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Universidad del País Vasco — Euskal Herriko Unibertsitatea

# Hausdorff dimension in groups acting on the $p$-adic tree

Amaia Zugadi Reizabal

PhD Thesis

University of the Basque Country
Bilbao, September 2011

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*"Voici mon secret. Il est très simple:*
*on ne voit bien qu'avec le cœur.*
*L'essentiel est invisible pour les yeux."*
Antoine de Saint Exupèry, Le Petit Prince

*"To be without some of the things you want*
*is an indispensable part of happiness."*
Bertrand Russell, The Conquest of Happiness

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This is one of those moments in life that one waits for, expects, dreams of so much that when it comes, one has no idea of how he/she managed to arrive to that point. Suddenly, a look back in time takes one through all along the way of big efforts, hard work, falls, deep dedication... but also through matchless learning, experiencing, small twinkles of success, satisfaction... It is exactly when one realizes that it is even harder to believe that all these things actually happened and that a life stage is going to its end, that something is going to break down into pieces to give way to a new beginning.

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Amaia Zugadi Reizabal
Leioa, 23th of September, 2011

Familiari, lagunei.

Y especialmente Vito, a vos.

# Contents

# Introduction

Subgroups of the group of automorphisms of a regular rooted tree have turned out to be a source of many interesting examples in group theory. Specially after pretty examples by Grigorchuk [Gri80] (the First and the Second Grigorchuk groups) and Gupta and Sidki [GS83] (the Gupta-Sidki groups), which were proved to be among the first counterexamples of the General Burnside Problem. More relevantly, Grigorchuk groups were the first examples of groups of intermediate word growth, amenable but not elementary amenable, and branch groups that are finitely generated.

We devote Chapter 2 to a well-known family that generalizes the First Grigorchuk group, the *spinal groups*. And Chapter 3 and 4 deal with *GGS-groups*, a generalization of the Second Grigorchuk group and the Gupta-Sidki groups.

The main of our subjects of study in this thesis is the order of the congruence quotients and, as a consequence, the Hausdorff dimension of (the closures of) these groups in $\Gamma$, where $\Gamma$ is the group of $p$-adic automorphisms. This is the goal in Chapters 2 and 3.

In Chapter 2, furthermore, we determine the set $S$ of all rational numbers that appear as Hausdorff dimensions of spinal groups (for primes $p > 2$). A key ingredient in our approach to this problem is provided by a general procedure for decomposing spinal groups as a semidirect product, which allows us to reduce to the case of 2-generator spinal groups.

As for Chapter 3, if the GGS-group $G$ is defined by the vector $\mathbf{e} = (e_1, \ldots, e_{p-1}) \in \mathbb{F}_p^{p-1}$, the determination of the order of $G_n$ is split into three cases, according as $\mathbf{e}$ is non-symmetric, non-constant symmetric, or constant. It is relevant that an important feature to solve this problem has been the theory of $p$-groups of maximal class.

In Chapter 4 we focus on non-symmetric GGS-groups and we describe them in terms of zeros of equations. The group $\Gamma$ of $p$-adic automorphisms of the $p$-adic rooted tree can naively be identified with the $p$-group $\mathbb{F}_p^{\mathbb{N}}$, using pointwise addition $+$ in the portraits of the automorphisms. We prove that the operation $+$ is also internal for all non-symmetric GGS-groups (as an example, the Gupta-Sidki group). In order to get this, we introduce the notion of equation, or pattern, for subgroups of $\Gamma$, and we describe all equations for these groups.

We proceed to give the details of the main results of this thesis.

Let $G$ be a countably based profinite group, and let $\{G(n)\}_{n\in\mathbb{N}}$ be a base of neighbourhoods of the identity consisting of open normal subgroups. If $H$ is a closed subgroup of $G$, then the value

$$\dim_G H = \liminf_{n\to\infty} \frac{\log_p |HG(n)/G(n)|}{\log_p |G/G(n)|}$$

gives a way of measuring the relative size of $H$ in $G$. For instance, provided that $G$ is infinite, we have $\dim_G H = 1$ if $H$ is open in $G$, and $\dim_G H = 0$ if $H$ is finite. As shown by Abercrombie [Abe94], and Barnea and Shalev [BS97], $\dim_G H$ coincides with the Hausdorff dimension of $H$ when $G$ is considered with the natural metric induced by the family $\{G(n)\}$. The set of all values of the Hausdorff dimension of the closed subgroups of $G$ is called the *spectrum* of $G$, and if we only consider the dimensions corresponding to a particular family $\Sigma$ of subgroups, we speak of the $\Sigma$-*spectrum* of $G$. In [BS97], Barnea and Shalev also show that the spectrum of a $p$-adic analytic pro-$p$ group consists only of rational numbers, if one works with the subgroups $G(n) = G^{p^n}$.

Let $\mathcal{T}$ be the $p$-adic tree, for a prime $p$ and let $\operatorname{Aut}\mathcal{T}$ be the group of automorphisms of $\mathcal{T}$.

Any $g \in \operatorname{Aut} \mathcal{T}$ can be completely determined by describing how $g$ sends the descendants of every vertex $u$ to the descendants of $g(u)$. This can be done by indicating, for every $x \in X = \{1, \ldots, p\}$, the element $\alpha(x) \in X$ such that $g(ux) = g(u)\alpha(x)$. Then $\alpha$ is a permutation of $X$, which we call the *label* of $g$ at $u$, and we denote by $g_{(u)}$. The set of all labels of $g$ constitutes the *portrait* of $g$. Thus $g$ is determined by its portrait.

An important automorphism of $\mathcal{T}$ is the automorphism that permutes the $p$ subtrees hanging from the root rigidly according to the permutation $\sigma = (1\ 2\ \ldots\ p)$. This is called the *rooted automorphism* corresponding to $\sigma$ and will be denoted by the letter $a$.

Let $\Gamma \subseteq \operatorname{Aut} \mathcal{T}$ be the set of all automorphisms that only have powers of $\sigma$ in their portraits. Then $\Gamma$ is a Sylow pro-$p$ subgroup of $\operatorname{Aut} \mathcal{T}$, and it is natural to take $\Gamma(n) = \operatorname{Stab}_\Gamma(n)$, the stabilizer in $\Gamma$ of all vertices in the $n$-th level of $\mathcal{T}$. Klopsch showed in [Klo99, Chapter VIII, Section 5] that the spectrum of all profinite branch groups is the full interval $[0, 1]$, and this applies in particular to $\Gamma$. (See Section 2.2 for the definition and [Gri00] for the basic theory of branch groups.) Later, Abért and Virág [AV05, Theorem 2] proved that every value $\lambda \in [0, 1)$ can be obtained as the Hausdorff dimension of a closed subgroup of $\Gamma$ which can be (topologically) generated by at most 3 elements. However, the probabilistic nature of their arguments does not provide explicit examples for every possible $\lambda$, and more specifically any examples for irrational $\lambda$. In the same paper, they also show that soluble subgroups of $\Gamma$ have dimension 0 (see the remark after Theorem 5). On the other hand, Bartholdi has proved [Bar06, Proposition 2.7] that a *regular* branch subgroup of $\Gamma$ has positive rational Hausdorff dimension.

In the recent paper [Sie08], Siegenthaler has considered the case $p = 2$, and has provided an explicit formula for the Hausdorff dimension of the closures of a special family of discrete subgroups of $\Gamma$, the spinal groups. As a consequence, he finds 3-generator spinal groups whose closure has irrational, even transcendental, Hausdorff dimension in $\Gamma$.

Spinal groups can be given in the form $\langle a, B \rangle$, where $a$ is as before, and where $B$ is an elementary abelian finite $p$-group consisting of automorphisms whose action is concentrated on a special subset of vertices of $\mathcal{T}$, which we call a spine. We refer the reader to Section 2.2 for details about spinal groups. In particular, spinal groups are branch if $p > 2$, but not necessarily regular branch.

The key ingredient for the construction of spinal groups is to consider a sequence $\Omega = (\omega_n)_{n\geq 1}$ of linear functionals of a finite-dimensional vector space $E$ over $\mathbb{F}_p$. We write $\mathrm{Spinal}(\Omega)$ for the spinal group $G$ constructed from $\Omega$. One of our main results in Chapter 2 is the determination, for $p > 2$, of a formula for the Hausdorff dimension in $\Gamma$ of the closure $\overline{G}$, in terms of the sequence $\Omega$.

**Theorem A.** *Let $G = \mathrm{Spinal}(\Omega)$ be a spinal group, where $p > 2$. Then:*

(i) *If $\omega_i = 0$ for some $i$, then $\dim_\Gamma \overline{G} = 0$.*

(ii) *If $\omega_i \neq 0$ for all $i$, let $m$ be the dimension of the subspace of $E^*$ generated by $\Omega$. For $n$ big enough and for every $i = 1, \ldots, m$, let $r_{n,i}$ be the minimum number of terms of the sequence $(\omega_{n-1}, \ldots, \omega_1)$, in that order, that are needed to generate a subspace of dimension $i$. Then,*

$$\dim_\Gamma \overline{G} = (p-1) \liminf_{n\to\infty} \left( \frac{1}{p^{r_{n,1}}} + \frac{1}{p^{r_{n,2}}} + \cdots + \frac{1}{p^{r_{n,m}}} \right).$$

By using Theorem A, we are able to determine the set of all values that are taken by the Hausdorff dimension for the family $\Sigma$ of the closures of all spinal subgroups of $\Gamma$. In other words, we calculate the $\Sigma$-spectrum of $\Gamma$, to which we refer as the *spinal spectrum*.

**Theorem B.** *If $p$ is odd, then the spinal spectrum of $\Gamma$ consists of $0$ and all numbers whose $p$-adic expansion is of the form $0.a_1 \ldots a_n$, where*

(i) *$a_i = 0$ or $p-1$ for every $i = 1, \ldots, n$.*

(ii) *$a_1 = p - 1$.*

*In particular, the spinal spectrum is contained in* $\mathbb{Q}$.

Thus, the situation for odd primes is dramatically different from that of the even prime. Note also that Theorems A and B generalize to all spinal groups (in the case of odd primes) a result of Šunić [Šun07, Theorem 2] dealing with a special class of spinal groups, for which the Hausdorff dimension $\lambda$ is always of the form $\lambda = 0.a_1 \dots a_n$ in base $p$, with all $a_i$ equal to $p-1$. A particular case, for $p = 3$, of the groups considered by Šunić is the so-called Fabrykowski-Gupta group. The Hausdorff dimension of this group in $\Gamma$ had been previously calculated by Bartholdi and Grigorchuk in [BG02]; according to Corollary 6.6 in there, the dimension is $2/3$, in agreement with Theorem A.

On the other hand, we want to point out that our proof of Theorem B is constructive, in the sense that it provides an algorithm which, given a number $\lambda$ whose $p$-adic expansion is of the appropriate type, yields a spinal group of Hausdorff dimension equal to $\lambda$.

For the proof of Theorem A, we need to calculate the orders of the quotient groups $G_n = G/\operatorname{Stab}_G(n)$ for every $n$. This is achieved in two steps: first, in Section 2.4, we get these orders for 2-generator spinal groups; and then, in Section 2.5, we obtain the formula for the general case. The key for this transition from 2-generator to arbitrary spinal groups is given by a general result about semidirect product decompositions of spinal groups. We think that these decompositions may have an independent interest, broader than just for the determination of the Hausdorff dimension. The result is valid for all primes, and reads as follows.

**Theorem C.** *Let* $G = \langle a, B \rangle$ *be a spinal group. Then, for every subgroup* $B_2$ *of* $B$*, there exists a complement* $B_1$ *in* $B$ *such that* $G = \langle a, B_1 \rangle \ltimes B_2^G$*. In particular, if* $B_2$ *is a maximal subgroup of* $B$*, then the normal closure* $B_2^G$ *has a complement in* $G$ *which is a 2-generator spinal group.*

As a matter of fact, if $G$ is constructed from a sequence $\Omega$ of linear functionals, then it is possible to give an explicit choice of $B_1$ in terms of $B_2$ and $\Omega$; details are given in Section 2.3.

In Chapter 3, we focus on GGS-groups.

Let $\mathcal{T}$ be the $m$-adic rooted tree. If an automorphism $g$ fixes a vertex $u$, then the restriction of $g$ to the subtree hanging from $u$ induces an automorphism $g_u$ of $\mathcal{T}$. In particular, if $g \in \mathrm{Stab}(1)$ then $g_i$ is defined for every $i = 1, \ldots, m$, and we can consider the map

$$\begin{array}{cccc} \psi & : & \mathrm{Stab}(1) & \longrightarrow & \operatorname{Aut}\mathcal{T} \times \overset{m}{\cdots} \times \operatorname{Aut}\mathcal{T} \\ & & g & \longmapsto & (g_1, \ldots, g_m). \end{array}$$

Clearly, $\psi$ is a group isomorphism.

Let $a$ be the rooted automorphism corresponding to $(1\ 2\ \ldots\ m)$. Since $a$ has order $m$, it makes sense to write $a^k$ for $k \in \mathbb{Z}/m\mathbb{Z}$. Now, given a non-zero vector $\mathbf{e} = (e_1, \ldots, e_{m-1}) \in (\mathbb{Z}/m\mathbb{Z})^{m-1}$, we can define recursively an automorphism $b$ of $\mathcal{T}$ via

$$\psi(b) = (a^{e_1}, \ldots, a^{e_{m-1}}, b).$$

We say that the subgroup $G = \langle a, b \rangle$ of $\operatorname{Aut}\mathcal{T}$ is the *GGS-group* corresponding to the *defining vector* $\mathbf{e}$. If $m = 2$ then there is only one GGS-group, which is isomorphic to $D_\infty$, the infinite dihedral group. The second Grigorchuk group is obtained by choosing $m = 4$ and $\mathbf{e} = (1, 0, 1)$, and the Gupta-Sidki group arises for $m$ equal to an odd prime and $\mathbf{e} = (1, -1, 0, \ldots, 0)$. The groups corresponding to $\mathbf{e} = (1, 0, \ldots, 0)$ and arbitrary $m$ have also deserved special attention. In the case $m = 3$, this group was introduced by Fabrykowski and Gupta in [FG85]. As a reference for GGS-groups, the reader can consult Section 2.3 of the monograph [BGŠ03] by Bartholdi, Grigorchuk, and Šunić, the habilitation thesis [Roz96] of Rozhkov, or the papers [Vov00] by Vovkivsky and [Per00, Per07] by Pervova.

Little is known about the orders of the congruence quotients $G_n$ when $G$ is a GGS-group. In the case that $\mathbf{e} = (1, 0, \ldots, 0)$ and $m = p$ is a prime, Šunić found in [Šun07] that, for every $n \geq 2$,

$$\log_p |G_n| = \begin{cases} p^{n-1} + 1, & \text{if } p \text{ is odd,} \\ 2^{n-2} + 2, & \text{if } p = 2. \end{cases}$$

Hence we may always assume that $m \geq 3$, as far as the problem of determining $|G_n|$ is concerned. To the best of our knowledge, the only other cases in which the order of $G_n$ has been determined for every $n$ correspond to $m = 3$. For the Gupta-Sidki group, Sidki himself (see [Sid87]) proved that

$$\log_3 |G_n| = 2 \cdot 3^{n-2} + 1, \quad \text{for every } n \geq 2.$$

On the other hand, for $\mathbf{e} = (1, 1)$, Bartholdi and Grigorchuk showed in [BG02] that

$$\log_3 |G_n| = \frac{3^n + 2n + 3}{4}, \quad \text{for every } n \geq 2.$$

Now, we assume that $m$ is equal to an odd prime $p$, and so $\mathcal{T}$ stands for the $p$-adic tree. The first of our main results is the determination of the order of $G_n$ for *all* GGS-groups under this assumption. Before giving the statement of the theorem, we introduce some notation. Given a vector $\mathbf{a} = (a_1, \ldots, a_n)$, we write $C(\mathbf{a})$ to denote the circulant matrix generated by $\mathbf{a}$, i.e. the matrix of size $n \times n$ whose first row is $\mathbf{a}$, and every other row is obtained from the previous one by applying a shift of length one to the right. In other words, the entries of $C(\mathbf{a})$ are $c_{ij} = a_{j-i+1}$, where $a_k$ is defined for every integer $k$ by reducing $k$ modulo $n$ to a number between 1 and $n$. If $\mathbf{e}$ is the defining vector of a GGS-group, then we write $C(\mathbf{e}, 0)$ for the circulant matrix $C(e_1, \ldots, e_{p-1}, 0)$ over $\mathbb{F}_p$. We say that $\mathbf{e}$ is *symmetric* if $e_i = e_{p-i}$ for all $i = 1, \ldots, p-1$.

**Theorem D.** *Let $G$ be a GGS-group over the $p$-adic tree, where $p$ is an odd prime, and let $\mathbf{e}$ be the defining vector of $G$. Then, for every $n \geq 2$, we have*

$$\log_p |G_n| = tp^{n-2} + 1 - \delta \, \frac{p^{n-2} - 1}{p - 1} - \varepsilon \, \frac{p^{n-2} - (n-2)p + n - 3}{(p-1)^2},$$

*where $t$ is the rank of the circulant matrix $C(\mathbf{e}, 0)$,*

$$\delta = \begin{cases} 1, & \text{if } \mathbf{e} \text{ is symmetric,} \\ 0, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \varepsilon = \begin{cases} 1, & \text{if } \mathbf{e} \text{ is constant,} \\ 0, & \text{otherwise.} \end{cases}$$

Observe that, under the assumption $m = p$ that we have made, all GGS-groups are subgroups of $\Gamma$. According to Theorem 1 of [Vov00], the requirement that $\mathbf{e}$ is non-zero implies that GGS-groups are infinite if $m = p$. Since they are countable groups, they cannot be closed in the pro-$p$ group $\Gamma$.

As an immediate consequence of Theorem A, we get the Hausdorff dimension of the closure of any GGS-group.

**Theorem E.** *Let $G$ be a GGS-group over the $p$-adic tree, where $p$ is an odd prime, and let $\mathbf{e}$ be the defining vector of $G$. Then*

$$\dim_\Gamma \overline{G} = \frac{(p-1)t}{p^2} - \frac{\delta}{p^2} - \frac{\varepsilon}{(p-1)p^2},$$

*where $t$ is the rank of the circulant matrix $C(\mathbf{e}, 0)$,*

$$\delta = \begin{cases} 1, & \text{if } \mathbf{e} \text{ is symmetric,} \\ 0, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \varepsilon = \begin{cases} 1, & \text{if } \mathbf{e} \text{ is constant,} \\ 0, & \text{otherwise.} \end{cases}$$

Our proof of Theorem D relies on finding some kind of branch structure inside a GGS-group $G$. In particular, if $\mathbf{e}$ is not constant, we show that $G$ is regular branch (see Section 3.3 for the definition). This result had been previously proved by Pervova and Rozhkov for *periodic* GGS-groups. On the other hand, it is worth mentioning that the theory of $p$-groups of maximal class plays also a crucial role in the proof of Theorem D, particularly in the case that $\mathbf{e}$ is constant.

In Chapter 4, as in the preceding chapters, we consider the $p$-adic rooted tree $\mathcal{T}$, for an odd prime $p$, and $\Gamma$, the Sylow pro-$p$ subgroup of $\operatorname{Aut} \mathcal{T}$ corresponding to $\sigma = (1 \, \ldots \, p) \in S_p$. Then $\Gamma$ is in one-to-one correspondence with $\mathbb{F}_p^{X^*}$, the set of infinite sequences of the form $(m_v)_{v \in X^*}$ with $m_v \in \mathbb{F}_p$, via portraits.

Roughly speaking, as it is explained in detail in Section 4.2, this correspondence allows us to describe every closed set, in particular closed subgroup $G$ of $\Gamma$ as the set of zeros of an ideal of polynomials. The polynomials are taken over the field $\mathbb{F}_p$ and the indeterminates are indexed by the vertices of the tree. We will say that these polynomials that vanish in $G$ are *equations for* $G$ or *patterns* [Gri05]. If such a polynomial has degree 1 we will say that it is a *linear equation* for $G$.

Section 4.2 introduces and looks more closely at all these concepts. In Section 4.4, we focus on GGS-groups and we explicitly describe a generating set for all the equations of non-symmetric GGS-groups.

The first of the two main results in this chapter can be summarized as follows.

**Theorem F.** *Let $G$ be a non-symmetric GGS-group. Then there are $p$ linear equations that generate all equations for $G$.*

We give the explicit expression of these $p$ linear equations in Theorem 4.4.6, and the way in which these linear equations "generate" all equations will be explained in Section 4.3.

It is interesting to know these equations explicitly for several reasons. First, we can describe the closure $\overline{G}$ (in the profinite topology of $\Gamma$) of such a group $G$ as the set of zeros of these equations and their translates, as it is shown in Theorem 4.4.6. Secondly, since these generating equations are linear and satisfy some extra conditions, we get to prove the second of the two main results in this chapter, Theorem G below. And finally, it also enriches the information contained in the Hausdorff dimension of the closures of these groups.

In Chapter 3 we compute the Hausdorff dimension of the closures of all GGS-groups. In this chapter, we recover the same values for non-symmetric GGS-groups in Corollary 4.4.7, another consequence of Theorem 4.4.6. Indeed, the Hausdorff dimension can be computed very easily if we know a convenient generating set of equations, as we show in Theorem 4.3.6. It is relevant to underline, anyway, that we actually rely on many of the results proved in Chapter 3.

Finally, Section 4.5 is devoted to the proof of the last significant result in this chapter, Theorem G, namely that non-symmetric GGS-groups possess another group operation that is abelian. In particular, we conclude that the Gupta-Sidki group has such a structure. The linearity and also the convenient construction of the polynomials of the generating set in Theorem F is important for the proof of this result.

**Theorem G.** *Let $G$ be a non-symmetric GGS-group. Pointwise addition in the portraits of elements gives $G$ the structure of an abelian group.*

We would like to point out that the consequences of the coexistence of these two group operations are yet to be explored. A reasonable direction to examine would be the relationship between the present work and Lie algebras, as we now explain.

The description of the elements of $\Gamma$ in terms of portraits is equivalent to a certain choice of a set-map

$$\pi : \Gamma \to A = \prod_{i=0}^{\infty} \operatorname{Stab}_\Gamma(i)/\operatorname{Stab}_\Gamma(i+1),$$

where $\prod$ denotes the unrestricted product. The group $A$ is an elementary abelian $p$-group with the operation inherited from $\Gamma$. This is exactly the sum of portraits. Now Theorem G can be rephrased as:

**Theorem G′.** *The image of $G$ under $\pi$ is a subgroup of $A$.*

One can compare this construction with the Lie algebra constructed by Magnus [Mag40]:

$$\mathcal{L}(G) = \bigoplus_{i=1}^{\infty} \gamma_i(G)/\gamma_{i+1}(G),$$

where $\gamma_i(G)$ is the $i$th term of the lower central series of $G$. The addition on $\mathcal{L}(G)$ is the operation induced by the group structure of $G$, and commutation in $G$ yields the Lie bracket. There is another similar construction, based on the dimension series, also known as the Brauer, Jennings [Jen41], Lazard [Laz53] or Zassenhaus [Zas40] series, which yields a restricted Lie algebra (see [Jac79] for the definition of restricted Lie algebras).

It would be interesting to investigate whether there is a map

$$\prod_{i=0}^{\infty} \gamma_i(G)/\gamma_{i+1}(G) \longrightarrow \prod_{i=0}^{\infty} \operatorname{Stab}_\Gamma(i)/\operatorname{Stab}_\Gamma(i+1)$$

which would enable to "read" the Lie algebra structure of $\mathcal{L}(G)$ directly on the portraits of the elements of $G$. Note that the Lie algebras associated to the Gupta-Sidki group have been explicitly described in [BG00], and the terms of the lower central series also admit a nice description in terms of portraits (see Theorem 4.2.4 in [Sie09]).

In this chapter we follow the approach developed in Olivier Siegenthaler's PhD thesis [Sie09]. We refer the reader to [Gri05], [Šun07], [Šun11] and the appendix in [AdlHKŠ07] for previous works on the subject.

# Chapter 1

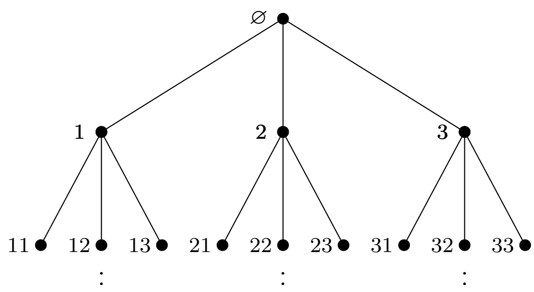
# Preliminaries

*"The best time to plant a tree*
*was 20 years ago.*
*The next best time is now."*
Chinese Proverb

## 1.1 The group of automorphisms of the $p$-adic rooted tree

### 1.1.1 Rooted trees

In graph theory, a *tree* is a connected graph with no cycles. There is a type of tree which is particularly interesting because of the rich group theoretical properties that appear in its group of isometries. We are refering to the *regular $d$-adic rooted tree*, denoted by $\mathcal{T}$; 'rooted' because it has a distinguished vertex and 'regular' and '$d$-adic' because every vertex has the same degree $d+1$ (except for the root, whose degree is $d$). If $X$ is an alphabet on $d$ letters, then the elements of $X^*$, the free monoid on $X$, can be identified with the vertices of the regular $d$-adic rooted tree. In such a way that the root corresponds to the empty word $\emptyset$ and two words $u, v \in X^*$ are connected by an edge if there exists $x \in X$ such that $u = vx$ or $v = ux$. We will take $X$ to be $\{1, \ldots, d\}$ in this work.




The 3-adic rooted tree with vertices labelled as words in $X = \{1, 2, 3\}$.

The set $X^n$ of words of length $n$ is the *nth level* of the tree, and if we consider the set $X^{\leq n}$ of all words of length $\leq n$, then we have a finite tree $\mathcal{T}_n$, which we refer to as *the tree* $\mathcal{T}$ *truncated at level* $n$. The set of right-infinite sequences over $X$ will be called the *boundary* of the tree and denoted by $X^\omega$. The elements of the boundary are the *infinite paths* or *ends* of the tree. If $u \in X^* \cup X^\omega$, then $v \in X^*$ is a *prefix* of $u$ if there is $w \in X^* \cup X^\omega$ such that $u = vw$. If $S \subseteq X^* \cup X^\omega$, we define $\mathrm{Prefix}(S)$ as the set of all prefixes of all elements in $S$. If $u \in X^*$, the *subtree* $uX^*$ of $X^*$ is the set of all the vertices $v \in X^*$ with $u$ as a prefix and the same edges as in $\mathcal{T}$. There is a canonical graph isomorphism between $uX^*$ and $X^*$ which corresponds to deleting $u$ from the beginning of the words, and so $uX^*$ is a regular $d$-adic rooted tree. The *children* of a vertex $u \in X^*$ are the $d$ vertices hanging from $u$, that is, $u1, u2, \ldots, ud$. And the $d$ subtrees $1X^*, \ldots, dX^*$ are usually called *the main subtrees* of $X^*$.

The graph structure of the tree induces a natural metric on the vertices: the distance between two vertices is the number of edges of the shortest *path* connecting them. In terms of words, the distance between $u, v \in X^*$ is defined by

$$d(u, v) = |u| + |v| - 2|u \wedge v|,$$

where $|u|$ denotes the length of the word $u$ and $u \wedge v$ is the longest common prefix of $u$ and $v$.

### 1.1.2 Automorphisms of the regular rooted tree

**Definition 1.1.1.** An *automorphism* of $\mathcal{T}$ is a bijection of the vertices that preserves incidence. The group of automorphisms of $\mathcal{T}$ will be denoted by $\operatorname{Aut}\mathcal{T}$.

Similarly, an automorphism of the truncated tree $\mathcal{T}_n$ is a bijection of its vertices preserving incidence, and $\operatorname{Aut}\mathcal{T}_n$ will be the group of automorphisms of $\mathcal{T}_n$.

Thus an automorphism is a graph isomorphism from $\mathcal{T}$ onto itself or, equivalently, an isometry of $\mathcal{T}$ with respect to the metric defined in the previous subsection.

The following properties of an automorphism $f$ are straightforward: $f$ fixes the root $\emptyset$; since $f$ is an isometry and the $n$th level is the sphere of radius $n$ centered at the root, then $f$ preserves the levels, and for the same reason every $f \in \operatorname{Aut}\mathcal{T}$ induces by restriction an element of $\operatorname{Aut}\mathcal{T}_n$; and the image of a vertex $u$ under $f$ determines the images of all its prefixes, i.e. the vertices in the path connecting $u$ to the root.

Next we define the portrait of an automorphism, which is another way of describing it, capturing the action of the automorphism on each vertex.

Any $f \in \operatorname{Aut}\mathcal{T}$ can be completely determined by describing how $f$ sends the children of every vertex $u$ to the corresponding children of $f(u)$. This can be done by indicating, for every $x \in X$, the element $\alpha(x) \in X$ such that $f(ux) = f(u)\alpha(x)$. Then $\alpha$ is a permutation of $X$, which we call the *label* of $f$ at $u$, and we denote by $f_{(u)}$. The set of all labels of $f$ constitutes the *portrait* of $f$. Thus $f$ is determined by its portrait. We have the following rules for labels under composition and inversion:

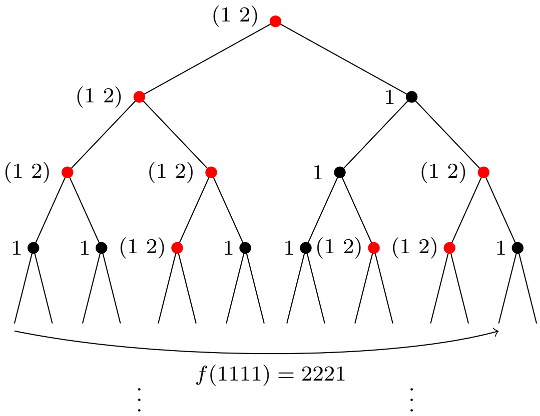
$$(fg)_{(u)} = f_{(u)}g_{(f(u))} \quad \text{and} \quad (f^{-1})_{(u)} = (f_{(f^{-1}(u))})^{-1}. \tag{1.1.1}$$

As a mnemonic for these rules, observe the similarity of these relations with the rules of derivation of real functions. Be also careful that $(f^{-1})_{(u)}$ is not the same permutation as $(f_{(u)})^{-1}$!

From these formulas we also get

$$(f^g)_{(g(u))} = (g_{(u)})^{-1} f_{(u)} g_{(f(u))}. \tag{1.1.2}$$

Note that even if we write images as $f(u)$, we write the composition of $f$ and $g$ as $fg$. Thus $(fg)(u) = g(f(u))$.




An automorphism $f$ of the dyadic tree given by means of its portrait.

As the reader might have noticed, given the portrait of an automorphism, if we want to know the image of a vertex $u$, we only need the permutations attached to the vertices of the path that goes from $u$ to $\emptyset$, i.e. the prefixes of $u$. In fact, the image of this path is exactly the path that goes from $f(u)$ to $\emptyset$. All these properties may be read off from the following formula:

$$f(x_1x_2\dots x_n) = f_{(\emptyset)}(x_1)f_{(x_1)}(x_2)\dots f_{(x_1\dots x_{n-1})}(x_n).$$

The *support* of an automorphism $f \in \operatorname{Aut}\mathcal{T}$ is the set of all vertices of $X^*$ with non-trivial label. If the support of $f$ is contained in $\{\emptyset\}$, we say that $f$ is the *rigid* or *rooted automorphism* corresponding to the permutation $f_{(\emptyset)}$.

**Definition 1.1.2.** If $f \in \operatorname{Aut}\mathcal{T}$ and $u \in X^*$, the *section* of $f$ at $u$ is the unique automorphism $f_u$ of $\mathcal{T}$ defined by

$$f(uv) = f(u)f_u(v)$$

for every $v \in X^*$.

A close look to this definition shows that $f_u$ is the automorphism of $\mathcal{T}$ whose portrait is a copy the labelling of $f$ in the subtree $uX^*$.

The same rules (1.1.1) and (1.1.2) that we have seen for labels apply if we want to obtain the sections of a composition, an inverse or a conjugate; simply erase parentheses where necessary.

### 1.1.3 The structure of $\operatorname{Aut}\mathcal{T}$

**Definition 1.1.3.** The subgroup $\operatorname{Stab}(n)$ of $\operatorname{Aut}\mathcal{T}$ consisting of the automorphisms that fix the $n$th level is called the *nth level stabilizer*. More generally, if $G \leq \operatorname{Aut}\mathcal{T}$ we define $\operatorname{Stab}_G(n) = \operatorname{Stab}(n) \cap G$.

**Remarks 1.1.4.** (i) An element in $\operatorname{Stab}(n)$ fixes all vertices of the truncated tree $\mathcal{T}_n$.

(ii) The subgroup $\operatorname{Stab}(n)$ is the kernel of the natural epimorphism $\pi_n : \operatorname{Aut}\mathcal{T} \to \operatorname{Aut}\mathcal{T}_n$ obtained by restriction. Hence $\operatorname{Stab}(n)$ is normal in $\operatorname{Aut}\mathcal{T}$ and $\operatorname{Aut}\mathcal{T}/\operatorname{Stab}(n) \cong \operatorname{Aut}\mathcal{T}_n$. In particular, $\operatorname{Stab}(n)$ has finite index in $\operatorname{Aut}\mathcal{T}$.

These stabilizers can be considered as natural congruence subgroups for $\operatorname{Aut}\mathcal{T}$. If $G$ is a subgroup of $\operatorname{Aut}\mathcal{T}$, then we refer to the quotient $G_n = G/\operatorname{Stab}_G(n)$ as the *nth congruence quotient* of $G$. Since the kernel of the action of $G$ on $\mathcal{T}_n$ is $\operatorname{Stab}_G(n)$, it follows that $G_n$ can be naturally seen as a subgroup of $\operatorname{Aut}\mathcal{T}_n$.

As a matter of fact, all subgroups of $\operatorname{Aut}\mathcal{T}_n$ arise as $G_n$ for some subgroup $G$ of $\operatorname{Aut}\mathcal{T}$. To see this, let us define, for every $f \in \operatorname{Aut}\mathcal{T}_n$, the *extension* $\operatorname{ext}(f)$ as the automorphism of the infinite tree $\mathcal{T}$ which has the same labels as $f$ in $\mathcal{T}_n$, and the rest of labels equal to 1. The map ext is a homomorphism, and so if $L$ is a subgroup of $\operatorname{Aut}\mathcal{T}_n$, then $G = \operatorname{ext}(L)$ is a subgroup of $\operatorname{Aut}\mathcal{T}$. Now, if we compose ext with the canonical map from $G$ to $G_n$, we obtain an isomorphism between $L$ and $G_n$ which preserves the action on $\mathcal{T}_n$.

Suppose now that an automorphism $f$ fixes a vertex $u$. Then the restriction of $f$ to the subtree hanging from $u$ induces the section automorphism $f_u$ of $\mathcal{T}$. If $f \in \mathrm{Stab}(1)$ then $f_i$ is defined for every $i = 1, \ldots, d$, and we can consider the map

$$\begin{array}{rccc} \psi & : & \mathrm{Stab}(1) & \longrightarrow & \mathrm{Aut}\,\mathcal{T} \times \overset{d}{\cdots} \times \mathrm{Aut}\,\mathcal{T} \\ & & f & \longmapsto & (f_1, \ldots, f_d). \end{array}$$

Clearly, $\psi$ is a group isomorphism.

In a similar way, the homomorphisms $\psi_n$ and $\overline{\psi}^n$ defined below are group isomorphisms, for every $n \in \mathbb{N}$.

$$\begin{array}{rccc} \psi_n & : & \mathrm{Stab}(n) & \longrightarrow & \mathrm{Aut}\,\mathcal{T} \times \overset{d^n}{\cdots} \times \mathrm{Aut}\,\mathcal{T} \\ & & f & \longmapsto & (f_v)_{v \in X^n}. \end{array}$$

$$\begin{array}{rccc} \overline{\psi}^n & : & \mathrm{Stab}_{\mathrm{Aut}\,\mathcal{T}_n}(1) & \longrightarrow & \mathrm{Aut}\,\mathcal{T}_{n-1} \times \overset{d}{\cdots} \mathrm{Aut}\,\mathcal{T}_{n-1} \\ & & f & \longmapsto & (f_1, \ldots, f_d). \end{array}$$

Observe that $f \in \mathrm{Aut}\,\mathcal{T}_n$ is determined by the images of the vertices of the $n$th level and so $\mathrm{Aut}\,\mathcal{T}_n$ can be seen as a subgroup of $S_{d^n}$. However, not all possible permutations of these vertices are allowed, in other words, $\mathrm{Aut}\,\mathcal{T}_n$ is a proper subgroup of $S_{d^n}$. On the other hand, $\mathrm{Aut}\,\mathcal{T}$ acts transitively on each of the levels of $\mathcal{T}$. In the same way, it acts transitively on the boundary of the tree.

**Theorem 1.1.5.** *If $\mathcal{T}$ is the $d$-adic rooted tree, then*

$$\mathrm{Aut}\,\mathcal{T}_n \cong \overbrace{S_d \wr (\ldots \wr (S_d \wr S_d) \ldots)}^{n},$$

*where $\wr$ denotes the permutational wreath product, and*

$$\mathrm{Aut}\,\mathcal{T} = \varprojlim \mathrm{Aut}\,\mathcal{T}_n \cong \ldots \wr (S_d \wr (S_d \wr S_d))$$

*is a profinite group, where $\{\mathrm{Stab}(n)\}_{n \geq 1}$ is a fundamental system of neighbourhoods of 1.*

In the remainder of this subsection, we assume that $d$ is equal to a prime $p$. This case is specially interesting for several group theoretical reasons.

**Theorem 1.1.6.** *Let $\mathcal{T}$ be the $p$-adic rooted tree and let $P_n$ be a Sylow $p$-subgroup of* $\operatorname{Aut}\mathcal{T}_n$. *Then*

$$P_n \cong \overbrace{C_p \wr (\ldots \wr (C_p \wr C_p)\ldots)}^{n},$$

*and if* $\Gamma$ *is a Sylow pro-$p$ subgroup of* $\operatorname{Aut}\mathcal{T}$, *then*

$$\Gamma = \varprojlim \Gamma_n \cong \ldots \wr (C_p \wr (C_p \wr C_p)).$$

Observe that, from the previous theorem, a Sylow $p$-subgroup of $\operatorname{Aut}\mathcal{T}_n$ is, at the same time, a Sylow $p$-subgroup of $S_{p^n}$.

There are some Sylow pro-$p$ subgroups of $\operatorname{Aut}\mathcal{T}$ that particularly interest us and are easy to visualize in terms of portraits.

**Lemma 1.1.7.** *Fix a $p$-cycle $\sigma \in S_p$ and let $\Gamma \subseteq \operatorname{Aut}\mathcal{T}$ be the set of all automorphisms that only have powers of $\sigma$ in their portraits. Then $\Gamma$ is a Sylow pro-$p$ subgroup of* $\operatorname{Aut}\mathcal{T}$.

If $\sigma \in S_p$ is a $p$-cycle, then the Sylow pro-$p$ subgroup of $\operatorname{Aut}\mathcal{T}$ constructed as in the previous lemma will be refereed as the Sylow pro-$p$ subgroup corresponding to $\sigma$. Throughout the thesis, unless otherwise stated, $\Gamma$ will denote the Sylow pro-$p$ subgroup of $\operatorname{Aut}\mathcal{T}$ corresponding to the $p$-cycle $\sigma = (1 \ldots p)$. In the literature, the group $\Gamma$ is sometimes called the group of $p$-adic automorphisms and denoted by $\operatorname{Aut}_p\mathcal{T}$.

Next, we define a family of maps defined over $\Gamma$ that turn out to be homomorphisms. They will be useful in Section 2.2.

For every level $n$, we have a product map $p_n : \Gamma \to \langle(1 \ldots p)\rangle$ given by

$$p_n(f) = \prod_{v \in X^n} f_{(v)}. \tag{1.1.3}$$

It follows from (1.1.1) that $p_n$ is a homomorphism. Similarly, the map

$$\begin{array}{rccc} p_n^i & : & \operatorname{Stab}_\Gamma(1) & \longrightarrow & \langle(1 \ldots p)\rangle \\ & & f & \longmapsto & \displaystyle\prod_{v \in iX^{n-1}} f_{(v)} \end{array} \tag{1.1.4}$$

is a homomorphism for every $i \in \{1, \ldots, p\}$. Observe that $p_n^i(f)$ is simply the product of all labels of $f$ at the vertices of the $n$th level of $\mathcal{T}$ which lie in the main subtree hanging from the vertex $i$.

