Id}\}$ or $\{\mathrm{Id}\}\times\mathrm{PSL}(2,\mathbf{C})$. In addition, all $\tau_{a_1}^*$, $\tau_{a_2}^*$ and $\tau_{a_3}^*$ lie in the same factor.*

Proof. The map $\tau_{a_i}^*$ acting on $\mathrm{H}^1(S;\rho)$ is an element of $\mathrm{SO}(4,\mathbf{C})$, whose square is $-\mathrm{Id}$, by (8.4). The image of $\tau_{a_i}^*$ under the isomorphism

$$\mathrm{SO}(4,\mathbf{C})\cong\mathrm{SL}(2,\mathbf{C})\times\mathrm{SL}(2,\mathbf{C})/\pm(\mathrm{Id},\mathrm{Id}),$$

is represented by a class $[(A,B)]$, whose square is the class $[(\mathrm{Id},-\mathrm{Id})]$, hence either $A^2=\mathrm{Id}$ and $B^2=-\mathrm{Id}$ or $A^2=-\mathrm{Id}$ and $B^2=\mathrm{Id}$. The first assertion is then deduced from the fact that a matrix of $\mathrm{SL}(2,\mathbf{C})$ whose square is the identity must be $\pm\mathrm{Id}$. For the last assertion, just use that $\tau_{a_1}^*\tau_{a_2}^*=\pm\tau_{a_3}^*$. □

Corollary 8.3.6. *Up to permuting $\mathcal{L}_-$ and $\mathcal{L}_+$, we get:*

1. *$\tau_{a_1}^*$, $\tau_{a_2}^*$ and $\tau_{a_3}^*$ act trivially on $\mathcal{L}_-$.*
2. *There is no point in $\mathcal{L}_+$ fixed by all $\tau_{a_1}^*$, $\tau_{a_2}^*$ and $\tau_{a_3}^*$.*

Proof. The first assertion is an immediate consequence of the lemma. For the second one, notice that the subgroup of $\mathrm{PSL}(2,\mathbf{C})$ consisting of the maps induced by the three involutions and the identity is isomorphic to $\mathbf{Z}/2\mathbf{Z}\oplus\mathbf{Z}/2\mathbf{Z}$ (also called the 4-Klein group), as each $\tau_{a_i}^*$ has order two and $\tau_{a_1}^*\tau_{a_2}^*=\pm\tau_{a_3}^*$. Now assume that there exists a point on $\mathbf{P}^1$ fixed by $\tau_{a_1}^*$, $\tau_{a_2}^*$ and $\tau_{a_3}^*$. Interpreting $\mathrm{PSL}(2,\mathbf{C})$ as the group of Möbius transformations, we can assume that the fixed point is infinity. We can also assume that $\tau_{a_1}^*$ fixes the origin as well; this implies that $\tau_{a_1}^*$ is $z\mapsto -z$. Since the involutions $\tau_{a_2}^*$ and $\tau_{a_3}^*$ both commute with $\tau_{a_1}^*$, they must fix the origin as well. Hence, $\tau_{a_1}^*=\tau_{a_2}^*=\tau_{a_3}^*$, which is not possible, for the group generated by the three involutions has order 4. □

Let Im denote the image. Since $\mathrm{Im}(i_1^*)\oplus\mathrm{Im}(i_2^*)=\mathrm{H}^1(S;\rho)$, from Proposition 8.3.4 (iii) we deduce:

Corollary 8.3.7. *Either both* $\mathrm{Im}(i_1^*)$ *and* $\mathrm{Im}(i_2^*)$ *belong to* $\mathcal{L}_-$*, or they both belong to* $\mathcal{L}_+$*.*

Either $\mathrm{Im}(i_2^*)$ belongs to $\mathcal{L}_-$ and, by Corollary 8.3.6, we may apply Proposition 8.2.3, or $\mathrm{Im}(i_2^*)$ belongs to $\mathcal{L}_+$. To get rid of this last case we will use a deformation argument. The idea is to deform the hyperbolic structure on M_2, so that it matches with another tangle which is invariant under the involutions τ_i. By Corollary 8.3.6, the tangle invariant by the involutions satisfies $\mathrm{Im}(i_1^*) \in \mathcal{L}_-$, hence $\mathrm{Im}(i_2^*) \in \mathcal{L}_-$ for the deformed structure on M_2, by Corollary 8.3.7. Then we shall use a continuity argument to have the same conclusion for the initial structure on M_2. Next subsection is devoted to this deformation argument.

