Stochastic modeling and solution schemes for immunization strategies

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Aitari
Apotono e!

To my father
Apotono e!

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General overview

Chapter 1

Motivation and objectives

1.1 Aims of the work

The work is organized in two autocontained parts. In the first part, the manuscript is oriented to the development of stochastic modeling in order to introduce new measures for risk management. We present an approach for stochastic modeling of different immunization strategies in fixed-income security portfolios under some sources of uncertainty, such as two-stage mean-risk (MR) immunization, two-stage and multistage Value-at-Risk (VaR) strategy, two-stage and multistage first order stochastic dominance constraints (SDC) strategy. These strategies are named averse ones as opposite to the risk neutral strategy whose aim is optimizing the objective function expected value alone, without taken care of the volatility of the function for some scenarios. Another risk averse measure for stochastic mixed 0-1 optimization is proposed as an alternative to the risk neutral one. The new measure is a mixture of the multistage VaR and stochastic dominance constraints strategies.

In this sense, Chapter 2 proposes several immunization strategies implemented in a pilot case of stochastic models for selecting financial portfolios in a market in which there are transaction costs and bonds with different credit ratings. There are two parameters whose random behavior must be taken into account, namely, trends in interest rates and the probabilities of default of the various institutions which issue the bonds. The validity of the proposed strategies is performed by using a case study, and the results that have been obtained seem to be reasonable.

In any case, the proposed model can become computationally difficult for real markets with many possible future scenarios in case of plain use of optimization engines. We may even have difficulties in solving it by using decomposition methods, in the case of a big cardinality of the set of profiles unless we expand some of the existing methods, see [39, 40] to exploit the modelling objects of the recent risk averse strategies being this a subject of our future research.

For these situations we are also alternatively considering to replace the strategy VaR&SDC with the strategy named MR&SDC that stands for Mean-Risk & Stochastic Dominance Constraints since it is a mixture of both. It consists of maximizing the VaR minus the sum of the weighted failure's probabilities of not reaching the set of thresholds imposed by the modeler. The model which implements the new strategy does not include scenario linking constraints, what is a good characteristic from a computational point of view. The validation of this other strategy will be the object of our future research.

Anyway, all these strategies append 0-1 variables which can make the model very difficult to solve in the multistage scheme. For that reason, we are also considering to extend the second order Stochastic Dominance Constrains strategy into the multistage. This strategy does not include 0-1 variables, which makes it very attractive from a computational point of view. The validation of this strategy is also a subject for our future research.

The second part of the memory is devoted to the development of solution schemes. Chapters 3 and 4 study methodologies and software technologies for the solution of large scale stochastic linear two-stage and multistage problems via scenario analysis by using decomposition techniques.

Chapter 3 presents an efficient scenario cluster decomposition approach for identifying tight feasibility cuts in Benders decomposition for solving medium-large and large scale two-stage stochastic problems where only continuous variables appear. In particular we extend the traditional two-stage Benders decomposition, and by using a cluster partitioning of the scenario tree, we propose a new algorithm named scenario Cluster Benders Decomposition (CBD) for dealing with the feasibility cut identification in the Benders method for solving large-scale two-stage stochastic linear problems. Some computational experience is presented, where we observe the favorable performance of the proposed Cluster Benders Decomposition (CBD) approach versus the performance of the Traditional single scenario Benders

Decomposition (TBD) approach.

Chapter 4 extends the proposed Cluster Benders Decompostion approach to the multistage linear case. An information structuring for scenario cluster partitioning of scenario trees is also presented, given the general model formulation of a multistage stochastic linear problem. The basic idea consists of explicitly rewriting the nonanticipativity constraints (NAC) of the variables in the stages with common information. As a result an assignment of the constraint matrix blocks into independent scenario cluster submodels is performed. This partitioning allows us to generate a new information structure to express the NAC which link the related clusters, such that the implicit NAC linking the submodels together is performed by a compact representation until a given break stage, and by a compact representation into each cluster submodel from that break stage until the end. Then, multistage problems can be represented as two blocks of stages models, and the proposed Cluster Benders Decomposition (CBD) can be used as an efficient tool for its solution. The validation via computational experimentation of this new algorithmic scheme will be also a subject for our future research.

1.2 Background and State-Of-The-Art

Mathematical optimization is actually one of the most reliable tools for decision-making. It has many real world applications and a wide range of problems in different areas such as distribution, finance, planning, power generation, air traffic, logistics, natural gas, oil and petrochemical designing and utilization, etc.

Two of its disciplines, deterministic linear and mixed integer optimizations have made this kind of optimization of linear and 0-1 mixed models much easier, at least for moderate dimension problems. However, since the 50's, it is well known that traditional deterministic optimization is not appropriate for capturing the uncertain behavior present in most real world applications.

Moreover, it was not until the 80's when Stochastic Optimization (SO) was broadly applied in real-world applications. Uncertainty is the key ingredient in many decision problems. There are several ways in which uncertainty can be formalized and over the past thirty years different approaches to optimization under uncertainty have been developed by using the risk neutral approach as opposed to the traditional deterministic approach where the uncertain parameters are replaced by their expected values. Although the risk neutral strategy is been currently replaced with the risk averse strategies for better risk management, the original SO developments were a real scientific break through.

The field of SO appears as a response to the need of incorporating uncertainty in mathematical models. Basically, it deals with mathematical models in which some parameters are random variables and, then, they are not controlled by the modeler. The need to incorporate uncertainty in mathematical programming models resulted in the field of SO, since it allows the management of the risk inherent to decision making due to uncertainty today in the main parameters of the problem. Early work started in 1955 with Beale [10] and Dantzig [24]. Although the first linear SO research approaches appear very early, only recently the advance in computer technology has made possible the solution of big size models, thus increasing the interest in SO and so producing an advance in mathematical theory. New problem formulations and algorithmic developments jointly with the inherent theoretical innovations appear almost every year and this variety is one of the strengths of the field.

Very frequently, mainly in problems with a given time horizon to exploit, some coefficients in the objective function and the right hand side (rhs) vector and in the constraint matrix are not known with certainty when the decisions are to be made, but some information is available. This circumstance allows us to use SO for solving multistage programs under uncertainty.

Stochastic programs have the reputation of being computationally difficult to solve. Many people faced with real-world problems are naturally inclined to solve simpler versions, for example to solve the deterministic program mentioned above that results from replacing all random variables with their expected values or to solve several deterministic programs, each corresponding to one particular scenario, and then to combine these different solutions by some heuristic rule. Computational experimentations through out the last decade have proved that in many cases these approaches can be totally inaccurate without guaranteing the solution optimality and, very frecuently, even providing infeasible solutions for many scenarios potentially to occur.

Computation in stochastic optimization has mainly focused on two-stage linear problems with a finite number of realizations. The general model is to choose some initial decision that optimizes the current objective function value plus the expected value of future recourse actions. With a finite number of second-stage realizations and all linear functions, we can always form the so-named Deterministic Equivalent Model (DEM) or extensive form, see the early work [90]. With many realizations, this form of the problem becomes quite large. So, methods that ignore the special structure of stochastic linear programs become quite inefficient. Taking advantage of structure is especially beneficial in stochastic programs and is the focus of much of the algorithmic work in this area.

Once the problem has been formulated by the corresponding DEM it can be solved by using Benders Decomposition (BD) (see [5, 11, 18, 21, 64]).

Other alternative to solve the DEM is Lagrangean Decomposition, see [20, 33, 51, 54, 73, 80, 81, 82] and, recently, [41, 43]. lagrangean procedures can also be applied to stochastic integer problems, in particular stochastic mixed 0-1 problems.

The simplest form of two-stage stochastic integer programs contains first-stage pure 0-1 variables and second stage continuous variables. Laporte and Louveaux [64] apply a branch-and-cut procedure for such problems, based on the Benders decomposition (BD) method. Alonso-Ayuso, Escudero and Ortuño [4] provide an efficient branch-and-fix coordination (BFC) methodology for solving a mixed 0-1 problems for two-stage environments. This methodology was used for solving a model in production planning applications. See also Alonso-Ayuso et al [2, 3], where a production plant dimensioning problem and a supply chain problem are solved. Carøe and Tind [21] generalize the BD to deal with stochastic programs having 0–1 mixed-integer recourse variables and either pure continuous or pure first-stage 0–1 variables.

When the first stage contains pure 0–1 variables, finite termination is readily justified by adopting search procedures that branch over the 0–1 first-stage variables. Sen and Sherali [83] propose decomposition algorithms based on a branch-and-cut approach for solving two-stage stochastic programs having first-stage pure 0–1 variables and 0–1 mixed-integer recourse variables, where a modified BD method is developed. Carøe and Schultz [20] and Hemmecke and Schultz [54] design a branch-and-bound algorithm for problem solving having mixed-integer variables in both stages. Escudero, Garín, Merino and Pérez, have developed a general algorithm to solve two-stage stochastic mixed 0-1 problems in their more general formulations, see [35, 36, 39].

Many operational and planning problems involve sequences of decisions over time. The decisions can respond to realizations of outcomes that are not known a priori. The resulting model for optimal decision making is a multistage stochastic program. In the general formulation of a multistage stochastic optimization, decisions on each stage have to be made stage-wise. At each stage, there are variables which correspond to decisions that have to be made without anticipation of some of future problem data, i.e., they take the same value under each scenario, i.e., the so-called nonanticipativity constraints must be satisfied, whose principle was stated by Wets [90] and restated in [76], see also [18] and many others.

In general, the methods for two-stage problems are generalized to the multistage case but include additional difficult modelling objects. The multistage stochastic linear problem with a finite number of possible future scenarios still has a deterministic equivalent linear program. However, the structure of this problem is somewhat more complex than that of the two-stage problem. The extensive form does not appear readily accesible to manipulations. This is one area of work to which this memory is oriented.

The methods that appear most promising are again based on decompositions, some form of Lagrangean decomposition, or Branch-and-Fix Coordination (BFC), see the series of works [34, 35, 37, 36, 38, 39, 40]. These approaches have in common the explotation of the submodels separability.

There are some efficient approaches to address multistage problems with all 0–1 and continuous variables, and where uncertainty appears only in the objective function coefficients and in the rhs (see [2, 3]). Moreover, there have been few attempts to solve up to optimality large-scale general mixed 0–1 multistage models, where both types of variables appear at any stage of the time horizon, and where uncertainty can appear anywhere in the problem. See [37, 38, 39, 40] for this extension and see also [33]. Concepts like Twin Node Family (TNF), common variables, candidates TNF or TNF integer sets are introduced in these BFC papers.

The use of the Lagrangean Substitution (LS) was introduced by Guignard [51] for bounding purposes in the branching candidate TNFs, and the utilization of the Augmented Lagrangean Decomposition (ALD) scheme ([68, 77]) for obtaining the continuous optimal solution for the so-named TNF integer sets.

Computational experimentation on different Lagrange multipliers updating schemes as the Subgradient method [53], the Volume Algorithm [9], the Progressive Hedging Algorithm [76], and the Dynamic Constrained cutting plane scheme [59] to be used in the LS and ALD approaches has been reported by Escudero, Garín, Pérez and Unzueta [41].

Most of the approaches deal with the optimization of the objective function expected value alone, as we have introduced above, the risk neutral strategy. However, as we have opposited it, the coherent (see in [8] the related features) risk averse strategies, scenario immunization [18, 32], semi-deviations [1, 69], excess probabilities [81], Value-at-Risk [45, 46], Conditional Value-at-Risk [74, 82] and first- and second-order stochastic dominance constraints [28, 29, 48, 49], and other approaches as well, allow to strong reduction of the risk of wrong decision making. See also [42, 43, 65, 79, 86], among others. Finally, another contribution to the previous line of research constitutes the study realized in the Chapter 2 of the present memory.

Part I

Models

Chapter 2

Stochastic models for immunization strategies

In this chapter we present a set of approaches for stochastic optimizing of immunization strategies based on risk averse measures as alternatives to the optimization of the objective function expected value, i.e., in the so-called risk neutral environment. The risk averse measures to consider, whose validity is analyzed in this work are as follows: min-max regret, mean-risk immunization, two-stage and multistage Value-at-Risk strategy, two-stage Conditional Value-at-Risk strategy, two-stage first and second order stochastic dominance, multistage first order stochastic dominance, and the new measure as a mixture of the multistage VaR & stochastic dominance at all stages. Most of these measures require from the modeler a threshold for the objective function value related to each scenario (the recent ones even allow a set of objective function so-called profiles) and a failure probability for not reaching the threshold. Uncertainty is represented by a scenario tree and is dealt with by both two-stage and multi-stage stochastic linear and mixed integer models with complete recourse. We will test the different risk averse strategies presented in the chapter by using, as a pilot case, the optimization of the immunization strategies in fixed-income security portfolios under some sources of uncertainty. The main difference in the bond portfolio strategies that are proposed in the work and the models that have been encountered in the literature is that we consider an investor who wishes to invest in a market with coupon bonds with a different level of risk of default.

2.1 Introduction

Stochastic programming models have been proposed and studied extensively since the 1950s, see the seminal papers by Beale [10], Dantzig [24], Charnes and Cooper [22], Van Slyke and Wets [87], Wets [89, 90], Dempster [27], Kall and Wallace [60], Birge and Louveaux [18], Wets and Ziemba [91], and Shapiro, Dentcheva and Ruszczynski [84], among some others. A stochastic vision is proposed for the financial models dealt with-in this work, rather than the traditional deterministic vision, such that uncertain parameters that are not controlled by the modeler are considered as random variables whose known or estimated probability distributions are independent of the decision variables.

The majority of the financial models proposed until the last decade are static and single-period. However, in cases where uncertainty prevails in all the stages of the planning horizon, then stochastic optimization models become more appropriate. Such models are not very common at present in practical financial applications due to their complexity and the complex requirement for input data. Nevertheless, some very interesting models have appeared in the literature in recent years. There are many ALM (Asset and Liability Management) stochastic optimization models, see [88, 93, 95], among others that are generally preferred by pension, insurance companies, wealthy individuals and hedge funds; see also [37, 94] among many others. One advantage of these scenario based models is that the parameters are not assumed to be known but are scenario dependent, hence they are uncertain. Bradley and Crane [19] present a multistage decision tree model for bond portfolio management. A novel feature was its ability to trace the bond movements from interest rate changes over time. This model is based on dynamic programming rather than stochastic optimization; hence its size grows faster than the latter with more periods and scenarios. Kusy and Ziemba [62] compare their stochastic optimization model for the Vancouver Savings Credit Union with that of Bradley and Crane, and argue in favour of the stochastic optimization model on computational and performance grounds. Both of these models are now easily solved with current optimization technology, see [88], for example. The Bradley and Crane model ushered in bond portfolio management and the management of fixed income securities in the literature; see, for example, [27, 50, 92]. See also [13, 23] for the current state of the literature on this subject. Since the early papers, computation methods have advanced spectacularly, so that large-scale linear optimization problems with continuous variables can, at least, now be solved efficiently. These advances have enabled to apply stochastic optimization increasingly more to real-life financial problems. Some of these financial applications are collected in [23, 78, 88, 93, 95, 96]. See also recent results in [12, 47], among others.

A significant contribution to the above line of research has been made possible thanks to the flexibility of stochastic optimization models to integrate through scenarios diverse multi-dimensional risk factors for risk management. However, the optimization models still become intractable when a large number of variables must be combined, particularly if they are 0-1 variables, with exponential increases in scenarios. In this case procedures are needed to break down the problem and reduce the number of scenarios, see [30, 31, 52, 56, 71], among others.

Most financial optimization models can be classified in-to two broad classes according to their primary objective, namely *risk management* and *financial engineering*. Risk management based models are used to select portfolios with specified exposure to different risks, such that it is concerned, firstly, with selecting which risk one is to be exposed to and which risk is to be immunized against. Secondly, it is concerned with assessing the risks of different securities, and, thirdly, with the construction and maintenance of portfolios with the specified risk-return characteristics. The focus of the optimization models is primarily on the third activity, but all of them are integrated and interdependent.

In this chapter we present an approach for stochastic modeling of different immunization strategies in fixed-income security portfolios under some sources of uncertainty, such as min-max regret, mean-risk immunization, two-stage and multistage Value-at-Risk strategy, two-stage Conditional Value-at-Risk strategy, two-stage and multistage stochastic dominance strategy and, in addition, the new two-stage and multistage mixture of VaR & stochastic dominance. We introduce different models that allow to consider transaction costs. Uncertainty is represented by a scenario tree and is dealt with by both two-stage and multistage stochastic linear and mixed integer models with complete recourse. The main difference between the bond portfolio strategies that are proposed and the models that have been encountered in the literature is that we consider an investor who wishes to invest in a market with coupon bonds with different level of risk of default. Then, there are two sources of uncertainty, or two risks, associated with the model, namely, interest rate risk and credit risk or risk of default. The latter is concerned with the solvency of their issuers and, therefore, the bonds themselves.

The remainder of the chapter is organized as follows. Section 2.2 introduces the classical deterministic linear model to fix notation and to set up the prototype to be improved by introducing uncertainty in the main parameters, namely, interest rate path and credit rating ranking (see Section 2.3). The traditional two-stage stochastic optimization approach is presented in Section 2.4. Section 2.5 is devoted to presenting the main two-stage risk averse strategies, whose validity is tested in this work, namely mean-risk immunization, value-at-risk (VaR) strategy , conditional Value-at-Risk (CVaR) strategy , stochastic dominance strategy and an extended VaR & stochastic dominance strategy . Section 2.6 presents our two-stage approach for fixed-income security portfolio immunization. Section 2.7 presents the approach as in Section 2.6 but considering a mixture of mean-risk and VaR measures. Section 2.8 presents our multistage stochastic scheme for portfolio optimization in a risk adverse environment by using the strategies based on maxmin and stochastic dominance as well as the new multistage strategy as the mixture of the VaR & stochastic dominance strategies. Section 2.9 introduces an illustrative example to show the performance of the different immunization strategies that have been presented in this work. Finally, Section 2.10 concludes.

2.2 The deterministic linear optimization model

In this section we introduce a deterministic optimization model used in risk management. In the following sections we introduce potential or actual extensions of this basic model. We now present the basic components and common notation of the mathematical prototype.

Let us consider a partition of the planning horizon (PH) into k subintervals of equal length $[t_0, t_1]$, $[t_1, t_2]$,..., $[t_{k-1}, t_k]$, being t_0 the starting and t_k the end of the PH. We also assume that portfolio rebalancing is only allowed at the beginning of each subinterval. $|\mathcal{T}| = k+1$ is the number of time periods, and t_k is the final period. So, set $\mathcal{T}$ is defined as the discretization or splitting of the time horizon, i.e., $\mathcal{T} = \{t_0, t_1, ..., t_k\}$. For the sake of simplicity, and without loss of generality (wlog), we assume that there are $|\mathcal{I}|$ different coupon bonds available at t_0, each of them maturing at t^i, such that $t^i \in \{t_1, ..., t_k\}$, but not necessarily for all the bonds, since there might be some bond i^* with period of maturity, t^{i*}, later than the final period of planning, i.e., $t^{i*} > t_k$. Then, coupon payments are also due at rebalancing points, where t^i is the maturity period of bond $i \in \mathcal{I}$, and $\mathcal{I}$ denotes the set of securities to be included in the portfolio.

Let I_0 denote the initial budget to be invested in the portfolio. As decision variables, x_{it}^+ denotes the volume of security $i \in \mathcal{I}$ purchased in period t, and x_{it}^- denotes the volume of security i sold in period t. Variable z_{it} denotes the volume of security i to be held in the portfolio following the transactions conducted in period t. And variable V_t is defined as the final value of the portfolio in period t, for $t_0 \leq t \leq t_k$. Let β be a parameter that denotes a fraction of the volume negotiated, which represents the transaction costs affecting each readjustment. Let us also assume that the nominal figure and the coupon payments do not generate transaction costs. P_{i0} denotes the unit price on the market of security i at the start of the planning horizon, i.e., initial period t_0, for $i \in \mathcal{I}$. Additionally, c_i is the annual coupon for security i, and C_{it} denotes the payment stream generated by one unit of security $i \in \mathcal{I}$ in period t, $t \in T - \{t_0\}$. This stream is expressed as follows,

$$C_{it} = \begin{cases} c_i \cdot h, & t_0 < t < t^i, \\ F_i + c_i \cdot h, & t = t^i, \\ 0, & t > t^i, \end{cases} \tag{2.1}$$

where h is the constant length of each sub-interval (fraction of a year) and F_i is the nominal value of security i, in this case, $F_i = P_{i,0}$, $\forall i \in \mathcal{I}$.

Finally, P_{it} denotes the unit price of security i at time period t if r is the interest rate at that period. If we consider the transaction costs as a fraction β then P_{it}^-, which denotes the unit selling price of security i at period t, is computed as $P_{it}^- = (1-\beta)P_{it}$; and P_{it}^+ is the unit purchase price of security i at period t, so that $P_{it}^+ = (1+\beta)P_{it}$.

The optimization model is a linear problem to maximize the final value of the portfolio,

$$(LO) \qquad \max V_{t_k} \tag{2.2}$$

s.t.

$$\begin{aligned}
x_{i0}^{+} &= z_{i0} \quad \forall i \in \mathcal{I} && (2.3)\\
\sum_{i\in\mathcal{I}} P_{i0}^{+} x_{i0}^{+} &= I_0 && (2.4)\\
z_{i,t-1} + x_{it}^{+} - x_{it}^{-} &= z_{it}, \quad \forall i \in \mathcal{I}, \quad t = t_1, \ldots, t_{k-1} && (2.5)\\
x_{i:t^i=t,t}^{-} &= z_{i:t^i=t,t-1}, \quad \forall i \in \mathcal{I}, t = t_1, \ldots, t_k && (2.6)\\
x_{it_k}^{-} &= z_{i,t_{k-1}}, \quad \forall i \in \mathcal{I} : t_k < t^i && (2.7)\\
\sum_{i\in\mathcal{I}:t<t^i} P_{it}^{+} x_{it}^{+} - \sum_{i\in\mathcal{I}:t<t^i} P_{it}^{+} x_{it}^{-} &= \sum_{i\in\mathcal{I}:t\le t^i} C_{it} z_{i,t-1}, \quad \forall t = t_1, \ldots, t_{k-1} && (2.8)\\
\sum_{i\in\mathcal{I}:t_k<t^i} P_{it_k}^{-} x_{it_k}^{-} + \sum_{i\in\mathcal{I}:t_k\le t^i} C_{it} z_{it_{k-1}} &= V_{t_k} && (2.9)
\end{aligned}$$

$$\begin{aligned}
0 &\le z_{i0}, x_{i0}^{+}, \quad \forall i \in \mathcal{I} && (2.10)\\
0 &\le x_{it}^{+}, \quad \forall i \in \mathcal{I}, \quad t = t_1, \ldots, t_{k-1} && (2.11)\\
0 &\le z_{it}, x_{it}^{-}, \quad \forall i \in \mathcal{I}, \quad t = t_1, \ldots, t_k && (2.12)\\
0 &\le V_{t_k}. && (2.13)
\end{aligned}$$

Equations (2.3)-(2.4) are the first period constraints representing the structural ones. In general these constraints establish that the portfolio must be formed at the initial period, t_0. The initial budget constraint assigns values to the initial investment in each security. In particular, equation (2.3) forces the portfolio to be built at the initial period. Equation (2.4) establishes the initial investment of the portfolio holder. Constraints (2.5)-(2.7) are balance equations that link the volume of securities purchased and sold in each period with the volume of securities in the portfolio. Equations (2.8) are the constraints that ensure that the portfolio is self-financing, i.e., the funds to purchase a new security at each period must be obtained by selling securities and from the yield from coupons on the bonds in the portfolio. Moreover, equation (2.9) forces the portfolio to be dismantled at the end of the process, so the final value is the sum of the cash obtained from selling securities and the yield obtained by the coupons paid on bonds.

Solution considerations: Given the state of the art in linear optimization solving, model (2.2)-(2.13) should not present major difficulties for problem solving, even for large scale instances. However, the optimal solution may not be suitable, given the uncertainty of the main parameters, namely the interest rate path and credit rating ranking.

2.3 Two sources of uncertainty: interest rate and credit risk

Uncertainty will be represented below in terms of random experiments with outcomes (i.e., scenarios) denoted by ω. The relevant set of outcomes is clearly problem-dependent. Also, it is not usually very important to be able to define those outcomes accurately because the focus is mainly on their impact on some (random) variables. The set of all outcomes is represented by Ω, that can be visualized as a set of scenarios to be structured in a tree below, and it can be dealt with by using stochastic optimization approaches. Each node in the scenario tree depicts a juncture for rendering decisions. A scenario is defined as a complete path from the root node to a leaf, and defines a single realization of all the uncertain parameters.

Throughout the chapter we will define a set of strategies from two independent sources of uncertainty, namely interest rates and ratings of the issuers of each bond. This is an innovative way to face the problem since we are going to deal with two completely different sources of uncertainty at the same time.

For the sake of simplicity and wlog we assume that there are n different coupon bonds available at t_0, each of them maturing at $t^1, t^2, ..., t^n$ respectively, that is, coupon payments are also due at rebalancing points. For each coupon bond in a given class j, $i \in \mathcal{I}_j$ with maturity period t^i, we will consider a replica $i' \in \mathcal{I}_{j'}$ in a different class j', with the same maturity period $t^i = t^{i'}$, but with a different level of risk of default. Then $\mathcal{J}$ is the set of classes of securities under consideration, where the securities in each class have the same maturity period but different credit risk. The risk in this case concerns the solvency of their issuers and therefore, the bonds themselves.

Interest rate risk

The price of a bond is directly related to the interest rate level all over the PH. The interest rate is a fraction of the par value or nominal (the money you invest) that you will recieve as payment when holding a bond. The interest rate depends on the maturity of the bond (when it will pay the par value) and it does not change in a linear way. In fact, the Term Structure of Interest Rates (for short, TSIR) changes much more in the short term and is more stable in the long term. In this way, interest rates may change independently for different bond maturities. As this is really difficult to modelize and following one of the assumptions of Khang's immunization theory, see [61], we will consider flat changes in interest rate structure. This means that every interest rate will change in a common way, whatever its maturity. So, if the annual interest rate is r, the corresponding interest rate for the maturity Δt ($r_{\Delta t}$) can be expressed,

$$r_{\Delta t} = \Delta t \cdot r,$$

where Δt is the time in years. We will assume that, at the time when the investment is made, i.e., initial period t_0, the interest rate is at a certain level r_{t_0}. In the next period it might change into any of the scenarios under consideration, see for example Figure 2.1.

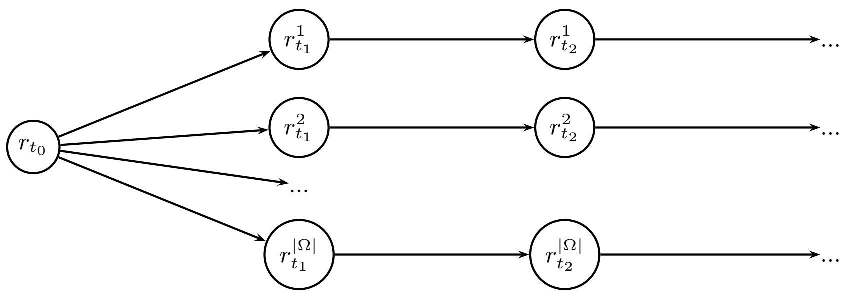


Figure 2.1: Two-stage scenario tree for the interest rate uncertainty

For simplicity, the tree assumes that change can only happen after the portfolio has been built.

Ths interest rate risk is related to the risk of this variable to change from one decision period to another. These changes may happen almost instantly and affect the whole market in the same way.

Credit risk

Anyway, the price of a bond depends not only on the risk free interest rate in force in the market, but also on the quality of the issuing company. That way, depending on the solvency of the bond issuer, its price will be higher or lower.

Moreover, the following definitions are provided to facilitate notation. [1]

[1] The definitions are taken from [57] and all the illustrative examples that will be used with the definitions are taken from[72].

Definition 1 *Credit rating means the rating given to an individual or a company to indicate their solvency as debtors in the issuing of short- or long-term securities. Rating agencies examine companies that issue bonds, as well as the situation of the bonds issued at regular intervals, and may upgrade or downgrade their ratings whenever they see fit.*

Table 2.1: S&P ranking definition

Rating	**Bond Quality**
AAA	Prime rating
AA	High grade rating
A	Upper medium grade rating
BBB	Lower medium grade rating
BB	Speculative
B	Highly speculative
CCC	Very highly speculative
CC	Extremely speculative
C	Very extremely speculative
D	Likely to default on capital, interest

Taking the S&P ranking as a reference point, bonds can be rated as shown in Table 2.1. The rating agency would examine the quality of the companies every year, so, their rating might change. As an example, Table 2.2 shows the global corporate average transition rates (1981-2009) in percentage.

Table 2.2: Examples of rate transition probabilities

	AAA	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**	**Default**
AAA	88.21	7.73	0.52	0.06	0.08	0.03	0.06	0
AA	0.56	86.6	8.10	0.55	0.06	0.09	0.02	0.02
A	0.04	1.95	87.05	5.47	0.40	0.16	0.02	0.08
BBB	0.01	0.14	3.76	84.16	4.13	0.70	0.16	0.26
BB	0.02	0.05	0.18	5.17	75.52	7.48	0.79	0.97
B	0.00	0.04	0.15	0.24	5.43	72.73	4.61	4.93
CCC	0.00	0.00	0.21	0.31	0.88	11.28	44.98	27.98

Definition 2 *Default means failure to pay back a loan on maturity, or when the terms of an agreement are fulfilled.*

The likelihood *risk of default* is closely linked to the credit rating of an organization at a given instant in time. By way of example, consider Table 2.3 which represents the risks of default, q_i, calculated as cumulative mean probabilities in percentage according to the S&P report.

Note that the higher the probability of default, the cheaper the bond will be. Somehow the bond is going to be penalized for its risk.

Table 2.3: Examples of risks of default

Credit	**Years since emission**						
Rating	**1**	**2**	**3**	**4**	**5**	**7**	**10**
AAA	0.00	0.03	0.14	0.26	0.39	0.58	0.82
AA	0.02	0.07	0.14	0.24	0.33	0.52	0.74
A	0.08	0.21	0.35	0.53	0.72	1.22	1.97
BBB	0.26	0.72	1.23	1.86	2.53	3.80	5.60
BB	0.97	2.94	5.27	7.49	9.51	13.19	17.45
B	4.93	10.76	15.65	19.46	22.30	26.47	30.82
CCC	27.98	36.95	42.40	45.57	48.05	50.26	53.41

Definition 3 *Recovery rate, z_i, is the proportion of the money owed that the issuer undertakes to pay to the purchaser in the case of default.*

Table 2.4: Examples of recovery rates

Type of instrument	**Recovery rate (%)**
Term Loans	**69.4**
Revolving credit	**78**
All loans	**73.8**
Senior secured bonds	**57.2**
Senior unsecured bonds	**43**
Senior subordinated bond	**28.3**
All other subordinated bond	**19.4**
All bonds	**37.4**

For example, S&P gives the figures shown in Table 2.4 for the period 1987-2009 and for the discounted recovery rates by instrument type.

Note that the lower the recovery rate is, the cheaper a bond would become.

According to these two concepts, we will define in the next sections the real interest rate as the interest rate that a company would pay in order to compensate its risk of default.

2.4 Traditional two-stage stochastic optimization approach

The stochastic version of the problem provides a general purpose-modeling framework, which embraces many real-world features, such as turnover constraints, transaction costs, limits on groups of assets, risk aversion, immunization constraints and other considerations. In particular, two-stage stochastic linear problems provide a suitable framework for modeling decision problems under uncertainty arising in several financial applications. The flexibility of these models is related to their dynamic nature, i.e., besides the first stage variables representing decisions made in face of uncertainty, the model includes second-stage decisions, i.e., recourse actions, which may be taken once a specific realization of the random parameters is observed.

The model for portfolio selection described below seeks to obtain the optimal readjustment, independently of changes in interest rates. This model enables transaction costs to be factored in, and takes into account different credit ratings for bonds. Thus, decision-makers can benefit the optimal portfolio taking into account the risk associated with the likelihood of bankruptcy or default on the part of the issuer of each bond and the weight that they themselves attribute to that risk.

Let the following notation for sets, parameters and variables which define the Deterministic Equivalent Model (DEM) for the stochastic scheme in the case of bonds portfolio optimization.

<u>Sets</u>:

$\mathcal{I}$, set of securities i to be included in the portfolio. $|\mathcal{I}|$ is the number of securities in set $\mathcal{I}$.

$\mathcal{J}$, set of classes of fixed-income security considered j, i.e., the set of different credit ratings considered.

$\mathcal{I}_j$, set of securities i that belong to class j, $\mathcal{I}_j \subset \mathcal{I}$, such that $\mathcal{I} = \cup_{j\in\mathcal{J}}\mathcal{I}_j$, and $\mathcal{I}_j \cap \mathcal{I}_g = \emptyset$, $j, g \in \mathcal{J} : j \neq g$. $|\mathcal{I}_j|$ is the number of securities in set $\mathcal{I}_j$, $j \in \mathcal{J}$.

$\mathcal{T}$, set of periods in PH plus the initial period t_0, such that it is the discretization or splitting of the time horizon, i.e., $\mathcal{T} = \{t_0, t_1, ..., t_k\}$.

Ω, set of scenarios ω under consideration, t_0 show the different joint situations for interest rates and default. $|\Omega|$ denotes the number of scenarios.

Parameters:

I_0, initial investment budget.

F_i, nominal value of security i, for $i \in \mathcal{I}$.

t^i, maturity period of security i, for $i \in \mathcal{I}$, such that $t^i \in \{t_1, ..., t_k, ..., t_n\}$.

β, fraction of the volume negotiated which represents the transaction costs affecting each readjustment. It is also considered that the nominal figure and the coupon payments do not generate transaction costs.

P_{i0}, unit price on the market of security i at the starting of PH, period t_0, for $i \in \mathcal{I}$.

w^ω, likelihood of scenario ω, given by the modeler.

r_t^ω, risk-free interest rate in period t under scenario ω. At period $t = t_0$, $r = r_{t_0}^\omega = r_{t_0}^{\omega'}$, for $\omega \neq \omega'$, $\omega, \omega' \in \Omega$.

q_{jt}^ω, measure of risk calculated at period t as the probability of default of the securities of class j under scenario ω. At period $t = t_0$, $q_j = q_{jt_0}^\omega = r_{jt_0}^{\omega'}$, for $\omega \neq \omega'$, $\omega, \omega' \in \Omega$.

z_i, recovery rate (as a fraction of one) for security i, for $i \in \mathcal{I}_j$, $j \in \mathcal{J}$.

R_{it}^ω, real interest rate (as a fraction of one) for security i of class j at time period t under scenario ω, for $i \in \mathcal{I}_j$, $j \in \mathcal{J}$, $t \in \mathcal{T}$ and $\omega \in \Omega$. Following [57], we calculate it as follows,

$$R_{it}^\omega = r_t^\omega + \frac{(1 + r_t^\omega)(1 - z_i)q_{jt}^\omega}{1 - (1 - z_i)q_{jt}^\omega}.$$

At period $t = t_0$, $R_i = R^{\omega}_{i0} = R^{\omega'}_{i0}$, for $\omega \neq \omega'$, $\omega, \omega' \in \Omega$. Moreover, $R^{\omega}_{it} = r^{\omega}_{t}$, for $i \in \mathcal{I}_j$, such that $q^{\omega}_{jt} = 0$.

c_i, annual coupon for security i. This coupon is calculated as $R_i \cdot F_i$, where R_i is the real interest rate at initial period t_0, and F_i is the nominal value of security i, for $i \in \mathcal{I}$.

C^{ω}_{it}, payment generated by one unit of security i in period t under scenario ω, for $i \in \mathcal{I}_j$,$j \in \mathcal{J}$, $t \in \mathcal{T} - \{t_0\}$, $\omega \in \Omega$. The stream is calculated as follows,

$$C^{\omega}_{it} = \begin{cases} c_i \cdot h, & t_0 < t < t^i, \quad i \in \mathcal{I}_j : q^{\omega}_{jt} < 1 \\ F_i + c_i \cdot h, & t = t^i, \quad i \in \mathcal{I}_j : q^{\omega}_{jt} < 1 \\ z_i \cdot F_i & t_0 < t \leq t^i, \quad i \in \mathcal{I}_j : q^{\omega}_{j,t-1} < 1 \quad \text{and} \quad q^{\omega}_{jt} = 1 \\ 0, & \text{otherwise}, \end{cases} \tag{2.14}$$

where h is the constant length of each sub-interval (fraction of a year) and q^{ω}_{jt} is the default probability for class j at period t under scenario ω. Notice that if there is a period t and a scenario ω such that $q^{\omega}_{jt} = 1$, it means that the class j of securities is in default, and then, $C^{\omega}_{it} = z_i \cdot F_i$, and $C^{\omega}_{i\tau} = 0$, $\forall \tau = t+1, \cdots, t_k$, for $i \in \mathcal{I}_j$. In another case, what happens is that for each period t and each scenario ω, $q^{\omega}_{jt} < 1$, i.e., the class of securities j is not in default, and the payment stream for all the securities in that class coincides with that defined in expression (2.1).

P^{ω}_{it}, unit price of security i at time period t under scenario ω, if R^{ω}_{it} is the real interest for security i at period t under scenario ω, for $i \in \mathcal{I}_j, j \in \mathcal{J}$, $t \in \mathcal{T} - \{t_0\}$, $\omega \in \Omega$. It is obtained as follows,

$$P^{\omega}_{it} = \sum_{t < \tau \leq t^i} C^{\omega}_{i\tau} ER^{\omega}_{i}(t, \tau),$$

where $ER^{\omega}_{i}(t,\tau) = (1 + R^{\omega}_{it} \cdot h)^{-(\tau - t + 1)}$.

$P^{+\omega}_{it}$, unit purchase price of security i at period t under scenario ω when the transaction cost β is considered.

$$P^{+\omega}_{it} = (1 + \beta) P^{\omega}_{it} \tag{2.15}$$

$P^{-\omega}_{it}$, unit selling price of security i at period t under scenario ω.

$$P^{-\omega}_{it} = (1 - \beta) P^{\omega}_{it} \tag{2.16}$$

<u>*Variables*</u>:

$x_{it}^{+\omega}$, volume of security i purchased in period t under scenario ω, for $i \in \mathcal{I}$, $t \in \mathcal{T}$, $\omega \in \Omega$.

$x_{it}^{-\omega}$, volume of security i sold in period t under scenario ω, for $i \in \mathcal{I}$, $t \in \mathcal{T} - \{t_0\}$, $\omega \in \Omega$.

z_{it}^{ω}, volume of security i to be held in the portfolio following the transactions conducted in period t under scenario ω, for $i \in \mathcal{I}$, $t \in \mathcal{T}$, $\omega \in \Omega$.

V_t^{ω}, final value of the portfolio in period t under scenario ω, for $t \in \mathcal{T}$, $\omega \in \Omega$.

The two-stage stochastic approach for maximizing the expected final value of the portfolio over the set of scenarios (i.e., in a risk neutral environment) is as follows,

$$(DEM1) \qquad \max \sum_{\omega \in \Omega} w^{\omega} V_{t_k}^{\omega} \tag{2.17}$$

s.t.

$$\begin{aligned}
x_{i0}^{+} &= z_{i0}, \quad \forall i \in \mathcal{I}, && (2.18)\\
\sum_{i \in \mathcal{I}} P_{i0}^{+} x_{i0}^{+} &= I_0 && (2.19)\\
z_{i,t-1}^{\omega} + x_{it}^{+\omega} - x_{it}^{-\omega} &= z_{it}^{\omega}, \forall i \in \mathcal{I}, t = t_1, \ldots, t_{k-1}, \omega \in \Omega && (2.20)\\
x_{i:t^i=t,t}^{-\omega} &= z_{i:t^i=t,t-1}^{\omega}, \ \forall i \in \mathcal{I}, t = t_1, ..., t_k, \\
& \qquad \omega \in \Omega && (2.21)\\
x_{i,t_k}^{-\omega} &= z_{i,t_{k-1}}^{\omega}, \ \forall i \in \mathcal{I} : t_k < t^i, \ \omega \in \Omega && (2.22)\\
\sum_{i \in \mathcal{I}: t < t^i} P_{it}^{+\omega} x_{it}^{+\omega} - \sum_{i \in \mathcal{I}: t < t^i} P_{it}^{+\omega} x_{it}^{-\omega} &= \sum_{i \in \mathcal{I}: t \leq t^i} C_{it}^{\omega} z_{i,t-1}^{\omega}, \forall t = t_1, ..., t_{k-1}, \\
& \qquad \omega \in \Omega && (2.23)\\
\sum_{i \in \mathcal{I}: t_k < t^i} P_{it_k}^{-\omega} x_{it_k}^{-\omega} + \sum_{i \in \mathcal{I}: t_k \leq t^i} C_{it}^{\omega} z_{i,t_{k-1}}^{\omega} &= V_{t_k}^{\omega}, \quad \forall \omega \in \Omega && (2.24)
\end{aligned}$$

$$\begin{aligned}
0 &\leq z_{i0}, x_{i0}^{+}, \quad \forall i \in \mathcal{I}, && (2.25)\\
0 &\leq x_{it}^{+\omega}, \quad \forall i \in \mathcal{I}, \quad t = t_1, \ldots, t_{k-1}, \quad \omega \in \Omega && (2.26)\\
0 &\leq z_{it}^{\omega}, x_{it}^{-\omega}, \quad \forall i \in \mathcal{I}, \quad t = t_1, \ldots, t_k, \quad \omega \in \Omega && (2.27)\\
0 &\leq V_{t_k}^{\omega}, \quad \forall \omega \in \Omega. && (2.28)
\end{aligned}$$

Solution considerations: It is well known that for a large set of scenarios, this kinds of problems become quite considerable. Methods that ignore the special structure of stochastic linear programs become quite inefficient. Taking benefit from the model's structure is specially required in stochastic programs. For large dimensions, perhaps the method most frequently used is based on building an outer linearization of the recourse cost function and a solution of the first stage problem (the primal problem) plus its linearization. This decomposition, given by Benders [11], has been widely extended in stochastic optimization to take care of feasibility questions, see its specialization given by Van Slyke and Wets [87], the well known L-shaped method; see also [18, 55, 60], among many others. However, generating feasibility cuts by the mere application of the scenario related feasibility, see the second part of this memory, maybe very inefficient for problem solving in some cases. Instead, we have proposed a cluster scenario decomposition approach for dealing with the feasibility problem, which generates tighter feasibility cuts to add to the master problem and, then, it improves the performance of the Traditional Benders Decomposition (TBD), see [6].

2.5 Risk averse models

Unfortunately, the real world is not a risk neutral environment in any case, and in order to introduce in the models the risk averseness inherent to any decision maker, several risk averse models have been studied in the literature; all of them in the two-stage scheme. In this section, we analyze some of the most important ones as well as introducing a new approach as a mixture of Value-at-Risk and stochastic dominance strategies.