### 1.1.4 Important classes of groups acting on rooted trees

There are several families of groups acting on rooted trees that have become important for their 'good' behaviour. Here we present some of them.

**Definition 1.1.8.** We say that a subgroup $G$ of $\operatorname{Aut}\mathcal{T}$ is *spherically transitive* if it acts transitively on all levels $X^n$.

**Definition 1.1.9.** A subgroup $G$ of $\operatorname{Aut}\mathcal{T}$ is said to be a *branch group* provided that:

(i) $G$ is spherically transitive.

(ii) For every $n \geq 1$, the image of $\mathrm{Stab}_G(n)$ under the map

$$\begin{array}{cccc} \psi_n & : & \mathrm{Stab}(n) & \longrightarrow & \operatorname{Aut}\mathcal{T} \times \overset{p^n}{\cdots} \times \operatorname{Aut}\mathcal{T} \\ & & f & \longmapsto & (f_v)_{v \in X^n} \end{array}$$

contains a subgroup of finite index of the form $L_n \times \overset{p^n}{\cdots} \times L_n$.

See [Gri00, Section 5] and the monograph [BGŠ03] for more information on the class of branch groups.

**Definition 1.1.10.** Let $G$ be a subgroup of $\operatorname{Aut}\mathcal{T}$. We say that $G$ is *self-similar* if every section of every element of $G$ is an element of $G$.

In particular, if $G \leq \operatorname{Aut}\mathcal{T}$ is self-similar, then the image of $\mathrm{Stab}_G(n)$ under $\psi_n$ is contained in $G \times \overset{p^n}{\cdots} \times G$.

**Definition 1.1.11.** Let $G$ be a self-similar spherically transitive group of automorphisms of $\mathcal{T}$, and let $K$ be a non-trivial subgroup of $\mathrm{Stab}_G(1)$. We say that $G$ is *weakly regular branch* over $K$ if

$$K \times \overset{p}{\cdots} \times K \subseteq \psi(K).$$

If furthermore $K$ has finite index in $G$, we say that $G$ is *regular branch* over $K$.

**Remark 1.1.12.** Let $G$ be a subgroup of $\operatorname{Aut}\mathcal{T}$. If $G$ is self-similar, then so is its closure in the profinite topology $\overline{G}$. And if $G$ is regular branch over $K$ and $K$ is a congruence subgroup, then $\overline{G}$ is regular branch over $\overline{K}$.

## 1.2 Hausdorff dimension

Our goal in this section is to introduce some background on Hausdorff dimension and to present some important results concerning Hausdorff dimension in groups acting on rooted trees, which are the starting point of the work done in Chapters 2 and 3.

Suppose we have a subgroup $H$ of a finite group $G$, and that we want to measure the relative size of $H$ with respect to $G$. We can use the quotient $|H|/|G|$, or even better $\log|H|/\log|G|$ if $G$ is a finite $p$-group. Indeed, if the orders of $G$ and $H$ are $p^a$ and $p^b$, respectively, then the number $|H|/|G| = p^b/p^a = p^{b-a}$ may hide the size relation between $H$ and $G$ for high values of $p$. That is why we are more interested in knowing the relation between $a$ and $b$ and why we consider $\log|H|/\log|G| = b/a$ instead.

If $G$ is infinite, the first problem is that both $|H|/|G|$ and $\log|H|/\log|G|$ are meaningless. If we rewrite $|H|/|G|$ as $1/|G:H|$ and interpret $1/\infty$ as 0, then we could make this choice for the dimension of $H$ in $G$, but it presents some problems: it does not distinguish subgroups of infinite index, and, intuitively, a subgroup of finite index of an infinite group should have dimension 1. On the other hand, the alternative of $\log|H|/\log|G|$ does not even allow a direct reinterpretation in the infinite setting.

Abercrombie proposed a way to overcome this situation in the case of profinite groups, using the concept of *Hausdorff dimension* of a metric space.

### 1.2.1 Hausdorff dimension in metric spaces

We introduce the definition of Hausdorff dimension over a metric space. The reader may find more extensive and complete information about the topic over the reals in Falconer's books [Fal85] and [Fal90].

Let $(X, d)$ be a metric space, $U \subseteq X$, and let $s$ be a non-negative number. For any $\delta > 0$ we define

$$\mathcal{H}^s_\delta(U) = \inf \sum_{i=1}^{\infty} (\operatorname{diam} U_i)^s,$$

where $\{U_i\}_{i=1}^{\infty}$ is a cover of $U$ by sets of diameter $\operatorname{diam} U_i \leq \delta$, and the infimum is taken over such covers. As $\delta$ decreases, the family of allowed covers of $U$ is reduced. Therefore, the infimum $\mathcal{H}^s_\delta(U)$ increases, and so approaches a limit as $\delta \to 0$, that we write as

$$\mathcal{H}^s(U) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(U).$$

We call $\mathcal{H}^s(U)$ the *$s$-dimensional Hausdorff measure of $U$*, and it can be proved that it actually is a measure on $X$.

On the other hand, for any given set $U \subseteq X$ and $\delta \leq 1$, $\mathcal{H}^s_\delta(U)$ is non-increasing with $s$, hence so is $\mathcal{H}^s(U)$. In fact, if $t > s$ and $\{U_i\}$ is a cover of $U$ by sets of diameter $\leq \delta$, we have

$$\sum_i (\operatorname{diam} U_i)^t = \sum_i (\operatorname{diam} U_i)^{t-s} (\operatorname{diam} U_i)^s \leq \\ \delta^{t-s} \sum_i (\operatorname{diam} U_i)^s \leq \sum_i (\operatorname{diam} U_i)^s.$$

Also $\mathcal{H}^t_\delta(U) \leq \delta^{t-s} \mathcal{H}^s_\delta(U)$, and letting $\delta \to 0$, we get the following lemma:

**Lemma 1.2.1.** *If $\mathcal{H}^s(U) < \infty$ for some $s$, then $\mathcal{H}^t(U) = 0$ for all $t > s$.*

Thus we see that there is a critical value of $s$ at which $\mathcal{H}^s(U)$ 'jumps' from $\infty$ to 0. This critical value is called the *Hausdorff dimension* of $U$, and written $\dim_X U$. In other words,

$$\dim_X U = \inf\{s \geq 0 \mid \mathcal{H}^s(U) = 0\} = \sup\{s \geq 0 \mid \mathcal{H}^s(U) = \infty\}.$$

### 1.2.2 Hausdorff dimension in countably based profinite groups

Suppose that the profinite group $G$ is countably based i.e. that there exists a descending chain $\{G(n)\}_{n\in\mathbb{N}}$ of open normal subgroups which form a base of neighbourhoods of the identity. This is the case, in particular, if $G$ is (topologically) finitely generated. In this situation, there is a natural metric in $G$, induced by $\{G(n)\}_{n\in\mathbb{N}}$:

$$d(x,y) = \inf\left\{\frac{1}{|G:G(n)|} : x \equiv y \pmod{G(n)}\right\}.$$

This gives $G$ the structure of a metric space and therefore we can compute the Hausdorff dimension of a subset of $G$ with respect to this metric. Note that the topology defined by this metric coincides with the original topology of $G$.

There is a nice formula due to Abercrombie [Abe94] and Barnea-Shalev [BS97] that provides the Hausdorff dimension of an arbitrary closed subgroup $H$ of $G$. Note that it is given in purely algebraic (and analytic) terms, in contrast with the (quite nasty) geometric definition given in the previous subsection:

$$\dim_G H = \liminf_{n\to\infty} \frac{\log_p |HG(n)/G(n)|}{\log_p |G/G(n)|}. \tag{1.2.1}$$

Observe the similarity with the finite case. The finite quotients $G/G(n)$ give approximations of the group $G$, which are better as $n$ increases. In the formula above, we project $H$ in these finite quotients and compute its relative size inside them, which is the quotient $\frac{\log |HG(n)/G(n)|}{\log |G/G(n)|}$. Finally, we take the limit when $n \to \infty$ to see the asymptotic behaviour of these numbers (the $\liminf$ is necessary since the limit need not exist).

In principle, the Hausdorff dimension of a closed subgroup of $G$ depends on the filtration $\{G(n)\}$ used to define the metric of $G$, and there are examples showing that this is so (see [BS97], Example 2.5). In any case, there is usually a natural choice for the system of neighbourhoods of the identity.

For example, for a finitely generated pro-$p$ group, we can take $G(n) = G^{p^n}$, and for the group of automorphisms of the $p$-adic rooted tree, we can consider the chain of the level stabilizers.

In the following proposition we show some easy properties of the Hausdorff dimension in a countably based profinite group.

**Proposition 1.2.2.** *Let $G$ be an infinite countably based profinite group and $H \leq_c G$. Then if we compute Hausdorff dimension with respect to the filtration $\{G(n)\}$, we have*

(i) $\dim_G H \in [0,1]$.

(ii) *If $K \leq_c G$ and $K \leq H$, then $\dim_G K \leq \dim_G H$. If in addition $|H:K| < \infty$, then $\dim_G K = \dim_G H$.*

(iii) *Open subgroups have Hausdorff dimension 1 and finite subgroups 0.*

(iv) *The Hausdorff dimension is invariant under conjugation.*

(v) *If $H$ is a (not necessarily closed) subgroup of $G$, then*

$$\dim_G \overline{H} = \liminf_{n\to\infty} \frac{\log |HG(n)/G(n)|}{\log |G/G(n)|}. \tag{1.2.2}$$

*Proof.* Part (i) and the first assertion of (ii) are clear because the expression in the limit itself does satisfy the corresponding inequalities. For the second part of (ii), note that

$$\begin{aligned}
\dim_G H &= \liminf_{n\to\infty} \frac{\log |HG(n)/G(n)|}{\log |G/G(n)|} \\
&= \liminf_{n\to\infty} \frac{\log (|HG(n)/G(n) : KG(n)/G(n)||KG(n)/G(n)|)}{\log |G/G(n)|} \\
&= \liminf_{n\to\infty} \frac{\log |HG(n)/G(n) : KG(n)/G(n)| + \log |KG(n)/G(n)|}{\log |G/G(n)|}.
\end{aligned}$$

Now $|HG(n)/G(n) : KG(n)/G(n)| = |HG(n) : KG(n)| \leq |H : K|$ is bounded for all $n$ and $\log |G/G(n)| \to \infty$, since $G$ is infinite. Consequently

$$\dim_G H = \liminf_{n\to\infty} \frac{\log |KG(n)/G(n)|}{\log |G/G(n)|} = \dim_G K.$$

Part (iii) follows from (ii) and the trivial observation that $\dim_G G = 1$ and $\dim_G\{1\} = 0$.

For (iv), suppose that $H, K \leq G$ and that there exists $g \in G$ such that $K = H^g$. Then

$$|KG(n)/G(n)| = |H^g G(n)/G(n)| = |(HG(n)/G(n))^g| = |HG(n)/G(n)|,$$

and the result holds.

Finally, as for (v), $HG(n) \leq \overline{H}G(n)$ is obvious. On the other hand, we have $G(n) \leq_o G$, and so $HG(n)$ is open and also closed in $G$. Then $\overline{H} \leq HG(n)$ and $\overline{H}G(n) \leq HG(n)$ as wanted. □

In the same way that the Hausdorff dimension changes with the filtration, it is not invariant under isomorphisms.

**Example 1.2.3.** Consider the $p$-adic rooted tree $\mathcal{T}$, $G = \operatorname{Aut}\mathcal{T}$ and the filtration of the stabilizers $G(n) = \operatorname{Stab}(n)$, and a closed subgroup $H$ of $G$. Let us consider the closed subgroup $K$ of $G$ which is obtained by hanging all the automorphisms of $H$ at the first main subtree of $\mathcal{T}$, and the identity at the rest of the main subtrees. Then $\psi(K) = H \times \{1\} \times \cdots \times \{1\}$, and so $K \cong H$. Also $|KG(n) : G(n)| = |K : \operatorname{Stab}_K(n)| = |H : \operatorname{Stab}_H(n-1)| = |HG(n-1) : G(n-1)|$, and $|G : G(n)| = (p!)^{1+p+\cdots+p^n} = p!|G : G(n-1)|^p$. So if we apply the formula given in (1.2.1), we get

$$\dim_G K = \frac{1}{p}\dim_G H.$$

**Definition 1.2.4.** Having fixed a filtration, the set $\operatorname{Spec}(G) = \{\dim_G H : H \leq_c G\}$ is the *spectrum* of $G$. If we only consider the dimensions corresponding to a particular family $\Sigma$ of subgroups, we speak of the $\Sigma$-*spectrum* of $G$.

The spectrum may be useful if we want to measure the 'complexity' of the subgroup structure of $G$.

**Theorem 1.2.5.** *(Barnea and Shalev, [BS97]) If $G$ is a $p$-adic analytic pro-$p$ group and $H \leq_c G$, and $\dim_G H$ denotes the Hausdorff dimension of $H$ with respect to the chain $G(n) = G^{p^n}$, then*

$$\dim_G H = \frac{\operatorname{Dim} H}{\operatorname{Dim} G},$$

*where $\operatorname{Dim} G$ denotes the dimension of $G$ as a $p$-adic Lie group. Therefore, if we write $d = \operatorname{Dim} G$, then*

$$\operatorname{Spec}(G) \subseteq \left\{0, \frac{1}{d}, \frac{2}{d}, \ldots, \frac{d-1}{d}, 1\right\}$$

*is finite and contains just rational numbers.*

### 1.2.3 Hausdorff dimension in $\Gamma$

Let us consider the profinite group $\operatorname{Aut} \mathcal{T}$ with $d = p$ a prime. We fix a $p$-cycle $\sigma$ and we consider $\Gamma$, the set of all automorphisms of $\mathcal{T}$ whose portrait only contains powers of $\sigma$. By Theorem 1.1.6 and Lemma 1.1.7, $\Gamma$ is a countably based pro-$p$ group.

The determination of the Hausdorff dimension of closed subgroups of $\Gamma$ has received special attention in the last few years (see [AV05, Sie08, Šun07]). The most natural choice is to calculate the Hausdorff dimension with respect to the metric induced by the filtration of $\Gamma$ given by the level stabilizers $\operatorname{Stab}_\Gamma(n)$.

Observe that as a consequence of Theorem 1.1.6 and Lemma 1.1.7 again, $\Gamma$ is the inverse limit of the finite $p$-groups $\Gamma_n$, and $\Gamma_n$ can be seen as a Sylow $p$-subgroup of the symmetric group on $p^n$ letters. In particular,

$$|\Gamma_n| = p^{(p^n-1)/(p-1)}.$$

If $H$ is a subgroup of $\Gamma$, it readily follows from (1.2.2) that

$$\dim_\Gamma \overline{H} = \liminf_{n\to\infty} \frac{\log_p |H_n|}{\log_p |\Gamma_n|} = (p-1) \liminf_{n\to\infty} \frac{\log_p |H_n|}{p^n}. \tag{1.2.3}$$

**Lemma 1.2.6.** *Let $H$ and $J$ be two subgroups of $\Gamma$ that are conjugate in* $\operatorname{Aut}\mathcal{T}$. *Then* $\dim_\Gamma \overline{H} = \dim_\Gamma \overline{J}$.

*Proof.* If $J = H^f$ with $f \in \operatorname{Aut}\mathcal{T}$, then $\overline{J} = \overline{H}^f$ and the result follows by the same proof as for the one given for (iv) of Proposition 1.2.2. (Note that we can not apply that result directly because in the present case the dimension is computed in $\Gamma$, and the subgroups are conjugate in $\operatorname{Aut}\mathcal{T}$.) $\square$

In the following theorem, we give an important result concerning the spectrum of $\Gamma$, proved by Abért and Virág in [AV05].

**Theorem 1.2.7.** *For every $\lambda \in [0,1]$, there exists $H \leq_c \Gamma$ (topologically) finitely generated by 3 elements such that* $\dim_\Gamma H = \lambda$. *Therefore,* $\operatorname{Spec}(\Gamma) = [0,1]$.

The methods used to prove this result are probabilistic and they do not give specific examples of 3-generated subgroups of an arbitrary Hausdorff dimension.

The problem is that, as is also proved in [AV05], subgroups of $\Gamma$ generated by three random elements have Hausdorff dimension 1 with probability 1. Obviously, this is an obstacle if we are interested in finding concrete examples.

As a first approximation, Siegenthaler [Sie08] succeeded in showing that there are subgroups of $\Gamma$ (when $p = 2$), among a family of subgroups that we call spinal groups, of transcendental Hausdorff dimension. See Chapter 2 for more information on spinal groups.

## 1.3 Circulant matrices over $\mathbb{F}_p$

Given a vector $\mathbf{a} = (a_1, \dots, a_n)$, we write $C(\mathbf{a})$ to denote the circulant matrix generated by $\mathbf{a}$, i.e. the matrix of size $n \times n$ whose first row is $\mathbf{a}$, and every other row is obtained from the previous one by applying a shift of length one to the right. In other words, the entries of $C(\mathbf{a})$ are $c_{ij} = a_{j-i+1}$, where $a_k$ is defined for every integer $k$ by reducing $k$ modulo $n$ to a number between 1 and $n$.

To conclude with this preliminary chapter, we give a lemma about the rank of circulant matrices. Part (i) can be useful in order to compute the rank of circulant matrices in particular examples, and (ii) and (iii) will be very useful in the proofs of several results in Chapters 3 and 4.

**Lemma 1.3.1.** *Let $p$ be a prime, and let $(a_0, \dots, a_{p-1}) \in \mathbb{F}_p^p$ be a non-zero vector. If $C = C(a_0, \dots, a_{p-1})$, then:*

(i) $\operatorname{rk} C = p - m$, *where $m$ is the multiplicity of 1 as a root of the polynomial $a(X) = a_0 + a_1 X + \cdots + a_{p-1} X^{p-1}$. As a consequence, we have $\operatorname{rk} C < p$ if and only if $\sum_{i=0}^{p-1} a_i = 0$.*

(ii) *If $\mathbf{1}$ denotes the column vector of length $p$ all of whose entries are equal to 1, then*

$$\operatorname{rk} C = \operatorname{rk} (C \mid \mathbf{1}) .$$

(iii) *The first $\operatorname{rk} C$ rows (columns) of $C$ are linearly independent.*

*Proof.* If we consider the quotient ring $V = \mathbb{F}_p[X]/(X^p - 1)$ as an $\mathbb{F}_p$-vector space, then both

$$\mathcal{B} = \{\overline{1}, \overline{X}, \dots, \overline{X^{p-1}}\}$$

and

$$\mathcal{B}' = \{\overline{1}, \overline{X-1}, \dots, \overline{(X-1)^{p-1}}\}$$

are bases of $V$. Multiplication by $\overline{a(X)}$ defines a linear map $\varphi : V \to V$, and the matrix of $\varphi$ with respect to $\mathcal{B}$ is $C$ (we construct the matrix by rows). Thus $\operatorname{rk} C = \operatorname{rk} \varphi$.

On the other hand, we can write $a(X) = (X-1)^m b(X)$, with $b(X) \in \mathbb{F}_p[X]$ and $b(1) \neq 0$. Let $b(X) = b_0 + b_1(X-1) + \cdots + b_{k-1}(X-1)^{k-1}$, where $k = p - m$ and $b_0 \neq 0$. Then the matrix of $\varphi$ with respect to $\mathcal{B}'$ is the block matrix

$$\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}, \quad \text{where} \quad B = \begin{pmatrix} b_0 & b_1 & \cdots & b_{k-2} & b_{k-1} \\ 0 & b_0 & \cdots & b_{k-3} & b_{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & b_0 \end{pmatrix}, \tag{1.3.1}$$

since $\overline{(X-1)^i} = \overline{0}$ in $V$ for all $i \geq p$. Thus

$$\operatorname{im}\varphi = \langle \overline{(X-1)^m}, \dots, \overline{(X-1)^{p-1}} \rangle, \tag{1.3.2}$$

and we also have $\operatorname{rk}\varphi = k$, from which (i) follows.

Let us now prove (ii). We first prove that

$$\operatorname{rk} C = \operatorname{rk} \begin{pmatrix} C \\ 1 \dots 1 \end{pmatrix}. \tag{1.3.3}$$

Since $C$ is the matrix of $\varphi$ with respect to $\mathcal{B}$ constructed by rows, it is clear that (1.3.3) is equivalent to $\overline{1 + X + \cdots + X^{p-1}}$ lying in the image of $\varphi$. Note that, since we are working with coefficients in $\mathbb{F}_p$, we have

$$1 + X + \cdots + X^{p-1} = (X-1)^{p-1},$$

and from (1.3.2), it follows that $\overline{(X-1)^{p-1}} \in \operatorname{im}\varphi$, as desired (since $(a_0, \dots, a_{p-1})$ is a non-zero vector, then $1 \leq \operatorname{rk} C \leq p$, and hence $0 \leq m \leq p-1$).

Now, since the transpose $C^t$ of $C$ is also a circulant matrix, we can apply (1.3.3) to $C^t$ and get

$$\operatorname{rk} C = \operatorname{rk} C^t = \operatorname{rk} \begin{pmatrix} C^t \\ 1 \dots 1 \end{pmatrix} = \operatorname{rk} \left( C \mid \mathbf{1} \right)^t = \operatorname{rk} \left( C \mid \mathbf{1} \right).$$

For the last part of the lemma, it suffices to prove that the first $k$ rows of $C$ are linearly independent: observe that the first $k$ columns of $C$ are the first $k$ rows of $C^t$, and we may apply the result for the rows of the circulant matrix $C^t$, and get the result for the columns of $C$. In (1.3.1), we see that the first $k$ rows of the matrix of $\varphi$ with respect to $\mathcal{B}'$ are linearly independent. Since $k$ is the full rank, this is equivalent to

$$\operatorname{im}\varphi = \langle \varphi(\overline{1}), \varphi(\overline{(X-1)}), \ldots, \varphi(\overline{(X-1)^{k-1}}) \rangle.$$

At the same time,

$$\begin{aligned}\operatorname{im}\varphi &= \langle \varphi(\overline{1}), \ldots, \varphi(\overline{(X-1)^{k-1}}) \rangle = \varphi\left( \langle \overline{1}, \ldots, \overline{(X-1)^{k-1}} \rangle \right) \\ &= \varphi\left( \langle \overline{1}, \overline{X}, \ldots, \overline{X^{k-1}} \rangle \right) = \langle \varphi(\overline{1}), \varphi(\overline{X}), \ldots, \varphi(\overline{X^{k-1}}) \rangle\end{aligned}$$

tells us that the first $k$ rows of the matrix of $\varphi$ with respect to $\mathcal{B}$ are linearly independent, i.e. the first $k$ rows of $C$ are linearly independent, as desired. □

*Notation.* The $i$th row and $j$th column of a matrix $C$ are denoted by $C_i$ and $C^j$, respectively.

# Chapter 2

# Hausdorff dimension of spinal groups

## 2.1 Introduction

Let $p$ be a prime, $\mathcal{T}$ the $p$-adic rooted tree, and $\Gamma$ the Sylow pro-$p$ subgroup corresponding to the $p$-cycle $(1\ \ldots\ p)$. Klopsch showed in [Klo99, Chapter VIII, Section 5] that the spectrum of all profinite branch groups is the full interval $[0,1]$, and this applies in particular to $\Gamma$. (See Section 1.1.4 for the definition and [Gri00] for the basic theory of branch groups.) Later, Abért and Virág [AV05, Theorem 2] proved that every value $\lambda \in [0,1)$ can be obtained as the Hausdorff dimension of a closed subgroup of $\Gamma$ which can be (topologically) generated by at most 3 elements. However, the probabilistic nature of their arguments does not provide explicit examples for every possible $\lambda$, and more specifically any examples for irrational $\lambda$. In the same paper, they also show that soluble subgroups of $\Gamma$ have dimension 0 (see the remark after Theorem 5). On the other hand, Bartholdi has proved [Bar06, Proposition 2.7] that a regular branch subgroup of $\Gamma$ has positive rational Hausdorff dimension (see Subsection 1.1.4 for the definition of regular branch groups).

In the recent paper [Sie08], Siegenthaler has considered the case $p = 2$, and has provided an explicit formula for the Hausdorff dimension of the closures of a special family of discrete subgroups of $\Gamma$, usually called *spinal groups* in the literature. As a consequence, he finds 3-generator spinal groups whose closure has irrational, even transcendental, Hausdorff dimension in $\Gamma$.

Spinal groups can be given in the form $\langle a, B\rangle$, where $a$ is the rooted automorphism corresponding to $(1\ 2\ \dots\ p)$, and where $B$ is an elementary abelian finite $p$-group consisting of automorphisms whose action is concentrated on a special subset of vertices of $\mathcal{T}$, which we call a spine. We refer the reader to Section 2.2 for details about spinal groups. In particular, spinal groups are branch if $p > 2$, but not necessarily regular branch.

The key ingredient for the construction of spinal groups is to consider a sequence $\Omega = (\omega_n)_{n\geq 1}$ of linear functionals of a finite-dimensional vector space $E$ over $\mathbb{F}_p$. We write $\mathrm{Spinal}(\Omega)$ for the spinal group $G$ constructed from $\Omega$. One of our main results is the determination, for $p > 2$, of a formula for the Hausdorff dimension in $\Gamma$ of the closure $\overline{G}$, in terms of the sequence $\Omega$.

**Theorem A.** *Let $G = \mathrm{Spinal}(\Omega)$ be a spinal group, where $p > 2$. Then:*

(i) *If $\omega_i = 0$ for some $i$, then $\dim_\Gamma \overline{G} = 0$.*

(ii) *If $\omega_i \neq 0$ for all $i$, let $m$ be the dimension of the subspace of $E^*$ generated by $\Omega$. For $n$ big enough and for every $i = 1, \dots, m$, let $r_{n,i}$ be the minimum number of terms of the sequence $(\omega_{n-1}, \dots, \omega_1)$, in that order, that are needed to generate a subspace of dimension $i$. Then,*

$$\dim_\Gamma \overline{G} = (p-1) \liminf_{n\to\infty} \left( \frac{1}{p^{r_{n,1}}} + \frac{1}{p^{r_{n,2}}} + \cdots + \frac{1}{p^{r_{n,m}}} \right).$$

By using Theorem A, we are able to determine the set of all values that are taken by the Hausdorff dimension for the family $\Sigma$ of the closures of all spinal subgroups of $\Gamma$. In other words, we calculate the $\Sigma$-spectrum of $\Gamma$, to which we refer as the *spinal spectrum*.

**Theorem B.** *If $p$ is odd, then the spinal spectrum of $\Gamma$ consists of $0$ and all numbers whose $p$-adic expansion is of the form $0.a_1 \dots a_n$, where*

(i) $a_i = 0$ *or* $p-1$ *for every* $i = 1, \ldots, n$.

(ii) $a_1 = p-1$.

*In particular, the spinal spectrum is contained in* $\mathbb{Q}$.

Thus, the situation for odd primes is dramatically different from that of the even prime. Note also that Theorems A and B generalize to all spinal groups (in the case of odd primes) a result of Šunić [Šun07, Theorem 2] dealing with a special class of spinal groups, for which the Hausdorff dimension $\lambda$ is always of the form $\lambda = 0.a_1 \ldots a_n$ in base $p$, with all $a_i$ equal to $p-1$. A particular case, for $p = 3$, of the groups considered by Šunić is the so-called Fabrykowski-Gupta group. The Hausdorff dimension of this group in $\Gamma$ had been previously calculated by Bartholdi and Grigorchuk in [BG02]; according to Corollary 6.6 in there, the dimension is $2/3$, in agreement with Theorem A.

On the other hand, we want to point out that our proof of Theorem B is constructive, in the sense that it provides an algorithm which, given a number $\lambda$ whose $p$-adic expansion is of the appropriate type, yields a spinal group of Hausdorff dimension equal to $\lambda$.

For the proof of Theorem A, we need to calculate the orders of the quotient groups $G_n = G/\operatorname{Stab}_G(n)$ for every $n$. This is achieved in two steps: first, in Section 2.4, we get these orders for 2-generator spinal groups; and then, in Section 2.5, we obtain the formula for the general case. The key for this transition from 2-generator to arbitrary spinal groups is given by a general result about semidirect product decompositions of spinal groups. We think that these decompositions may have an independent interest, broader than just for the determination of the Hausdorff dimension. The result is valid for all primes, and reads as follows.

**Theorem C.** *Let* $G = \langle a, B \rangle$ *be a spinal group. Then, for every subgroup* $B_2$ *of* $B$, *there exists a complement* $B_1$ *in* $B$ *such that* $G = \langle a, B_1 \rangle \ltimes B_2^G$. *In particular, if* $B_2$ *is a maximal subgroup of* $B$, *then the normal closure* $B_2^G$ *has a complement in* $G$ *which is a* 2*-generator spinal group.*

As a matter of fact, if $G$ is constructed from a sequence $\Omega$ of linear functionals, then it is possible to give an explicit choice of $B_1$ in terms of $B_2$ and $\Omega$; details are given in Section 2.3.

In this chapter, we deal (as in many cases in the literature) with spinal groups for which the corresponding spines have only one vertex at every level. However, it is also interesting to allow more than one vertex, and this class of spinal groups (which we call *multi-edge spinal groups*) have also received special attention. For example, several well-known and relevant groups, such as the Grigorchuk and Gupta-Sidki groups, fall within this family. In Chapter 3 we study the Hausdorff dimension of the so-called *GGS-groups*, an important type of 2-generator multi-edge spinal groups which is modelled by the second Grigorchuk group and the Gupta-Sidki group. Again, we work with the $p$-adic rooted tree for an odd prime $p$. In that setting, we determine the Hausdorff dimension of all GGS-groups.

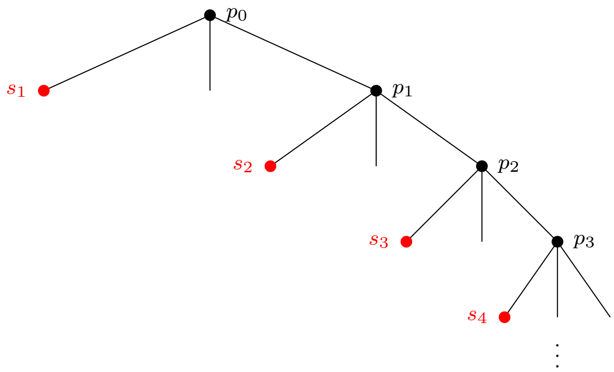
This chapter is adapted from of the already published paper [FAZR11], written by the author and her advisor.

*Notation.* In this chapter, we use brackets to enclose every countable collection of elements for which it is important to know how the elements are ordered. Thus we write sequences (finite or infinite), and bases of a finite-dimensional space with brackets. Of course, ordinary sets are written by using curly brackets.

## 2.2 Basic theory of spinal groups

Spinal groups have been considered in a number of research papers ([BGŠ03, BŠ01, Gri00] are basic references), sometimes with small differences in the definitions. Particular attention has been devoted to the case when the spine has one vertex at every level, and when the spinal group is contained in $\Gamma$. We also work under these hypotheses here but, as will be explained below, our approach is more general than in previous accounts. Because of this greater generality, it is convenient to give an exposition of the basic theory of spinal groups in some detail, and we do so in this section.

Let $P = (p_n)_{n\geq 0}$ be an infinite path in $\mathcal{T}$ beginning at the root. If we consider, for every $n \geq 1$, an immediate descendant $s_n$ of $p_{n-1}$ not lying in $P$, we say that the sequence $S = (s_n)_{n\geq 1}$ is a *spine* of $\mathcal{T}$.




A spine $(s_n)_{n\geq 1}$ (in red) in the 3-adic rooted tree, associated to the path $(p_n)_{n\geq 0}$.

An automorphism $b \in \Gamma$ is said to be *spinal* over $S$ if its support is contained in $S$. Note that we do not exclude the possibility that $b$ has a trivial label at a vertex of $S$. It is clear from (1.1.1) that a non-trivial spinal automorphism has order $p$, and that any two spinal automorphisms defined over the same spine commute. A *spinal group* defined over $S$ is a subgroup $G = \langle a, B \rangle$ of $\Gamma$, where $B$ is a finite subgroup consisting of spinal automorphisms corresponding to $S$, and $a$ is the rooted automorphism corresponding to $(1 \ldots p)$. Note that $B$ is an elementary abelian $p$-group, and so it can be seen as a vector space over $\mathbb{F}_p$. Obviously, $|G : G'| \leq p\,|B|$ is finite. In the sequel, we write $A$ to denote the subgroup generated by $a$.

If $G = \langle A, B \rangle$ is a spinal group over $S$, then $G$ is completely determined by the labels of every $b \in B$ at all vertices $s_n \in S$. Since $b \in \Gamma$, all these labels are powers of the cycle $(1 \ldots p)$. Then we can write

$$b_{(s_n)} = (1 \ldots p)^{\omega_n(b)}$$

for some exponent $\omega_n(b)$, which can be seen as an element of $\mathbb{F}_p$. This way, we obtain a sequence of *label maps* $\omega_n : B \to \mathbb{F}_p$, which are actually homomorphisms (or linear functionals), i.e. elements of the dual space $B^*$. Clearly, we have

$$\bigcap_{n\in\mathbb{N}} \ker \omega_n = 1. \tag{2.2.1}$$

Conversely, if we want to *construct* a spinal group over $S$, then we can choose

(i) a finite-dimensional vector space $E$ over $\mathbb{F}_p$, and

(ii) a sequence $\Omega = (\omega_n)_{n\in\mathbb{N}}$ of elements of the dual space $E^*$.

Then we can define a representation $\mathfrak{X} : E \longrightarrow \Gamma$ by assigning to each $e \in E$ the spinal automorphism labelled with the permutation

$$(1 \ldots p)^{\omega_n(e)}$$

at the vertex $s_n$, for every $n \geq 1$, and with 1 otherwise. If we put $B = \mathfrak{X}(E)$, then $G = \langle A, B\rangle$ is a spinal group, which we denote by $\mathrm{Spinal}(\Omega)$. (Note that the vector space $E$ is implicit in $\Omega$.) If $\mathfrak{X}$ is faithful, then we can identify $B$ with $E$, and then the label maps of $B$ can be identified with the sequence $\Omega$.

Clearly, all spinal groups arise by using this construction. Observe that our definition is more general than some others in the literature, in that we do not require that $\omega_n \neq 0$ for all $n \geq 1$ or that $\mathfrak{X}$ should be faithful (that is, that the sequence $\Omega$ should fulfil condition (2.2.1)). As we will see in Proposition 2.2.5, if $\omega_n = 0$ for some $n$, then $G$ is finite. On the other hand, if $\mathfrak{X}$ is not faithful, it suffices to replace $E$ with $E/\ker\mathfrak{X}$ in order to get a faithful representation. Thus, it seems that the larger degree of freedom of our definition does not make a big difference. However, when one is trying to prove results about spinal groups, it does. As will become clear below, if $G = \mathrm{Spinal}(\Omega)$ is a spinal group, one has usually to deal also with groups of the form $\widehat{G} = \mathrm{Spinal}(\widehat{\Omega})$, where $\widehat{\Omega} = (\omega_n)_{n\geq t+1}$ for some integer $t \geq 0$. According to our definition, these are all again spinal groups. However, if we require that the representation should be faithful, then we need the condition

$$\bigcap_{n \geq t+1} \ker \omega_n = 1$$

for all $t$, and not only (2.2.1). This is a restriction which is usually found in the literature. Of course, $\widehat{G}$ can be written via a faithful representation by changing $E$, but we want both $G$ and $\widehat{G}$ to come *from the same* $E$. On the other hand, it also turns out that it is better not to banish the groups with some $\omega_n$ equal to 0 from the realm of spinal groups. This way, we may always assert that the complement $\langle A, B_1 \rangle$ of Theorem C is again a spinal group. (See the remark after Corollary 2.3.3.) As a matter of fact, as we will see in Sections 2.4 and 2.5, the key to the determination of the Hausdorff dimension of a spinal group is the examination of where the linear functionals become 0 in $\langle A, B_1 \rangle$, when this complement is a 2-generator spinal group.

Spines and spinal groups can be defined in the same way over the truncated trees $\mathcal{T}_n$. In that case, we use sequences of linear functionals of length $n-1$, instead of infinite sequences. For every spinal group $G = \mathrm{Spinal}(\Omega)$ defined over $\mathcal{T}$, with corresponding representation $\mathfrak{X} : E \longrightarrow \Gamma$, the group $G_n$ acting on $\mathcal{T}_n$ is also spinal, via the representation $\mathfrak{X}_n : E \longrightarrow \Gamma_n$ which is naturally induced from $\mathfrak{X}$. Thus if we write $\Omega_n = (\omega_1, \dots, \omega_{n-1})$, then we have $G_n = \mathrm{Spinal}(\Omega_n)$. All spinal subgroups of $\mathrm{Aut}\, \mathcal{T}_n$ arise in this way, as explained in Subsection 1.1.3 of the Preliminaries. More precisely, assume that $L$ is spinal over $\mathcal{T}_n$, defined via a sequence $\Phi$. Let $\mathrm{ext}(\Phi)$ denote the sequence which is obtained by extending $\Phi$ with an infinite sequence of zeros (the zero linear functional). Then we can identify $L$ with $H_n$, where $H = \mathrm{Spinal}(\mathrm{ext}(\Phi))$.

In the literature, spinal groups usually appear associated to the particular spine $U$ consisting of the vertices $u_n = p \overset{n-1}{\ldots} p 1$, for $n \geq 1$. Our first theorem shows that, as far as Hausdorff dimension is concerned, this is not a real restriction.

**Theorem 2.2.1.** *Let $G$ be a spinal group. Then $G$ is conjugate in* $\operatorname{Aut}\mathcal{T}$ *to a spinal group $J$ defined over $U$. Hence,* $\dim_\Gamma \overline{G} = \dim_\Gamma \overline{J}$.

*Proof.* Let $G = \langle A, B \rangle$ be defined over the spine $S = (s_n)_{n\geq 1}$, and let $P = (p_n)_{n\geq 0}$ be the path corresponding to $S$. Let us write $p_{n+1} = p_n x_n$ and $s_{n+1} = p_n y_n$ for every $n \geq 0$, with $x_n, y_n \in X$. For every $n \geq 0$, let $\sigma_n \in S_p$ be defined by means of the following two conditions:

(i) $\sigma_n(x_n) = p$ and $\sigma_n(y_n) = 1$.

(ii) $(1 \, \dots \, p)^{\sigma_n}$ is a power of $(1 \, \dots \, p)$.

Observe that there is one (in fact, only one) such permutation in $S_p$: since

$$(1 \, \dots \, p)^{\sigma_n} = (\sigma_n(1) \, \dots \, \sigma_n(p))$$

and the positions of 1 and $p$ in this last tuple are predetermined by the images of $x_n$ and $y_n$, there is only one way to choose the rest of the images if we want to obtain a power of $(1 \, \dots \, p)$.

Now consider the automorphism $f$ of $\mathcal{T}$ having the label $\sigma_n$ at all vertices of the $n$th level, for every $n \geq 0$. Observe that, according to condition (i) of the definition of $\sigma_n$, we have $f(S) = U$. If $b \in B$, then it follows from (1.1.2) that $(b^f)_{(f(v))}$ is the conjugate of $b_{(v)}$ by $\sigma_n$ for all $v \in X^n$. By condition (ii), and since $b$ is spinal over $S$, we deduce that $b^f$ is spinal over $U$. For the same reason, $a^f$ is a non-trivial power of $a$. Thus $G^f = \langle a^f, B^f \rangle = \langle a, B^f \rangle$ is a spinal group over $U$, and we can take $J = G^f$.

Finally, by Lemma 1.2.6 we conclude that $\dim_\Gamma \overline{G} = \dim_\Gamma \overline{J}$. □

Note that, if $G$ is defined by using a sequence $\Omega$, the sequence $\Omega'$ corresponding to the group $J$ in the last theorem is not necessarily equal to $\Omega$: we have $\omega'_n = m_n \omega_n$ for all $n \geq 1$, where $m_n$ is the exponent such that $(1 \dots p)^{\sigma_n} = (1 \dots p)^{m_n}$. In the sequel, all spinal groups considered are defined over the particular spine $U$.