8.3.3 A deformation argument

Let $A = \overline{\partial M_2 \setminus S}$ be the pair of annuli, one around each arc of $K \cap B_2$. The pair (M_2, A) is a *pared* manifold.

Definition. A *pared* manifold is a pair (N, P), where N is a compact oriented 3-manifold, and $P \subset \partial N$ is a union of tori and annuli such that:

1. no two components of P are isotopic in ∂N,
2. every abelian noncyclic subgroup of $\pi_1(N)$ is conjugate to a subgroup of a component or P, and
3. there are no essential annuli $(S^1 \times [0,1], S^1 \times \partial[0,1]) \to (M, P)$.

We say that a pared manifold (N, P) is hyperbolic when the interior of N admits a complete hyperbolic structure with cusps at P. The rank of the cusp is one for an annulus, and two for a torus.

Lemma 8.3.8. *There exists a pared manifold* (M_3, A')*, such that*

1. (M_3, A') *is obtained from a 2-tangle: namely* M_3 *is the exterior of two properly embedded arcs in a 3-ball,* A' *are the annuli around the arcs of the tangle, and* $A' \cup S = \partial M_3$*.*
2. *For* $i = 1, 2, 3$, $\tau_i : S \to S$ *extends to an involution of* (M_3, A')*.*
3. *The pared manifolds* (M_3, A') *and* $(M_2 \cup M_3, A \cup A')$ *are both hyperbolic.*

Proof of Lemma 8.3.8. We take M_3 to be the exterior of a simple 2-tangle that is symmetric with respect to τ_1, τ_2 and τ_3. Here simple means that M_3 is irreducible, ∂-irreducible, atoroidal and anannular. In [Wu96], Wu gives a criterion for deciding when a rational tangle is simple and provides examples of simple tangles with the required symmetries. In particular, the pared manifold (M_3, A') admits a hyperbolic structure with totally geodesic boundary in $S = \partial M_3 \setminus A'$ (and rank one cusps in A'). Since (M_3, A') is simple and (M_2, A) hyperbolic, standard arguments in 3-dimensional topology prove that $(M', T') = (M_2 \cup M_3, A \cup A')$ is irreducible, acylindrical, atoroidal and not Seifert fibered (S should be horizontal in a Seifert fibration), hence hyperbolic. □

The variety of representations of $\pi_1(M_2)$ to $\mathrm{SL}(2,\mathbf{C})$ is denoted by

$$R(M_2) = \hom(\pi_1(M_2), \mathrm{SL}(2,\mathbf{C})),$$

and it is an algebraic subset of affine space $\mathbf{C}^N$.

Lemma 8.3.9. *If* $\operatorname{trace}(\rho(\mu)) = -2$*, then* $\operatorname{Im}(i_2^*)$ *belongs to* $\mathcal{L}_-$*.*

Proof. Let us write $\rho_2 = \rho \circ i_{2*}$, where $i_2\colon M_2 \to M$ is the inclusion. We connect $\rho_2 \in R(M_2)$ to $\rho_2' \in R(M_2)$, a lift of the holonomy representation of M_2 that matches with the tangle M_3 of Lemma 8.3.8, which is a symmetric tangle. Namely we want to find a path or representations

$$\begin{array}{rcl} [0,1] & \to & R(M_2) \\ t & \mapsto & \varphi_t \end{array}$$

that satisfies:

(i) $\varphi_0 = \rho_2$.

(ii) $\forall t \in [0,1]$, φ_t is the lift of the holonomy of a hyperbolic structure on M_2, with rank one cusps at the arcs $K \cap B_2$, and satisfying

$$\operatorname{trace}(\varphi_t(\mu)) = -2.$$

(iii) $\forall t \in [0,1]$, $\dim H_1(M_2;\varphi_t) = 2$.

(iv) $\varphi_1 = \rho_2'$ is the lift of the holonomy of a hyperbolic structure on M_2 that matches with M_3 in Lemma 8.3.8.