Mean-risk immunization strategy

Let us consider the general two-stage stochastic linear problem for the maximization of the objective function expected value,

$$\begin{array}{rll} \mathcal{Q}_E = & \max & cx + E_\xi[\min w^\omega (q^{\omega T} y^\omega)] \\ & \text{s.t.} & \\ & & b_1 \le Ax \le b_2 \\ & & h_1^\omega \le T^\omega x + W^\omega y^\omega \le h_2^\omega \quad \forall \omega \in \Omega \\ & & x, y^\omega \ge 0 \quad \forall \omega \in \Omega, \end{array} \tag{2.29}$$

where c is a known vector of the objective function coefficients for the x variables in the first stage, b_1 and b_2 are the left and right hand side vectors for the first stage constraints, respectively, and A is the known matrix for the first stage constraints; w^ω is the likelihood attributed to scenario ω, h_1^ω and h_2^ω are the left and right hand side vectors for the second stage constraints, respectively, and q^ω is the vector of the objective function coefficients for the y variables, while T^ω and W^ω are the technology matrices under scenario ω, for $\omega \in \Omega$.

Putting together the stochastic components of the problem, we have the vector $\xi^\omega = (q^\omega, h_1^\omega, h_2^\omega, T^\omega, W^\omega)$. Finally, E_ξ represents the mathematical expectation with respect to ξ over the set of scenarios Ω.

The main criticism that can be made against this very popular mean strategy is that it ignores the variance on the objective function value over the scenarios and, in particular, the "left" queue of the non-wanted scenarios. However, there are some other approaches that in addition deal with new risk measures (in a risk averse environment) by also considering, e.g., semi-deviations [69, 32], excess probabilities [81], Value-at-Risk [45, 46], conditional Value-at-Risk [8, 75, 82] and first- and second-order stochastic dominance constrains [29, 28, 49, 48]. Those approaches are more amenable than the classical mean-variance, see [66, 67], mainly in the presence of 0-1 variables.

In this work, we can use the excess probability approach, such that,

$$\mathcal{Q}_P = P(\omega \in \Omega : cx + q^\omega y^\omega > \phi), \tag{2.30}$$

where ϕ is a prescribed threshold for the excess probability $\mathcal{Q}_P$. So, alternatively to maximize $\mathcal{Q}_E$, where

$$\mathcal{Q}_E = cx + \sum_{\omega \in \Omega} w^\omega q^\omega y^\omega \tag{2.31}$$

the mean-risk function to maximize is

$$\mathcal{Q}_E + \psi \mathcal{Q}_P \tag{2.32}$$

where ψ is a positive weighting parameter. A more amenable expression of (2.32) for computational purposes, at least, can be as follows, see [81],

$$\begin{array}{lll} (Mean - Risk) & \max & \mathcal{Q}_E + \psi \sum_{\omega \in \Omega} w^\omega (1 - \nu^\omega) \\ & \text{s.t.} & \\ & & cx + q^\omega y^\omega + M\nu^\omega \geq \phi \quad \forall \omega \in \Omega \\ & & b_1 \leq Ax \leq b_2 \\ & & h_1^\omega \leq T^\omega x + W^\omega y^\omega \leq h_2^\omega \quad \forall \omega \in \Omega \\ & & x, y^\omega \geq 0, \quad \forall \omega \in \Omega \\ & & \nu^\omega \in \{0,1\}, \quad \forall \omega \in \Omega \end{array} \tag{2.33}$$

where ν^ω is a 0-1 variable such that its value is 1 if the objective function value $cx + q^\omega y^\omega$ for scenario ω is below threshold ϕ and otherwise, is 0; and M is the given smallest parameter that still allows any feasible solution to the original problem.

Following this formulation, we introduce a mean-risk model by changing the objective function, and appending a constraint and the 0-1 variable per each scenario ω in model (DEM1) (2.17)-(2.28). So, ν^ω will take value 1 if the final wealth under scenario ω is below threshold ϕ and 0 otherwise, then,

$$\nu^\omega = \begin{cases} 1 & \text{for } V_{t_k}^\omega < \phi \\ 0 & \text{otherwise.} \end{cases}$$

Then, the new mean-risk model becomes a parametric model defined as follows,

$$\begin{aligned} (DEM-MR) \qquad & \max \sum_{\omega\in\Omega} w^\omega V_{t_k}^\omega + \psi \sum_{\omega\in\Omega} w^\omega (1-\nu^\omega) && (2.34)\\ \text{s.t.} \qquad & \text{constraints } \ (2.18)-(2.28) \text{ and} \\ & V_{t_k}^\omega + M\nu^\omega \geq \phi \quad \forall \omega \in \Omega, && (2.35)\\ & \nu^\omega \in \{0,1\} \quad \forall \omega\in\Omega. && (2.36) \end{aligned}$$

Notice that the parameters ψ, ϕ and M must be adequately chosen to calibrate the model at the first two, and M for reducing the computational effort.

Value-at-Risk (VaR) strategy

Recent theoretical research on risk management suggests that the measures based on quantiles are good functions to measure the risk. Between them, the Value-at-Risk (VaR) has become a reference for many financial applications, see e.g. [45, 46] among others.

The VaR approach is very attractive since it is easy to interpret: it measures up in monetary units and can be used to carry out an estimation of the necessary volume of own funds to cover the market risk in business activities developed by the financial institutions. A very useful approximation in the context of stochastic optimization is, precisely, the VaR optimization in the model.

$$
\begin{array}{lll}
(VaR) & \max VaR & \\
\text{s.t.} & & \\
& cx + q^\omega y^\omega + M\nu^\omega \geq VaR \quad \forall \omega \in \Omega & \\
& b_1 \leq Ax \leq b_2 & \\
& h_1^\omega \leq T^\omega x + W^\omega y^\omega \leq h_2^\omega \quad \forall \omega \in \Omega & \\
& \sum_{\omega \in \Omega} w^\omega \nu^\omega \leq \alpha & \\
& x, y^\omega \geq 0 \quad \forall \omega \in \Omega & \\
& VaR \geq 0 & \\
& \nu^\omega \in \{0, 1\} \quad \forall \omega \in \Omega, &
\end{array}
\tag{2.37}
$$

where ν^ω is a 0-1 variable such that its value is 1 if the objective function value $cx + q^\omega y^\omega$ for scenario ω is below VaR and otherwise, is 0 and M is the given parameter presented above. Notice that the optimization is realized over $(1 - \alpha)\%$ of the scenario, α being the accepted probability of the scenarios to occur whose function value $cx + q^\omega y^\omega$ is smaller than VaR.

The advantage of the VaR strategy over the traditional maxmin strategy, see model (2.42), is that this requires that the objective function value should be not smaller than VaR for all the scenarios, no matter how representative and how many they are. So, its value may not be too high given that very restrictive constraint.

Conditional Value-at-Risk (CVaR) strategy

The advantage of the VaR approach over the maxmin strategy and the expected value strategy (2.29) is obvious, as it takes into account the probability of bad scenarios. However, it does not consider how bad accepted scenarios can be. The so-named CVaR strategy takes into account the conditional expectation of the objective function value $cx + q^\omega y^\omega$ above VaR. Let us consider model,

$$
\begin{array}{lll}
(CVaR) & \max & VaR + \mathcal{B} \sum_{\omega \in \Omega} w^\omega (cx + q^\omega y^\omega - VaR)_+ \\
\text{s.t.} & & \\
& & cx + q^\omega y^\omega + M\nu^\omega \geq VaR \quad \forall \omega \in \Omega \\
& & b_1 \leq Ax \leq b_2 \\
& & h_1^\omega \leq T^\omega x + W^\omega y^\omega \leq h_2^\omega \quad \forall \omega \in \Omega \\
& & \sum_{\omega \in \Omega} w^\omega \nu^\omega \leq \alpha \\
& & x, y^\omega \geq 0, \quad \forall \omega \in \Omega \\
& & VaR \geq 0 \\
& & \nu^\omega \in \{0, 1\} \quad \forall \omega \in \Omega,
\end{array}
\tag{2.38}
$$

where ν^ω is a 0-1 variable with the same meaning as above, i.e., its value is 1 if the function value $cx + q^\omega y^\omega$ for scenario ω is below VaR and otherwise, is 0, and M and $\mathcal{B}$ are given parameters. In the objective function, $(z)_+$ gives the positive value of the variable z.

As it can be seen in [82, 75] the traditional CVaR that has numerous applications, where the VaR is maximized in [75] and the objective function expected value and the weighted VaR are maximized in [82]. And, in both of them, the conditional expectation of the objective function value below VaR is minimized in a composite weighted way.

First order Stochastic Dominance strategy

"The concept of Stochastic Dominance Constraints (SDC) aims to identify acceptable solutions for the problem under uncertainty and optimizing over them. The random variable X is said to be stochastically smaller in first order (respectively in second order) than a random variable Y, i.e., $X \preceq_1 Y$ (resp. $X \preceq_2 Y$) iff $Eh(X) \leq Eh(Y)$ for all nondecreasing (respectively nondecreasing convex) functions h for which both expectations exist", see [49] (respectively, [48]).

Let us consider the following model whose aim is to maximize an objective function, whose vector of coefficients could be given by a general function $g(x, y)$ that, in our case, has been defined as the objective function expected value over the set of scenarios, such that the function value $cx + q^\omega y^\omega$ is not below the threshold, say ϕ^p for a finite number of thresholds, say $|\mathcal{P}|$, and a probability of failure, say α^p given for threshold ϕ^p, for $p \in \mathcal{P}$, where $\mathcal{P}$ is the so-named set of profiles. So, the first order stochastic dominance constraint strategy (1SDC) can be implemented as follows,

$$
\begin{array}{lll}
(1SDC) & \max & cx + \sum_{\omega\in\Omega} w^\omega q^\omega y^\omega \\
\text{s.t.} & & \\
 & & cx + q^\omega y^\omega + M\nu^{\omega p} \geq \phi^p \quad \forall \omega \in \Omega, \quad p \in \mathcal{P} \\
 & & b_1 \leq Ax \leq b_2 \\
 & & h_1^\omega \leq T^\omega x + W^\omega y^\omega \leq h_2^\omega \quad \forall \omega \in \Omega \\
 & & \displaystyle\sum_{\omega\in\Omega} w^\omega \nu^{\omega p} \leq \alpha^p \quad \forall p \in \mathcal{P} \\
 & & x, y^\omega \geq 0, \quad \forall \omega \in \Omega \\
 & & \nu^{\omega p} \in \{0,1\} \quad \forall \omega \in \Omega, \quad p \in \mathcal{P},
\end{array}
\tag{2.39}
$$

where the 0-1 variable $\nu^{\omega p}$ is defined as the above 0-1 variables, i.e., it takes the value 1 provided that the function value $cx+q^\omega y^\omega$ does not reach the threshold ϕ^p for the pth profile under scenario ω, for $\omega \in \Omega$, $p \in \mathcal{P}$.

Second order Stochastic Dominance strategy

The second order stochastic dominance is very similar to the first order in terms of the leading strategy, although there is a very important difference, namely, second order stochastic dominance does not append 0-1 variables, which makes it computationally more attractive.

The second-order stochastic dominance constraints strategy (2SDC) requires a set of profiles given by the pairs (ϕ^p, e^p) $\forall p \in \mathcal{P}$, where e^p is the upper bound of the expected deficit of the objective value over the scenarios on reaching the threshold ϕ^p. It can be implemented as follows,

$$\begin{array}{lll} (2SDC) & \max & cx + \sum_{\omega\in\Omega} w^\omega q^\omega y^\omega \\ & \text{s.t.} & \\ & & \phi^p - \left(cx + q^\omega y^\omega\right) \le v^{\omega p} \quad \forall \omega \in \Omega, \quad p \in \mathcal{P} \\ & & b_1 \le Ax \le b_2 \\ & & h_1^\omega \le T^\omega x + W^\omega y^\omega \le h_2^\omega \quad \forall \omega \in \Omega \\ & & \sum\limits_{\omega\in\Omega} w^\omega v^{\omega p} \le e^p \quad \forall p \in \mathcal{P} \\ & & x, y^\omega \ge 0, \quad \forall \omega \in \Omega \\ & & v^{\omega p} \ge 0, \quad \forall \omega \in \Omega, \quad p \in \mathcal{P}, \end{array} \tag{2.40}$$

such that $v^{\omega p}$ is a non-negative variable equal to the difference (if it is positive) between the threshold ϕ^p and the objective value for scenario ω, so named objective value deficit on reaching threshold ϕ^p.

Notice that both stochastic dominance strategies, $1SDC$ and $2SDC$, i.e., models (2.39) and (2.40), have the advantage over the other strategies presented above, in that they force the function $cx + q^\omega y^\omega$ not to be smaller than given thresholds all over the potential values of the function for given probabilities of failure.

As an innovation in the two-stage environment, we present a combination of the strategies VaR & SDC.

Value-at-Risk & Stochastic Dominance strategy

Let us now consider the following model whose aim is to maximize the weighted sum of the potential values-at-risk, for the given profiles, such that the function value $cx + q^\omega y^\omega$ is not below each of them for given probability of failure, α^p such that the value-at-risk for profile p is lower bounded by profile ϕ^p, for $p \in \mathcal{P}$.

$$
\begin{array}{rll}
(E-VaR) & \max & \sum_{p\in\mathcal{P}} \gamma^p V^p \\
& \text{s.t.} & \\
& & cx + q^\omega y^\omega + M\nu^{\omega p} \geq V^p \quad \forall \omega \in \Omega, \quad p \in \mathcal{P} \\
& & b_1 \leq Ax \leq b_2 \\
& & h_1^\omega \leq T^\omega x + W^\omega y^\omega \leq h_2^\omega \quad \forall \omega \in \Omega \\
& & \sum\limits_{\omega\in\Omega} w^\omega \nu^{\omega p} \leq \alpha^p \quad \forall p \in \mathcal{P} \\
& & x, y^\omega \geq 0 \quad \forall \omega \in \Omega \\
& & V^p \geq \phi^p \quad p \in \mathcal{P} \\
& & \nu^{\omega p} \in \{0,1\} \quad \forall \omega \in \Omega, \quad p \in \mathcal{P},
\end{array}
\tag{2.41}
$$

where γ^p is the weight attributed to profile p in the objective function to maximize, and the 0-1 variable $\nu^{\omega p}$ takes the value 1 provided that the function value $cx+q^\omega y^\omega$ does not reach the minimum value V^p, such that $V^p \geq \phi^p$ under scenario ω, for $\omega \in \Omega$, $p \in \mathcal{P}$. Notice that for $|\mathcal{P}| = 1$, $(E-VaR)$ model (2.41) coincides with (VaR) model (2.37).

Solution considerations: We must point out that the models (2.37), (2.38), (2.39) and (2.41) have the computational disadvantage (being stronger for models (2.39) and (2.41) for $|\mathcal{P}| > 1$ than for models (2.37) and (2.38)) that model (2.33) has not, since they have constraints linking 0-1 variables from different scenarios. In any case, a decomposition approach must be used for problem solving in large-scale instances. So, we propose to use a Branch-and-Fix Coordination approach that has proved to provide good results, see [36, 38].

In order to test the different risk averse strategies, that we have taken into consideration, we consider the portfolio optimization problem presented in Section 2.4 in order to build a portfolio immunization model that will deal with two sources of uncertainty.

2.6 Two stage fixed-income security portfolio immunization

Traditional immunization is a portfolio strategy based on the maxmin approach to be used to match interest-rate risk of an asset portfolio against future streams of liabilities, in order to achieve net zero market exposure. There is a considerable amount of literature on portfolio immunization, see for instance [14, 15], among many others. A bond portfolio is said to be immunized under parallel interest-rate shifts when the market value of the portfolio is, at least, as great as the present value (V) of the liabilities to be satisfied and the portfolio duration is matched with the liability duration. Bierwag and Khang [15] prove that immunization can be seen as a maximization strategy throughout all states of the art in which the objective of investors is to achieve a guaranteed minimum return over the PH. According to Dantzig [25], this maximum solution can be found by solving an equivalent linear problem that depends on the TSIR hypothesis.

With this terminology, *immunization* is the maxmin strategy obtained as the optimal solution of the following deterministic linear model,

$$\begin{aligned} & \max V \\ \text{s.t.} \quad & \\ & \sum_{i=1}^{n} v_i^{\omega} x_i \geq V \quad \forall \omega \in \Omega \\ & \sum_{i=1}^{n} x_i = I_0 \\ & V, x_i \geq 0 \quad \forall i = 1, ..., n, \end{aligned} \tag{2.42}$$

where the variable x_i indicates the percentage of investment I_0 that has to be assigned to buy asset i, v_i^{ω} denotes the portfolio value at the end of the PH if an amount of $\mathcal{I}$ budget is invested at the starting of the PH in asset i and, shortly afterwards, the interest rate becomes r^{ω} remaining unchanged until the end of the PH; and, finally, V is the minimum portfolio value guaranteed at the end of the PH.

In any case, it must be pointed out that the solution given by the DEM model (2.42) is not exactly an immunized portfolio but an approximation to it. This is due to the fact that only a finite number of scenarios (i.e., interest rate shifts) is considered. One way to obtain a more accurate result is to include not only the biggest interest rate fluctuations but also, sufficiently small shifts in interest rates. One of the most important factors for the development of portfolio strategies to combat interest rate risk is the so-named Dynamic Global Portfolio Immunization Theorem put forward by Khang [61], under which the optimum strategy for ensuring the final value of a portfolio at the end of a time period, regardless of changes in interest rates, is that which matches Macaulay's term planning horizon[2] for that portfolio at all times. Given the nature of the term of a portfolio, this strategy would imply continuous readjustment. In any event, the optimality of the strategy is based on the following assumptions, among others:

a. The interest rate structure has parallel changes, i.e., if the structure changes from $g(t)$ to $g^*(t,\beta)$, then

$$g^*(t,\beta) = g(t) + \beta.$$

b. There are no transaction costs. This assumption is extremely important, since if there are transaction costs and those costs are high, then the continuous readjustment of the portfolio would not be viable.

c. The yield on bonds depends only on the current interest rate, with no need to take into account any risk of default.

The first of these assumptions avoids the risk of underestimating the performance of the structure, known as the *Risk of Immunization*, see [44]. The assumption of no transaction costs is crucial in a dynamic context, since the strategy of continuous readjustment may not be optimum if transaction costs are considered, given the high costs that it would entail. Furthermore, given the third assumption, there is no analysis in the context of bonds with different credit ratings and therefore, with a positive associated risk of default. We are going to break these last two assumptions to check whether they are crucial to its validity.

[2]The Macaulay duration is the weighted-average term to maturity of the cash flows from a bond, where the weights are the present value of the cash flow divided by the price, such that

$$d_i = \frac{\sum_{s=1}^{i}(t_s - t_0)Coupon \cdot Discount}{\sum_{s=1}^{i} Coupon \cdot Discount}$$

In a zero-coupon bond the Macaulay duration is equal to the bonds maturity.

A two-stage stochastic extension of the immunization model is presented in model (DEM2), but, previously, the following notation is introduced:

v_{i0}^{ω}, final value of an investment of P_{i0} monetary units in security i made at the initial period, t_0, if the instantaneous spot interest rate changes, shortly afterwards from R_{it_0} to $R_{it_1}^{\omega}$ and no additional unexpected interest rate change takes place until the end of the PH under scenario ω. It is calculated as follows,

$$v_{i0}^{\omega} = \frac{\sum_{\tau=t_1}^{t_k} C_{i\tau}^{\omega} ER_i^{\omega}(t_0,\tau)}{ER_i^{\omega}(t_0,t_k)}, \tag{2.43}$$

where $i \in \mathcal{I}$, $\omega \in \Omega$, and $ER_i^{\omega}(t_0,\tau) = \prod_{s=t_0}^{\tau}(1 + R_{is}^{\omega} \cdot h)^{-1}$.

Then, the immunization at period t_0 is formulated by the constraints

$$\sum_{i\in\mathcal{I}} v_{i0}^{\omega} z_{i0} \quad \geq \quad V_0 \quad \forall \omega \in \Omega,$$

such that it is the big substantive difference with the two-stage stochastic model introduced in Section 4.

So, the two-stage maxmin DEM is as follows,

$$
\begin{aligned}
(DEM2) \qquad & \max V_0 + \sum_{\omega\in\Omega} w^{\omega} V_{t_k}^{\omega} && (2.44)\\
\text{s.t.} \qquad & \text{constraints } (2.18) - (2.28) \text{ and} && \\
& \sum_{i\in\mathcal{I}} v_{i0}^{\omega} z_{i0} \geq V_0 \quad \forall \omega \in \Omega && (2.45)\\
& V_0 \geq 0, && (2.46)
\end{aligned}
$$

where the objective function (2.44) to maximize gives the investment value at the end of the initial period t_0 and the expected final value of the portfolio over the set of scenarios. Additionally, constraints (2.45) are the key to the immunization strategy at the initial period, since they ensure a minimum wealth under all the future uncertainties.

2.7 A new approach for mean-risk immunization

As a new two-stage approach in the risk adverse environment, we will introduce a hybrid mean-risk and Value-at-Risk (VaR) model for optimizing immunization strategies over the scenarios, such that the portfolio's value at the end of the initial period is not smaller than VaR with a given probability of failure α, over the scenarios. For this purpose, we will change the objective function (2.44) and constraints (2.45). Firstly, we consider a 0-1 variable per scenario, ν^ω, such that, as before, it will take value 1 if the value of the portfolio at the end of the initial period under scenario ω, $\sum_{i\in\mathcal{I}} v_{i0}^\omega z_{i0}$, is smaller than V_0, and 0, otherwise. We point out that we introduce these 0-1 variables in order to control by some means the strict behavior of constraints (2.45). The new set of constraints that replace (2.45) and since now they are second stage constraints, they can be expressed as follows,

$$\sum_{i\in\mathcal{I}} z_{i,0} v_{i,0}^\omega + M\nu^\omega \geq V_0 \quad \forall\omega \in \Omega,$$

where M is a parameter to calibrate the model. We have experimented with the choice of M as the maximum value between V_0 and $V_{t_k}^\omega$, $\omega \in \Omega$, and the results are satisfactory.

We optimize the model subject to the immunization constraints over the scenarios, such that the sum of their failure probabilities is less than α which depends on the risk averseness of the investor. To ensure this, the constraint to introduce can be expressed,

$$\sum_{\omega\in\Omega} w^\omega \nu^\omega \leq \alpha.$$

Then, the new DEM is as follows,

$$(DEM3) \qquad \max V_0 + \sum_{\omega\in\Omega} w^\omega V_{t_k}^\omega \tag{2.47}$$

$$\text{s.t.} \quad \text{constraints } (2.18) - (2.28) \text{ and}$$

$$\sum_{i\in\mathcal{I}} v_{i0}^\omega z_{i0} + M\nu^\omega \geq V_0 \quad \forall\omega \in \Omega \tag{2.48}$$

$$\sum_{\omega\in\Omega} w^\omega \nu^\omega \leq \alpha \tag{2.49}$$

$$V_0 \geq 0 \tag{2.50}$$

$$\nu^\omega \in \{0,1\} \;\; \forall\omega \in \Omega. \tag{2.51}$$

Note: The parameters M and α must be adequately chosen to calibrate the model.

Solution considerations: Model (DEM3) is a two-stage stochastic mixed 0-1 optimization and then, a further computational effort must be made, since plain use of even state-of-the-art optimization engines cannot be affordable given the large-scale nature of the problem. Alternatively, we propose some types of decomposition approaches based on Benders decomposition, Branch-and-Fix Coordination and others that have been proved to provide good results in the field, see [35, 36, 38].

The following section constitutes one of the main contributions of this memory.

2.8 A more accurate solution: a multistage stochastic scheme

Multistage maxmin immunization strategy

In the previous sections we have presented two-stage models where the immunization condition is imposed on precisely at the initial period, t_0. However, the immunization strategy should prevail through out all the periods of the PH, given the uncertainty of the main parameters all over the horizon. So, furthermore, some corrective action should be implemented between periods t and $t+1$, then multistage stochastic models become more appropriate.

In the general formulation of a multistage model, decisions on each stage have to be made stage-wise. For example, first stage variables are selected before observing the realization of uncertain parameters, through the occurrence of the scenarios. After having decided on first stage and having observed each realization of uncertain parameters, the second stage (or recourse) decision has to be made. After having decided on a second stage and having observed each realization of uncertain parameters, the next (third) stage decision has to be made, and so on. At each stage, there are variables which correspond to decisions that have to be made without anticipation of some of future problem data, i.e., they take the same value under each scenario that belong to the same group for each stage (i.e., the so-named nonanticipativity constraints must be satisfied).

Each node, say $\mathbf{g}$, in Figure 2.2 represents a point in time where a decision can be made. Once a decision is made, some contingencies may occur, and

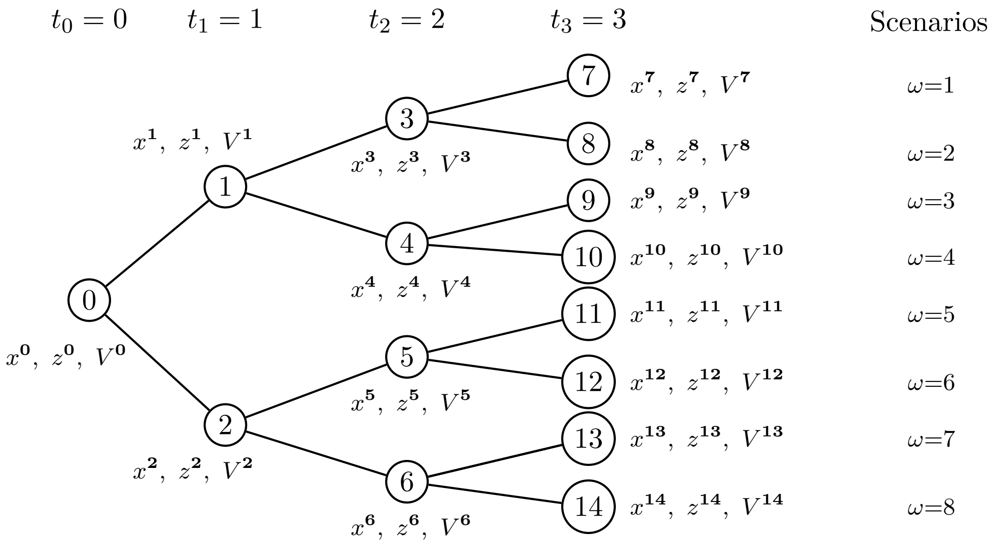


Figure 2.2: Multistage scenario tree

information related to these contingencies is available at the beginning of each stage. In this context, a stage is a point in time where a decision is made and, in some cases, can be included by a subset of consecutive time periods. In this example, there are $|\mathcal{T}| = 4$ stages, and $\mathcal{T} = \{t_0 = 0, \cdots, t_k = 3\}$. At each stage, there are some types of vectors of decision variables, namely, x, z and V. Let also $\mathcal{G}$ denote the set of scenario groups, and $\mathcal{G}_t$ the subset of scenario groups that belong to stage t, such that $\mathcal{G} = \cup_{t\in\mathcal{T}}\mathcal{G}_t$. The structure of this information is visualized as a tree where each root-to-leaf path represents one specific scenario, and corresponds to one realization of the whole set of the uncertain parameters. In this example, we have $|\Omega| = 8$ scenarios and $|\mathcal{G}| = 15$ scenario groups.

Notice that the scenario group concept corresponds to the node concept in the underlying scenario tree. However, it can sometimes be useful to work with scenario groups instead of nodes if, for example, we want to split the set of scenarios into different subsets. Two scenarios belong to the same group in a given stage provided that they have the same realizations of the uncertain parameters up to the stage. Ω_g denotes the set of scenarios that belong to group $\mathbf{g}$, for $\mathbf{g} \in \mathcal{G}$.

With this notation, for example, scenario group $\mathbf{g} = 5$ corresponds to the scenario set $\Omega_{\mathbf{5}} = \{5, 6\}$. Notice also that there is an identity between the scenario groups of the last stage and the scenarios, then scenario group $\mathbf{g} = \mathbf{7}$ corresponds to the first scenario, $\Omega_{\mathbf{7}} = \{1\}$, and scenario group $\mathbf{g} = \mathbf{14}$ is exactly the last scenario, $\Omega_{\mathbf{14}} = \{8\}$.

We will use the boldface letter to denote the corresponding group and distinguish it from the scenarios. $t(\mathbf{g})$ denotes the stage of scenario group $\mathbf{g}$, such that, $\mathbf{g} \in \mathcal{G}_{t(\mathbf{g})}$. $\pi(\mathbf{g})$ indicates the immediate ancestor group of group $\mathbf{g}$, such that $\pi(\mathbf{g}) \in \mathcal{G}_{t(\mathbf{g})-1}$, for $\mathbf{g} \in \mathcal{G} - \{\mathcal{G}_{\mathbf{0}}\}$. And, finally, $S_{\mathbf{g}}$, denotes the set of successor groups of group $\mathbf{g}$, where, $s(\mathbf{g}) \in S_{\mathbf{g}} \subseteq \mathcal{G}_{t(\mathbf{g})+1}$.

The following additional notation is used in the model.

<u>*Parameters*</u>

$w^{\mathbf{g}}$, likelihood of scenario group $\mathbf{g}$, for $\mathbf{g} \in \mathcal{G}_t$, $t \in \mathcal{T}$, computed as $\sum_{\omega \in \Omega_{\mathbf{g}}} w^{\omega}$.

$r^{\mathbf{g}}$, risk-free interest rate under scenario group $\mathbf{g}$, for $\mathbf{g} \in \mathcal{G}_t$, $t \in \mathcal{T}$.

$q_j^{\mathbf{g}}$, measure of risk calculated as the probability of default of the securities of class j under scenario group $\mathbf{g}$, for $\mathbf{g} \in \mathcal{G}_t$, $t \in \mathcal{T}$, $j \in \mathcal{J}$.

$R_i^{\mathbf{g}}$, real interest rate (as a fraction of one) for security i of class j under scenario group $\mathbf{g}$, for $\mathbf{g} \in \mathcal{G}_t$, $t \in \mathcal{T}$, $i \in \mathcal{I}_j$, $j \in \mathcal{J}$. It is calculated as follows,

$$R_i^{\mathbf{g}} = r^{\mathbf{g}} + \frac{(1 + r^{\mathbf{g}})(1 - z_i)q_j^{\mathbf{g}}}{1 - (1 - z_i)q_j^{\mathbf{g}}}.$$

$C_i^{\mathbf{g}}$, payment stream generated by one unit of security i under scenario group $\mathbf{g}$, for $\mathbf{g} \in \mathcal{G}_t$, $t \in \mathcal{T} - \{t_0\}$, $i \in \mathcal{I}_j$, $j \in \mathcal{J}$. It is calculated as follows,

$$C_i^{\mathbf{g}} = \begin{cases} c_i \cdot h, & t(\mathbf{g}) < t < t^i, \quad i \in \mathcal{I}_j : q_j^{\mathbf{g}} < 1 \\ F_i + c_i \cdot h, & t = t^i, \quad i \in \mathcal{I}_j : q_j^{\mathbf{g}} < 1 \\ z_i \cdot F_i & t = t(\mathbf{g}), \quad i \in \mathcal{I}_j : q_j^{\pi(g)} < 1 \quad \text{and} \quad q_j^{\mathbf{g}} = 1 \\ 0, & \text{otherwise,} \end{cases} \tag{2.52}$$

where h is the constant length of each sub-interval (fraction of a year) and $q_j^{\mathbf{g}}$ is the default probability for class j under scenario group g. Notice that $q_j^{\mathbf{g}} = 1$ means that in stage $t(\mathbf{g})$, the class of securities j is in default, and then $C_i^{\mathbf{g}} = z_i F_i$, at period $t = t(\mathbf{g})$, and $C_{i\tau}^{\mathbf{g}} = 0$, for $\tau = t(\mathbf{g}) + 1, \ldots, t_k$, for all the securities $i \in \mathcal{I}_j$.

$P_i^{\mathbf{g}}$, unit price of security i under scenario group $\mathbf{g}$, for $\mathbf{g} \in \mathcal{G}_t$, $t \in \mathcal{T} - \{t_0\}$, $i \in \mathcal{I}_j$, $j \in \mathcal{J}$. It is calculated as follows,

$$P_i^{\mathbf{g}} = \sum_{t(\mathbf{g})<\tau\leq t^i} C_{i\tau}^{\mathbf{g}} ER_i^{\mathbf{g}}(t(\mathbf{g}), \tau),$$

where $t(\mathbf{g})$ is the period to which scenario group $\mathbf{g}$ belongs to, i.e., $g \in \mathcal{G}_{t(\mathbf{g})}$, and τ, denotes the corresponding element in the payment stream $C_i^{\mathbf{g}}$. Finally, $ER_i^{\mathbf{g}}(t(\mathbf{g}), \tau) = (1 + R_i^{\mathbf{g}})^{-(\tau - t(\mathbf{g})+1)}$.

$P_i^{+\mathbf{g}}$, unit purchase price of security i under scenario group $\mathbf{g}$, for $i \in \mathcal{I}_j$, $j \in \mathcal{J}$, $\mathbf{g} \in \mathcal{G}_t$, $t \in \mathcal{T} - \{t_k\}$, where $P_i^{+\mathbf{g}} = (1 + \beta)P_i^{\mathbf{g}}$.

$P_i^{+\mathbf{g}}$, unit selling price of security i under scenario group $\mathbf{g}$, for $i \in \mathcal{I}_j$, $j \in \mathcal{J}$, $\mathbf{g} \in \mathcal{G}_t$, $t \in \mathcal{T} - \{t_0\}$, where $P_i^{-\mathbf{g}} = (1 - \beta)P_i^{\mathbf{g}}$.

$v_{i,t(\mathbf{g})}^{\omega}$, final value under scenario group $\mathbf{g}$ of an investment of $P_i^{\mathbf{g}}$ monetary units in security i made at period $t(\mathbf{g})$, if the instantaneous real interest rate changes, just afterwards, from $R_i^{\mathbf{g}}$ until R_i^{ω} at the end of the PH, for $i \in \mathcal{I}$, $g \in \mathcal{G}_{t(\mathbf{g})}$, $\omega \in \Omega_g$. It is calculated as follows,

$$v_{i,t(\mathbf{g})}^{\omega} = \frac{\sum_{\tau=t(\mathbf{g})+1}^{t_k} C_{i,\tau}^{\mathbf{g}} ER_i^{\omega}(t(\mathbf{g}), \tau)}{ER_i^{\omega}(t(\mathbf{g}), t_k)}, \tag{2.53}$$

where $ER_i^{\omega}(t(\mathbf{g}), \tau) = \prod_{s=t(\mathbf{g})}^{\tau}(1 + R_{i,s}^{\omega} \cdot h)^{-1}$, $\omega \in \Omega_{\mathbf{g}}$.

Variables:

$x_i^{+\mathbf{g}}$, volume of security i purchased under scenario group $\mathbf{g}$, for $\mathbf{g} \in \mathcal{G}_t$, $t \in \mathcal{T}$, $i \in I$.

$x_i^{-\mathbf{g}}$, volume of security i sold under scenario group $\mathbf{g}$, for $\mathbf{g} \in \mathcal{G}_t$, $t \in \mathcal{T} - \{t_0\}$, $i \in I$.

$z_i^{\mathbf{g}}$, volume of security i to be held in the portfolio following the transactions conducted under scenario group $\mathbf{g}$, for $\mathbf{g} \in \mathcal{G}_t$, $t \in \mathcal{T}$, $i \in I$.

$V^{\mathbf{g}}$, final value of the portfolio under scenario group $\mathbf{g}$ for $\mathbf{g} \in \mathcal{G}_t$, $t \in \mathcal{T}$.

The multistage DEM that includes the immunization constraints under each scenario group to maximize the investment value at the end of the initial period t_0 and the expected final value portfolio over the set of scenarios in a multistage approach and risk adverse environment with the maxmin strategy, is given as follows,

$$(DEM4) \qquad \max V^{\mathbf{0}} + \sum_{t=t_1}^{t_k} \sum_{\mathbf{g}\in\mathcal{G}_t} w^{\mathbf{g}} V^{\mathbf{g}} \tag{2.54}$$

s.t.

$$\begin{aligned}
x_i^{\mathbf{0}} &= z_i^{\mathbf{0}} \quad \forall i \in \mathcal{I} && (2.55)\\
\sum_{i\in\mathcal{I}} P_i^{+\mathbf{0}} x_i^{+\mathbf{0}} &= I_0 && (2.56)\\
\sum_{i\in\mathcal{I}} v_{i0}^{\omega} z_i^{\mathbf{0}} &\geq V^{\mathbf{0}} \quad \forall \omega \in \Omega_{\mathbf{0}} = \Omega && (2.57)\\
z_i^{\pi(\mathbf{g})} + x_i^{+\mathbf{g}} - x_i^{-\mathbf{g}} &= z_i^{\mathbf{g}} \quad \forall i \in \mathcal{I} \quad \mathbf{g} \in \mathcal{G}_t, \quad t = t_1, \ldots, t_{k-1} && (2.58)\\
x_{i:t^i=t}^{-\mathbf{g}} &= z_{i:t^i=t}^{\pi(\mathbf{g})} \quad \forall i \in \mathcal{I}, \quad \mathbf{g} \in \mathcal{G}_t, \quad t = t_1 \ldots, t_k && (2.59)\\
x_i^{-\mathbf{g}} &= z_i^{\pi(\mathbf{g})} \quad \forall i \in \mathcal{I} : t_k < t^i, \quad \mathbf{g} \in \mathcal{G}_{t_k} && (2.60)\\
\sum_{i\in\mathcal{I}:t<t^i} v_{i,t(\mathbf{g})}^{\omega} z_i^{\mathbf{g}} &\geq V^{\mathbf{g}} \quad \forall \omega \in \Omega_{\mathbf{g}}, \quad \mathbf{g} \in \mathcal{G}_t, \quad t = t_1, \ldots, t_{k-1} && (2.61)\\
\sum_{i\in\mathcal{I}:t<t^i} P_i^{+\mathbf{g}} x_i^{+\mathbf{g}} - \sum_{i\in\mathcal{I}:t<t^i} P_i^{+\mathbf{g}} x_i^{-\mathbf{g}} &= \sum_{i\in\mathcal{I}:t\leq t^i} C_i^{\mathbf{g}} z_i^{\pi(\mathbf{g})} \quad \forall \mathbf{g} \in \mathcal{G}_t, \quad t = t_1, \ldots, t_{k-1} && (2.62)\\
\sum_{i\in\mathcal{I}:t_k<t^i} P_i^{-\mathbf{g}} x_i^{-\mathbf{g}} + \sum_{i\in\mathcal{I}:t_k\leq t^i} C_i^{\mathbf{g}} z_i^{\pi(\mathbf{g})} &= V^{\mathbf{g}} \quad \forall \mathbf{g} \in \mathcal{G}_{t_k} && (2.63)\\
0 &\leq x_i^{+\mathbf{g}} \quad \forall i \in \mathcal{I}, \quad \mathbf{g} \in \mathcal{G}_t, \quad t = t_0, \ldots, t_{k-1} && (2.64)\\
0 &\leq z_i^{\mathbf{g}}, x_i^{-\mathbf{g}} \quad \forall i \in \mathcal{I}, \quad \mathbf{g} \in \mathcal{G}_t, \quad t = t_0, \ldots, t_k && (2.65)\\
0 &\leq V^{\mathbf{g}} \quad \forall \mathbf{g} \in \mathcal{G}_t, \quad t = t_0, \ldots, t_k. && (2.66)
\end{aligned}$$

Equations (2.55)-(2.55) are the same as (2.3)-(2.4) in model (DEM1), but for the multistage environment. Constraints (2.57) are the key to the immunization strategy at initial period, i.e., they ensure a minimum wealth under all the future uncertainties. Constraints (2.58)-(2.60) are balance equations for each scenario group that link the volume of securities purchased and sold in each period with the volume of securities in the portfolio. Constraints (2.61) are the key to the immunization strategy at each scenario group of each decision period, that ensure a minimum value under all the future scenarios. Equations (2.62)-(2.63) are also the same as in model (DEM1), but also for the multistage environment.

Notice that model (DEM4) is an extension of the two-stage maxmin strategy given by model (DEM2) to the multistage scheme, where $\alpha = 0$, i.e., over the 100% of the scenarios. In following sections we introduce a more flexible scheme that allows an optimization of the VaR strategy given a selected class of potential (and variable) thresholds defined under different scenario groups and given probabilities of failure.

Multistage Stochastic Dominance Constraints (SDC) strategy

The extension of the two-stage first order stochastic dominance strategy to a multistage scheme is based on the choice of a threshold p from profile class $\mathcal{P}$ to satisfy given probabilities of failure, α^p, $p \in \mathcal{P}$. Moreover a $0-1$ variable $\nu^{\omega p}$ for each pair of threshold and scenario (notice that the scenario groups of last stage are the scenarios) is introduced in order to optimize the stochastic dominance strategy. It is defined as follows,

$$\nu^{\omega p} = \begin{cases} 1 & \text{for } V_{t_k}^{\omega} < \phi^p \\ 0 & \text{otherwise,} \end{cases}$$

for $\omega \in \Omega, p \in \mathcal{P}$. The DEM to maximize the expected final value portfolio over the set of scenarios in a multistage environment is as follows,

$$\begin{aligned}
(DEM5) \qquad & \max \sum_{g \in \mathcal{G}_{t_k}} w^{\mathbf{g}} V^{\mathbf{g}} = \sum_{\omega \in \Omega} w^{\omega} V_{t_k}^{\omega} && (2.67) \\
\text{s.t.} \qquad & && (2.68) \\
& \text{constraints } \ (2.55), (2.56), (2.58) - (2.60), (2.62) - (2.66) \text{ and} && \\
& V^{\mathbf{g}} + M\nu^{\mathbf{g}p} \geq \phi^p \ \ \forall \mathbf{g} \in \mathcal{G}_{t_k}, \quad p \in \mathcal{P} && (2.69) \\
& \sum_{g \in \mathcal{G}_{t_k}} w^{\mathbf{g}} \nu^{\mathbf{g}p} \leq \alpha^p \ \ \forall p \in \mathcal{P} && (2.70) \\
& V^{\mathbf{g}} \geq 0 \ \ \forall \mathbf{g} \in \mathcal{G}_{t_k} && (2.71) \\
& \nu^{\mathbf{g}p} \in \{0, 1\} \ \ \forall \mathbf{g} \in \mathcal{G}_{t_k}, \quad p \in \mathcal{P}. && (2.72)
\end{aligned}$$

Notice that model (DEM5) does not consider the maxmin strategy related constraints (2.57) and (2.61) from model (DEM4), since instead it considers the stochastic dominance constraints (2.69)-(2.70).