In studying a spinal group $G = \langle A, B \rangle$, it is usually necessary to work with the sections of the elements of $G$ at a particular vertex of the tree.

For example, if $b \in B$, then the section $\widehat{b}$ at the vertex $p.\overset{n}{\ldots}.p$ is again a spinal automorphism. If $G$ is constructed from a sequence $\Omega = (\omega_i)_{i\geq 1}$, and $\mathfrak{X} : E \longrightarrow \Gamma$ is the corresponding representation, then we can write $b = \mathfrak{X}(e)$ for some $e \in E$, and we have $\widehat{b} = \widehat{\mathfrak{X}}(e)$, where $\widehat{\mathfrak{X}}$ is the representation of $E$ associated to the sequence $\widehat{\Omega} = (\omega_i)_{i\geq n+1}$. Thus $\widehat{b}$ lies in the spinal group $\widehat{G} = \mathrm{Spinal}(\widehat{\Omega})$, which is constructed from the same abstract vector space as $G$, but with a different representation.

More generally, the section of $b \in B$ at any vertex $v$ of the $n$th level is

$$b_v = \begin{cases} a^{\omega_n(b)}, & \text{if } v = p\overset{n-1}{\ldots}p1, \\ \widehat{b}, & \text{if } v = p.\overset{n}{\ldots}.p, \\ 1, & \text{otherwise,} \end{cases} \tag{2.2.2}$$

and again $b_v$ belongs to $\widehat{G}$.

At this point, it is convenient to introduce the following notation.

**Definition 2.2.2.** If $\Omega = (\omega_i)_{i\geq 1}$ is a sequence and $n \geq 0$ is an integer, then the sequence $\sigma^n\Omega = (\omega_i)_{i\geq n+1}$ is called the *shift* of $\Omega$ of step $n$.

The first statement of the next proposition is a direct consequence of the formula for the section of a composition. On the other hand, the two properties of part (ii) can be proved simultaneously by induction on $n$. (See the proof of Theorem 5 in [Gri00], and the paragraph before Lemma 4.1 in [BŠ01].)

**Proposition 2.2.3.** *Let $G = \mathrm{Spinal}(\Omega)$ be a spinal group, and let $v$ be a vertex in the $n$th level of $\mathcal{T}$. Let us write $\psi_v$ for the map sending each $g \in G$ to $g_v$, and $\widehat{G} = \mathrm{Spinal}(\sigma^n\Omega)$. Then:*

(i) *$\psi_v(G)$ is contained in $\widehat{G}$.*

(ii) *If $\omega_1, \ldots, \omega_n$ are all different from 0, then $\psi_v(\mathrm{Stab}_G(n)) = \widehat{G}$, and $G$ acts transitively on the $n+1$-st level of the tree $\mathcal{T}$.*

**Remark 2.2.4.** If we write $\widehat{B}$ for $\widehat{\mathfrak{X}}(E)$, then we have $\widehat{G} = \langle A, \widehat{B} \rangle$. Now, even if $\widehat{B} = \{\widehat{b} \mid b \in B\}$, it is erroneous to write $\widehat{G} = \{\widehat{g} \mid g \in G\}$. First of all, note that we have only defined $\widehat{g}$ for $g \in B$, and that it is not clear how we should define $\widehat{g}$ for an arbitrary $g \in G$. After all, every section $g_v$ with $v \in X^n$ lies in $\widehat{G}$ and, contrary to the case of an element of $B$, there is no special reason to choose a particular section as $\widehat{g}$ instead of the others. On the other hand, and more importantly, if $\omega_n = 0$ then it follows from (1.1.1) that every $g \in G$ has a trivial label at all vertices of $X^n$. Thus the sections $g_v$ with $v \in X^n$ all lie in $\mathrm{Stab}_{\widehat{G}}(1)$ and cannot cover the whole of $\widehat{G}$.

As we next see, the condition $\omega_n = 0$ has strong effects on $G$.

**Proposition 2.2.5.** *Let $G = \mathrm{Spinal}(\Omega)$ be a spinal group, and suppose that $\omega_n = 0$ for some $n \geq 1$. Then $G$ is finite.*

*Proof.* Let $g \in G$, and let $v \in X^n$. If $g = a_1 b_1 \dots a_k b_k$ with $a_i \in A$ and $b_i \in B$, then we have $g_v = (b_1)_{v_1} \dots (b_k)_{v_k}$ for some $v_i \in X^n$. Since $\omega_n = 0$, it follows from (2.2.2) that $(b_i)_{v_i} = 1$ or $\widehat{b}_i$ for every $i = 1, \dots, k$. Thus $g_v \in \widehat{B}$, and we can consider the injective map

$$\begin{array}{cccc} \psi_n & : & \mathrm{Stab}_G(n) & \longrightarrow & \widehat{B} \times \overset{p^n}{\cdots} \times \widehat{B} \\ & & g & \longmapsto & (g_v)_{v \in X^n}. \end{array}$$

Thus $|\mathrm{Stab}_G(n)| \leq |\widehat{B}|^{p^n}$ is finite. Since $|G : \mathrm{Stab}_G(n)|$ is also finite, we conclude that $G$ is finite. $\square$

Our next proposition deals with the case in which all linear functionals $\omega_n$ are non-trivial. See Section 1.1.4 for the definition of a branch group.

**Proposition 2.2.6.** *Let $G = \mathrm{Spinal}(\Omega)$ be a spinal group, and suppose that $p > 2$ and $\omega_n \neq 0$ for all $n \geq 1$. Then $G$ is a branch group.*

*Proof.* By Proposition 2.2.3, we know that $G$ satisfies condition (i) in the definition of a branch group. Let us see that also (ii) holds. Put $\widehat{G}^{(n)} = \mathrm{Spinal}(\sigma^n \Omega)$ for every $n \geq 1$ and, for simplicity, write $\widehat{G}$ instead of $\widehat{G}^{(1)}$.

Note that $\psi_n(\mathrm{Stab}_G(n)) \subseteq \widehat{G}^{(n)} \times \overset{p^n}{\cdots} \times \widehat{G}^{(n)}$, again by Proposition 2.2.3. We are going to show that

$$(\widehat{G}^{(n)})' \times \overset{p^n}{\cdots} \times (\widehat{G}^{(n)})' \subseteq \psi_n(G' \cap \mathrm{Stab}_G(n)). \tag{2.2.3}$$

Since $|\widehat{G} : \widehat{G}'|$ is finite, it will follow from here that $G$ is a branch group.

Let us then prove (2.2.3), by induction on $n$. Consider first the case $n = 1$. Since $\widehat{G} = \langle a, \widehat{b} \mid b \in B \rangle$, we have $\widehat{G}' = \langle [a, \widehat{b}]^h \mid b \in B, h \in \widehat{G} \rangle$. There exists $c \in B$ such that $\omega_1(c) = 1$, since $\omega_1 \neq 0$. Then $\psi(c) = (a, 1, \ldots, 1, \widehat{c})$, and consequently $\psi([c, b^a]) = ([a, \widehat{b}], 1, \ldots, 1)$. (We need $p > 2$ for this.) Now, by (ii) of Proposition 2.2.3, for every $h \in \widehat{G}$ we can find $f \in \mathrm{Stab}_G(1)$ such that the first component of $\psi(f)$ is $h$. Thus, if $g = [c, b^a]^f \in G'$, we have $\psi(g) = ([a, \widehat{b}]^h, 1, \ldots, 1)$. By considering conjugates of the form $g^{a^i}$, we can place the element $[a, \widehat{b}]^h$ in any other component of the tuple, and the rest of the components will be 1. This proves that the whole direct product $\widehat{G}' \times \overset{p}{\cdots} \times \widehat{G}'$ lies in the image of $G'$ under $\psi$.

Consider now the general case of the induction. Given a tuple $(h_1, \ldots, h_{p^n})$ in $(\widehat{G}^{(n)})' \times \overset{p^n}{\cdots} \times (\widehat{G}^{(n)})'$, we have to see that it is the image under $\psi_n$ of some element in $G'$. By the inductive hypothesis, there exist elements $f_1, \ldots, f_p$ in $\widehat{G}'$ such that

$$\psi_{n-1}(f_i) = (h_{(i-1)p^{n-1}+1}, \ldots, h_{ip^{n-1}}),$$

for all $i = 1, \ldots, p$. Now, by the case $n = 1$, we can find $g \in G'$ such that $\psi(g) = (f_1, \ldots, f_p)$. Hence

$$\psi_n(g) = (h_1, \ldots, h_{p^n}),$$

which completes the induction, and the proof of the proposition. □

Clearly, spinal groups admit a natural decomposition as a semidirect product.

**Proposition 2.2.7.** *Let $G = \langle A, B \rangle$ be a spinal group. Then $G = A \ltimes B^G$, and $B^G = \mathrm{Stab}_G(1) = \langle b^{a^i} \mid b \in B,\ i = 0, \ldots, p-1 \rangle$.*

If $G = \langle A, B\rangle$ is a spinal group and $g = a_1b_1 \dots a_kb_k$ is an element of $G$, with $a_i \in A$ and $b_i \in B$, then Bartholdi and Šunić have shown [BŠ01, Lemma 4.7] that the product $b_1 \dots b_k$ is independent of the factorization of $g$. In the following lemma we present an alternative proof of this result, and we generalize it in the case of an element $g \in \mathrm{Stab}_G(1)$ to show that, given any decomposition $g = b_1^{a_1} \dots b_k^{a_k}$, it is not only the product $b_1 \dots b_k$ which is well-defined, but also the product of those $b_i$ that appear conjugated by the same element of $A$.

**Lemma 2.2.8.** *Let $G = \langle A, B\rangle$ be a spinal group. Then:*

(i) *The map $p_G : G \longrightarrow B$ given by*

$$p_G(a_1b_1 \dots a_kb_k) = b_1 \dots b_k$$

*is a well-defined homomorphism.*

(ii) *For every $i \in \{0, \dots, p-1\}$, the map $p_G^i : \mathrm{Stab}_G(1) \longrightarrow B$ given by*

$$p_G^i(b_1^{a_1} \dots b_k^{a_k}) = \prod_{j=1}^{k} b_j^{\varepsilon_{ij}},$$

*where*

$$\varepsilon_{ij} = \begin{cases} 1, & \text{if } a_j = a^i, \\ 0, & \text{otherwise,} \end{cases}$$

*is a well-defined homomorphism.*

*Proof.* (i) First of all, observe that an automorphism $b \in B$ has at most one non-trivial label in every level, and thus, by (1.1.3), $p_n(b)$ is nothing but the value of that label. Consequently, $b$ is completely determined by the sequence $(p_n(b))_{n\geq1}$. In our situation, if $g = a_1b_1 \dots a_kb_k$, then we have $p_n(g) = p_n(b_1 \dots b_k)$ for all $n \geq 1$, since $p_n$ is a homomorphism and $p_n(a) = 1$. Since $p_n(g)$ only depends on $g$, this means that the product $b_1 \dots b_k$ is independent of the factorization of $g$.

(ii) This can be proved as in (i), by using the product maps $p_1^{i+1}$ and $p_n^i$ for $n \geq 2$, where $i$ and $i + 1$ are to be reduced to the interval $[1, p]$ modulo $p$. □

## 2.3 Spinal groups as semidirect products

In Proposition 2.2.7 of the previous section, we have seen that a spinal group $G = \langle A, B \rangle$ can be decomposed as a semidirect product in the form $A \ltimes B^G$. On the other hand, Bartholdi and Šunić [BŠ01, Proposition 4.9] have given a decomposition of the form $\langle A, B_1 \rangle \ltimes B_2^G$, where $B_1$ is a particular subgroup of index $p$ in $B$, and $B_2$ is an arbitrary complement of $B_1$ in $B$. The goal of this section is to provide a broad generalization of these facts by showing that, for every subgroup $B_2$ of $B$, we can always find an appropriate complement $B_1$ in $B$ such that the semidirect product decomposition $G = \langle A, B_1 \rangle \ltimes B_2^G$ holds. This result will be given in Theorem 2.3.2, and it will allow us to reduce the study of the Hausdorff dimension of spinal groups to the case of 2-generator groups.

If we want to produce semidirect product decompositions as those above for spinal groups, it is convenient to have a way of handling subgroups of $B$ easily. Suppose that $G$ is defined via a sequence $\Omega$ of linear functionals, and let $\mathfrak{X} : E \longrightarrow \Gamma$ be the corresponding representation. Then the subgroups of $B$ are all epimorphic images of the subspaces of $E$ under $\mathfrak{X}$. By standard linear algebra, the subspaces of $E$ are in one-to-one correspondence with the subspaces of $E^*$ by taking null spaces, where the null space $\Theta^\perp$ of a subset $\Theta$ of $E^*$ is defined to be the intersection $\bigcap_{\vartheta \in \Theta} \ker \vartheta$ of the kernels of all linear functionals in $\Theta$. For the properties of this correspondence, we refer the reader to [BML77, pages 211-213]. Of course, we have the equality $\langle \Theta \rangle^\perp = \Theta^\perp$. With a little abuse of terminology, we say that the subgroup $\mathfrak{X}(\Theta^\perp)$ is the null space of $\Theta$ in $B$. (Note that the action of an element $\vartheta \in \Theta$ on $B$ is not well-defined unless $\ker \mathfrak{X}$, i.e. the null space of $\Omega$, is contained in $\ker \vartheta$.) Thus all subgroups of $B$ arise as null spaces of subsets (or subspaces) of $E^*$, and if necessary, we may assume that the subset is linearly independent.

If $\widetilde{B}$ is the null space of $\Theta$ in $B$, then the spinal group $\widetilde{G} = \langle A, \widetilde{B} \rangle$ can be written in the form $\mathrm{Spinal}(\widetilde{\Omega})$, where $\widetilde{\Omega} = (\widetilde{\omega}_i)_{i \geq 1}$ consists of the restrictions to the subspace $\widetilde{E} = \Theta^{\perp}$ of the elements of $\Omega$. On occasions, it may be interesting to express $\widetilde{\omega}_i$ in terms of a basis of $\widetilde{E}^*$. For this purpose, assume that $\Theta = \{\vartheta_1, \dots, \vartheta_r\}$ is linearly independent, and extend it to a basis $\mathcal{B} = (\vartheta_1, \dots, \vartheta_m)$ of $E^*$. Then the restrictions $(\widetilde{\vartheta}_{r+1}, \dots, \widetilde{\vartheta}_m)$ form a basis of $\widetilde{E}^*$, and if we decompose each $\omega_i$ with respect to $\mathcal{B}$,

$$\omega_i = \lambda_{i,1}\vartheta_1 + \cdots + \lambda_{i,r}\vartheta_r + \lambda_{i,r+1}\vartheta_{r+1} + \cdots + \lambda_{i,m}\vartheta_m,$$

then we have

$$\widetilde{\omega}_i = \lambda_{i,r+1}\widetilde{\vartheta}_{r+1} + \cdots + \lambda_{i,m}\widetilde{\vartheta}_m.$$

Thus $\widetilde{\omega}_i$ can be obtained as the restriction to $\widetilde{E}$ of the projection of $\omega_i$ to the subspace $\langle \vartheta_{r+1}, \dots, \vartheta_m \rangle$, with respect to the basis $\mathcal{B}$.

Let $G = \langle A, B \rangle$ be a spinal group defined via a sequence $\Omega$, and consider the quotient $G_n = G / \operatorname{Stab}_G(n)$ as a subgroup of $\operatorname{Aut} \mathcal{T}_n$. If we want to produce a subgroup of $B_n$, we can take a subgroup $\widetilde{B}$ of $B$ and consider its image $\widetilde{B}_n$ in $G_n$. As already mentioned, $\widetilde{B}$ can be obtained as the null space in $B$ of a subspace $U$ of $E^*$, and thus we have $\widetilde{B}_n = \mathfrak{X}_n(U^{\perp})$. (Here, as in Section 2.2, $\mathfrak{X}_n$ denotes the representation of $E$ in $\Gamma_n$ naturally induced by $\mathfrak{X}$.) Now, if $H = \langle A, C \rangle$ is another spinal group, generated by a sequence $\Delta$ such that $\Omega_n = \Delta_n$ (thus, in particular, the elements of $\Delta$ also lie in $E^*$), then we have $H_n = G_n$. So we may produce a subgroup $C_n$ of $G_n$ by the same procedure as above, i.e. by taking the null space in $C$ of another subspace $V$ of $E^*$. Clearly, if $U = V$ then we have $\widetilde{B}_n = \widetilde{C}_n$. In the next proposition, we see that the same conclusion holds if $U$ and $V$ only coincide in the subspace generated by $\Omega_n$, the part of $\Omega$ which is 'visible' in the action of $G_n$.

**Proposition 2.3.1.** *Let $G = \langle A, B \rangle$ and $H = \langle A, C \rangle$ be spinal groups, defined by two sequences $\Omega$ and $\Delta$ such that $\Omega_n = \Delta_n$. Suppose that $U$ and $V$ are subspaces of $E^*$, and let $\widetilde{B}$ and $\widetilde{C}$ be their null spaces in $B$ and $C$, respectively. Then, the following two conditions are equivalent:*

(i) $\widetilde{B}_n = \widetilde{C}_n$*, i.e. the subgroups* $\widetilde{B}$ *and* $\widetilde{C}$ *have the same image in* $G_n$.

(ii) $U \cap \langle \Omega_n \rangle = V \cap \langle \Omega_n \rangle$.

*Proof.* Let $\mathfrak{X}$ and $\mathfrak{Y}$ be the representations corresponding to $G$ and $H$, respectively. Since $\Omega_n = \Delta_n$, we have $\mathfrak{X}_n = \mathfrak{Y}_n$. By the paragraph before this proposition, we have to prove that $\mathfrak{X}_n(U^\perp) = \mathfrak{X}_n(V^\perp)$. This is equivalent to $U^\perp + \ker \mathfrak{X}_n = V^\perp + \ker \mathfrak{X}_n$ or, in other words, to $U^\perp + \langle \Omega_n \rangle^\perp = V^\perp + \langle \Omega_n \rangle^\perp$. By the properties of null spaces of subspaces of $E^*$, this amounts to asking that $U \cap \langle \Omega_n \rangle = V \cap \langle \Omega_n \rangle$. $\square$

We are now ready for the main result of this section. In the proof of part (ii), we will apply the previous proposition in the case that $\Delta = \mathrm{ext}(\Omega_n)$, for which the condition $\Omega_n = \Delta_n$ trivially holds. Recall that $H$ is then naturally isomorphic to $H_n$, which is in turn equal to $G_n$.

**Theorem 2.3.2.** *Let $G = \mathrm{Spinal}(\Omega)$ be a spinal group, and let $B_2$ be an arbitrary subgroup of $B$. Then, there exists a complement $B_1$ of $B$ such that, if we put $H = \langle a, B_1 \rangle$ and $K = B_2^G$, we have:*

(i) *$G = H \ltimes K$.*

(ii) *$G_n = H_n \ltimes K_n$, as groups of automorphisms of the truncated tree $\mathcal{T}_n$. As a consequence, $|G_n| = |H_n||K_n|$.*

*More precisely, $B_1$ can be given explicitly as follows. If we write $B_2$ as the null space of a linearly independent subset $\Theta$ of $E^*$, then we can choose $B_1$ to be the null space of the subset $\Theta'$ of $\Omega$ which is constructed according to the following rule: $\omega_i$ belongs to $\Theta'$ if and only if it is linearly independent with $\Theta \cup \{\omega_1, \dots, \omega_{i-1}\}$.*

*Proof.* Write $\Theta = \{\vartheta_1, \dots, \vartheta_r\}$ and $\Theta' = \{\vartheta_{r+1}, \dots, \vartheta_m\}$, and set $E_1 = (\Theta')^\perp$ and $E_2 = \Theta^\perp$, i.e.

$$E_1 = \bigcap_{i=r+1}^{m} \ker \vartheta_i, \qquad \text{and} \qquad E_2 = \bigcap_{i=1}^{r} \ker \vartheta_i.$$

Thus, if $\mathfrak{X} : E \longrightarrow \Gamma$ is the representation corresponding to the spinal group $G$, we have $B_1 = \mathfrak{X}(E_1)$ and $B_2 = \mathfrak{X}(E_2)$.

(i) By the construction of $\Theta$, the subspaces $\langle\Theta\rangle$ and $\langle\Theta'\rangle$ of $E^*$ have trivial intersection. By taking null spaces, it follows that $E = E_1 + E_2$. Hence $B = B_1 B_2$, and consequently

$$G = \langle A, B\rangle = \langle A, B_1, B_2\rangle = \langle A, B_1\rangle B_2^G = HK.$$

Thus, it suffices to prove that $H \cap K = 1$. Before proceeding, we make some considerations.

If $\Theta'$ is empty, then $\Theta$ is a basis of the subspace $\langle\Omega\rangle$, and consequently $E_2 = \Theta^\perp = \Omega^\perp = \ker\mathfrak{X}$. So $K = B_2^G = 1$, and there is nothing to prove. Hence we may assume that $\vartheta_{r+1}$ is defined. Let $t \geq 0$ be the integer such that $\vartheta_{r+1} = \omega_{t+1}$. Thus $\omega_{t+1}$ is the first linear functional of the sequence $\Omega$ which is linearly independent from $\Theta$, i.e. we have $\omega_1, \ldots, \omega_t \in \langle\vartheta_1, \ldots, \vartheta_r\rangle$ but $\omega_{t+1} \notin \langle\vartheta_1, \ldots, \vartheta_r\rangle$.

Put $\widehat{G} = \mathrm{Spinal}(\sigma^t\Omega)$, and let $\widehat{\mathfrak{X}} : E \longrightarrow \Gamma$ denote the representation of $E$ corresponding to $\widehat{G}$. Write also $\widehat{B} = \widehat{\mathfrak{X}}(E)$, $\widehat{B}_1 = \widehat{\mathfrak{X}}(E_1)$, and $\widehat{B}_2 = \widehat{\mathfrak{X}}(E_2)$.

*Claim 1.* $\widehat{B}_1 \cap \widehat{B}_2 = 1$.

If $\widehat{b} = \widehat{\mathfrak{X}}(e_1) = \widehat{\mathfrak{X}}(e_2)$ with $e_1 \in E_1$ and $e_2 \in E_2$, we have to prove that $\widehat{b} = 1$. For this purpose, it suffices to show that $\omega_i(e_2) = 0$ for all $i \geq t+1$. If $\omega_i \in \Theta'$ then $\omega_i(e_1) = 0$, since $E_1 = (\Theta')^\perp$. Since $\widehat{\mathfrak{X}}(e_1) = \widehat{\mathfrak{X}}(e_2)$, it follows that also $\omega_i(e_2) = 0$. Let us assume now that $\omega_i \notin \Theta'$. By the construction of $\Theta'$, we know that $\omega_i$ is a linear combination of the form

$$\omega_i = \lambda_1\vartheta_1 + \cdots + \lambda_r\vartheta_r + \lambda_{r+1}\vartheta_{r+1} + \cdots + \lambda_j\vartheta_j,$$

where the linear functionals $\vartheta_{r+1}, \ldots, \vartheta_j \in \Theta'$ appear in the sequence $\omega_{t+1}, \ldots, \omega_{i-1}$. Thus, as argued above, we have

$$(\lambda_{r+1}\vartheta_{r+1} + \cdots + \lambda_j\vartheta_j)(e_2) = 0.$$

On the other hand, since $E_2 = \bigcap_{i=1}^r \ker\vartheta_i$, we also have

$$(\lambda_1\vartheta_1 + \cdots + \lambda_r\vartheta_r)(e_2) = 0.$$

Hence $\omega_i(e_2) = 0$ also in this case, which proves the claim.

*Claim 2.* $K$ is contained in $\mathrm{Stab}(t+1)$.

Since $K = B_2^G$, it suffices to see that $b \in \mathrm{Stab}(t+1)$ for every $b \in B_2$. Let us write $b = \mathfrak{X}(e)$ with $e \in E_2$. Then $\vartheta_i(e) = 0$ for $i = 1, \ldots, r$, and since $\omega_1, \ldots, \omega_t \in \langle \vartheta_1, \ldots, \vartheta_r \rangle$, we also have $\omega_i(e) = 0$ for $i = 1, \ldots, t$. Thus the label of $b$ at the vertex $p \overset{i-1}{\ldots} p1$ is 1 for all $i = 1, \ldots, t$. Consequently, $b$ fixes all vertices of the tree at level $t+1$, and the claim is proved.

Now let $h$ be an element of $H \cap K$, and let us prove that $h = 1$. As the elements of $K$ fix all the vertices of the $t$ first levels of the tree, it suffices to see that $h_v = 1$ for every $v \in X^t$. By Proposition 2.2.3, we have $h_v \in \widehat{H}$. Since $h \in \mathrm{Stab}(t+1)$, it follows that

$$h_v \in \mathrm{Stab}_{\widehat{H}}(1) = \langle \widehat{b}^{a^i} \mid \widehat{b} \in \widehat{B}_1,\ i = 0, \ldots, p-1 \rangle.$$

Observe that every $\widehat{b} \in \widehat{B}_1$ has a trivial label at the vertex 1, since $\omega_{t+1}(e) = 0$ for every $e \in E_1$. As a consequence, all the conjugates $\widehat{b}^{a^i}$ have disjoint support, and so they commute with each other. Hence the group $\mathrm{Stab}_{\widehat{H}}(1)$ is abelian, and we can write the section $h_v$ as a product of conjugates of some $\widehat{b}$ by powers of $a$, ordered in such a way that we first have the elements which appear conjugated by $a^0$, then those conjugated by $a$, and so on, until we finally have the elements conjugated by $a^{p-1}$. According to the definition of the homomorphisms $p_G^i$ given in Lemma 2.2.8, we can express this fact as follows:

$$h_v = \prod_{i=0}^{p-1} p_{\widehat{G}}^i(h_v)^{a^i}.$$

We are going to see that $h_v = 1$ by proving that $p_{\widehat{G}}^i(h_v) = 1$ for all $i = 0, \ldots, p-1$. Observe that $p_{\widehat{G}}^i(h_v) \in \widehat{B}_1$. Then the proof of (i) will be completed once we prove the following claim, since we know that $\widehat{B}_1 \cap \widehat{B}_2 = 1$, by Claim 1.

*Claim 3.* $p_{\widehat{G}}^i(h_v) \in \widehat{B}_2$.

Since $h \in K = B_2^G$, we can write $h = b_1^{g_1} \dots b_k^{g_k}$, with $b_i \in B_2$ and $g_i \in G$. Then

$$h_v = (b_1^{g_1})_{w_1} \dots (b_k^{g_k})_{w_k} \tag{2.3.1}$$

for some $w_i \in X^t$. If $b \in B_2$, $g \in G$, and $w \in X^t$, then by the formula for the section of a conjugate, we have

$$(b^g)_w = (g_u)^{-1} b_u g_u, \tag{2.3.2}$$

where $u = g^{-1}(w)$, since $b \in \mathrm{Stab}(t)$. If we write $b = \mathfrak{X}(e)$ with $e \in E_2$, then $\omega_t(e) = 0$, and consequently $b_u = 1$ or $\widehat{b}$ by (2.2.2). By (2.3.1) and (2.3.2), the section $h_v$ is a subproduct (i.e. a product of some of the factors, in the same order) of

$$(\widehat{b}_1)^{f_1} \dots (\widehat{b}_k)^{f_k},$$

where

$$f_i = (g_i)_{g_i^{-1}(w_i)}$$

belongs to $\widehat{G}$. Let us write $f_i = a_i q_i$, for some $a_i \in A$ and some $q_i \in \mathrm{Stab}_{\widehat{G}}(1)$. Then $h_v$ is a subproduct of

$$(\widehat{b}_1^{a_1})^{q_1} \dots (\widehat{b}_k^{a_k})^{q_k}.$$

By applying the homomorphism $p_{\widehat{G}}^i$ to this last expression, it follows that $p_{\widehat{G}}^i(h_v)$ is a subproduct of $\widehat{b}_1 \dots \widehat{b}_k$, which proves that $p_{\widehat{G}}^i(h_v) \in \widehat{B}_2$, as desired.

(ii) Let $P = \langle A, C \rangle$ be the spinal group over $\mathcal{T}$ defined by the sequence $\mathrm{ext}(\Omega_n)$, and let us apply part (i) to $P$. For this purpose, construct $\Theta''$ from $\Theta$ and $\mathrm{ext}(\Omega_n)$ in the same way as $\Theta'$ is obtained from $\Theta$ and $\Omega$. Then we have a semidirect product decomposition $P = Q \ltimes R$, where $Q = \langle A, C_1 \rangle$ and $R = C_2^P$, and where $C_1$ and $C_2$ are the null spaces in $C$ of $\Theta''$ and $\Theta$, respectively.

Since all the automorphisms in $P$ have label 1 in all vertices at or below the $n$th level of $\mathcal{T}$, it follows that we can identify $P$ with $P_n$, and thus we have $P_n = Q_n \ltimes R_n$. On the other hand, we have $G_n = P_n$ and, since $B_2$ and $C_2$ have the same images in $G_n$, also $K_n = R_n$. Thus we only need to prove that $H_n = Q_n$, which will be true if we see that $B_1$ and $C_1$ have the same image in $G_n$. Recall that $B_1$ is the null space of $\Theta'$ in $B$, and that $C_1$ is the null space of $\Theta''$ in $C$. By construction, we have $\Theta'' = \Theta' \cap \Omega_n$, $\langle\Theta\rangle \cap \langle\Theta'\rangle = 0$, and $\langle\Theta, \Omega_n\rangle = \langle\Theta, \Theta''\rangle$. It follows that

$$\langle\Theta''\rangle = \langle\Theta''\rangle \cap \langle\Omega_n\rangle \subseteq \langle\Theta'\rangle \cap \langle\Omega_n\rangle$$

and, on the other hand,

$$\langle\Theta'\rangle \cap \langle\Omega_n\rangle \subseteq \langle\Theta'\rangle \cap (\langle\Theta\rangle + \langle\Theta''\rangle) = (\langle\Theta'\rangle \cap \langle\Theta\rangle) + \langle\Theta''\rangle = \langle\Theta''\rangle.$$

Thus

$$\langle\Theta'\rangle \cap \langle\Omega_n\rangle = \langle\Theta''\rangle \cap \langle\Omega_n\rangle,$$

and we deduce that $B_1$ and $C_1$ have the same image in $G_n$ by using Proposition 2.3.1. $\square$

The semidirect product decomposition given by Bartholdi and Šunić in Proposition 4.9 of [BŠ01] is a special case of the previous theorem, if we take as $\Theta$ any subset which, together with $\omega_1$, forms a basis of $E^*$. (We may assume that $\omega_1 \neq 0$, since the result is trivial otherwise.) Following the notation of Theorem 2.3.2, we then have $\Theta' = \{\omega_1\}$. This amounts to saying that $B_1$ is the null space of $\omega_1$ in $B$, and that $B_2$ can be any complement of $B_1$ in $B$.

We are interested in the opposite situation, when we choose $\Theta$ small. If $\Theta$ is empty, then we have a trivial decomposition, so we consider the case when $\Theta = \{\vartheta\}$ consists of only one non-trivial linear functional. Then the null space $B_1$ of $\Theta'$ in $B$ is cyclic, $B_2$ is the null space of $\vartheta$, and $G = \langle A, B_1\rangle \ltimes B_2^G$. Recall that the spinal group $H = \langle A, B_1\rangle$ can be written as $\mathrm{Spinal}(\widetilde{\Omega})$, where the sequence $\widetilde{\Omega}$ consists of the restrictions of the elements of $\Omega$ to the subspace $\widetilde{E} = (\Theta')^\perp$. We put this in more detail in the following corollary.

**Corollary 2.3.3.** *Let $G = \mathrm{Spinal}(\Omega)$ be a spinal group, and let $\Theta = \{\vartheta\} \subseteq E^*$, with $\vartheta \neq 0$. Define $\Theta'$ as in Theorem 2.3.2, and consider a basis $\mathcal{B}$ of*

*$E^*$ which contains $\Theta \cup \Theta'$. For every $i \geq 1$, let $\lambda_i \vartheta$ be the projection of $\omega_i$ to the subspace $\langle \vartheta \rangle$, with respect to the basis $\mathcal{B}$. If $\widetilde{\Omega} = (\widetilde{\omega}_i)_{i \geq 1}$ is the sequence of restrictions of the $\omega_i$ to $\widetilde{E} = (\Theta')^{\perp}$, then we have:*

(i) *$\widetilde{\omega}_i = \lambda_i \widetilde{\vartheta}$, for all $i \geq 1$.*

(ii) *$H = \mathrm{Spinal}(\widetilde{\Omega})$ is a 2-generator subgroup of $G$.*

(iii) *If $B_2$ is the null space of $\vartheta$ in $B$ and we put $K = B_2^G$, then $G = H \ltimes K$ and $G_n = H_n \ltimes K_n$ for every $n \geq 1$.*

Even if $\omega_i \neq 0$ for every $i \geq 1$, it might happen that some $\widetilde{\omega}_i$ is the null map. This is the main reason why we allow the possibility that $\omega_i = 0$ for some $i$ in the definition of a spinal group.

## 2.4 Hausdorff dimension: the 2-generator case

In this section, we deal with spinal groups in which $B = \langle b \rangle$ is cyclic. Thus $G = \langle a, b \rangle$ is generated by 2 elements. Our goal is to determine the order of $G_n$ for every $n$ in terms of the values of the sequence $\Omega$. If we write $b_i = b^{a^i}$ for $i = 0, \ldots, p-1$, then we have $G = \langle a \rangle \ltimes \langle b_0, \ldots, b_{p-1} \rangle$.

**Theorem 2.4.1.** *Let $G = \langle a, b \rangle$ be a two-generator spinal group, and suppose that $\omega_1 = 0$. If $\ell$ is the first index for which $\omega_\ell \neq 0$, then for every positive integer $n \geq 1$ we have*

$$\log_p |G_n| = \begin{cases} 1, & \text{if } n \leq \ell; \\ p+1, & \text{if } n > \ell. \end{cases}$$

*(Note that we may have $\ell = \infty$.)*

*Proof.* Since $\omega_1 = 0$, the support of $b$ is contained in the subtree $pX^*$. It follows that $b_0, \ldots, b_{p-1}$ have disjoint support, and consequently they commute with each other. Hence $B^G = \langle b_0 \rangle \times \cdots \times \langle b_{p-1} \rangle$. If we use the bar notation in the quotient $G_n$, it follows that $|G_n| = p\, o(\overline{b})^p$. If $n \leq \ell$, then $b$

acts trivially on the truncated tree $\mathcal{T}_n$. So $\overline{b} = \overline{1}$ and $|G_n| = p$. If $n > \ell$, then $\overline{b}$ has order $p$, and we have $|G_n| = p^{p+1}$. $\square$

Next we deal with the more complicated case where $\omega_1 \neq 0$. Under this assumption, we give the value of $\log_p |G_n|$ in Theorem 2.4.5. The idea is to work by induction on $n$, and to use the relation

$$|G_n| = |G_{n-1}| |\operatorname{Stab}_{G_n}(n-1)|.$$

Thus the main task is to determine the order of the stabilizer $\operatorname{Stab}_{G_n}(n-1)$.

Since $G/\operatorname{Stab}_G(1) = \langle \overline{a} \rangle \cong C_p$, the following result is clear.

**Lemma 2.4.2.** *If we write an element $g \in G$ in the form $g = a^{r_1} b^{s_1} \ldots a^{r_k} b^{s_k}$, with $r_i, s_i \in \mathbb{Z}$, then $g \in \operatorname{Stab}_G(1)$ if and only if $r_1 + \cdots + r_k$ is a multiple of $p$.*

On the other hand, by Proposition 2.2.3, there is an embedding

$$\begin{array}{cccc} \psi : & \operatorname{Stab}_G(1) & \longrightarrow & \widehat{G} \times \overset{p}{\cdots} \times \widehat{G} \\ & g & \longmapsto & (g_u)_{u \in X}, \end{array}$$

where $\widehat{G} = \operatorname{Spinal}(\sigma\Omega)$. Note that, in this case, $g_u$ is simply the restriction of $g$ to the subtree $uX^*$, viewed in a natural way as an automorphism of the whole tree $\mathcal{T}$.

Let $\widehat{a}$ and $\widehat{b}$ be the sections of $b$ at the vertices 1 and $p$, respectively. If $\omega_1 = 0$, then $\widehat{a}$ is the trivial automorphism, and consequently

$$\psi(b_0) = (1, 1, \ldots, 1, \widehat{b}), \psi(b_1) = (\widehat{b}, 1, \ldots, 1, 1), \ldots,$$
$$\psi(b_{p-1}) = (1, 1, \ldots, \widehat{b}, 1). \quad (2.4.1)$$

Thus $\psi$ maps $\operatorname{Stab}_G(1)$ onto the direct product $\widehat{B} \times \overset{p}{\cdots} \times \widehat{B}$, where $\widehat{B} = \langle \widehat{b} \rangle$, and $\operatorname{Stab}_{G_n}(n-1)$ corresponds to $\operatorname{Stab}_{\widehat{B}_{n-1}}(n-2) \times \overset{p}{\cdots} \times \operatorname{Stab}_{\widehat{B}_{n-1}}(n-2)$. Let $\ell$ be, as in Theorem 2.4.1, the first index for which $\omega_\ell \neq 0$. Then $\widehat{b}$ fixes the vertices at level $n-2$ if and only if $\ell \geq n-1$. If that is the case, then the only value for which the image of $\widehat{b}$ in $B_{n-1}$ is non-trivial is for $\ell = n-1$. Hence

$$|\operatorname{Stab}_{G_n}(n-1)| = \begin{cases} 1, & \text{if } n \neq \ell+1, \\ p^p, & \text{if } n = \ell+1. \end{cases} \tag{2.4.2}$$

Observe that Theorem 2.4.1 is a direct consequence of this result. Of course, the proof given before is shorter, but (2.4.2) will also be necessary in order to obtain Theorem 2.4.5 (see the proof of Lemma 2.4.4).

Suppose now that $\omega_1 \neq 0$. Then $\widehat{a}$ is a non-trivial power of $a$, and by replacing $b$ with an appropriate power of $b$, we may assume that $\widehat{a} = a$. In this case, we have

$$\psi(b_0) = (a, 1, 1 \ldots, 1, \widehat{b}), \psi(b_1) = (\widehat{b}, a, 1, \ldots, 1, 1), \ldots,$$
$$\psi(b_{p-1}) = (1, 1, 1, \ldots, \widehat{b}, a). \tag{2.4.3}$$

Let us next see what the elements of the stabilizer $\operatorname{Stab}_G(2)$ look like. If $g \in \operatorname{Stab}_G(2)$, then in particular $g \in B^G$, and $g$ can be written as a word in $b_0, \ldots, b_{p-1}$. Of course, if $\omega_1 = 0$ then $\operatorname{Stab}_G(2)$ is the whole of $B^G$.

**Lemma 2.4.3.** *If $\omega_1 \neq 0$, then $g \in \operatorname{Stab}_G(2)$ if and only if the weight of each $b_i$ in a word representing $g$ is a multiple of $p$.*

*Proof.* We have $g \in \operatorname{Stab}_G(2)$ if and only if $\psi(g) \in \operatorname{Stab}_{\widehat{G}}(1) \times \overset{p}{\cdots} \times \operatorname{Stab}_{\widehat{G}}(1)$. If we look at the $i$th component of $\psi(g)$, we find from (2.4.3) that the only non-trivial contributions come from $b_{i-1}$ and $b_i$, which yield $a$ and $\widehat{b}$, respectively. (The indices in $b_{i-1}$ and $b_i$ are taken as residues modulo $p$ between 0 and $p-1$.) Now, by Lemma 2.4.2, a word in $a$ and $\widehat{b}$ lies in $\operatorname{Stab}_{\widehat{G}}(1)$ if and only if the weight of $a$ is a multiple of $p$, and the result follows. $\square$

**Lemma 2.4.4.** *Let $G = \langle a, b \rangle$ be a two-generator spinal group, and suppose that $p > 2$ and $\omega_1 \neq 0$. Then:*

(i) *If $\omega_2 = 0$, and $\ell$ is the first index greater than 2 such that $\omega_\ell \neq 0$, then* *for every* $n \geq 3$,

$$|\operatorname{Stab}_{G_n}(n-1)| = \begin{cases} 1, & \textit{if } n \neq \ell+1, \\ p^{p(p-1)}, & \textit{if } n = \ell+1. \end{cases} \tag{2.4.4}$$

*Here, we take* $\ell = \infty$ *if* $\omega_i = 0$ *for all* $i \geq 2$.

