Tesis Doctoral

2018

Caracterización de Soluciones de Problemas de Equilibrio Vectoriales

Juan Luis Ródenas Pedregosa

Programa de Doctorado en
Tecnologías Industriales

Director: Vicente Novo Sanjurjo

Codirector: César Gutiérrez Vaquero

A mis padres.

Agradecimientos / Acknowledgements

En primer lugar, quiero agradecer a los profesores Vicente Novo y César Gutiérrez la confianza depositada en mí para realizar esta tesis; su esfuerzo, apoyo y dedicación a lo largo de todo este tiempo; por despertar en mí la curiosidad e interés en el campo de la investigación y por haberme hecho crecer como matemático y como persona.

Quiero expresar mi gratitud a todos mis compañeros del Departamento de Matemática Aplicada I por su cálida acogida. En particular, a Lidia Huerga y a Estibalitz Durand, con quienes he compartido la mayor parte de mis horas en este largo recorrido y siempre han tenido una gran predisposición para resolver cualquier tipo de problema. Asimismo a los profesores Bienvenido Jiménez y Esther Gil, con quienes he compartido docencia agradablemente y siempre se han mostrado muy colaborativos.

I want to express my sincere gratitude to Professor Tamaki Tanaka, from the Niigata University, for taking very good care of me during my first stay abroad, and for making me curious about Japanese culture. I learned a lot from our mathematical discussions and it was very helpful for the development of this thesis.

I would like also to thank Professor Gábor Kassay, from the Babeş-Bolyai University, for his friendly reception in Cluj-Napoca during my second stay. Now I know much better the beautiful country of Romania. His valuable comments were very useful for this thesis.

Por último, quiero agradecer el inestimable apoyo que he recibido durante todo este tiempo de mis padres y mi hermano, por su paciencia y por acompañarme desde la distancia a lo largo de este difícil camino, así como por sus valiosos consejos en los mejores y peores momentos.

Preface

This work is submitted in order to aim for the PhD degree with international mention by the National Distance Education University (UNED) in the PhD Program in Industrial Technologies. Since English is the most habitual way to communicate in Mathematical Research, this work is written in this language almost entirely, according to the UNED regulations.

My interest in Operations Research began when I was studying for a Mathematics degree at the University of Alicante, whose contents were considerably focused on this area. In addition, I completed a Master's Degree at the Complutense University of Madrid, in which I learned some advanced topics in Functional Analysis as, for instance, the Ekeland Variational Principle.

In Spain, the *Vector Optimization Group* (UNED), leaded by Professor Vicente Novo, was one of the most interesting research groups to develop a PhD thesis in Optimization with a strong component in Functional Analysis. In 2013 I applied for the PhD courses *Multiobjective programming* and *Multiobjective optimization* of UNED, in which Professors Vicente Novo and Bienvenido Jiménez introduced me to vector optimization and vector equilibrium problems.

In 2014 I started my PhD thesis with Professors Vicente Novo (UNED) and César Gutiérrez (University of Valladolid), and I became a member of their research team inside the project *Optimization of Vector Functions and Set-Valued Maps* (reference MTM2012-30942) with a PhD fellowship (Spanish FPI Fellowship Programme, reference BES-2013-066316). Recently, our group obtained a new project named *Optimization and Equilibrium Problems with Vector and Set-Valued Mappings* (reference MTM2015-68103-P) from the Spanish Ministry of Economy and Competitiveness with the excellent rating.

The Spanish FPI Fellowship Programme has a mobility subprogramme which allowed me to enjoy two brief research stays in other universities. In 2015, I stayed 3 months in the Graduate School of Science and Technology of Niigata University of Japan (reference EEBB-I-15-10365) with the support of Professor Tamaki Tanaka. In 2016, I stayed 4 months in the Faculty of Mathematics and Computer Science of Babeş-Bolyai University (reference EEBB-I-16-11560) with the support of Professor Gábor Kassay. Both stays had a significant value to develop this PhD thesis.

In today's global society, mathematics plays directly or indirectly an important role. Scientific and industrial needs are increasingly complex and we have to provide better mathematical tools and results in order to handle with any hypothetical situation. This work contributes to such aim with new meaningful improvements and extensions of several popular results in the literature from alternative points of view. Moreover, we focus on equilibrium problems since in this way we deal together with many different mathematical problems, and hence the obtained results can be applied to a wide range of real situations.