Multistage VaR & SDC

Following the risk averse two-stage models introduced in Section 2.5, now we introduce an extension of the multistage hybrid VaR strategy. Notice that the multistage maxmin model (DEM4) provides a 0% VaR strategy (i.e., over all the scenarios), by introducing an immunization set of constraints for each scenario group at each decision period.

Let us consider the class of threshold profiles $\mathcal{P}$ as the set of minimum threshold profiles ϕ^p to be satisfied. In the most ambitious situation of modeling, we could choose this set as potential minimum thresholds to satisfy under each scenario group at each decision stage, i.e., $\mathcal{P} = \{\phi^{\mathbf{g}}, \mathbf{g} \in \mathcal{G}_t, t = t_0, \cdots, t_{k-1}\}$. So, in this case, $|\mathcal{P}| = |\mathcal{G}_{t_0}| + |\mathcal{G}_{t_1}| + \cdots + |\mathcal{G}_{t_{k-1}}|$ and, then, the minimum threshold, $\phi^{\mathbf{g}}$, is reached by V^p with probability of failure α^p. Then, the set of immunization constraints defined for each $p \in \mathcal{P}$ are as follows,

$$\sum_{i \in \mathcal{I}: t < t^i} v^{\omega}_{i,t(p)} z^p_i + M\nu^{\omega p} \geq V^p \quad \forall \omega \in \Omega_{\mathbf{g}}, \quad p = \mathbf{g}, \quad \mathbf{g} \in \mathcal{G}_t, \quad t = t_0, \ldots, t_{k-1},$$

where the $0-1$ variable $\nu^{\omega p}$ is defined as follows,

$$\nu^{\omega p} = \begin{cases} 1 & \text{for} \quad \sum_{i \in \mathcal{I}: t < t^i} v^{\omega}_{i,t(p)} z^p_i < V^p \quad \forall \omega \in \Omega_{\mathbf{g}} \quad p = \mathbf{g}, \quad \mathbf{g} \in \mathcal{G}_t, \quad t = t_0, \ldots, t_{k-1} \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the constraint

$$\sum_{\omega \in \Omega_p} w^{\omega} \nu^{\omega p} \leq \alpha^p \quad \forall p \in \mathcal{P}$$

forces the optimization with a given probability of failure, α^p. The new DEM is defined as follows,

$$\begin{aligned}
(DEM6) \quad & \max \sum_{p \in \mathcal{P}} \gamma^p V^p + \sum_{g \in \mathcal{G}_{t_k}} w^{\mathbf{g}} V^{\mathbf{g}} && (2.73) \\
\text{s.t.} \quad & \text{constraints} \quad (2.55), (2.56), (2.58) - (2.60), (2.62) - (2.66) \text{ and} && \\
& \sum_{i \in \mathcal{I}: t < t^i} v^{\omega}_{i,t(p)} z^p_i + M\nu^{\omega p} \geq V^p \forall \omega \in \Omega_{\mathbf{g}} p = \mathbf{g}, \mathbf{g} \in \mathcal{G}_t, t = t_0, \ldots, t_{k-1} && (2.74) \\
& \sum_{\omega \in \Omega_p} w^{\omega} \nu^{\omega p} \leq \alpha^p \quad \forall p \in \mathcal{P} && (2.75) \\
& \nu^{\omega p} \in \{0, 1\} \quad \forall \omega \in \Omega_{\mathbf{g}} \quad p = \mathbf{g}, \quad \mathbf{g} \in \mathcal{G}_t, \quad t = t_0, \ldots, t_{k-1} && (2.76) \\
& V^p \geq \phi^p \quad \forall p \in \mathcal{P}. && (2.77)
\end{aligned}$$

With this choice of class $\mathcal{P}$, there are $|\Omega| \cdot (|\mathcal{T}| - 1)$ $0-1$ variables and immunization constraints (2.74), and $|\mathcal{P}|$ constraints (2.75) to force the optimization for the different probabilities of failure. Notice that the cardinality of class $\mathcal{P}$ may be reduced and then, the computational effort for the solution of (DEM6), by selecting a subset of scenario groups in a number which is not as ambitious as that given above. We consider that this reduction can be model dependent.

In the particular case of $\phi^p = 0$ and the probable $v^{\omega}_{i,t(p)} z^p_i \geq 0, \forall p \in \mathcal{P}$ (or for $|\mathcal{P}| = 0$), then model (DEM6) (2.73)-(2.77) is as model (DEM4) (2.54)-(2.66), i.e, the multistage maxmin strategy. Alternatively, model (DEM6) for $|\mathcal{P}| = 1$ is the so-named multistage VaR strategy.

Solution considerations: It is also well known that for a large set of scenarios and a large set of securities the kind of models as (DEM5) and (DEM6) become large-scale multistage mixed 0-1 linear problems. In this case, we have to consider alternative decomposition methods to the traditional plain using of the state-of-the-art optimization engine of choice. We propose to use an adaptation of our Cluster-based Benders Decomposition to the multistage environment and to integrate it in our Branch-and-Fix Coordination approach for solving large scale multi-stage stochastic mixed 0-1 linear models, that has been proved to produce good results in reasonable computing time, see [6, 36, 39, 40].

2.9 Case study

We proceed to illustrate the main strategies (see a summary in Tables 2.25 and 2.26) introduced in the previous sections by applying them to an illustrative case, assuming that the planning horizon (18 months) is divided into three periods of equal length (6 months), and the portfolio rebalancing is only allowed at the beginning of each sub-interval, i.e., $\mathcal{T} = \{t_0 = 0, t_1 = 0.5, t_2 = 1, t_3 = 1.5\}$ and $k = 3$. We assume that at the beginning of the PH the current interest rate is 1.5% (compounded semiannually) and the interest rates may move upwards and downwards by 100 basic points, see Figure 2.3. Also, the expected interest rate outstanding at the beginning of each sub-interval is the current interest rate. The initial budget is 1 million of monetary units.

We will assume that the transaction costs incurred at each portfolio rearrangement are a percentage β of the volume trade at each period. Principal

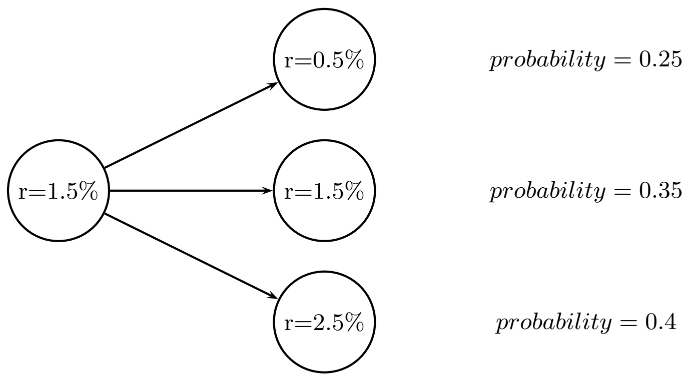


Figure 2.3: Outcomes for the interest rate variation

and coupon repayments do not generate any transaction cost although other assumptions could easily be implemented. We will suppose that there are proportional transaction costs (β) of 0.15%.

We set up a market comprising eight different bond clases, four with a high credit rating (AAA) -government bonds, for instance- and the other four from a financial institution with a high credit risk (for example, BB, B or CCC). We consider eight possible classes of assets, seven of them are shown in Tables 2.1, 2.2 and 2.3 plus the class of Default (D), the last one is shown in Table 2.1, now denoted by $\mathcal{I}_8$.

Table 2.5: Case study data

Class j	**Credit rating**	**Default Prob:** q	**Recovery Rate:** z	**Annual Coupon:** c
1	AAA	0.0000	0.694	1.5%
5	BB	0.0097	0.374	2.11857%
6	B	0.0493	0.374	4.77999%
7	CCC	0.2798	0.374	23.3719%

In order to analyze what an investor would do in different situations and different markets, we will consider three instances in the case study. In Instance 1, we consider bonds i for $i \in \mathcal{I}_1 = \{1, 2, 3, 4\}$, i.e., they are bonds in the first class (high credit rating, AAA), and bonds i for $i \in \mathcal{I}_7 = \{1', 2', 3', 4'\}$, i.e., bonds in the class CCC (very high credit risk). In Instance 2, instead, we consider bonds i in the first class (high credit rating, AAA), i.e., $i \in \mathcal{I}_1 =$

$\{1, 2, 3, 4\}$, and bonds i for $i \in \mathcal{I}_6 = \{1', 2', 3', 4'\}$, i.e., bonds in the class B (quite high credit risk). In Instance 3 we consider bonds in the first class and bonds i for $i \in \mathcal{I}_5 = \{1', 2', 3', 4'\}$, i.e., bonds in the class BB (some credit risk).

We also assume that the market interest rate in force is 1.5% and each bond will have half-yearly ($h = 0.5$) coupon payments according to its *real* interest rate level. Some of the characteristics of these two classes of assets are taken from Tables 2.3 and 2.4, and are shown in Table 2.5.

Table 2.6: Market to consider in the case study

Asset $i \in \mathcal{I}_j$	**Maturity time:** t^i	**Instance 1 class:** j	**Instance 2 class:** j	**Instance 3 class:** j
1	0.5	1	1	1
1'	0.5	7	6	5
2	1	1	1	1
2'	1	7	6	5
3	1.5	1	1	1
3'	1.5	7	6	5
4	2	1	1	1
4'	2	7	6	5

Deterministic linear model

Table 2.7 shows the optimal solution of the deterministic model (LO) (2.2)-(2.13), where the aim is to maximize the final value of the portfolio with bonds AAA and CCC in Instance 1 and bonds AAA and B in Instance 2.

Table 2.7: Deterministic optimal solution of (LO) model

Instance	s	t_s	z_{1t_s}	$z_{1't_s}$	z_{2t_s}	$z_{2't_s}$	z_{3t_s}	$z_{3't_s}$	z_{4t_s}	$z_{4't_s}$
	0	0	0	0	0	0	0	9985.02	0	0
1	1	0.5	-	-	0	0	0	11134.2	0	0
	2	1	-	-	-	-	0	12415.7	0	0
	s	t_s	z_{1t_s}	$z_{1't_s}$	z_{2t_s}	$z_{2't_s}$	z_{3t_s}	$z_{3't_s}$	z_{4t_s}	$z_{4't_s}$
	0	0	0	0	0	0	0	9985.02	0	0
2	1	0.5	-	-	0	0	0	10220.9	0	0
	2	1	-	-	-	-	0	10462.4	0	0

In both instances, the future posibilities are not taken into account, so, none of the risks actually affect the optimal solution, since it is a deterministic strategy. It consists of investing, in the initial period $t_0 = 0$, all in the risky bond that matures on the PH, $i = 3'$. And then, reinvesting all the coupon payments in the same bond throughout all the decision stages $t_1 = 1$ and $t_2 = 1.5$. In this way we can receive all the coupon payments but avoiding transaction costs. In any case, it must be stressed that all the securities in which this strategy proposes to invest, in both instances, are risky securities. Remind that the normal numbering for the securities denotes government bonds and the prime numbering denotes bonds issued by financial institutions. This is not a surprise given that risky securities offer higher returns, and the objective function does not penalize risk.

Two-stage scheme

Let us consider a two-stage scheme which assumes that changes may happen at the beginning of the second decision stage, and will stay in that way until the end of the PH. It has its advantages and disadvantages. On the one hand, one might think that it is better to behave in this way (as in finance the new information is so important) and build another two-stage model at the second stage in a rolling horizon approach. But, on the other hand, this scheme can produce very myopic decisions, since it makes decisions while taking into account very little, the future possibilities.

We assume that default probabilities may change over time and can be different depending on the scenario. Recovery rates, however will be constant for each class in our case study, i.e, $z_j = z_i = z_l$, $\forall i, l \in \mathcal{I}_j$. The recovery rate values for the government bonds, AAA, is taken from Table 2.5 being $z_1 = 0.694$. For the second classes in both instances, i.e., the financial bonds, the values are also taken from Table 2.5, they are $z_5 = 0.374$, $z_6 = 0.374$ and $z_7 = 0.374$, respectively. Table 2.6 shows the market to consider in this section in both cases.

In order to structure the possibilities in the second source of uncertainty, we know that it may change in different ways. Each one of these changes has a transition probability, which is the conditional probability, $p(S_2|S_1)$, of transiting to state S_2, given the state S_1. We will assume that each of the asset classes will have several possibilities for the next maturity period.

The AAA rated bonds may have the two possibilities depicted in Figure

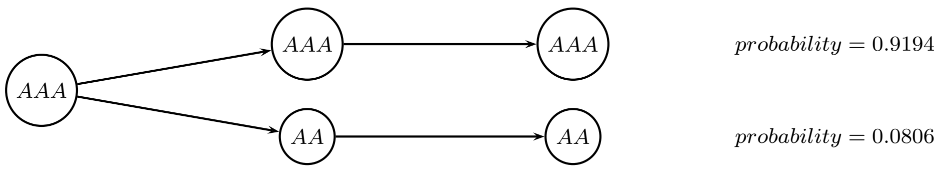


Figure 2.4: Rating variation for AAA bonds in the case study

2.4. This means that a AAA rated bond (class $j = 1$) is much more likely to stay in the same rating than changing. But in unusual circumstances it would change, and would go down to another quite safe stage. In particular, $p(AAA/AAA) = 0.9194$, and $p(AA/AAA) = 0.0806$. Moreover, as shown in Table 2.3, the probabilities of default in these classes are $q_{1,t} = 0.00$, and $q_{2,t} = 0.02$, $\forall t = t_1, ..., t^i$, $i \in \mathcal{I}_j$, $j = 1, 2$.

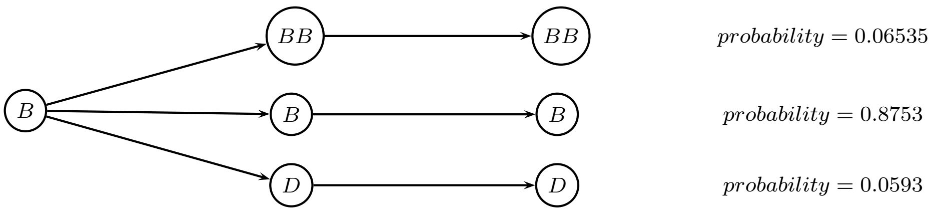


Figure 2.5: Rating variation for B bonds in the case study

The B rated bonds (class $j = 6$) may have the possibilities in the future depicted in Figure 2.5. In particular, the transition probabilities are $p(BB/B) = 0.06535$, $p(B/B) = 0.8753$ and $p(D/B) = 0.0593$. As shown in Table 2.3, the probabilities of default in these classes are $q_{5,t} = 0.0097$, $q_{6,t} = 0.0493$ and $q_{8,t} = 1$, $\forall t = t_1, ..., t^i$, $i \in \mathcal{I}_j$, $j = 5, 6, 8$.

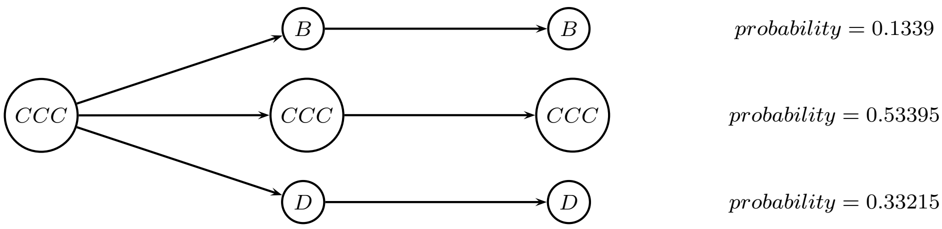


Figure 2.6: Rating variation for CCC bonds in the case study

The CCC rated bonds (class $j = 7$) may have also the possibilities in the future depicted in Figure 2.6. In particular, the transition probabilities are $p(B/CCC) = 0.1339$, $p(CCC/CCC) = 0.53395$ and $p(D/CCC) = 0.33215$. Notice that a CCC rated bond is much more likely to change into default, D than the B rated bonds ($p(D/B) = 0.0593$). Moreover, as shown in Table 2.3, the probabilities of default in these classes are $q_{5,t} = 0.097$, $q_{6,t} = 0.0493$, $q_{7,t} = 0.2798$ and $q_{8,t} = 1$, $\forall t = t_1, ..., t^i$, $i \in \mathcal{I}_j$, $j = 6, 7, 8$.

Table 2.8: Generic state description.

Instance	S_1	S_2
1	CCC	B
2	B	BB

We consider two different instances with similar shape. We have a safe asset and a risky one. In order to describe all the instances in one go, we will define the generic states S_1 and S_2 that will be $S_1 = CCC$ and $S_2 = B$ for Instance 1, and $S_1 = B$ and $S_2 = BB$ for Instance 2 as it is shown in Table 2.8.

Figure 2.7 depicts the 18 scenarios considered in the two stage models, where the classes of bonds are AAA ($j = 1$) and S_1. In order to build the scenario tree we have considered three aspects in which the outcomes might change, namely, $J^1 = \{1, 2\}$ (the possible rating of the first asset class), $J^j = \{j-1, j, 8\}$ (the possible rating of the second asset class), and $r = \{0.5\%, 1.5\%, 2.5\%\}$ (the risk free interest rate level). So we define the set of scenarios as the cartesian product of those three outcomes, ($\Omega = J^1 \times J^j \times r$), that for simplicity we assume independent risk. Then, the probability w^ω of scenario ω is computed as the product of the probabilities of the three independent outcomes.

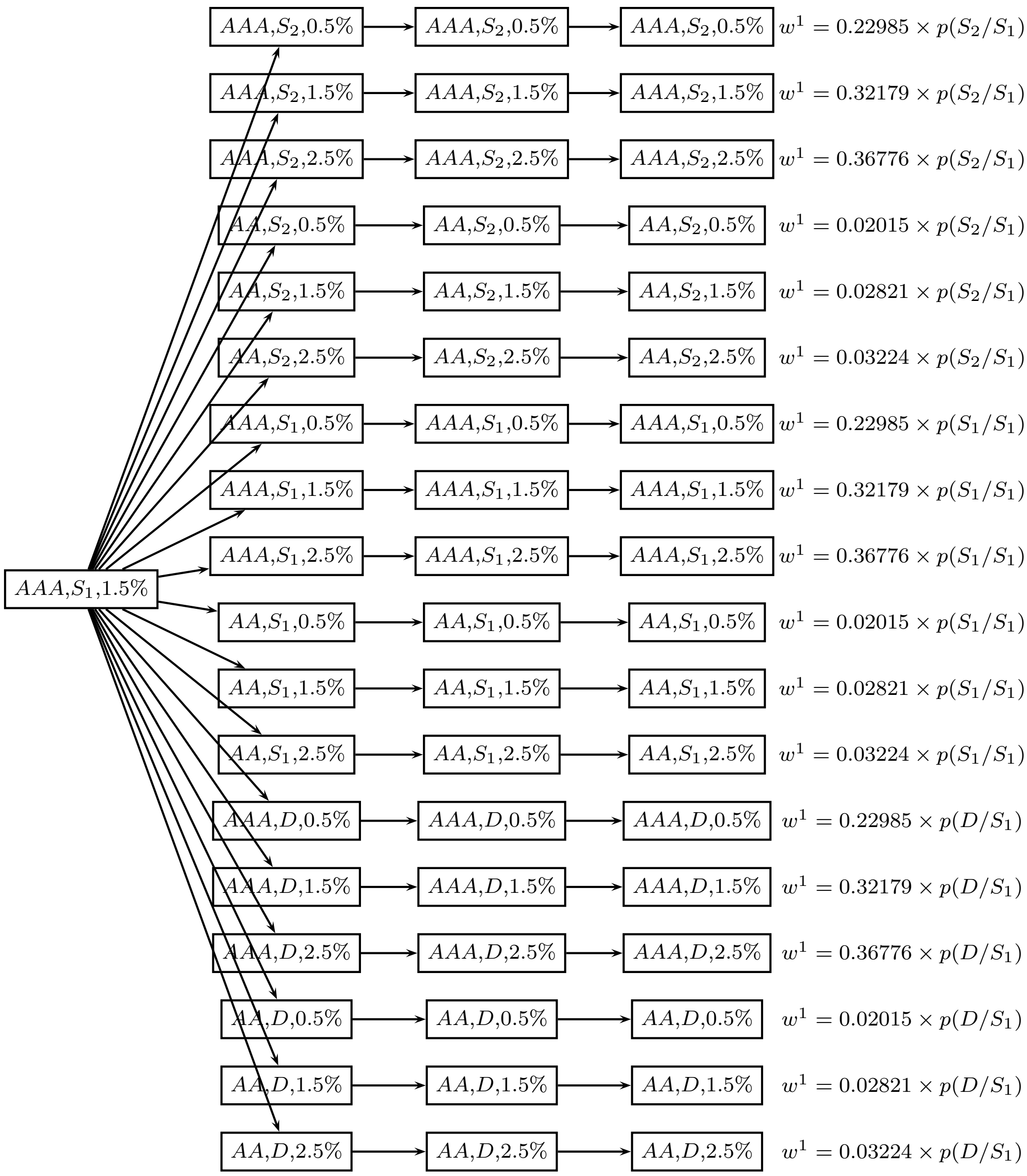


Figure 2.7: Two-stage scenario tree for the rating and interest rate variation. Instances 1 and 2

Two-stage stochastic model (DEM1)

Table 2.9 shows the stochastic optimal solution of the two stage model (DEM1) (2.17)-(2.28) for optimizing strategies under the two sources of uncertainty and considering the two-stage scenario tree information given in Figure 2.7, with

AAA and CCC bonds, i.e., the Instance 1 of the case study. The aim of model (DEM1) is to maximize the expected final value of the portfolio, so, in a risk neutral environment.

Table 2.9: Stochastic optimal solution for model (DEM1). Instance 1

ω	s	t_s	$z^\omega_{1t_s}$	$z^\omega_{1't_s}$	$z^\omega_{2t_s}$	$z^\omega_{2't_s}$	$z^\omega_{3t_s}$	$z^\omega_{3't_s}$	$z^\omega_{4t_s}$	$z^\omega_{4't_s}$
	0	0	9985.02	0	0	0	0	0	0	0
1, 4	1	0.5	-	-	0	0	0	8453.20	0	0
	2	1	-	-	-	-	0	9341.70	0	0
2, 5	1	0.5	-	-	0	0	0	8534.94	0	0
	2	1	-	-	-	-	0	9436.58	0	0
3, 6	1	0.5	-	-	0	0	0	8617.03	0	0
	2	1	-	-	-	-	0	9531.93	0	0
7, 10	1	0.5	-	-	0	0	0	9941.53	0	0
	2	1	-	-	-	-	0	11079.50	0	0
8, 11	1	0.5	-	-	0	0	0	10044.80	0	0
	2	1	-	-	-	-	0	11200.90	0	0
9, 12	1	0.5	-	-	0	0	0	10148.60	0	0
	2	1	-	-	-	-	0	11323.00	0	0
13	1	0.5	-	-	0	0	9945.76	0	0	0
	2	1	-	-	-	-	10019.90	0	0	0
14	1	0.5	-	-	0	0	10044.80	0	0	0
	2	1	-	-	-	-	10120.10	0	0	0
15	1	0.5	-	-	0	0	10144.40	0	0	0
	2	1	-	-	-	-	10220.80	0	0	0
16	1	0.5	-	-	0	0	10007.00	0	0	0
	2	1	-	-	-	-	10081.80	0	0	0
17	1	0.5	-	-	0	0	10107.00	0	0	0
	2	1	-	-	-	-	10182.90	0	0	0
18	1	0.5	-	-	0	0	10207.50	0	0	0
	2	1	-	-	-	-	10284.60	0	0	0

In this case, the optimal strategy at least deals with the risk of default at the initial period. As it is possible to have a defaulted bond in t_1, the optimal solution ensures a minimum wealth at the beginning (by investing everything in the sure asset $i = 1$ that matures in the next period $t_1 = 1$, thus avoiding all the transaction costs). After that, the strategy invests in the sure or the risky asset depending on the probability of default of the asset classes in the corresponding scenario. In those scenarios in which any of the bonds in the market may default, it invests everything in the sure bond that matures on the

PH, $i = 3$. In the other case, this strategy invests everything in the risky asset that also matures at the PH, $i = 3'$. Notice also that the optimal solution proposes the same strategy under different pairs of scenarios, $\omega = 1, 4;\ 2, 5;\ 3, 6;\ 7, 10;\ 8, 11$ or $9, 12$.

The optimal solution of model (DEM1) for Instance 2 (i.e., by considering the two-stage scenario tree with $S_1 = B$ and $S_2 = BB$), shows a similar behaviour. In this case, the optimal solution is also to invest safe and short at the beginning and reinvest long in the next periods, investing in safe or risky depending on the risk of default of the classes in the corresponding scenario. The difference between both strategies comes from the fact that the risky asset pays different coupons. Notice, in any case, that the reinvestment of those payments is exactly the same.

Mean-risk immunization model (DEM-MR)

The results obtained by implementing the approach given in model (DEM-MR) (2.34)-(2.36) are the same as those obtained in the optimization of the two-stage stochastic model (DEM1).

The 0-1 variables reach their optimum value depending on the parameter ϕ. If ϕ is bigger than approximately the V-s involved, they all take value 0. In the other case, their optimum value is 1. This result is due to the aim of this approach, that is to ensure the minimum wealth that we would get in the model to immunize the portfolio. As in our specific case study we have modelized the default as a particular scenario, we are already dealing in a certain way with the risk of default and avoiding those decisions that do not exceed minimum wealth.

In any case, independently of the value of parameter ϕ, the optimal strategy consists of investing safe and short at the beginning and reinvesting long in the next periods, investing in safe or risky depending on the risk of default of the classes in the corresponding scenario.

Two-stage immunization model (DEM2)

Table 2.10 shows the optimal solution to the two-stage stochastic model (DEM2) (2.44)-(2.46) for optimizing immunization strategies under the two sources of uncertainty at the initial period $t_0 = 0$, for Instance 1 (i.e., the two-stage scenario tree information given in Figure 2.7 with $S_1 = CCC$ and

Table 2.10: Stochastic optimal solution for model (DEM2). Instance 1

ω	s	t_s	$z^{\omega}_{1t_s}$	$z^{\omega}_{1't_s}$	$z^{\omega}_{2t_s}$	$z^{\omega}_{2't_s}$	$z^{\omega}_{3t_s}$	$z^{\omega}_{3't_s}$	$z^{\omega}_{4t_s}$	$z^{\omega}_{4't_s}$
	0	0	0	0	0	0	9758.12	0	226.91	0
1	1	0.5	-	-	0	0	0	8525.02	0	0
	2	1	-	-	-	-	0	9421.08	0	0
2	1	0.5	-	-	0	0	0	8522.23	0	0
	2	1	-	-	-	-	0	9422.53	0	0
3	1	0.5	-	-	0	0	0	8519.44	0	0
	2	1	-	-	-	-	0	9423.98	0	0
4	1	0.5	-	-	0	0	0	8472.64	0	0
	2	1	-	-	-	-	0	9363.18	0	0
5	1	0.5	-	-	0	0	0	8469.61	0	0
	2	1	-	-	-	-	0	9364.35	0	0
6	1	0.5	-	-	0	0	0	8466.60	0	0
	2	1	-	-	-	-	0	9365.52	0	0
7	1	0.5	-	-	0	0	0	10026.00	0	0
	2	1	-	-	-	-	0	11173.70	0	0
8	1	0.5	-	-	0	0	0	10029.90	0	0
	2	1	-	-	-	-	0	11184.30	0	0
9	1	0.5	-	-	0	0	0	10033.70	0	0
	2	1	-	-	-	-	0	11194.80	0	0
10	1	0.5	-	-	0	0	0	9964.39	0	0
	2	1	-	-	-	-	0	11105.00	0	0
11	1	0.5	-	-	0	0	0	9967.96	0	0
	2	1	-	-	-	-	0	1115.20	0	0
12	1	0.5	-	-	0	0	0	9971.45	0	0
	2	1	-	-	-	-	0	11125.30	0	0
13	1	0.5	-	-	0	0	9832.15	0	226.91	0
	2	1	-	-	-	-	9907.11	0	226.91	0
14	1	0.5	-	-	0	0	9832.89	0	226.91	0
	2	1	-	-	-	-	9908.23	0	226.91	0
15	1	0.5	-	-	0	0	9833.63	0	226.91	0
	2	1	-	-	-	-	9909.35	0	226.91	0
16	1	0.5	-	-	0	0	9832.61	0	226.91	0
	2	1	-	-	-	-	9907.80	0	226.91	0
17	1	0.5	-	-	0	0	9833.35	0	226.91	0
	2	1	-	-	-	-	9908.93	0	226.91	0
18	1	0.5	-	-	0	0	9834.10	0	226.91	0
	2	1	-	-	-	-	9910.06	0	226.91	0

$S_2 = B$). The strategy for the first period is observed to be very close to that proposed by Khang [61]. Notice that, as it is an immunization strategy, the aim is to ensure a minimal value, so it advises to invest in $t_0 = 0$ in the sure assets in any of the possible scenarios. In addition, it invests in those sure assets that fit their duration as was proposed by Khang in his immunization theorem. But once it simply immunizes the initial period, the optimal strategy for the other periods is very similar to the strategy that we obtained in the two-stage optimization model (DEM1), i.e., to invest in the bond that matures at the PH, which will or not be risky depending on the scenario.

Table 2.11: Stochastic optimal solution for model (DEM2). Instance 2

ω	s	t_s	$z^{\omega}_{1t_s}$	$z^{\omega}_{1't_s}$	$z^{\omega}_{2t_s}$	$z^{\omega}_{2't_s}$	$z^{\omega}_{3t_s}$	$z^{\omega}_{3't_s}$	$z^{\omega}_{4t_s}$	$z^{\omega}_{4't_s}$
	0	0	3234.58	0	0	0	0	0	6750.45	0

Table 2.11 shows the optimal solution in the initial period t_0 for the two stage model (DEM2) (2.44)-(2.46) for optimizing immunization strategies by considering the two-stage scenario tree information given in Figure 2.7 with $S_1 = B$ and $S_2 = BB$ (i.e. Instance 2). The solution in this case is quite similar to that obtained for Instance 1, see Table 2.10. It also immunizes at period $t_0 = 0$ by equalling the maturity of the portfolio with its duration. Anyway, instead of taking two long term bonds, it invests something in the short term and adjusts the duration by investing more in the longest term. After that the strategy is quite similar to the previous one.

It is worth noting that when immunizing we only take into account non risky bonds. This is due to the fact that we are maximizing the expected value of the minimum wealth the portfolio would obtain under any future realization of the uncertain variables, no matter whether they are very likely to happen or not. We would like to improve the performance of this model by adjusting it so as to maximize the expected value of the minimum wealth the portfolio would obtain under as many as the most probable scenarios.

Two-stage VaR model (DEM3)

When considering the two-stage scenario tree information given in Figure 2.7, with AAA and CCC bonds (i.e., Instance 1), the "bad" scenarios are so probable that for any probability $\alpha \in (0\%, 10\%)$ we would obtain the same solutions optimizing model (DEM2) over the 100% and over the 90% of the scenarios, respectively. Notice that model (DEM2) always has to take into account at least one of the "bad" scenarios when immunizing.

Moreover, when considering the scenario tree information given in Figure 2.7, with *AAA* and *B* bonds (i.e., Instance 2) with probability $\alpha = 5\%$, the optimal strategy is also exactly the same as we had in model (DEM2). In this case, as the probabilities of the "bad" scenarios are higher than 5%, it is absolutely necessary to take into account one of them, at least, and as a result the decision is not to invest in risky bonds from the beginning.

Table 2.12: Stochastic optimal solution for model (DEM3). Instance 2. $\alpha = 6\%$

ω	s	t_s	$z^{\omega}_{1t_s}$	$z^{\omega}_{1't_s}$	$z^{\omega}_{2t_s}$	$z^{\omega}_{2't_s}$	$z^{\omega}_{3t_s}$	$z^{\omega}_{3't_s}$	$z^{\omega}_{4t_s}$	$z^{\omega}_{4't_s}$
	0	00	0	0	0	0	0	9252.99	0	732.03
1,4	1	0.5	-	-	0	0	0	9480.72	0	732.03
	2	1	-	-	-	-	0	9717.74	0	732.03
2,5	1	0.5	-	-	0	0	0	9482.98	0	732.03
	2	1	-	-	-	-	0	9721.24	0	732.03
3,6	1	0.5	-	-	0	0	0	9485.25	0	732.03
	2	1	-	-	-	-	0	9724.75	0	732.03
7,10	1	0.5	-	-	0	0	0	9486.55	0	732.03
	2	1	-	-	-	-	0	9726.76	0	732.03
8,11	1	0.5	-	-	0	0	0	9488.9	0	732.03
	2	1	-	-	-	-	0	9730.37	0	732.03
9,12	1	0.5	-	-	0	0	0	9491.25	0	732.03
	2	1	-	-	-	-	0	9734	0	732.03
13	1	0.5	-	-	0	0	3692.02	0	0	0
	2	1	-	-	-	-	3719.53	0	0	0
14	1	0.5	-	-	0	0	3728.81	0	0	0
	2	1	-	-	-	-	3756.73	0	0	0
15	1	0.5	-	-	0	0	3765.77	0	0	0
	2	1	-	-	-	-	3794.11	0	0	0
16	1	0.5	-	-	0	0	3714.76	0	0	0
	2	1	-	-	-	-	3742.53	0	0	0
17	1	0.5	-	-	0	0	3751.89	0	0	0
	2	1	-	-	-	-	3780.07	0	0	0
18	1	0.5	-	-	0	0	3789.19	0	0	0
	2	1	-	-	-	-	3817.8	0	0	0

Table 2.12 gives the optimal strategy obtained by solving model (DEM3) (2.47)-(2.51), with $\alpha = 6\%$ and the two-stage scenario tree information given in Figure 2.7 (i.e., Instance 2). In this case, the strategy changes completely in

comparison with the solution that has been obtained in model (DEM2). This is due to the fact that those scenarios in which default may happen are not sufficiently probable for the decision maker. So, the optimal strategy consists of investing everything in risky bonds that match the maturity and the duration of the portfolio. After the initial period $t_0 = 0$, the optimal solution depends on the scenario. In those "good" scenarios (in which there are no defaulted bonds) the portfolio is built up with risky bonds. It is worth pointing out that it is not exactly the strategy it followed that has been scenarios ω =4, 5 and 6. The reason is that as they have such a small probability, they do not have a high weight in the objective function and then, they simply do whatever is easiest. So, the optimal strategy is to invest in risky bonds for the "good" scenarios. The optimal solution for the "bad" scenarios, however, is to invest everything in safe assets.

Multistage scheme

Firstly, we introduce the main aspects for the construction of the multi-stage scenario tree adapted to our illustrative example, with market conditions similar to those in the two-stage environment. In the case of the interest rate changes, we consider the same outcomes but with different probabilities, see Figure 2.8.

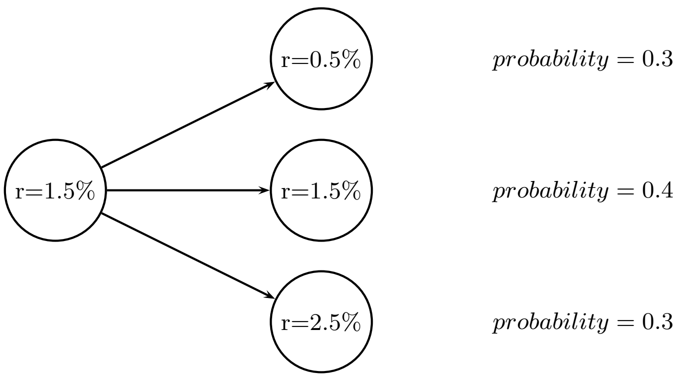


Figure 2.8: Outcomes for the interest rate variation

Given the number of decision periods in the case study, the scenario tree has been designed as a three-stage one, still considering Instances 1 and 2, but a third instance is created by considering a new class of bonds, the speculative class $j = 5$, BB. The BB rated bonds may have the outlooks depicted in Figure 2.9. In particular, the transition probabilities are $p(BBB/BB) = 0.0633$, $p(BB/BB) = 0.9248$ and $p(D/BB) = 0.0119$. Moreover, as is shown in Table 2.3, the probabilities of default in these classes are $q_{4,t} = 0.0026$, $q_{5,t} = 0.0097$ and $q_{8,t} = 1$, $\forall t = t^1, ..., t^i$, $i \in \mathcal{I}_j$, $j = 4, 5, 8$.

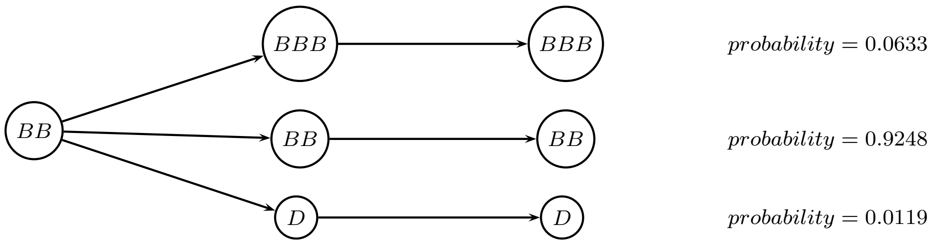


Figure 2.9: Rating variation for BB bonds in the case study

We now consider three different instances with similar shape. We have a safe asset and a risky one. In order to describe all the instances at the same time, we define the generic states S_1 and S_2, such that $S_1 = CCC$ and $S_2 = B$ (Instance 1), $S_1 = B$ and $S_2 = BB$ (Instance 2), and $S_1 = BB$ and $S_2 = BBB$ (Instance 3).

Figure 2.10 depicts the 27 scenarios considered for the three-stage and the three instances.

For Instance 1 the classes of bonds AAA ($j = 1$) and CCC ($j = 7$) are considered. In order to build a simple but representative three-stage scenario tree we have considered different aspects at the different stages. In stage $t_1 = 0.5$, the outcomes result from $J^7 \times r$, where $J^7 = \{6, 7, 8\}$ (the possible rating of the second asset class) and $r = \{0.5\%, 1.5\%, 2.5\%\}$ (the risk free interest rate level). So, a set of nine scenarios is defined as the cartesian product of those two sets of outcomes. In the next stage $t_2 = 1$, the scenarios are defined as the cartesian product given by ($\Omega = J^7 \times r \times J^1$), since for simplicity they are assumed as independent risks, and where $J^1 = \{1, 2\}$ (the possible rating of the first asset class). Then, the probability w^ω of scenario ω is computed as the product of the probabilities of the three independent outcomes.

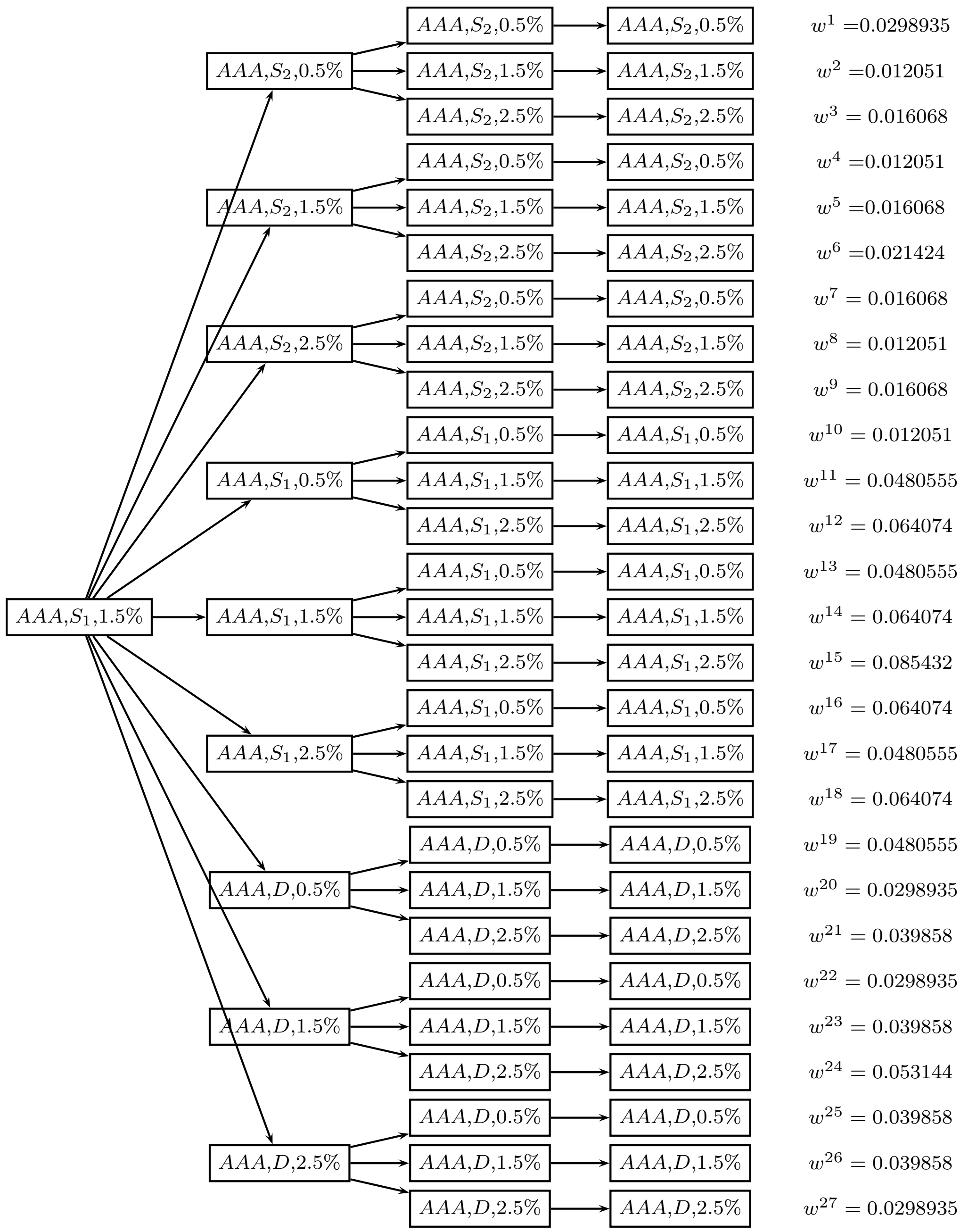


Figure 2.10: Three-stage scenario tree for rating and interest rate variation. Instances 1,2,3

For Instance 2 the classes of bonds AAA ($j = 1$) and B ($j = 6$) are considered. In stage $t_1 = 0.5$, the outcomes result from $J^6 \times r$, where $J^6 = \{5, 6, 8\}$ (the possible rating of asset class $j = 6$), and $r = \{0.5\%, 1.5\%, 2.5\%\}$ (the risk free interest rate level). So, a set of nine scenarios is defined as the cartesian product of those two set of outcomes. In the next stage $t_2 = 1$, the scenarios are defined as the cartesian product given by ($\Omega = J^6 \times r \times J^1$), that for simplicity are assumed as independent risks, and where $J^1 = \{1, 2\}$ (the possible rating of asset class $j = 1$). Additionally, w^ω represents the probability of scenario ω.