Assuming we have this path of representations, then since M_3 is τ_1 and τ_2-invariant, the image of $i_3^*\colon \mathrm{H}^1(M_3;\varphi_1) \to \mathrm{H}^1(S;\varphi_1|_{\pi_1(S)})$ is a subspace $\tau_{a_i}^*$-invariant. Hence the image of i_3^* must be contained in $\mathcal{L}_-$, by Corollary 8.3.6 (ii). This implies that for this hyperbolic structure

$$\operatorname{Im}\left(i_2^*\colon \mathrm{H}^1(M_2;\rho_2') \to \mathrm{H}^1(S;\rho_2'|_S)\right) \in \mathcal{L}_-,$$

by Corollary 8.3.7. Now, since there exists the path φ_t, the ruled quadric of $\mathrm{H}^1(S_0;\varphi_t)$ is also deformed continuously (notice that as $\varphi_t|_{S_0}$ is irreducible and φ_t of a meridian has trace -2, by (iii), Lemmas 8.2.1, 8.3.1, and 8.3.3 apply to $\mathrm{H}^1(S_0;\varphi_t)$). Hence along the deformation, the image of i_2^* is contained in $\mathcal{L}_-$, as $\mathcal{L}_+ \cap \mathcal{L}_- = \emptyset$. Hence

$$\operatorname{Im}\left(i_2^*\colon \mathrm{H}^1(M_2;\rho_2) \to \mathrm{H}^1(S;\rho)\right) \in \mathcal{L}_-,$$

as claimed.

Let us justify the existence of the path ϕ_t between ρ_2 and ρ_2'. If both $\rho_2(\pi_1(M_2))$ and $\rho_2'(\pi_1(M_2))$ are geometrically finite, then they can be connected along the space of geometrically finite structures of the pared manifold, because by Ahlfors-Bers theorem this space is isomorphic to the Teichmüller space of S, cf. [Ota98]. In addition, this is an open subset of the variety of representations, and since the dimension of de cohomology is upper semi-continuous (it can only jump in a Zariski closed subset), (iii) can be achieved by avoiding a proper Zariski closed subset (hence of real codimension ≥ 2). If any of $\rho_2(\pi_1(M_2))$ and $\rho_2'(\pi_1(M_2))$ is not geometrically finite, then it lies in the closure of geometrically finite structures (cf. [Ota96] though this is a particular case of the density theorem), thus there is still a path in the space of representations satisfying (ii) and (iii). $\square$

By Lemma 8.3.9, Corollary 8.3.7 and Proposition 8.2.3,

$$\tau(M,\rho) = \pm\,\tau(M^\tau,\rho^\tau).$$

We shall prove that there is also equality of signs.

Proposition 8.3.10. *If* $\mathrm{trace}(\rho(\mu)) = -2$, *then*

$$\tau(M,\rho) = \tau(M^\tau,\rho^\tau).$$

Proof. To remove the sign ambiguity, we use again the deformation φ_t of the proof of Lemma 8.3.9. Since $\forall t \in [0,1]$, φ_t satisfies the sufficiency criterion of Proposition 8.2.3, the eigenvalues of $\tau_{a_i}^*$ restricted to the image of i_2^* belong to $\{\pm i\}$, and they do not change as we deform t, hence the determinant of $\tau_{a_i}^*$ restricted to the image of i_2^* is $+1$, because this holds for $\rho_2' = \varphi_1$ (as M_3 is τ_i-invariant). $\square$

8.4 Invariance when trace = +2.

When $\mathrm{trace}(\rho(\mu)) = 2$, Lemma 8.3.1 does not apply, and hence we cannot use the argument of Section 8.3. Recall that

$$R(M) = \hom(\pi_1(M), \mathrm{SL}(2,\mathbf{C}))$$

denotes the variety of representations of $\pi_1(M)$ in $\mathrm{SL}(2,\mathbf{C})$. We will consider representations $\rho_n \in R(M)$ for which the arguments of Section 8.3 apply and such that ρ_n converges to ρ in $R(M)$, as $n \to \infty$.