For any suggestion or comment concerning about this work, the reader can write to the following email address:

jlrodenas@ind.uned.es

Madrid, March 2018

Juan Luis Ródenas Pedregosa

Index

Chapter 1

Introduction

It is well known that mathematics is permanently involved in our daily life. We have automated many actions in which we solve different problems in a simple way, without noticing about it and even without having advanced studies. For instance, the millennial decimal system is transmitted and elementary operations are taught in modern societies. Everybody uses this system and, moreover, everybody actually needs it. In the past, the Roman numeral system was established in Europe for many centuries since it allows to add and subtract with relative easiness, while the decimal system was not widespread. However, experts in mathematics were required to multiply and divide (even computing the current VAT of a product would not be a simple problem if only Roman numerals are known). Currently, our global society implicitly implements mathematics in the day to day as well as scientists discover them.

Nowadays, mathematicians are focused on solving more increasingly complex problems due to the current needs of the society. Uncountable models for our day to day are introduced by physicists, chemists, engineers, economists and so on in their respective areas, and they have to be studied from a mathematical point of view. For these goals, mathematical classic problems are considered in different ways by means of proper adaptations and extensions, although they were originally motivated for a different situations or settings.

The intrinsic transversality of mathematics allows to apply results from one area into other ones, and then, the theory that was developed in a natural way may be inherited by the other ones. Therefore, mathematical models that allow to work in several areas at the same time and to transfer results between them automatically, are interesting to be studied.

1.1 Scalar equilibrium problems

In 1994, E. Blum and W. Oettli published the work titled *From optimization and variational inequalities to equilibrium problems* [28], in which several classic mathematical problems from different areas were generalized and unified in only one: *equilibrium problems*. After that, this concept was popularized in such a way that work [28] has more than 800 citations according to MathSciNet database, which is a very considerable figure in the field of mathematics.

The term of *equilibrium* is defined as *a state of rest or balance due to the equal action of opposing forces*, so that is very common in other areas as Physics, Chemistry, Biology or Economics. In mathematics, the notion of equilibrium problem is actually a Ky Fan inequality [59], and it was originally proposed to establish the existence of equilibriums in Game Theory. In [147] and [150], the *equilibrium problem* is defined before than in [28], but is not the main topic of any of these two works.

Let us recall the original framework given in [28]. Consider a real topological linear space X, a nonempty closed convex subset $S \subset X$ and a bifunction $f\colon S \times S \to \mathbb{R}$. The *equilibrium problem* (EP in short form) is the following one: Find $\bar{x} \in S$ such that

$$f(\bar{x}, x) \geq 0 \quad \forall x \in S. \tag{1.1}$$

A point satisfying this condition is said to be a solution of EP and it is denoted as $\bar{x} \in \mathrm{E}(f, S)$.

Next, we will show some examples of classic mathematical problems that are particular cases of equilibrium problems (see [28, 57, 122]). Here X^* denotes the topological dual space of X. Assume that X^* is topologized so that the canonical bilinear form $\langle \cdot, \cdot \rangle$ is continuous on $X^* \times X$.

Problem 1.1. (Optimization problems)
Let $g\colon S \to \mathbb{R}$ be a function. Consider the problem of finding $\bar{x} \in S$ such that

$$g(\bar{x}) \leq g(x) \quad \forall x \in S. \tag{1.2}$$

We will refer to it as OP, and we will denote its set of solutions as $\operatorname{argmin}(g, S)$ or $\mathrm{O}(g, S)$. By taking

$$f(x, y) := g(y) - g(x),$$

it is easy to check that $\bar{x} \in \mathrm{E}(f, S)$ if and only if $\bar{x} \in \mathrm{O}(g, S)$.

Problem 1.2. (Variational inequalities)
Let $T\colon S \to X^*$ be a mapping. It is requested to find $\bar{x} \in S$ such that

$$\langle T(\bar{x}), x - \bar{x} \rangle \geq 0 \quad \forall x \in S. \tag{1.3}$$

We will refer to it as VIP, and we will denote its set of solutions as $\mathrm{V}(T, S)$. By considering

$$f(x, y) := \langle T(x), y - x \rangle,$$

it is clear that $\bar{x} \in \mathrm{E}(f, S)$ if and only if $\bar{x} \in \mathrm{V}(T, S)$.