For Instance 3 the classes of bonds AAA ($j = 1$) and BB ($j = 5$) are considered. Similarly as the above instances, in stage $t_1 = 0.5$, the outcomes result from $J^5 \times r$, where $J^5 = \{4, 5, 8\}$ (the possible rating of asset class $j = 5$) and $r = \{0.5\%, 1.5\%, 2.5\%\}$ (the risk free interest rate level). So, a set of nine scenarios is defined as the cartesian product of those two set of outcomes. In the next stage $t_2 = 1$, the scenarios are defined as the cartesian product given by ($\Omega = J^5 \times r \times J^1$), since for simplicity they are assumed as independent risks, and where $J^1 = \{1, 2\}$ (the possible rating of asset class $j = 1$).

Multi-stage maxmin immunization model (DEM4)

Table 2.13: Optimal solution at period t_0 for model (DEM4)

Instance 1	$\mathbf{g}$	$z_1^{\mathbf{g}}$	$z_{1'}^{\mathbf{g}}$	$z_2^{\mathbf{g}}$	$z_{2'}^{\mathbf{g}}$	$z_3^{\mathbf{g}}$	$z_{3'}^{\mathbf{g}}$	$z_4^{\mathbf{g}}$	$z_{4'}^{\mathbf{g}}$
	0	0	0	4886.05	0	0	0	5098.97	0
Instance 2	$\mathbf{g}$	$z_1^{\mathbf{g}}$	$z_{1'}^{\mathbf{g}}$	$z_2^{\mathbf{g}}$	$z_{2'}^{\mathbf{g}}$	$z_3^{\mathbf{g}}$	$z_{3'}^{\mathbf{g}}$	$z_4^{\mathbf{g}}$	$z_{4'}^{\mathbf{g}}$
	0	0	0	4886.05	0	0	0	5098.97	0
Instance 3	$\mathbf{g}$	$z_1^{\mathbf{g}}$	$z_{1'}^{\mathbf{g}}$	$z_2^{\mathbf{g}}$	$z_{2'}^{\mathbf{g}}$	$z_3^{\mathbf{g}}$	$z_{3'}^{\mathbf{g}}$	$z_4^{\mathbf{g}}$	$z_{4'}^{\mathbf{g}}$
	0	0	0	4886.05	0	0	0	5098.97	0

The optimal solutions to the multi-stage model (DEM4) (2.54)-(2.66) for Instances 1, 2 and 3, at the initial period t_0 are shown in Table 2.13. All of them are equal and consist of the same strategy, i.e., investing part of the portfolio in the safe asset that matures before the end of PH and the rest of the portfolio on the safe asset with the longest maturity. As the portfolio is made up almost half and half, it matches the duration with the PH, so it is following in some way the strategy proposed by Khang [61]. It is also important to note that it invests in safe assets, thus, avoiding the risk of default. So it is dealing with both risks.

Table 2.14: Optimal solution at period t_1 for model (DEM4). Instance 1

g	Ω_g	$z_2^{\mathbf{g}}$	$z_{2'}^{\mathbf{g}}$	$z_3^{\mathbf{g}}$	$z_{3'}^{\mathbf{g}}$	$z_4^{\mathbf{g}}$	$z_{4'}^{\mathbf{g}}$
1	$\{1,2,3\}$	0	9320.31	0	0	0	0
2	$\{4,5,6\}$	0	0	0	7714.91	0	862.44
3	$\{7,8,9\}$	0	0	0	0	0	8115.14
4	$\{10,11,12\}$	0	10075.40	0	0	0	0
5	$\{13,14,15\}$	0	10030.70	0	0	0	0
6	$\{16,17,18\}$	0	9987.34	0	0	0	0
7	$\{19,20,21\}$	10094.50	0	0	0	0	0
8	$\{22,23,24\}$	4960.57	0	0	0	5098.97	0
9	$\{25,26,27\}$	0	0	0	0	10092.00	0

The optimal strategies at each scenario group in stage t_1, $g \in \mathcal{G}_{t_1}$, are shown in Tables 2.14 and 2.15 for Instances 1 and 2, respectively. The strategy to follow for Instance 3 is exactly the same as for Instance 2. In all the cases, the strategies to follow depend on the scenario groups in period t_1, in any case, they are very similar. In any of the scenarios in which default may not occur, the optimal solution consists of changing all the investments in risky bonds (since it does not consider the possibility of default in the next periods). However, if there is a probability of default, it invests in safe assets, reorganizing the portfolio depending on the interest rate.

Table 2.15: Optimal solution at period t_1 for model (DEM4). Instance 2

g	ω	$z_2^{\mathbf{g}}$	$z_{2'}^{\mathbf{g}}$	$z_3^{\mathbf{g}}$	$z_{3'}^{\mathbf{g}}$	$z_4^{\mathbf{g}}$	$z_{4'}^{\mathbf{g}}$
1	$\Omega_1 = 1,2,3$	0	9953.04	0	0	0	0
2	$\Omega_2 = 4,5,6$	0	0	0	9551.31	0	227.76
3	$\Omega_3 = 7,8,9$	0	0	0	0	0	9705.83
4	$\Omega_4 = 10,11,12$	0	10079.10	0	0	0	0
5	$\Omega_5 = 13,14,15$	0	0	0	9785.87	0	239.46
6	$\Omega_6 = 16,17,18$	0	0	0	0	0	10059.00
7	$\Omega_7 = 19,20,21$	10094.5	0	0	0	0	0
8	$\Omega_8 = 22,23,24$	4960.57	0	0	0	5098.97	0
9	$\Omega_9 = 25,26,27$	0	0	0	0	10092.00	0

In those scenarios for which the risky asset improves or maintains its level, the strategy is very similar to that in the case of default, investing in risky assets instead of investing in those safe ones. This strategy is different depending on the current interest rate. If interest rates have fallen (they are in 0.5), it considers that they can maintain their level or change into 1.5 or 2.5, so, they would be bigger in any case and thus, the optimal solution consists of investing everything short. If the interest rates maintain their level (1.5), they can go down or up and, thus, the optimal solution consists of matching the duration with the PH. Finally, if interest rates go up to 2.5, the possibilities considered in the scenario tree are to stay or to fall and, thus, the optimal strategy is to invest long.

Table 2.16: Optimal solution at period t_2 for model (DEM4). Instance 1

$\mathbf{g}$	$\Omega_{\mathbf{g}}$	$z_3^{\mathbf{g}}$	$z_{3'}^{\mathbf{g}}$	$z_4^{\mathbf{g}}$	$z_{4'}^{\mathbf{g}}$
10	1	0	9476.65	0	0
11	2	0	9524.64	0	0
12	3	0	9572.64	0	0
13	4	0	8556.32	0	862.44
14	5	0	8560.58	0	862.44
15	6	0	8564.84	0	862.44
16	7	0	796.068	0	8115.14
17	8	0	800.099	0	8115.14
18	9	0	804.131	0	8115.14
19	10	0	11074.00	0	0
20	11	0	11134.60	0	0
21	12	0	11194.70	0	0
22	13	0	11025.30	0	0
23	14	0	11085.20	0	0
24	15	0	11145.00	0	0
25	16	0	10977.60	0	0
26	17	0	11037.20	0	0
27	18	0	11096.90	0	0
28	19	10104.60	0	0	0
29	20	10155.00	0	0	0
30	21	10205.40	0	0	0
31	22	5003.78	0	5098.97	0
32	23	5028.74	0	5098.97	0
33	24	5053.69	0	5098.97	0
34	25	75.20	0	10092.00	0
35	26	75.57	0	10092.00	0
36	27	75.95	0	10092.00	0

The main difference between the strategies to follow in Instances 1 and 2 depends on the risky asset maintaining its rating.

So, for Instance 2, the optimal solution is also to invest short if the interest rate can only rise, matching the duration with the HP if the interest rate is still 1.5, and investing long if the interest rate can just fall.

For Instance 1, the optimal strategy is always to invest short in those scenarios where the risky asset is still CCC. Notice that we can invest in risky asset and, as their reinvestments would give such big returns (real interest rates are around $20-22\%$), the different strategies barely affect the optimality of the solution, so, anything that might be done is profitable.

Table 2.16 shows the optimal strategies at period t_2 for Instance 1. We should point out that the strategy does not change for the other instances. As in the three-stage scenario tree considered, see Figure 2.10, there is no change after period t_2, so, the optimal solution is to invest everything in the asset that matures in the PH but, sometimes it can leave something remaining for the other assets, thereby, avoiding transaction costs.

Anyway, in those scenarios with no default it invests in risky assets, and invests in safe assets for the "bad" scenarios.

Multi-stage stochastic dominance model (DEM5)

Let us summarize some of the results obtained by solving the three-stage extension to the first order stochastic dominance strategy, model (DEM5) (2.67)-(2.72). We have considered two different situations: one in which we ask for a minimum wealth -always the same- that is higher than that obtained with the recovery rate; and another in which we ask for two minimum values: one higher than we get with safe assets and another higher than the recovery rate. In both of these, the probability has been taken as $\alpha = 0.06$.

After solving several situations, we can conclude that these strategies are very dependent on the values of the potential profiles, ϕ^p, $p \in \mathcal{P}$. It is very important to choose them in order to have the best results or those that best match the needs of the investor. This may not be so easy a priori, since the decision maker does not necessarily need to know what could be obtained in different situations at the beginning of the study.

Case 1: $|\mathcal{P}| = 1$, $\phi^1 = 5 \cdot 10^5$, $\alpha^1 = 0.06$

Let us summarize the optimal strategy corresponding to Instance 1. For Instances 2 and 3, the results are very similar.

Table 2.17: Optimal solution at t_0 for model (DEM5). Instance 1. Case 1.

$\mathbf{g}$	$z_1^{\mathbf{g}}$	$z_{1'}^{\mathbf{g}}$	$z_2^{\mathbf{g}}$	$z_{2'}^{\mathbf{g}}$	$z_3^{\mathbf{g}}$	$z_{3'}^{\mathbf{g}}$	$z_4^{\mathbf{g}}$	$z_{4'}^{\mathbf{g}}$
0	9985.02	0	0	0	0	0	0	0

Table 2.17 shows the optimal strategy at period t_0. It consists of investing all the portfolio in the asset with the smallest maturity. So, it invests everything in the short term. It is also important to note that it invests in safe assets, thus avoiding the risk of default. But it is not immunizing against interest rate changes, due to the fact that it is not matching the duration of the portfolio with the PH.

Table 2.18: Optimal solution at t_1 for model (DEM5). Instance 1. Case 1.

$\mathbf{g}$	ω	$z_2^{\mathbf{g}}$	$z_{2'}^{\mathbf{g}}$	$z_3^{\mathbf{g}}$	$z_{3'}^{\mathbf{g}}$	$z_4^{\mathbf{g}}$	$z_{4'}^{\mathbf{g}}$
1	$\Omega_1 = 1, 2, 3$	0	9241.81	0	0	0	0
2	$\Omega_2 = 4, 5, 6$	0	0	0	8648.04	0	0
3	$\Omega_3 = 7, 8, 9$	0	0	0	0	0	8205.94
4	$\Omega_4 = 10, 11, 12$	0	9990.59	0	0	0	0
5	$\Omega_5 = 13, 14, 15$	0	10044.80	0	0	0	0
6	$\Omega_6 = 16, 17, 18$	0	10099.10	0	0	0	0
7	$\Omega_7 = 19, 20, 21$	9994.99	0	0	0	0	0
8	$\Omega_8 = 22, 23, 24$	0	0	10044.30	0	0	0
9	$\Omega_9 = 25, 26, 27$	0	0	0	0	10189.40	0

Table 2.18 shows the optimal strategy for each scenario group that belongs to period t_1. In the scenarios where default may not occur, the optimal solution consists of changing all the investments in risky bonds (as they do not consider the possibility of default in the next periods). In the case of default, it would keep the safe assets instead, reorganizing them depending on the interest rate.

As the three-stage scenario tree does not consider any changes after period t_2, the optimal solution in this case is to invest everything in the asset that matures in the PH but, sometimes, in order to avoid not paying transaction costs, it can leave something remaining for other assets. Anyway, in those scenarios with no default it invests in risky assets and it invests in safe assets for the "bad" scenarios.

Case 2: $|\mathcal{P}| = 2$, $\phi^1 = 5 \cdot 10^5$, $\phi^2 = 1.05 \cdot 10^6$, $\alpha^1 = \alpha^2 = 0.06$

Let us summarize the optimal strategy corresponding to Instance 2. For Instances 1 and 3, the results are very similar.

Table 2.19: Optimal solution at t_0 for model (DEM5). Instance 2. Case 2.

$\mathbf{g}$	$z_1^{\mathbf{g}}$	$z_{1'}^{\mathbf{g}}$	$z_2^{\mathbf{g}}$	$z_{2'}^{\mathbf{g}}$	$z_3^{\mathbf{g}}$	$z_{3'}^{\mathbf{g}}$	$z_4^{\mathbf{g}}$	$z_{4'}^{\mathbf{g}}$
0	3205.13	0	0	0	0	0	1895.68	4884.22

Table 2.19 shows the optimal solution at period t_0. It consists of investing part of the portfolio in assets that mature before the end of PH and the rest in the assets with the longest maturity. The investment strategy shares between risky and safe assets in order to ensure, on the one hand, returns that are at least ϕ^1 in the most probable cases but on the other hand, it ensures a minimum wealth (ϕ^2) in situation of default. As the portfolio is made up almost half and half, it matches the duration with the PH, so it is following in some way the strategy proposed by Khang [61]. It is also important to notice that it invests in risky assets since the failure scenarios are not considered sufficiently probable.

Table 2.20: Optimal solution at t_1 for model (DEM5). Instance 2. Case 2.

$\mathbf{g}$	ω	$z_2^{\mathbf{g}}$	$z_{2'}^{\mathbf{g}}$	$z_3^{\mathbf{g}}$	$z_{3'}^{\mathbf{g}}$	$z_4^{\mathbf{g}}$	$z_{4'}^{\mathbf{g}}$
1	$\Omega_1 = 1, 2, 3$	0	10217.9	0	0	0	0
2	$\Omega_2 = 4, 5, 6$	0	0	0	5113.17	0	4884.22
3	$\Omega_3 = 7, 8, 9$	0	0	0	0	0	9980.67
4	$\Omega_4 = 10, 11, 12$	0	10166.5	0	0	0	0
5	$\Omega_5 = 13, 14, 15$	0	0	0	5240.53	0	4884.22
6	$\Omega_6 = 16, 17, 18$	0	0	0	0	0	10166.1
7	$\Omega_7 = 19, 20, 21$	6946.08	0	0	0	0	0
8	$\Omega_8 = 22, 23, 24$	0	0	5062.21	0	1895.68	0
9	$\Omega_9 = 25, 26, 27$	0	0	0	0	7031.04	0

Table 2.20 shows the optimal strategy for each scenario group in period t_1. The strategy to follow is very similar no matter if the risky asset changes its

rating or not. The only difference is that in defaulted scenarios the optimal solution would be to invest safe, and risky in other cases.

In any case the strategy is different depending on the current interest rate. If interest rates have fallen (they are in the 0.5), it considers that they can maintain their level or change into 1.5 or 2.5, so they would be bigger in any case and, thus, the optimal solution consists of investing everything short. If the interest rates maintain their level (1.5), they can go down or up and, thus, the optimal solution consists of matching the duration with the PH. Both risks are treated. Finally, if interest rates go up to 2.5, the possibilities considered in the scenario tree are to stay or to fall and, thus, the optimal strategy is to invest long.

It is worth pointing out that the three-stage scenario tree does not consider any changes after period t_2, so, the optimal solution in this case is to invest everything in the asset that matures in the PH but, sometimes, in order to avoid paying transaction costs, it can leave something remaining for the other assets. Anyway, in those scenarios with no default invests in risky assets. On the other hand, it invests in safe assets for the "bad" scenarios.

Multi-stage VaR & stochastic dominance constraint model (DEM6)

Finally, we summarize the results of the optimization of model (DEM6) (2.73)-(2.77).

Case 1: $\mathcal{P} = \{g \in \mathcal{G}_t : t = t_0, \cdots, t_{k-1}\}$ and $\phi^p = 0$ for all $p \in \mathcal{P}$

Let us summarize the optimal solutions for the different stages in Instance 3 with $\alpha = 0.05$.

Table 2.21: Optimal solution at t_0 for model (DEM6). Instance 3. Case 1

$\mathbf{g}$	$z_1^{\mathbf{g}}$	$z_{1'}^{\mathbf{g}}$	$z_2^{\mathbf{g}}$	$z_{2'}^{\mathbf{g}}$	$z_3^{\mathbf{g}}$	$z_{3'}^{\mathbf{g}}$	$z_4^{\mathbf{g}}$	$z_{4'}^{\mathbf{g}}$
0	0	0	0	4858.33	0	0	5126.69	0

Table 2.21 shows the optimal solution at period t_0. It consists of matching the duration with the HP in order to avoid immunization risk, but investing in risky assets. It seems that default has such a small probability that the model does not take them into account when optimizing.

Table 2.22: Optimal solution at t_1 for model (DEM6). Instance 3. Case 1

$\mathbf{g}$	ω	$z_2^{\mathbf{g}}$	$z_{2'}^{\mathbf{g}}$	$z_3^{\mathbf{g}}$	$z_{3'}^{\mathbf{g}}$	$z_4^{\mathbf{g}}$	$z_{4'}^{\mathbf{g}}$
1	$\Omega_1 = 1,2,3$	0	10147.90	0	0	0	0
2	$\Omega_2 = 4,5,6$	0	4963.71	0	0	0	5126.69
3	$\Omega_3 = 7,8,9$	0	0	0	0	0	10100.30
4	$\Omega_4 = 10,11,12$	0	10125.70	0	0	0	0
5	$\Omega_5 = 13,14,15$	0	4940.94	0	23.01	0	5126.69
6	$\Omega_6 = 16,17,18$	0	0	0	0	0	10121.80
7	$\Omega_7 = 19,20,21$	3710.30	4858.33	0	0	0	5126.69
8	$\Omega_8 = 22,23,24$	0	4858.33	3700.56	0	28.03	5126.69
9	$\Omega_9 = 25,26,27$	0	4858.33	0	0	3782.48	5126.69

Table 2.22 gives the optimal solution at period t_1. It is clear that this is a much riskier strategy since in those failure scenarios we would get very little return. But, in terms of expected return, this is the best strategy due to the fact that failure has a such small probability.

When implementing this model in Instance 1 with $\alpha \in (0, 0.10)$, the optimal strategies are exactly those obtained with model (DEM4). This is because default is so probable that it is important to take it into account when optimizing.

The same happens with Instance 2 if $\alpha \leq 0.5$. Even for Instance 2 with $\alpha = 0.6$ the optimal solution does not change. It is worth pointing out that, since the default scenarios in period t_0 do not reach in probability the value of α, the improvement that would achieve in V_0 is compensated by what it would lose in V_g for $g \in \mathcal{G}_t, t = t_1, \cdots, t_k$.

Case 2: $\mathcal{P} = \{g \in \mathcal{G}_t : t = t_0, \cdots, t_{k-1}\}$ **and** $\phi^p = 500000$ **for all** $p \in \mathcal{P}$

In order to ensure minimum wealth for even those failure scenarios, let us consider a threshold of 500000 monetary units for all potential profiles, $\mathcal{P}$. The optimal solution obtained for Instances 1 and 2, is the same as that obtained in Case 1. This happens since the expected wealth is higher than ϕ^p, $p \in \mathcal{P}$. For high values of this potential profiles, for example 1.200.000, the problem becomes infeasible.

Finally, let us summarize the optimal solution for the different stages in Instance 3 with $\alpha = 0.05$

Table 2.23: Optimal solution at t_0 for model (DEM6). Instance 3. Case 2

$\mathbf{g}$	$z_1^{\mathbf{g}}$	$z_{1'}^{\mathbf{g}}$	$z_2^{\mathbf{g}}$	$z_{2'}^{\mathbf{g}}$	$z_3^{\mathbf{g}}$	$z_{3'}^{\mathbf{g}}$	$z_4^{\mathbf{g}}$	$z_{4'}^{\mathbf{g}}$
0	0	0	922.25	3932.18	0	0	1028.56	4102.04

Table 2.23 presents the optimal solution at period t_0. It consists of investing part of the portfolio in assets that mature before the PH and the rest in the assets with the longest maturity. The investment strategy shares between risky and safe assets in order to ensure, on the one hand, as high as possible the minimum return over all the periods but, in the other hand, to ensure minimum wealth (ϕ^p) in situation of default. As the portfolio is made up almost half and half, it matches the duration with the PH, so it is following in some way the strategy proposed by Khang [61]. It is also important to note that it invests in risky assets since the failure scenarios are not considered sufficiently probable.

The optimal solution at period t_1 is shown in Table 2.24. It consists of investing short if the interest rates are going to rise, long if they are going to fall, and following the strategy proposed by Khang when both possibilities exist. In any case, it invests in risky assets in no failure scenarios, and it invests in safe assets in scenarios of default.

Table 2.24: Optimal solution at t_1 for model (DEM6). Instance 3. Case 2

$\mathbf{g}$	ω	$z_2^{\mathbf{g}}$	$z_{2'}^{\mathbf{g}}$	$z_3^{\mathbf{g}}$	$z_{3'}^{\mathbf{g}}$	$z_4^{\mathbf{g}}$	$z_{4'}^{\mathbf{g}}$
1	$\Omega_1 = 1, 2, 3$	922.25	9212.82	0	0	0	0
2	$\Omega_2 = 4, 5, 6$	922.25	3932.18	0	217.24	909.58	4102.04
3	$\Omega_3 = 7, 8, 9$	0	0	0	0	1028.56	9063.62
4	$\Omega_4 = 10, 11, 12$	0	10116.90	0	0	0	0
5	$\Omega_5 = 13, 14, 15$	0	3932.18	0	2044.50	0	4102.04
6	$\Omega_6 = 16, 17, 18$	0	0	0	0	0	10112.30
7	$\Omega_7 = 19, 20, 21$	4957.83	0	0	0	0	0
8	$\Omega_8 = 22, 23, 24$	978.62	0	2958.37	0	1028.56	0
9	$\Omega_9 = 25, 26, 27$	0	0	0	0	5014.98	0

It is important to note that, we are maximizing over all the periods of time with this strategy, but we will only get the final value if we maintain the strategy until the end of the PH.

Models comparison

By considering the advantages and disadvantages of the different models tested in the case study, see Tables 2.25 and 2.26, we can observe that the models that offer better immunization strategies are the models (DEM4) (100% multistage VaR strategy), (DEM5) (multistage stochastic dominance constraints strategy), and (DEM6) (VaR & multistage stochastic dominance constraints strategy) for good choices of the potential profiles. Each of them has advantages and disadvantages, and the decision maker should choose among the three strategies depending on its preferences.

Table 2.25: Comparing the different two-stage models as tested in the case study

Stg	**Model**	**Strategy**	**Pros**	**Cons**
	LP	• Invest everything in the risky asset that matures at the PH	• Easy	• Myopic • No risks
Two	**DEM1**	• $\underline{t_0}$: safe & short • $\underline{t > t_0}$: "Good" scen.: risky mat. PH "Bad" scen.: safe mat. PH	• Risk of def. treated	• Myopic • Interest Rate risk not treated
	DEM- -MR	• $\underline{t_0}$: safe & short • $\underline{t > t_0}$: "Good" scen.: risky mat. PH "Bad" scen.: safe mat. PH	• Risk of def. treated	• Myopic • Interest Rate risk not treated
	DEM2	• $\underline{t_0}$: safe & D=PH • $\underline{t > t_0}$: "Good" scen.: risky mat. PH "Bad" scen.: safe mat. PH	• Both risks treated	• Myopic • Risk averse- -ness not treat. (too risk adv.)
	DEM3	• $\underline{t_0}$: safe & D=PH (Instance 1) risky & D=PH (Instance 2) • $\underline{t > t_0}$: "Good" scen.: risky mat. PH "Bad" scen.: safe mat. PH	• Both risks treated • Risk averse model	• Myopic • May big lose in worst cases
Notation: D=PH means the strategy that matches the duration with the PH.				

Model (DEM5) would be better for those investors who are only interested in the final value of the portfolio and they also know a priori which possible thresholds can be more interesting for their purposes.

On the contrary, model (DEM6) would be better for those investors with no initial expectations and are also interested in optimizing the portfolio all over the PH. It could be very interesting, for example, for those decision makers who would like to dissolve the portfolio at any time in order to use new information in the market. Model (DEM4), instead, could be very interesting for a very risk adverse investor.

Table 2.26: Comparing the different multistage models as tested in the case study

Stg	Model	Strategy	Pros	Cons
Multi	**DEM4**	• t_0: safe & D=PH • $t > t_0$: risky in "good" ω & safe else r will rise: short r will fall: long r can fall or rise: D=PH	• Not myopic • Both risks treated	• Risk av. not treated (too risk av.)
	DEM5	$\boxed{\lvert\mathcal{P}\rvert = 1}$ well chosen • t_0: safe & short • $t > t_0$: risky in "good" ω & safe else short, long or D=PH depending on r	• Not myopic •Risk averse model	• Interest Rate risk not treated •ϕ dependent • CNT
		$\boxed{\lvert\mathcal{P}\rvert = 2}$ well chosen • t_0: safe & short (Instance 1) share (safe-risky); D=PH (Inst. 2, 3) • $t > t_0$: risky in "good" ω & safe else short, long or D=PH dep. on r	• Not myopic • Both risks • Risk averse • Bad sit. cov.	•ϕ dependent • CNT
	DEM6	$\boxed{\phi = 0}$ • t_0: safe & short (Inst. 1, 2) risky & short (Inst. 3) • $t > t_0$: risky in "good" ω & safe else short, long or D=PH dep. on r	• Not myopic • Both risks • Risk averse • Not ϕ dep.	• May big lose in worst cases • Max. $\forall\ V_t$, get just V_{t_k} • CNT
		$\boxed{\phi \neq 0}$ • t_0: safe & short (Inst. 1, 2) share (safe-risky); D=PH (Inst. 3) • $t > t_0$: risky in "good" ω & safe else short, long or D=PH dep. on r	• Not myopic • Both risks • Risk averse • Not so ϕ dep. • Bad sit. cov.	• Max. $\forall\ V_t$, get just V_{t_k} • CNT
Notation: D=PH means the strategy that matches the duration with the PH. CNT means that it **can become** Computationally Not Treatable or very hard to solve.				

In any case, models (DEM5) and (DEM6) can become computationally non treatable for real markets with many possible future scenarios, in case of using plain state-of-the-art optimization engines instead of using proven decomposition approaches, meanwhile (DEM4) can easily be solved.

2.10 Conclusions

This chapter proposes several stochastic models for selecting portfolios in a market in which there are transaction costs and bonds with different credit ratings. In particular, new concepts and modelings have been introduced and tested. We have also extended some of them from the two-stage formulation to the general multistage case. The intention is to check whether the assumptions made in the dynamic immunization theorem put forward by Khang [61] are crucial to its validity. Another aim is to check wether the theoretical models proposed in the literature and developed in this chapter are suitable to optimize immunization strategies in fixed-income security portfolios under both sources of uncertainty.

The inclusion of transaction costs is observed to affect the optimality of the strategy proposed by Khang, since the continual readjustment of the portfolio that these costs entail results in additional costs which are too high. This means that the immune strategy ceases to be optimal.

Uncertainty is introduced into the model through a scenario analysis scheme. In this case there are two parameters whose random behavior must be taken into account, namely, the trends in interest rates and the probabilities of default of the various institutions which issue the bonds.

Different immunization strategies are considered. The validity of the proposed strategies is performed by using an illustrative case study. No definitive conclusions can be drawn from the case study (the aim of the chapter has merely been to present the immunization strategies of choice), but the results that have been obtained seem to be reasonable. Based on them we favor the multistage immunization strategies given by the models (DEM4) (multistage 100% VaR strategy), (DEM5) (stochastic dominance constraints strategy alone) and (DEM6) (mixture of the VaR strategy and the stochastic dominance constraints strategy).

As a future work we are planning an extensive computational experience with large scale cases in fixed-income security portfolios, scenarios and profiles to test the validity of our BFC decomposition algorithm for dealing with risk averse measures, see [7] in stochastic mixed integer optimization in this type of financial application. See also in [7] the extension of the CVaR and second stage SDC into the multistage scheme.

Part II

Algorithms: Cluster Benders Decomposition

Chapter **3**

Two-stage scheme

The optimization of stochastic linear problems, via scenario analysis, based on Benders decomposition requires appending feasibility and/or optimality cuts to the master problem until the iterative procedure reaches the optimal solution. The cuts are identified by solving the auxiliary submodels attached to the scenarios. In this chapter, we propose the algorithm so-named scenario Cluster Benders Decomposition (CBD) for dealing with the feasibility cut identification in the Benders method for solving large-scale two-stage stochastic linear problems. The scenario tree is decomposed into a set of scenario clusters and tighter feasibility cuts are obtained by solving the auxiliary submodel for each cluster instead of each individual scenario. Then, the scenario cluster based scheme allows us to identify tighter feasibility cuts that yield feasible second stage decisions in reasonable computing time. Some computational experience is reported by using CPLEX as the solver of choice for the auxiliary linear optimization submodels at each iteration of the algorithm CBD. The results that are reported show the favorable performance of the new approach over the traditional single scenario based Benders decomposition (TBD); it also outperforms the plain use of CPLEX for medium-large and large size instances.

3.1 Introduction

Two-stage stochastic linear problems provide a suitable framework for modeling decision problems under uncertainty arising in several applications. The flexibility of these models is related to their dynamic nature, i.e., besides the first stage variables that represent decisions made in the face of uncertainty, the model considers second stage decisions, i.e., recourse actions, which can be taken once a specific realization of the random parameters is observed. For an introduction to two-stage stochastic programming models and solution procedures based on scenario analysis, see [18, 60, 89]. Moreover, many applications require an excessive number of scenarios and then, this kind of problems become quite large. So, methods that ignore the special structure of stochastic linear programs become quite inefficient. However, taking advantage of this structure is especially beneficial in stochastic programs. Perhaps, the method that is most frequently used is based on building an outer linear relaxation of the recourse cost function around a solution of the first stage problem. One of the alternative decomposition procedures is known as the Dantzig-Wolfe approach [26] that solves the dual of the problem. Another decomposition, known as Benders method [11] solves the primal problem. This latter method has been widely used in stochastic programming approaches to take care of the feasibility cut generation, see [87]. As is well known, when the number of scenarios is finite, the Benders decomposition method converges to an optimal solution in a finite number of iterations when it exists, or proves the infeasibility of problem (3.1). Moreover, the structure of the stochastic programs clearly allows us to modify the phase of cuts generation. As in the multicut version, see [16, 17, 63], one cut per realization in the second stage is placed. However, in the feasibility phase, the submodel can be integrated by a set of realizations i.e., a scenario cluster, and a tighter cut could be generated and appended to the master problem and it is basically, the proposal of this chapter.

Let a scenario cluster be defined as a subset of scenarios where the nonanticipativity constraints, see its definition and introduction in [90] are considered while solving the related two-stage stochastic submodel. So, the main contribution of this chapter consists of proposing a scenario cluster decomposition approach for dealing with the feasibility problem in the Benders method for solving large-scale two-stage stochastic linear problems. We computationally compare the performance of the new approach, named scenario Cluster Benders Decomposition (for short, CBD) versus the Traditional Benders Decomposition (TBD) for different choices of the number of scenario clusters and the contents of each cluster. Given the scenario cluster partitioning, the saving in computation time is remarkable in the large-scale cases with which we have experimented. Additionally, when comparing decomposition based procedures with the plain use of a state-of-the-art optimization engine (open source codes or commercial ones) we can verify that for stochastic problems with only continuous variables, the decomposition approaches, in particular the algorithm CBD, bet the plain use of the solver CPLEX usually for small number of clusters. The successful results that we have obtained with the CBD approach may open up the possibility of tightening the lower bound of the solution value at the Twin Node Family submodels in the exact Branch-and-Fix Coordination (BFC) scheme for solving two-stage stochastic mixed 0-1 problems [35, 36, 38]. The combination of that strategy with our CBD approach would be a subject to future research.

The remainder of the chapter is organized as follows. Section 3.2 briefly outlines the Benders decomposition method for two-stage stochastic problems. Section 3.3 deals with an illustrative example. Section 3.4 presents the innovation of the proposed scenario cluster decomposition scheme. Section 3.5 introduces in detail the CBD approach that is proposed. Section 3.6 reports some computational results, mainly for large-scale instances that show the good performance of the new approach. Section 3.7 concludes.

3.2 Benders decomposition for two-stage stochastic problems

Let us consider the Deterministic Equivalent Model (DEM) to the two-stage stochastic linear problem in *compact* representation

$$\begin{array}{llll} (LO): & z_{LO} & = \min & c^T x + E_\psi[\min w^\omega (q^{\omega T} y^\omega)] \\ & s.t. & & \\ & & & b_1 \leq Ax \leq b_2 \\ & & & h_1^\omega \leq T^\omega x + W^\omega y^\omega \leq h_2^\omega \quad \forall \omega \in \Omega \\ & & & x, y^\omega \geq 0 \quad \forall \omega \in \Omega, \end{array} \tag{3.1}$$

where x is the n_x-vector of the first stage variables, y^ω is the n_y-vector of the second stage variables for scenario ω, for $\omega \in \Omega$, where Ω is the set of scenarios to consider, c is a known vector of the objective function coefficients for the x variables, b_1 and b_2 are the left and right hand side vectors for the first stage constraints, respectively, A is the first stage constraint matrix, w^ω is the likelihood attributed to scenario ω, h_1^ω and h_2^ω are the left and right hand side vectors for the second stage constraints, respectively, and q^ω is the vector of the objective function coefficients for the y variables, while T^ω is the technology matrix and W^ω is the recourse matrix under scenario ω, for

$\omega \in \Omega$. Putting together the stochastic components of the problem, we have the vector $\psi^\omega = (q^\omega, h_1^\omega, h_2^\omega, T^\omega, W^\omega)$. Finally, E_ψ represents the mathematical expectation with respect to ψ over the set of scenarios Ω.

The structure of the uncertain information in the two-stage stochastic linear model (3.1) can be visualized as a tree, where each root-to-leaf path represents one specific scenario, ω, and corresponds to one realization of the whole set of the uncertain parameters.

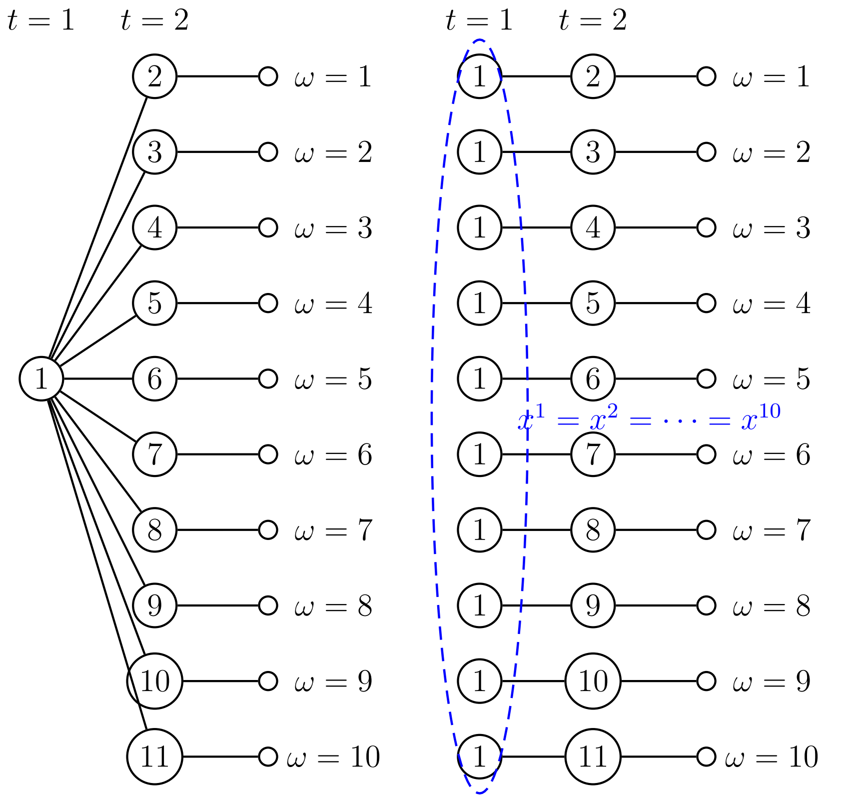
In the example depicted in Figure 3.1, there are $|\Omega| = 10$ root-to-leaf possible paths, i.e., scenarios. Following the nonanticipativity principle stated in [90] and restated in [76], see [18] among many others, all scenarios should have the same value for the related first stage variables in the two-stage problem. The left part of Figure 3.1 implicitly represents the nonanticipativity constraints (NAC, for short). This is the compact representation shown in model (3.1). The right part of Figure 3.1 gives the same information as the compact representation but using a splitting variable scheme. Notice that it explicitly represents the NAC (i.e., imposing the equality) on the first stage variables x^ω for all the scenarios in Ω.

The two-stage linear problem (3.1) can be decomposed and its optimal solution can be iteratively obtained by identifying extreme points and rays based cuts from the optimization of the so-named *Auxiliary Program* (AP). So, the cuts are appended to the so-named *Relaxed Master Program* (RMP) that can be expressed, see [11],

$$
\begin{array}{rcl}
\overline{z}_{LO} & = & \min c^T x + \theta \\
\text{s.t.} & & \\
b_1 & \leq & Ax \leq b_2 \\
0 & \geq & \nu_{j1}^{\omega T} \left[\begin{pmatrix} h_1^\omega \\ -h_2^\omega \end{pmatrix} + T^\omega x \right] \quad \forall \nu_{j1}^\omega \in \overline{\mathcal{J}}^{e_f} \\
\theta & \geq & \sum_{\omega \in \Omega} w^\omega \nu_{j2}^{\omega T} \left[\begin{pmatrix} h_1^\omega \\ -h_2^\omega \end{pmatrix} + T^\omega x \right] \quad \forall \nu_{j2}^\omega \in \overline{\mathcal{J}}^{e_o} \\
 & & x \geq 0, \theta \in I\!R,
\end{array}
$$

where $\overline{\mathcal{J}}^{e_o} \subseteq \mathcal{J}^{e_o}$ and $\overline{\mathcal{J}}^{e_f} \subseteq \mathcal{J}^{e_f}$ are the subsets of the extreme points and extreme rays already identified, respectively, see [87].

We first give a presentation of the Traditional Benders Decomposition



Compact representation Splitting variable representation

Figure 3.1: Scenario tree

method (TBD) taken from [18].

Primal Scenario procedure

Step 0: Set $k := e_o := e_f := 0$, where e_o and e_f are to count the number of optimality and feasibility cuts along the iterations of the algorithm, respectively.

Step 1: Set $k := k + 1$. Solve the program (RMP) (with $\theta = 0$ if $e_o = 0$).

$$
\begin{aligned}
(RMP) \qquad & \min \quad c^T x + \theta \\
& \text{s.t.} \\
& b_1 \leq Ax \leq b_2
\end{aligned}
$$

$$0 \geq \hat{\nu}_{j_1}^{\omega T} \begin{pmatrix} h_1^\omega + T^\omega x \\ -h_2^\omega + T^\omega x \end{pmatrix} \quad \forall j_1 = 0, ..., e_f \tag{3.2}$$

$$\theta \geq \sum_{\omega \in \Omega} w^\omega \hat{\nu}_{j_2}^{\omega T} \begin{pmatrix} h_1^\omega + T^\omega x \\ -h_2^\omega + T^\omega x \end{pmatrix} \quad \forall j_2 = 0, ..., e_o \tag{3.3}$$

$$x \geq 0, \theta \in I\!R,$$

where $\hat{\nu}_{j_1}$ and $\hat{\nu}_{j_2}$ are the values of the corresponding dual variables (i.e., simplex multipliers) obtained in the feasibility (Step 2) and auxiliary primal (Step 3) problems, respectively.

Save the optimal solution $\hat{x}$ and $\hat{\theta}$ of the primal variables x and θ, respectively.

Step 2: For each scenario $\omega \in \Omega$, solve the following feasibility problem

$$\begin{array}{ll} (FEAS) & z_{FEAS}^\omega = \min e^T v_1^{+\omega} + e^T v_1^{-\omega} + e^T v_2^{+\omega} + e^T v_2^{-\omega} \\ & s.t. \\ & W^\omega y^\omega - I u^{-\omega} + I v_1^{+\omega} - I v_1^{-\omega} = h_1^\omega - T^\omega \hat{x} \\ & W^\omega y^\omega + I u^{+\omega} - I v_2^{+\omega} + I v_2^{-\omega} = h_2^\omega - T^\omega \hat{x} \\ & y^\omega, v_1^{+\omega}, v_1^{-\omega}, v_2^{+\omega}, v_2^{-\omega}, u^{+\omega}, u^{-\omega} \geq 0. \end{array} \tag{3.4}$$

If there exists a scenario ω, such that $z_{FEAS}^\omega \neq 0$ (infeasible scenario problem), set $e_f := e_f + 1$, $\phi^\omega = +\infty$, save the values $\hat{\nu}_{e_f}^\omega$ of the dual variables ν^ω, define the feasibility cut (3.2) and go to Step 1.

If $z_{FEAS}^\omega = 0$ (feasible) $\forall \omega \in \Omega$, go to Step 3.

Step 3: For each scenario $\omega \in \Omega$, solve the auxiliary primal problem

$$\begin{array}{llll} (OPT) & \phi^\omega & = & \min q^{\omega T} y^\omega \\ & & & s.t. \\ & & & \begin{pmatrix} W^\omega \\ -W^\omega \end{pmatrix} y^\omega \geq \begin{pmatrix} h_1^\omega - T^\omega \hat{x} \\ -h_2^\omega + T^\omega \hat{x} \end{pmatrix} \\ & & & y^\omega \geq 0. \end{array} \tag{3.5}$$

Save the objective function value, ϕ^ω and the simplex multipliers associated with the optimal solution of problem (3.5), $\hat{\nu}_{e_o}^\omega$, and define the optimality cut.

Set $\phi := \sum_{\omega \in \Omega} w^\omega \phi^\omega$. If $\phi \leq \hat{\theta}$ then stop, since the optimal solution has been found in k-th iteration.

In other case, set $e_o := e_o + 1$, add the new cut to the constraint set (3.3) and return to Step 1.

As it is well known, when Ω is finite, this method finitely converges to an optimal solution when it exists or proves the infeasibility of problem (3.1).