Let $\rho \in R(M)$ be a lift of the holonomy with $\mathrm{trace}(\rho(\mu)) = 2$. By Thurston's hyperbolic Dehn filling and for $n \in \mathbf{N}$ large enough, the orbifold with underlying space S^3, branching locus K and ramification index n is hyperbolic. It induces a representation of $\pi_1(M)$ in $\mathrm{PSL}(2,\mathbf{C})$ that lifts to $\rho_n \in R(M)$. The lift satisfies $\mathrm{trace}(\rho_n(\mu)) = \pm 2\cos(\pi/n)$, and there is precisely one lift for every choice of sign. We chose the lift satisfying

$$\mathrm{trace}(\rho_n(\mu)) = +2\cos(\pi/n).$$

Proposition 8.4.1 ([Thu])**.** *For $n \in \mathbf{N}$ large enough, there exist $\rho_n \in R(M)$ which is a lift of the holonomy of the orbifold with underlying space S^3, branching locus K and ramification index n, so that $\rho_n \to \rho$.*

These orbifolds can also be considered for the mutant knot, and there exist the corresponding mutant representations
$$\rho_n^\tau \in R(M^\tau).$$
Namely, the lifts of the holonomies of the orbifold structures on K^τ are the "mutant representations" of ρ_n. Moreover, $\rho_n^\tau \to \rho^\tau$.

Lemma 8.4.2. *For $n \in \mathbf{N}$ large enough, $\mathrm{H}^*(M;\rho_n) \cong \mathrm{H}^*(M^\tau;\rho_n^\tau) = 0$ and*
$$\tau(M,\rho_n) = \tau(M^\tau,\rho_n^\tau).$$

Proof. We use upper semi-continuity of cohomology (Proposition 4.2.2) to say that
$$\mathrm{H}^*(M;\rho_n) \cong \mathrm{H}^*(M^\tau;\rho_n^\tau) = 0.$$
More precisely, the dimension of cohomology is an upper semi-continuous function on $R(M)$, and, as ρ and ρ^τ are acyclic, then all representations in a Zariski open subset containing ρ and ρ^τ are acyclic, and so are ρ_n and ρ_n^τ, as claimed. Since ρ_n and ρ_n^τ are acyclic, $\dim H_1(M_2;\rho_n) = \dim H_1(M_2;\rho_n^\tau) = 2$. In addition, up to conjugation,
$$\rho_n(\mu) \sim \begin{pmatrix} e^{\pi i/n} & 0 \\ 0 & e^{-\pi i/n} \end{pmatrix}.$$
Hence $\mathbf{C}^2$ has no $\rho_n(\mu)$-invariant proper subspaces and Lemma 8.3.1 applies. Thus the pairing (8.6), Lemma 8.3.3 and all the results of Section 8.3.1 hold true for ρ_n. To conclude, for the deformation argument we use that ρ_n is the lift of the holonomy of an orbifold. Instead of working with pared structures on (M_2, A), we work with orbifold structures with underlying space the ball B_2, branching locus $K \cap B_2$ and branching index n. The results on the space of hyperbolic structures (geometrically finite or infinite) apply, and we may use the deformation argument of Lemma 8.3.9. □

By Proposition 8.4.1 and Lemma 8.4.2, by taking the limit when $n \to \infty$ we get:

Corollary 8.4.3. *If $\mathrm{trace}(\rho(\mu)) = +2$, then*
$$\tau(M,\rho) = \tau(M^\tau,\rho^\tau).$$

The proofs for trace $= 2$ and trace $= -2$ are quite different, because $\mathrm{H}^1(\partial S;\rho)$ is non zero when the $\mathrm{trace}(\rho(\mu))$ is $+2$, and vanishes when it is -2. The generic case is $\mathrm{trace}(\rho(\mu)) \neq 2$. The proof of Section 8.3 applies to the following situation.

Proposition 8.4.4. *Let $\rho\colon \pi_1(M) \to \mathrm{SL}(2,\mathbf{C})$ be a representation satisfying:*

1. $\mathrm{H}^1(M;\rho) = 0$;
2. $\mathrm{trace}(\rho(\mu)) \neq 2$;
3. *ρ restricted to $\pi_1(S)$ is irreducible;*

4. *the representation ρ is in the same irreducible component of $R(M)$ as some representation such that the image of i_2^* is contained in Q_-.*

Then $\tau(M,\rho) = \tau(M^\tau,\rho)$.