Notice that if $g\colon X \to \mathbb{R}$ is convex and Gâteaux differentiable in Problem 1.1, with Gâteaux differential $Dg(x) \in X^*$ at $x \in X$, then we have another way to deal with convex differentiable optimization problems via variational inequalities. In fact, it is known from Convex Analysis that $\bar{x} \in \mathrm{V}(Dg, S)$ if and only if $\bar{x} \in \operatorname{argmin}(g, S)$.

Problem 1.3. (Saddle point problems)
Suppose that $S_1 \subset X$ and $S_2 \subset X$ are nonempty convex closed sets and let $g\colon S_1 \times S_2 \to \mathbb{R}$ be a bifunction. A point $(\bar{x}_1, \bar{x}_2) \in S_1 \times S_2$ is said to be a saddle point of g if

$$g(\bar{x}_1, x_2) \leq g(x_1, \bar{x}_2) \quad \forall x_1 \in S_1, x_2 \in S_2. \tag{1.4}$$

By setting $S := S_1 \times S_2$ and $f\big((x_1, x_2), (y_1, y_2)\big) := g(y_1, x_2) - g(x_1, y_2)$, we have that $(\bar{x}_1, \bar{x}_2) \in E(f, S)$ if and only if $(\bar{x}_1, \bar{x}_2)$ is a solution of (1.4).

Problem 1.4. (Fixed points)

Consider a mapping $T\colon S \to S$. It is said that $\bar{x} \in S$ is a fixed point of T if satisfies

$$T(\bar{x}) = \bar{x}. \tag{1.5}$$

Consider $\bar{x} \in S$ and set $f(x,y) := \langle x - T(x), y - x\rangle$. We have that $\bar{x} \in \mathrm{E}(f,S)$ if and only if $\bar{x}$ satisfies (1.5). Indeed, if $\bar{x}$ is a fixed point of T, then $f(\bar{x},x) = 0$ for all $x \in S$, so $\bar{x} \in \mathrm{E}(f,S)$. Conversely, if $\bar{x}$ is a solution of EP, we have that $f\big(\bar{x}, T(\bar{x})\big) = -\|\bar{x} - T(\bar{x})\|^2 \geq 0$, and then $T(\bar{x}) = \bar{x}$ necessarily.

Problem 1.5. (Nash equilibria in noncooperative games)

Suppose that there are $n \in \mathbb{N}$ players and each one j has associated a nonempty convex closed strategy set, S_j. Set $S := \prod_{j=1}^{n} S_j$, the set of all strategies, and let $f_j\colon S \to \mathbb{R}$ be the loss function of the j-th player. Given a play $x = (x_1, \ldots, x_n) \in S$, the play of all players except $i \in \{1, \ldots, n\}$ is denoted by $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. A point $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n) \in S$ is said to be a Nash equilibrium if for every player $j = 1, \ldots, n$ it is verified that

$$f_j(\bar{x}) \leq f_j(\bar{x}_{-j}, y_j) \quad \forall y_j \in S_j. \tag{1.6}$$

That is, $\bar{x}$ is a Nash equilibrium if no player can reduce his loss modifying uniquely his own strategy. By setting $f\colon S \times S \to \mathbb{R}$ as

$$f(x,y) = \sum_{j=1}^{n} \big(f_j(x_{-j}, y_j) - f_j(x)\big),$$

there holds that $\bar{x} \in S$ satisfies (1.6) if and only if $\bar{x} \in \mathrm{E}(f,S)$. Indeed, if $\bar{x}$ is a Nash equilibrium, then $f_j(\bar{x}_{-j}, x_j) - f_j(\bar{x}) \geq 0$ for all $j = 1, \ldots, n$ and $x \in S$, and so $f(\bar{x}, x) \geq 0$ for all $x \in S$. Conversely, if $\bar{x} \in \mathrm{E}(f,S)$, we choose an arbitrary player $k \in \{1, \ldots, n\}$ and $x \in S$ such that $x_{-k} = \bar{x}_{-k}$. Hence, for each $x \in S$

$$0 \leq f(\bar{x}, x) = \sum_{j=1}^{n} \big(f_j(\bar{x}_{-j}, x_j) - f_j(\bar{x})\big) = f_k(\bar{x}_{-k}, x_k) - f_k(\bar{x}),$$

and so $f_k(\bar{x}_{-k}, x_k) \geq f_k(\bar{x})$ for all $x_k \in S_k$. Since k is an arbitrary player, it follows that $\bar{x}$ is a Nash equilibrium.