(ii) *If* $\omega_2 \neq 0$, *then*

$$|\operatorname{Stab}_{G_n}(n-1)| = \begin{cases} p^{p(p-1)}, & \textit{if } n = 3, \\ |\operatorname{Stab}_{\widehat{G}_{n-1}}(n-2)|^p, & \textit{if } n \geq 4, \end{cases} \tag{2.4.5}$$

*where* $\widehat{G} = \operatorname{Spinal}(\sigma\Omega)$.

*Proof.* Put $\widehat{b}_i = \widehat{b}^{a^i}$ for $i = 0, \ldots, p-1$. First of all, we prove that

$$\psi(\operatorname{Stab}_G(2)) = L \times \overset{p}{\cdots} \times L,$$

where $L$ is the subgroup of $\widehat{G}$ consisting of all elements which, written as a word in $a$ and $\widehat{b}$, satisfy that the weight of both $a$ and $\widehat{b}$ is divisible by $p$. Equivalently, $L$ consists of the elements that can be represented as a word in the $\widehat{b}_i$ whose total length is a multiple of $p$. (This is clear if we collect all occurrences of $a$ to the left in the expression of an element of $L$ as a word in $a$ and $\widehat{b}$.) The inclusion $\subseteq$ is a direct consequence of Lemma 2.4.3, taking into account the values of $\psi(b_0), \ldots, \psi(b_{p-1})$, given in (2.4.3). Let us now prove the reverse inclusion. It suffices to see that $L \times \overset{p}{\cdots} \times L \subseteq \operatorname{im}\psi$. Since $\psi$ is a homomorphism, we only need to see that $(h, 1, \ldots, 1) \in \operatorname{im}\psi$ for all $h \in L$ (the same argument applies if $h$ is in a different component). Let us write $h$ as a word in $a$ and $\widehat{b}$, say $h = a^{r_1}\widehat{b}^{s_1} \ldots a^{r_k}\widehat{b}^{s_k}$, with $r_i, s_i \in \mathbb{Z}$. By the definition of $L$, both $r = r_1 + \cdots + r_k$ and $s = s_1 + \cdots + s_k$ are divisible by $p$. If $g = b_0^{r_1} b_1^{s_1} \ldots b_0^{r_k} b_1^{s_k}$, it follows from (2.4.3) that the first component of $\psi(g)$ is exactly $h$, the second component is $a^s$, the last component is $\widehat{b}^r$, and the rest of the components are 1. (Note that it is at this point where we use the condition that $p > 2$.) Since $r$ and $s$ are divisible by $p$, we conclude that $\psi(g) = (h, 1, \ldots, 1)$, as desired.

As a consequence, we have $|\operatorname{Stab}_{G_n}(n-1)| = |\operatorname{Stab}_{L_{n-1}}(n-2)|^p$ for all $n \geq 3$.

(i) Assume that $\omega_2 = 0$. If $n \neq \ell + 1$, then we have $\operatorname{Stab}_{\widehat{G}_{n-1}}(n-2) = 1$ by (2.4.2) and *a fortiori* $\operatorname{Stab}_{L_{n-1}}(n-2) = 1$. Hence $\operatorname{Stab}_{G_n}(n-1) = 1$. If $n = \ell + 1$, then $\widehat{b}$ fixes all vertices of the $n-2$-nd level, and consequently

$$\operatorname{Stab}_{\widehat{G}}(n-2) = \langle \widehat{b}_0 \rangle \times \cdots \times \langle \widehat{b}_{p-1} \rangle$$

is elementary abelian of order $p^p$. By using the natural identification of $\operatorname{Stab}_{\widehat{G}}(n-2)$ with the vector space $\mathbb{F}_p^p$, the subspace $\operatorname{Stab}_L(n-2)$ of words in the $\widehat{b}_i$ of total length divisible by $p$ corresponds to the hyperplane $x_0 + \cdots + x_{p-1} = 0$. Hence $|\operatorname{Stab}_{L_{n-1}}(n-2)| = p^{p-1}$ and $|\operatorname{Stab}_{G_n}(n-1)| = p^{p(p-1)}$, which concludes the proof of (i).

(ii) Assume now that $\omega_2 \neq 0$. If $n = 3$, then $\operatorname{Stab}_{\widehat{G}_2}(1)$ is the direct product of the subgroups generated by the images of the $\widehat{b}_i$, and we can argue as in the last part of (i) to prove that $|\operatorname{Stab}_{G_3}(2)| = p^{p(p-1)}$.

Assume now that $n \geq 4$. Since $\omega_2 \neq 0$, it follows from Lemma 2.4.3 that $\operatorname{Stab}_{\widehat{G}}(2) \subseteq L$. Hence $|\operatorname{Stab}_{G_n}(n-1)| = |\operatorname{Stab}_{\widehat{G}_{n-1}}(n-2)|^p$, as desired. $\square$

**Theorem 2.4.5.** *Let $G = \langle a, b \rangle$ be a two-generator spinal group, and suppose that $\omega_1 \neq 0$. Let $k$ be the first index for which $\omega_k = 0$ and let $\ell$ be the first index greater than $k$ such that $\omega_\ell \neq 0$ (if there are not such indices, put $k = \infty$ or $\ell = \infty$). Then, for every positive integer $n \geq 1$, we have*

$$\log_p |G_n| = \begin{cases} 1, & \text{if } n = 1, \\ p^{n-1} + 1, & \text{if } 1 < n \leq k, \\ p^{k-1} + 1, & \text{if } k < n \leq \ell, \\ p^k + 1, & \text{if } n > \ell. \end{cases}$$

*Proof.* We use induction on $n$. The cases $n = 1$ and $n = 2$ are obvious, so suppose that $n \geq 3$. Since $|G_n| = |G_{n-1}||\operatorname{Stab}_{G_n}(n-1)|$, the result will follow immediately if we prove that

$$\log_p |\operatorname{Stab}_{G_n}(n-1)| = \begin{cases} p^{n-1} - p^{n-2}, & \text{if } 3 \le n \le k, \\ 0, & \text{if } k < n \le \ell, \\ p^k - p^{k-1}, & \text{if } n = \ell + 1, \\ 0, & \text{if } n > \ell + 1. \end{cases}$$

Suppose first that $3 \le n \le k$. Since $\omega_i \ne 0$ for $1 \le i \le n-2$, we may apply $n-3$ times the recurrence relation (2.4.5) of Lemma 2.4.4, to get

$$\log_p |\operatorname{Stab}_{G_n}(n-1)| = p^{n-3} \log_p |\operatorname{Stab}_{\widehat{G}_3}(2)|,$$

where $\widehat{G} = \operatorname{Spinal}(\sigma^{n-3}\Omega)$. Now we also have $\omega_{n-2}, \omega_{n-1} \ne 0$, so we may still apply Lemma 2.4.4 to the group $\widehat{G}$ to conclude that

$$\log_p |\operatorname{Stab}_{G_n}(n-1)| = p^{n-2}(p-1).$$

If $n > k$, then we apply $k-2$ times (2.4.5). It follows that

$$\log_p |\operatorname{Stab}_{G_n}(n-1)| = p^{k-2} \log_p |\operatorname{Stab}_{\widehat{G}_{n-k+2}}(n-k+1)|,$$

where now $\widehat{G} = \operatorname{Spinal}(\sigma^{k-2}\Omega)$. Since $\omega_k = 0$, we find that $\widehat{G}$ satisfies the conditions of part (i) of Lemma 2.4.4. Then we may use directly (2.4.4) to obtain, as desired, that

$$\log_p |\operatorname{Stab}_{G_n}(n-1)| = \begin{cases} 0, & \text{if } n \ne \ell + 1, \\ p^{k-1}(p-1), & \text{if } n = \ell + 1. \end{cases}$$

$\square$

## 2.5 Hausdorff dimension: the general case

In this section, we will get a formula for the Hausdorff dimension in $\Gamma$ of the closure of a spinal group $G = \operatorname{Spinal}(\Omega)$ in terms of the sequence $\Omega$, provided that $p > 2$. If we have $\omega_n = 0$ for some $n \geq 1$, then $G$ is finite by Proposition 2.2.5. Hence the closure $\overline{G}$ coincides with $G$, and is also finite. Consequently, $\dim_\Gamma \overline{G} = 0$ in this case. For this reason, in the results of this section we make the assumption that all the linear functionals in the sequence $\Omega$ are non-trivial. We follow to a great extent the arguments used by Siegenthaler in [Sie08] for the case $p = 2$.

**Proposition 2.5.1.** *Let* $G = \mathrm{Spinal}(\Omega)$ *be a spinal group, where* $p > 2$ *and* $\omega_i \neq 0$ *for all* $i$*. Provided that* $\dim\langle \Omega_n \rangle \geq 2$*, let* $t \in \{1, \ldots, n-2\}$ *be the largest integer such that* $\omega_1, \ldots, \omega_t$ *are proportional to* $\omega_1$*. Then, we have*

$$\log_p |G_n| = 1 + p^t(\log_p |\widehat{G}_{n-t}| - \delta(n)),$$

*where* $\widehat{G} = \mathrm{Spinal}(\sigma^t \Omega)$*, and*

$$\delta(n) = \begin{cases} 0, & \text{if } \omega_1 \text{ is linearly independent from } \omega_{t+1}, \ldots, \omega_{n-1}, \\ 1, & \text{otherwise.} \end{cases}$$

*Proof.* We apply Corollary 2.3.3 to the groups $G = \mathrm{Spinal}(\Omega)$ and $\widehat{G} = \mathrm{Spinal}(\sigma^t \Omega)$, with $\vartheta = \omega_1$. In principle, we would need to consider two bases $\mathcal{B}$ and $\widehat{\mathcal{B}}$ of $E^*$, one for each case, since the subset $\Theta'$ might be different for $G$ and for $\widehat{G}$. However, if we follow the procedure for constructing $\Theta'$, we can see that we get the same $\Theta'$ for both groups, because we have chosen $\vartheta = \omega_1$. Thus we may assume that $\mathcal{B} = \widehat{\mathcal{B}}$.

For every $i \geq 1$, let $\widetilde{\omega}_i$ be the restriction of $\omega_i$ to the subspace $(\Theta')^\perp$ of $E$, and put $\widetilde{\Omega} = (\widetilde{\omega}_i)_{i \geq 1}$. As indicated in Corollary 2.3.3, we have $\widetilde{\omega}_i \in \langle \widetilde{\omega}_1 \rangle$ for all $i$, and $\widetilde{\omega}_i = 0$ if and only if $\omega_1$ does not appear in the decomposition of $\omega_i$ with respect to the basis $\mathcal{B}$. By the definition of $\mathcal{B}$ and $t$, it follows that $\widetilde{\omega}_i \neq 0$ for $i = 1, \ldots, t$, and $\widetilde{\omega}_{t+1} = 0$.

If $B_2$ is the null space of $\omega_1$ in $B$, then the null space of $\omega_1$ in $\widehat{B}$ is $\widehat{B}_2$. Thus, if we apply Corollary 2.3.3, we get semidirect product decompositions $G_n = H_n \ltimes K_n$, and $\widehat{G}_n = \widehat{H}_n \ltimes L_n$ for every $n \geq 1$, where $H = \mathrm{Spinal}(\widetilde{\Omega})$, $K = B_2^G$ and $L = \widehat{B}_2^{\widehat{G}}$. (Here, it is important that $\Theta'$ is the same for $G$ and $\widehat{G}$ in order to know that the first subgroup in the decomposition of $\widehat{G}_n$ is $\widehat{H}_n$.)

Now, we have $|G_n| = |H_n||K_n|$, and so it suffices to calculate the orders of $H_n$ and $K_n$. Since $H = \mathrm{Spinal}(\widetilde{\Omega})$ is a 2-generator spinal group with $\widetilde{\omega}_1 \neq 0$, we may apply Theorem 2.4.5 to calculate $|H_n|$. The first trivial term of the sequence $\widetilde{\Omega}$ is $\widetilde{\omega}_{t+1}$. Let $\ell$ be the first index greater than $t+1$ for which $\widetilde{\omega}_\ell \neq 0$. One readily checks that $\ell$ is also the first index for which $\omega_1$ is linearly dependent with $\omega_{t+1}, \dots, \omega_\ell$. Hence

$$\log_p |H_n| = p^{t+\delta(n)} + 1, \tag{2.5.1}$$

where $\delta(n)$ is as in the statement of the proposition.

On the other hand, since the first functional of $\sigma^t \widetilde{\Omega}$ is $\widetilde{\omega}_{t+1} = 0$, we may use Theorem 2.4.1 to get

$$\log_p |\widehat{H}_{n-t}| = \delta(n)p + 1. \tag{2.5.2}$$

Next, we relate $K = B_2^G$ with $L = \widehat{B}_2^{\widehat{G}}$. Consider the injective homomorphism

$$\begin{array}{cccc} \psi_t & : & \mathrm{Stab}_G(t) & \longrightarrow & \widehat{G} \times \overset{p^t}{\cdots} \times \widehat{G} \\ & & g & \longmapsto & (g_u)_{u \in X^t}. \end{array}$$

We have $\psi_t(B_2) = \{1\} \times \cdots \times \{1\} \times \widehat{B}_2$, and we claim that

$$\psi_t(B_2^G) = \widehat{B}_2^{\widehat{G}} \times \cdots \times \widehat{B}_2^{\widehat{G}}. \tag{2.5.3}$$

Since

$$(b^g)_u = (b_{g^{-1}(u)})^{g_{g^{-1}(u)}}$$

for every $b \in B_2$, $g \in G$, and $u \in X^t$, the inclusion $\subseteq$ is clear. For the reverse inclusion, put $v = p.\overset{t}{\ldots}.p$. By Proposition 2.2.3, for every $f \in \widehat{G}$ we can find $g \in \mathrm{Stab}_G(t)$ such that $\psi_v(g) = f$, and then $\psi_t(b^g) = (1, \dots, 1, \widehat{b}^f)$ for every $b \in B_2$. This proves that

$$\psi_t(R) = \{1\} \times \cdots \times \{1\} \times \widehat{B}_2^{\widehat{G}}$$

for some subgroup $R$ of $B_2^G$. Now, let $u$ be an arbitrary vertex of $X^t$. We know by Proposition 2.2.3 that $G$ acts transitively on $X^t$. Let us choose

$g \in G$ such that $g(v) = u$. Then

$$\psi_t(R^g) = \{1\} \times \cdots \times \{1\} \times \widehat{B}_2^{\widehat{G}} \times \{1\} \times \cdots \times \{1\},$$

where the non-trivial component is in the position of $u$. It follows that $\widehat{B}_2^{\widehat{G}} \times \cdots \times \widehat{B}_2^{\widehat{G}}$ is contained in $\psi_t(B_2^G)$, which concludes the proof of claim (2.5.3).

As a consequence, we have $|K_n| = |L_{n-t}|^{p^t}$. Hence, by using (2.5.1) and (2.5.2), we conclude that

$$\begin{aligned} \log_p |G_n| = \log_p |H_n| + \log_p |K_n| &= 1 + p^{t+\delta(n)} + p^t \log_p |L_{n-t}| \\ &= 1 + p^{t+\delta(n)} + p^t(\log_p |\widehat{G}_{n-t}| - \log_p |\widehat{H}_{n-t}|) \\ &= 1 + p^t(\log_p |\widehat{G}_{n-t}| + p^{\delta(n)} - \delta(n)p - 1) \\ &= 1 + p^t(\log_p |\widehat{G}_{n-t}| - \delta(n)). \end{aligned}$$

□

**Proposition 2.5.2.** *Let $G = \mathrm{Spinal}(\Omega)$ be a spinal group, where $p > 2$ and $\omega_i \neq 0$ for all $i$. Provided that $\dim\langle\Omega_n\rangle \geq 2$, let $k \in \{1, \ldots, n-2\}$ be the smallest integer such that $\omega_k$ is linearly independent from $\omega_{k+1}, \ldots, \omega_{n-1}$. Then*

$$\log_p |G_n| = 1 + p^k \log_p |\widehat{G}_{n-k}|,$$

*where $\widehat{G} = \mathrm{Spinal}(\sigma^k\Omega)$.*

*Proof.* We first make a partition of the first $k$ terms of the sequence $\Omega$ in blocks of proportional linear functionals, where each block is as long as possible:

$$\overbrace{\omega_1, \ldots, \omega_{s_1}}^{t_1 \text{ prop. to } \omega_1}, \overbrace{\omega_{s_1+1}, \ldots, \omega_{s_2}}^{t_2 \text{ prop. to } \omega_{s_1+1}}, \overbrace{\omega_{s_2+1}, \ldots, \omega_{s_3}}^{t_3 \text{ prop. to } \omega_{s_2+1}}, \ldots, \overbrace{\omega_{s_{\ell-1}+1}, \ldots, \omega_{s_\ell}}^{t_\ell \text{ prop. to } \omega_{s_{\ell-1}+1}}.$$

Here, $s_i = t_1 + \cdots + t_i$ for every $i = 1, \ldots, \ell$, and $s_\ell = k$. For convenience, we also put $s_0 = t_0 = 0$.

Put $\widehat{G}^{(i)} = \operatorname{Spinal}(\sigma^{s_i}\Omega)$ for $i = 0, \ldots, \ell$. Thus $\widehat{G}^{(0)} = G$. If we apply Proposition 2.5.1 to $\widehat{G}^{(i)}$ for $i = 0, \ldots, \ell - 1$, we have

$$\log_p |\widehat{G}^{(i)}_{n-s_i}| = \begin{cases} 1 + p^{t_{i+1}}(\log_p |\widehat{G}^{(i+1)}_{n-s_{i+1}}| - 1), & \text{if } i = 0, \ldots, \ell - 2, \\ 1 + p^{t_\ell} \log_p |\widehat{G}^{(\ell)}_{n-s_\ell}|, & \text{if } i = \ell - 1. \end{cases} \tag{2.5.4}$$

Now, if we multiply equation (2.5.4) by $p^{t_i}$ for all $i = 0, \ldots, \ell - 1$ and sum all the results, we get the desired equality. $\square$

**Theorem 2.5.3.** *Let $G = \operatorname{Spinal}(\Omega)$ be a spinal group, where $p > 2$ and $\omega_i \neq 0$ for all $i$. Provided that $\dim\langle\Omega_n\rangle = m$, let $k_i$ be the smallest integer such that $\dim\langle\omega_{k_i+1}, \ldots, \omega_{n-1}\rangle = i$, for every $i = 1, \ldots, m$. (Note that $k_m = 0$.) Then, we have*

$$\log_p |G_n| = \sum_{i=1}^{m} p^{k_i} + p^{n-1} = 1 + p^{k_{m-1}} + p^{k_{m-2}} + \cdots + p^{k_1} + p^{n-1}.$$

*Proof.* If $m = 1$, then $G_n$ is generated by 2 elements, and $\log_p |G_n| = 1 + p^{n-1}$ by Theorem 2.4.5. Thus the result holds in this case.

Assume now that $m \geq 2$, and let $\widehat{G}^{(i)} = \operatorname{Spinal}(\sigma^{k_i}\Omega)$, for $i = 1, \ldots, m$. Thus $\widehat{G}^{(m)} = G$. By the previous proposition, we have

$$\log_p |\widehat{G}^{(i)}_{n-k_i}| = 1 + p^{k_{i-1}-k_i} \log_p |\widehat{G}^{(i-1)}_{n-k_{i-1}}|, \quad \text{for } i = 2, \ldots, m.$$

On the other hand, by arguing as in the case $m = 1$, we have

$$\log_p |\widehat{G}^{(1)}_{n-k_1}| = 1 + p^{n-k_1-1}.$$

By putting all these equalities together, we get the desired result. $\square$

**Theorem 2.5.4.** *Let $G = \operatorname{Spinal}(\Omega)$ be a spinal group, where $p > 2$ and $\omega_i \neq 0$ for all $i$. If $\dim\langle\Omega\rangle = m$, let $n_0$ be the first integer such that $\dim\langle\Omega_{n_0}\rangle = m$. For every $n \geq n_0$ and $i = 1, \ldots, m$, let $r_{n,i}$ be the minimum number of terms of the sequence $(\omega_{n-1}, \ldots, \omega_1)$, in that order, that are needed to generate a subspace of dimension $i$. (For fixed $i$, the number $r_{n,i}$ may vary with $n$, but we always have $r_{n,1} = 1$.) Then,*

$$\dim_\Gamma \overline{G} = (p-1) \liminf_{n\to\infty} \left( \frac{1}{p^{r_{n,1}}} + \frac{1}{p^{r_{n,2}}} + \cdots + \frac{1}{p^{r_{n,m}}} \right).$$

*Proof.* This is an immediate consequence of (1.2.3) and the previous theorem: note that, for a fixed value of $n$, we have $r_{n,i} = n - k_{i-1}$ for $i = 1, \ldots, m$, if we put $k_0 = n - 1$. $\square$

## 2.6 The spinal spectrum

In this final section, we determine completely the set of values which can be taken by the Hausdorff dimension of the closure of spinal groups, provided that $p > 2$. We begin by introducing some useful notation.

To every finite sequence $\Psi = (\psi_1, \ldots, \psi_q)$ of elements of $E^*$, we associate a number $\lambda(\Psi) \in [0, 1]$ as follows. Let us define

$$m_i = \dim\langle \psi_1, \ldots, \psi_i \rangle, \quad \text{for every } i = 1, \ldots, q,$$

and

$$n_i = \begin{cases} m_1, & \text{if } i = 1, \\ m_i - m_{i-1}, & \text{if } 1 < i \leq q. \end{cases}$$

Note that $n_i = 0$ or $1$ for every $i$. Then we put

$$\lambda(\Psi) = 0.n_1 \ldots n_q,$$

where the expression is taken in base $p$. Also, we write $\Psi(i)$ for the sequence

$$(\psi_{i+1}, \ldots, \psi_{i+q}),$$

where the subindices are reduced modulo $q$ to a value between 1 and $q$. Thus $\Psi(0) = \Psi$ and $\Psi(i) = \Psi(i + q)$ for every $i \geq 0$.

**Lemma 2.6.1.** *Let $G = \mathrm{Spinal}(\Omega)$ be a spinal group, where $p > 2$ and $\omega_i \neq 0$ for all $i$. Suppose that the sequence $\Omega$ is periodic, with period $\Pi = (\pi_1, \ldots, \pi_q)$, and let $\Psi = (\pi_q, \ldots, \pi_1)$. Then,*

$$\dim_\Gamma \overline{G} = (p - 1) \min\{\lambda(\Psi(0)), \ldots, \lambda(\Psi(q - 1))\}.$$

*Proof.* Put $m = \dim\langle\Omega\rangle = \dim\langle\Pi\rangle$. According to Theorem 2.5.4, we have

$$\dim_\Gamma \overline{G} = (p-1) \liminf_{n\to\infty} \lambda_n,$$

where

$$\lambda_n = \frac{1}{p^{r_{n,1}}} + \frac{1}{p^{r_{n,2}}} + \cdots + \frac{1}{p^{r_{n,m}}}$$

is defined for $n > q$. Clearly, the sequence $(\lambda_n)_{n>q}$ is periodic with period of length $q$, and so we have

$$\liminf_{n\to\infty} \lambda_n = \min\{\lambda_{q+1}, \ldots, \lambda_{2q}\}.$$

Also, by the definition of $r_{n,i}$, we have

$$\lambda_n = \lambda(\omega_{n-1}, \ldots, \omega_{n-q})$$

for all $n > q$. It follows that the set $\{\lambda_{q+1}, \ldots, \lambda_{2q}\}$ coincides (not in the same order) with

$$\{\lambda(\Psi(0)), \ldots, \lambda(\Psi(q-1))\},$$

and we are done. $\square$

Now we prove Theorem B.

**Theorem 2.6.2.** *If $p$ is odd, then the spinal spectrum of $\Gamma$ consists of $0$ and all numbers whose $p$-adic expansion is of the form $0.a_1 \ldots a_n$, where*

(i) *$a_i = 0$ or $p-1$ for every $i = 1, \ldots, n$.*

(ii) *$a_1 = p-1$.*

*In particular, the spinal spectrum is contained in $\mathbb{Q}$.*

*Proof.* By Proposition 2.2.5, if $G = \mathrm{Spinal}(\Omega)$ and $\omega_i = 0$ for some $i$, then $\dim_\Gamma \overline{G} = 0$. Thus it suffices to prove that, in the case that $\omega_i \neq 0$ for all $i$, the set of values taken by the Hausdorff dimension consists of all numbers in the interval $[0,1]$ whose $p$-adic expansion is of the form specified above.

On the one hand, let $G = \mathrm{Spinal}(\Omega)$ be a spinal group for which $\omega_i \neq 0$ for all $i$. Put $m = \dim\langle\Omega\rangle$ and

$$\lambda_n = \frac{1}{p^{r_{n,1}}} + \frac{1}{p^{r_{n,2}}} + \cdots + \frac{1}{p^{r_{n,m}}},$$

where $r_{n,i}$ is defined as in Theorem 2.5.4, and in particular $r_{n,1} = 1$. Then $\dim_\Gamma \overline{G} = (p-1)\lambda$, where $\lambda = \liminf_{n\to\infty} \lambda_n$. Now, the $p$-adic expansion of every $\lambda_n$ has $m$ non-zero digits (one of which is the first digit), and they are all equal to 1. It follows that the same is true for the $p$-adic expansion of $\lambda$, with the only exception that it may have $m$ or fewer non-zero digits.

Conversely, let $\mu = 0.a_1 \dots a_n$ be as in the statement of the theorem (of course, we may assume $a_n = p-1$), and let us see that there exists $G = \mathrm{Spinal}(\Omega)$ such that $\dim_\Gamma \overline{G} = \mu$. More precisely, we prove that we can choose $G$ such that the sequence $\Omega$ of linear functionals is periodic. In the next two paragraphs, we explain how to construct the period $\Pi$ for $\Omega$. By Lemma 2.6.1, it is more convenient to define $\Pi$ backwards; thus, we construct a sequence $\Psi = (\psi_1, \dots, \psi_q)$ and then put $\Pi = (\psi_q, \dots, \psi_1)$.

Put $\lambda = \mu/(p-1)$, and write $\lambda = 0.b_1 \dots b_n$ (so that $b_n = 1$). Let $m$ be the number of ones in the $p$-adic expansion of $\lambda$, and choose a vector space $E$ of dimension $m$ over $\mathbb{F}_p$. Let $\Theta = (\vartheta_1, \dots, \vartheta_m)$ be a basis of $E^*$. If $n = m$, then we can simply take $\Psi = \Theta$. Assume then that $n > m$, i.e. that there is at least one zero in the $p$-adic expansion $0.b_1 \dots b_n$. We can see this expansion as formed by alternating blocks of the form $1 \dots 1$ and $0 \dots 0$ (beginning and ending with ones). Let $k_1, \dots, k_r$ be the lengths of the blocks of ones, so that $k_1 + \cdots + k_r = m$. This decomposition of $m$ gives rise to a partition of $\Theta$ into blocks of lengths $k_1, \dots, k_r$, which we denote by $\Theta_1, \dots, \Theta_r$. We are going to obtain the sequence $\Psi$ from $\Theta$ by inserting some carefully chosen elements of $E^*$ in between these blocks, and possibly also after the last block $\Theta_r$. More precisely, if $\ell_1, \dots, \ell_{r-1}$ are the lengths of the blocks of zeros in the expansion of $\lambda$, we insert exactly $\ell_i$ linear functionals between the blocks $\Theta_i$ and $\Theta_{i+1}$. The number of elements to be inserted after the last block will be clear later on.

For simplicity, let us write $k$ and $\ell$ for $k_1$ and $\ell_1$. We begin by inserting the whole sequence

$$\vartheta_1, \ldots, \vartheta_k \tag{2.6.1}$$

$\lceil \ell/k \rceil$ times between the blocks of $\Theta$, taking care of inserting at each position the number of linear functionals that has been indicated. (It might happen that we had to put some elements after the last block of $\Theta$.) Then we introduce the whole sequence

$$\vartheta_1 + \vartheta_{k+1}, \ldots, \vartheta_k + \vartheta_{k+1}, \vartheta_{k+1} \tag{2.6.2}$$

at least once, and so many times as to guarantee that all the 'holes' in between the blocks have been filled. (Again, it might be necessary to put elements after the last block.) This completes the construction of $\Theta$, and hence also of $\Pi$ and $\Omega$.

In order to make sure that $\dim_\Gamma \overline{G} = \mu$ for $G = \mathrm{Spinal}(\Omega)$ (and to understand the construction of $\Psi$), note that we have $\lambda(\Psi) = 0.b_1 \ldots b_n$, since all the elements that we have introduced between the blocks of $\Theta$ are linearly dependent with the linear functionals that appear before. According to Lemma 2.6.1, we only need to prove that $\lambda(\Psi(i)) \geq \lambda(\Psi)$ for $i = 1, \ldots, q-1$, where $q$ is the length of $\Psi$. Let $\Psi'$ be the sequence which is obtained from $\Psi$ by deleting the blocks $\Theta_2, \ldots, \Theta_r$. Then we are only left with some repetitions of the sequence (2.6.1), followed by some repetitions of (2.6.2). As a consequence, we have the following two properties:

(i) Any sequence of $k$ consecutive elements of $\Psi'$ is linearly independent.

(ii) Any sequence of $k+1$ consecutive elements of $\Psi'$ which contains at least one element from (2.6.2) is linearly independent.

Here, $\Psi'$ should be considered as a cycle, so that '$k+1$ consecutive elements' also covers the case when we choose $t < k+1$, and we consider the last $t$ elements of $\Psi'$ together with the first $k+1-t$ elements. Observe that (i) and (ii) imply that any sequence of $k+1$ consecutive elements of $\Psi$ containing at least one element of the blocks $\Theta_2, \ldots, \Theta_r$ is linearly independent. This property, together with (ii), yields that the $p$-adic expansion of $\lambda(\Psi(i))$ begins with $k+1$ ones for $i = \ell, \ldots, q-1$. Thus $\lambda(\Psi(i)) > \lambda(\Psi)$ in this case. Now, suppose that $1 \leq i \leq \ell-1$. Then, $\Psi(i)$ begins with the $k$ linearly independent elements $\vartheta_1, \ldots, \vartheta_k$ (probably in a different order), followed by $\ell - i$ of these same elements, and then $\vartheta_{k+1}$. It follows that

$$\lambda(\Psi(i)) = 0.1 \overset{k}{\ldots} 10 \overset{\ell-i}{\ldots} 01 \ldots$$

is also greater than $\lambda(\Psi)$, which concludes the proof. $\square$

Actually, the proof of the previous theorem provides an algorithm which, given a number $\mu \in [0,1]$ with an appropriate $p$-adic expansion, constructs a periodic sequence $\Omega$ such that $G = \mathrm{Spinal}(\Omega)$ satisfies $\dim_\Gamma \overline{G} = \mu$. Let us illustrate this with a couple of examples.

**Examples 2.6.3.** (i) Let us produce a period $\Pi$ which yields a spinal group with Hausdorff dimension $(p-1)\lambda$, where $\lambda = 0.111100101$ in base $p$. We choose a vector space $E$ of dimension 6 over $\mathbb{F}_p$, and a basis $\{\vartheta_1, \ldots, \vartheta_6\}$ of $E^*$. By following the steps of the last proof, we take

$$\Psi = (\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_1, \vartheta_2, \vartheta_5, \vartheta_3, \vartheta_6, \vartheta_4, \vartheta_1 + \vartheta_5, \vartheta_2 + \vartheta_5, \vartheta_3 + \vartheta_5, \vartheta_4 + \vartheta_5, \vartheta_5),$$

and we let $\Pi$ be the same sequence written in reverse order.

(ii) Let us now obtain a group with Hausdorff dimension $(p-1)\lambda$, with $\lambda = 0.11101$. In this case, it suffices to choose $E$ of dimension 4. If $\{\vartheta_1, \ldots, \vartheta_4\}$ is a basis of $E^*$, we can take

$$\Psi = (\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_1, \vartheta_4, \vartheta_2, \vartheta_3, \vartheta_1 + \vartheta_4, \vartheta_2 + \vartheta_4, \vartheta_3 + \vartheta_4, \vartheta_4).$$

Observe that the simpler choice

$$\Psi = (\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_1, \vartheta_4, \vartheta_2, \vartheta_3, \vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4)$$

is not valid, even if $\lambda(\Psi) = \lambda$. The reason is that, in this case, $\lambda(\Psi(5)) = 0.111001$ is smaller than $\lambda$. This shows that we cannot use the sequence

$$\vartheta_1, \ldots, \vartheta_k, \vartheta_{k+1}$$

instead of (2.6.2) in the proof of Theorem 2.6.2, and explains why we have had to add $\vartheta_{k+1}$ in all but the last component.

# Chapter 3

# GGS-groups: order of congruence quotients and Hausdorff dimension

## 3.1 Introduction

The second of the Grigorchuk groups and the Gupta-Sidki group are particular instances of the family of *GGS-groups* (GGS after Grigorchuk, Gupta, and Sidki, a term coined by Gilbert Baumslag), to which this chapter is devoted. We work over the $p$-adic rooted tree, where $p$ is an odd prime, and we determine the order of all congruence quotients of GGS-groups; these are the automorphism groups induced by GGS-groups on the finite trees which are obtained by truncating the $p$-adic tree at every level. As a consequence, we also obtain the Hausdorff dimension of the closures of GGS-groups.

Let $\mathcal{T}$ be the $d$-adic rooted tree, by now, with vertices indexed by $X^*$, the free monoid on the alphabet $X = \{1, \ldots, d\}$ and let us define $a$ as the rooted automorphism corresponding to $(1\ 2\ \ldots\ d)$. Since $a$ has order $d$, it makes sense to write $a^k$ for $k \in \mathbb{Z}/d\mathbb{Z}$. Now, given a non-zero vector $\mathbf{e} = (e_1, \ldots, e_{d-1}) \in (\mathbb{Z}/d\mathbb{Z})^{d-1}$, we can define recursively an automorphism

$b$ of $\mathcal{T}$ via

$$\psi(b) = (a^{e_1}, \dots, a^{e_{d-1}}, b).$$

We say that the subgroup $G = \langle a, b \rangle$ of $\operatorname{Aut} \mathcal{T}$ is the *GGS-group* corresponding to the *defining vector* $\mathbf{e}$. If $d = 2$ then there is only one GGS-group, which is isomorphic to $D_\infty$, the infinite dihedral group. The second Grigorchuk group is obtained by choosing $d = 4$ and $\mathbf{e} = (1, 0, 1)$, and the Gupta-Sidki group arises for $d$ equal to an odd prime and $\mathbf{e} = (1, -1, 0, \dots, 0)$. The groups corresponding to $\mathbf{e} = (1, 0, \dots, 0)$ and arbitrary $d$ have also deserved special attention. In the case $d = 3$, this group was introduced by Fabrykowski and Gupta in [FG85]. As a reference for GGS-groups, the reader can consult Section 2.3 of the monograph [BGŠ03] by Bartholdi, Grigorchuk, and Šunić, the habilitation thesis [Roz96] of Rozhkov, or the papers [Vov00] by Vovkivsky and [Per00, Per07] by Pervova.

Little is known about the orders of the congruence quotients $G_n$ when $G$ is a GGS-group. In the case that $\mathbf{e} = (1, 0, \dots, 0)$ and $d = p$ is a prime, Šunić found in [Šun07] that, for every $n \geq 2$,

$$\log_p |G_n| = \begin{cases} p^{n-1} + 1, & \text{if } p \text{ is odd,} \\ 2^{n-2} + 2, & \text{if } p = 2. \end{cases}$$

Hence we may always assume that $d \geq 3$, as far as the problem of determining $|G_n|$ is concerned. To the best of our knowledge, the only other cases in which the order of $G_n$ has been determined for every $n$ correspond to $d = 3$. For the Gupta-Sidki group, Sidki himself (see [Sid87]) proved that

$$\log_3 |G_n| = 2 \cdot 3^{n-2} + 1, \quad \text{for every } n \geq 2.$$

On the other hand, for $\mathbf{e} = (1, 1)$, Bartholdi and Grigorchuk showed in [BG02] that

$$\log_3 |G_n| = \frac{3^n + 2n + 3}{4}, \quad \text{for every } n \geq 2.$$

From now onwards, we assume that $d$ is equal to an odd prime $p$, and so $\mathcal{T}$ stands for the $p$-adic tree. The first of our main results in this chapter is the determination of the order of $G_n$ for *all* GGS-groups under this assumption. Before giving the statement of the theorem, we introduce some notation. Recall first the definition of a circulant matrix (see Section 1.3). If $\mathbf{e}$ is the defining vector of a GGS-group, then we write $C(\mathbf{e}, 0)$ for the circulant matrix $C(e_1, \ldots, e_{p-1}, 0)$ over $\mathbb{F}_p$ and we say that $\mathbf{e}$ is *symmetric* if $e_i = e_{p-i}$ for all $i = 1, \ldots, p-1$.

**Theorem D.** *Let $G$ be a GGS-group over the $p$-adic tree, where $p$ is an odd prime, and let $\mathbf{e}$ be the defining vector of $G$. Then, for every $n \geq 2$, we have*

$$\log_p |G_n| = tp^{n-2} + 1 - \delta\, \frac{p^{n-2} - 1}{p - 1} - \varepsilon\, \frac{p^{n-2} - (n-2)p + n - 3}{(p-1)^2},$$

*where $t$ is the rank of the circulant matrix $C(\mathbf{e}, 0)$,*

$$\delta = \begin{cases} 1, & \textit{if } \mathbf{e} \textit{ is symmetric,} \\ 0, & \textit{otherwise,} \end{cases} \qquad \textit{and} \qquad \varepsilon = \begin{cases} 1, & \textit{if } \mathbf{e} \textit{ is constant,} \\ 0, & \textit{otherwise.} \end{cases}$$

Let us define $\Gamma$ as usual in this work and observe that, under the assumption $d = p$ that we have made, all GGS-groups are subgroups of $\Gamma$. According to Theorem 1 of [Vov00], the requirement that $\mathbf{e}$ is non-zero implies that GGS-groups are infinite if $d = p$. Since they are countable groups, their Hausdorff dimension is 0 inside the uncountable group $\Gamma$.

Our second main result is related to the Hausdorff dimension of the closures of GGS-groups.

As an immediate consequence of Theorem D, we get the Hausdorff dimension of the closure of any GGS-group.

**Theorem E.** *Let $G$ be a GGS-group over the $p$-adic tree, where $p$ is an odd prime, and let $\mathbf{e}$ be the defining vector of $G$. Then*

$$\dim_\Gamma \overline{G} = \frac{(p-1)t}{p^2} - \frac{\delta}{p^2} - \frac{\varepsilon}{(p-1)p^2},$$

*where $t$ is the rank of the circulant matrix $C(\mathbf{e}, 0)$,*

$$\delta = \begin{cases} 1, & \textit{if } \mathbf{e} \textit{ is symmetric,} \\ 0, & \textit{otherwise,} \end{cases} \qquad \textit{and} \qquad \varepsilon = \begin{cases} 1, & \textit{if } \mathbf{e} \textit{ is constant,} \\ 0, & \textit{otherwise.} \end{cases}$$

Our proof of Theorem D relies on finding some kind of branch structure inside a GGS-group $G$. In particular, if $\mathbf{e}$ is not constant, we show that $G$ is regular branch (see Subsection 1.1.4 for the definition). This result had been previously proved by Pervova and Rozhkov for *periodic* GGS-groups. On the other hand, it is worth mentioning that the theory of $p$-groups of maximal class plays also a crucial role in the proof of Theorem D, particularly in the case that $\mathbf{e}$ is constant.

In this chapter we adapt the paper [FAZR], that has already been submitted, and whose authors are Gustavo A. Fernández-Alcober and the author of this dissertation.