Corollary 8.4.5. *For a generic representation ρ of the irreducible component of $R(M)$ that contains a lift of the holonomy, $\tau(M,\rho) = \tau(M^\tau,\rho^\tau)$.*

8.5 The Kinoshita-Terasaka and Conway mutants

Let KT and C be the Kinoshita-Terasaka knot and the Conway knot respectively. It is well known that they are mutant hyperbolic knots. Using the Snap program [CGHN00], based of J. Weeks' SnapPea [Wee], we have obtained all the necessary information to compute their torsion.

The fundamental groups of these knots have the following presentations:

$$\begin{aligned} \pi_1(S^3 \setminus C) &= \langle abc \mid abACbcbacBCABaBc, aBcBCABacbCbAbacbc\rangle, \\ \pi_1(S^3 \setminus KT) &= \langle abc \mid aBCbABBCbaBcbbcABcbbaB, abcACaB\rangle. \end{aligned}$$

As usual, capital letters denote inverse.

The image of the holonomy representation is contained in $\mathrm{PSL}(2,\mathbf{Q}(\omega))$ where $\mathbf{Q}(\omega)$ is the number field generated by a root ω of the following polynomial:

$$p(x) = x^{11} - x^{10} + 3x^9 - 4x^8 + 5x^7 - 8x^6 + 8x^5 - 5x^4 + 6x^3 - 5x^2 + 2x - 1.$$

The torsions are then elements of $\mathbf{Q}(\omega)$. In order to express elements in $\mathbf{Q}(\omega)$, we use the $\mathbf{Q}$-basis $(\omega^{10},\omega^9,\cdots,\omega,1)$. Tables 8.1 and 8.2 give the coefficients of the torsions of KT and C with respect to this $\mathbf{Q}$-basis. On each table, the first column gives the element of the basis. We let n denote the dimension of the irreducible representation of $\mathrm{SL}(2,\mathbf{C})$ used to compute the torsion, and the tables show the values for $n = 2$ (i.e. the standard representation), but also $n = 4$ and $n = 6$. In order to compare them, the coefficients of the torsion for Kinoshita-Terasaka (KT) and Conway (C) knots are tabulated side by side. We give a table for each lift of the holonomy, one when the trace of the meridian is 2 (Table 8.1) and another when it is -2 (Table 8.2).

Of course, for $n = 2$ and for any lift of the holonomy, the torsion of KT and the torsion of C is the same. Notice that for the 4-dimensional representation, they are also the same for one lift but different for the other, and that they differ for both lifts when we use the 6-dimensional representation.

As said in the introduction, when $n = 2$, these had been computed by Dunfield, Friedl and Jackson in [DFJ]. They computed numerically a twisted Alexander invariant (which are not mutation invariant) for all knots up to 15 crossings, and the torsions computed here are just the evaluations at ± 1.

	$n=2$		$n=4$		$n=6$	
	KT	C	KT	C	KT	C
ω^{10}	356	356	11112880	11112880	676803770859632	662357458754672
ω^9	−620	−620	−38963592	−38963592	−640579476284656	−579216259622896
ω^8	636	636	36107416	36107416	212555254795952	153724448856752
ω^7	−864	−864	−31579196	−31579196	−990061444305088	−943617945204928
ω^6	1228	1228	60889040	60889040	1004678681648016	908722528184976
ω^5	−1080	−1080	−58195768	−58195768	−444238765345264	−349679698188784
ω^4	780	780	36555000	36555000	482101712163904	424247992815424
ω^3	−628	−628	−31740272	−31740272	−371824600930944	−320894530449024
ω^2	428	428	21313180	21313180	51168266257072	15655188602032
ω^1	−188	−188	−8829332	−8829332	−165869512283168	−152117462516768
1	124	124	7476160	7476160	−37602419304496	−50452054740016

Table 8.1: Torsions for the lift of the holonomy with trace of the meridian 2. The table gives the coefficients of the torsion of n-dimensional representation (with respect to a $\mathbf{Q}$-basis for $\mathbf{Q}(\omega)$).