Problem 1.6. (Complementarity problems)
This is a special case of Problem 1.2. Let S be a closed convex cone and $S^\circ := \{x^* \in X^* : \langle x^*, x\rangle \geq 0 \text{ for all } x \in S\}$ be its positive polar cone. Consider a mapping $T\colon S \to X^*$. The problem is to find $\bar{x} \in S$ such that

$$\begin{cases} T(\bar{x}) \in S^\circ, \\ \langle T(\bar{x}), \bar{x}\rangle = 0. \end{cases} \tag{1.7}$$

Then $\bar{x}$ is a solution of (1.7) if and only if $\bar{x} \in \mathrm{V}(T, S)$. Indeed, if (1.7) holds, then $T(\bar{x}) \in S^\circ$ and so $\langle T(\bar{x}), x\rangle \geq 0$ for all $x \in S$. Moreover, as $\langle T(\bar{x}), \bar{x}\rangle = 0$, we have that $\langle T(\bar{x}), x - \bar{x}\rangle \geq 0$ by linearity, so $\bar{x} \in \mathrm{V}(T, S)$. Conversely, if $\bar{x} \in \mathrm{V}(T, S)$, by considering $x = 2\bar{x} \in S$ in (1.3), it follows that $\langle T(\bar{x}), \bar{x}\rangle \geq 0$, and by taking $x = 0 \in S$ in (1.3), we obtain that $\langle T(\bar{x}), -\bar{x}\rangle \geq 0$. Hence $\langle T(\bar{x}), \bar{x}\rangle = 0$ and $0 \leq \langle T(\bar{x}), y - \bar{x}\rangle = \langle T(\bar{x}), y\rangle$ for all $y \in S$. Therefore $\bar{x}$ satisfies (1.7).

Problem 1.7. (Variational inequalities with set-valued mappings)
Let $T\colon S \rightrightarrows X^*$ be a set-valued mapping with nonempty convex compact values. It is requested to find $\bar{x} \in S$ and $\bar{\xi} \in T(\bar{x})$ such that

$$\langle \bar{\xi}, y - \bar{x}\rangle \geq 0 \quad \forall y \in S. \tag{1.8}$$

By setting $f(x, y) := \max_{\xi \in T(x)} \langle \xi, y - x\rangle$, it is verified that $\bar{x} \in \mathrm{E}(f, S)$ if and only if $\bar{x}$ together with some $\bar{\xi} \in T(\bar{x})$ satisfy (1.8). Indeed, if $\bar{x} \in \mathrm{E}(f, S)$, then

$$f(\bar{x}, x) = \max_{\xi \in T(\bar{x})} \langle \xi, x - \bar{x}\rangle \geq 0 \quad \forall x \in S.$$

We will show by contradiction that there exists some suitable $\bar{\xi} \in T(\bar{x})$ such that $\bar{x}$ and $\bar{\varepsilon}$ satisfy (1.8). Suppose that for each $\xi \in T(\bar{x})$, there exists $x_\xi \in S$ and $\varepsilon_\xi > 0$ such that $\langle \xi, x_\xi - \bar{x}\rangle < -\varepsilon_\xi$. Hence, the open sets

$$S(x, \varepsilon) := \{\xi \in T(\bar{x}) : \langle \xi, x - \bar{x}\rangle < -\varepsilon\} \quad (x \in S, \varepsilon > 0)$$

cover the set $T(\bar{x})$. By compactness, there exists a finite subcover of $T(\bar{x})$,

$$\{S(x_j, \varepsilon_j)\}_{j=1}^n.$$

Define $\varepsilon := \min_{j=1,\ldots,n} \{\varepsilon_j\}$. Since $T(\bar{x}) \subset \bigcup_{j=1}^n S(x_j, \varepsilon_j) \subset \bigcup_{j=1}^n S(x_j, \varepsilon)$, then

$$\min_{j=1,\ldots,n} \langle \xi, x_j - \bar{x}\rangle \leq -\varepsilon \quad \forall \xi \in T(\bar{x}).$$