However, as we will illustrate with the example shown in the next section, the generation of feasibility cuts by a mere utilization of the scenario related feasibility problem to be solved at Step 2 of the procedure, may not be efficient in large-scale stochastic instances. Indeed, we propose a scenario cluster decomposition approach for dealing with the feasibility problem, which generates tighter feasibility cuts to add to the relaxed master problem. This new scheme provides a more efficient procedure for solving large-scale two-stage stochastic problems as we report in Section 3.6.

3.3 Illustrative example

As can be seen in the previous section, Step 2 of the TBD method consists of determining whether a first stage decision, x, is also second stage feasible. This step can be extremely time-consuming. It requires the solution of up to $|\Omega|$ phase-one problems of the form (3.4). The process may have to be iteratively repeated to obtain successive candidate first stage decisions.

To illustrate the feasibility cuts generation, consider the following example taken also from [18]:

$$\begin{array}{rcl}
\min 3x_1 + 2x_2 & - & \sum_{\omega=1}^{|\Omega|} w^\omega (15y_1^\omega + 12y_2^\omega) \\
\text{s.t.} & & \\
0 & \leq & x_1 - 3y_1^\omega - 2y_2^\omega \quad \forall \omega \in \Omega \\
0 & \leq & x_2 - 2y_1^\omega - 5y_2^\omega \quad \forall \omega \in \Omega \\
0.8 \cdot u_1^\omega & \leq & y_1^\omega \leq u_1^\omega \quad \forall \omega \in \Omega \\
0.8 \cdot u_2^\omega & \leq & y_2^\omega \leq u_2^\omega \quad \forall \omega \in \Omega \\
x_1 \geq 0, & & x_2 \geq 0, \quad y_1^\omega \geq 0, y_2^\omega \geq 0 \quad \forall \omega \in \Omega,
\end{array} \tag{3.6}$$

where, independently, u_1=4 or 6 and u_2= 4 or 8 with probability $\frac{1}{2}$ each, and $u = (u_1, u_2)^T$. Then,

$$\{(u_1^\omega, u_2^\omega) : (4,4), (4,8), (6,4), (6,8)\}, \quad w^\omega = \frac{1}{4}, \quad \omega = \{1,2,3,4\}.$$

If at the first iteration of Step 1, as in this example, there is no system of constraints in x, an initial feasible solution is needed. Starting from the initial

Table 3.1: Feasibility cuts generated for the illustrative example by TBD

Iteration	Scenario	Feasibility cut
1	1	$x_1 \geq 6.4$
2	1	$x_2 \geq 6.4$
3	1	$0.272727x_1 + 0.090909x_2 \geq 6.4$
4	1	$0.2x_2 \geq 4.48$
5	2	$0.2x_2 \geq 7.68$
6	1	$0.333333x_1 \geq 5.333333$
7	2	$0.333333x_1 \geq 7.46666$
8	4	$0.333333x_1 \geq 9.06666$
9	4	$0.2x_2 \geq 8.32$

solution $\hat{x}^1 = (x_1, x_2)^1 = (0, 0)$, Table 3.1 shows the feasibility cuts that are generated.

By appending these nine cuts to the *RMP* model, the first stage solution is as follows,

$$\hat{x}^{10} = (27.2, 41.6),$$

which is feasible for the second stage decisions, see Table 3.1.

3.4 Scenario Cluster Benders Decomposition scheme innovation

In addition to the two formulations presented in Figure 3.1, we propose a scenario-cluster partitioning to allow for a mixture of compact and splitting variable representations into and inter the scenario cluster submodels, respectively. As an illustrative example, let us consider again the scenario tree depicted in Figure 3.1. Figure 3.2 shows the problem decomposition in $\hat{p} = 5$ (left tree) and $\hat{p} = 2$ (right tree) scenario clusters into which the set of scenarios is split. Observe that the NAC for the first stage vectors of variables are given by $\mathbf{x}^1 = \cdots = \mathbf{x}^5$ for the left part of the figure, and they are given by $\mathbf{x}^1 = \mathbf{x}^2$ for the right part of the figure, where $\mathbf{x}^p$ is the vector x of the first stage variables for scenario cluster p.

Let Ω^p denote the set of scenarios included in cluster p, for $p = 1, ..., \hat{p}$. In the left tree of Figure 3.2, there are five scenario clusters given by $\Omega^1 = \{1, 2\}$, $\Omega^2 = \{3, 4\}$, $\Omega^3 = \{5, 6\}$, $\Omega^4 = \{7, 8\}$ and $\Omega^5 = \{9, 10\}$. In the right tree, there are two scenario clusters, given by $\Omega^1 = \{1, 2, 3, 4, 5\}$ and $\Omega^2 = \{6, 7, 8, 9, 10\}$.

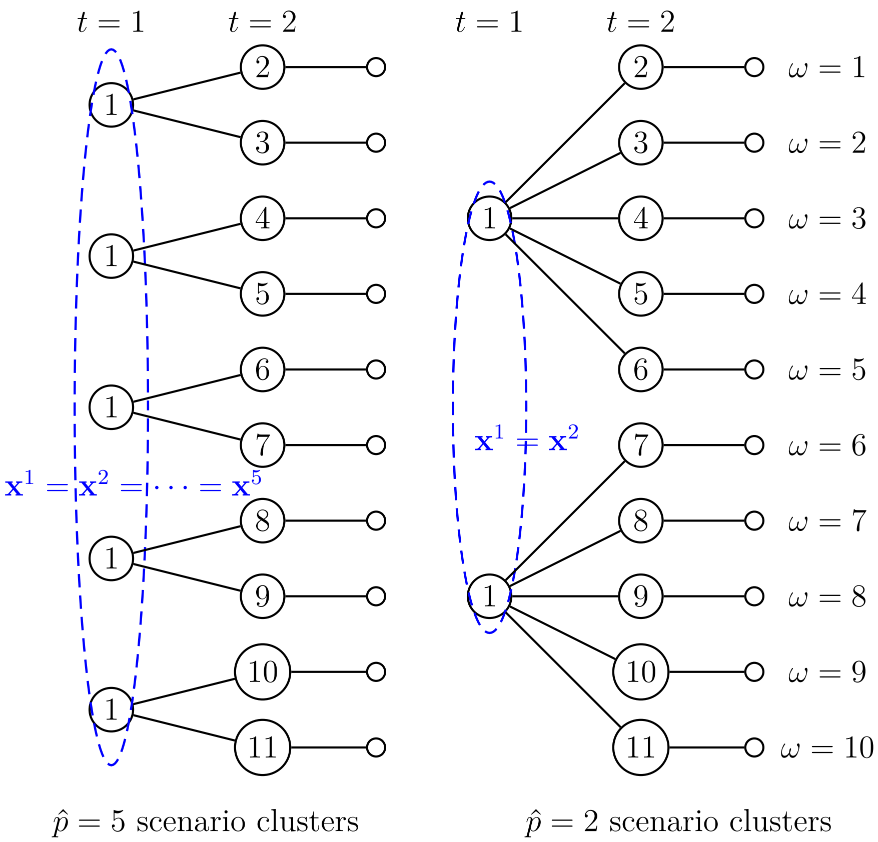


Figure 3.2: Scenario cluster partitioning

The criterion for scenario clustering in the sets, say, $\Omega^1, \ldots, \Omega^{\hat{p}}$, where $\hat{p}$ is the number of clusters to consider, is instance dependent. Given the scenario cluster partitioning, the initial model (3.1) can be decomposed into $\hat{p}$ smaller problems. By slightly abusing the notation, the problem to consider for scenario cluster p can be expressed (in compact representation) as follows,

$$
\begin{array}{lll}
(LO^p): \; z^p_{LO} & = \min & c^T x^p + \sum_{\omega \in \Omega^p} w^\omega (q^{\omega T} y^\omega) \\
s.t. & & \\
 & & b_1 \le A x^p \le b_2 \\
 & & h^\omega_1 \le T^\omega x^p + W^\omega y^\omega \le h^\omega_2 \quad \forall \omega \in \Omega^p \\
 & & x^p, y^\omega \ge 0 \quad \forall \omega \in \Omega^p,
\end{array}
\tag{3.7}
$$

where $p = 1, ..., \hat{p}$. The $\hat{p}$ problems (3.7) are linked by the NAC for the first stage variables,

$$x^p_i = x^{p'}_i \tag{3.8}$$

for all $p \neq p'$, $p, p' = 1, ..., \hat{p}$ and $i = 1, ..., n_x$. See [38].

For simplicity and without loss of generality, we can select the number of scenario clusters, $\hat{p}$, as a divisor of the number of scenarios, $|\Omega|$. In this case, $|\Omega^p| = \frac{|\Omega|}{\hat{p}} = l$, where $|\Omega^p|$ defines the size of scenario cluster p, i.e., the number of scenarios that belong to the corresponding cluster, for $p = 1, ..., \hat{p}$. This choice forces all the scenario clusters to have the same size, l. Then, the scenario clusters are defined in terms of blocks of l-consecutive scenarios, $\Omega^1 = \{1, ..., l\}$, $\Omega^2 = \{l + 1, ..., 2 \cdot l\}$,..., $\Omega^{\hat{p}} = \{(\hat{p} - 1) \cdot l + 1, ..., (\hat{p} - 1) \cdot l + l\}$. In a more general case, the number of scenario clusters can be chosen as any value $1 \le \hat{p} \le |\Omega|$, such that the total number of the scenarios in each cluster along the set of clusters, must be equal to the total number of scenarios.

A simple look at the feasibility problem (3.4) reveals that its objective function coefficients do not depend of any specific scenario, so, we can consider a cluster of scenarios instead of one scenario alone. The objective function in problem (3.4) depends on the set of artificial variables v^+, v^-, whose dimension is the total number of second stage constraints. Then, this feasibility model can be globally formulated for a set of scenarios as a minimization problem in the variables $v^{+\omega}_1, v^{-\omega}_1$ and $v^{+\omega}_2, v^{-\omega}_2$, for $\omega \in \Omega^p$. Then, we can define the following feasibility scenario cluster model for each iteration,

$$
\begin{array}{lll}
z^p_{FEASC} & = & \min \sum_{\omega \in \Omega^p} w^\omega (e^T v^{+\omega}_1 + e^T v^{-\omega}_1 + e^T v^{+\omega}_2 + e^T v^{-\omega}_2) \\
s.t. & & \\
 & & W^\omega y^\omega - I u^{-\omega} + I v^{+\omega}_1 - I v^{-\omega}_1 = h^\omega_1 - T^\omega \hat{x} \quad \forall \omega \in \Omega^p \\
 & & W^\omega y^\omega + I u^{+\omega} - I v^{+\omega}_2 + I v^{-\omega}_2 = h^\omega_2 - T^\omega \hat{x} \quad \forall \omega \in \Omega^p \\
 & & y^\omega, v^{+\omega}_1, v^{-\omega}_1, v^{+\omega}_2, v^{-\omega}_2, u^{+\omega}, u^{-\omega} \ge 0 \quad \forall \omega \in \Omega^p,
\end{array}
\tag{3.9}
$$

where the dimension of $e^T = (1, ..., 1)$ is the number of constraints for scenario cluster p, and $v_1^{+\omega}, v_1^{-\omega}$ and $v_2^{+\omega}, v_2^{-\omega}$ are the artificial variables for the left and right hand side second stage constraints. These variables are introduced to generate a problem which detects the infeasibility in model (3.1), given a fixed value, say $\hat{x}$, of the variable vector x. Additionally, notice that in the feasibility model (3.9), the slack and excess variables for the second stage inequality constraints, $u^{+\omega}$ and $u^{-\omega}$, respectively, can take any value different to zero. These variables appear in problems (1) with inequality constraints. However, to ensure the feasibility of problem (3.1), given a fixed vector $\hat{x}$, the solution value of problem (3.9), i.e., variables v_1^+, v_1^-, v_2^+ and v_2^- for each scenario cluster must be equal to zero.

Table 3.2: Feasibility cuts generated for the illustrative example by CBD for a given scenario cluster partition

Iteration	Scenario cluster	Feasibility cut
1	1	$x_1 \geq 8$
2	1	$x_2 \geq 8$
3	1	$0.272727x_1 + 0.090909x_2 \geq 8$
4	1	$0.2x_2 \geq 6.08$
5	1	$0.2x_2 \geq 7.68$
6	1	$0.333333x_1 \geq 7.46666$
7	2	$0.333333x_1 \geq 9.06666$
8	2	$0.2x_2 \geq 8.32$

If there is a scenario cluster p, such that $z_{FEAS}^p \neq 0$ (infeasible scenario cluster problem), there is a new feasibility cut and we must increase the counter, i.e., $e_{fc} := e_{fc} + 1$, save the corresponding values $\hat{\nu}_{e_{fc}}^{\omega}$ of the dual variables ν^{ω}, $\omega \in \Omega^p$, define the cluster feasibility cut (3.10) and go to Step 1.

Let us consider again the example presented in Section 3.3, with $\hat{p} = 2$ scenario clusters, where the first cluster is included by the two first scenarios, $\Omega^1 = \{1, 2\}$, and the second cluster by the last two, $\Omega^2 = \{3, 4\}$. Starting again from the initial solution $\hat{x}^1 = (x_1, x_2)^1 = (0, 0)$, Table 3.2 shows the feasibility cuts that are generated by the algorithm CBD. In this case, eight cuts are generated, six cuts to satisfy the feasibility of the second stage constraints in the first cluster, scenarios 1 an 2, and two cuts to satisfy the feasibility of the second stage constraints in the second cluster, scenarios 3 and 4.

Table 3.3: Feasibility cuts generated for the illustrative example by CBD for all scenario cluster

Iteration	Scenario cluster	Feasibility cut
1	1	$x_1 \geq 8.8$
2	1	$x_2 \geq 8.8$
3	1	$0.272727x_1 + 0.090909x_2 \geq 8.8$
4	1	$0.2x_2 \geq 4.48$
5	1	$0.2x_2 \geq 7.04$
6	1	$0.878787x_1 + 0.181818x_2 \geq 27.7333$
7	1	$0.272727x_1 + 0.290909x_2 \geq 18.8888$
8	1	$0.333333x_1 \geq 9.06666$
9	1	$0.2x_2 \geq 8.32$

Finally, let us consider, $\hat{p} = 1$ scenario cluster, it is included by the four scenarios, $\Omega^1 = \{1, 2, 3, 4\}$. By using the same initial solution $\hat{x}^1 = (\hat{x}_1, \hat{x}_2)^1 = (0, 0)$, Table 3.3 shows the set of feasibility cuts that are generated. Notice that Tables 3.1 and 3.2 report different sets of cuts.

3.5 Algorithm CBD

In order to gain computational efficiency, we present our proposed scenario cluster based scheme to be used in Benders decomposition. Before executing the proposed algorithm for solving the original two-stage stochastic linear problem, we are required to fix some data structuring. A decision has to be made on fixing the number of scenario clusters $\hat{p}$, for considering the splitting variable representation and, consequently, the set of scenarios in each of the clusters, Ω^p, $\forall p = 1, ..., \hat{p}$.

Then, the feasibility model is solved at Step 2 for each scenario cluster, p, i.e., the set of scenarios in Ω^p, such that the new feasibility model is to be optimized for a given scenario cluster instead of a particular scenario.

Let e_{fc} denote the new counter of feasibility cuts generated from the scenario cluster feasibility models ($FEASC$). Moreover, once the feasibility cut has been identified and appended, the algorithm goes back to Step 1 in order to solve the new relaxed master program. Notice that in this case, the number of iterations can-not be the same for the distinct partitions into clusters of scenarios.

Primal Scenario Cluster procedure

Step 0: Set $k := 0$, $p := 0$, $e_o := 0$, $e_{fc} := 0$.

Step 1: Solve the relaxed master program RMP (for $\theta = 0$ if $k = 0$). $k := k + 1$.

$$\begin{aligned}
&\min c^T x + \theta \\
&\text{s.t.} \\
&b_1 \leq Ax \leq b_2 \\
&-\hat{\nu}_j^{\omega T} T^\omega x \geq \hat{\nu}_j^{\omega T} \begin{pmatrix} h_1^\omega \\ -h_2^\omega \end{pmatrix} \quad \forall \omega \in \Omega^p \quad j = 0, ..., e_{fc} \qquad (3.10) \\
&- \sum_{\omega \in \Omega^p} w^\omega \hat{\nu}_{j_2}^{\omega T} T^\omega x + \theta \geq \sum_{\omega \in \Omega^p} w^\omega \hat{\nu}_{j_2}^{\omega T} \begin{pmatrix} h_1^\omega \\ -h_2^\omega \end{pmatrix} \forall j_2 = 0, ..., e_o \quad (3.11) \\
&x \geq 0, \theta \in I\!R
\end{aligned}$$

Save the values $\hat{x}$ and $\hat{\theta}$ of the primal variables x and θ.

Step 2: Set $p := p + 1$. Solve the feasibility problem for scenario cluster p,

$$\begin{aligned}
(FEASC): \quad & z_{FEASC}^p = \min \sum_{\omega \in \Omega^p} w^\omega (e^T v_1^{+\omega} + e^T v_1^{-\omega} + e^T v_2^{+\omega} + e^T v_2^{-\omega}) \\
& s.t. \\
& W^\omega y^\omega - I u^{-\omega} + I v_1^{+\omega} - I v_1^{-\omega} = h_1^\omega - T^\omega \hat{x} \quad \forall \omega \in \Omega^p \qquad (3.12) \\
& W^\omega y^\omega + I u^{+\omega} - I v_2^{+\omega} + I v_2^{-\omega} = h_2^\omega - T^\omega \hat{x} \quad \forall \omega \in \Omega^p \\
& y^\omega, v_1^{+\omega}, v_1^{-\omega}, v_2^{+\omega}, v_2^{-\omega}, u^{+\omega}, u^{-\omega} \geq 0 \quad \forall \omega \in \Omega^p.
\end{aligned}$$

If $z_{FEASC}^p \neq 0$ (infeasible scenario cluster problem): Set $e_{fc} := e_{fc} + 1$, $\phi^\omega = +\infty \;\; \forall \omega \in \Omega^p$, save the values $\hat{\nu}_{e_{fc}}^\omega$ of the dual variables $\hat{\nu}^\omega$, $\omega \in \Omega^p$ and define the feasibility cut (3.10). Go to Step 1.

If $z_{FEASC}^p = 0$ (feasible scenario cluster problem) and $p < \hat{p}$, go to Step 2.

Step 3: Solve the auxiliary primal problem for all scenarios ω, $\omega \in \Omega$,

$$\begin{aligned}
(OPT) \quad \phi^\omega \quad = \quad & \min q^{\omega T} y^\omega \\
& s.t. \\
& \begin{pmatrix} W^\omega \\ -W^\omega \end{pmatrix} y^\omega \geq \begin{pmatrix} h_1^\omega - T^\omega \hat{x} \\ -h_2^\omega + T^\omega \hat{x} \end{pmatrix} \qquad (3.13) \\
& y^\omega \geq 0.
\end{aligned}$$

Set $e_o := e_o + 1$, save ϕ^ω and the values $\hat{\nu}^\omega_{e_o}$ of the dual variables ν^ω, reset $\theta := 0$ and define the optimality cut (3.11).

Step 4: Set $\phi := \sum_{\omega\in\Omega} w^\omega \phi^\omega$. If $\phi \leq \theta$ then stop, since the optimal solution has been found in k-th iteration.

Save $\theta := \theta + c\hat{x}$ and go to Step 1.

The dimensions of the cluster-based dual vector to be used for identifying the feasibility cut, problem $(FEASC)$ (3.12), are clearly greater than the dimensions for one scenario based scheme, problem (3.4). In effect, problem (3.12) has $2\cdot|\Omega^p|$ blocks of constraints for each scenario cluster p while there are two blocks of constraints for each scenario feasibility problem (3.4). However, notice that the solution to the feasibility cut identification problem for each scenario cluster forces the feasibility in more scenarios than by using the scheme for each individual scenario. Then, the scenario cluster based scheme allows us to identify tighter feasibility cuts than when using a scenario based procedure.

3.6 Computational experience

Our testbed of instances has a similar structure as the example (3.6) used in Section 3.3, but we have added some additional variables and constraints and increased as well the number of scenarios. So, we have generated a set of medium and large-scale instances that require a big number of feasibility cuts.

Let us use the following notation for the variables and constraints, where x_i denotes a first stage variable, for $i = 1, ..., n_x$ and, similarly, y^ω_j represents a second stage variable under a given scenario, for $j = 1, ..., n_y$ and $\omega \in \Omega$. Now, our problem can be written as follows,

$$\begin{array}{rcl}
z_{LO} & = & \min \sum_{i=1}^{n_x} c_i x_i + \sum_{\omega\in\Omega} w^\omega \sum_{j=1}^{n_y} q^\omega_j y^\omega_j \\
s.t. & & \\
0 & \leq & T^\omega \begin{pmatrix} x_1 \\ \vdots \\ x_{n_x} \end{pmatrix} + W^\omega \begin{pmatrix} y^\omega_1 \\ \vdots \\ y^\omega_{n_y} \end{pmatrix} \quad \forall \omega \in \Omega \\
0.8 h^\omega_j & \leq & y^\omega_j \leq h^\omega_j \quad \forall j = 1, ..., n_y, \quad \omega \in \Omega \\
0 & \leq & x_i, \quad \forall i = 1, ..., n_x,
\end{array}$$

where w^ω will be considered equiprobable, i.e., $w^\omega = \frac{1}{|\Omega|}$. The c_i coefficient is integer and uniformly distributed over $[3, 12]$ for $i = 1, ..., \frac{1}{2}n_x$, and over $[2, 11]$

for $i = \frac{1}{2}n_x + 1, ..., n_x$. The q_j^ω coefficient was randomly generated with the expression $-(28 + \frac{2\omega}{|\Omega|} \cdot a)$ for $j = 1, ..., \frac{1}{2}n_y$, and the expression $-(14 + \frac{4\omega}{|\Omega|} \cdot a)$ for $j = \frac{1}{2}n_y + 1, ..., n_y$, where a is an integer uniformly distributed over $[1, 9]$.

For each scenario $\omega \in \Omega$, the matrices T^ω and W^ω have m rows and have also been randomly generated. The $m \cdot n_x$ elements of matrix T^ω are integer and uniformly distributed over $[1, 15]$. The $m \cdot n_y$ elements of matrix W^ω have been generated with the expression $-(21 + \frac{\omega}{|\Omega|} \cdot a)$ for $j = 1, ..., \frac{1}{2}n_y$, and the expression $-(16 + \frac{\omega}{|\Omega|} \cdot a)$ for $j = \frac{1}{2}n_y + 1, ..., n_y$.

The upper bound h_{2j}^ω for the second stage variable y_j^ω was generated with the expression $(4 + \frac{\omega}{|\Omega|})$ for $j = 1, ..., \frac{1}{2}n_y$, and the expression $(6 + \frac{\omega}{10|\Omega|})$ for $j = \frac{1}{2}n_y + 1, ..., n_y$.

We report the results of the computational experience obtained while optimizing our testbed of randomly generated instances. Our algorithmic approach has been implemented in a C++ experimental code (Visual C++ 2008 Express Edition) by using CPLEX v12.2 [85] as a solver of the linear optimization relaxed master problem and auxiliary submodels at each iteration within the open source engine COIN-OR [58, 70]. For comparison purposes, plain CPLEX is used for solving the original linear full model. The computations were carried out on a HP Pavillon DV3 computer under Windows 7 operating system with 32 bits, 2.26GHz, 4Gb of RAM and 2 cores. All the programs have been attached to the memory on a CD and are listed in Appendix A with a brief description.

Tables 3.4 shows the DEM dimensions in compact representation of the instances in our testbed. The headings are as follows: nc, number of constraints, computed as $(m + n_y)|\Omega|$; nv, number of variables computed as $n_x + n_y|\Omega|$; n_x, number of first stage variables; n_y, number of second stage variables per scenario; nel, number of nonzero coefficients in the constraint matrix; and $dens$, constraint matrix density. The number of scenarios varies in the testbed, being $\{10, 100, 200, 400, 500\}$.

The headings of Tables 3.5, 3.6 and 3.7 are as follows: $\hat{p}$, number of clusters into which the set of scenarios is partitioned; $|\Omega^p|$, number of scenarios per cluster (which is the same for each cluster p in our testbed); $\#fc$, number of feasibility cuts that have been identified; $\#it$, number of iterations that are needed to reach the optimal solution; T_{CBD} and T_{CPLEX}, total computation time (secs.) for obtaining the optimal solution of the original stochastic

Table 3.4: Model dimensions. Compact representation

Instance	nc	nv	n_x	n_y	nel	$dens$	$\|\Omega\|$
P1	1200	660	60	60	72600	0.0916	10
P2	2000	1100	100	100	201000	0.0913	10
P3	320000	4040	40	40	324000	2.5e-04	100
P4	400000	5050	50	50	405000	2.0e-04	100
P5	480000	6060	60	60	486000	1.6e-04	100
P6	640000	8040	40	40	648000	1.25e-04	200
P7	800000	10050	50	50	810000	1.00e-04	200
P8	960000	12060	60	60	972000	8.39e-05	200
P9	1280000	16040	40	40	1296000	6.31e-05	400
P10	1600000	20040	40	40	1602000	4.99e-05	500
P11	2000000	25020	50	50	2025000	4.04e-05	500

problem by using the CBD algorithm for each choice of the number of clusters that we have been experimented with and by plain use of CPLEX, respectively.

Table 3.5: Performance of CBD scheme for the small instances P1 and P2

P1	$\|\Omega\| = 10$		T_{CPLEX} :	3.50	P2	$\|\Omega\| = 10$		T_{CPLEX} :	10.19
$\hat{p}$	$\|\Omega^p\|$	$\#fc$	$\#it$	T_{CBD}	$\hat{p}$	$\|\Omega^p\|$	$\#fc$	$\#it$	T_{CBD}
1	10	114	154	12.22	1	10	95	137	33.83
2	5	162	210	13.40	2	5	145	190	33.25
5	2	289	337	18.42	5	2	286	334	46.14
10	1	307	347	17.15	10	1	390	432	50.72

We can observe in Table 3.5 that the performance of the plain use of CPLEX for the small instances is better than the performance of the CBD approach, as expected. However, we are interested on testing the required number of feasibility cut iterations for obtaining the optimal solution by using the CBD approach when choosing different scenario cluster partitioning. We can also observe that the optimal choice of the number of clusters, $\hat{p}$, is the smallest one. Moreover, when the number of clusters increases, more feasibility cuts are needed to be appended to the relaxed master problem and then, more iterations in the CBD algorithm to obtain the optimal solution. Notice that when the number of clusters is the number of scenarios, $\hat{p} = |\Omega|$, the CBD scheme coincides with the TBD scheme.

Table 3.6: Performance of CBD scheme for the small-medium instances P3, P4 and P5

P3	$\|\Omega\| = 100$		T_{CPLEX} :	12.63
$\hat{p}$	$\|\Omega^p\|$	$\#fc$	$\#it$	T_{CBD}
1	100	10	12	7.17
2	50	18	20	6.16
4	25	28	30	6.32
10	10	54	56	8.18
50	2	108	110	11.04
100	1	130	132	7.49

P4	$\|\Omega\| = 100$		T_{CPLEX} :	20.921
$\hat{p}$	$\|\Omega^p\|$	$\#fc$	$\#it$	T_{CBD}
1	100	26	28	16.12
2	50	59	61	20.64
4	25	98	100	22.81
10	10	203	205	35.94
50	2	446	448	66.40
100	1	497	499	35.88
P5	$\|\Omega\| = 100$		T_{CPLEX} :	33.99
$\hat{p}$	$\|\Omega^p\|$	$\#fc$	$\#it$	T_{CBD}
1	100	20	32	21.12
2	50	34	46	22.39
4	25	54	66	33.76
10	10	103	115	40.93
50	2	218	230	62.50
100	1	265	277	77.22

We can observe in Tables 3.5-3.8 that an appropriate partitioning of the set of scenarios into clusters produces much tighter feasibility cuts identification and then, smaller computation time to obtain the optimal solution of the original stochastic problem. This result is quite remarkable, since we have obtained computation times smaller in one order of magnitude for the CBD scheme at least, than the required computation time by using the TBD scheme.

Table 3.7: Performance of CBD scheme for medium-large instances P6, P7 and P8

P6	$\|\Omega\| = 200$		T_{CPLEX} :	31.48
$\hat{p}$	$\|\Omega^p\|$	$\#fc$	$\#it$	T_{CBD}
1	200	39	41	29.57
2	100	60	62	31.10
4	50	109	111	44.17
10	20	241	243	96.16
50	4	533	535	157.23
100	2	631	633	157.23
200	1	622	624	158.03

P7	$\|\Omega\| = 200$		T_{CPLEX} :	69.06
$\hat{p}$	$\|\Omega^p\|$	$\#fc$	$\#it$	T_{CBD}
1	200	44	46	60.51
2	100	65	67	58.56
4	50	117	119	67.16
10	20	296	298	135.71
50	4	647	649	179.35
100	2	695	697	171.94
200	1	701	703	96.56
P8	$\|\Omega\| = 200$		T_{CPLEX} :	65.65
$\hat{p}$	$\|\Omega^p\|$	$\#fc$	$\#it$	T_{CBD}
1	200	16	25	37.78
2	100	30	39	38.94
4	50	55	64	46.82
10	20	102	111	77.93
50	4	269	278	131.51
100	2	308	317	142.19
200	1	338	347	160.61

We can also observe in Tables 3.6-3.8 that for the small-medium, medium-large and large size instances, the optimal choice of the number of clusters, $\hat{p}$, is the smallest, since the smaller the number of required feasibility cuts, the smaller number of iterations in the CBD scheme to obtain the optimal solution. For all those instances the required computation time by using the CBD algorithm is smaller than that required by plain use of CPLEX.

Table 3.8: Performance of CBD scheme for the large instances P9, P10 and P11

P9	$\|\Omega\| = 400$		T_{CPLEX} :	113.868
$\hat{p}$	$\|\Omega^p\|$	$\#fc$	$\#it$	T_{CBD}
1	400	18	21	41.19
2	200	27	35	45.33
8	50	163	171	108.27
20	20	306	309	196.33
50	8	615	618	343.84
100	2	703	711	390.94
400	1	759	768	383.68

P10	$\|\Omega\| = 500$		T_{CPLEX} :	85.72
$\hat{p}$	$\|\Omega^p\|$	$\#fc$	$\#it$	T_{CBD}
1	500	12	14	36.29
2	250	30	32	40.94
5	100	71	73	60.09
10	50	122	124	83.23
50	10	439	441	280.93
100	5	544	546	297.38
250	2	646	648	359.31
500	1	715	717	378.30
P11	$\|\Omega\| = 500$		T_{CPLEX} :	130.99
$\hat{p}$	$\|\Omega^p\|$	$\#fc$	$\#it$	T_{CBD}
1	500	12	14	47.24
2	250	28	30	54.25
5	100	68	70	74.62
10	50	120	122	183.23
50	10	452	454	332.66
100	5	625	627	415.41
250	2	798	800	494.47
500	1	809	811	511.38

3.7 Conclusions

We have proposed in this chapter an efficient scenario cluster decomposition approach for identifying tight feasibility cuts in Benders decomposition for solving medium-large and large scale two-stage stochastic problems where only continuous variables appear. Some computational experience is presented, where we observe the favorable performance of the proposed Cluster Benders Decomposition (CBD) approach versus the performance of the Traditional single scenario Benders Decomposition (TBD) approach.

We point out that the state-of-the-art optimization engine CPLEX requires more computation time to obtain the optimal solution than the CBD approach does in 9 out of 11 instances (i.e., the largest ones) in our testbed for a small number of clusters (in particular, $\hat{p} = 1, 2$).

So, although more computational experience is required, the new approach seems to be very promising based on our provisional results. Moreover, for a big number of clusters (in particular, $\hat{p} = |\Omega|$, i.e., the singleton cluster TBD approach), plain use of CPLEX outperforms our CBD approach.

Chapter **4**

Multistage scheme

The multistage stochastic linear problem with a finite number of possible future scenarios still has a Deterministic Equivalent Model. However, the structure of this problem is somewhat more complex than that of the two-stage problem. The extensive form does not appear readily accesible to manipulations. The aim in this chapter is to extend the proposed Cluster Benders Decompostion approach to the multistage linear case. An information structuring for scenario cluster partitioning of scenario trees is also presented, given the general model formulation of the DEM in a multistage stochastic linear problem. The basic idea consists of explicitly rewriting the nonanticipativity constraints of the variables in the stages with common information. As a result an assignment of the constraint matrix blocks into independent scenario cluster submodels is performed. Then, multistage problems can be represented as two-blocks of stages models, and the proposed Cluster Benders Decomposition (CBD) can be used as an efficient tool for its solution.

4.1 Introduction

In this chapter we present a stochastic linear optimization modeling approach and a Multistage Cluster Benders Decomposition (for short MCBD) algorithm, for solving general multistage linear optimization problems under uncertainty via scenario tree analysis. The main feature with respect to the two-stage scheme is the information structuring for generating, naming and manipulating the scenario cluster submodels.

As a result, an assigment of the constraint matrix blocks into a two blocks of stages model formulation in performed. We present a compact representation with implicit NAC for linking the submodels with common information until a given break stage; and a compact representation for each cluster submodel, to treat the implicit NAC related to each of the scenario clusters from the break stage until the last one.

The remainder of the chapter is organized as follows. Section 4.2 introduces the multistage DEM both in compact and splitting variable representation. Section 4.3 proposes a scenario-cluster partitioning to allow a combination of compact and splitting variable representations in the different stages of the problem. Section 4.4 deals with an illustrative example. Section 4.5 introduces the main concepts of scenario cluster submodels. Section 4.6 presents the innovation of the proposed two blocks of stages decomposition scheme for any multistage problem. Section 4.7 introduces in detail the multistage CBD approach that is proposed. Section 4.8 concludes.

4.2 Multistage DEM

Without loss of generality and for the sake of simplicity we will consider, in particular, the following multistage deterministic linear model

$$
\begin{aligned}
\min & \sum_{t \in \mathcal{T}} c_t x_t \\
\text{s.t. } & b_{11} \leq A_1 x_1 \leq b_{21} \\
& b_{1t} \leq A'_t x_{t-1} + A_t x_t \leq b_{2t} \quad && \forall t \in \mathcal{T} - \{1\} \\
& x_t \in I\!R^+ && \forall t \in \mathcal{T},
\end{aligned}
\tag{4.1}
$$

where $\mathcal{T}$ is the set of stages, c_t is the vector of the objective function coefficients, b_{1t} and b_{2t} are the left and right hand side vectors (for short lhs and rhs, respectively), and A'_t and A_t are the constraint matrices, respectively, for stage t. Finally, x_t is the n_{x_t} dimensional vector of continuos variables for stage t, for $t \in \mathcal{T}$.

To extend the deterministic model (4.1) to introduce uncertainty in the parameters, we will use a *scenario analysis* approach. In our case, the uncertainty can appear anywhere in the model, that is in the objective function, the left or right hand sides and the constraint matrix coefficients. To illustrate some additional concepts let us consider the different representations of a scenario tree given in Figure 4.1.

Let us consider the decision tree in the left part of the figure. It corresponds to the compact representation of the stochastic version, see *DEM* (4.2). Each node, say g, in the figure represents a point in time where a decision can be made. Once a decision is made, some contingencies may occur (e.g., in this example the number of contingencies is two for each stage t), and information related to these contingencies is available at the beginning of each stage. In this context, a stage is a point in time where a decision is made and, in some cases, can be included by a subset of consecutive time periods.

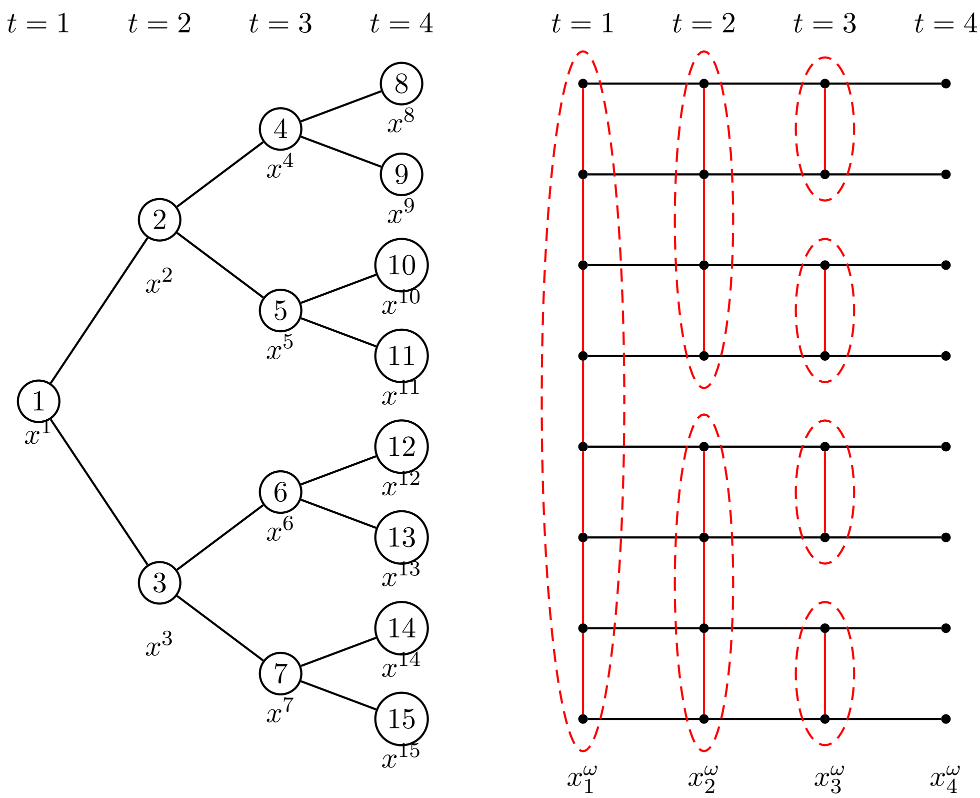


Figure 4.1: Scenario tree. Compact and splitting variable representations.

We will denote by $\mathcal{T}$, the set of stages, and by $T = |\mathcal{T}|$, their number. In this example, there are $T = 4$ stages. At each stage, there is a type of decision, vector of continuous variables x. Let also $\mathcal{G}$ denote the set of scenario groups, and $\mathcal{G}_t$ the subset of scenario groups that belong to stage t, such that $\mathcal{G} = \cup_{t \in \mathcal{T}} \mathcal{G}_t$.

The structure of this information is visualized as a tree, where each root-to-leaf path represents one specific scenario, ω, and corresponds to one realization of the whole set of the uncertain parameters. In this sense, Ω will denote the set of scenarios and w^ω will denote the likelihood or probability assigned by the modeler to scenario ω, such that $\sum_{\omega \in \Omega} w^\omega = 1$. Two scenarios belong to the same group in a given stage provided that they have the same realizations of the uncertain parameters up to the stage.

In the example of Figure 4.1, there are $|\Omega| = 8$ root-to-leaf possible paths. Moreover, each node in the tree can be associated with a scenario group, g, where $\mathcal{G}$ represents the set of scenario groups.

Following the *nonanticipativity* principle, see [76, 90] and also [18], among many others, both scenarios should have the same value for the related variables with the time index up to the given stage. Some of the elements and concepts introduced in this chapter have been taken from [39, 40].

Let us assume that the vector of the objective function values, c, the *lhs* and *rhs* vectors b_1 and b_2 respectively and the constraint matrix coefficients, A, depend on the scenario groups. So the compact representation of the linear *DEM* of the stochastic version with complete recourse of multistage problem (4.1) can be expressed

$$
\begin{aligned}
\min & \sum_{g \in \mathcal{G}} w_g (c^g x^g) \\
\text{s.t. } & b_1^1 \leq A_1 x^1 \leq b_2^1 \\
& b_1^g \leq A'_g x^{\pi(g)} + A_g x^g \leq b_2^g \quad && \forall g \in \mathcal{G} - \{1\} \\
& x^g \in I\!R^+ && \forall g \in \mathcal{G},
\end{aligned}
\tag{4.2}
$$

where w_g is the likelihood assigned by the modeler to scenario group g, such that $w_g = \sum_{\omega \in \Omega_g} w^\omega$, and the vectors and matrices of parameters c^g, b_1^g, b_2^g, A'_g and A_g depend now on each group g, for $g \in \mathcal{G}$, $\pi(g)$ is the scenario group related to the immediate ancestor node of node g associated with scenario group g in the scenario tree, such that $\pi(g) \in \mathcal{G}_{t(g)-1}$, for $g \in \mathcal{G} - \{1\}$, where $t(g)$ is the stage to which scenario group g belongs to, such that $g \in \mathcal{G}_{t(g)}$, and x^g is the copy of the x vector of variables for scenario group g. Without loss of generality, x^g will denote the n_x dimensional vector of the continuous variables.

The right part of Figure 4.1 gives the same information as the compact representation but this time using a splitting variable scheme. At each stage t we have the presentation of the nonanticipativity constraints (i.e., imposing the equality) on the variables x_t^ω for the scenarios ω that belong to the same group Ω_g, $g \in \mathcal{G}_t$, $t \in \mathcal{T}^-$. Model (4.4) gives the splitting variable representation where the constraints explicitly appear in the model. Then, the *nonanticipativity* principle can derive the set

$$\{(x_t^\omega) : x_t^\omega = x_t^{\omega'} \ \forall \omega, \omega' \in \Omega_g, \omega \neq \omega', g \in \mathcal{G}_t, t \leq T-1\}. \tag{4.3}$$

Following the nonanticipativity principle, the corresponding equalities must be satisfied for the stage t.

$$c^g = c_t^\omega = c_t^{\omega'}, A'_g = A'^{\omega_t} = A_t'^{\omega'}, A_g = A_t^\omega = A_t^{\omega'}, b_1^g = b_{1t}^\omega = b_{1t}^{\omega'}, b_2^g = b_{2t}^\omega = b_{2t}^{\omega'},$$
$$\forall \omega, \omega' \in \Omega_g, \omega \neq \omega', g \in \mathcal{G}_t, t \leq T-1$$

Observe that for a given stage t, $A_t'^\omega$ and A_t^ω are the technology matrices for the x_t variables, and these variables satisfy the nonanticipativity constraints (4.3) for each stage t.

Given that $\Omega_1 = \Omega$, it results that at first stage all the parameters and variables in the model must take the same value under each scenario. Then, wlog, we will denote this same value by c_1, b_{11}, b_{21}, A_1, A'_1 and x_1 with independece of scenario ω.