## 3.2 General properties of GGS-groups

Throughout this chapter, $a$ and $b$ denote the canonical generators of a GGS-group $G$, and $b_i = b^{a^i}$ for every integer $i$. Note that $b_i = b_j$ if $i \equiv j \pmod p$. The images of the elements $b_i$ under the map $\psi$ of the introduction can be easily described:

$$\begin{aligned} \psi(b_0) &= (a^{e_1}, a^{e_2}, \dots, a^{e_{p-1}}, b), \\ \psi(b_1) &= (b, a^{e_1}, \dots, a^{e_{p-2}}, a^{e_{p-1}}), \\ &\vdots \\ \psi(b_{p-1}) &= (a^{e_2}, a^{e_3}, \dots, b, a^{e_1}). \end{aligned} \tag{3.2.1}$$

We begin with some easy facts about GGS-groups.

**Theorem 3.2.1.** *If $G = \langle a, b \rangle$ is a GGS-group, then:*

(i) $\mathrm{Stab}_G(1) = \langle b \rangle^G = \langle b_0, \dots, b_{p-1} \rangle$ *and* $G = \langle a \rangle \ltimes \mathrm{Stab}_G(1)$.

(ii) $\mathrm{Stab}_G(2) \le G' \le \mathrm{Stab}_G(1)$.

(iii) $|G : G'| = p^2$ *and* $|G : \gamma_3(G)| = p^3$.

*Proof.* One can easily check the equalities in part (i). Thus $G/\operatorname{Stab}_G(1)$ is cyclic and $G' \le \operatorname{Stab}_G(1)$.

The quotient $G/G' = \langle aG', bG' \rangle$ is elementary abelian of order at most $p^2$. It follows that $G'/\gamma_3(G) = \langle [a,b]\gamma_3(G) \rangle$ has order at most $p$. If $G' = \gamma_3(G)$ then $\gamma_i(G) = G'$ for every $i \ge 3$. On the other hand, since $G$ is residually a finite $p$-group, the intersection of all the $\gamma_i(G)$ is trivial. Consequently $G' = 1$, which is a contradiction, since $b^a \neq b$ by (3.2.1). We conclude that $|G' : \gamma_3(G)| = p$. Now, if $|G : G'| \le p$ then $G/G'$ is cyclic, and $G' = \gamma_3(G)$. Hence we necessarily have $|G : G'| = p^2$, and (iii) follows.

It only remains to prove that $N = \operatorname{Stab}_G(2)$ is contained in $G'$. Since $|G : G'| = p^2$, it suffices to prove that $|G/N : (G/N)'| = p^2$. If $|G/N : (G/N)'| \le p$ then $G/N$, being a finite $p$-group, must be cyclic. This is a contradiction, since $\langle aN \rangle$ and $\langle bN \rangle$ are two different subgroups of order $p$ in $G/N$. (Note that $\langle bN \rangle$ is contained in $\operatorname{Stab}_G(1)/N$ while $\langle aN \rangle$ is not.) □

Now if $g \in \operatorname{Stab}_G(1)$, it readily follows from (3.2.1) and the previous theorem that $g_i \in G$ for all $i = 1, \ldots, p$. Thus the image of $\operatorname{Stab}_G(1)$ under $\psi$ is actually contained in $G \times \overset{p}{\cdots} \times G$, and so

$$\psi(\operatorname{Stab}_G(k)) \subseteq \operatorname{Stab}_G(k-1) \times \overset{p}{\cdots} \times \operatorname{Stab}_G(k-1) \tag{3.2.2}$$

for all $k \ge 1$. Another important property of the map $\psi$ is the following.

**Proposition 3.2.2.** *If $G$ is a GGS-group, then the composition of $\psi$ with the projection on any component is surjective from* $\operatorname{Stab}_G(1)$ *onto $G$.*

*Proof.* Let us fix a position $i \in \{1, \ldots, p\}$, and let $j \in \{1, \ldots, p-1\}$ be such that $e_j \neq 0$. It follows from (3.2.1) that $\psi(b_{i-j})$ and $\psi(b_i)$ have the entries $a^{e_j}$ and $b$ in the $i$th component. Since $G = \langle a, b \rangle = \langle a^{e_j}, b \rangle$, the result follows. □

Recall that for every positive integer $n$, we can define an isomorphism $\overline{\psi}^n$ from the stabilizer of the first level in $\operatorname{Aut} \mathcal{T}_n$ to the direct product $\operatorname{Aut} \mathcal{T}_{n-1} \times \overset{p}{\cdots} \times \operatorname{Aut} \mathcal{T}_{n-1}$, in the same way as $\psi$ is defined. Since $G_n$ can be seen as a subgroup of $\operatorname{Aut}\mathcal{T}_n$, we can consider the restriction of $\overline{\psi}^n$ to $\operatorname{Stab}_{G_n}(1)$. It follows from (3.2.2) that

$$\overline{\psi}^n(\operatorname{Stab}_{G_n}(k)) \subseteq \operatorname{Stab}_{G_{n-1}}(k-1) \times \overset{p}{\cdots} \times \operatorname{Stab}_{G_{n-1}}(k-1).$$

Obviously, $G_1$ is of order $p$, generated by the image $\overline{a}$ of $a$. Next we deal with $G_2$. Let us write $\tilde{g}$ for the image of an element $g \in G$ in $G_2$. Since $G_2 = \langle \tilde{a} \rangle \ltimes \operatorname{Stab}_{G_2}(1)$, it suffices to understand $\operatorname{Stab}_{G_2}(1) = \langle \tilde{b}_0, \ldots, \tilde{b}_{p-1} \rangle$. Observe that $\overline{\psi}^2$ sends $\operatorname{Stab}_{G_2}(1)$ into $G_1 \times \overset{p}{\cdots} \times G_1$, which can be identified with $\mathbb{F}_p^p$ under the linear map

$$(\overline{a}^{i_1}, \ldots, \overline{a}^{i_p}) \longmapsto (i_1, \ldots, i_p).$$

This allows us to consider $\operatorname{Stab}_{G_2}(1)$ as a vector space over $\mathbb{F}_p$.

Before analyzing $G_2$ in the next theorem, we need the following lemma (see Exercise 4 in Section 1 of the book [Ber08]) about finite $p$-groups of maximal class, which will be also used at some other places in this chapter.

**Lemma 3.2.3.** *Let $P$ be a finite $p$-group such that $|P : P'| = p^2$. If $P$ has an abelian maximal subgroup $A$, then $P$ is a group of maximal class. Furthermore, if $g_0 \in P \setminus A$, then:*

(i) *If $a \in A \setminus \gamma_2(P)$, then $\gamma_2(P)/\gamma_3(P)$ is generated by the image of $[a, g_0]$.*

(ii) *If $i \geq 2$ and $a \in \gamma_i(P) \setminus \gamma_{i+1}(P)$, then $\gamma_{i+1}(P)/\gamma_{i+2}(P)$ is generated by the image of $[a, g_0]$.*

**Theorem 3.2.4.** *Let $G$ be a GGS-group with defining vector $\mathbf{e}$, and put $C = C(\mathbf{e}, 0)$. Then:*

(i) *The dimension of $\operatorname{Stab}_{G_2}(1)$ coincides with the rank $t$ of $C$.*

(ii) *$G_2$ is a $p$-group of maximal class of order $p^{t+1}$.*

*Proof.* (i) If $\tilde{g} \in \operatorname{Stab}_{G_2}(1)$ and $\overline{\psi}^2(\tilde{g}) = (\overline{a}^{i_1}, \ldots, \overline{a}^{i_p})$, where we consider the exponents $i_1, \ldots, i_p$ as elements of $\mathbb{F}_p$, we define

$$\overline{\Psi}^2(\tilde{g}) = (i_1, \ldots, i_p) \in \mathbb{F}_p^p.$$

Observe that $\overline{\Psi}^2$ is injective.

By (3.2.1),

$$\overline{\Psi}^2(\tilde{b}_0) = (e_1, e_2, \ldots, e_{p-1}, 0) = (\mathbf{e}, 0)$$

coincides with the first row of $C$. Since the components of the rest of the $b_i$ are obtained by permuting cyclically those of $b_0$, and since $C = C(\mathbf{e}, 0)$, it follows that $\overline{\Psi}^2(\tilde{b}_i)$ is the $(i+1)$st row of $C$. Thus the dimension of $\operatorname{Stab}_{G_2}(1)$ coincides with the dimension of the subspace of $\mathbb{F}_p^p$ generated by the rows of $C$, i.e. with the rank $t$ of the matrix $C$.

(ii) We have

$$|G_2| = |G_2 : \operatorname{Stab}_{G_2}(1)||\operatorname{Stab}_{G_2}(1)| = p \cdot p^t = p^{t+1}.$$

On the other hand, it follows from (ii) and (iii) of Theorem 3.2.1 that $|G_2 : G_2'| = p^2$. Since $\operatorname{Stab}_{G_2}(1)$ is an abelian maximal subgroup of $G_2$, we conclude from Lemma 3.2.3 that $G_2$ is a $p$-group of maximal class. $\square$

As a consequence, we can improve part (ii) of Theorem 3.2.1.

**Corollary 3.2.5.** *If $G$ is a GGS-group, then* $\operatorname{Stab}_G(2) \leq \gamma_3(G)$.

*Proof.* Since the defining vector $\mathbf{e}$ of $G$ is different from $(0, \ldots, 0)$, it is clear that the rank $t$ of the matrix $C(\mathbf{e}, 0)$ is at least 2. It follows from the previous theorem that $G_2 = G/\operatorname{Stab}_G(2)$ is a $p$-group of maximal class of order greater than or equal to $p^3$. Thus $|G_2 : \gamma_3(G_2)| = p^3 = |G : \gamma_3(G)|$, and consequently $\operatorname{Stab}_G(2)$ is contained in $\gamma_3(G)$. $\square$

We have seen in Theorem 3.2.1 that $G' \leq \operatorname{Stab}_G(1)$. Next we want to characterize which elements of $\operatorname{Stab}_G(1)$ belong to $G'$. This goal will be achieved in Theorem 3.2.10. If $g \in \operatorname{Stab}_G(1) = \langle b_0, \ldots, b_{p-1} \rangle$, then we can write $g$ as a word in $b_0, \ldots, b_{p-1}$, i.e. we can write $g = \omega(b_0, \ldots, b_{p-1})$, where $\omega = \omega(x_0, \ldots, x_{p-1})$ is a group word in the $p$ variables $x_0, \ldots, x_{p-1}$.

**Definition 3.2.6.** Let $\omega$ be a group word in the variables $x_0, \ldots, x_{p-1}$, where $p$ is a prime. Then:

(i) The *partial p-weight* of $\omega$ with respect to a variable $x_i$, with $0 \leq i \leq p-1$, is the sum of the exponents of $x_i$ in the expression for $\omega$, considered as an element of $\mathbb{F}_p$.

(ii) The *total p-weight* of $\omega$ is the sum of all its partial $p$-weights.

It is not difficult to give examples showing that the representation of an element $g \in \mathrm{Stab}_G(1)$ as a word in $b_0, \ldots, b_{p-1}$ is not unique. Our first step towards the proof of Theorem 3.2.10 will be to see that, however, the partial and total $p$-weights are the same for all word representations.

Let $g = \omega(b_0, \ldots, b_{p-1})$ be an arbitrary element of $\mathrm{Stab}_G(1)$, and suppose that the partial $p$-weight of $\omega$ with respect to $x_i$ is $r_i$, for $i = 0, \ldots, p-1$. It follows from (3.2.1) that

$$\psi(g) = (a^{m_1}\omega_1(b_0, \ldots, b_{p-1}), \ldots, a^{m_p}\omega_p(b_0, \ldots, b_{p-1})), \tag{3.2.3}$$

where each $\omega_i$ is a word of total $p$-weight $r_i$ (and where $r_p$ is to be understood as $r_0$), and

$$m_i = (r_0 \; r_1 \; \ldots \; r_{p-1})C^i. \tag{3.2.4}$$

**Theorem 3.2.7.** *Let $G$ be a GGS-group, and let $g \in \mathrm{Stab}_G(1)$. Then the partial and total p-weights are the same for all representations of $g$ as a word in $b_0, \ldots, b_{p-1}$.*

*Proof.* It suffices to see that, if $\omega$ is a word such that $\omega(b_0, \ldots, b_{p-1}) = 1$, then the total $p$-weight of $\omega$ is 0, and the partial $p$-weight $r_i$ of $\omega$ with respect to $x_i$ is equal to 0, for every $i = 0, \ldots, p-1$. Obviously, the second assertion implies the first one, but the proof will go the other way around.

As in (3.2.3), we write

$$\psi(\omega(b_0, \ldots, b_{p-1})) = (a^{m_1}\omega_1(b_0, \ldots, b_{p-1}), \ldots, a^{m_p}\omega_p(b_0, \ldots, b_{p-1})). \tag{3.2.5}$$

Since this element is equal to 1, it follows that $m_i = 0$ for $i = 1, \ldots, p$. According to (3.2.4), this means that

$$(r_0 \; r_1 \; \ldots \; r_{p-1})C = (0 \; 0 \; \ldots \; 0).$$

Now, since $\operatorname{rk} C = \operatorname{rk}(C \mid \mathbf{1})$ by Lemma 1.3.1, we also have $(r_0\, r_1\, \ldots\, r_{p-1})\mathbf{1} = 0$, that is,

$$r_0 + r_1 + \cdots + r_{p-1} = 0.$$

This proves that the total $p$-weight of $\omega$ is 0.

Now we return to (3.2.5). Since $\omega(b_0, \ldots, b_{p-1}) = 1$ by hypothesis, then we also have $\omega_i(b_0, \ldots, b_{p-1}) = 1$ for all $i = 1, \ldots, p$. Now, since the total $p$-weight of $\omega_i$ is $r_i$, it follows from the previous paragraph that $r_i = 0$. $\square$

The independence of the partial and total $p$-weights from the word representation allows us to give the following definition.

**Definition 3.2.8.** Let $G$ be a GGS-group, and let $g \in \operatorname{Stab}_G(1)$. We define the *partial weight* of $g$ with respect to $b_i$, and the *total weight* of $g$, as the corresponding $p$-weights for any word $\omega$ representing $g$.

We prefer to speak simply about weights instead of $p$-weights in the case of an element $g \in \operatorname{Stab}_G(1)$, since all elements $b_i$ (with respect to which the weights are considered) have order $p$. Now the following result is clear.

**Theorem 3.2.9.** *Let $G$ be a GGS-group. Then the maps from $\operatorname{Stab}_G(1)$ to $\mathbb{F}_p$ sending every $g \in \operatorname{Stab}_G(1)$ to its partial weight with respect to one of the $b_i$ or to its total weight are well-defined homomorphisms.*

**Theorem 3.2.10.** *Let $G$ be a GGS-group. Then the derived subgroup $G'$ consists of all the elements of $\operatorname{Stab}_G(1)$ whose total weight is equal to $0$.*

*Proof.* The map $\vartheta$ sending each element of $\operatorname{Stab}_G(1)$ to its total weight is a homomorphism onto the abelian group $\mathbb{F}_p$, and consequently $G' \leq \ker \vartheta$. Since $|G : G'| = p^2$ and $|G : \operatorname{Stab}_G(1)| = |\operatorname{Stab}_G(1) : \ker \vartheta| = p$, the equality follows. $\square$

**Definition 3.2.11.** Let $G$ be a GGS-group. If $g \in \operatorname{Stab}_G(1)$ has partial weight $r_i$ with respect to $b_i$ for $i = 0, \ldots, p-1$, we say that $(r_0, \ldots, r_{p-1}) \in \mathbb{F}_p^p$ is the *weight vector* of $g$.

As we next see, we can analyze the subgroups $\mathrm{Stab}_G(2)$ and $\mathrm{Stab}_G(3)$ by using the weight vector.

**Theorem 3.2.12.** *Let $G$ be a GGS-group with defining vector $\mathbf{e}$, and put $C = C(\mathbf{e}, 0)$. If the weight vector of $g \in \mathrm{Stab}_G(1)$ is $(r_0, \ldots, r_{p-1})$, then:*

(i) *We have $g \in \mathrm{Stab}_G(2)$ if and only if $(r_0 \ \ldots \ r_{p-1})C = (0 \ \ldots \ 0)$.*

(ii) *If $g \in \mathrm{Stab}_G(3)$ then $(r_0, \ldots, r_{p-1}) = (0, \ldots, 0)$.*

*Proof.* (i) If we write $\psi(g)$ as in (3.2.3), then $g \in \mathrm{Stab}_G(2)$ if and only if $m_i = 0$ in $\mathbb{F}_p$ for every $i = 1, \ldots, p$. Now, by (3.2.4), this is equivalent to the condition $(r_0 \ \ldots \ r_{p-1})C = (0 \ \ldots \ 0)$.

(ii) Again we use the expression in (3.2.3). If $g \in \mathrm{Stab}_G(3)$ then $\omega_i(b_0, \ldots, b_{p-1}) \in \mathrm{Stab}_G(2)$ for all $i = 1, \ldots, p$. As mentioned above, $\omega_i(b_0, \ldots, b_{p-1})$ is an element of total weight $r_i$. Let $(s_0, \ldots, s_{p-1})$ be the weight vector of this element, so that $r_i = s_0 + \cdots + s_{p-1}$. Then, by (i), we have $(s_0 \ \ldots \ s_{p-1})C = (0 \ \ldots \ 0)$. Since $\mathrm{rk}\, C = \mathrm{rk}(C \mid \mathbf{1})$ by Lemma 1.3.1, it follows that $r_i = s_0 + \cdots + s_{p-1} = 0$, as desired. □

One may wonder whether the converse holds in (ii) of the previous theorem, i.e. if the weight vector of an element is $(0, \ldots, 0)$, does it lie in $\mathrm{Stab}_G(3)$? We make things clearer in the following theorem.

**Theorem 3.2.13.** *Let $G$ be a GGS-group. Then $\mathrm{Stab}_G(1)'$ consists of all elements of $\mathrm{Stab}_G(1)$ whose weight vector is $(0, \ldots, 0)$. Furthermore, we have $|G : \mathrm{Stab}_G(1)'| = p^{p+1}$.*

*Proof.* The map $\rho$ which sends every element of $\mathrm{Stab}_G(1)$ to its weight vector is a homomorphism onto $\mathbb{F}_p^p$. Thus $|\mathrm{Stab}_G(1) : \ker \rho| = p^p$. Since $\mathbb{F}_p^p$ is abelian, it follows that $\mathrm{Stab}_G(1)' \leq \ker \rho$. On the other hand, since $\mathrm{Stab}_G(1) = \langle b_0, \ldots, b_{p-1} \rangle$ and every $b_i$ has order $p$, we have $|\mathrm{Stab}_G(1) : \mathrm{Stab}_G(1)'| \leq p^p$. Hence $\ker \rho = \mathrm{Stab}_G(1)'$ and $|\mathrm{Stab}_G(1) : \mathrm{Stab}_G(1)'| = p^p$. Since $|G : \mathrm{Stab}_G(1)| = p$, we are done. □

In particular, we have $\operatorname{Stab}_G(3) \leq \operatorname{Stab}_G(1)'$. Once we prove Theorem D, it will follow that $|G : \operatorname{Stab}_G(3)| = p^{tp+1-\delta}$, where $t$ is the rank of $C(\mathbf{e}, 0)$ and $\delta$ is 1 or 0, according as $\mathbf{e}$ is symmetric or not. Since $t$ is always at least 2, we have $|G : \operatorname{Stab}_G(3)| > p^{p+1}$ in every case. Hence $\operatorname{Stab}_G(3)$ is always a proper subgroup of $\operatorname{Stab}_G(1)'$, and the converse of (ii) in Theorem 3.2.12 does not hold.

Next we prove a result which will allow us to reduce, for the calculation of the order of congruence quotients and of the Hausdorff dimension, to the case of GGS-groups with defining vectors of the form $\mathbf{e} = (1, e_2, \ldots, e_{p-1})$. We need the following lemma.

**Lemma 3.2.14.** *Let $p$ be a prime, and let $\sigma = (1\ 2\ \ldots\ p)$. Assume that $\alpha \in S_p$ satisfies the following two conditions:*

(i) *$\alpha$ normalizes the subgroup $\langle \sigma \rangle$.*

(ii) *$\alpha(p) = p$.*

*Then, for every $i = 1, \ldots, p-1$, if $\alpha(i) = j$ we have $\alpha(p-i) = p-j$.*

*Proof.* If we think of $S_p$ as the set of permutations of the field $\mathbb{F}_p$, then $\sigma$ corresponds to the map $\ell \mapsto \ell+1$, and the normalizer of $\langle \sigma \rangle$ in $S_p$ corresponds to the affine group over $\mathbb{F}_p$ (see Lemma 14.1.2 of [Cox04]). Thus $\alpha(\ell) = a\ell + b$ for some $a \in \mathbb{F}_p^\times$ and $b \in \mathbb{F}_p$. Since $\alpha(p) = p$, it follows that $b = 0$, and so $\alpha(\ell) = a\ell$ for every $\ell \in \mathbb{F}_p$. Hence $\alpha$ is a linear map and, as a consequence,

$$\alpha(p-i) = \alpha(-i) = -\alpha(i) = -j = p-j.$$

□

We say that an automorphism $f$ of $\mathcal{T}$ has *constant portrait* if $f$ has the same label at all vertices of $\mathcal{T}$. By formula (1.1.1) for the labels of a composition, the set of all automorphisms of constant portrait is a subgroup of $\operatorname{Aut} \mathcal{T}$.

**Theorem 3.2.15.** *Let $G$ be a GGS-group with defining vector $\mathbf{e} = (e_1, \ldots, e_{p-1})$, and assume that $e_k \neq 0$. Then there exists $f \in \operatorname{Aut} \mathcal{T}$ of constant portrait such that $L = G^f$ is a GGS-group whose defining vector $\mathbf{e}' = (e'_1, \ldots, e'_{p-1})$ satisfies:*

(i) *$\mathbf{e}'$ is a permutation of the vector $\mathbf{e}/e_k$, that is, there exists $\alpha \in S_{p-1}$ such that $e'_i = e_{\alpha(i)}/e_k$ for all $i = 1, \ldots, p-1$.*

(ii) *$\alpha(1) = k$, and so $e'_1 = 1$.*

(iii) *If $\alpha(i) = j$ then $\alpha(p-i) = p-j$. In other words, two values which are placed in symmetric positions of $\mathbf{e}$ are moved (after division by $e_k$) to symmetric positions of $\mathbf{e}'$. Thus $\mathbf{e}'$ is symmetric if and only if $\mathbf{e}$ is.*

(iv) *$\operatorname{rk} C(\mathbf{e}, 0) = \operatorname{rk} C(\mathbf{e}', 0)$.*

*Furthermore, we have $|G_n| = |L_n|$ for every $n$, and $\dim_\Gamma \overline{G} = \dim_\Gamma \overline{L}$.*

*Proof.* Observe that there exists a permutation $\beta \in S_p$, in fact only one, that normalizes the subgroup $\langle \sigma \rangle$ and such that $\beta(k) = 1$ and $\beta(p) = p$. Indeed, since $\sigma^\beta = (\beta(1) \ \ldots \beta(p))$ and the positions of 1 and $p$ are already fixed in this last tuple, there is only one way to choose the rest of the images of $\beta$ if we want to obtain a power of $\sigma$. Let $r$ be defined by the condition $\sigma^\beta = \sigma^r$, and set $\alpha = \beta^{-1}$. Note that $\alpha(1) = k$ and that, by Lemma 3.2.14, if $\alpha(i) = j$ then $\alpha(p-i) = p-j$.

Now we define an automorphism $f$ of $\mathcal{T}$ by choosing the labels at all vertices of $\mathcal{T}$ equal to $\beta$. We claim that $L = G^f$ satisfies the properties of the statement of the theorem. We have

$$(g^f)_{(v)} = \beta^{-1} g_{(f^{-1}(v))} \beta$$

for every $g \in G$ and every vertex $v$ of the tree. It readily follows that $a^f = a^r$. We now consider $c = b^f$. Let $S$ be the set of all vertices of the form $p.\overset{n}{\ldots}.pi$, where $n \geq 0$ and $1 \leq i \leq p-1$. If $v \in S$, then we have $f(v) = p.\overset{n}{\ldots}.p\beta(i)$, and consequently $f^{-1}(v) = p.\overset{n}{\ldots}.p\alpha(i)$. Thus

$$c_{(v)} = \beta^{-1} b_{(p \ldots p\alpha(i))} \beta = (\sigma^{e_{\alpha(i)}})^\beta = \sigma^{r e_{\alpha(i)}}$$

in this case. On the other hand, if $v \notin S$, then also $f^{-1}(v) \notin S$, and so we have $b_{(f^{-1}(v))} = 1$ and $c_{(v)} = 1$. Thus $c$ is the automorphism given by the recursive relation

$$\psi(c) = (a^{re_{\alpha(1)}}, \ldots, a^{re_{\alpha(p-1)}}, c).$$

Now, let $\ell$ be the inverse of $re_{\alpha(1)}$ modulo $p$, and put $b' = c^{\ell}$. Then $L = \langle a, b' \rangle$, where $b'$ is the automorphism defined by

$$\psi(b') = (a^{e'_1}, \ldots, a^{e'_{p-1}}, b'),$$

i.e. $L$ is the GGS-group with defining vector $\mathbf{e}'$. This proves (i), (ii), and (iii).

Let us now check (iv). If $C = C(\mathbf{e}, 0)$, $C' = C(\mathbf{e}', 0)$ and we define $e_p = 0$, then

$$c'_{ij} = e_{\alpha(j-i+1)}/e_k = e_{\alpha(j)-\alpha(i)+\alpha(1)}/e_k = c_{\alpha(i)-\alpha(1)+1,\alpha(j)}/e_k,$$

since we know that $\alpha$ is a homomorphism by the proof of Lemma 3.2.14. (Here, all indices are taken modulo $p$ between 1 and $p$.) By observing that the maps $i \mapsto \alpha(i) - \alpha(1) + 1$ and $j \mapsto \alpha(j)$ are permutations of $\mathbb{F}_p$, we conclude that $\operatorname{rk} C = \operatorname{rk} C'$.

Finally, since $G$ and $L$ are conjugate, we obviously have $|G_n| = |L_n|$, and by Lemma 1.2.6, also $\dim_\Gamma \overline{G} = \dim_\Gamma \overline{L}$. □

We want to stress the fact that the automorphism $f$ conjugating $G$ to $L$ in the previous theorem has constant portrait. This has nice consequences, such as the following one.

**Proposition 3.2.16.** *Let $J$ and $K$ be two subgroups of* $\operatorname{Aut}\mathcal{T}$*, where $J$ is contained in* $\operatorname{Stab}(1)$*. If $f \in \operatorname{Aut}\mathcal{T}$ has constant portrait, then we have*

$$K \times \overset{p}{\cdots} \times K \subseteq \psi(J)$$

*if and only if*

$$K^f \times \overset{p}{\cdots} \times K^f \subseteq \psi(J^f).$$

*Proof.* Since $f^{-1}$ is also an automorphism of constant portrait, it suffices to prove the 'only if' part. Let $\beta$ be the permutation appearing at all labels of $f$. Then we can write $f = ch$, where $c$ is the rooted automorphism corresponding to $\beta$ and $h \in \mathrm{Stab}(1)$ is such that $\psi(h) = (f, \ldots, f)$.

Let us now consider an arbitrary tuple $(k_1, \ldots, k_p)$, with $k_i \in K$ for every $i = 1, \ldots, p$. By hypothesis, there exists $j \in J$ such that $\psi(j) = (k_1, \ldots, k_p)$. Then $\psi(j^c) = (k_{\beta^{-1}(1)}, \ldots, k_{\beta^{-1}(p)})$, and consequently

$$\psi(j^f) = \psi(j^c)^{\psi(h)} = (k_{\beta^{-1}(1)}, \ldots, k_{\beta^{-1}(p)})^{(f,\ldots,f)} = (k^f_{\beta^{-1}(1)}, \ldots, k^f_{\beta^{-1}(p)}).$$

Clearly, this implies that $K^f \times \cdots \times K^f \subseteq \psi(J^f)$. $\square$

The previous proposition will be useful when we want to find a branch structure in a GGS-group. The same can be said about the following result.

**Proposition 3.2.17.** *Let $G$ be a GGS-group, and let $L$ and $N$ be two normal subgroups of $G$. If $L = \langle X \rangle^G$ for a subset $X$ of $G$, and $(x, 1, \ldots, 1) \in \psi(N)$ for every $x \in X$, then*

$$L \times \overset{p}{\cdots} \times L \subseteq \psi(N).$$

*Proof.* By Proposition 3.2.2, if $g \in G$ there exists $h \in \mathrm{Stab}_G(1)$ such that the first component of $\psi(h)$ is $g$. Since $(x, 1, \ldots, 1) \in \psi(N)$ and $N$ is normal in $G$, it follows that $(x^g, 1, \ldots, 1) \in \psi(N)$ for every $x \in X$ and $g \in G$. Hence

$$L \times \{1\} \times \overset{p-1}{\cdots} \times \{1\} \subseteq \psi(N),$$

since $L = \langle x^g \mid x \in X,\ g \in G \rangle$.

Now, if $\psi(n) = (\ell_1, \ell_2, \ldots, \ell_p)$ then $\psi(n^a) = (\ell_p, \ell_1, \ldots, \ell_{p-1})$. As a consequence,

$$\{1\} \times \cdots \times \{1\} \times L \times \{1\} \times \cdots \times \{1\} \subseteq \psi(N),$$

where $L$ may appear at any position. The result follows. $\square$

## 3.3 GGS-groups with non-constant defining vector

In this section we prove Theorems D and E in the case that the defining vector $\mathbf{e}$ of the GGS-group $G$ is not constant. As it turns out, the key is to prove that $G$ has a certain branch structure. Recall the concepts given in Subsection 1.1.4.

It is well-known (and an immediate consequence of Proposition 3.2.2) that every GGS-group $G$ is self-similar and spherically transitive. We next see that, if $\mathbf{e}$ is not constant, then $G$ is regular branch over $\gamma_3(G)$.

**Lemma 3.3.1.** *Let $G$ be a GGS-group with non-constant defining vector $\mathbf{e}$. Then*

$$\psi(\gamma_3(\mathrm{Stab}_G(1))) = \gamma_3(G) \times \overset{p}{\cdots} \times \gamma_3(G).$$

*In particular,*

$$\gamma_3(G) \times \overset{p}{\cdots} \times \gamma_3(G) \subseteq \psi(\gamma_3(G))$$

*and $G$ is a regular branch group over $\gamma_3(G)$.*

*Proof.* Since $\psi(\mathrm{Stab}_G(1))$ is contained in $G \times \overset{p}{\cdots} \times G$, it clearly suffices to prove the inclusion $\supseteq$. By Theorem 3.2.15 and Proposition 3.2.16, we may assume that $\mathbf{e} = (1, e_2, \ldots, e_{p-1})$. If $e_{p-1} = 0$ then

$$\psi(b) = (a, \ldots, a^{e_{p-2}}, 1, b),$$

and consequently

$$\psi([b_0, b_1, b_0]) = ([a, b, a], 1, \ldots, 1)$$

and

$$\psi([b_0, b_1, b_1]) = ([a, b, b], 1, \ldots, 1).$$

Since $G = \langle a, b\rangle$, it follows that $\gamma_3(G) = \langle [a, b, a], [a, b, b]\rangle^G$, and then by Proposition 3.2.17, we have $\gamma_3(G) \times \cdots \times \gamma_3(G) \subseteq \psi(\gamma_3(\mathrm{Stab}_G(1)))$. Thus we may assume that $e_{p-1} \neq 0$.

Now we consider the following two cases separately:

(i) There exists $k \in \{2, \ldots, p-2\}$ such that $(e_{k-1}, e_k)$ and $(e_k, e_{k+1})$ are not proportional.

(ii) $(e_{k-1}, e_k)$ and $(e_k, e_{k+1})$ are proportional for all $k = 2, \ldots, p-2$.

Observe that if $p = 3$ then case (ii) vacuously holds.

(i) Let us put

$$g_k = b_{p-k+1}^{e_k} b_{p-k}^{-e_{k-1}}$$

for $2 \leq k \leq p-2$, so that

$$\psi(g_k) = (a^{e_k^2 - e_{k-1}e_{k+1}}, \ldots, 1).$$

(The intermediate values represented by the dots are not necessarily 1 in this case.) Since $(e_{k-1}, e_k)$ and $(e_k, e_{k+1})$ are not proportional, we have $e_k^2 - e_{k-1}e_{k+1} \neq 0$. Hence there is a power $g$ of $g_k$ such that

$$\psi(g) = (a, \ldots, 1).$$

On the other hand, since

$$\psi(b_1 b_{p-1}^{-e_{p-1}}) = (ba^{-e_2 e_{p-1}}, \ldots, 1),$$

with the help of $g$ we can get an element $h \in \mathrm{Stab}_G(1)$ such that

$$\psi(h) = (b, \ldots, 1).$$

Consequently,

$$\psi([b_0, b_1, g]) = ([a, b, a], 1, \ldots, 1)$$

and

$$\psi([b_0, b_1, h]) = ([a, b, b], 1, \ldots, 1),$$

and the result follows as before from Proposition 3.2.17.

(ii) Since $e_1 = 1$, it follows that $e_i = e_2^{i-1}$ for every $i = 1, \ldots, p-1$. (Note that this is valid all the same if $p = 3$.) Hence $\mathbf{e} = (1, m, m^2, \ldots, m^{p-2})$ with

$m \neq 1$, because $\mathbf{e}$ is not constant. Since $e_{p-1} \neq 0$, we also have $m \neq 0$, and consequently $m^{p-1} = 1$. Then

$$\psi(b_0 b_1^{-m}) = (ab^{-m}, 1, \dots, 1, ba^{-1})$$

and

$$\psi(b_1 b_2^{-m}) = (ba^{-1}, ab^{-m}, 1, \dots, 1).$$

Hence

$$\psi([b_0, b_1, b_1 b_2^{-m}]) = ([a, b, ba^{-1}], 1, \dots, 1)$$

and

$$\psi([b_2^m, b_1, b_0 b_1^{-m}]) = ([a, b, ab^{-m}], 1, \dots, 1).$$

Now, since $G' = \langle [a, b] \rangle^G$ and $\langle ab^{-m}, ba^{-1} \rangle = \langle b^{1-m}, ba^{-1} \rangle$ is the whole of $G$ (at this point, it is essential that $m \neq 1$), it follows that

$$\gamma_3(G) = \langle [a, b, ab^{-m}], [a, b, ba^{-1}] \rangle^G.$$

Thus the result is again a consequence of Proposition 3.2.17. □

As a consequence of the previous lemma, we can show that, for $\mathbf{e}$ non-constant and $n \geq 3$, there is a close relation between $\mathrm{Stab}_G(n)$ and $\mathrm{Stab}_G(n-1)$ in a GGS-group $G$.

**Lemma 3.3.2.** *Let $G$ be a GGS-group with non-constant defining vector $\mathbf{e}$. Then, for every $n \geq 3$ we have*

$$\psi(\mathrm{Stab}_G(n)) = \mathrm{Stab}_G(n-1) \times \overset{p}{\cdots} \times \mathrm{Stab}_G(n-1)$$

*and*

$$\overline{\psi}^{n+1}(\mathrm{Stab}_{G_{n+1}}(n)) = \mathrm{Stab}_{G_n}(n-1) \times \overset{p}{\cdots} \times \mathrm{Stab}_{G_n}(n-1).$$

*Proof.* Clearly, it suffices to prove the first equality. By using Corollary 3.2.5 and Lemma 3.3.1, we have

$$\mathrm{Stab}_G(2) \times \overset{p}{\cdots} \times \mathrm{Stab}_G(2) \subseteq \gamma_3(G) \times \overset{p}{\cdots} \times \gamma_3(G) \subseteq \psi(\gamma_3(G)).$$

Thus $\mathrm{Stab}_G(n-1) \times \cdots \times \mathrm{Stab}_G(n-1)$ is contained in the image of $\mathrm{Stab}_G(1)$ under $\psi$ for all $n \geq 3$, and the result follows. □

If the vector $\mathbf{e}$ is non-symmetric, we can improve Lemma 3.3.1 as follows.

**Lemma 3.3.3.** *Let $G$ be a GGS-group with non-symmetric defining vector. Then*

$$\psi(\mathrm{Stab}_G(1)') = G' \times \overset{p}{\cdots} \times G'.$$

*In particular,*

$$G' \times \overset{p}{\cdots} \times G' \subseteq \psi(G')$$

*and $G$ is a regular branch group over $G'$.*

*Proof.* By Theorem 3.2.15 and Proposition 3.2.16, we may assume that $e_1 = 1$ and $e_{p-1} \neq 1$, since $\mathbf{e}$ is non-symmetric. Let us write $m$ for $e_{p-1}$.

By using (3.2.1), we get

$$\begin{aligned}
\psi([b_0, b_1]) &= ([a,b], 1, \ldots, 1, [b, a^m]) \\
&\equiv ([a,b], 1, \ldots, 1, [a,b]^{-m}) \pmod{\gamma_3(G) \times \overset{p}{\cdots} \times \gamma_3(G)}, \\
\psi([b_{p-1}, b_0]^m) &= (1, \ldots, 1, [b, a^m]^m, [a,b]^m) \\
&\equiv (1, \ldots, 1, [a,b]^{-m^2}, [a,b]^m) \pmod{\gamma_3(G) \times \overset{p}{\cdots} \times \gamma_3(G)}, \\
&\vdots \\
\psi([b_1, b_2]^{m^{p-1}}) &= ([b, a^m]^{m^{p-1}}, [a,b]^{m^{p-1}}, 1, \ldots, 1) \\
&\equiv ([a,b]^{-m^p}, [a,b]^{m^{p-1}}, 1, \ldots, 1) \pmod{\gamma_3(G) \times \overset{p}{\cdots} \times \gamma_3(G)}.
\end{aligned}$$

Since $m^p = m$ (recall that $m \in \mathbb{F}_p$), if we multiply together all the expressions above, we obtain that

$$\psi([b_0, b_1][b_{p-1}, b_0]^m \ldots [b_1, b_2]^{m^{p-1}}) \equiv ([a,b]^{1-m}, 1, \ldots, 1) \pmod{\gamma_3(G) \times \overset{p}{\cdots} \times \gamma_3(G)}.$$

If we use the inclusion

$$\gamma_3(G) \times \overset{p}{\cdots} \times \gamma_3(G) \subseteq \psi(\mathrm{Stab}_G(1)'),$$

which is a consequence of Lemma 3.3.1, we get

$$([a,b]^{1-m}, 1, \ldots, 1) \in \psi(\mathrm{Stab}_G(1)').$$

Now, since $G = \langle a, b\rangle$ and $m \neq 1$, it follows that $G'$ is the normal closure of $[a,b]^{1-m}$. By Proposition 3.2.17, we conclude that $G' \times \cdots \times G' \subseteq \psi(\mathrm{Stab}_G(1)')$. $\square$

Now we can proceed to calculate the order of $G_n$ for every $n \geq 1$, and as a consequence, to obtain the Hausdorff dimension of $\overline{G}$ in $\Gamma$, provided that the defining vector $\mathbf{e}$ is not constant. We deal separately with the following two cases: (i) $\mathbf{e}$ is not symmetric; (ii) $\mathbf{e}$ is symmetric and not constant. In both cases, the key is to determine the order of $\mathrm{Stab}_{G_3}(2)$ and to use Lemma 3.3.2. We begin by case (i).

**Theorem 3.3.4.** *Let $G$ be a GGS-group with non-symmetric defining vector $\mathbf{e}$. Then*

$$|\,\mathrm{Stab}_{G_3}(2)| = p^{t(p-1)},$$

*where $t$ is the rank of $C(\mathbf{e},0)$.*

*Proof.* By Theorem 3.2.15, we may assume that $e_1 = 1$ and $e_{p-1} \neq 1$. For simplicity, let us write $C$ for $C(\mathbf{e},0)$. Since $\mathrm{Stab}_{G_3}(2) = \mathrm{Stab}_G(2)/\mathrm{Stab}_G(3)$, we are going to study the image of $\mathrm{Stab}_G(2)$ under the canonical epimorphism $\pi_3 : \mathrm{Aut}\,\mathcal{T} \to \mathrm{Aut}\,\mathcal{T}_3$ that takes $G$ onto $G_3$.