	$n=2$		$n=4$		$n=6$	
	KT	C	KT	C	KT	C
ω^{10}	7352	7352	−106244812	−84923788	−5089618734386048	−5181970358958464
ω^9	12100	12100	−40892392	−98464552	26333637242897408	26767528167113984
ω^8	−18868	−18868	135740632	176373400	−26132678464882128	−26556943437149136
ω^7	−16	−16	81031412	30483572	18961525460403712	19282331500463872
ω^6	−19124	−19124	70025564	154082012	−41268295304316624	−41948393922548432
ω^5	29448	29448	−188927128	−264857368	41815776250571680	42495766908786848
ω^4	−14272	−14272	71097428	118825172	−25207995553964480	−25621419777084608
ω^3	13576	13576	−71628932	−116091140	22311420427155024	22676270315709264
ω^2	−13352	−13352	98553148	124139068	−15990083236426320	−16248280122238544
ω^1	2780	2780	−4562444	−18136844	5898804809613840	5996288593045520
1	−5812	−5812	48068144	56560304	−5891958922292320	−5986195442605152

Table 8.2: Torsions for the lift of the holonomy with trace of the meridian −2, for the n-dimensional representations. Again the table gives the coefficients with respect to a $\mathbf{Q}$-basis for $\mathbf{Q}(\omega)$.

8.6 Other mutations and other representations

The mutation considered in this paper is called $(0,4)$-mutation, because the involved surface is planar and has 4 boundary components. By tubing along invariant arcs of the knot, this is a particular case of the so called $(2,0)$-mutation, namely, the mutation along a closed surface of genus 2 and using the hyperelliptic involution.

In [Rub87], Ruberman proved that a $(2,0)$-mutant of a hyperbolic manifold is again hyperbolic. The behavior of invariants under $(2,0)$-mutation has been investigated by many authors, see [CL99, DGST10, MR09] for instance. Unfortunately, our arguments do not apply, as in Section 8.3 we require two involutions, and in genus two mutation we can use only the hyperelliptic involution. So we arise:

Question. Is $\tau(M,\rho)$ invariant under genus two mutation?

The three dimensional representation of $\mathrm{SL}(2,\mathbf{C})$ is conjugate to the adjoint representation in the automorphism group of the Lie algebra $\mathfrak{sl}(2,\mathbf{C})$. The representation $Ad\rho$ is not acyclic, but a natural choice of basis for homology has been given in [Por97], hence its torsion is well defined. Moreover, we have:

Proposition 8.6.1 ([Por97])**.** *The torsion $\tau(M, Ad\rho)$ is invariant under $(2,0)$-mutation.*

The proof is straightforward, as $\mathrm{H}^1(S; Ad\rho)$ is the cotangent space to the variety of characters of S, and the action of the hyperelliptic involution is trivial on the variety of characters of S.

We have seen that if we compose the lift of the holonomy with the 6-dimensional representation of $\mathrm{SL}(2,\mathbf{C})$ (or the 4-dimensional one when the trace of the meridian is -2), the torsion is not invariant under $(2,0)$-mutation, as it is not invariant under $(0,4)$-mutation, see the example of the previous section.

Question. Working with the lift of the holonomy with trace of the meridian 2, is the torsion of the 4-dimensional representation invariant under $(0,4)$-mutation?

To conclude, we notice that our arguments do not apply if we tensorize $\rho : \pi_1(M) \to \mathrm{SL}(2,\mathbf{C})$ with the abelianization map $\pi_1(M) \twoheadrightarrow \mathbf{Z} = \langle t \mid \rangle$. This torsion gives the twisted polynomial in $\mathbf{C}[t^{\pm}]$ studied in [DFJ], where it is proved not to be mutation invariant. If we could apply the arguments of this paper, then we would be in the generic situation of Proposition 8.4.4 and Section 8.3, because $\mathrm{trace}(\rho(\mu)) = \pm(t+1/t) \neq 2$. Notice that two of the involutions in Figure 8.1 reverse the orientation of the meridian, and only one preserves them. Thus we can use only a single involution, and at least two involutions are required in our argument from Section 8.3, more precisely in Corollary 8.3.6(ii).

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