So, the *splitting variable* representation of the linear *DEM* of the stochastic version with complete recourse of the deterministic multistage problem (4.1) can be expressed

$$\begin{aligned}
&\min c_1 x_1 + \sum_{\omega \in \Omega} \sum_{t \in \mathcal{T}-\{1\}} w^\omega \left(c_t^\omega x_t^\omega \right) \\
\text{s.t. } & b_{11} \leq A_1 x_1 \leq b_{21} \\
& b_{1t}^\omega \leq A_t'^\omega x_{t-1}^\omega + A_t^\omega x_t^\omega \leq b_{2t}^\omega \quad \forall \omega \in \Omega, t \in \mathcal{T} - \{1\} \\
& x_t^\omega - x_t^{\omega'} = 0 \quad \forall \omega, \omega' \in \Omega_g : \omega \neq \omega', g \in \mathcal{G}_t, t \leq T-1 \\
& x_t^\omega \in I\!R^+ \quad \forall \omega \in \Omega, t \in \mathcal{T},
\end{aligned} \tag{4.4}$$

where w^ω is the likelihood assigned to scenario ω, c_t^ω is the row vector of the objective function coefficients, A_t^ω and $A_t'^\omega$ are the constraint matrices, and b_{1t}^ω and b_{2t}^ω are the *lhs* and *rhs* vectors respectively, for $\omega \in \Omega, t \in \mathcal{T}$.

Notice, x_t^ω is the replica of the n_x dimensional vector of continuous variable x at stage t under scenario ω.

We can see that the relaxation of the *nonanticipativity* constraints, $x_t^\omega - x_t^{\omega'} = 0$, $\forall \omega, \omega' \in \Omega_g : \omega \neq \omega', g \in \mathcal{G}_t, t \leq T-1$ in model (4.4) results in a set of $|\Omega|$ independent mixed 0-1 models, where (4.5) is the model for scenario $\omega \in \Omega$.

$$\begin{aligned} z^\omega =& \min c_1 x_1 + \sum_{t \in \mathcal{T}-\{1\}} w^\omega \left(c_t^\omega x_t^\omega \right) \\ \text{s.t. } & b_{11} \leq A_1 x_1 \leq b_{21} \\ & b_{1t}^\omega \leq A_t'^\omega x_{t-1}^\omega + A_t^\omega x_t^\omega \leq b_{2t}^\omega \qquad \forall t \in \mathcal{T} - \{1\} \\ & x_t^\omega \in I\!R^+ \qquad \forall t \in \mathcal{T}. \end{aligned} \tag{4.5}$$

In general, the information about up to the stage where the scenario submodels have common information and then, up to what stage the NAC must be explicit is saved in the subsets $\mathcal{G}_t$ and Ω_g, $g \in \mathcal{G}_t$, $t \in T$ for any multistage stochastic problem with T stages and $|\Omega|$ scenarios.

4.3 Scenario cluster partitioning

In this section we propose a scenario-cluster partitioning to allow for a combination of compact and splitting variable representations in the different stages of the problem, depending on the scenario cluster partition of choice.

It is clear that the explicit representation of the nonaticipativity constraints (4.3) is not desirable for all pairs of scenarios in order to reduce the dimensions of model. In fact, we can represent implicitly the NAC for some pairs of scenarios in order to gain computational efficiency. We will decompose the scenario tree into a subset of scenario clusters subtrees $\mathcal{P}$, each one for a scenario cluster, p, for $p \in \mathcal{P}$. Let q denote the number of scenario *clusters* to consider, i.e., $q = |\mathcal{P}|$. Let Ω^p denote the set of scenarios that belongs to a generic cluster p, where $p \in \mathcal{P}$ and $\sum_{p=1}^{q} |\Omega^p| = |\Omega|$. It is clear that the criterion for scenario clustering in the sets, say, $\Omega^1, \ldots, \Omega^q$ is instance dependent. Moreover, we favor the approach that shows higher scenario clustering for greater number of scenario groups in common. In any case, notice that $\Omega^p \bigcap \Omega^{p'} = \emptyset$, $p, p' = 1, \ldots, q : p \neq p'$ and $\Omega = \cup_{p=1}^{q} \Omega^p$.

Let also $\mathcal{G}^p \subset \mathcal{G}$ denote the set of scenario groups for cluster p, such that $\Omega_g \cap \Omega^p \neq \emptyset$ means that $g \in \mathcal{G}^p$. So, $\mathcal{G}_t^p = \mathcal{G}_t \cap \mathcal{G}^p$ denote the set of scenario groups for cluster $p \in \mathcal{P}$ in stage $t \in \mathcal{T}$.

The number of clusters, q, can be selected as a divisor of $|\Omega|$, then we have that $1 \leq |\Omega^p| = \frac{|\Omega|}{q} \leq |\Omega|$, where Ω^p gives the set of scenarios in cluster p for $p = 1, \ldots, q$. And, then, the scenario clusters are $\Omega^1 = \{1, ..., \ell\}$, $\Omega^2 = \{1 + \ell, ..., 2 \cdot \ell\}$,..., $\Omega^q = \{1 + (q-1) \cdot \ell, \ldots, q \cdot \ell\}$, where $\ell = \frac{|\Omega|}{q}$.

As we will see below, the value q will be associated with the number of stages with explicit NAC between scenario cluster submodels.

Definition 4 *A break stage t^* is a stage t such that the number of scenario clusters is $q = |\mathcal{G}_{t^*+1}|$, where $t^*+1 \in \mathcal{T}$. Observe that, in this case, any cluster $p \in \mathcal{P}$ is induced by a group $g \in \mathcal{G}_{t^*+1}$ and contains all scenarios belonging to that group, i.e., $\Omega^p = \Omega_g$.*

Definition 5 *The scenario cluster models are those that result from the relaxation of NAC until some break stage t^* in model (4.4), called t^*-decomposition.*

Notice that the choice of $t^* = 0$ corresponds to the full model and $t^* = T-1$ corresponds to the scenario partitioning.

Notice that the LO^p submodels (see below submodel (4.6)) are expressed in compact representation, for each $p \in \{1, 2, \ldots, q\}$ and contain $|\Omega^p| = \frac{|\Omega|}{q}$ scenarios. That is, fixing the q scenario clusters to decompose the problem implies fixing the stages where the nonanticipativity constraints are to be explicitly modeled and then, it implies fixing the dimensions of each cluster that themselves are modeled by a compact representation.

4.4 An illustrative example

In the example depicted in Figure 4.1 we have $|\Omega| = 2^3 = 8$ scenarios and $|\mathcal{G}| = 15$ scenario groups. Three cases will be considered for defining the t^*-

decomposition and generating the q scenario clusters where q can be chosen from the set of divisors of $|\Omega| = 8$, i.e., $q \in \{2, 4, 8\}$.

For each selection of t^*, and then of q, problems (4.2)-(4.4) will be decomposed into the scenario cluster submodels (4.6). They have equal dimensions if the branching factor of the original scenario tree is constant at each node. In any case, they are explicitly linked by the nonanticipativity constraints (4.3).

Figure 4.2 shows the problem decomposition for $t^* = 1$ i.e., in $q = 2$ scenario clusters (left tree), for $t^* = 2$ i.e., in $q = 4$ scenario clusters (central tree) and for $t^* = 3$ i.e., in $q = 8$ scenario clusters (right tree).

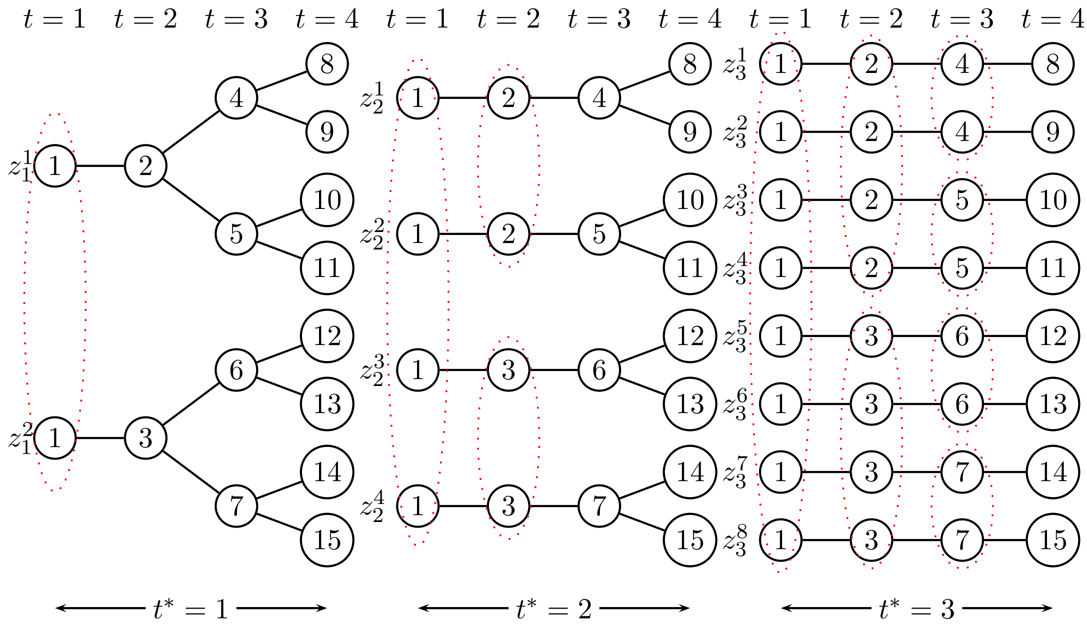


Figure 4.2: Scenario cluster partitioning for $t^* = 1$, $t^* = 2$ and $t^* = 3$

Let us analyze its left part, where the break stage is $t^* = 1$. In this case there are two clusters, namely $p = 1$ where the subset of scenario groups is $\mathcal{G}^1 = \{1, 2, 4, 5, 8, \ 9, 10, 11\}$ and $p = 2$ where the corresponding subset of scenario groups is $\mathcal{G}^2 = \{1, 3, 6, 7, 12, 13, 14, 15\}$. The scenarios in each set are $\Omega^1 = \{1, 2, 3, 4\}$ and $\Omega^2 = \{5, 6, 7, 8\}$ and they are linked by the nonanticipativity constraints related only to stage $t = 1$, i.e., $\mathbf{x}_1^p$, $p = 1, 2$ in constraints (4.3). The corresponding objective function value for cluster p in model (4.6) given this choice of the break stage is $z_{t^*}^p = z_1^p$, for $p = 1, 2$.

The central part of the figure gives the problem decomposition related to $q = 4$; i.e., by considering as break stage $t^* = 2$. In this case four clusters are considered, namely, $p = 1$, defined by the nodes $\mathcal{G}^1 = \{1, 2, 4, 8, 9\}$; $p = 2$, defined by $\mathcal{G}^2 = \{1, 2, 5, 10, 11\}$; $p = 3$, defined by $\mathcal{G}^3 = \{1, 3, 6, 12, 13\}$; and $p = 4$, defined by $\mathcal{G}^4 = \{1, 3, 7, 14, 15\}$. The corresponding set of scenarios in each cluster are, $\Omega^1 = \{1, 2\}$, $\Omega^2 = \{3, 4\}$, $\Omega^3 = \{5, 6\}$ and $\Omega^4 = \{7, 8\}$. These scenario cluster submodels are linked by the nonanticipativity constraints related to stages $t = 1, 2$, such that $\mathbf{x}_1^1 = \mathbf{x}_1^2 = \mathbf{x}_1^3 = \mathbf{x}_1^4 = \mathbf{x}_1$ and, explicitly for stage $t = 2$, $\mathbf{x}_2^1 = \mathbf{x}_2^2$ and $\mathbf{x}_2^3 = \mathbf{x}_2^4$. The corresponding objective function value of cluster p in model (4.6) given this choice of the break stage is $z_{t^*}^p = z_2^p$, for $p = 1, 2, 3, 4$.

Finally, the right part of the figure gives the problem decomposition for $q = 8$ scenario clusters, where each cluster is included by one scenario, being linked by the nonanticipativity constraints up to stage $t^* = 3$. Now, eight clusters are considered, namely, $p = 1$, defined by the set $\mathcal{G}^1 = \{1, 2, 4, 8\}$; $p = 2$, defined by $\mathcal{G}^2 = \{1, 2, 4, 9\}$; $p = 3$, defined by $\mathcal{G}^3 = \{1, 2, 5, 10\}$; $p = 4$, defined by $\mathcal{G}^4 = \{1, 2, 5, 11\}$; $p = 5$, defined by $\mathcal{G}^5 = \{1, 3, 6, 12\}$; $p = 6$, defined by $\mathcal{G}^6 = \{1, 3, 6, 13\}$; $p = 7$, defined by $\mathcal{G}^7 = \{1, 3, 7, 14\}$; and $p = 8$, defined by $\mathcal{G}^8 = \{1, 3, 7, 15\}$. The corresponding sets of scenarios in each cluster are the singleton scenario set, $\Omega^1 = \{1\}$, $\Omega^2 = \{2\}$,..., $\Omega^8 = \{8\}$. These scenario cluster submodels are linked by the nonanticipativity constraints up to stage $t^* = 3$, such that $\mathbf{x}_1^1 = \cdots = \mathbf{x}_1^8 = \mathbf{x}_1$ and, explicitly for stage $t = 2$, $\mathbf{x}_2^1 = \mathbf{x}_2^2 = \mathbf{x}_2^3 = \mathbf{x}_2^4$ and $\mathbf{x}_2^5 = \mathbf{x}_2^6 = \mathbf{x}_2^7 = \mathbf{x}_2^8$, and stage $t = 3$, $\mathbf{x}_3^1 = \mathbf{x}_3^2$ and $\mathbf{x}_3^3 = \mathbf{x}_3^4$. The corresponding objective function value of cluster p in model (4.6) given this choice of the break stage is $z_{t^*}^p = z_3^p$, for $p = 1, \cdots, 8$.

4.5 Scenario cluster submodels

Let us assume that we have broken down the scenario set into q clusters. Now, let us formulate the clusters submodels, and next the full mixed 0-1 DEM, first, via splitting variable representation, so that the submodels are linked by the explicit NAC in a first step up to stage t^*, and, in a second step, via compact representation, such that the NAC up to the break stage are implicit.

From the elements in model (4.1), and having taken into account the choice of the break stage t^*, we can define a new structure of variables and constraints to express each scenario cluster in compact representation. In order to do this, let $\mathbf{x}_t^p$ denote the vectors of the continuous variables, for scenario cluster $p \in \mathcal{P}$

and stage $t \in \mathcal{T}$. Let also nx_t^p denote the number of continuous variables for pair (p, t), respectively.

By using the two following concepts we will define from the splitting variable representation of the full model, the corresponding linear optimization submodel for each scenario cluster.

Definition 6 *The representative scenario for scenario group g at stage t in cluster p is the first ordered scenario in the scenario group,* $\omega_g^p = min\{\omega \in \Omega_g\},\ g \in \mathcal{G}_t^p,\ p \in \mathcal{P},\ t \in \mathcal{T}$.

Definition 7 *The last ordered scenario for scenario group g at stage t in cluster p is defined as* $\overline{\omega}_g^p = \max\{\omega \in \Omega_g\},\ g \in \mathcal{G}_t^p,\ p \in \mathcal{P},\ t \in \mathcal{T}$.

The set of constraints for each scenario cluster can be split into two blocks. The first block represents the constraints related to the vectors of variables until stage $t^* + 1$, (i.e., stages with explicit NAC in the full model) that must be linked with their own replicas in all of the other clusters. It includes the block of first stage constraints. The second block of constrants represents the constraints related to the vector of variables from stage $t^* + 2$, i.e., stages with implicit NAC in the full model. Accordingly, the linear optimization submodel for cluster $p \in \mathcal{P}$ can be formulated as follows,

$$
\begin{array}{lll}
(LO^p) \quad z^p = \min \sum_{t=1}^{T} \mathbf{w}_t^p \mathbf{c}_t^p \mathbf{x}_t^p & & \\
\text{s.t.} \quad \mathbf{b}_{11}^p \leq \mathbf{A}_1^p \mathbf{x}_1^p \leq \mathbf{b}_{21}^p, & & \\
\quad \mathbf{b}_{1t}^p \leq \mathbf{A}_t^{\prime p} \mathbf{x}_{t-1}^p + \mathbf{A}_t^p \mathbf{x}_t^p \leq \mathbf{b}_{2t}^p & 2 \leq t \leq t^* + 1, & \\
\quad [\mathbf{b}_{1t}]^p \leq [\mathbf{A}_t]^{\prime p} \mathbf{x}_{t-1}^p + [\mathbf{A}_t]^p \mathbf{x}_t^p \leq [\mathbf{b}_{2t}]^p & t^* + 1 < t \leq T & \\
\quad \mathbf{x}_t^p \in I\!R^{+nx_t^p} & \forall t \in \mathcal{T} &
\end{array}
\tag{4.6}
$$

The first block of constraint matrices ($\mathbf{A}_t^{\prime p}$, $\mathbf{A}_t^p$) is related to the vectors of variables $\mathbf{x}_t^p$, whose *lhs* and *rhs* are, respectively, $\mathbf{b}_{1t}^p$ and $\mathbf{b}_{2t}^p$, for $2 \leq t^* + 1$. We can define the blocks of the matrices by stages. Obviously, the matrices and vectors for the first block at first stage are as follows: $\mathbf{A}_1 := A_1$, $\mathbf{b}_{11} := b_{11}$ and $\mathbf{b}_{21} := b_{21}$.

For the stages $2 \leq t \leq t^* + 1$, the matrices for the first block are as follows: $\mathbf{A}_t^{\prime p} := A_t^{\prime \omega_g^p}$, $\mathbf{A}_t^p := A_t^{\omega_g^p}$, $\mathbf{b}_{1t}^p := b_{1t}^{\omega_g^p}$, and $\mathbf{b}_{2t}^p := b_{2t}^{\omega_g^p}$

For cluster p and stage t for $t \leq t^* + 1$, the weight $\mathbf{w}_t^p$ can be expressed as $\mathbf{w}_t^p = \sum_{\omega \in \Omega_g \,:\, g \in \mathcal{G}_t^p} w^\omega$. Similarly, we can define the objective function coefficients $\mathbf{c}_t^p$.

The second block represents the constraints for stages from $t^* + 2$ until the last stage T. In all of these stages, the nonanticipativity principle is implicity taken into account, since the submodel for each cluster is formulated via a compact representation. The constraint matrices $[\mathbf{A}_t']^p$ and $[\mathbf{A}_t]^p$ can be split into the $|\mathcal{G}_{t-1}^p|$ and $|\mathcal{G}_t^p|$ submatrices related to the scenarios groups in a given cluster p, respectively. Let the representative scenario $\omega_{g_i}^p$, see Definition 6, define the related block of matrices for scenario group $g_i \in \mathcal{G}_t^p$, $i \in \{1, \dots, |\mathcal{G}_t^p|\}$ (at stage t) in cluster p.

Notice that the matrices $[\mathbf{A}_t']^p$ have $|\mathcal{G}_{t-1}^p|$ vertical blocks, while the matrices $[\mathbf{A}_t]^p$ have $|\mathcal{G}_t^p|$ vertical blocks. It can be observed that if there are explicit NAC in stage $t - 1$, then $[\mathbf{A}_t']^p$ would be block diagonal matrices with the same number of vertical blocks as $[\mathbf{A}_t]^p$ that is, $|\mathcal{G}_t^p|$, see (4.7). But, since the NAC are implicitly considered, then the matrices become grouped matrices by columns, such that they will have in the same column the matrices $A_t'^{\omega}$ and $A_t'^{\omega'}$ for $\mathbf{x}_{t-1}^p$, respectively, where $x_{t-1}^\omega = x_{t-1}^{\omega'}$ $\forall \omega, \omega' \in \Omega_g : \omega \neq \omega',\ g \in \mathcal{G}_t^p,\ t^* + 1 < t \leq T$. Notice that these matrices can easily lose the diagonal block structure, see below.

$$[\mathbf{A}_t']^p = \underbrace{\begin{pmatrix} A_t^{\prime \omega_{g_1}^p} & 0 & \dots & 0 \\ 0 & A_t^{\prime \omega_{g_2}^p} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_t^{\prime \omega_{g_{|\mathcal{G}_{t-1}^p|}}^p} \end{pmatrix}}_{|\mathcal{G}_{t-1}^p| \text{ vertical blocks}}, \tag{4.7}$$

$$[\mathbf{A}_t]^p = \underbrace{\begin{pmatrix} A_t^{\omega_{g_1}^p} & 0 & 0 & \dots & 0 \\ 0 & A_t^{\omega_{g_2}^p} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_t^{\omega_{g_{|\mathcal{G}_t^p|}}^p} \end{pmatrix}}_{|\mathcal{G}_t^p| \text{ vertical blocks}} \tag{4.8}$$

For cluster p and stage t for $t > t^* + 1$, let the vectors

$$\mathbf{x}_t^p = \begin{pmatrix} x_t^{\omega_{g_1}^p} \\ x_t^{\omega_{g_2}^p} \\ \vdots \\ x_t^{\omega_{g_{|\mathcal{G}_t^p|}}^p} \end{pmatrix}, \mathbf{b}_{1t}^p = \begin{pmatrix} b_{1t}^{\omega_{g_1}^p} \\ b_{1t}^{\omega_{g_2}^p} \\ \vdots \\ b_{1t}^{\omega_{g_{|\mathcal{G}_t^p|}}^p} \end{pmatrix}, \mathbf{b}_{2t}^p = \begin{pmatrix} b_{2t}^{\omega_{g_1}^p} \\ b_{2t}^{\omega_{g_2}^p} \\ \vdots \\ b_{2t}^{\omega_{g_{|\mathcal{G}_t^p|}}^p} \end{pmatrix}, \tag{4.9}$$

Notice that these vectors have the dimension $|\mathcal{G}_t^p|$. So, the weight vector $\mathbf{w}_t^p$ is as follows,

$$\mathbf{w}_t^p = \left(\sum_{\omega=\omega_{g_1}^p}^{\omega_{g_2}^p - 1} w^\omega, \sum_{\omega=\omega_{g_2}^p}^{\omega_{g_3}^p - 1} w^\omega, \dots, \sum_{\omega=\omega_{g_{|\mathcal{G}_t^p|}}^p}^{\omega=\overline{\omega}_t^p} w^\omega \right), \tag{4.10}$$

where, by slightly abusing the notation, $\overline{\omega}_t^p$ denotes the last ordered scenario in cluster p at stage t. Similarly, we can define the objective function coefficients vector $\mathbf{c}_t^p$.

The q cluster submodels (4.6) are linked by the NAC, that now can be formulated as follows,

$$\mathbf{x}_t^p - \mathbf{x}_t^{p'} = 0, \quad \forall p, p' \in \mathcal{P}_g, p \neq p', g \in \mathcal{G}_t, t \leq t^* \tag{4.11}$$

where $\mathcal{P}_g$ is the set of scenario clusters that share scenario group g, i.e., $p, p' \in \mathcal{P}_g$ provided that $g \in \mathcal{G}^p \cap \mathcal{G}^{p'}$.

It is very important to point out that the expression of model (4.6) allows the so-named nonsymetric scenario trees. This type of trees allows to have a different number of succesor scenario groups for the groups of a given stage and, then, opening for the possibility that the number of constraints and variables be different from one scenario group to another for the stages after a break one, see [40]. This type of scenario trees are very common in practice but they have been considered up to now, in the open literature, probably due to its very complex treatment.

Let us consider the example depicted in the central part of Figure 4.2, where $T = 4$, $|\Omega| = 8$ and $|\mathcal{G}| = 15$. In this case, $t = 2$ is the stage t^*-decomposition and, so, $q = 4$ clusters, whose scenario groups are given in Table 4.1.

Table 4.1: Scenario groups for $q = 4$. Illustrative example

$\mathcal{G}_t^p$	$p = 1$	$p = 2$	$p = 3$	$p = 4$
$t = 1$	{1}	{1}	{1}	{1}
$t = 2$	{2}	{2}	{3}	{3}
$t = 3$	{4}	{5}	{6}	{7}
$t = 4$	{8,9}	{10,11}	{12,13}	{14,15}

The subset of scenario groups for cluster p and stage t, $\mathcal{G}_t^p$ can be determined. In this case, until $t^* + 1 = 3$ all of these subsets have a singleton element and for $t^* + 2 = 4$ these subsets have one or more elements.

Before defining the blocks of the matrices by stages, we must determine the representative scenario w_g^p, for each of the scenario groups, $g \in \mathcal{G}_t^p$ in cluster $p \in \mathcal{P}$. This information appears in Table 4.2.

Obviously, the matrices and vectors for the first stage $t = 1$ and, then, the $q = 4$ cluster models are: $\mathbf{A}_1 := A_1$, $\mathbf{b}_{11} := b_{11}$ and $\mathbf{b}_{21} := b_{21}$. For first block and stages $2 \leq t \leq t^* + 1 = 3$, they are as follows:

1. For scenario cluster $p = 1$: $\mathbf{A}_t'^{1} := A_t'^{\omega_g^1}$, $\mathbf{A}_t^1 := A_t^{\omega_g^1}$, $\mathbf{b}_{1t}^1 := b_{1t}^{\omega_g^1}$, and $\mathbf{b}_{2t}^1 := b_{2t}^{\omega_g^1}$ for $2 \leq t \leq 3$, where the representative scenario for $t = 2$, $g \in \mathcal{G}_2^1 = \{2\}$ is $\omega_2^1 = min\{\omega \in \Omega_2\} = 1$ and for $t = 3$, $g \in \mathcal{G}_3^1 = \{4\}$ is $\omega_4^1 = min\{\omega \in \Omega_4\} = 1$, see Tables 4.1 and 4.2.

2. For scenario cluster $p = 2$: $\mathbf{A}_t'^{2} := A_t'^{\omega_g^2}$, $\mathbf{A}_t^2 := A_t^{\omega_g^2}$, $\mathbf{b}_{1t}^2 := b_{1t}^{\omega_g^2}$, and $\mathbf{b}_{2t}^2 := b_{2t}^{\omega_g^2}$, for $2 \leq t \leq 3$, where the representative scenario for $t = 2$,

Table 4.2: Representative scenario for scenario group g in cluster p. 2-decomposition

w_g^p	$p=1$	$p=2$	$p=3$	$p=4$
$g=1$	$\min\{w \in \Omega_1\}=\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
$g=2$	$\min\{w \in \Omega_2\}=\{1\}$	$\{1\}$	–	–
$g=3$	–	–	$\min\{w \in \Omega_3\}=\{5\}$	$\{5\}$
$g=4$	$\min\{w \in \Omega_4\}=\{1\}$	–	–	–
$g=5$	–	$\min\{w \in \Omega_5\}=\{3\}$	–	–
$g=6$	–	–	$\min\{w \in \Omega_6\}=\{5\}$	–
$g=7$	–	–	–	$\min\{w \in \Omega_7\}=\{7\}$
$g=8$	$\min\{w \in \Omega_8\}=\{1\}$	–	–	–
$g=9$	$\min\{w \in \Omega_9\}=\{2\}$	–	–	–
$g=10$	–	$\min\{w \in \Omega_{10}\}=\{3\}$	–	–
$g=11$	–	$\min\{w \in \Omega_{11}\}=\{4\}$	–	–
$g=12$	–	–	$\min\{w \in \Omega_{12}\}=\{5\}$	–
$g=13$	–	–	$\min\{w \in \Omega_{13}\}=\{6\}$	–
$g=14$	–	–	–	$\min\{w \in \Omega_{14}\}=\{7\}$
$g=15$	–	–	–	$\min\{w \in \Omega_{15}\}=\{8\}$

$g \in \mathcal{G}_2^2 = \{2\}$ is $\omega_2^2 = min\{\omega \in \Omega_2\} = 1$ and for $t=3$, $g \in \mathcal{G}_3^2 = \{5\}$ is $\omega_5^2 = min\{\omega \in \Omega_5\} = 3$, see Tables 4.1 and 4.2.

3. For scenario cluster $p=3$: $\mathbf{A}_t'^3 := A_t'^{\omega_g^3}$, $\mathbf{A}_t^3 := A_t^{\omega_g^3}$, $\mathbf{b}_{1t}^3 := b_{1t}^{\omega_g^3}$, and $\mathbf{b}_{2t}^3 := b_{2t}^{\omega_g^3}$, $2 \leq t \leq 3$, where the representative scenario for $t=2$, $g \in \mathcal{G}_2^3 = \{3\}$ is $\omega_3^3 = min\{\omega \in \Omega_3\} = 5$ and for $t=3$, $g \in \mathcal{G}_3^3 = \{6\}$ is $\omega_6^3 = min\{\omega \in \Omega_6\} = 5$, see Tables 4.1 and 4.2.

4. For scenario cluster $p=4$: $\mathbf{A}_t'^4 := A_t'^{\omega_g^4}$, $\mathbf{A}_t^4 := A_t^{\omega_g^4}$, $\mathbf{b}_{1t}^4 := b_{1t}^{\omega_g^4}$, and $\mathbf{b}_{2t}^4 := b_{2t}^{\omega_g^4}$, $2 \leq t \leq 3$, where the representative scenario for $t=2$, $g \in \mathcal{G}_2^4 = \{3\}$ is $\omega_3^4 = min\{\omega \in \Omega_3\} = 5$ and for $t=3$, $g \in \mathcal{G}_3^4 = \{7\}$ is $\omega_7^4 = min\{\omega \in \Omega_7\} = 7$, see Tables 4.1 and 4.2.

The matrices for the second block and for stage $t^*+1 = 3 < t \leq 4$, $[\mathbf{A}_t']^p$ and $[\mathbf{A}_t]^p$ are as follows :

1. For scenario cluster $p=1$: For scenario group $g_i \in \mathcal{G}_4^1 = \{8,9\}$, $i \in \{1,\ldots,|\mathcal{G}_4^1|\} = \{1,2\}$, the representative scenario $\omega_{g_i}^1 = min\ \{\omega \in \Omega_{g_i}\}$ for group g_i is $\omega_8^1 = min\ \{\omega \in \Omega_8\} = 1$ for group $g_1 = 8$ and $\omega_9^1 = min\ \{\omega \in \Omega_9\} = 2$ for group $g_2 = 9$.

Due to the NAC in stage $t = 3$, $x_3^1 = x_3^2 = \mathbf{x}_3^1$, the corresponding block of coefficients for that stage has only one column and is defined as: $[\mathbf{A}_4']^1 = \begin{pmatrix} A_4'^1 \\ A_4'^2 \end{pmatrix}$. The corresponding vector of the x variables for stage $t = 4$ is, $\mathbf{x}_4^1 = \begin{pmatrix} x_4^1 \\ x_4^2 \end{pmatrix}$ and the matrix of coefficients for the corresponding block is $[\mathbf{A}_4]^1 = \begin{pmatrix} A_4^1 & 0 \\ 0 & A_4^2 \end{pmatrix}$.

2. For scenario cluster $p = 2$: For each scenario group $g_i \in \mathcal{G}_4^2 = \{10, 11\}$, $i \in \{1, \ldots, |\mathcal{G}_4^2|\} = \{1, 2\}$, the representative scenarios are $\omega_{10}^2 = min\ \{\omega \in \Omega_{10}\} = 3$ for group $g_1 = 10$, and $\omega_{11}^2 = min\ \{\omega \in \Omega_{11}\} = 4$ for group $g_1 = 11$. Then, the corresponding blocks of matrices are, $[\mathbf{A}_4']^2 = \begin{pmatrix} A_4'^3 \\ A_4'^4 \end{pmatrix}$, and $[\mathbf{A}_4]^2 = \begin{pmatrix} A_4^3 & 0 \\ 0 & A_4^4 \end{pmatrix}$. Due to the NAC in stage $t = 3$, $x_3^3 = x_3^4 = \mathbf{x}_3^2$, and the corresponding vectors of the x variables are $\mathbf{x}_3^2 = x_3^3$, and $\mathbf{x}_4^2 = \begin{pmatrix} x_4^3 \\ x_4^4 \end{pmatrix}$.

3. For scenario cluster $p = 3$: For scenario group $g_i \in \mathcal{G}_4^3 = \{12, 13\}$, $i \in \{1, \ldots, |\mathcal{G}_4^3|\} = \{1, 2\}$, the representative scenarios are $\omega_{12}^3 = min\ \{\omega \in \Omega_{12}\} = 5$ for group $g_1 = 12$, and $\omega_{13}^3 = min\ \{\omega \in \Omega_{13}\} = 6$ for group $g_1 = 13$. The corresponding blocks are $[\mathbf{A}_4']^3 = \begin{pmatrix} A_4'^5 \\ A_4'^6 \end{pmatrix}$, $[\mathbf{A}_4]^3 = \begin{pmatrix} A_4^5 & 0 \\ 0 & A_4^6 \end{pmatrix}$ and the corresponding vectors of the x variables are $\mathbf{x}_3^3 = x_3^5 = x_3^6$ and $\mathbf{x}_4^3 = \begin{pmatrix} x_4^5 \\ x_4^6 \end{pmatrix}$.

4. For scenario cluster $p = 4$: For scenario group $g_i \in \mathcal{G}_4^4 = \{14, 15\}$, $i \in \{1, \ldots, |\mathcal{G}_4^1|\} = \{1, 2\}$, the representative scenarios are $\omega_{14}^4 = min\ \{\omega \in \Omega_{14}\} = 7$ for group $g_3 = 14$ and $\omega_{15}^4 = min\ \{\omega \in \Omega_{15}\} = 8$ for group $g_1 = 15$.

   Due to NAC at stage $t = 3$, $\mathbf{x}_3^3 = x_3^7 = x_3^8$, and at stage $t = 4$, $\mathbf{x}_4^3 = \begin{pmatrix} x_4^7 \\ x_4^8 \end{pmatrix}$. The corresponding blocks of matrices are, $[\mathbf{A}_4']^4 = \begin{pmatrix} A_4'^7 \\ A_4'^8 \end{pmatrix}$, and $[\mathbf{A}_4]^4 = \begin{pmatrix} A_4^7 & 0 \\ 0 & A_4^8 \end{pmatrix}$.

The constraint matrix structure of the q cluster submodels in the $t^* = 2$-decomposition is given by the system

$$
\begin{pmatrix} b_{11} \\ b_{12}^1 \\ b_{13}^1 \\ b_{14}^1 \\ b_{14}^2 \end{pmatrix}
\leq
\begin{pmatrix}
A_1 & & & & \\
A_2^{\prime 1} & A_2^1 & & & \\
 & A_3^{\prime 1} & A_3^1 & & \\
 & & A_4^{\prime 1} & A_4^1 & \\
 & & A_4^{\prime 2} & & A_4^2
\end{pmatrix}
\cdot
\begin{pmatrix} x_1 \\ x_2^1 \\ x_3^1 \\ x_4^1 \\ x_4^2 \end{pmatrix}
\leq
\begin{pmatrix} b_{21} \\ b_{22}^1 \\ b_{23}^1 \\ b_{24}^1 \\ b_{24}^2 \end{pmatrix}
\tag{4.12}
$$

$$
\begin{pmatrix} b_{11} \\ b_{12}^1 \\ b_{13}^3 \\ b_{14}^3 \\ b_{14}^4 \end{pmatrix}
\leq
\begin{pmatrix}
A_1 & & & & \\
A_2^{\prime 1} & A_2^1 & & & \\
 & A_3^{\prime 3} & A_3^3 & & \\
 & & A_4^{\prime 3} & A_4^3 & \\
 & & A_4^{\prime 4} & & A_4^4
\end{pmatrix}
\cdot
\begin{pmatrix} x_1 \\ x_2^1 \\ x_3^3 \\ x_4^3 \\ x_4^4 \end{pmatrix}
\leq
\begin{pmatrix} b_{21} \\ b_{22}^1 \\ b_{23}^3 \\ b_{24}^3 \\ b_{24}^4 \end{pmatrix}
\tag{4.13}
$$

$$
\begin{pmatrix} b_{11} \\ b_{12}^5 \\ b_{13}^5 \\ b_{14}^5 \\ b_{14}^6 \end{pmatrix}
\leq
\begin{pmatrix}
A_1 & & & & \\
A_2^{\prime 5} & A_2^5 & & & \\
 & A_3^{\prime 5} & A_3^5 & & \\
 & & A_4^{\prime 5} & A_4^5 & \\
 & & A_4^{\prime 6} & & A_4^6
\end{pmatrix}
\cdot
\begin{pmatrix} x_1 \\ x_2^5 \\ x_3^5 \\ x_4^5 \\ x_4^6 \end{pmatrix}
\leq
\begin{pmatrix} b_{21} \\ b_{22}^1 \\ b_{23}^1 \\ b_{24}^5 \\ b_{24}^6 \end{pmatrix}
\tag{4.14}
$$

$$
\begin{pmatrix} b_{11} \\ b_{12}^5 \\ b_{13}^7 \\ b_{14}^7 \\ b_{14}^8 \end{pmatrix}
\leq
\begin{pmatrix}
A_1 & & & & \\
A_2^{\prime 5} & A_2^5 & & & \\
 & A_3^{\prime 7} & A_3^7 & & \\
 & & A_4^{\prime 7} & A_4^7 & \\
 & & A_4^{\prime 8} & & A_4^8
\end{pmatrix}
\cdot
\begin{pmatrix} x_1 \\ x_2^5 \\ x_3^7 \\ x_4^7 \\ x_4^8 \end{pmatrix}
\leq
\begin{pmatrix} b_{21} \\ b_{22}^5 \\ b_{23}^7 \\ b_{24}^7 \\ b_{24}^8 \end{pmatrix}
\tag{4.15}
$$

The notation in (4.12) - (4.15) is $A_t^{\prime\omega}$, A_t^ω: scenario matrices, b_{1t}^ω, b_{2t}^ω: scenario *lhs* and *rhs* for the corresponding stage and x_t^ω: scenario continuous variables, respectively.

The t^*-decomposition in scenario clusters of the DEM (4.2) can be given alternatively by a splitting variable representation, for explicitly satisfying the NAC between the cluster submodels until stage t^*, or by a compact representation, when the NAC are implicit, as in case of (4.12) - (4.15).

4.6 Multistage full model with two blocks of stages structure

By using the compact representation of the t^*-descompostion in scenario clusters given in the example (4.12) - (4.15) and reordering some of the rows, we obtain the following two blocks of stages structure matrix coefficients for the full model,

$$
\begin{pmatrix} b_{11} \\ b_{12}^1 \\ b_{12}^5 \\ b_{13}^1 \\ b_{13}^3 \\ b_{13}^5 \\ b_{13}^7 \end{pmatrix}
\leq
\begin{pmatrix}
A_1 & & & & & & \\
A_2^{\prime 1} & A_2^1 & & & & & \\
A_2^{\prime 5} & & A_2^5 & & & & \\
 & A_3^{\prime 1} & & A_3^1 & & & \\
 & A_3^{\prime 3} & & & A_3^3 & & \\
 & & A_3^{\prime 5} & & & A_3^5 & \\
 & & A_3^{\prime 7} & & & & A_3^7
\end{pmatrix}
\cdot
\begin{pmatrix} x_1 \\ x_2^1 \\ x_2^5 \\ x_3^1 \\ x_3^3 \\ x_3^5 \\ x_3^7 \end{pmatrix}
\leq
\begin{pmatrix} b_{21} \\ b_{22}^1 \\ b_{22}^5 \\ b_{23}^1 \\ b_{23}^3 \\ b_{23}^5 \\ b_{23}^7 \end{pmatrix}
$$

$$
\begin{pmatrix} b_{14}^1 \\ b_{14}^2 \\ b_{14}^3 \\ b_{14}^4 \\ b_{14}^5 \\ b_{14}^6 \\ b_{14}^7 \\ b_{14}^8 \end{pmatrix}
\leq
\begin{pmatrix}
A_4^{\prime 1} & A_4^1 & & & & & & & & & & \\
A_4^{\prime 2} & & A_4^2 & & & & & & & & & \\
 & & & A_4^{\prime 3} & A_4^3 & & & & & & & \\
 & & & A_4^{\prime 4} & & A_4^4 & & & & & & \\
 & & & & & & A_4^{\prime 5} & A_4^5 & & & & \\
 & & & & & & A_4^{\prime 6} & & A_4^6 & & & \\
 & & & & & & & & & A_4^{\prime 7} & A_4^7 & \\
 & & & & & & & & & A_4^{\prime 8} & & A_4^8
\end{pmatrix}
\cdot
\begin{pmatrix} x_3^1 \\ x_4^1 \\ x_4^2 \\ x_3^3 \\ x_4^3 \\ x_4^4 \\ x_3^5 \\ x_4^5 \\ x_4^6 \\ x_3^7 \\ x_4^7 \\ x_4^8 \end{pmatrix}
\leq
\begin{pmatrix} b_{24}^1 \\ b_{24}^2 \\ b_{24}^3 \\ b_{24}^4 \\ b_{24}^5 \\ b_{24}^6 \\ b_{24}^7 \\ b_{24}^8 \end{pmatrix}
$$

Observe in the first block of constraints that we have just considered the matrices of coefficients for some of the scenario clusters for stages $t \leq t^* + 1$; clusters $p = 1$ and 3 in the case of the example. New elements must be defined in order to generalize the formulation of the multistage DEM in terms of these two blocks. Notice that $\mathcal{P}$ is the set of the q scenario clusters.

Definition 8 *A representative scenario cluster, say p_g^t for scenario group g at stage t is the first ordered scenario cluster associated with group g, then,* $p_g^t = \{\min\ p \mid g \in \mathcal{G}_t^p,\ p \in \mathcal{P}\}$

Let us consider the representative cluster set, $\mathcal{P}^t$ for stage $t \in \mathcal{T}$. Each element in set $\mathcal{P}^t$ is the representative scenario cluster of the clusters that belong to any group g at stage t, such that $\mathcal{P}^t = \{p_1^t,\ p_2^t, \dots p_{|\mathcal{G}_t|}^t\}$. See also that $\mathcal{P}^t = \mathcal{P}, \forall t > t^*$. For stages $t \in \{1, \dots, t^*\}$, the number of elements in such set coincides with the number of scenario groups, i.e., $|\mathcal{P}^t| = |\mathcal{G}_t|$, in particular, $\mathcal{P}^1 = \{1\}$. Moreover, each set is included in the corresponding set for the next stage, $\mathcal{P}^1 \subset \mathcal{P}^2 \subset \dots \subset \mathcal{P}^{t^*} \subset \mathcal{P}^{t^*+1} \subseteq \dots \subseteq \mathcal{P}^T = \mathcal{P}$.

In the illustrative example depicted in in the central part of Figure 4.2, the set of representative clusters are: $\mathcal{P}^1 = \{1\}$, $\mathcal{P}^2 = \{1, 3\}$ and $\mathcal{P}^3 = \mathcal{P}^4 = \mathcal{P} = \{1, 2, 3, 4\}$.