Let $g$ be an arbitrary element of $\mathrm{Stab}_G(1)$, and let $(r_0, \ldots, r_{p-1})$ denote the weight vector of $g$. By Theorem 3.2.12, we have $g \in \mathrm{Stab}_G(2)$ if and only if

$$(r_0\ r_1\ \ldots\ r_{p-1})C = (0\ 0\ \ldots\ 0).$$

Since the rank of $C$ is $t$, this system has $p^{p-t}$ solutions, which we denote by

$$r^{(i)} = (r_0^{(i)}, \ldots, r_{p-1}^{(i)}),$$

for $i = 1, \ldots, p^{p-t}$. We may assume that $r^{(1)} = (0, \ldots, 0)$.

Each solution $r^{(i)}$ determines a subset $R^{(i)}$ of $\mathrm{Stab}_G(2)$, consisting of all the elements whose weight vector is $r^{(i)}$. Put $S^{(i)} = \pi_3(R^{(i)})$. By the discussion in the previous paragraph, we know that $\mathrm{Stab}_{G_3}(2)$ is the union of all the $S^{(i)}$ for $i = 1, \ldots, p^{p-t}$. We will prove the following:

(i) If $i \neq j$ then $S^{(i)}$ and $S^{(j)}$ are disjoint. (By Theorem 3.2.7, we know that $R^{(i)}$ and $R^{(j)}$ are disjoint, but we have to rule out the possibility that an element in $R^{(i)}$ and an element in $R^{(j)}$ have the same image in $G_3$.)

(ii) $|S^{(i)}| = p^{p(t-1)}$ for all $i = 1, \dots, p^{p-t}$.

Once (i) and (ii) are proved, it readily follows that $|\operatorname{Stab}_{G_3}(2)| = p^{t(p-1)}$, as desired.

We begin by proving (i). For this purpose, assume that $g \in R^{(i)}$ and $h \in R^{(j)}$ are two elements with the same image in $G_3$. Then $gh^{-1} \in \operatorname{Stab}_G(3)$ and, by Theorem 3.2.12, the weight vector of $gh^{-1}$ is $(0, \dots, 0)$. Since the weight vector defines a homomorphism from $\operatorname{Stab}_G(1)$ to $\mathbb{F}_p^p$, it follows that $r^{(i)} = r^{(j)}$, and so $i = j$, as desired.

Now we proceed to the proof of (ii). By definition, each $S^{(i)}$ is non-empty. If $h_i$ is an element of $S^{(i)}$, then it is clear that $S^{(i)} = h_i S^{(1)}$. Thus $|S^{(i)}| = |S^{(1)}|$, and it suffices to see that $S^{(1)}$ has the desired cardinality. Let $g$ be an arbitrary element of $\operatorname{Stab}_G(2)$. According to (3.2.3), we have $g \in R^{(1)}$ if and only if each component of $\psi(g)$ has total weight equal to 0. By Theorem 3.2.10, this is equivalent to $\psi(g)$ lying in $G' \times \cdots \times G'$. On the other hand, since $G' \leq \operatorname{Stab}_G(1)$, we have $\psi^{-1}(G' \times \cdots \times G') \leq \operatorname{Stab}(2)$. Hence

$$R^{(1)} = G \cap \psi^{-1}(G' \times \cdots \times G'). \tag{3.3.1}$$

Note that this equality is valid for any defining vector $\mathbf{e}$. Now, since we are working under the assumption that $\mathbf{e}$ is non-symmetric, we have $G' \times \cdots \times G' \leq \psi(G')$ by Lemma 3.3.3. Thus we conclude that $R^{(1)} = \psi^{-1}(G' \times \cdots \times G')$ in this case or, equivalently, that

$$\psi(R^{(1)}) = G' \times \cdots \times G'.$$

We consider now the following commutative diagram:

$$\begin{array}{ccc} R^{(1)} & \xrightarrow{\pi_3} & S^{(1)} \\ \psi\Big\downarrow & & \Big\downarrow\overline{\psi}^3 \\ G' \times \cdots \times G' & \xrightarrow{\pi_2\times\cdots\times\pi_2} & \dfrac{G'}{\operatorname{Stab}_G(2)} \times \cdots \times \dfrac{G'}{\operatorname{Stab}_G(2)}, \end{array} \tag{3.3.2}$$

where $\pi_2$ denotes reduction modulo $\operatorname{Stab}_G(2)$. (Take into account that $G'$ contains $\operatorname{Stab}_G(2)$ by Theorem 3.2.1.) By the discussion of the preceding paragraph, the left vertical arrow of the diagram is surjective. Consequently, the right vertical arrow is also surjective, and since it is obviously injective, it follows that it is a bijective map. In particular,

$$|S^{(1)}| = |G' : \operatorname{Stab}_G(2)|^p.$$

Now, by Theorems 3.2.1 and 3.2.4, we have $|G : G'| = p^2$ and $|G : \operatorname{Stab}_G(2)| = p^{t+1}$. Thus $|G' : \operatorname{Stab}_G(2)| = p^{t-1}$, and we conclude that $|S^{(1)}| = p^{p(t-1)}$, as desired. $\square$

**Theorem 3.3.5.** *Let $G$ be a GGS-group with non-symmetric defining vector $\mathbf{e}$. Then*

$$\log_p |G_n| = tp^{n-2} + 1, \quad \textit{for every } n \geq 2,$$

*where $t$ is the rank of $C(\mathbf{e}, 0)$, and*

$$\dim_\Gamma \overline{G} = \frac{(p-1)t}{p^2}.$$

*Proof.* We argue by induction on $n \geq 2$. By Theorem 3.2.4, we have $|G_2| = p^{t+1}$. Suppose now that $n > 2$ and that the result is true for $n-1$. By using Lemma 3.3.2, we have

$$|\operatorname{Stab}_{G_n}(n-1)| = |\operatorname{Stab}_{G_{n-1}}(n-2)|^p = \cdots = |\operatorname{Stab}_{G_3}(2)|^{p^{n-3}}.$$

Since $|\operatorname{Stab}_{G_3}(2)| = p^{t(p-1)}$ by Theorem 3.3.4, we conclude that

$$|G_n| = |G_{n-1}||\operatorname{Stab}_{G_n}(n-1)| = p^{tp^{n-3}+1} \cdot p^{tp^{n-3}(p-1)} = p^{tp^{n-2}+1},$$

as desired. Finally, the value of $\dim_\Gamma \overline{G}$ follows directly from (1.2.3). $\square$

Next we consider the case when the vector $\mathbf{e}$ is non-constant and symmetric. As in the non-symmetric case, the key is to obtain the order of $\mathrm{Stab}_{G_3}(2)$ and to use Lemma 3.3.2.

**Theorem 3.3.6.** *Let $G$ be a GGS-group with symmetric non-constant defining vector $\mathbf{e}$. Then*

$$|\,\mathrm{Stab}_{G_3}(2)| = p^{t(p-1)-1},$$

*where $t$ is the rank of $C(\mathbf{e},0)$.*

*Proof.* Let $\pi$, $R^{(i)}$ and $S^{(i)}$ for $i = 1,\ldots,p^{p-t}$ be as in the proof of Theorem 3.3.4. The plan of the proof is the same as in that theorem. The difference is that, in this case, we need to see that

$$|S^{(1)}| = p^{p(t-1)-1}.$$

For that purpose, it suffices to prove that the image of $S^{(1)}$ under the injective map $\psi_3$ is a subgroup of index $p$ of

$$\frac{G'}{\mathrm{Stab}_G(2)} \times \cdots \times \frac{G'}{\mathrm{Stab}_G(2)}.$$

We know from (3.3.1) that $R^{(1)} = G \cap \psi^{-1}(G' \times \cdots \times G')$ consists of all elements of $G$ whose weight vector is $(0,\ldots,0)$. According to Theorem 3.2.13, we have $R^{(1)} = \mathrm{Stab}_G(1)'$. Hence

$$R^{(1)} = \langle [b_i, b_j]^h \mid 0 \le i,j \le p-1,\ h \in \mathrm{Stab}_G(1)\rangle. \tag{3.3.3}$$

Let us consider again the commutative diagram in (3.3.2). Since

$$\ker(\tilde{\pi} \times \cdots \times \tilde{\pi}) = \mathrm{Stab}_G(2) \times \cdots \times \mathrm{Stab}_G(2) = \psi(\mathrm{Stab}_G(3))$$

by Lemma 3.3.2, and since $\mathrm{Stab}_G(3) \le R^{(1)}$ by Theorem 3.2.12, it follows that the index

$$\left|\frac{G'}{\mathrm{Stab}_G(2)} \times \cdots \times \frac{G'}{\mathrm{Stab}_G(2)} : \psi_3(S^{(1)})\right| =$$
$$|(\tilde{\pi} \times \cdots \times \tilde{\pi})(G' \times \cdots \times G') : (\tilde{\pi} \times \cdots \times \tilde{\pi})(\psi(R^{(1)}))|$$

is the same as

$$|G' \times \cdots \times G' : \psi(R^{(1)})|.$$

Thus it suffices to prove that this last index is $p$.

Let $\overline{\psi}$ the map from $R^{(1)}$ to $G'/\gamma_3(G) \times \overset{p}{\cdots} \times G'/\gamma_3(G)$ which is obtained by first applying $\psi$ and then reducing every component modulo $\gamma_3(G)$. Observe that $G'/\gamma_3(G) \times \overset{p}{\cdots} \times G'/\gamma_3(G)$ can be seen as a vector space of dimension $p$ over $\mathbb{F}_p$, since $|G' : \gamma_3(G)| = p$. Since we may assume that $e_1 = 1$, and since $e_{p-1} = e_1$, we have

$$\psi([b_i, b_{i+1}]) = (1, \ldots, 1, [b, a], [a, b], 1, \ldots, 1), \quad \text{for } i = 1, \ldots, p-1,$$

where $[b, a]$ appears at the $i$th position. Now, $G'/\gamma_3(G)$ is generated by the image of $[b, a]$, and so it readily follows that the dimension of $\overline{\psi}(R^{(1)})$ is at least $p-1$. Hence

$$|G' \times \cdots \times G' : \psi(R^{(1)})(\gamma_3(G) \times \cdots \times \gamma_3(G))| = 1 \text{ or } p.$$

Since $\gamma_3(G) \times \cdots \times \gamma_3(G) \leq \psi(R^{(1)})$ by Lemma 3.3.1 and (3.3.1), we get

$$|G' \times \cdots \times G' : \psi(R^{(1)})| = 1 \text{ or } p.$$

Thus it suffices to see that $([a, b], 1, \ldots, 1) \notin \psi(R^{(1)})$ in order to conclude that $|G' \times \cdots \times G' : \psi(R^{(1)})| = p$, as desired.

Let $\lambda : \mathrm{Stab}_G(1) \longrightarrow \mathbb{F}_p$ be the homomorphism given by

$$g \longmapsto \sum_{i=0}^{p-1} i r_i,$$

where $(r_0, \ldots, r_{p-1})$ is the weight vector of $g$. If $g \in \mathrm{Stab}_G(1)$ then the weight vector of $g^b$ is also $(r_0, \ldots, r_{p-1})$, and the weight vector of $g^a$ is $(r_{p-1}, r_0, \ldots, r_{p-2})$. Hence $\lambda(g^b) = \lambda(g)$, and if $g \in G'$, then furthermore

$$\lambda(g^a) = \sum_{i=0}^{p-1} i r_{i-1} = \sum_{i=0}^{p-1} r_{i-1} + \sum_{i=0}^{p-1} (i-1) r_{i-1} = \lambda(g),$$

since $r_0 + \cdots + r_{p-1} = 0$ by Theorem 3.2.10. It follows that $\lambda(g^h) = \lambda(g)$ for every $g \in G'$ and $h \in G$.

Now we define $\Lambda : G' \times \cdots \times G' \longrightarrow \mathbb{F}_p$ by means of

$$\Lambda(g_1, \ldots, g_p) = \lambda(g_1) + \cdots + \lambda(g_p).$$

By the preceding paragraph, we have

$$\Lambda(g^h) = \Lambda(g), \quad \text{for all } g \in G' \times \cdots \times G' \text{ and } h \in G \times \cdots \times G.$$

Hence $\ker \Lambda$ is a normal subgroup of $G \times \cdots \times G$.

For every $1 \leq i < j \leq p$, we have

$$\begin{aligned}\psi([b_i, b_j]) = (1, \ldots, 1, [b, a^{e_{i-j}}], 1, \ldots, 1, [a^{e_{j-i}}, b], 1, \ldots, 1) = \\ (1, \ldots, 1, b_0^{-1} b_{e_{i-j}}, 1, \ldots, 1, b_{e_{j-i}}^{-1} b_0, 1, \ldots, 1),\end{aligned}$$

where the non-trivial components are at positions $i$ and $j$. Since $\mathbf{e}$ is symmetric, we have $e_{i-j} = e_{j-i}$, and consequently

$$\Lambda(\psi([b_i, b_j])) = e_{i-j} - e_{j-i} = 0.$$

Hence $\psi([b_i, b_j]) \in \ker \Lambda$, and since $\ker \Lambda$ is a normal subgroup of $G \times \cdots \times G$, it follows from (3.3.3) that $\psi(R^{(1)}) \leq \ker \Lambda$. Since

$$\Lambda([a, b], 1, \ldots, 1) = \Lambda(b_1^{-1} b_0, 1, \ldots, 1) = -1,$$

we deduce that $([a, b], 1, \ldots, 1) \notin \psi(R^{(1)})$, which completes the proof. □

**Theorem 3.3.7.** *Let $G$ be a GGS-group with a non-constant symmetric defining vector $\mathbf{e}$. Then*

$$\log_p |G_n| = tp^{n-2} + 1 - \frac{p^{n-2} - 1}{p - 1}, \quad \textit{for every } n \geq 2,$$

*where $t$ is the rank of $C(\mathbf{e}, 0)$, and*

$$\dim_\Gamma \overline{G} = \frac{(p-1)t - 1}{p^2}.$$

*Proof.* The proof is completely similar to that of Theorem 3.3.5. □

## 3.4 GGS-groups with constant defining vector

In this section, we deal with the case where the defining vector is constant, say $\mathbf{e} = (e, \dots, e)$, where $e \in \mathbb{F}_p^\times$. Let $m$ be the inverse of $e$ in $\mathbb{F}_p^\times$, and $b^* = b^m$. Then $G = \langle a, b^* \rangle$, and $\psi(b^*) = (a, \dots, a, b^*)$. For this reason, we may assume in the remainder of this section that $\mathbf{e} = (1, \dots, 1)$.

We begin by defining a sequence of elements of $G$ that will be fundamental in the sequel. We put $y_0 = ba^{-1}$ and, more generally, $y_i = y_0^{a^i}$ for every integer $i$. Thus $y_i^{a^j} = y_{i+j}$ for all $i, j \in \mathbb{Z}$. Also,

$$y_i^b = y_i^{aa^{-1}b} = y_{i+1}^{y_1}. \tag{3.4.1}$$

Observe that $y_i = y_j$ if $i \equiv j \pmod p$, so that the set $\{y_0, \dots, y_{p-1}\}$ already contains all the $y_i$. In the following lemma, we collect some important properties of the elements $y_i$. We adopt the following convention: given a vector $v$ of length $p$ and an integer $i$, not lying in the range $\{1, \dots, p\}$, the $i$th position of $v$ is to be understood as the $j$th position, where $j \in \{1, \dots, p\}$ and $i \equiv j \pmod p$.

**Lemma 3.4.1.** *Let $G$ be a GGS-group with constant defining vector. Then:*

(i) $y_{p-1}y_{p-2}\dots y_1y_0 = 1$.

(ii) *If $z_i$ is the tuple of length $p$ having $y_2$ at position $i-2$, $y_1^{-1}$ at position $i-1$, and $1$ elsewhere, then*

$$\psi([y_i, y_j]) = z_i z_j^{-1}, \quad \textit{for every } i \textit{ and } j. \tag{3.4.2}$$

(iii) *We have*

$$[y_i, y_j] = [y_i, y_{i-1}][y_{i-1}, y_{i-2}] \dots [y_{j+1}, y_j], \quad \textit{for every } i > j. \tag{3.4.3}$$

*Proof.* (i) We have

$$\begin{aligned} y_{p-1}y_{p-2}\dots y_1y_0 &= a^{-(p-1)}ba^{p-2}\cdot a^{-(p-2)}ba^{p-3}\dots a^{-1}b\cdot ba^{-1} \\ &= a^{-(p-1)}b^pa^{-1} = 1. \end{aligned}$$

(ii) Clearly, it is enough to see the result for $i > j$. On the other hand, since both sequences $(y_i)$ and $(z_i)$ are periodic of period $p$, we may assume that $i$ and $j$ lie in the set $\{3,\dots,p+2\}$. If $r = j-3$ and $k = i-r$, then

$$[y_i, y_j] = [y_k^{a^r}, y_3^{a^r}] = [y_k, y_3]^{a^r},$$

and so $\psi([y_i, y_j])$ is the result of applying to $\psi([y_k, y_3])$ the permutation which moves every element $r$ positions to the right. It readily follows that it suffices to prove (3.4.2) for $[y_k, y_3]$ with $4 \le k \le p+2$.

Since $y_i = a^{-i}ba^{i-1} = a^{-1}b_{i-1}$ for every $i$, we have

$$[y_k, y_3] = b_{k-1}^{-1}ab_2^{-1}b_{k-1}a^{-1}b_2 = b_{k-1}^{-1}b_1^{-1}b_{k-2}b_2 = (b_1^{-1}b_{k-2})^{b_{k-1}}(b_{k-1}^{-1}b_2). \tag{3.4.4}$$

Now, it follows from (3.2.1) that

$$\begin{aligned} \psi((b_1^{-1}b_{k-2})^{b_{k-1}}) &= (y_1^{-1}, 1, \overset{k-4}{\dots}, 1, y_1, 1, \dots, 1)^{(a, \overset{k-2}{\dots}, a, b, a, \dots, a)} \\ &= \begin{cases} (y_2^{-1}, 1, \overset{k-4}{\dots}, 1, y_2, 1, \dots, 1), & \text{if } 4 \le k \le p+1, \\ (y_1^{-1}y_2^{-1}y_1, 1, \dots, 1, y_2), & \text{if } k = p+2. \end{cases} \end{aligned}$$

Here, we have used that $y_1^b = y_2^{y_1}$ by (3.4.1). Similarly,

$$\psi(b_{k-1}^{-1}b_2) = \begin{cases} (1, y_1, 1, \overset{k-4}{\dots}, 1, y_1^{-1}, 1, \dots, 1), & \text{if } 4 \le k \le p+1, \\ (y_1^{-1}, y_1, 1, \dots, 1), & \text{if } k = p+2. \end{cases}$$

By taking these values to (3.4.4), we obtain that $\psi([y_k, y_3]) = z_kz_3^{-1}$, as desired.

(iii) This follows immediately from (ii), since

$$\begin{aligned} \psi([y_i, y_j]) &= (z_iz_{i-1}^{-1})(z_{i-1}z_{i-2}^{-1})\dots(z_{j+1}z_j^{-1}) \\ &= \psi([y_i, y_{i-1}])\psi([y_{i-1}, y_{i-2}])\dots\psi([y_{j+1}, y_j]) \\ &= \psi([y_i, y_{i-1}][y_{i-1}, y_{i-2}]\dots[y_{j+1}, y_j]). \end{aligned}$$

□

Next we introduce a maximal subgroup $K$ of $G$ that will play a key role in the determination of the order of $G_n$ in the case that $\mathbf{e}$ is constant.

**Lemma 3.4.2.** *Let $G$ be a GGS-group with constant defining vector, and let $K = \langle ba^{-1}\rangle^G$. Then:*

(i) $G' \leq K$ *and* $|G : K| = p$.

(ii) $K = \langle y_0, y_1, \ldots, y_{p-1}\rangle$ *and* $K' = \langle [y_1, y_0]\rangle^G$.

(iii) $K' \times \overset{p}{\cdots} \times K' \subseteq \psi(K') \subseteq \psi(G') \subseteq K \times \overset{p}{\cdots} \times K$. *In particular, $G$ is a weakly regular branch group over $K'$.*

(iv) *If $L = \psi^{-1}(K' \times \overset{p}{\cdots} \times K')$ (which, by (iii), is contained in $K'$), then the conjugates $[y_{i+1}, y_i]^{b^j}$, where $0 \leq i, j \leq p-1$, generate $K'$ modulo $L$.*

*Proof.* (i) Since $[a, ba^{-1}] = [a, b]^{a^{-1}} \in K$ and $K$ is normal in $G$, it follows that $G'$ is contained in $K$. Then $|G : K| = |G/G' : K/G'| = p$.

(ii) Let us first prove that $K = \langle y_0, y_1, \ldots, y_{p-1}\rangle$. For this purpose, it suffices to see that $N = \langle y_0, y_1, \ldots, y_{p-1}\rangle$ is a normal subgroup of $G$. This is clear, since $y_i^a = y_{i+1}$ and $y_i^b = y_{i+1}^{y_1}$ for every $i$.

It follows that

$$K' = \langle [y_i, y_j] \mid 0 \leq j < i \leq p-1\rangle^K = \langle [y_i, y_j] \mid 0 \leq j < i \leq p-1\rangle^G,$$

where the second equality holds because $K'$ is normal in $G$. By (3.4.3), every commutator $[y_i, y_j]$ with $0 \leq j < i \leq p-1$ can be expressed in terms of the $[y_k, y_{k-1}]$ with $k = 1, \ldots, p-1$. Since $[y_k, y_{k-1}] = [y_1, y_0]^{a^{k-1}}$, we conclude that $K' = \langle [y_1, y_0]\rangle^G$.

(iii) Let us first prove the inclusion $\psi(G') \subseteq K \times \overset{p}{\cdots} \times K$. We have

$$\begin{aligned}\psi([b,a]) = \psi(b^{-1}b^a) &= (a^{-1}, a^{-1}, \ldots, a^{-1}, b^{-1})(b, a, \ldots, a, a) \\ &= (a^{-1}b, 1, \ldots, 1, b^{-1}a) \in K \times \overset{p}{\cdots} \times K.\end{aligned}$$

Now, since $K$ is normal in $G$, it readily follows that

$$\psi([b,a]^g) \in K \times \overset{p}{\cdots} \times K, \quad \text{for every } g \in G.$$

This proves the desired inclusion.

Now we focus on proving that $K' \times \overset{p}{\cdots} \times K' \subseteq \psi(K')$. By Proposition 3.2.17 and (ii), it suffices to see that

$$([y_1, y_0], 1, \ldots, 1) \in \psi(K').$$

We consider separately the cases $p \geq 5$ and $p = 3$.

Suppose first that $p \geq 5$. By using (3.4.2), we have

$$\psi([y_1, y_2]) = (y_1, 1, \ldots, 1, y_2, y_1^{-1}y_2^{-1})$$

and

$$\psi([y_3, y_4]) = (y_2, y_1^{-1}y_2^{-1}, y_1, 1, \ldots, 1).$$

If $k = [[y_3, y_4], [y_1, y_2]]$, it follows that

$$\psi(k) = ([y_2, y_1], 1, \ldots, 1),$$

since $p \geq 5$. Hence

$$([y_1, y_0], 1, \ldots, 1) = \psi(k^{b^{-1}}) \in \psi(K'),$$

as desired.

Assume now that $p = 3$. We have

$$\psi([y_1, y_0]) = (y_1 y_0, y_0^{-1}, y_1^{-1}),$$

since $y_2 y_1 y_0 = 1$, by (i) of Lemma 3.4.1. Hence

$$\begin{aligned}\psi([y_0, y_1]^b) &= (y_0^{-1}y_1^{-1}, y_0, y_1)^{(a,a,b)} = (y_1^{-1}y_2^{-1}, y_1, y_1^b)\\ &= ((y_2 y_1)^{-1}, y_1, y_2^{y_1}) = (y_0, y_1, (y_0^{-1}y_1^{-1})^{y_1})\\ &= (y_0, y_1, y_1^{-1}y_0^{-1}),\end{aligned}$$

and

$$([y_1, y_0], 1, 1) = \psi([y_0, y_1]^{ba}[y_1, y_0]) \in \psi(K'),$$

which completes the proof.

(iv) Let us consider an arbitrary element $g \in G$, and let us write $g = ha^i b^j$, for some $i, j \in \mathbb{Z}$, $h \in G'$. Then

$$[y_1, y_0]^g = ([y_1, y_0][y_1, y_0, h])^{a^i b^j} \equiv [y_1, y_0]^{a^i b^j} = [y_{i+1}, y_i]^{b^j} \pmod{L},$$

since $\psi([y_1, y_0, h]) \in \psi(G'') \subseteq K' \times \overset{p}{\cdots} \times K'$ by (iii). Now, since the conjugates $[y_1, y_0]^g$ generate $K'$ by (ii), the result follows. $\square$

In the following results, we consider the action of an element of $G$ by conjugation as an endomorphism of $K/K'$, which allows us to multiply several conjugates of an element of $K$, modulo $K'$, by adding the elements by which we are conjugating. This gives a meaning to expressions like $g^{1+a+\cdots+a^{p-1}} \in K'$ for an element $g \in K$.

**Lemma 3.4.3.** *Let $G$ be a GGS-group with constant defining vector, and let $K = \langle ba^{-1} \rangle^G$. If $g \in K$ then*

$$g^{1+a+\cdots+a^{p-1}} \in K'.$$

*Proof.* The map $R$ sending $g \in K$ to $g^{1+a+\cdots+a^{p-1}}K'$ is a well-defined homomorphism from $K$ to $K/K'$, and we want to see that $R$ is the trivial homomorphism. Since $K = \langle y_0, \dots, y_{p-1} \rangle$ by (ii) of Lemma 3.4.2, it suffices to check that $y_i \in \ker R$ for every $i$. Now,

$$R(y_i) = y_i y_{i+1} \dots y_{p-1} y_0 \dots y_{i-1} K' = y_{p-1} y_{p-2} \dots y_1 y_0 K' = K'$$

by (i) of Lemma 3.4.1, and we are done. $\square$

**Lemma 3.4.4.** *Let $G$ be a GGS-group with constant defining vector, and let $K = \langle ba^{-1} \rangle^G$. If $g \in K'$ and we write $\psi(g) = (g_1, \dots, g_p)$, then:*

(i) $g_p g_{p-1} \dots g_1 \in K'$.

(ii) $\prod_{i=1}^{p-1} g_i^{a+a^2+\cdots+a^i} \in K'$.

*Similarly, if* $g \in K' \operatorname{Stab}_G(n)$ *for some* $n \geq 1$*, then both* $g_p g_{p-1} \dots g_1$ *and* $\prod_{i=1}^{p-1} g_i^{a+a^2+\cdots+a^i}$ *lie in* $K' \operatorname{Stab}_G(n-1)$.

*Proof.* We first deal with the case that $g \in K'$. Let us consider the following two maps:

$$\begin{array}{rcccc} P & : & K \times \overset{p}{\cdots} \times K & \longrightarrow & K/K' \\ & & (g_1, \dots, g_p) & \longmapsto & g_p \dots g_1 K', \end{array}$$

and

$$\begin{array}{rcccc} Q & : & K \times \overset{p}{\cdots} \times K & \longrightarrow & K/K' \\ & & (g_1, \dots, g_p) & \longmapsto & \prod_{i=1}^{p-1} g_i^{a+a^2+\cdots+a^i} K'. \end{array}$$

Clearly, $P$ and $Q$ are homomorphisms. By (iii) of Lemma 3.4.2, $\psi(K')$ is contained in the domain of $P$ and $Q$, and our goal is to prove that it is actually in the kernels of these maps. Since the image of $K' \times \overset{p}{\cdots} \times K'$ is trivial, it suffices to see that $\psi(g) \in \ker P$ and $\psi(g) \in \ker Q$ for every $g$ in a system of generators of $K'$ modulo $L$, where $L = \psi^{-1}(K' \times \overset{p}{\cdots} \times K')$. By (iv) of Lemma 3.4.2, the conjugates $[y_{i+1}, y_i]^{b^j}$, for $i, j = 0, \dots, p-1$ constitute such a set of generators.

Let $c \in \Gamma$ be defined by means of $\psi(c) = (a, a, \dots, a)$. We claim that

$$g^b \equiv g^c \pmod{L}, \quad \text{for every } g \in K'. \tag{3.4.5}$$

Indeed, we have $\psi(b) = \psi(c)(1, \dots, 1, a^{-1}b)$, and so

$$\begin{aligned} \psi(g^b) = \psi(g^c)^{(1,\dots,1,a^{-1}b)} &= \psi(g^c)[\psi(g^c), (1, \dots, 1, a^{-1}b)] \\ &\equiv \psi(g^c) \pmod{K' \times \overset{p}{\cdots} \times K'}, \end{aligned}$$

since $\psi(g^c) \in K \times \overset{p}{\cdots} \times K$ and $a^{-1}b \in K$.

As a consequence of (3.4.5), it suffices to see that $\psi([y_{i+1}, y_i]^{c^j})$ lies in both $\ker P$ and $\ker Q$. Since

$$P(\psi([y_{i+1}, y_i]^{c^j})) = P(\psi([y_{i+1}, y_i]))^{a^j}$$

and

$$Q(\psi([y_{i+1}, y_i]^{c^j})) = Q(\psi([y_{i+1}, y_i]))^{a^j},$$

we have reduced ourselves to proving that $\psi([y_{i+1}, y_i])$ is in the kernel of $P$ and $Q$ for every $i$. According to (3.4.2), we have $\psi([y_{i+1}, y_i]) = z_{i+1}z_i^{-1}$, with $z_i$ as defined in Lemma 3.4.1. Now, one can easily check that

$$P(z_i) = y_1^{-1}y_2K' \quad \text{and} \quad Q(z_i) = y_2^{-1}K' \quad \text{for every } i,$$

where in the case of $Q$ and $i = 1$ we need to use that

$$y_2^{a+a^2+\cdots+a^{p-1}} \equiv y_2^{-1} \pmod{K'},$$

by Lemma 3.4.3. It readily follows that $\psi([y_{i+1}, y_i])$ lies in both $\ker P$ and $\ker Q$, as desired.

Assume now that $g \in K' \operatorname{Stab}_G(n)$, and let us write $g = fh$, with $f \in K'$ and $h \in \operatorname{Stab}_G(n)$. Put $\psi(f) = (f_1, \ldots, f_p)$ and $\psi(h) = (h_1, \ldots, h_p)$. Since $h_1, \ldots, h_p \in \operatorname{Stab}_G(n-1)$, which is a normal subgroup of $G$, we have

$$g_p \ldots g_1 = f_p h_p \ldots f_1 h_1 = f_p \ldots f_1 h^*,$$

for some $h^* \in \operatorname{Stab}_G(n-1)$. Since $f \in K'$, we already know that $f_p \ldots f_1 \in K'$, and so we conclude that $g_p \ldots g_1 \in K' \operatorname{Stab}_G(n-1)$, as desired. The second assertion can be proved in a similar way. □

**Theorem 3.4.5.** *Let $G$ be a GGS-group with constant defining vector, and let $K = \langle ba^{-1}\rangle^G$ and $L = \psi^{-1}(K' \times \overset{p}{\cdots} \times K')$. Then the following isomorphisms hold:*

$$K'/L \cong K/K' \times \overset{p-2}{\cdots} \times K/K',$$

*and*

$$K' \operatorname{Stab}_G(n)/L \operatorname{Stab}_G(n) \cong K/K' \operatorname{Stab}_G(n-1) \times \overset{p-2}{\cdots} \times K/K' \operatorname{Stab}_G(n-1),$$

*for every $n \geq 3$.*

*Proof.* Let $\pi^*$ be the map given by

$$\begin{array}{ccc} K \times \overset{p}{\cdots} \times K & \longrightarrow & K/K' \times \overset{p-2}{\cdots} \times K/K' \\ (g_1, \ldots, g_p) & \longmapsto & (g_1K', \ldots, g_{p-2}K'), \end{array}$$

and let $R$ be the composition of $\psi : K' \longrightarrow K \times \overset{p}{\cdots} \times K$ with $\pi^*$. If we see that $R$ is surjective, and that $\ker R = L$, then the first isomorphism of the statement follows.

Let $g \in K'$ be an element lying in $\ker R$. If $\psi(g) = (g_1, \ldots, g_p)$, then we have $g_1, \ldots, g_{p-2} \in K'$. By (ii) of Lemma 3.4.4, it follows that

$$g_{p-1}^{a+\cdots+a^{p-1}} \in K',$$

and by applying Lemma 3.4.3, we get $g_{p-1} \in K'$. Now, (i) of Lemma 3.4.4 immediately yields that also $g_p \in K'$. This proves that $\ker R = L$.

Now we prove that

$$K/K' \times \{\bar{1}\} \times \cdots \times \{\bar{1}\} \subseteq R(K'). \tag{3.4.6}$$

Then, by arguing as in the proof of Proposition 3.2.17, it follows that $R$ is surjective. By (3.4.2), we have

$$\psi([y_1, y_2]) = (y_1, 1, \ldots, 1, h_{p-1}, h_p)$$

for some elements $h_{p-1}, h_p \in K$. Hence

$$\psi([y_1, y_2]^{b^{i-1}}) = (y_i, 1, \ldots, 1, h^*_{p-1}, h^*_p)$$

for every $i$, and we are done, since $K = \langle y_0, \ldots, y_{p-1} \rangle$.

The second isomorphism can be proved in a similar way. Observe that the condition $n \geq 3$ guarantees that $\mathrm{Stab}_G(n-1) \leq G' \leq K$, so that it makes sense to write $K/K' \,\mathrm{Stab}_G(n-1)$. Consider this time the homomorphism

$$\begin{array}{cccc} \pi_n^* & : & K \times \overset{p}{\cdots} \times K & \longrightarrow & K/K'\,\mathrm{Stab}_G(n-1) \times \overset{p-2}{\cdots} \times K/K'\,\mathrm{Stab}_G(n-1) \\ & & (g_1, \ldots, g_p) & \longmapsto & (g_1K'\,\mathrm{Stab}_G(n-1), \ldots, g_{p-2}K'\,\mathrm{Stab}_G(n-1)), \end{array}$$

and let $R_n$ be the composition of $\psi : K' \longrightarrow K \times \overset{p}{\cdots} \times K$ with $\pi_n^*$. Observe that the surjectiveness of $R$ already implies that $R_n$ is surjective. Let us prove that $\ker R_n = L\,\mathrm{Stab}_G(n) \cap K'$. The same proof as above, but using the last part of Lemma 3.4.4, shows that

$$\begin{aligned}\psi(\ker R_n) &= (K'\,\mathrm{Stab}_G(n-1) \times \overset{p}{\cdots} \times K'\,\mathrm{Stab}_G(n-1)) \cap \psi(K')\\ &= (K' \times \overset{p}{\cdots} \times K')(\mathrm{Stab}_G(n-1) \times \overset{p}{\cdots} \times \mathrm{Stab}_G(n-1)) \cap \psi(K').\end{aligned}$$

Since $K' \times \overset{p}{\cdots} \times K' \subseteq \psi(K')$, we can apply Dedekind's Law to get

$$\psi(\ker R_n) = (K' \times \overset{p}{\cdots} \times K')\big((\mathrm{Stab}_G(n-1) \times \overset{p}{\cdots} \times \mathrm{Stab}_G(n-1)) \cap \psi(K')\big).$$

Now, since $n \geq 3$, we have

$$\begin{aligned}(\mathrm{Stab}_G(n-1) \times \overset{p}{\cdots} \times \mathrm{Stab}_G(n-1)) \cap \psi(K') &= \psi(\mathrm{Stab}_G(n)) \cap \psi(K')\\ &= \psi(\mathrm{Stab}_G(n) \cap K'),\end{aligned}$$

and it follows that

$$\begin{aligned}\psi(\ker R_n) = (K' \times \overset{p}{\cdots} \times K')\psi(\mathrm{Stab}_G(n) \cap K') &= \psi(L)\psi(\mathrm{Stab}_G(n) \cap K')\\ &= \psi(L(\mathrm{Stab}_G(n) \cap K')).\end{aligned}$$

Hence

$$\ker R_n = L(\mathrm{Stab}_G(n) \cap K') = L\,\mathrm{Stab}_G(n) \cap K',$$

as claimed.

Now, we can readily obtain the desired isomorphism:

$$\begin{aligned}K'\,\mathrm{Stab}_G(n)/L\,\mathrm{Stab}_G(n) &\cong K'/(L\,\mathrm{Stab}_G(n) \cap K') = K'/\ker R_n\\ &\cong R_n(K') = K/K'\,\mathrm{Stab}_G(n-1) \times \overset{p-2}{\cdots} \times K/K'\,\mathrm{Stab}_G(n-1).\end{aligned}$$

□

**Theorem 3.4.6.** *Let $G$ be a GGS-group with constant defining vector, and let $K = \langle ba^{-1}\rangle^G$. Then, for every $n \geq 2$, the quotient $G/K'\,\mathrm{Stab}_G(n)$ is a $p$-group of maximal class of order $p^{n+1}$.*

*Proof.* For simplicity, let us write $T_n = K' \operatorname{Stab}_G(n)$, $Q_n = G/T_n$ and $A_n = K/T_n$ (take into account that $\operatorname{Stab}_G(2) \le G' \le K$). Since $|Q_n : Q_n'| = |G : G'| = p^2$ and $A_n$ is an abelian maximal subgroup of $Q_n$, it follows from Lemma 3.2.3 that $Q_n$ is a $p$-group of maximal class. As a consequence, if we want to prove that $|Q_n| = p^{n+1}$, it suffices to see that the nilpotency class of $Q_n$ is $n$.

We need an auxiliary result. Let $(x_i)_{i\ge1}$ be a sequence of elements of $G$ such that $\{x_1, x_2\} = \{a, b\}$ and $x_i \in \{a, b\}$ for every $i \ge 3$. We claim that, for every $i \ge 2$, the section $\gamma_i(Q_n)/\gamma_{i+1}(Q_n)$ is generated by the image of the commutator $[x_1, x_2, \dots, x_i]$. We argue by induction on $i$. If $i = 2$ then we have to show that the image of $[a, b]$ generates $\gamma_2(Q_n)/\gamma_3(Q_n)$. This follows immediately from (i) in Lemma 3.2.3, since $[a, b] = [a, a^{-1}b]$, where $bT_n \in Q_n \setminus A_n$ and $a^{-1}bT_n = (ba^{-1}T_n)^a \in A_n \setminus \gamma_2(Q_n)$. Now, if we assume that the result holds for $i-1$, we get it for $i$ by using (ii) of Lemma 3.2.3.

Let us now prove that the class of $Q_n$ is $n$, by induction on $n$. Assume first that $n = 2$. We have

$$\psi([b, a]) = (a^{-1}b, 1, \dots, 1, b^{-1}a)$$

and

$$\psi([b, a, b]) = ([a^{-1}b, a], 1, \dots, 1, [b^{-1}a, b]) = ([b, a], 1, \dots, 1, [a, b]),$$

so that $[b, a, b] \in \operatorname{Stab}_G(2)$. It follows that the image of $[b, a, b]$ in $Q_2$ is trivial. By the previous paragraph, we necessarily have $\gamma_3(Q_2) = \gamma_4(Q_2)$. Hence $\gamma_3(Q_2) = 1$, and the class of $Q_2$ is at most 2. If $Q_2$ is of class 1, then $[b, a] \in K' \operatorname{Stab}_G(2)$ and, by Lemma 3.4.4, $a^{-1}b \in K' \operatorname{Stab}_G(1)$. Hence $a^{-1} \in \operatorname{Stab}_G(1)$, which is a contradiction. Thus $Q_2$ is of class 2.