Definition 9 *The ancestor scenario cluster, say ϕ_t^p, of cluster $p \in \mathcal{P}$ for $t \in \mathcal{T}$ is the representative cluster in stage $t-1$ for the ancestor scenario group of the group in stage t for which cluster p is the representative scenario cluster.*

By using the previous elements and given the break stage t^*, the full multistage DEM can be formulated in a compact representation, as follows,

$$
\begin{aligned}
(DEM)\quad z =& \min \mathbf{c}_1\mathbf{x}_1 + \sum_{t=2}^{t^*} \sum_{p=p_1^t}^{p_{|\mathcal{P}^t|}^t} \mathbf{w}_t^p(\mathbf{c}_t^p\mathbf{x}_t^p) + \sum_{t=t^*+1}^{T} \sum_{p=1}^{q} \mathbf{w}_t^p(\mathbf{c}_t^p\mathbf{x}_t^p) \\
\text{s.t.}\quad & \mathbf{b}_{11} \le \mathbf{A}_1\mathbf{x}_1 \le \mathbf{b}_{21}, \\
& \mathbf{b}_{1t}^p \le \mathbf{A}_t'^p\mathbf{x}_{t-1}^{\phi_t^p} + \mathbf{A}_t^p\mathbf{x}_t^p \le \mathbf{b}_{2t}^p \quad \forall p \in \mathcal{P}^t, \quad 2 \le t \le t^*+1, \\
& [\mathbf{b}_{1t}]^p \le [\mathbf{A}_t]'^p\mathbf{x}_{t-1}^p + [\mathbf{A}_t]^p\mathbf{x}_t^p \le [\mathbf{b}_{2t}]^p \quad \forall p \in \mathcal{P}, \quad t^*+1 < t \le T, \\
& \mathbf{x}_t^p \in I\!R^{+nx_t^p} \quad \forall p \in \mathcal{P}, \quad t \in \mathcal{T},
\end{aligned}
\tag{4.16}
$$

In order to present the new scheme of the Cluster Benders Decomposition for multistage problems, we will use a similar notation to the two-stage model (3.1). The generic first stage matrix coefficients in a two stage problem is denoted by $\mathbf{A}$. In the illustrative example with the 2−decompostion we will consider,

$$\mathbf{A} = \begin{pmatrix} A_1 & & & & & & \\ A_2^{'1} & A_2^1 & & & & & \\ A_2^{'5} & & A_2^5 & & & & \\ & A_3^{'1} & & A_3^1 & & & \\ & A_3^{'3} & & & A_3^3 & & \\ & & A_3^{'5} & & & A_3^5 & \\ & & A_3^{'7} & & & & A_3^7 \end{pmatrix} =$$

$$= \begin{pmatrix} \mathbf{A}_1 & & & & & & \\ \mathbf{A}_2^{'p=1} & \mathbf{A}_2^{p=1} & & & & & \\ \mathbf{A}_2^{'p=3} & & \mathbf{A}_2^{p=3} & & & & \\ & \mathbf{A}_3^{'p=1} & & \mathbf{A}_3^{p=1} & & & \\ & \mathbf{A}_3^{'p=2} & & & \mathbf{A}_3^{p=2} & & \\ & & \mathbf{A}_3^{'p=3} & & & \mathbf{A}_3^{p=3} & \\ & & \mathbf{A}_3^{'p=4} & & & & \mathbf{A}_3^{p=4} \end{pmatrix}$$

corresponding to the vectors of the first block of variables,

$$\begin{pmatrix} x_1 \\ x_2^1 \\ x_2^5 \\ x_3^1 \\ x_3^3 \\ x_3^5 \\ x_3^7 \end{pmatrix} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2^{p=1} \\ \mathbf{x}_2^{p=3} \\ \mathbf{x}_3^{p=1} \\ \mathbf{x}_3^{p=2} \\ \mathbf{x}_3^{p=3} \\ \mathbf{x}_3^{p=4} \end{pmatrix}$$

The generic second stage technology matrix coefficients in a two stage problem is denoted by $\mathbf{T}^p$, for $p \in \mathcal{P}$. It is the matrix of coefficientes for constraints of stage $t^* + 2$ corresponding to variables of the predecesor stage, $t^* + 1$, where t^* is the break stage considered.

Notice that each row in matrix T corresponds to each cluster block p, and the nonnegative elements of matrix T^p are denoted by $T^{*p} = [A_{t^*+2}]'^{\phi^p_{t^*+2}}$, where $p \in \mathcal{P}$, t^* is the break stage and ϕ^p_t denotes the ancestor cluster of cluster p in stage $t-1$.

In our example, we consider

$$\mathbf{T} = \begin{pmatrix} A_4'^1 & & & \\ A_4'^2 & & & \\ & A_4'^3 & & \\ & A_4'^4 & & \\ & & A_4'^5 & \\ & & A_4'^6 & \\ & & & A_4'^7 \\ & & & A_4'^8 \end{pmatrix} = \begin{pmatrix} [\mathbf{A}_4]'^{p=1} & & & \\ & [\mathbf{A}_4]'^{p=2} & & \\ & & [\mathbf{A}_4]'^{p=3} & \\ & & & [\mathbf{A}_4]'^{p=4} \end{pmatrix} =$$

$$= \begin{pmatrix} \mathbf{T}^{p=1} \\ \mathbf{T}^{p=2} \\ \mathbf{T}^{p=3} \\ \mathbf{T}^{p=4} \end{pmatrix}$$

corresponding to the t^*+1 stage variables, where $p = 1 = \phi_4^1$ is the ancestor of cluster 1 in stage 3, $p = 2 = \phi_4^2$ is the ancestor of cluster 2 in stage 3, $p = 3 = \phi_4^3$ is the ancestor of cluster 3 in stage 3 and $p = 4 = \phi_4^4$ is the ancestor of cluster 4 in stage 3. In our case,

$$\begin{pmatrix} x_3^1 \\ x_3^3 \\ x_3^5 \\ x_3^7 \end{pmatrix} = \begin{pmatrix} \mathbf{x}_3^{p=1} \\ \mathbf{x}_3^{p=2} \\ \mathbf{x}_3^{p=3} \\ \mathbf{x}_3^{p=4} \end{pmatrix}.$$

And the generic second stage recourse matrix coefficients in a two stage problem is denoted by $\mathbf{W}^p$, for $p \in \mathcal{P}$. It is the matrix of coefficientes corresponding to the block of variables from stage t^*+2 until T, where t^* is the break stage considered. In our case it is defined by,

$$\mathbf{W} = \begin{pmatrix} A_4^1 & & & & & & & \\ & A_4^2 & & & & & & \\ & & A_4^3 & & & & & \\ & & & A_4^4 & & & & \\ & & & & A_4^5 & & & \\ & & & & & A_4^6 & & \\ & & & & & & A_4^7 & \\ & & & & & & & A_4^8 \end{pmatrix} =$$

$$= \begin{pmatrix} [\mathbf{A}_4]^{p=1} & & & \\ & & & \\ & [\mathbf{A}_4]^{p=2} & & \\ & & [\mathbf{A}_4]^{p=3} & \\ & & & [\mathbf{A}_4]^{p=4} \end{pmatrix}$$

corresponding to the second block variables, in our case,

$$\begin{pmatrix} x_4^1 \\ x_4^2 \\ x_4^3 \\ x_4^4 \\ x_4^5 \\ x_4^6 \\ x_4^7 \\ x_4^8 \end{pmatrix} = \begin{pmatrix} \mathbf{x}_4^{p=1} \\ \mathbf{x}_4^{p=2} \\ \mathbf{x}_4^{p=3} \\ \mathbf{x}_4^{p=4} \end{pmatrix}.$$

All these matrices can be represented in the multistage tree of the example as in Figure 4.6.

Given a break stage t^*, the full multistage DEM can always be formulated

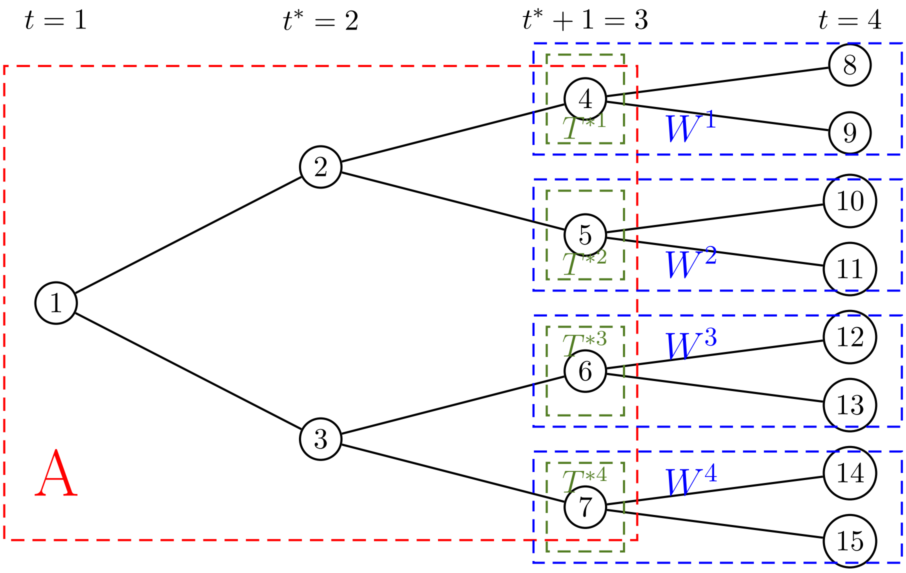


Figure 4.3: Illustrative example. Specification of the matrices.

in a compact representation as a two-blocks formulation, given by,

$$
\begin{aligned}
(DEM) \quad & z = \min \mathbf{c}^t\mathbf{x} + E_\psi[\mathbf{w}^p(\mathbf{q}^p\mathbf{y}^p)] \\
\text{s.t.} \quad & \mathbf{B}_1 \leq \mathbf{A}\mathbf{x} \leq \mathbf{B}_2, \\
& \mathbf{h}_1^p \leq \mathbf{T}^p\mathbf{x} + \mathbf{W}^p\mathbf{y}^p \leq \mathbf{h}_2^p \quad \forall p \in \mathcal{P}, \\
& \mathbf{x}, \mathbf{y}^p \in I\!R^+ \quad \forall p \in \mathcal{P},
\end{aligned}
\tag{4.17}
$$

where $\mathbf{x}$ denotes the vector of the first block of variables, defined in this case by

$$
\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2^{p=1\in\mathcal{P}^2} \\ \dots \\ \mathbf{x}_2^{p=3=|\mathcal{P}^2|} \\ \dots \\ \mathbf{x}_{t^*+1}^{p=p_1^{t^*+1}} \\ \dots \\ \mathbf{x}_{t^*+1}^{p=|\mathcal{P}^{t^*+1}|} \end{pmatrix}.
$$

$\mathbf{c}$ is the corresponding vector of the function coefficients for the $\mathbf{x}$ variables, such that, $\mathbf{c}^t\mathbf{x} = \sum_{t=1}^{t^*}\sum_{p=p_1^t}^{p_{|\mathcal{P}^t|}^t} \mathbf{w}_t^p(\mathbf{c}_t^p\mathbf{x}_t^p)$, $\mathbf{B}_1$ and $\mathbf{B}_2$ are the left and right hand side vectors for the first block constraints, respectively, $\mathbf{A}$ is the first block constraint matrix and $\mathbf{y}^p$, for $p \in \mathcal{P}$, denotes the vector of the second block variables, given in a general t^*-decomposition by,

$$\begin{pmatrix} \mathbf{x}_{t^*+1}^{p=1} \\ \mathbf{x}_{t^*+1}^{p=2} \\ \dots \\ \mathbf{x}_{t^*+1}^{p=|\mathcal{P}|} \\ \dots \\ \mathbf{x}_T^{p=1} \\ \mathbf{x}_T^{p=2} \\ \dots \\ \mathbf{x}_T^{p=|\mathcal{P}|} \end{pmatrix}.$$

Notice that $E_\psi[\mathbf{w}^p(\mathbf{q}^p\mathbf{y}^p)] = \sum_{t=t^*+1}^{T}\sum_{p=1}^{q} \mathbf{w}_t^p(\mathbf{c}_t^p\mathbf{x}_t^p)$ is the mathematical expectation with respect to ψ over the set of scenario clusters $\mathcal{P}$, $\mathbf{h}_1^p$ and $\mathbf{h}_2^p$ denote the left and right hand side vectors of the second block of constraints, respectively, for each scenario cluster $p \in \mathcal{P}$, $\mathbf{w}^p$ is the likelihood attibuted to scenario cluster p, $\mathbf{q}^p$ is the vector of objective function coefficients for the y variables, $\mathbf{T}^p$ is the technology matrix and $\mathbf{W}^p$ is the recourse matrix under scenario cluster $p \in \mathcal{P}$. Given a generic t^*-decomposition, they are defined as,

$$\mathbf{T}^p = \begin{pmatrix} \underbrace{0 \ \dots \ 0}_{\text{clusters } 1 \dots \text{p-1}} & [\mathbf{A}_{t^*+2}]'^{\phi_{t^*+2}^p} & \underbrace{0 \ \dots \ 0}_{\text{clusters p+1} \dots |\mathcal{P}|} \end{pmatrix}$$

$$\mathbf{W}^p = \begin{pmatrix} [\mathbf{A}_{t^*+2}]^p & & & & \\ [\mathbf{A}_{t^*+3}]'^p & [\mathbf{A}_{t^*+3}]^p & & & \\ & & \dots & & \\ & & & [\mathbf{A}_T]'^p & [\mathbf{A}_T]^p \end{pmatrix}$$

for each scenario cluster $p \in \mathcal{P}$.

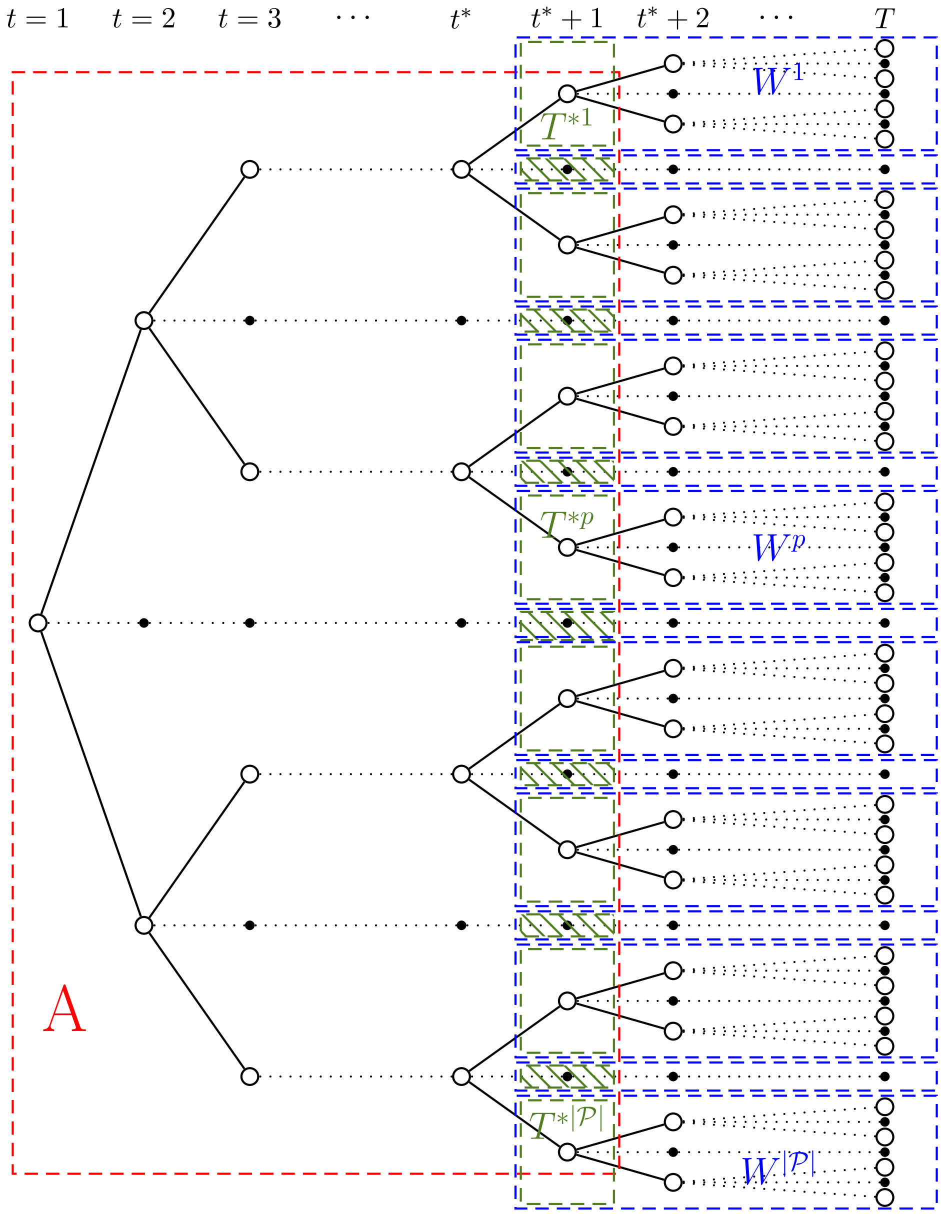


Figure 4.4: Two-blocks decomposition. Specification of the matrices, where T^{*p} is the nozero submatrix of T^p.

4.7 Multistage Cluster Benders Decomposition

Before executing the proposed algorithm for solving the original multistage stochastic linear problem, it is neccesary to fix the data structuring in order to build a two block of stages structure, see Table 4.3.

Table 4.3: Data structuring

Step 0:	Define the scenario tree: $\mathcal{T}$, Ω, $\mathcal{G}$, $\mathcal{G}_t$, $\forall t \in \mathcal{T}$, Ω_g, $\forall g \in \mathcal{G}$ and w^ω, $\forall \omega \in \Omega$.
Step 1:	Decide the break stage t^* and then, the number of scenario clusters $q = \lvert\mathcal{G}_{t^*+1}\rvert$.
Step 2:	Define $\mathcal{G}^p$, $\mathcal{G}_t^p$, Ω^p, ω_g^p, $\mathcal{P}^t$, ϕ_t^p and n_{xt}^p $\forall t \in \mathcal{T}, p \in \mathcal{P}$.
Step 3:	Generate the cluster models (4.6)
Step 4:	Generate the full model (4.16)

The *Multistage Cluster Benders Decomposition* (MCBD) algorithm works over a structure in which we have split the set $\mathcal{T}$ of stages into two parts or blocks, the first one includes the stages $t = 1, ..., t^* + 1$, and the second part includes the other stages in set $\mathcal{T}$, from $t^* + 2$ until T. Variables in stage $t^* + 1$ link both blocks, where t^* is the break stage. Then, the scenario tree is decomposed into a two-blocks scenario cluster tree, where the NAC are implicitly satisfied until stage t^* in the full model, since they are used a compact representation for these stages, and it is implicitly satisfied into each scenario cluster model from stage $t^* + 1$ until the last stage, T.

Multistage CBD procedure

Step 0: Set $k := e_o := e_f := 0$, where e_o and e_f are for counting the number of optimality and feasibility cuts along the iterations of the algorithm, respectively.

Step 1: Set $k := k + 1$. Solve the program (RMP) (with $\theta = 0$ if $e_o = 0$).

$$
\begin{aligned}
(RMP) \quad & \min \quad \mathbf{c}^T\mathbf{x} + \theta \\
& \text{s.t.} \\
& \mathbf{B}_1 \leq \mathbf{A}\mathbf{x} \leq \mathbf{B}_2 \\
& 0 \geq \hat{\nu}_{j_1}^{pT} \begin{pmatrix} \mathbf{h}_1^p + \mathbf{T}^p\mathbf{x} \\ -\mathbf{h}_2^p + \mathbf{T}^p\mathbf{x} \end{pmatrix} \quad \forall j_1 = 0, ..., e_f & (4.18) \\
& \theta \geq \sum_{p \in \mathcal{P}} \mathbf{w}^p \hat{\nu}_{j_2}^{pT} \begin{pmatrix} \mathbf{h}_1^p + \mathbf{T}^p\mathbf{x} \\ -\mathbf{h}_2^p + \mathbf{T}^p\mathbf{x} \end{pmatrix} \quad \forall j_2 = 0, ..., e_o & (4.19) \\
& \mathbf{x} \geq 0, \theta \in I\!R,
\end{aligned}
$$

where $\hat{\nu}_{j_1}$ and $\hat{\nu}_{j_2}$ are the values of the corresponding dual variables (i.e., simplex multipliers) obtained in the feasibility (Step 2) and auxiliary primal (Step 3) problems, respectively.

Save the optimal solution $\hat{\mathbf{x}}$ and $\hat{\theta}$ of the primal variables $\mathbf{x}$ and θ, respectively.

Step 2: For each scenario cluster $p \in \mathcal{P}$, solve the following feasibility problem

$$
\begin{aligned}
(FEAS) \quad & z_{FEAS}^p = \min e^T v_1^{+p} + e^T v_1^{-p} + e^T v_2^{+p} + e^T v_2^{-p} \\
& s.t. \\
& \mathbf{W}^p\mathbf{y}^p - Iu^{-p} + Iv_1^{+p} - Iv_1^{-p} = \mathbf{h}_1^p - \mathbf{T}^p\hat{\mathbf{x}} & (4.20) \\
& \mathbf{W}^p\mathbf{y}^p + Iu^{+p} - Iv_2^{+p} + Iv_2^{-p} = \mathbf{h}_2^p - \mathbf{T}^p\hat{\mathbf{x}} \\
& \mathbf{y}^p, v_1^{+p}, v_1^{-p}, v_2^{+p}, v_2^{-p}, u^{+p}, u^{-p} \geq 0.
\end{aligned}
$$

If there is a scenario cluster p, such that $z_{FEAS}^p \neq 0$ (infeasible cluster problem), set $e_f := e_f + 1$, $\phi^p = +\infty$, save the values $\hat{\nu}_{e_f}^p$ of the dual variables ν^p, define the feasibility cut (4.18) and go to Step 1.

If $z_{FEAS}^p = 0$ (feasible) $\forall p \in \mathcal{P}$, go to Step 3.

Step 3: For each scenario cluster $p \in \mathcal{P}$, solve the auxiliary primal problem

$$
\begin{aligned}
(OPT) \quad \phi^p \; = \; & \min \mathbf{q}^{pT}\mathbf{y}^p \\
& s.t. \\
& \begin{pmatrix} \mathbf{W}^p \\ -\mathbf{W}^p \end{pmatrix} \mathbf{y}^p \geq \begin{pmatrix} \mathbf{h}_1^p - \mathbf{T}^p\hat{\mathbf{x}} \\ -\mathbf{h}_2^p + \mathbf{T}^p\hat{\mathbf{x}} \end{pmatrix} & (4.21) \\
& \mathbf{y}^p \geq 0.
\end{aligned}
$$

Save the objective function value, ϕ^p and the simplex multipliers associated with the optimal solution of problem (4.21), $\hat{\nu}_{e_o}^p$, and define the optimality cut.

Set $\phi := \sum_{p\in\mathcal{P}}(\sum_{t=t^*+1}^{T} w_t^p)\phi^p$. If $\phi \leq \hat{\theta}$ then stop, since the optimal solution has been found in k-th iteration.

In other case, set $e_o := e_o + 1$, add the new cut to the constraint set (4.19) and return to Step 1.

4.8 Conclusions

In this chapter a general multistage stochastic modeling approach and an algorithmic framework, named *Multistage Cluster Benders Decomposition*, have been proposed for solving multistage problems with continuous variables under uncertainty in the parameters, being the uncertainty represented by (symetric or nonsymetric) scenarios trees. It can appear in any coefficient of the objective function, constraint matrix and left- and right-hand-sides at any stage. The approach treats the uncertainty by scenario cluster analysis.

The proposed algorithm works over a structured information in which we have split the set of stages into two blocks of stages; the first includes the stages $t = 1, ..., t^* + 1$, and the second includes the other stages in set $\mathcal{T}$, since $t^* + 2$ until T. Variables in stage $t^* + 1$ link both blocks, where t^* is the break stage. Then, the scenario tree is decomposed into a two-blocks scenario cluster tree, where the NAC are implicitly satisfied until stage t^*, since they are represented in compact representation in the full model for these stages, and they are implicitly satisfied into each scenario cluster model from stage $t^* + 1$ until T.

As a future work we are considering a computational comparison between our purpose and the traditional Nested Benders Decomposition as a decomposition tool for iteratively solving multistage models. Another subject for future research derives from the observation of the independent character of the scenario cluster submodels, such that it paves the way for a parallelized algorithm.

Conclusions

Chapter **5**

Conclusions, original contributions and future research

5.1 Conclusions

Chapter 2: Stochastic models for immunization strategies

This chapter proposes several stochastic models for selecting portfolios in a market in which there are transaction costs and bonds with different credit ratings. In particular, new concepts and modelings have been introduced and tested. We have also extended some of them from the two-stage formulation to the general multistage case. The intention is to check whether the assumptions made in the dynamic immunization theorem put forward by Khang are crucial to its validity and to check wether the theoretical models proposed in the literature and developed in this chapter are suitable to optimize immunization strategies in fixed-income security portfolios under both sources of uncertainty.

According to Khang, and his Dynamic Global Portfolio Immunization Theorem [61], in case of parallel changes in the interest rate structure, ausence of trasaction costs and no other source of incertainty apart from the interest rate risk, the immunized portfolio would be the one that matches Macaulay's duration with the planning horizon all over the time. The inclusion of transaction costs, nevertheless, is observed to affect the optimality of the proposed strategy, since the continual readjustment of the portfolio that these costs entail results in additional costs which are too high. This means that the immune strategy ceases to be optimal.

Furthermore, introducing a second source of uncertainty also affect the optimality of the theoretical result due to the fact that the problem to be faced changes in a signifficant way. This is an innovative way to face the problem since we deal with two completely different sources of uncertainty at the same time: trends in interest rates and the probabilities of default of the various institutions which issue the bonds. Uncertainty is introduced into the model through a scenario analysis scheme.

Different immunization strategies are considered, such as, min-max regret, mean-risk immunization, two-stage and multistage Value-at-Risk strategy, two-stage Conditional Value-at-Risk strategy, two-stage first and second order stochastic dominance and multistage first order stochastic dominance strategies, and the new measure as a mixture of the multistage VaR & stochastic dominance at all stages. The validity of the proposed strategies is performed by using an illustrative case study. No definitive conclusions can be drawn from the case study (the aim of the chapter has merely been to present the immunization strategies of choice), but the results that have been obtained seem to be reasonable.

Based on them we favor the multistage immunization strategies given by the models (DEM4) (using multistage 100% VaR), (DEM5) (using the stochastic dominance constraints alone) and (DEM6) (using the mixture of the VaR strategy and the stochastic dominance constraints).

Each of them has advantages and disadvantages, and the decision maker should choose between the three depending on its preferences. Model (DEM5) would be better for those investors who are only interested in the final value of the portfolio and they also know a priori which possible thresholds can be more interesting for their purposes. On the contrary, model (DEM6) would be better for those investors with no initial expectations and are also interested in optimizing the portfolio all over the PH. It could be very interesting, for example, for those decision makers who would like to dissolve the portfolio at any time in order to use the new information in the market. Model (DEM4), instead, could be very interesting for a very risk averse investor. In any case, models (DEM5) and (DEM6) can become computationally non treatable for real markets with many possible future scenarios in case of using plain state-of-the-art optimization engines instead of using proven decomposition approaches, meanwhile (DEM4) can easily be solved.

Chapter 3: Two-stage scheme

We have proposed in this chapter an efficient scenario cluster decomposition approach for identifying tight feasibility cuts in Benders decomposition for solving medium-large and large scale two-stage stochastic problems where only continuous variables appear. Some computational experience is presented, where we observe the favorable performance of the proposed Cluster Benders Decomposition (CBD) approach versus the performance of the Traditional single scenario Benders Decomposition (TBD) approach.

In the Benders decomposition, the two-stage linear problem (3.1) can be decomposed and its optimal solution can be iteratively obtained by identifying extreme points and rays based cuts from the optimization of the so-named *Auxiliary Program* (AP). So, the cuts are appended to the so-named *Relaxed Master Program* (RMP). The TBD solves a feasibility auxiliar problem for scenario and creates a cut, if needed, each time. The CBD solves a feasibility auxiliar problem for each cluster of scenarios, so it solves less subproblems and creates tighter feasibility cuts.

We point out that the state-of-the-art optimization engine CPLEX requires more computation time to obtain the optimal solution than the CBD approach does in 9 out of 11 instances (i.e., the largest ones) in our testbed for a small number of clusters (in particular, $\hat{p} = 1, 2$).

So, although more computational experience is required, the new approach seems to be very promising based on our provisional results. Moreover, for a big number of clusters (in particular, $\hat{p} = |\Omega|$, i.e., the singleton cluster TBD approach), plain use of CPLEX outperforms our CBD approach.

Chapter 4: Multistage scheme

In this chapter a general stochastic multistage modeling approach and an algorithmic framework, named *Multistage Cluster Benders Decomposition* (MCBD), have been proposed for solving multistage problems under uncertainty in the parameters, being the uncertainty represented by scenarios trees. It can appear in any coefficient of the objective function, constraint matrix and left- and right-hand-sides at any stage. The approach treats the uncertainty by scenario cluster analysis.

The algorithm proposed works over a structure information in which we have split the set of stages into two blocks of stages, the first one includes the stages $t = 1, ..., t^* + 1$, and the second one includes the other stages in set $\mathcal{T}$, since $t^* + 2$ until T. Variables in stage $t^* + 1$ link both blocks, where t^* is the break stage. Then, the scenario tree are decomposed into a two-blocks scenario cluster tree, where the NAC are implicitly satisfied until stage t^*, since they are used a compact representation in the full model for these stages, and they are implicitly satisfied into each scenario cluster model from stage $t^* + 1$ until T.

In this way it is possible to define a *Relaxed Master Program* in which we do not deal just with the first stage variables; we deal with the whole first block of stages variables. Then, we can define an *Auxiliary Problem* for each cluster of the second block of stages creating an structure that reminds the *Traditional Benders Decomposition* in the two stage environment.

Although computational experience is required, the theoretical development of the algorithm seems to be reasonable and promising.

5.2 Contributions visibility

Certain main results of this memory have lead to several publications:

- **Working paper series Biltoki**. *On solving two-stage stochastic linear problems by using a new approach, Cluster Benders Decomposition.* (L. Aranburu, L. F. Escudero, M.A. Garín and G. Pérez).

  http://econpapers.repec.org/paper/ehubiltok/201008.htm

- **Journal TOP**.*A so-called study on our Cluster Benders Decomposition approach for solviong two stage stochastic linear problems.* (L. Aranburu, L. F. Escudero, M.A. Garín and G. Pérez). Ed. Springer. *Submitted for publication.*

- **Book titled: Stochastic Programming: Applications to Finance, Energy Planning and Logistics**. *Stochastic models for optimizing immunization strategies in fixed-income security portfolios under some sources of uncertainty.* (L. Aranburu, L. F. Escudero, M.A. Garín and G. Pérez). Ed. H. Gassmann, S. W. Wallace and Y. Zhao. *Submitted for publication.*

- **OR2011 Proceedings** *Modern multistage risk measures for immunizing fixed-income portfolios under uncertainty.* (L. Aranburu, L. F. Escudero, M.A. Garín and G. Pérez). Ed. Springer. *Submitted for publication.*

Furthermore, the main results of this memory have also been presented in several national and international meetings:

- **XII Encuentro de Economía Aplicada**. *Modelos estocásticos de selección de carteras de renta fija con costes de transacción y distintas calificaciones crediticias.* (L. Aranburu, M.A. Garín and G. Pérez). Madrid (Spain). June 2009.

- **XXIII European Conference on Operations Research (EURO)**. *A stochastic model for mixed-income securities portfolio selection with transaction costs and default probabilities.* (L. Aranburu, L. F. Escudero, M.A. Garín and G. Pérez). Bonn (Germany). July 2009. *Invited talk.*

- **II Jornadas de Investigación de la Facultad de Ciencia y Tecnología (UPV/EHU)**. *Programación estocástica multietapa 0-1. Algoritmos y aplicaciones.* (L. Aranburu, L. F. Escudero, M.A. Garín, M. Merino, G. Pérez and A. Unzueta). Leioa (Spain). March 2010.

- **IX Workshop on Advances in Continuous Optimization (EUROPT)**. *Risk averse strategies in Stochastic Optimization.* (L. Aranburu, L. F. Escudero, M.A. Garín and G. Pérez). Ballarat (Australia). July 2011.

- **International Conference on Operational Research (OR2011)**. *Modern multistage risk measures for immunizing fixed-income portfolios under uncertainty.* (L. Aranburu, L. F. Escudero, M.A. Garín and G. Pérez). Zürich (Switzerland). August 2011.

- **Annual Conference of the Serbian Operations Research Society**. *Risk management in optimization under uncertainty: models, algorithms and applications.* (L. F. Escudero. Joint work with L. Aranburu, M.A. Garín, M. Merino, G. Pérez and A. Unzueta). Zlatibor mountain (Serbia). October 2011. *Keynote talk in the innauguration session.*

- **Workshop on Optimization Issues in Energy Efficient Distributes Systems (OPTIM2011)**. *Risk averse strategies in stochastic* *optimization.* (L. Aranburu, L. F. Escudero, M.A. Garín, M. Merino, G. Pérez and A. Unzueta). Bellaterra (Spain). October 2011. *Invited talk.*

5.3 Future research

The first part of this study, is more related to stochastic modelling. In this sense we have proposed several risk averse measure to deal with the immunization of fixed-income securities portfolio problem in different ways. Some of the proposed models can become computationally difficult for real markets with many possible future scenarios in case of plain use of optimization engines. We may even have difficulties for solving them by using decomposition methods, in some cases, in case of a big cardinality of the set of profiles.

For these situations we are considering different strategies whose validation will be a piece of future research:

- *Model without scenario linking constraints:* MR&SDC that stands for Mean-Risk & Stochastic Dominance Constraints since it is a mixture of both. It consists of maximizing the multistage VaR minus the sum of the weighted failure's probabilities of not reaching the set of thresholds imposed by the modeler. The model that implements the new strategy does no include scenario linking constraints, what is a good characteristic from a computational point of view and it could replace the strategy VaR&SDC.
- *Model without 0-1 variables:* Multistage Second order Stochastic Dominance, which is the extension to the multistage of the strategy described in Section 2.5 in Chapter 2. This strategy does not include 0-1 variables, which makes it very attractive from a computational point of view.

The validation of the following variations of our new approach (multistage VaR&SDC) are also a piece of future research:

- Mixture of multistage VaR and first order SDC
- Mixture of multistage VaR and second order SDC

- Mixture of multistage CVaR and first order SDC
- Mixture of multistage CVaR and second order SDC

The second part of the manuscript is more related to the algorithmic approach, more specifically to the development of two-stage and multistage Cluster Benders Decomposition (CBD). In this sense we are considering the following extensions:

- The combination of the exact *Branch-and-Fix Coordination* scheme for solving two-stage stochastic mixed 0-1 problems [35, 36, 38], with the CBD approach in order to tighten the lower bound of the solution value at the Twin Node Family.
- Test the validity of the algorith MCBD by a significant case study.
- A computational comparison between our proposed MCBD and the traditional Nested Benders Decomposition as a decomposition tool for iteratively solving multistage models.
- Another point of future research derives from the observation of the independent character of the scenario cluster submodels, such that it paves the way for a parallelized algorithm.

As a future work we are also planning an extensive computational experience with large scale cases in fixed-income security portfolios, scenarios and profiles to test the validity of, not only CBD and MCBD decomposition algorithm, but also our stochastic mixed integer optimization algorithm approaches for problem solving in this type of financial application.

In that sense, we are planning to use different computational algorithms that would make easier the solution of them:

- *CBD* described in Chapter 3 to solve the two stage immunization model (DEM2).
- *MCBD* described in Chapter 4 to solve the multistage immunization model (DEM4).

- *Multistage Branch-and-Fix Coordination (BFC-MS)* described in [34] and [39] for multistage mixed 0-1 models.
- *Mixture of BFC-MS and Lagrangean Decomposition* for multistage mixed 0-1 models with scenario linking constraints, such as, (DEM5) and (DEM6).

Another objective pursued by future research is to study the validity of the new risk averse measures and the algorithmic approaches proposed in this memory for other applications, not necessarily in the field of finance.

Appendix A

CBD programming codes

A brief description of the programming code corresponding to the implementation of the CBD and used in the computational experience povided in Section 3.6 in Chapter 3 is described below. It has been implemented in a C++ experimental code (Visual C++ 2008 Express Edition) by using CPLEX v12.2 [85] as a solver of the linear optimization Relaxed Master Problem (RMP) and Auxiliary Problems (AP) at each iteration within the open source engine COIN-OR [58, 70]. The content of this programming code is included in the attached CD that contains eleven self-executable folders (named from P1 to P11), with the routines, each with a different two-stage so-named problem from those listed in Table 3.4. Each program can be executed for any number of clusters $\hat{p}$, being $\hat{p}$ a divisor of the number of scenarios $|\Omega|$. The files are organized in a friendly way for creating a new example with a different number of first stage (n_x) and second stage (n_y) variables, as well as different number of scenarios ($|\Omega|$).

The main program is as follows,

benders-cplex.cpp, main program that obtains the solution of the DEM using the Cluster Benders Decomposition algorithm described in Section 3.5 of the Chapter 3. It uses the following external functions:

- **inicioprimal.cpp,** external function of benders.cpp corresponding to the definition of the initial *Relaxed Master Program* (RMP) that generates an objective function and initial values for the first stage variables (x_i).

auxiliarprimal1.cpp, external function of benders.cpp that generates the feasibility *Auxiliar Subproblem* (3.12) for the variables y_i^p, for each scenario cluster $p \in \mathcal{P}$.

auxiliarprimal2.cpp, external function of benders.cpp that generates the optimality *Auxiliar Subproblem* (3.13) for the variables y_i^ω, for each scenario $\omega \in \Omega$.

maestroprimal1.cpp, external function of benders.cpp that generates the feasibility cut to be appended to the *Relaxed Master Program* at each iteration.

maestroprimal2.cpp, external function of benders.cpp that generates the optimality cut to be appended to the *Relaxed Master Program* at each iteration.

modelosBL.cpp, external function of benders.cpp that generates each one of the two-stage problems with the related deterministic parameters as well as those which are uncertain under each scenario.

nirea.cpp, external function of benders.cpp that generates random numbers.

The main program also uses the header files described below:

constantesBL.h, header file for imputting profile constant values such as n_x, n_y, $|\Omega|$, $\hat{p}$ and several tolerances for different examples. Additionally, it calculates some other constant values derived from the other ones, such as the dimensions number of variables, constrains and nonzero elements of the full model as well as the scenario and scenario cluster submodels that will be solved of the CBD procedure.

pm.h, header file that includes the header files of the open source engines COIN-OR and CPLEX that are needed for the correct solution of the main program.

Appendix B

Resumen

B.1 Introducción

El contenido del trabajo está organizado en dos partes. En una primera parte (Capítulo 2), el manuscrito está orientado al desarrollo de la modelización estocástica necesaria para introducir nuevas medidas de gestión de riesgo en problemas de optimización matemática bajo incertidumbre, partiendo de modelos de dos etapas y aportando su extensión al caso general de múltiples etapas. La incertidumbre en los parámetros del problema es tratada vía la metodología conocida como Análisis de Escenarios.

Se revisan algunas de las medidas de gestión del riesgo conocidas de la literatura para modelos de dos etapas, entre las que están: la inmunización media-riesgo (mean-risk immunization), el valor en riesgo en dos etapas (two-stage value at risk), valor en riesgo condicional en dos etapas (two-stage conditional value at risk) o las condiciones de dominancia estocástica de primer y segundo orden también en dos etapas (two-stage first and second order stochastic dominance).

Se analizan las distintas estrategias aversas al riesgo utilizando como caso piloto un modelo de optimización de estrategias de inmunización para carteras de renta fija con dos fuentes de incertidumbre.

Además de generalizar al caso multietapa varias de las estrategias citadas, se propone una nueva medida de aversión al riesgo, definida como una mixtura de las estrategias valor en riesgo (VaR) y dominancia estocástica (SDC).

La segunda parte de esta memoria (Capítulos 3 y 4) está dedicada al desarrollo de esquemas algorítmicos de solución. Se proponen en particular innovadoras metodologías y tecnologías computacionales para la resolución de problemas estocásticos lineales (dos etapas y multietapa) a gran escala, basadas en técnicas de descomposición.

Tomando como punto de partida la descomposición tradicional de Benders para modelos estocásticos lineales de dos etapas, se propone en el Capítulo 3 un nuevo esquema que resuelve en cada iteración un modelo auxiliar de factibilidad más fuerte, definido a la vez para un cluster o racimo de escenarios, en lugar del tradicional que es definido uno a uno para cada escenario. Se propone el procedimiento denominado CBD: Cluster Benders Decomposition. Los cortes de factibilidad generados con el nuevo esquema son más fuertes, reduciendo sustancialmente el número de iteraciones necesarias para la obtención del óptimo del problema. Se aportan los resultados obtenidos en la experiencia computacional sobre un conjunto de casos generados aleatoriamente como prueba de la eficiencia del procedimiento propuesto.

Los modelos de optimización estocástica multietapa son tradicionalmente más difíciles de manejar que los de dos etapas. En este sentido, el Capítulo 4 proporciona un esquema de representación de un modelo general de optimización estocástica multietapa, como un modelo en dos bloques de etapas, estructura ésta última que recuerda a la tradicional de un modelo de dos etapas.

Una vez generados los elementos necesarios para tal representación, el modelo resultante, puede ser tratado con el esquema de descomposición de Benders por clusters o racimos de escenarios. Se propone así el esquema algorítmico, Multistage CBD.

B.2 Modelización estocástica de estrategias de inmunización

Los modelos de optimización estocástica han sido ampliamente estudiados desde los años cincuenta. En la memoria completa se revisa el estado del arte citando varios de los trabajos más interesantes publicados en la literatura.

La mayoría de los modelos financieros propuestos y utilizados en las últimas dos decadas han sido estáticos y de un sólo periodo, aunque también son tratados modelos de dos etapas. Sin embargo, en los casos en los que la incertidumbre prevalece en todas las etapas del horizonte de planificación, los modelos de optimización multietapa se convierten en más apropiados. En la actualidad, estos modelos no son muy comunes en la práctica en aplicaciones financieras dada su complejidad y el alto y complejo requerimiento de introducción y manejo de los datos. Aun así, en los últimos años han aparecido en la literatura modelos de optimización muy interesantes, básicamente para gestión de activos.

En la presente memoria se analizan algunas de las estrategias aversas al riesgo utilizando como caso piloto un modelo de optimización de estrategias de inmunización para carteras de renta fija con dos fuentes de incertidumbre. Así, el modelo financiero presentado considera que hay parámetros inciertos que no son controlados por el modelizador, pero son conocidas o estimadas sus distribuciones de probabilidad, y tales parámetros pueden ser considerados como variables aleatorias independientes a las variables de decisión del problema. Introducimos diferentes modelizaciones que permiten considerar costes de transacción. La incertidumbre se representa mediante un árbol de escenarios multietapa.

La mayor diferencia entre el modelo de gestión de bonos propuesta en este trabajo y los modelos encontrados en la literatura, es que consideramos un inversor que quiere invertir en un mercado con bonos con diferentes niveles de riesgo de impago (*default*). Por lo tanto, se consideran dos fuentes de incertidumbre, o dos riesgos, asociados al modelo, denominados riesgo de tipo de interés y riesgo de impago, respectivamente. Este último está relacionado con la solvencia del emisor del bono y, por tanto, del mismo bono.