Now we assume the result for $n-1$, and we prove it for $n$. We have

$$\psi([b, a, b, \overset{n-1}{\ldots}, b]) = ([b, a, \overset{n-1}{\ldots}, a], 1, \dots, 1, [a, b, \overset{n-1}{\ldots}, b]),$$

and

$$[b, a, \overset{n-1}{\ldots}, a], [a, b, \overset{n-1}{\ldots}, b] \in K' \operatorname{Stab}_G(n-1),$$

since $Q_{n-1}$ has class $n-1$ by the induction hypothesis. Thus

$$\psi([b,a,b,\overset{n-1}{\ldots},b]) \in K' \operatorname{Stab}_G(n-1) \times \overset{p}{\cdots} \times K' \operatorname{Stab}_G(n-1). \tag{3.4.7}$$

Now,

$$\begin{aligned}
(K' \operatorname{Stab}_G(n-1) &\times \overset{p}{\cdots} \times K' \operatorname{Stab}_G(n-1)) \cap \psi(G) \\
&= (K' \times \overset{p}{\cdots} \times K')(\operatorname{Stab}_G(n-1) \times \overset{p}{\cdots} \times \operatorname{Stab}_G(n-1)) \cap \psi(G) \\
&\subseteq \psi(K')(\operatorname{Stab}_G(n-1) \times \overset{p}{\cdots} \times \operatorname{Stab}_G(n-1)) \cap \psi(G) \\
&= \psi(K')(\operatorname{Stab}_G(n-1) \times \overset{p}{\cdots} \times \operatorname{Stab}_G(n-1) \cap \psi(G)) \\
&= \psi(K')\psi(\operatorname{Stab}_G(n)) = \psi(K' \operatorname{Stab}_G(n)).
\end{aligned}$$

It follows that $[b,a,b,\overset{n-1}{\ldots},b] \in K' \operatorname{Stab}_G(n)$, and so this commutator becomes trivial in $Q_n$. Since the image of this commutator generates the quotient $\gamma_{n+1}(Q_n)/\gamma_{n+2}(Q_n)$, we have $\gamma_{n+1}(Q_n) = 1$. Hence the class of $Q_n$ is at most $n$.

If $Q_n$ has class strictly less than $n$, then since the image of $[b,a,b,\overset{n-2}{\ldots},b]$ generates $\gamma_n(Q_n)/\gamma_{n+1}(Q_n)$, it follows that

$$[b,a,b,\overset{n-2}{\ldots},b] \in K' \operatorname{Stab}_G(n).$$

Since

$$\psi([b,a,b,\overset{n-2}{\ldots},b]) = ([b,a,\overset{n-2}{\ldots},a],1,\ldots,1,[a,b,\overset{n-2}{\ldots},b]),$$

it follows from Lemma 3.4.4 that

$$[b,a,\overset{n-2}{\ldots},a] \in K' \operatorname{Stab}_G(n-1).$$

This is a contradiction, since $Q_{n-1}$ is of class $n-1$, and $\gamma_{n-1}(Q_{n-1})/\gamma_n(Q_{n-1})$ is generated by the image of $[b,a,\overset{n-2}{\ldots},a]$. Thus we conclude that the nilpotency class of $Q_n$ is $n$, which completes the proof of the theorem. □

**Theorem 3.4.7.** *Let $G$ be a GGS-group with a constant defining vector. Then*

$$\log_p |G_n| = p^{n-1} + 1 - \frac{p^{n-2}-1}{p-1} - \frac{p^{n-2}-(n-2)p+n-3}{(p-1)^2},$$

*for every $n \geq 2$, and*

$$\dim_\Gamma \overline{G} = \frac{p-2}{p-1}.$$

*Proof.* As on previous occasions, the formula for the Hausdorff dimension of $\overline{G}$ is immediate once we obtain $\log_p |G_n|$. For that purpose, we argue by induction on $n$. If $n = 2$, then by Theorem 3.2.4, we have $\log_p |G_2| = t + 1$, where $t$ is the rank of the matrix $C = C(1, \overset{p-1}{\ldots}, 1, 0)$. By Lemma 1.3.1, $p - t$ is the multiplicity of 1 as a root in $\mathbb{F}_p$ of the polynomial $X^{p-2} + \cdots + X + 1$. Thus $t = p$ and $\log_p |G_2| = p + 1$, as desired.

Assume now that $n \geq 3$. Let $K = \langle ba^{-1} \rangle^G$, and $L = \psi^{-1}(K' \times \overset{p}{\cdots} \times K')$. Then we have the following decomposition of the order of $G_n$:

$$|G_n| = |G : K' \operatorname{Stab}_G(n)||K' \operatorname{Stab}_G(n) : L \operatorname{Stab}_G(n)||L \operatorname{Stab}_G(n) : \operatorname{Stab}_G(n)|. \tag{3.4.8}$$

By Theorem 3.4.6, we know that $|G : K' \operatorname{Stab}_G(n)| = p^{n+1}$. On the other hand, since

$$K' \operatorname{Stab}_G(n)/L \operatorname{Stab}_G(n) \cong K/K' \operatorname{Stab}_G(n-1) \times \overset{p-2}{\cdots} \times K/K' \operatorname{Stab}_G(n-1)$$

by Theorem 3.4.5, and since $|K/K' \operatorname{Stab}_G(n-1)| = p^{n-1}$ (again by Theorem 3.4.6), it follows that

$$|K' \operatorname{Stab}_G(n) : L \operatorname{Stab}_G(n)| = p^{(n-1)(p-2)}.$$

Finally,

$$\begin{aligned}
|L \operatorname{Stab}_G(n) : \operatorname{Stab}_G(n)| &= |L : \operatorname{Stab}_L(n)| = |\psi(L) : \psi(\operatorname{Stab}_L(n))| \\
&= |K' \times \overset{p}{\cdots} \times K' : \operatorname{Stab}_{K'}(n-1) \times \overset{p}{\cdots} \times \operatorname{Stab}_{K'}(n-1)| \\
&= |K' : \operatorname{Stab}_{K'}(n-1)|^p = |K' \operatorname{Stab}_G(n-1) : \operatorname{Stab}_G(n-1)|^p \\
&= |G/\operatorname{Stab}_G(n-1)|^p / |G/K' \operatorname{Stab}_G(n-1)|^p \\
&= |G_{n-1}|^p p^{-np}.
\end{aligned}$$

Now, from (3.4.8) we get

$$\begin{aligned}
\log_p |G_n| &= p \log_p |G_{n-1}| + n + 1 + (n-1)(p-2) - np \\
&= p \log_p |G_{n-1}| - n - p + 3,
\end{aligned}$$

and the result follows by applying the induction hypothesis to $G_{n-1}$. $\square$

# Chapter 4

# The equations satisfied by GGS-groups and the abelian group structure of the Gupta-Sidki group

## 4.1 Introduction

As in the preceding chapters, let us consider the $p$-adic rooted tree $\mathcal{T}$ for an odd prime $p$, and $\Gamma$, the Sylow pro-$p$ subgroup of $\operatorname{Aut}\mathcal{T}$ corresponding to $\sigma = (1 \ldots p) \in S_p$. Then $\Gamma$ is in one-to-one correspondence with $\mathbb{F}_p^{X^*}$, the set of infinite sequences of the form $(m_v)_{v\in X^*}$ with $m_v \in \mathbb{F}_p$, via portraits.

Roughly speaking, as it is explained in detail in Section 4.2, this correspondence allows us to describe every closed set, in particular closed subgroup $G$ of $\Gamma$ as the set of zeros of an ideal of polynomials. The polynomials are taken over the field $\mathbb{F}_p$ and the indeterminates are indexed by the vertices of the tree. We will say that these polynomials that vanish in $G$ are *equations for $G$* or *patterns* [Gri05]. If such a polynomial has degree 1 we will say that it is a *linear equation* for $G$.

Section 4.2 introduces and looks more closely at all these concepts. In Section 4.4, we focus on GGS-groups and we explicitly describe a generating set for all the equations of non-symmetric GGS-groups. Recall that the GGS-group with defining vector $\mathbf{e} = (e_1, \ldots, e_{p-1}) \in \mathbb{F}_p^{p-1}$ is the group $G \subseteq \Gamma$ generated by $a$, the rooted automorphism corresponding to $\sigma$, and the automorphism $b$ that is recursively defined as $\psi(b) = (a^{e_1}, \ldots, a^{e_{p-1}}, b)$. If $e_i \neq e_{p-i}$ for some $1 \leq i \leq (p-1)/2$, we say that the GGS-group is non-symmetric.

The first of the two main results in this chapter can be summarized as follows.

**Theorem F.** *Let $G$ be a non-symmetric GGS-group. Then there are $p$ linear equations that generate all equations for $G$.*

We give the explicit expression of these $p$ linear equations in Theorem 4.4.6, and the way in which these linear equations "generate" all equations will be explained in Section 4.3.

It is interesting to know these equations explicitly for several reasons. First, we can describe the closure $\overline{G}$ (in the profinite topology of $\Gamma$) of such a group $G$ as the set of zeros of these equations and their translates, as it is shown in Theorem 4.4.6. Secondly, since these generating equations are linear and satisfy some extra conditions, we get to prove the second of the two main results in this chapter, Theorem G below. And finally, it also enriches the information contained in the Hausdorff dimension of the closures of these groups.

In Chapter 3 we have computed the Hausdorff dimension of the closures of all GGS-groups. In this chapter, we recover the same values for non-symmetric GGS-groups in Corollary 4.4.7, another consequence of Theorem 4.4.6. Indeed, the Hausdorff dimension can be computed very easily if we know a convenient generating set of equations, as we show in Theorem 4.3.6.

It is relevant to underline, anyway, that we actually rely on many of the results proved in Chapter 3.

Finally, Section 4.5 is devoted to the proof of the last significant result in this chapter, Theorem G, namely that non-symmetric GGS-groups possess another group operation that is abelian. In particular, we conclude that the Gupta-Sidki group has such a structure. The linearity and also the convenient construction of the polynomials of the generating set in Theorem F is important for the proof of this result.

**Theorem G.** *Let $G$ be a non-symmetric GGS-group. Pointwise addition in the portraits of elements gives $G$ the structure of an abelian group.*

We would like to point out that the consequences of the coexistence of these two group operations are yet to be explored. A reasonable direction to examine would be the relationship between the present work and Lie algebras, as we now explain.

The description of the elements of $\Gamma$ in terms of portraits is equivalent to a certain choice of a set-map

$$\pi : \Gamma \to A = \prod_{i=0}^{\infty} \operatorname{Stab}_{\Gamma}(i)/\operatorname{Stab}_{\Gamma}(i+1),$$

where $\prod$ denotes the unrestricted product. The group $A$ is an elementary abelian $p$-group with the operation inherited from $\Gamma$. This is exactly the sum of portraits. Now Theorem G can be rephrased as:

**Theorem G′.** *The image of $G$ under $\pi$ is a subgroup of $A$.*

One can compare this construction with the Lie algebra constructed by Magnus [Mag40]:

$$\mathcal{L}(G) = \bigoplus_{i=1}^{\infty} \gamma_i(G)/\gamma_{i+1}(G),$$

where $\gamma_i(G)$ is the $i$th term of the lower central series of $G$. The addition on $\mathcal{L}(G)$ is the operation induced by the group structure of $G$, and commutation in $G$ yields the Lie bracket. There is another similar construction, based on the dimension series, also known as the Brauer, Jennings [Jen41], Lazard [Laz53] or Zassenhaus [Zas40] series, which yields a restricted Lie algebra (see [Jac79] for the definition of restricted Lie algebras).

It would be interesting to investigate whether there is a map

$$\prod_{i=0}^{\infty} \gamma_i(G)/\gamma_{i+1}(G) \longrightarrow \prod_{i=0}^{\infty} \operatorname{Stab}_\Gamma(i)/\operatorname{Stab}_\Gamma(i+1)$$

which would enable to “read” the Lie algebra structure of $\mathcal{L}(G)$ directly on the portraits of the elements of $G$. Note that the Lie algebras associated to the Gupta-Sidki group have been explicitly described in [BG00], and the terms of the lower central series also admit a nice description in terms of portraits (see Theorem 4.2.4 in [Sie09]).

In this chapter we follow the approach developed in Olivier Siegenthaler's PhD thesis [Sie09]. We refer the reader to [Gri05], [Šun07], [Šun11] and the appendix in [AdlHKŠ07] for previous works on the subject.

This is an extended version of the paper [SZR], that has been accepted in the Eur. J. Combin. and has been written by Olivier Siegenthaler and the author.

*Notation.* In this chapter, we have chosen to use the letter $g$ for elements in $\Gamma$, as we give $f$ another use. On the other hand, in some of the proofs below, we have a GGS-group $G$ and we need to work in $G_n = G/\operatorname{Stab}_G(n)$. In these cases, for economy in the notation, we use the same letters $a, b_0, \ldots, b_{p-1}$ to denote $\pi_n(a), \pi_n(b_0), \ldots, \pi_n(b_{p-1})$, i.e. the images of the elements in $G$ under $\pi_n$. We believe that it is clear from the context where the elements belong in each case.

## 4.2 Algebraic geometry in $\Gamma$

In this subsection we introduce and develop some of the ordinary algebraic geometry in $\Gamma$. We rely on Chapters 1 and 2 of the PhD thesis of Siegenthaler

[Sie09]. As it is shown in Proposition 4.2.8, and its preceding lemmas, the situation is quite peculiar.

Let $g$ be an element of $\Gamma$ and let us think of $g$ as the infinite sequence of permutations $(g_{(v)})_{v\in X^*}$ (i.e. its portrait). In the same way, since $g_{(v)} = \sigma^{m_v}$ where $m_v \in \mathbb{F}_p$ for $v \in X^*$, we can also choose to think of $g$ as the infinite sequence $(m_v)_{v\in X^*} \in \mathbb{F}_p^{X^*}$. In other words, we are giving a correspondence, as sets, between $\Gamma$ and $\mathbb{F}_p^{X^*}$. Let us state these concepts properly.

We define the following map

$$\begin{array}{rcccl} \mathrm{Log} & : & \langle\sigma\rangle & \longrightarrow & \mathbb{F}_p \\ & & \sigma^m & \longmapsto & m, \end{array}$$

which is clearly a homomorphism, and for $v \in X^*$, we write $[v]$ for the following function:

$$\begin{array}{rcccl} [v] & : & \Gamma & \longrightarrow & \mathbb{F}_p \\ & & g & \longmapsto & \mathrm{Log}(g_{(v)}). \end{array}$$

These maps can be added and multiplied together and they also admit the product by a scalar (by pointwise operations). In other words, we can construct functions $F([v_1], [v_2], \dots, [v_k])$, where $v_1, \dots, v_k \in X^*$ and $F$ is a polynomial in $k$ indeterminates. Let $\mathcal{A}$ be the set of all possible such functions:

$$\mathcal{A} = \{F([v_1], [v_2], \dots, [v_k]) \mid F \text{ a polynomial, } v_1, \dots, v_k \in X^* \text{ and } k \in \mathbb{N}\}.$$

More than a set, $\mathcal{A}$ has an $\mathbb{F}_p$-algebra structure, coming from the algebra structure of $\mathbb{F}_p$. Similarly, if $n \in \mathbb{N}$, and we define $[v]$ going from $\Gamma_n$ to $\mathbb{F}_p$, we define

$$\mathcal{A}_n = \{F([v_1], [v_2], \dots, [v_k]) \mid F \text{ a polynomial, } v_1, \dots, v_k \in X^{\leq n-1} \text{ and } k \in \mathbb{N}\}.$$

As before, $\mathcal{A}_n$ has an $\mathbb{F}_p$-algebra structure. On the other hand, the algebras $\mathcal{A}_n$, together with the natural injections, form a direct system whose direct limit is precisely $\mathcal{A} = \cup\mathcal{A}_n$.

The definitions of $\mathcal{A}$ and $\mathcal{A}_n$ are equivalent to the ones in [Sie09], as it is stated in Corollary 2.1.2 of the same work:

**Lemma 4.2.1.** *$\mathcal{A}$ consists of all continuous functions $\Gamma \to \mathbb{F}_p$.*

As a trivial consequence, $\mathcal{A}_n$ is generated, as an $\mathbb{F}_p$-vector space, by the characteristic functions $\{\chi_g\}_{g\in\Gamma_n}$ where

$$\chi_g(s) = \begin{cases} 1, & \text{if } s = g; \\ 0, & \text{otherwise.} \end{cases}$$

We will say that the *depth* of $[v] \in \mathcal{A}$ is $|v| + 1$, where $|v|$ is the length of $v$ as a word in $X$. In the case of a function $f \in \mathcal{A}$, the depth of $f$ is defined as

$$\mathrm{depth}(f) = \min_{f=F([v_1],[v_2],\ldots,[v_k])} \left( \max_{1\le i\le k} \mathrm{depth}([v_i]) \right),$$

where the minimum is taken over all polynomials so that $f = F([v_1], \ldots, [v_k])$ ($k$ and the $v_i$ may vary with $F$), and the depth of a constant function is set to 0.

If $f \in \mathcal{A}$ is such that $\mathrm{depth}(f) = n$ and $g \in \Gamma$, then the value $f(g)$ only depends on $g$ modulo the $n$th stabilizer $\mathrm{Stab}_\Gamma(n)$, i.e.

$$f(gh) = f(g) \tag{4.2.1}$$

for every $h \in \mathrm{Stab}_\Gamma(n)$.

If $V$ is a subset of $\Gamma$, we let $\mathcal{I}(V) \subseteq \mathcal{A}$ denote the annihilator of $V$, i.e. the set of polynomial functions vanishing on $V$:

$$\mathcal{I}(V) = \{f \in \mathcal{A} \mid f(g) = 0 \text{ for all } g \in V\}.$$

If $I$ is a subset of $\mathcal{A}$, we let $\mathcal{V}(I)$ be the annihilator of $I$, i.e. the set

$$\mathcal{V}(I) = \{g \in \Gamma \mid f(g) = 0 \text{ for all } f \in I\}.$$

We will say that $f \in \mathcal{I}(V)$ is an *equation* for $V$. Note that the name of equation makes sense for an element in $\mathcal{I}(V)$, since it really is what we usually call an equation in an algebraic setting, that is, a polynomial. If $f = F([v_1], [v_2], \ldots, [v_k]) \in \mathcal{I}(V)$ is a linear combination of its variables, then we say that $f$ is a *linear equation* for $V$. This kind of equations will play an important role in this chapter.

Replacing $\Gamma$ by $\Gamma_n$ and $\mathcal{A}$ by $\mathcal{A}_n$, we get the corresponding definitions of the maps $\mathcal{V}$ and $\mathcal{I}$ in the case of truncated trees.

If $v \in X^*$ and $f \in \mathcal{A}$, we define $v * f \in \mathcal{A}$ as follows

$$\begin{array}{rcrcl} v * f & : & \Gamma & \longrightarrow & \mathbb{F}_p \\ & & g & \longmapsto & f(g_v). \end{array}$$

Observe that if $v \in X^*$ and $f = F([v_1], [v_2], \ldots, [v_k]) \in \mathcal{A}$, then

$$v * f = F([vv_1], [vv_2], \ldots, [vv_k]) \in \mathcal{A}.$$

**Remarks 4.2.2.** (i) The algebra $\mathcal{A}_n$ can naturally be identified with the subalgebra of $\mathcal{A}$ formed by functions of depth $\leq n$ (more precisely, we identify $f \in \mathcal{A}_n$ with $f \circ \pi_n \in \mathcal{A}$, where the domain of $\pi_n$ is restricted to $\Gamma$).

(ii) If we have an equation $f$ for $G_n$ for some $n \in \mathbb{N}$, then, by (i), we may think of $f$ as an equation for $G$. Hence $\mathcal{I}(G_n) \subseteq \mathcal{I}(G)$ holds for all $n \in \mathbb{N}$.

(iii) From the previous two remarks we deduce that if we get all the equations for $G_n$ for all $n \in \mathbb{N}$, then we have all equations for $G$. At the same time, we get all equations of the closure in the profinite topology $\overline{G}$ of $G$, since $\pi_n(\overline{G}) = \pi_n(G)$. In other words,

$$\mathcal{I}(G) = \mathcal{I}(\overline{G}) = \bigcup_{n \in \mathbb{N}} \mathcal{I}(G_n).$$

We shall see in Proposition 4.3.3 that, in certain cases, it even suffices to know one specific $\mathcal{I}(G_d)$, if we want to describe $\mathcal{I}(G)$.

But firstly in Proposition 4.2.8 we prove several properties of the maps $\mathcal{I}$ and $\mathcal{V}$ that will be useful in some of our results. Let us give some results that are interesting by themselves but which will also help to prove the mentioned proposition.

**Definition 4.2.3.** We call a subset $V \subseteq \Gamma$ *Zariski-closed* if there is $I \subseteq \mathcal{A}$ such that $V = \mathcal{V}(I)$.

It turns out that finite unions and arbitrary intersections of Zariski-closed sets are again Zariski-closed, hence these sets form the closed sets of a topology, the *Zariski topology*. We show that this topology coincides with the profinite topology of $\Gamma$.

**Lemma 4.2.4.** *The Zariski and the profinite topology of* $\Gamma$ *coincide.*

*Proof.* Consider $I \subseteq \mathcal{A}$ and let us prove that $\mathcal{V}(I)$ is closed in the profinite topology. Let us define $I_n = I \cap \mathcal{A}_n = \{f \in I \mid \operatorname{depth}(f) \leq n\}$. Note that $\mathcal{V}(I_n)$ is a closed subset of $\Gamma$, and therefore so is $\mathcal{V}(I) = \cap \mathcal{V}(I_n)$. Conversely, if $V$ is closed in $\Gamma$, then

$$V = \bigcap_{n \geq 1} V \operatorname{Stab}_\Gamma(n).$$

Observe that, for each $n \in \mathbb{N}$, we can write $V \operatorname{Stab}_\Gamma(n) = V_n \operatorname{Stab}_\Gamma(n)$, where $V_n \subseteq \Gamma_n$, thanks to the decomposition $\Gamma = \Gamma_n \ltimes \operatorname{Stab}_\Gamma(n)$. Now, since the Zariski topology in $\Gamma_n$ coincides with the discrete topology, there exists $\mathcal{I}_n \subseteq \mathcal{A}_n$ such that $V_n = \mathcal{V}(I_n)$. Hence, by (4.2.1), $V \operatorname{Stab}_\Gamma(n) = \mathcal{V}(I_n)$ is Zariski-closed for every $n \in \mathbb{N}$ and so is $V$. $\square$

Recall that an ideal $I$ is *radical* if whenever $f^n$ is in $I$ for some $n \geq 0$, then $f$ also belongs to $I$.

**Lemma 4.2.5.** *All ideals of* $\mathcal{A}$ *are radical.*

*Proof.* The identities $Z^{p^m} = Z$ hold in $\mathbb{F}_p$ for all $m \geq 0$. Therefore they also hold in $\mathcal{A}$. As a consequence, if $f^n$ is in the ideal $I$ for some $n \geq 0$, then $f^{p^m} \in I$ for some $m \geq 0$, and thus $f$ belongs to $I$. $\square$

**Lemma 4.2.6.** *All prime ideals of* $\mathcal{A}$ *are maximal.*

*Proof.* For a proper ideal $I$ of $\mathcal{A}$ and for each $n \in \mathbb{N}$, let us define $I_n = \{f \in I \mid \operatorname{depth}(f) \leq n\} = I \cap \mathcal{A}_n$, which is an ideal of $\mathcal{A}_n$. Then there exists $k \in \mathbb{N}$ such that $I_n = \mathcal{A}_n$ for all $n < k$ and $I_k \neq \mathcal{A}_k$. It is an easy exercise to prove that $I$ is prime (maximal) in $\mathcal{A}$ if and only if $I_n$ is prime (maximal) in $\mathcal{A}_n$ for all $n \geq k$. Therefore, it suffices to prove that prime ideals of $\mathcal{A}_n$ coincide with those that are maximal. Recall that $\mathcal{A}_n$ is generated by the characteristic functions $\{\chi_g\}_{g\in\Gamma_n}$. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{A}_n$ and consider $f \in \mathcal{A}_n \setminus \mathfrak{p}$. We will see that $\chi_g \in \mathfrak{p} + (f)$ for all $g \in \Gamma_n$. Let $g \in \Gamma_n$ be such that $\chi_g \notin \mathfrak{p}$. Since $\mathfrak{p}$ is a prime ideal, $f \cdot \chi_g \notin \mathfrak{p}$. On the other hand, $f \cdot \chi_g = \lambda\chi_g$, with $\lambda = f(g) \in \mathbb{F}_p$, and from the previous assertion $\lambda \neq 0$. Therefore $\chi_g \in (f)$, which concludes the proof. □

The following lemma corresponds to the Weak Nullstellensatz.

**Lemma 4.2.7.** *If $\mathfrak{m}$ is a proper ideal of $\mathcal{A}$, then*

$$\mathcal{V}(\mathfrak{m}) \neq \emptyset.$$

*As a consequence, $\mathfrak{m}$ is maximal in $\mathcal{A}$ if and only if $\mathfrak{m} = \mathcal{I}(g)$ for every $g \in \mathcal{V}(\mathfrak{m})$.*

*Proof.* Suppose, by way of contradiction, that $\mathcal{V}(\mathfrak{m}) = \emptyset$. Consider $n \geq 0$ and for every $g \in \Gamma_n$, pick a lift $\tilde{g}$ of $g$ in $\Gamma$, and a function $f_g \in \mathfrak{m}$ such that $f_g(\tilde{g}) \neq 0$. Then $f_g \cdot \chi_g = \lambda\chi_g$ is in $\mathfrak{m}$, with $\lambda = f_g(\tilde{g}) \neq 0$. Thus $\chi_g$ is in $\mathfrak{m}$ for all $g \in \Gamma_n$, and $\mathcal{A}_n \subseteq \mathfrak{m}$. Since this reasoning holds for all $n \geq 0$, the assumption $\mathcal{V}(\mathfrak{m}) = \emptyset$ implies $\mathfrak{m} = \mathcal{A}$, a contradiction to the properness of $\mathfrak{m}$.

Now, let us prove the second assertion of the lemma. It is clear that $\mathcal{I}(g)$ is maximal in $\mathcal{A}$ for every $g \in \Gamma$, being the kernel of the following $\mathbb{F}_p$-algebra homomorphism:

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathbb{F}_p \\ f & \longmapsto & f(g). \end{array}$$

For the converse, let $\mathfrak{m}$ be a maximal ideal of $\mathcal{A}$ and since $\mathcal{V}(\mathfrak{m}) \neq \emptyset$, let us consider $g \in \mathcal{V}(\mathfrak{m})$. It follows that $\mathfrak{m} \subseteq \mathcal{I}(g)$, and therefore $\mathfrak{m} = \mathcal{I}(g)$, because $\mathfrak{m}$ is maximal and $\mathcal{I}(g) \neq \mathcal{A}$. □

**Proposition 4.2.8.** *Consider the sets $I \subseteq \mathcal{A}$ and $V \subseteq \Gamma$. Then*

(i) *$\mathcal{V}(I)$ is closed in $\Gamma$ and $\mathcal{I}(V)$ is an ideal of $\mathcal{A}$.*

(ii) $\mathcal{I}(\mathcal{V}(I))$ *is the ideal generated by* $I$*, and* $\mathcal{V}(\mathcal{I}(V))$ *is the closure of* $V$*.*

(iii) *The maps* $\mathcal{I}$ *and* $\mathcal{V}$ *define order-reversing bijections which are inverse to one another, between the closed subsets of* $\Gamma$ *and the ideals of* $\mathcal{A}$*.*

*Proof.* The first assertion of (i) is a direct consequence of Lemma 4.2.4 and the second one is trivial. At the same time, part (iii) can be easily deduced from (ii). Now, it is routine to prove that $\mathcal{V}(\mathcal{I}(V))$ is the Zariski closure of $V$, and then by Lemma 4.2.4, it coincides with the closure in the profinite topology of $V$. Let $I$ be an ideal of $\mathcal{A}$ and let us prove now that $\mathcal{I}(\mathcal{V}(I)) = I$. If we show that $\mathcal{I}(\mathcal{V}(I))$ is equal to the radical $\mathrm{Rad}(I)$ of $I$ (an element $f$ is in $\mathrm{Rad}(I)$ if some power $f^n$ belongs to $I$), we will have the desired equality, due to Lemma 4.2.5. It is a well-known fact that

$$\mathrm{Rad}(I) = \bigcap_{\mathfrak{p} \text{ prime},\, I \subseteq \mathfrak{p}} \mathfrak{p}.$$

On the other hand,

$$\mathcal{I}(\mathcal{V}(I)) = \bigcap_{g \in \mathcal{V}(I)} \mathcal{I}(g) = \bigcap_{\mathfrak{m} \text{ maximal},\, I \subseteq \mathfrak{m}} \mathfrak{m},$$

by Lemma 4.2.7. Finally, from Lemma 4.2.6, we get $\mathcal{I}(\mathcal{V}(I)) = \mathrm{Rad}(I)$. $\square$

**Remark 4.2.9.** For any $n \in \mathbb{N}$, the proposition also holds if we replace $\mathcal{A}$ and $\Gamma$, by $\mathcal{A}_n$ and $\Gamma_n$. Note that, in this case, the topology is the discrete topology in $\Gamma_n$.

## 4.3 Branching of ideals and Hausdorff dimension through equations

Our goal in Section 4.4 is to get to know $\mathcal{I}(G)$ when $G \leq \Gamma$ is a non-symmetric GGS-group. For this purpose, we have some concepts and results available that will be useful.

If $G \leq \Gamma$ is self-similar and $f$ is an equation for $G$, it is obvious from the definitions that $v * f$ is again an equation for $G$, for every $v \in X^*$. So when $G$ is self-similar there will be many 'redundant' elements in $\mathcal{I}(G)$. The following definitions are motivated by this fact.

**Definition 4.3.1.** Let $I \subseteq \mathcal{A}$ be an ideal. We say that $I$ is

(i) *branching* if $v * f \in I$ for all $f \in I$;

(ii) *generated by* $S \subseteq I$ *as a branching ideal* if $I$ is generated by $\{v * s \mid v \in X^* \text{ and } s \in S\}$ as an ideal.

We have shown one direction of the lemma below, which can be found in [Sie09].

**Lemma 4.3.2.** *Let $G$ be a closed subgroup of $\Gamma$. Then $G$ is self-similar if and only if the ideal $\mathcal{I}(G)$ is branching.*

Our next proposition is Corollary 2.2.8 in [Sie09], and it gives more detail about the branching structure of $\mathcal{I}(G)$ when $G$ is regular branch. It is also one of the directions of the equivalences that Šunić proves in Theorem 3 of [Šun07].

**Proposition 4.3.3.** *Let $G \leq \Gamma$ be regular branch over $K$, and suppose that $K$ contains $\mathrm{Stab}_G(d-1)$. Then $\mathcal{I}(G)$ is generated by $\mathcal{I}(G_d)$ as a branching ideal.*

In the last part of this section we show how the Hausdorff dimension of a closed subgroup $G$ of $\Gamma$ can be read off a nice generating set of $\mathcal{I}(G)$.

**Definition 4.3.4.** An element $f \in \mathcal{A}$ of depth $n$ is *nice* if there is $f_1 \in \mathcal{A}_{n-1}$ and a linear polynomial $f_2 = F_2([v_1], \ldots, [v_k]) \neq 0$ where $v_1, \ldots, v_k \in X^{n-1}$ such that $f = f_1 + f_2$. The linear part $f_2$ of $f$ will be denoted by $L(f)$.

**Definition 4.3.5.** Let $G \leq \Gamma$ be a self-similar group, $T \subset \mathcal{A}$ a set of nice functions and for $n \in \mathbb{N}$, let us define $T_n = \{f \in T \mid \operatorname{depth}(f) = n\}$ and $S_n = \{L(f) \mid f \in T_n\}$. We will say that $T$ is a *nice generating set* for the ideal $\mathcal{I}(G)$ if for each $n \in \mathbb{N}$, it satisfies the following properties:

(i) $T_1 \cup \ldots \cup T_n$ generates $\mathcal{I}(G_n)$;

(ii) $x * T_n \subseteq T_{n+1}$ for all $x \in X$;

(iii) $|T_n| = |S_n|$, and $S_n$ is linearly independent.

This definition differs slightly from the one given in Section 2.3 of [Sie09], being suited for self-similar groups only. Following the proof of Proposition 2.3.3 of the same work, we can show that for every self-similar group $G$, the ideal $\mathcal{I}(G)$ admits a nice generating set.

**Theorem 4.3.6.** *Let $G$ be a closed self-similar subgroup of $\Gamma$, $T$ a nice generating set for $\mathcal{I}(G)$, and $d_n$ the number of functions of depth $n$ in $T$. Then*

$$\dim_\Gamma G = 1 - \sum_{n\geq 0} \frac{r_n}{p^n},$$

*where $r_n = d_{n+1} - pd_n$ for $n \geq 0$.*

*Proof.* Let $T_n$ and $S_n$ be as in Definition 4.3.5, and let us compute $\log_p |G_n|$ for $n \in \mathbb{N}$. We have the relations

$$\log_p |G_n| = \log_p |G_{n-1}| + \log_p |\operatorname{Stab}_{G_n}(n-1)| = \sum_{i=1}^{n} \log_p |\operatorname{Stab}_{G_i}(i-1)|$$

(note that $\operatorname{Stab}_{G_1}(0) = G_1$). Now, for each $i \geq 1$, let us consider $\operatorname{Stab}_{G_i}(i-1)$ as a linear subspace of $\mathbb{F}_p^{p^{i-1}}$. Since all functions of $T$ are nice and condition (i) of the definition is satisfied, it follows that the subspace $\operatorname{Stab}_{G_i}(i-1)$ is exactly the set of zeros of the linear functions of $S_i$. As $S_i$ is a linearly independent family, and $|S_i| = |T_i| = d_i$, we get

$$\log_p |\operatorname{Stab}_{G_i}(i-1)| = p^{i-1} - d_i,$$

for $i \in \mathbb{N}$. Therefore,

$$\log_p |G_n| = 1 + p + \cdots + p^{n-1} - (d_1 + \cdots + d_n) = \log_p |\Gamma_n| - \sum_{i=1}^{n} d_i$$

and by (1.2.3),

$$\dim_\Gamma G = 1 - \limsup_{n\to\infty} \underbrace{\frac{p-1}{p^n} \sum_{i=1}^{n} d_i}_{a_n}.$$

On the other hand, since $T$ is a nice generating set, it also satisfies the second condition of Definition 4.3.5 and hence the numbers $r_n = d_{n+1} - pd_n$ are non-negative (note that $d_0 = 0$). Using these relations until getting rid of all the $d_i$ we get

$$\sum_{i=1}^{n} d_i = \sum_{i=0}^{n-1} \frac{r_i}{p^i} \cdot \frac{p^n - p^i}{p-1},$$

and then we have

$$a_n = \frac{p-1}{p^n} \sum_{i=1}^{n} d_i = \sum_{i=0}^{n-1} \frac{r_i}{p^i} \cdot \frac{p^n - p^i}{p^n}.$$

If we define

$$b_n = \sum_{i=0}^{n-1} \frac{r_i}{p^i},$$

for $n \in \mathbb{N}$, it is clear that $a_n \leq b_n$ for all $n \in \mathbb{N}$. We will prove that the difference

$$b_n - a_n = \frac{1}{p^n} \sum_{i=0}^{n-1} r_i,$$

goes to 0, when $n$ tends to infinity.

Using the relation $r_i = d_{i+1} - pd_i$ for $i = 0, \ldots, n-1$, we can write

$$\sum_{i=0}^{n-1} r_i = d_n - (p-1) \sum_{i=1}^{n-1} d_i, \qquad b_n = \sum_{i=0}^{n-1} \frac{r_i}{p^i} = \frac{d_n}{p^{n-1}}. \tag{4.3.1}$$

The sequence $b_n$ is clearly non-decreasing and hence it has a limit, $\lambda$. Let us fix $\varepsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that for all $i \geq n_0$ we have

$$\lambda - \varepsilon \leq \frac{d_i}{p^{i-1}} \leq \lambda. \tag{4.3.2}$$

Now, combining (4.3.1) and (4.3.2), we get

$$\sum_{i=0}^{n-1} r_i \leq d_n-(p-1)\sum_{i=n_0}^{n-1} d_i \leq \lambda p^{n-1}-(\lambda-\varepsilon)(p-1)\sum_{i=n_0}^{n-1} p^{i-1} = \varepsilon p^{n-1}+(\lambda-\varepsilon)p^{n_0-1}.$$

Summarizing, if we fix $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have

$$0 \leq \frac{1}{p^n}\sum_{i=0}^{n-1} r_i \leq \frac{\varepsilon p^{n-1} + (\lambda-\varepsilon)p^{n_0-1}}{p^n}.$$

Hence

$$0 \leq \lim_{n\to\infty} \frac{1}{p^n}\sum_{i=0}^{n-1} r_i \leq \frac{\varepsilon}{p},$$

for all $\varepsilon > 0$. Therefore

$$\lim_{n\to\infty}(b_n - a_n) = \lim_{n\to\infty}\frac{1}{p^n}\sum_{i=0}^{n-1} r_i = 0,$$

and since $b_n$ also has a limit, we deduce that

$$\limsup_{n\to\infty} a_n = \limsup_{n\to\infty} b_n = \lim_{n\to\infty} b_n,$$

which completes the proof. □

**Remark 4.3.7.** By Lemma 4.3.2, $G$ self-similar implies $\mathcal{I}(G)$ branching. Hence $r_n$ measures the number of nice functions of depth $n+1$ in a nice generating set of $\mathcal{I}(G)$, after removing the functions we obtain by using the fact that $\mathcal{I}(G)$ is branching.

## 4.4 Equations for GGS-groups

The goal of this section is to describe $\mathcal{I}(G)$ when $G$ is a non-symmetric GGS-group. In the first theorem, given an arbitrary GGS-group $G$, we describe a family $\{R_i\}$ of linear equations of depth 2 for $G$. While in Theorem 4.4.5, we do the same with $\{P_j\}$, a set of equations of depth 3. Finally, we see that if $G$ is non-symmetric, then these linear equations are basically **all** the equations. This is proved in Theorem 4.4.6.

We denote by $\mathbf{0}$ the column vector (of the appropriate length) all of whose entries are equal to 0.

**Theorem 4.4.1.** *Let $G$ be a GGS-group with defining vector $\mathbf{e}$, $C = C(\mathbf{e}, 0)$ and $t = \operatorname{rk} C$. Let us consider the vector space $\operatorname{Null} C = \{r \in \mathbb{F}_p^p \mid Cr^t = \mathbf{0}\}$ and a basis $\{r^i = (r_1^i, r_2^i, \dots, r_p^i)\}_{i=1}^{p-t}$ of $\operatorname{Null} C$. Then*

$$R_i = \sum_{j=1}^{p} r_j^i[j]$$

*for $i = 1, \dots, p-t$ are linearly independent equations of depth 2 for $G$. Moreover, they generate $\mathcal{I}(G_2)$ as an ideal.*

*Proof.* Put $A = \langle a \rangle$ and $N = \operatorname{Stab}_{G_2}(1) = \langle b_0, b_1, \dots, b_{p-1} \rangle$ (we use the same letters $a, b_0, \dots, b_{p-1}$ to denote their images under $\pi_2$). Since $G_2 = N \rtimes A$ and $(na^j)_{(x)} = n_{(x)} a^j_{(n(x))} = n_{(x)}$ for all $n \in N$, $j \in \mathbb{F}_p$ and $x \in X$, we first need to check that actually $R_i \in \mathcal{I}(N)$ for $i = 1, \dots, p-t$. Take into account that $N$ can be identified with the linear space of dimension $t$ spanned by the rows $C_j$ of $C$ (look at (3.2.1) and remember we work modulo $\operatorname{Stab}_G(2)$). Therefore, since $\{r^i = (r_1^i, r_2^i, \dots, r_p^i)\}_{i=1}^{p-t} \subseteq \operatorname{Null} C$, then $R_i \subseteq \mathcal{I}(N)$ for $i = 1, \dots, p-t$, and hence $(R_i \mid i = 1, \dots, p-t) \subseteq \mathcal{I}(G_2)$. Using Remark 4.2.9, we have

$$\mathcal{V}(R_i \mid i = 1, \dots, p-t) \supseteq \mathcal{V}(\mathcal{I}(G_2)) = G_2.$$

Now, from the choice of $\{r^i\}$ to be a basis, the $R_i$ are linearly independent and we also have $|\mathcal{V}(R_i \mid i = 1, \dots, p-t)| = p^{p+1-(p-t)} = p^{t+1}$. On the other hand, we also have $|G_2| = p^{t+1}$, by part (ii) of Theorem 3.2.4. Therefore

$$|\mathcal{V}(R_i \mid i = 1, \dots, p-t)| \leq |G_2|,$$

and so $\mathcal{V}(R_i \mid i = 1, \dots, p-t) = G_2$. Finally, we apply the map $\mathcal{I}$ and use (ii) of Proposition 4.2.8 and Remark 4.2.9, we get $\mathcal{I}(G_2) = (R_i \mid i = 1, \dots, p-t)$, as desired. $\square$

Our next step is to get linear equations of depth 3 for a GGS-group.