Uno de los resultados más importantes en el campo de la inmunización de carteras de renta fija, es el llamado Teorema de Inmunización Global Dinámica enunciada por Khang [61]. Según esta teoría, si la estructura de los tipos de interés tiene cambios paralelos, no hay costes de transacción en el mercado y la única fuente de incertidumbre que afecta a los activos de renta fija es el riesgo de tipo de interés, una cartera inmunizada sería aquella cuya duración de Macaulay[1] igualase al horizonte de planificación en cada momento del tiempo.

[1] La duración de Macaulay de un activo es una media ponderada de la rentabilidad del bono en cada instante, donde la ponderación se elige como las desviaciones de cada instante con respecto al inicial:

$$d_{it} = \frac{\sum_{s=1}^{i} (t_s - t_0) Cupon \cdot Descuento}{\sum_{s=1}^{i} Cupon \cdot Descuento}$$

Los modelos propuestos en esta memoria permiten incluir costes de transacción y tienen en cuenta diferentes calificaciones creditícias en los bonos. De esta forma, se pretende no sólo comparar y elegir entre las distintas medidas de riesgo, sino también comprobar si los supuestos en los que se basa el Teorema de Inmunización Global Dinámica de Khang [61] son cruciales para su validez.

En este resumen nos centramos en describir con mayor detalle la modelización de la nueva medida de riesgo propuesta, definida como una mixtura de las estrategias VaR (Value-at-Risk) y SDC (Stochastic Dominance Constraints) multietapa.

Descripción del problema y fuentes de incertidumbre

Consideramos una partición del horizonte de planificación (PH) en k subintervalos de igual longitud $[t_0, t_1]$, $[t_1, t_2]$,..., $[t_{k-1}, t_k]$, siendo t_0 el inicio y t_k el final del PH. También asumimos que el reajuste de la cartera sólo está permitido al inicio de cada subintervalo. $|\mathcal{T}| = k + 1$ es el número de periodos de tiempo, y t_k es el periodo final. Así que, se define el conjunto $\mathcal{T}$ como la discretización del horizonte de planificación, es decir, $\mathcal{T} = \{t_0, t_1, ..., t_k\}$. Por simplicidad y sin pérdida de generalidad, asumimos que hay $|I|$ bonos cupon diferentes disponibles en t_0, cada uno de ellos con madurez en t^i, tal que $t^i \in \{t_1, ..., t_k\}$, pero no necesariamente para todos los bonos, ya que puede haber algún bono i^* con madurez, t^{i*}, posterior al periodo de planificación final, es decir, $t^{i*} > t_k$. Así, los pagos de los cupones se producen el los puntos de reajuste, donde t^i es el periodo de madurez del bono $i \in I$, e I denota el conjunto de bonos a incluir en la cartera.

Asumimos un presupuesto inicial dado para invertir en la cartera. Las variables de decisión más importantes son el volumen de cada título a comprar y vender en cada periodo. Se asumen los costes de transacción que afectan a cada reajuste. El objetivo del problema consiste en reajustar la cartera de renta fija a lo largo del horizonte de planificación bajo varias fuentes de incertidumbre como se describe en la siguiente subsección. La incertidumbre en los parámetros más importantes es representada por un árbol de escenarios. Nuestra estrategia de inmunización, que se describe a continuación, está basada en dos fuentes de incertidumbre independientes, los tipos de interés y las calificaciones de las compañías emisoras de los bonos, respectivamente. El riesgo en éste último caso, concierne a la solvencia del emisor y del propio bono. Se proporcionan las siguientes definiciones para facilitar la notación.

Credit rating denota la calificación dada a un individuo o a una compañía para indicar su solvencia como deudores en títulos de corto o largo plazo. Las agencias calificadoras examinan las compañías emisoras de bonos, al igual que la situación de los mismos bonos a intervalos regulares y pueden subir o bajar su calificación cuando quiera que lo vean conveniente. *Default o Impago*[2] denota el incumplimiento a la hora de pagar un préstamo al llegar al periodo de maduración, o cuando se cumplan los términos del contrato. La probabilidad *riesgo de default o riesgo de impago* está muy relacionada con la calificación de crédito de una entidad en un instante del tiempo dado. La *tasa de recuperación*, z_i, es la proporción del dinero debido que el deudor se compromete a pagar en caso de default.

El modelo de selección de carteras descrito a continuación trata de obtener el reajuste óptimo, independientemente de los cambios en los tipos de interés, es decir, trata de obtener una cartera inmunizada. Permite incluir costes de transacción y tiene en cuenta diferentes calificaciones creditícias en los bonos. Por tanto, la cartera óptima tiene en cuenta el riesgo asociado con la probabilidad de bancarrota por parte del emisor, y el peso que el inversor atribuye a este riesgo.

Modelo Determinista Equivalente

En Optimización Estocástica es bien conocido que la optimización del valor esperado de la función objetivo (en nuestro caso piloto, definida como la riqueza esperada final a lo largo de los periodos de planificación y bajo todos los escenarios) proporciona una estrategia neutral al riesgo, dado que no considera la variabilidad a lo largo de los periodos y bajo cada uno de los escenarios. Por tanto, asumiendo la ventaja desde el punto de vista del coste computacional, la literatura está considerando una variedad de medidas aversas al riesgo para reducir la probabilidad de ocurrencia de los escenarios no deseados. Algunas de las medidas más interesantes son el VaR (valor en riesgo), CVaR (valor en riesgo condicionada), media-riesgo, la reciente dominancia estocástica (SDC), y por último nuestra propuesta en dos etapas, denotada como VaR&SDC como mezcla de las estrategias VaR y SDC.

[2] Abusando del lenguage, se utiliza la terminología anglosajona para referirse a un determinado concepto

La memoria completa analiza las ventajas y los inconvenientes, tanto computacionales como de otro tipo, de todas estas estrategias. Además presenta a modo de ejemplo y haciendo uso de una pequeña aplicación financiera algunas conclusiones al respecto. En dicho ejemplo se han obtenido algunos de los mejores resultados con el modelo determinista equivalente (DEM) implementando la estrategia VaR&SDC descrita a continuación. Requiere la siguiente notación adicional:

Conjuntos

J, conjunto de clases de activos de renta fija j considerados, es decir, el conjunto de distintas calificaciones creditícias considerados.

I_j, conjunto de activos i que pertenece a la clase j, $I_j \subset I$.

Parámetros

I_0, inversión inicial.

F_i, valor nominal del activo i, para $i \in I$.

t^i, periodo de madurez del activo i, tal que $t^i \in \{t_1, ..., t_k, ..., t_n\}$, para $i \in I$.

β, porcentaje del volumen negociado que representa los costes de transacción que afectan a cada reajuste. También se considera que el pago del valor nominal no genera costes de transacción.

P_{i0}, precio unitario de mercado del activo i al inicio del PH, periodo t_0, para $i \in I$.

w^ω, probabilidad del escenario ω, dado por el modelizador.

w^g, probabilidad del grupo de escenarios g, para $g \in \mathcal{G}_t$, $t \in \mathcal{T}$, calculado como $\sum_{\omega \in \Omega_g} w^\omega$.

r^g, tipo de interés libre de riesgo bajo el grupo de escenarios g, para $g \in \mathcal{G}_t$, $t \in \mathcal{T}$.

q_j^g, medida de riesgo calculado como la probabilidad de default de los activos de la clase j bajo el grupo de escenarios g, para $g \in \mathcal{G}_t$, $t \in \mathcal{T}$, $j \in J$.

R_i^g, tipo de interés real (como fracción de uno) para el activo i de la clase j bajo el grupo de escenarios g, para $g \in \mathcal{G}_t$, $t \in \mathcal{T}$, $i \in I_j$, $j \in J$. Su calculo está basado en r^g, el tipo de recuperación z_i y q_j^g.

C_i^g, cadena de pagos generados por una unidad del activo i bajo el grupo de escenarios g, para $g \in \mathcal{G}_t$, $t \in \mathcal{T} - \{t_0\}$, $i \in I_j$, $j \in J$. Su cálculo está basado en F_i,, el tipo de recuperación z_i y q_j^g.

P_i^g, precio unitario del activo i bajo el grupo de escenarios g, para $g \in \mathcal{G}_t$, $t \in \mathcal{T} - \{t_0\}$, $i \in I_j$, $j \in J$. Su cálculo está basado en C_i^g y C_i^g.

P_i^{+g}**,** P_i^{-g} precios de compra y venta unitarios del activo i, respectivamente, bajo el grupo de escenarios g, para $i \in I_j$, $j \in J$, $g \in \mathcal{G}_t$, $t \in \mathcal{T} - \{t_k\}$, donde $P_i^{+g} = (1+\beta)P_i^g$ y $P_i^{-g} = (1-\beta)P_i^g$.

$v_{i,t(g)}^{\omega}$, valor final bajo el grupo de escenarios g de una inversión de $P_i^{\mathbf{g}}$ unidades monetarias en el activo i hecha en el periodo, digamos, $t(g)$ a donde pertenece el grupo g, si el tipo de interés real instantáneo cambia, justo después, de $R_i^{\mathbf{g}}$ a R_i^{ω} *al final* del PH, para $i \in I$, $g \in \mathcal{G}_{t(g)}$, $\omega \in \Omega_g$. Su cálculo está basado en C_i^g y C_i^g.

Variables

x_i^{+g}**,** x_i^{-g} volúmenes del activo i comprados y vendidos, respectivamente, bajo el grupo de escenarios g, para $g \in \mathcal{G}_t$, $t \in \mathcal{T}$, $i \in I$.

z_i^g, volumen del activo i contenido en la cartera bajo el grupo de escenarios g, para $g \in \mathcal{G}_t$, $t \in \mathcal{T}$, $i \in I$.

V^g, valor final de la cartera bajo el grupo de escenarios g, para $g \in \mathcal{G}_t$, $t \in \mathcal{T}$.

Consideremos el conjunto de objetivos preestablecidos, digamos, $\mathcal{P}$ como el conjunto de umbrales ϕ^g a satisfacer por V^g bajo el grupo de escenarios g (en cada etapa de decisión), es decir, $\mathcal{P} = \{\phi^g, g \in \mathcal{G}_t, t = t_0, \cdots, t_{k-1}\}$ para probabilidades de incumplimiento dados, digamos, α^g y pesos, digamos, γ^g, para $g \in \mathcal{P}$.

Por tanto, el DEM en representación compacta es el que sigue,

$$\text{(VAR\&DSC)} \qquad \max \sum_{g\in\mathcal{P}} \gamma^g V^g + \sum_{g\in\mathcal{G}_{t_k}} w^g V^g \tag{B.1}$$

$$\text{s.t.} \quad x_i^0 = z_i^0 \quad \forall i \in I \tag{B.2}$$

$$\sum_{i\in\mathcal{I}} P_i^{+0} x_i^{+0} = I_0 \tag{B.3}$$

$$z_i^{\pi(g)} + x_i^{+g} - x_i^{-g} = z_i^g \quad \forall i \in I, \quad g \in \mathcal{G}_t, \quad t = t_1, \dots, t_{k-1} \tag{B.4}$$

$$x_{i:t^i=t}^{-g} = z_{i:t^i=t}^{\pi(g)} \quad \forall i \in I, \quad g \in \mathcal{G}_t, \quad t = t_1 \dots, t_k \tag{B.5}$$

$$x_i^{-g} = z_i^{\pi(g)} \quad \forall i \in I : t_k < t^i, \quad g \in \mathcal{G}_{t_k} \tag{B.6}$$

$$\sum_{i\in I:t<t^i} P_i^{+g} x_i^{+g} - \sum_{i\in I:t<t^i} P_i^{+g} x_i^{-g} = \sum_{i\in I:t\le t^i} C_i^g z_i^{\pi(g)} \quad \forall g \in \mathcal{G}_t, \quad t = t_1, \dots, t_{k-1} \tag{B.7}$$

$$\sum_{i\in I:t_k<t^i} P_i^{-g} x_i^{-g} + \sum_{i\in I:t_k\le t^i} C_i^g z_i^{\pi(g)} = V^g \quad \forall g \in \mathcal{G}_{t_k} \tag{B.8}$$

$$\sum_{i\in I:t<t^i} v_{i,t(g)}^{\omega} z_i^g + M\nu^{\omega g} \ge V^g \quad \forall \omega \in \Omega_g, g \in \mathcal{G}_t, t = t_0, \dots, t_{k-1} \tag{B.9}$$

$$\sum_{\omega\in\Omega_g} w^{\omega} \nu^{\omega g} \le \alpha^g \quad \forall g \in \mathcal{P} \tag{B.10}$$

$$V^g \ge \phi^g \quad \forall g \in \mathcal{P} \tag{B.11}$$

$$\nu^{\omega g} \in \{0,1\} \qquad \forall \omega \in \Omega_g, g \in \mathcal{P}. \tag{B.12}$$

La función objetivo (B.1) consiste en la maximización de la mixtura ponderada de los VaR en cada grupo de escenarios y el valor esperado final. Las ecuaciones (B.2)-(B.3) son restricciones de primera etapa, representando las condiciones estructurales. Las restricciones (B.4)-(B.6) son ecuaciones de balance que unen los activos que se compran y venden en cada periodo con los activos a mantener en la cartera. Las ecuaciones (B.9) son la llave de la estrategia de inmunización para asegurar un valor mínimo bajo todos los escenarios futuros para los grupos de escenarios. Las ecuaciones (B.7) aseguran que la cartera sea autofinanciada. Las ecuaciones (B.8) fuerzan que la cartera se deshaga al final del PH. Las restricciones (B.10) fuerzan la optimización con una probabilidad de incumplimiento dado.

La validez de las estrategias propuestas se lleva a cabo mediante un caso de estudio ilustrativo del que no se pueden sacar conclusiones definitivas (el propósito de este capítulo ha sido simplemente presentar las estrategias entre las que elegir) pero los resultados parecen razonables.

En cuanto a la validez del Teorema de Inmunización Global Dinámica propuesta por Khang, podemos decir que los supuestos en los que se basa son absolutamente necesarios. Por un lado, la inclusión de los costes de transacción afecta a la optimalidad de la estrategia propuesta, dado que el reajuste continuo de la cartera que ésta implica, supone costes adicionales demasiado elevados. Por otro lado, la introducción de la segunda fuente de incertidumbre cambia totalmente el planteamiento del problema, por lo que la estrategia propuesta por Khang vuelve a no ser óptima en este contexto.

A la hora de elegir entre los distintos modelos de gestión de riesgos propuestos en la memoria, se puede decir que cada uno de ellos tiene sus vetajas y desventajas y, dependiendo del perfil del inversor, se podrían preferir unas u otras.

En cualquier caso, para un inversor averso al riesgo cuyo objetivo es mantener una cartera inmunizada frente a los dos riesgos durante el horizonte de planificación, podríamos decir que la mejor estrategia es la llamada (VaR&SDC) descrita en las ecuaciones (B.1)-(B.10). Esta estrategia permite inmunizar la cartera frente a las dos fuentes de incertidumbre en los escenarios más probables (se consideran más o menos escenarios dependiendo de la aversión al riesgo del inversor), durante todo el horizonte de planificación. Pero, además, permite asegurar una ganancia mínima en aquellos escenarios menos probables.

Aun así, el modelo (VaR&DSC) puede complicarse mucho para mercados reales con gran número de escenarios futuros, alcanzando una alta complejidad para la aplicación de solvers estándar de optimización. Incluso podríamos tener dificultades para resolverlo utilizando técnicas de descomposición, en el caso de una gran cardinalidad del conjunto de umbrales.

Para estas situaciones estamos considerando reemplazar la estrategia VaR&DSC por una nueva estrategia llamada MR&DSC como la mixtura de las estrategias media-riesgo y SDC multietapa. Consiste en maximizar el VaR menos la suma de las probabilidades ponderadas de no superar los umbrales propuestos. La ventaja es que este modelo no incluye restricciones que mezclan escenarios.

B.3 Modelos lineales dos etapas. Descomposición de Benders

El procedimiento iterativo de optimización de problemas estocásticos (representados mediante análisis de escenarios) de dos etapas lineales, basado en la descomposición de Benders, requiere tras la definicón del subproblema maestro, la identificación de cortes de factibilidad y de optimalidad que garanticen la optimalidad de la solución, y por ende la convergencia del procedimiento. Los cortes de factibilidad y de optimalidad se generan resolviendo los submodelos auxiliares correspondientes ligados a cada escenario aislado.

La estructura que presentan en particular los problemas auxiliares de factibilidad, permiten definirlos para un conjunto de escenarios. En este caso se prueba la factibilidad de la solución bajo varios escenarios simultáneamente. El conjunto de escenarios se divide en clusters o racimos de escenarios, y se itera en este número de clusters en lugar de iterar sobre el número de escenarios. Los cortes de factibilidad generados son además más fuertes y producen decisiones de segunda etapa factibles en un tiempo computacional razonable, requiriendo menos iteraciones del procedimiento para alcanzar el óptimo.

Así, se propone el algoritmo denominado CBD: Descomposición de Benders por Clusters o racimos de escenarios como un esquema eficiente para la solución de modelos estocásticos lineales bietapa de grandes dimensiones.

La memoria completa reporta experiencia computacional utilizando CPLEX como herramienta de software para los submodelos lineares auxiliares así como para la solución del submodelo maestro, en cada iteración del algoritmo CBD. Los resultados obtenidos, sobre problemas a media y gran escala, muestran el favorable rendimiento de la nueva estrategia en comparación con la descomposición tradicional de Benders; mejorando incluso el uso directo de esta herramienta sobre el modelo completo sin descomponer.

Descomposición de Benders tradicional (TBD)

Consideramos el Modelo Determinista Equivalente (DEM) del problema estocástico lineal de dos etapas en representación *compacta*,

$$\begin{array}{lll} (LO): & z_{LO} = \min & c^T x + E_\psi[\min w^\omega (q^{\omega T} y^\omega)] \\ & s.t. & \\ & & b_1 \leq Ax \leq b_2 \\ & & h_1^\omega \leq T^\omega x + W^\omega y^\omega \leq h_2^\omega \quad \forall \omega \in \Omega \\ & & x, y^\omega \geq 0 \quad \forall \omega \in \Omega, \end{array} \tag{B.13}$$

donde x es el vector de longitud n_x de las variables de primera etapa, y^ω es el vector de longitud n_y de las variables de segunda etapa bajo el escenario ω, para $\omega \in \Omega$, donde Ω es el conjunto de escenarios a considerar; c es un vector de coeficientes conocidos de las variables x en la función objetivo, b_1 y b_2 son los vectores de cotas inferiores y superiores, respectivemente, de las restricciones de primera etapa; A es la matriz de coeficientes de las restricciones de primera etapa; w^ω es la probabilidad de ocurrrencia del escenario ω; h_1^ω y h_2^ω son los vectores de cotas inferiores y superiores, respectivamente, de las restricciones de segunda etapa bajo el escenario ω y q^ω es el vector de coeficientes de la función objetivo para las variables de segunda etapa y bajo cada escenario; mientras que T^ω es la matriz de coeficientes de las variables x en las restricciones de segunda etapa y W^ω es la matriz de coeficientes de las variables y en las restricciones de segunda etapa para el escenario ω, para $\omega \in \Omega$.

El problema lineal de dos etapas (B.13) se puede descomponer y su solución óptima se puede obtener de forma iterativa, mediante la identificación de puntos y rayos extremos basada en la generación de cortes creados por la optimización del llamado *Programa auxiliar* (AP). Así, estos cortes se introducen en el llamado *Problema maestro relajado* (RMP) que se puede expresar como sigue, ver [11],

$$
\begin{array}{rcl}
\overline{z}_{LO} & = & \min c^T x + \theta \\
\text{s.t.} & & \\
b_1 & \leq & Ax \leq b_2 \\
0 & \geq & \nu_{j1}^{\omega T} \left[\begin{pmatrix} h_1^\omega \\ -h_2^\omega \end{pmatrix} + T^\omega x \right] \quad \forall \nu_{j1}^\omega \in \overline{\mathcal{J}}^{e_f} \\
\theta & \geq & \sum_{\omega \in \Omega} w^\omega \nu_{j2}^{\omega T} \left[\begin{pmatrix} h_1^\omega \\ -h_2^\omega \end{pmatrix} + T^\omega x \right] \quad \forall \nu_{j2}^\omega \in \overline{\mathcal{J}}^{e_o} \\
 & & x \geq 0, \theta \in I\!R,
\end{array}
$$

Una vez resuelto en la primera iteración el modelo RMP (**Paso 1**) para las variables de primera etapa, se resuelve el siguiente problema de optimalidad en las variables de segunda etapa (**Paso 2**), para cada escenario $\omega \in \Omega$:

$$\begin{array}{ll} (FEAS) & z^{\omega}_{FEAS} = \min e^T v_1^{+\omega} + e^T v_1^{-\omega} + e^T v_2^{+\omega} + e^T v_2^{-\omega} \\ & s.t. \\ & W^{\omega} y^{\omega} - I u^{-\omega} + I v_1^{+\omega} - I v_1^{-\omega} = h_1^{\omega} - T^{\omega}\hat{x} \\ & W^{\omega} y^{\omega} + I u^{+\omega} - I v_2^{+\omega} + I v_2^{-\omega} = h_2^{\omega} - T^{\omega}\hat{x} \\ & y^{\omega}, v_1^{+\omega}, v_1^{-\omega}, v_2^{+\omega}, v_2^{-\omega}, u^{+\omega}, u^{-\omega} \geq 0. \end{array} \tag{B.14}$$

Si la solución óptima no es 0 para algún escenario, significa que la solución no es factible bajo dicho escenario. Se define en este caso un corte de factibilidad que se añade al RMP y se vuelve al Paso 1. En caso de que $z^{\omega}_{FEAS} = 0$, para todos los escenarios $\omega \in \Omega$, no hay cortes de factibilidad para esa solución y podemos pasar a chequear la optimalidad de la solución mediante el siguiente problema de optimalidad (**Paso 3**):

$$\begin{array}{lll} (OPT) \quad \phi^{\omega} & = & \min q^{\omega T} y^{\omega} \\ & & s.t. \\ & & \begin{pmatrix} W^{\omega} \\ -W^{\omega} \end{pmatrix} y^{\omega} \geq \begin{pmatrix} h_1^{\omega} - T^{\omega}\hat{x} \\ -h_2^{\omega} + T^{\omega}\hat{x} \end{pmatrix} \\ & & y^{\omega} \geq 0. \end{array} \tag{B.15}$$

Una vez resuelto el problema auxiliar de optimalidad para cada escenario, se calcula su valor esperado $\phi := \sum_{\omega\in\Omega} w^{\omega}\phi^{\omega}$. Se compara este valor esperado con $\hat{\theta}$, solución obtenida en la anterior iteración del problema maestro (Paso 1). Si $\phi \leq \hat{\theta}$, el procedimiento para, pues hemos encontrado el óptimo. En caso contrario, se genera y añade al problema maestro un corte de optimalidad y se vuelve al Paso 1.

Este método obtiene el óptimo en un número finito de iteraciones o prueba que el problema inicial es infactible siempre que Ω sea finito.

Innovación: esquema CBD

Se propone a continuación un esquema basado en la descomposición del conjunto de escenarios en clusters o racimos. La particular estructura del problema (B.14) muestra que su función objetivo no depende de ningún escenario en particular, por lo que se podría definir para un cluster o conjunto de escenarios en lugar de para uno solo. Así se propone el siguiente método de descomposición de Benders por clusters de escenarios (CBD),

Paso 0: Fijar $k := 0$, $p := 0$, $e_o := 0$, $e_{fc} := 0$.

Paso 1: Resolver el problema maestro relajado RMP (con $\theta = 0$ si $k = 0$). $k := k + 1$.

$$\min c^T x + \theta$$
$$\text{s.t.}$$
$$b_1 \le Ax \le b_2$$
$$-\hat{\nu}_j^{\omega T} T^\omega x \ge \hat{\nu}_j^{\omega T} \begin{pmatrix} h_1^\omega \\ -h_2^\omega \end{pmatrix} \quad \forall \omega \in \Omega^p \quad j = 0, ..., e_{fc} \tag{B.16}$$
$$-\sum_{\omega \in \Omega^p} w^\omega \hat{\nu}_{j_2}^{\omega T} T^\omega x + \theta \ge \sum_{\omega \in \Omega^p} w^\omega \hat{\nu}_{j_2}^{\omega T} \begin{pmatrix} h_1^\omega \\ -h_2^\omega \end{pmatrix} \forall j_2 = 0, ..., e_o \tag{B.17}$$
$$x \ge 0, \theta \in I\!R$$

Guardar los valores $\hat{x}$ y $\hat{\theta}$ de las variables x y θ.

Paso 2: Fijar $p := p + 1$. Resolver el problema auxiliar de factibilidad para el cluster de escenarios p,

$$
\begin{array}{ll}
(FEASC): & z_{FEASC}^p = \min \sum_{\omega \in \Omega^p} w^\omega (e^T v_1^{+\omega} + e^T v_1^{-\omega} + e^T v_2^{+\omega} + e^T v_2^{-\omega}) \\
& s.t. \\
& W^\omega y^\omega - Iu^{-\omega} + Iv_1^{+\omega} - Iv_1^{-\omega} = h_1^\omega - T^\omega \hat{x} \quad \forall \omega \in \Omega^p \\
& W^\omega y^\omega + Iu^{+\omega} - Iv_2^{+\omega} + Iv_2^{-\omega} = h_2^\omega - T^\omega \hat{x} \quad \forall \omega \in \Omega^p \\
& y^\omega, v_1^{+\omega}, v_1^{-\omega}, v_2^{+\omega}, v_2^{-\omega}, u^{+\omega}, u^{-\omega} \ge 0 \quad \forall \omega \in \Omega^p.
\end{array} \tag{B.18}
$$

Si $z_{FEASC}^p \ne 0$, la solución obtenida en el Paso 1 no es factible en ese cluster, y hay que definir y añadir el corte de factibilidad (B.16) e ir al Paso 1.

En caso contrario, volver al Paso 2 hasta llegar al último cluster. En este caso, la solución obtenida es factible y vamos al Paso 3.

Paso 3: Resolver el problema auxiliar de optimalidad para cada escenario ω, con $\omega \in \Omega$,

$$\begin{array}{lcl} (OPT) \quad \phi^\omega & = & \min q^{\omega T} y^\omega \\ & & s.t. \\ & & \begin{pmatrix} W^\omega \\ -W^\omega \end{pmatrix} y^\omega \geq \begin{pmatrix} h_1^\omega - T^\omega \hat{x} \\ -h_2^\omega + T^\omega \hat{x} \end{pmatrix} \\ & & y^\omega \geq 0. \end{array} \tag{B.19}$$

Calcular el valor esperado $\phi := \sum_{\omega \in \Omega} w^\omega \phi^\omega$. Comparar este valor esperado con $\hat{\theta}$. Si $\phi \leq \hat{\theta}$, entonces parar, pues hemos encontrado el óptimo. En caso contrario, se genera y añade al problema maestro un corte de optimalidad (B.17) y se vuelve al Paso 1.

Señalar que con el procedimiento propuesto se resuelven problemas auxiliares de factibilidad de mayores dimensiones en cada iteración pero a cambio se obtienen cortes de factibilidad más ajustados lo cual nos puede llevar a conseguir el óptimo en menos iteraciones.

En la experiencia computacional llevada a cabo se observa el buen rendimiento de la descomposición de Benders por clusters (CBD) propuesta, en comparación con el rendimiento de la descomposición de Benders tradicional. Además, se ha observado que para un número pequeño de clusters, el método propuesto mejora incluso el tiempo conseguido por una de las más eficientes herramientas de la actualidad, como es CPLEX. Por tanto, aunque haría falta más experiencia computacional, los resultados obtenidos por el algoritmo CBD parecen prometedores.

B.4 Modelos lineales multietapa

Un problema estocástico lineal multietapa con un número finito de escenarios futuros también tiene un Modelo Determinista Equivalente (DEM). Aun así, la estructura de este problema es bastante más complicada que la del problema dos etapas. La formulación extendida o en variables divididas no es en absoluto fácil de manipular.

El objetivo del Capítulo 4 es extender el método de descomposición de Benders por clusters al caso multietapa. Para ello, se presenta inicialmente una novedosa estructuración de la información, de forma que el modelo multietapa se puede representar como un modelo con dos bloques de etapas. Finalmente, sobre esta formulación se puede utilizar una estrategia similar a la CBD presentada anteriormente.

Etapa de rotura y estructura en dos bloques de etapas

Por simplicidad, vamos a considerar un modelo multietapa en la que las variables de cada etapa están relacionadas sólo con variables de la misma etapa o la anterior. De esta forma y siguiendo la representación por grupos de escenarios (entendiendo por grupo de escenario cada uno de los nodos del árbol multietapa) podríamos escribir nuestro problema de la siguiente forma.

$$\begin{aligned} \min & \sum_{g \in \mathcal{G}} w_g(c^g x^g) & \\ \text{s.t. } & b_1^1 \leq A_1 x^1 \leq b_2^1 & \\ & b_1^g \leq A'_g x^{\pi(g)} + A_g x^g \leq b_2^g & \forall g \in \mathcal{G} - \{1\} \\ & x^g \in I\!R^+ & \forall g \in \mathcal{G}, \end{aligned} \tag{B.20}$$

donde g representa cada nodo del árbol B.1 o grupo de escenarios, $\pi(g)$ representa el nodo predecesor de g y el resto de vectores y matrices son los habituales.

Así, proponemos una partición del conjunto de escenarios en clusters o racimos de escenarios que identificará de forma única la denominada etapa de rotura.

Descomponemos el árbol de escenarios en un subconjunto de subárboles de clusters de escenarios $\mathcal{P}$, cada uno para un cluster de escenarios, p, para $p \in \mathcal{P}$. Sea q el número de clusters a considerar, i.e., $q = |\mathcal{P}|$. Sea Ω^p el conjunto de escenarios que pertenecen a un cluster genérico p donde $p \in \mathcal{P}$ y $\sum_{p=1}^q |\Omega^p| = |\Omega|$. Sea $\mathcal{G}^p \subset \mathcal{G}$ el conjunto de grupos de escenarios para el cluster p. Así que, $\mathcal{G}_t^p = \mathcal{G}_t \cap \mathcal{G}^p$ denota el conjunto de grupos de escenarios para el cluster $p \in \mathcal{P}$ en la etapa $t \in \mathcal{T}$.

Sin pérdida de generalidad, el número de clusters q se puede elegir como divisor de $|\Omega|$, en cuyo caso, el número de escenarios en cada cluster es el mismo y viene dado por, $1 \leq |\Omega^p| = \frac{|\Omega|}{q} \leq |\Omega|$.

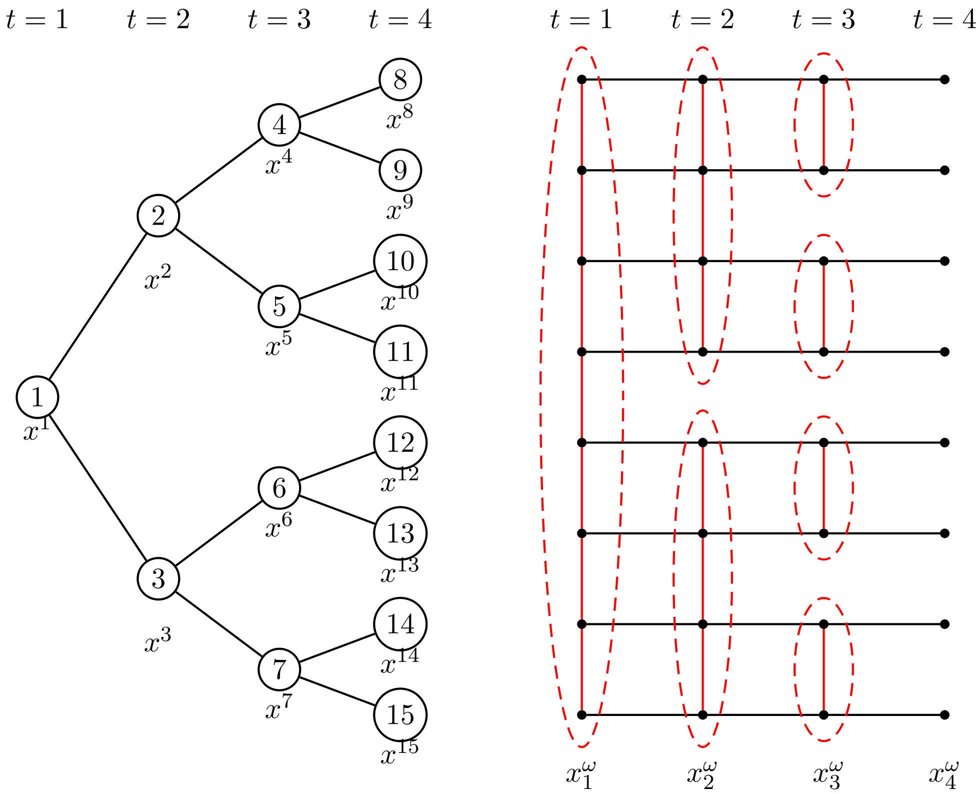


Figure B.1: Árbol de escenarios en formulación compacta y extendida.

Definición 1 *La etapa de rotura t^* es una etapa t tal que el número de clusters de escenarios es $q = |\mathcal{G}_{t^*+1}|$, donde $t^* + 1 \in \mathcal{T}$. Observar que en este caso, cualquier cluster $p \in \mathcal{P}$ se incluye en algún grupo $g \in \mathcal{G}_{t^*+1}$ y contiene todos los escenarios que corresponden a ese grupo, i.e., $\Omega^p = \Omega_g$.*

Una vez determinada la etapa de rotura, se divide el conjunto de etapas del problema en dos bloques. El primero hasta la etapa $t^* + 1$ y el segundo de la etapa $t^* + 2$ hasta la última. Con esta partición, se puede reformular el problema, de forma que en el primer bloque de etapas se formula la no-anticipatividad de forma implícita para el modelo completo, y el segundo bloque de etapas, se formula de forma implícita para cada cluster de escenarios.

Obtenemos las siguiente representación general de los modelos multietapa en dos bloques de etapas,

$$\begin{aligned} (DEM) \quad z = & \min \mathbf{c}^t\mathbf{x} + E_\psi[\mathbf{w}^p(\mathbf{q}^p\mathbf{y}^p)] \\ \text{s.t.} \quad & \mathbf{B}_1 \leq \mathbf{A}\mathbf{x} \leq \mathbf{B}_2, \\ & \mathbf{h}_1^p \leq \mathbf{T}^p\mathbf{x} + \mathbf{W}^p\mathbf{y}^p \leq \mathbf{h}_2^p \quad \forall p \in \mathcal{P}, \\ & \mathbf{x}, \mathbf{y}^p \in I\!R^+ \quad \forall p \in \mathcal{P}, \end{aligned} \tag{B.21}$$

donde $\mathbf{x}$ es el vector de variables del primer bloque de etapas, i.e., es el vector de variables formado por todas las variables desde la etapa 1 al $t^* + 1$ e $\mathbf{y}^p$ denota el vector de variables del segundo bloque de etapas en el cluster p. Las matrices identifican para el ejemplo detallado en el Capítulo 4 de la memoria completa en la Figura B.2.

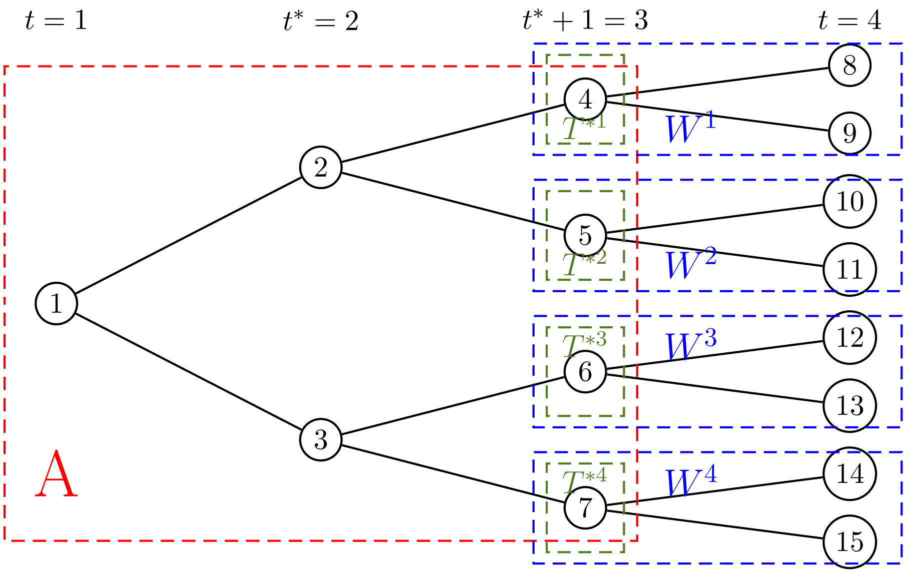


Figure B.2: Especificación de las matrices en un ejemplo.

El primer bloque incluye las etapas $t = 1, \cdots, t^* + 1$ y el segundo incluye el resto. Las variables de la etapa $t^* + 1$ son las variables de enganche entre ambos bloques, siendo t^* la etapa de rotura.

Esquema MCBD

El método propuesto y denotado como MCBD trabaja sobre una estructura en la que hemos dividido el conjunto de etapas en dos partes o bloques.

Necesitamos por lo tanto de un proceso de estructuración y generación de los datos del nuevo modelo en dos bloques de etapas. Los pasos más destacados de este procedimiento preliminar aparecen en la Tabla B.1.

Table B.1: Estructuración de los datos

Paso 0:	Definir el árbol de escenarios: $\mathcal{T}$, Ω, $\mathcal{G}$, $\mathcal{G}_t$, $\forall t \in \mathcal{T}$, Ω_g, $\forall g \in \mathcal{G}$ y w^ω, $\forall \omega \in \Omega$.
Paso 1:	Decidir la etapa de rotura t^* y, por tanto, el número de clusters de escenarios $q = \|\mathcal{G}_{t^*+1}\|$.
Paso 2:	Definir $\mathcal{G}^p$, $\mathcal{G}^p_t$, Ω^p, ω^p_g, $\mathcal{P}^t$, ϕ^p_t y n^p_{xt} $\forall t \in \mathcal{T}, p \in \mathcal{P}$.
Paso 3:	Generar los submodelos por cluster
Paso 4:	Generar el modelo completo

De esta forma, el árbol de escenarios se descompone en un árbol de clusters de escenarios. El algoritmo propuesto trabaja sobre la estructura compacta de la información del modelo completo para las etapas en el primer bloque, y también sobre la representación compacta de cada cluster para las etapas del segundo bloque.

Paso 0: Fijar $k := 0$, $p := 0$, $e_o := 0$, $e_{fc} := 0$.

Paso 1: Resolver el problema maestro relajado *RMP* (con $\theta = 0$ si $k = 0$). $k := k + 1$.

$$
\begin{aligned}
(RMP) \qquad & \min \quad \mathbf{c}^T\mathbf{x} + \theta \\
& \text{s.t.} \\
& \mathbf{B}_1 \le \mathbf{A}\mathbf{x} \le \mathbf{B}_2 \\
& 0 \ge \hat{\nu}^{pT}_{j_1} \begin{pmatrix} \mathbf{h}^p_1 + \mathbf{T}^p\mathbf{x} \\ -\mathbf{h}^p_2 + \mathbf{T}^p\mathbf{x} \end{pmatrix} \qquad \forall j_1 = 0, ..., e_f
\end{aligned} \tag{B.22}
$$

$$\begin{array}{ll} \theta \geq \sum_{p\in\mathcal{P}} \mathbf{w}^p \hat{\upsilon}_{j_2}^{pT} \begin{pmatrix} \mathbf{h}_1^p + \mathbf{T}^p\mathbf{x} \\ -\mathbf{h}_2^p + \mathbf{T}^p\mathbf{x} \end{pmatrix} & \forall j_2 = 0, ..., e_o \\ \mathbf{x} \geq 0, \theta \in I\!R, & \end{array} \tag{B.23}$$

Guardar los valores $\hat{x}$ y $\hat{\theta}$ de las variables x y θ.

Paso 2: Resolver el problema auxiliar de factibilidad para cada cluster de escenarios p,

$$\begin{array}{ll} (FEAS) & z_{FEAS}^p = \min e^T v_1^{+p} + e^T v_1^{-p} + e^T v_2^{+p} + e^T v_2^{-p} \\ & s.t. \\ & \mathbf{W}^p\mathbf{y}^p - Iu^{-p} + Iv_1^{+p} - Iv_1^{-p} = \mathbf{h}_1^p - \mathbf{T}^p\hat{\mathbf{x}} \\ & \mathbf{W}^p\mathbf{y}^p + Iu^{+p} - Iv_2^{+p} + Iv_2^{-p} = \mathbf{h}_2^p - \mathbf{T}^p\hat{\mathbf{x}} \\ & \mathbf{y}^p, v_1^{+p}, v_1^{-p}, v_2^{+p}, v_2^{-p}, u^{+p}, u^{-p} \geq 0. \end{array} \tag{B.24}$$

Si $z_{FEASC}^p \neq 0$, la solución obtenida en el Paso 1 no es factible en ese cluster, y hay que definir y añadir el corte de factibilidad (B.22) e ir al Paso 1.

En caso contrario, volver al Paso 2 hasta llegar al último cluster. En este caso, la solución obtenida es factible y vamos al Paso 3.

Paso 3: Para cada escenario cluster del segundo bloque de etapas, resolver el problema auxiliar de optimalidad ω, con $\omega \in \Omega$,

$$\begin{array}{lll} (OPT) & \phi^p = & \min \mathbf{q}^{pT}\mathbf{y}^p \\ & & s.t. \\ & & \begin{pmatrix} \mathbf{W}^p \\ -\mathbf{W}^p \end{pmatrix} \mathbf{y}^p \geq \begin{pmatrix} \mathbf{h}_1^p - \mathbf{T}^p\hat{\mathbf{x}} \\ -\mathbf{h}_2^p + \mathbf{T}^p\hat{\mathbf{x}} \end{pmatrix} \\ & & \mathbf{y}^p \geq 0. \end{array} \tag{B.25}$$

Calcular el valor esperado $\phi := \sum_{\omega\in\Omega} w^\omega \phi^\omega$. Comparar este valor esperado con $\hat{\theta}$. Si $\phi \leq \hat{\theta}$, entonces parar, pues hemos encontrado el óptimo. En caso contrario, se genera y añade al problema maestro un corte de optimalidad (B.23) y se vuelve al Paso 1.

Como trabajo futuro más inmediato, se está considerando la implementación del procedimiento propuesto así como una comparación entre el modelo propuesto y la tradicional descomposición de Benders anidada como herramienta de descomposición para la resolución de modelos multietapa.

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