**Lemma 4.4.2.** *Let $G$ and $K$ be two groups, and consider a map $\varphi : G \to K$. Suppose that $G$ has a semidirect product decomposition $N \rtimes H$. If*

(i) *$\varphi|_H$ and $\varphi|_N$ are homomorphisms;*

(ii) *$\varphi(nh) = \varphi(n)\varphi(h)$ for all $n \in N$ and $h \in H$; and*

(iii) *$\varphi(n^h) = \varphi(n)^{\varphi(h)}$ for all $n \in N$ and $h \in H$;*

*then $\varphi$ is a homomorphism.*

*Proof.* It is an easy exercise. □

**Theorem 4.4.3.** *Let $G$ be a GGS-group with defining vector $\mathbf{e}$ and $C = C(\mathbf{e}, 0)$. Then there exists a function $\beta \in \mathcal{A}$ of the form*

$$\beta = \sum_{i=1}^{p} \lambda_i [i] \tag{4.4.1}$$

*whose restriction to $G$ is a homomorphism with $\beta(a) = 0$ and $\beta(b) = 1$. The tuple of coefficients $\lambda = (\lambda_1, \ldots, \lambda_p)$ is any that satisfies $C\lambda^t = \mathbf{1}$. We will say that $\beta$ is a* counter *of $G$.*

*Proof.* It suffices to prove the result in $G_2$. Put $A = \langle a \rangle$ and $N = \mathrm{Stab}_{G_2}(1)$. Since $G = N \rtimes A$ and by Lemma 4.4.2, it suffices to find a linear combination (4.4.1) satisfying

(i) $\beta(b) = 1$, and

(ii) $\beta(na^j) = \beta(n) = \beta(n^{a^j})$ for all $n \in N$ and $j \in \mathbb{F}_p$.

(Note that the conditions $\beta(a) = 0$, and $\beta|_A$ and $\beta|_N$ being homomorphisms are automatically satisfied from the choice of $\beta$ as a linear combination of the $[i]$ with $i \in X$.) Let $\mathbf{e}$ be the defining vector of $G$, $C = C(\mathbf{e}, 0)$ and $t = \mathrm{rk}\, C$. We claim that there exists a linear combination (4.4.1) such that

$$\beta(b_0) = \beta(b_1) = \ldots = \beta(b_{p-1}) = 1. \tag{4.4.2}$$

We will prove this later, but suppose for a moment that such a $\beta$ exists. Condition (i) is clearly satisfied. Observe also that the first equality in (ii) is obvious since multiplying $a$ to the right only changes the portrait at the root. On the other hand, let us write an element $g \in G$ in the form $g = \omega(b_0, b_1, \ldots, b_{p-1})a^j$ with $\omega$ a word in $p$ variables and $j \in \mathbb{F}_p$. Then $\beta(g)$ gives us the total weight in $\mathbb{F}_p$ of the word $\omega$. In other words, the existence of $\beta$ proves that the total weight of an element $g \in G$ is well-defined. Note that this last observation proves the second equality in (ii), since then

$$\beta(\omega(b_0, b_1, \ldots, b_{p-1})^a) = \beta(\omega(b_1, b_2, \ldots, b_0)) = \beta(\omega(b_0, b_1, \ldots, b_{p-1})).$$

Let us then prove that there exists such a linear combination $\beta$ satisfying (4.4.2). Using (3.2.1), the $p$ equalities in (4.4.2) are equivalent to the system of equations

$$\begin{pmatrix} e_1 & e_2 & \cdots & 0 \\ 0 & e_1 & \cdots & e_{p-1} \\ \vdots & & \vdots & \vdots \\ e_2 & e_3 & \cdots & e_1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_p \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}. \tag{4.4.3}$$

So the existence of $\beta$ is equivalent to the existence of a solution for (4.4.3), and this is equivalent to (ii) in Lemma 1.3.1. $\square$

As noted in the proof of the previous theorem, if we write $g \in G$ in the form $g = \omega(b_0, b_1, \ldots, b_{p-1})a^j$ with $\omega$ a word in $p$ variables and $j \in \mathbb{F}_p$, then $\beta(g)$ computes the total weight in $\mathbb{F}_p$ of the word $\omega$. In other words, it proves that the total weight of an element $g \in G$ is well-defined. This is already proved in Theorem 3.2.9 (although it is stated for elements in $\mathrm{Stab}_G(1)$). In the same theorem we prove that the partial weights are also well-defined homomorphisms $\mathrm{Stab}_G(1) \to \mathbb{F}_p$. Recall that the $i$th partial weight of an element $g \in \mathrm{Stab}_G(1)$ is the weight of the $i$th variable $b_i$ in a word $\omega$ representing $g$. We may also define the $i$th partial weight for $i \in \mathbb{Z}$, as the $j$th partial weight, taking $j \equiv i \pmod p$ in the range $[0, 1, \ldots, p-1]$. In the following corollary we give the same result but, as for the total weight, we give an explicit expression for the homomorphisms involved.

**Corollary 4.4.4.** *Let $G$ be a GGS-group and $\beta$ a counter of $G$. Then, for every $i \in \{1, 2, \dots, p\}$, the function*

$$\beta_i = i * \beta : \Gamma \to \mathbb{F}_p$$

*restricted to* $\mathrm{Stab}_G(1)$ *is the $i$th partial weight.*

*Proof.* Let $g \in \mathrm{Stab}_G(1)$ and $\omega$ a word representing $g$. Then

$$\psi(g) = \psi(\omega(b_0, \dots, b_{p-1})) = (\omega_1(b_0, \dots, b_{p-1})a^{m_1}, \dots, \omega_p(b_0, \dots, b_{p-1})a^{m_p}),$$

where $\omega_i$ is a word in $p$ variables and $m_i \in \mathbb{F}_p$, for $i = 1, \dots, p$. Note that the $i$th partial weight of $g$ is exactly the total weight of the $i$th component of $\psi(g)$, by (3.2.1). $\square$

**Theorem 4.4.5.** *Let $G$ be a GGS-group with defining vector* $\mathbf{e}$ *and let $\beta$ be a counter of $G$. Then*

$$P_j = [j] - \sum_{i=1}^{p} e_{j-i}(i * \beta)$$

*for $j = 1, \dots, p$ are equations of depth 3 for $G$. (The indices of the $e_{j-i}$ are taken modulo $p$ between 0 and $p-1$ and we set $e_0 = 0$.)*

*Proof.* Let $g \in \mathrm{Stab}_G(1)$ and let us compute the images of $g$ under the maps $[j]$ for $j = 1, \dots, p$, with respect to the images of $g$ under the partial weights $\beta_1, \dots, \beta_p$:

$$\begin{aligned}
[1](g) &= e_1\beta_p(g) + e_{p-1}\beta_2(g) + \cdots + e_2\beta_{p-1}(g) \\
&\vdots \\
[p](g) &= e_{p-1}\beta_1(g) + e_{p-2}\beta_2(g) + \cdots + e_1\beta_{p-1}(g).
\end{aligned}$$

(These relations come from (3.2.1).) Now by Corollary 4.4.4 we have $\beta_i = i * \beta$ and the result follows. $\square$

Let us focus now our attention on GGS-groups with non-symmetric defining vector.

**Theorem 4.4.6.** *Let $G$ be a non-symmetric GGS-group with defining vector $\mathbf{e}$ and let $t$ be the rank of $C = C(\mathbf{e}, 0)$. Let us define $R_i$ as in Theorem 4.4.1, for $i = 1, \dots, p-t$, and $P_j$ as in Theorem 4.4.5, for $j = 1, \dots, p$. Then, the set*

$$S = \{R_i, P_j \mid i = 1, \dots, p-t,\ j = 1, \dots, t\}$$

*generates $\mathcal{I}(G)$ as a branching ideal. Therefore,*

$$\overline{G} = \mathcal{V}(\{v * Q \mid v \in X^*,\ Q \in S\}).$$

*Proof.* By Theorems 4.4.1 and 4.4.5 we have $S \subseteq \mathcal{I}(G)$. Since $G$ is non-symmetric, then $G$ is a self-similar, regular branch group over $G'$ and $\mathrm{Stab}_G(2) \leq G'$ (by Theorem 3.2.1). Hence we apply Proposition 4.3.3, and we obtain that $\mathcal{I}(G_3)$ generates $\mathcal{I}(G)$ as a branching ideal. So the problem reduces to understanding $\mathcal{I}(G_3)$. Let us define

$$\widetilde{S} = \{R_i, x * R_i, P_j \mid i = 1, \dots, p-t,\ x \in X,\ j = 1, \dots, t\}$$

and prove that $\widetilde{S}$ generates $\mathcal{I}(G_3)$. First we prove that the functions in $\widetilde{S}$ are linearly independent. Let us consider a linear combination of the elements in $\widetilde{S}$ that is equal to zero,

$$Q = \sum_{i=1}^{p-t} a_i R_i + \sum_{x \in X} \sum_{i=1}^{p-t} b_i^x (x * R_i) + \sum_{j=1}^{t} d_j P_j = 0, \tag{4.4.4}$$

and let us prove that all the coefficients are equal to zero. Recall that $C = (c_{ij})$ with $c_{ij} = e_{j-i+1}$ (here we consider the indices modulo $p$ between 1 and $p$, and $e_p = 0$). For $k = 1, \dots, p$ let us define $g_k \in \Gamma_3$ by means of its portrait as follows

$$[v](g_k) = \begin{cases} c_{ki}, & \text{if } v = i \in X, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$Q(g_k) = \sum_{j=1}^{t} d_j P_j(g_k) = \sum_{j=1}^{t} d_j [j](g_k) = \sum_{j=1}^{t} d_j c_{kj} = 0,$$

for $k = 1, \ldots, p$. These $p$ conditions are equivalent to the following equality between matrices:

$$\begin{pmatrix} c_{11} & \cdots & c_{1t} \\ \vdots & \vdots & \vdots \\ c_{p1} & \cdots & c_{pt} \end{pmatrix} \begin{pmatrix} d_1 \\ \vdots \\ d_t \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{4.4.5}$$

By part (iii) of Lemma 1.3.1, the first $t$ columns of $C$ are linearly independent. Thus (4.4.5) implies $d_i = 0$ for $i = 1, \ldots, t$. Now, taking into account (4.4.4) and $d_i = 0$ for $i = 1, \ldots, t$, we have

$$Q = \sum_{i=1}^{p-t} a_i R_i + \sum_{x \in X} \sum_{i=1}^{p-t} b_i^x (x * R_i) = 0.$$

But $\{R_i, x * R_i \mid i = 1, \ldots, p-t,\ x \in X\}$ is clearly linearly independent and hence $a_i = 0$ and $b_i^x = 0$ for $i = 1, \ldots, p-t$ and $x \in X$. This proves that $\widetilde{S}$ is actually linearly independent. Therefore we have $|\widetilde{S}| = p - t + p(p-t) + t = p(p-t+1)$ independent equations for a vector space of dimension $\log_p |\Gamma_3| = p^2 + p + 1$, hence $\log_p |\mathcal{V}(\widetilde{S})| \leq p^2 + p + 1 - p(p-t+1) = pt + 1$. By Theorem 3.3.5, we also have $\log_p |G_3| = pt + 1$, and then

$$\log_p |\mathcal{V}(\widetilde{S})| = \log_p |G_3|.$$

On the other hand, $(\widetilde{S}) \subseteq \mathcal{I}(G_3)$ and using Remark 4.2.9 we get

$$\mathcal{V}(\widetilde{S}) \supseteq \mathcal{V}(\mathcal{I}(G_3)) = G_3.$$

From the inequality above and this inclusion, we have $\mathcal{V}(\widetilde{S}) = G_3$ and applying the map $\mathcal{I}$ to this equality we deduce $(\widetilde{S}) = \mathcal{I}(G_3)$, by (ii) in Proposition 4.2.8 and Remark 4.2.9. Therefore $\widetilde{S}$ generates $\mathcal{I}(G)$ as a branching ideal, and thus $S$ generates $\mathcal{I}(G)$ as a branching ideal.

For the last assertion of the theorem, we only need to take into account part (ii) of Proposition 4.2.8. □

Note that $P_j \in \mathcal{I}(G)$ for $j = t+1, \ldots, p$. They are not necessary as generators but they may be useful for some calculations.

As a consequence of the previous theorem, we give the Hausdorff dimension of the closure of any non-symmetric GGS-group. These values match the ones obtained in Chapter 3, in Theorem 3.3.5.

**Corollary 4.4.7.** *Let $G$ be a non-symmetric GGS-group with defining vector $\mathbf{e}$ and let $t$ be the rank of $C(\mathbf{e}, 0)$. Then*

$$\dim_\Gamma \overline{G} = 1 - \frac{p-t}{p} - \frac{t}{p^2} = \frac{(p-1)t}{p^2}.$$

*Proof.* By Theorem 4.4.6, the following subset of $\mathcal{A}$ generates the ideal $\mathcal{I}(G)$:

$$T = \{v * Q \mid v \in X^*,\, Q \in S\},$$

where $S = \{R_i, P_j \mid i = 1, \dots, p-t,\, j = 1, \dots, t\}$. It is easy to see that $T$ is a nice generating set for $\mathcal{I}(G)$. Then by Theorem 4.3.6 and Remark 4.3.7,

$$\dim_\Gamma \overline{G} = 1 - \sum_{n \geq 0} \frac{r_n}{p^n},$$

where $r_n$ is the number of polynomials of depth $n+1$ in $S$. In our case, $r_0 = 0$, $r_1 = p - t$, $r_2 = t$ and $r_n = 0$ for $n \geq 3$, and the result follows. $\square$

**Example 4.4.8.** Let $G$ be the Gupta-Sidki group for $p \geq 3$, i.e. let $G$ be the GGS-group with defining vector $\mathbf{e} = (1, -1, 0, \dots, 0)$. The rank of $C(\mathbf{e}, 0)$ is $p-1$, and the following is a counter of $G$:

$$\beta = -\sum_{i=1}^{p} i[i].$$

Then the $p$ functions

$$R = \sum_{i=1}^{p} [i] \quad \text{and} \quad P_j = [j] - (j-1) * \beta + (j-2) * \beta,$$

for $j = 1, \dots, p-1$, generate $\mathcal{I}(G)$ as a branching ideal. We also have that the Hausdorff dimension of the closure of $G$ is

$$\dim_\Gamma \overline{G} = 1 - \frac{1}{p} - \frac{p-1}{p^2} = \left(\frac{p-1}{p}\right)^2.$$

## 4.5 Addition in non-symmetric GGS-groups

When we define spinal automorphisms in Chapter 2, we do not include rooted automorphisms in the definition and hence not all automorphisms of a spinal group are spinal. On the other hand, the product of two automorphisms corresponding to two different spines is not a spinal automorphism.

Next, we generalize the definitions of spine, spinal automorphism and spinal group to g-spine, g-spinal automorphism and g-spinal group respectively (note that the *g-* stands for generalized), in such a way that we solve these inconveniences. The new definitions are due to Siegenthaler [Sie09], although he uses the terms spine and spinal for the generalized versions.

**Definition 4.5.1.** An element $g \in \operatorname{Aut} \mathcal{T}$ is *finitary* if there is $n \in \mathbb{N}$ such that $g_v = 1$ for all $v \in X^n$. The minimal such $n$ is called the *depth* of $g$.

**Definition 4.5.2.** An element $g \in \operatorname{Aut} \mathcal{T}$ is called *g-spinal* if there exists a finite set $S \subseteq X^\omega$ (possibly empty) such that for every $v \in X^* \setminus \operatorname{Prefix}(S)$, the element $g_v$ is finitary. The minimal such $S$ is the set of *g-spines* of $g$ and will be denoted by $\operatorname{gSpines}(g)$.

**Remarks 4.5.3.** (i) Note that $\operatorname{gSpines}(g) = \emptyset$ if $g \in \operatorname{Aut} \mathcal{T}$ is finitary. So we also include finitary automorphisms in the definition of g-spinal automorphisms. In particular, rooted automorphisms are g-spinal automorphisms.

(ii) Moreover, the product of two g-spinal automorphisms is g-spinal and the inverse of a g-spinal automorphism is so again (see Subsection 2.7 in [Sie09]).

These remarks yield the following definition.

**Definition 4.5.4.** The subgroup of $\operatorname{Aut} \mathcal{T}$ of g-spinal automorphisms is denoted by Sp and the subgroups of Sp are called *g-spinal groups.*

Observe that spinal groups are in particular g-spinal groups, and so are GGS-groups. Note also that if $G \leq \mathrm{Sp}$, this does **not** mean that the closure $\overline{G}$ of $G$ is also inside Sp. It is wrong in particular for any GGS-group.

Our goal is to give a new and suitable description of the elements of non-symmetric GGS groups. This will be done in Theorem 4.5.12. Let us start by giving a couple of technical definitions.

**Definition 4.5.5.** Two sequences $v_1, v_2 \in X^\omega$ are *cofinal* if they are of the form

$$v_1 = u_1 v, \qquad v_2 = u_2 v,$$

for some $v \in X^\omega$ and words $u_1, u_2 \in X^*$ of the same length.

It is easy to check that being cofinal is an equivalence relation.

**Definition 4.5.6.** If $v \in X^\omega$, the equivalence class of $v$ with respect to being cofinal is called the *cofinality class* of $v$ and denoted by $\mathrm{Cof}(v)$.

In the following lemma, we see that if $G$ is a GGS-group and $g \in G$, every g-spine of $g$ is cofinal with $p^\infty$.

**Lemma 4.5.7.** *Let $G$ be a GGS-group and $g \in G$. Then*

$$\mathrm{Cof}(s) = \mathrm{Cof}(p^\infty)$$

*for every $s \in \mathrm{gSpines}(g)$.*

*Proof.* We prove the statement by induction on the length $n$ of $g$. Write $g = \omega(a, b)$, and let $n$ be the length of the word $\omega$. We can assume $\omega$ is a positive word, since $a$ and $b$ have finite order. Clearly, if $n = 1$, i.e. $g = a$ or $g = b$, the assertion is true. Suppose it true for all elements in $G$ of length $n - 1$ and let $g$ be of length $n$. There are two possible cases:

(i) $g = ah$ with $h \in G$ of length $n - 1$.

(ii) $g = bh$ with $h \in G$ of length $n - 1$.

Write $U = \mathrm{gSpines}(h)$. Suppose we are in the first case and let $v \in X^*$. Then

$$(ah)_v = a_v h_{a(v)} = \begin{cases} ah, & \text{if } v = \emptyset; \\ h_{a(v)}, & \text{if } v \neq \emptyset. \end{cases}$$

Here we see that if $a(v) \notin \mathrm{Prefix}(U)$, then $(ah)_v$ is finitary. Or equivalently, if $v \notin a^{-1}(\mathrm{Prefix}(U)) = \mathrm{Prefix}(a^{-1}(U))$, then $(ah)_v$ is finitary. In other words,

$$\mathrm{gSpines}(ah) \subseteq a^{-1}(U).$$

Suppose we are in case (ii) now. For $v \in X^*$, we have

$$(bh)_v = b_v h_{b(v)} = \begin{cases} bh_v, & \text{if } v = p^m \text{ for some } m \geq 0; \\ a^{e_i} h_v, & \text{if } v = p^m i \text{ for some } m \geq 0, \text{ and } i = 1, \dots, p-1; \\ h_{b(v)}, & \text{otherwise.} \end{cases}$$

In this case, if $v \neq p^m$ for every $m \geq 0$, $v \notin \mathrm{Prefix}(U)$ and $v \notin b^{-1}(\mathrm{Prefix}(U)) = \mathrm{Prefix}(b^{-1}(U))$, then we can assure that $(bh)_v$ finitary. Hence

$$\mathrm{gSpines}(bh) \subseteq \{p^\infty\} \cup U \cup b^{-1}(U).$$

In any case,

$$\mathrm{Cof}(s) = \mathrm{Cof}(p^\infty),$$

for all $s \in \mathrm{gSpines}(g)$, as desired. To see this, we only need to check that $\mathrm{Cof}(s) = \mathrm{Cof}(p^\infty)$ for all $s \in a^{-1}(U)$ and $s \in b^{-1}(U)$. Let $u \in X^\omega$ be such that $\mathrm{Cof}(u) = \mathrm{Cof}(p^\infty)$. Then there exists $u_1 \in X^*$ such that $u = u_1 p^\infty$. Then the following is true in any of the two cases $c = a$ or $c = b$:

$$c^{-1}(u) = c^{-1}(u_1 p^\infty) = c^{-1}(u_1)(c^{-1})_{u_1}(p^\infty) = c^{-1}(u_1) x p^\infty,$$

for some $x \in X$ depending on $c$ and $u_1$. Therefore, both $a^{-1}(u)$ and $b^{-1}(u)$ are cofinal with $p^\infty$. $\square$

**Definition 4.5.8.** Let $G$ be a g-spinal group. Let us define the following subset of $\overline{G}$

$$G^* = \{\bar{g} \in \overline{G} \cap \mathrm{Sp} \mid \mathrm{Cof}(s) = \mathrm{Cof}(p^\infty) \text{ for all } s \in \mathrm{gSpines}(\bar{g})\}.$$

Observe that in the definition of $G^*$ are also included finitary automorphisms, i.e. automorphisms $\bar{g} \in \overline{G}$ such that $\mathrm{gSpines}(\bar{g}) = \emptyset$.

Lemma 4.5.7 proves that for a GGS-group $G$, we have

$$G \subseteq G^*.$$

The surprising fact is that the other inclusion also holds for non-symmetric GGS-groups. We state this result in Theorem 4.5.12, which will be proved with the help of some preliminary lemmas.

**Lemma 4.5.9.** *Let $G$ be a GGS-group. Then all finitary automorphisms in $\overline{G}$ are rooted automorphisms, and consequently belong to $G$.*

*Proof.* Let $\bar{g} \in \overline{G}$ be a finitary automorphism. Then there exists $n \in \mathbb{N}$ such that $\bar{g}_v = 1$ for all $v \in X^n$. If the depth of $\bar{g}$ is 1 then $\bar{g}$ is a rooted automorphism and there is nothing to prove. Suppose, by way of contradiction, that the depth $n$ is $\geq 2$. Let $u \in X^{n-2}$. By Theorem 4.4.5 and since $\overline{G}$ is self-similar it follows that for $j = 1, \ldots, p$,

$$(u * P_j)(\bar{g}) = P_j(\bar{g}_u) = 0.$$

Since all labels of $\bar{g}$ in the $n$th level of the tree are trivial, we get

$$0 = P_j(\bar{g}_u) = [j](\bar{g}_u) = [uj](\bar{g}),$$

for all $j = 1, \ldots, p$. To summarize, we get $[uj](\bar{g}) = 0$ for any $u \in X^{n-2}$ and $j = 1, \ldots, p$, i.e. $\bar{g}_w = 1$ for all $w \in X^{n-1}$. But this means that the depth of $\bar{g}$ is $n-1$, which is a contradiction. □

We will focus our attention now on the non-symmetric case.

**Lemma 4.5.10.** *Let $G$ be a non-symmetric GGS-group and suppose that $\bar{g} \in \overline{G}$ satisfies $\bar{g}_i \in G$ for all $i = 1, \ldots, p$. Then*

$$\bar{g} \in G.$$

*Proof.* Let $\bar{g} \in \overline{G}$ be such that $\bar{g}_i \in G$ for all $i = 1, \ldots, p$. By multiplying an appropriate power $a^j \in G$ of $a$, we have

$$\bar{h}_1 = \bar{g} a^j \in \mathrm{Stab}_\Gamma(1).$$

The first level sections of $\bar{h}_1$ are still in $G$, then let $\gamma_i \in \mathbb{F}_p$ be the total weight of the $i$th section of $\bar{h}_1$, for $i = 1, \dots, p$. Now, if we write $\bar{h}_2 = \bar{h}_1 b_0^{-\gamma_p} b_1^{-\gamma_1} \cdots b_{p-1}^{-\gamma_{p-1}}$, we know there exist $c_1, \dots, c_p \in G'$ such that

$$\psi(\bar{h}_2) = (a^{\alpha_1} c_1, \dots, a^{\alpha_p} c_p),$$

where $\alpha_i \in \mathbb{F}_p$ for $i = 1, \dots, p$ (here we are using Theorem 3.2.10). Now, by Lemma 3.3.3,

$$G' \times \overset{p}{\cdots} \times G' \leq \psi(G') \leq \psi(\mathrm{Stab}_G(1)) \tag{4.5.1}$$

and so there exists $c \in \mathrm{Stab}_G(1)$ such that $\psi(c) = (c_1^{-1}, \dots, c_p^{-1})$. If we write $\bar{h}_3 = \bar{h}_2 c$, then

$$\psi(\bar{h}_3) = (a^{\alpha_1}, \dots, a^{\alpha_p}).$$

Hence $\bar{h}_3 \in \overline{G}$ is finitary and by Lemma 4.5.9, we deduce that $\bar{h}_3 = 1 \in G$. Going all the way back, there exists $g \in G$ such that

$$\bar{g} = \bar{h}_3 g \in G.$$

□

**Corollary 4.5.11.** *Let $G$ be a non-symmetric GGS-group, set $n \in \mathbb{N}$ and suppose that $\bar{g} \in \overline{G}$ satisfies $\bar{g}_v \in G$ for all $v \in X^n$. Then*

$$\bar{g} \in G.$$

*Proof.* It follows by using induction on $n$ and the lemma above. □

**Theorem 4.5.12.** *Let $G$ be a non-symmetric GGS-group. Then*

$$G = G^*.$$

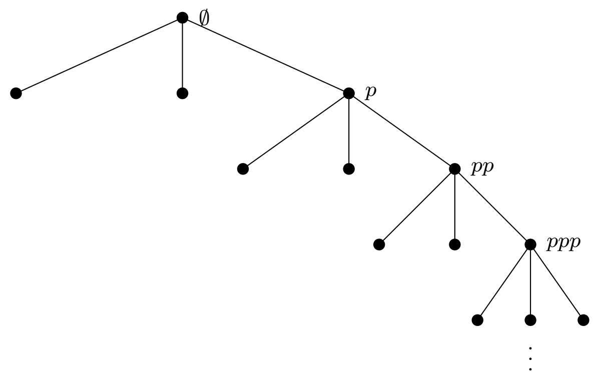
*Proof.* The inclusion $G \subseteq G^*$ is given in Lemma 4.5.7 and it is true for all GGS-groups. Let us prove the other inclusion. Let $\bar{g} \in \overline{G}$ be a g-spinal automorphism such that

$$\mathrm{Cof}(s) = \mathrm{Cof}(p^\infty),$$

for all $s \in \text{gSpines}(\bar{g})$. From this fact and since $\text{gSpines}(\bar{g}) \subseteq X^\omega$ is a finite set, there exists $n \in \mathbb{N}$ such that for every $s \in \text{gSpines}(\bar{g})$,

$$s = vp^\infty$$

for some $v \in X^n$. If $\bar{g}_v \in G$ for all $v \in X^n$, we have $\bar{g} \in G$ by Corollary 4.5.11, and we are done. Let $v \in X^n$ and let us prove that, in fact, $\bar{g}_v \in G$. To simplify notation, let us write $h = \bar{g}_v$ from this point onwards. If $v \notin \text{Prefix}(\text{gSpines}(\bar{g}))$, then $h$ is finitary and by Lemma 4.5.9 belongs to $G$. Suppose now that $v \in \text{Prefix}(\text{gSpines}(\bar{g}))$. Then from the choice of $n$, it is clear that $\text{gSpines}(h) = \{p^\infty\}$. For this reason, by Lemma 4.5.9, the activity of $h$ will be concentrated in the vertices $p^m$ and $p^m i$, where $m \geq 0$ and $i = 1, \ldots, p$, as is shown in the following picture.




The vertices marked with black dots are the ones that may have activity for $h$.

Our goal is to prove that $h \in G$, so multiplying by an appropriate power of $a$, we may assume that $h \in \text{Stab}(1)$. We will see that $h = b^k$ for some $k \in \mathbb{F}_p$, but let us first prove that

$$h_{(p^m)} = 1,$$

for all $m \geq 0$. For $m = 0$, this is true since $h \in \text{Stab}(1)$. By way of contradiction, suppose there exists $m \geq 1$ such that $h_{(p^m)} \neq 1$. Then since $h_{p^{m-1}} \in \overline{G}$ (if $G$ is self-similar, then $\overline{G}$ is also self-similar), in particular we have

$$0 = P_p(h_{p^{m-1}}) = \left( [p] - \sum_{i=1}^{p} e_{p-i}(i * \beta) \right) (h_{p^{m-1}}). \tag{4.5.2}$$

Since $h_{(p^m)} \neq 1$, then $[p](h_{p^{m-1}}) = \mathrm{Log}(h_{(p^m)}) \neq 0$. Now, using this fact and (4.5.2), we deduce that there exists $jk \in X^2$ such that $j \neq p$ and $[jk](h_{p^{m-1}}) \neq 0$. But this is a contradiction because we already saw that there is no activity at distance 2 from the g-spine $p^\infty$.

Let us prove now that $h = b^k$ for some $k \in \mathbb{F}_p$. Let us write $r_i = [i](h)$, for $i = 1, \dots, p-1$. Since $h \in \overline{G}$, in particular $P_i(h) = 0$. Equivalently,

$$r_i = e_i(p * \beta)(h).$$

In other words, there exists $k = (p * \beta)(h) \in \mathbb{F}_p$ such that

$$r_i = ke_i, \tag{4.5.3}$$

for $i = 1, \dots, p-1$. We will prove that $[i](h_{p^m}) = ke_i$ for all $i = 1, \dots, p-1$ and $m \geq 0$, by induction on the level $m$. As we already proved it for $m = 0$, suppose that $m \geq 1$ and

$$[i](h_{p^{m-1}}) = ke_i,$$

for all $i = 1, \dots, p-1$. If we apply the argument we used for $h$, to $h_{p^m}$ this time, we get

$$ke_i = e_i(p * \beta)(h_{p^{m-1}}),$$

for all $i = 1, \dots, p-1$. On the other hand, applying the same argument to $h_{p^{m-1}}$, we get

$$[i](h_{p^m}) = le_i, \tag{4.5.4}$$

for some $l \in \mathbb{F}_p$ and $i = 1, \dots, p-1$. Since, by (4.5.4), the activity in the first level of $h_{p^m}$ and $b^l$ is the same, then so is their image under $\beta$. Therefore,

$$ke_i = e_i(p * \beta)(h_{p^{m-1}}) = e_i\beta(h_{p^m}) = e_i\beta(b^l) = le_i,$$

for all $i = 1, \dots, p-1$. Now, there exists some $e_i \neq 0$ and then we get $k = l$. Consequently, $[i](h_{p^m}) = ke_i$ for all $i = 1, \dots, p-1$ and $m \geq 0$, and since the activity of $h$ is concentrated at distance 1 from $p^\infty$, we conclude that $h = b^k$, as desired. $\square$

**Theorem 4.5.13.** *Let $G$ be a non-symmetric GGS-group. Then the following binary operation:*

$$(g+h)_{(v)} = g_{(v)}h_{(v)},$$

*for $g, h \in G$ and $v \in X^*$, gives $G$ the structure of an abelian group.*

*Proof.* Clearly, $\Gamma$ with the operation $+$ is an abelian group. We need to check that, in fact, $G$ is a subgroup of $(\Gamma, +)$, i.e. that $+$ is an operation in $G$ and that inverses are still in $G$.

From Theorems 4.4.6 and 4.5.12, the automorphism $g \in \Gamma$ belongs to $G$ if and only if

(i) $(v*R_i)(g) = (v*P_j)(g) = 0$, for all $v \in X^*$, $i = 1, \ldots, p-t$, $j = 1, \ldots, t$; and

(ii) $g \in \mathrm{Sp}$ with $\mathrm{Cof}(s) = \mathrm{Cof}(p^\infty)$, for all $s \in \mathrm{gSpines}(g)$.

Let $g, h \in G$. Since $v * R_i$ and $v * P_j$ are both a linear combination of their variables, and taking into account the definition of $+$, we have

$$(v * R_i)(g+h) = (v * R_i)(g) + (v * R_i)(h) = 0,$$

for all $v \in X^*$ and $i = 1, \ldots, p-t$ (and similarly for the $P_j$). On the other hand, observe that if $g_v$ and $h_v$ are finitary for some $v \in X^*$, then $(g+h)_v = g_v + h_v$ is finitary. In other words, if

$$\begin{aligned} v \in (X^* \setminus \mathrm{Prefix}(\mathrm{gSpines}(g))) &\cap (X^* \setminus \mathrm{Prefix}(\mathrm{gSpines}(h))) \\ &= X^* \setminus (\mathrm{Prefix}(\mathrm{gSpines}(g)) \cup \mathrm{Prefix}(\mathrm{gSpines}(h))), \end{aligned} \tag{4.5.5}$$

then $g+h$ is finitary, and so

$$\mathrm{gSpines}(g+h) \subseteq \mathrm{gSpines}(g) \cup \mathrm{gSpines}(h).$$

Therefore, $g+h$ satisfies both conditions (i) and (ii), and hence $g+h \in G$. As for the inverse $-g \in \Gamma$ of an element $g \in G$, note that it has to be defined as

$$[v](-g) = -[v](g).$$

Similarly to how we have done for $g+h$, we get $-g \in G$. □

It is clear that this pointwise addition $+$ is a group operation in the whole of $\Gamma$. The relevance of this theorem is that when $G < \Gamma$ is a non-symmetric GGS-group, and we operate with $+$, we still fall down to the same set $G$.

**Examples 4.5.14.** The Gupta-Sidki group, given by the defining vector $\mathbf{e} = (1, -1, 0, \dots, 0)$, and the Fabrykowski-Gupta group, with $\mathbf{e} = (1, 0, 0)$, acquire the structure of abelian groups with respect to the pointwise addition stated in the theorem.

# Bibliography

[Abe94] Alexander G. Abercrombie, *Subgroups and subrings of profinite rings*, Math. Proc. Camb. Phil. Soc. **116** (1994), 209–222.

[AdlHKŠ07] Goulnara Arzhantseva, Pierre de la Harpe, Delaram Kahrobaei, and Zoran Šunić, *The true prosoluble completion of a group: examples and open problems*, Geom. Dedicata **124** (2007), 5–26, With an appendix by Arzhantseva and Šunić.

[AV05] Miklós Abért and Bálint Virág, *Dimension and randomness in groups acting on rooted trees*, J. Amer. Math. Soc. **18** (2005), 157–192.

[Bar06] Laurent Bartholdi, *Branch rings, thinned rings, tree enveloping rings*, Israel J. Math. **154** (2006), 93–139.

[Ber08] Yakov Berkovich, *Groups of Prime Power Order*, vol. 1, de Gruyter, 2008.

[BG00] Laurent Bartholdi and Rostislav I. Grigorchuk, *Lie methods in growth of groups and groups of finite width*, Computational and geometric aspects of modern algebra (Edinburgh, 1998), London Math. Soc. Lecture Note Ser., vol. 275, Cambridge Univ. Press, Cambridge, 2000, pp. 1–27.

[BG02] ———, *On parabolic subgroups and Hecke algebras of some fractal groups*, Serdica Math. J. **28** (2002), 47–90.

[BGŠ03] Laurent Bartholdi, Rostislav I. Grigorchuk, and Zoran Šunić, *Branch groups*, Handbook of Algebra, vol. 3, North-Holland, 2003, pp. 989–1112.

[BML77] Garrett Birkhoff and Saunders Mac Lane, *A Survey of Modern Algebra*, Macmillan Publishing Co., 1977.

[BS97] Yiftach Barnea and Aner Shalev, *Hausdorff dimension, pro-p groups, and Kac-Moody algebras*, Trans. Amer. Math. Soc. **349** (1997), 5073–5091.

[BŠ01] Laurent Bartholdi and Zoran Šunić, *On the word and period growth of some groups of tree automorphisms*, Comm. Algebra **29** (2001), 4923–4964.

[Cox04] David A. Cox, *Galois Theory*, Wiley-Interscience, 2004.

[Fal85] Kenneth J. Falconer, *The Geometry of Fractal sets*, Cambridge Univ. Press, 1985.

[Fal90] ———, *Fractal Geometry - Mathematical Foundations and Applications*, Wiley, 1990.

[FAZR] Gustavo A. Fernández-Alcober and Amaia Zugadi-Reizabal, *GGS-groups: order of congruence quotients and Hausdorff dimension*, submitted, available at `http://arxiv.org/abs/1108.2289`.

[FAZR11] ———, *Spinal groups: semidirect product decompositions and Hausdorff dimension*, J. Group Theory **14** (2011), 491–519.

[FG85] Jacek Fabrykowski and Narain Gupta, *On groups with subexponential growth functions*, J. Indian Math. Soc. **49** (1985), 249–256.

[Gri80] Rostislav I. Grigorchuk, *On Burnside's problem on periodic groups*, Funktsional. Anal. i Prilozhen. **14** (1980), 53–54, in Russian.

[Gri00] ———, *New Horizons in Pro-p Groups*, ch. Just infinite branch groups, pp. 121–179, Birkhäuser, 2000.

[Gri05] ———, *Infinite groups: geometric, combinatorial and dynamical aspects*, Progr. Math., vol. 248, ch. Solved and unsolved problems around one group, pp. 117–218, Birkhäuser, 2005.

[GS83] Narain Gupta and Said Sidki, *On the Burnside problem for periodic groups*, Math. Z. **182** (1983), 385–388.

[Jac79] Nathan Jacobson, *Lie algebras*, Dover Publications Inc., New York, 1979, Republication of the 1962 original.

[Jen41] Stephen A. Jennings, *The structure of the group ring of a p-group over a modular field*, Trans. Amer. Math. Soc. **50** (1941), 175–185.

[Klo99] Benjamin Klopsch, *Substitution Groups, Subgroup Growth and Other Topics*, Ph.D. thesis, University of Oxford, 1999.

[Laz53] Michel Lazard, *Sur les groupes nilpotents et les anneaux de Lie*, Ann. École Norm. Sup. **71** (1953), no. 3, 101–190.

[Mag40] Wilhelm Magnus, *Über Gruppen und zugeordnete Liesche Ringe*, J. Reine Angew. Math. **182** (1940), 142–149.

[Per00] Ekaterina Pervova, *Profinite topologies in just infinite branch groups*, 154, preprint of the Max Planck Institute for Mathematics, Bonn, Germany.

[Per07] ———, *Profinite completions of some groups acting on trees*, J. Algebra **310** (2007), 858–879.

[Roz96] Alexander V. Rozhkov, *Finiteness conditions in groups of tree automorphisms*, Habilitation thesis, Chelyabinsk, 1996, in Russian.

[Sid87] Said Sidki, *On a 2-generated infinite 3-group: subgroups and automorphisms*, J. Algebra **110** (1987), 24–55.

[Sie08] Olivier Siegenthaler, *Hausdorff dimension of some groups acting on the binary tree*, J. Group Theory **11** (2008), 555–567.

[Sie09] ———, *Discrete and profinite groups acting on regular rooted trees*, Ph.D. thesis, Georg-August-Universität Göttingen, 2009, available at `https://portal.d-nb.de/opac.htm?method=simpleSearch&query=olivier+siegenthaler`.

[Šun07] Zoran Šunić, *Hausdorff dimension in a family of self-similar groups*, Geom. Dedicata **124** (2007), 213–236.

[Šun11] ———, *Pattern closure of groups of tree automorphisms*, Bulletin of Mathematical Sciences **1** (2011), 1–13.

[SZR] Olivier Siegenthaler and Amaia Zugadi-Reizabal, *The equations satisfied by GGS-groups and the abelian group structure of the Gupta-Sidki group*, Eur. J. Combin., accepted.

[Vov00] Taras Vovkivsky, *Infinite torsion groups arising as generalizations of the second grigorchuk group*, Proceedings of the International Algebraic Conference on the Occasion of the 90th Birthday of A.G. Kurosh, de Gruyter, 2000, pp. 357–377.

[Zas40] Hans Zassenhaus, *Ein Verfahren, jeder endlichen p-Gruppe einen Lie-Ring mit der Charakteristik p zuzuordnen*, Abh. Math. Sem. Univ. Hamburg **13** (1940), 200–207.