By applying Gordan's alternative theorem (see, for instance, [46, Theorem 3.4.2]), we deduce that there exist nonnegative real numbers $\lambda_1, \ldots, \lambda_n$ such that $\sum_{j=1}^{n} \lambda_j = 1$ and

$$\sum_{j=1}^{n} \lambda_j \langle \xi, x_j - \bar{x} \rangle \leq -\varepsilon \quad \forall \xi \in T(\bar{x}).$$

Since S is convex, then $\hat{x} := \sum_{j=1}^{n} \lambda_j x_j \in S$. Moreover, it is clear that

$$\langle \xi, \hat{x} - \bar{x} \rangle \leq -\varepsilon \quad \forall \xi \in T(\bar{x}).$$

Therefore, $f(\bar{x}, \hat{x}) < 0$, which contradicts that $\bar{x} \in \mathrm{E}(f, S)$. Conversely, if $\bar{x} \in S$ and $\bar{\xi} \in T(\bar{x})$ satisfy (1.8), then

$$f(\bar{x}, x) \geq \langle \bar{\xi}, x - \bar{x} \rangle \geq 0 \quad \forall x \in S,$$

and so $\bar{x} \in \mathrm{E}(f, S)$.

Remark 1.8. Let $g\colon S \to \mathbb{R}$ be a locally Lipschitz function and denote by $\partial^\circ g(x)$ and $g^\circ(x, v)$ the Clarke generalized gradient of g at x, and the Clarke derivative of g at x in the direction $v \in X$, respectively (see [44]). Then, if we consider $T = \partial^\circ g$ in Problem 1.7, we see that $f(x, y) = g^\circ(x, y - x)$.

It is clear from the previous examples that equilibrium problems encompass many different problems. Then, a natural line of research concerns with the generalization of important results of these particular problems to the general equilibria framework, so that these results can be applied to the other reunified problems. For example, some optimality conditions for nonsmooth optimization problems were extended to equilibrium problems, and so, they were automatically applied to variational inequalities (see [81]).

In [28] it is assumed on a standard equilibrium problem that f satisfies the *diagonal null property*, that is,

$$f(x, x) = 0 \quad \forall x \in S,$$

and that f is *monotone*, that is,

$$f(x, y) + f(y, x) \leq 0 \quad \forall x, y \in S. \tag{1.9}$$

Diagonal null property and monotone type assumptions play an important role in equilibrium problems both in the scalar case, and in the vector-valued case. Moreover, diagonal null assumption and triangle inequality property (see (1.12)), have a direct impact in Ekeland Variational Principles for bifunctions, as we will see further in Chapter 4. However, many authors did not consider these assumptions in the initial setting of a standard equilibrium problem in order to deal with a more general problem.

Another natural research line similar to the previous ones deals with some mathematical classic tools to be applied to equilibrium problems, as Knaster-Kuratowski-Mazurkiewicz theorems [58, 121], Fan theorems [7, 59] or Ekeland variational principles [54–56]. For instance, Kalmoun and Riahi [115] obtained a generalized Knaster-Kuratowski-Mazurkiewicz theorem, which was applied to equilibrium problems, and after that, existence results for maximal elements with respect to a preference relation, fixed point and saddle point theorems were achieved.

The classic Weierstrass theorems allows to show the existence of solutions in optimization problems. It is based on compactness and continuity assumptions, but there are many generalizations in which these assumptions have been weakened by using different semicontinuity notions or some closedness hypotheses on the sublevel sets, and other weakest statements that may replace the compactness requirement. In a similar way, in the literature there is a very wide diversity of existence results for equilibrium problems. The ones obtained by Blum and Oettli [28] can not be derived by using Weierstrass theorem on a product space, since different assumptions are required for each coordinate of the bifunction. However, existence results for optimization problems can be deduced as particular cases of existence results in equilibrium problems.

Approximate solutions are a basic concept to deal with real world problems and so they are a requested research topic in several areas from long time ago. The current models hold much information, which requires to handle thousands (or millions) of variables and, consequently, its complexity is very high. Then, from a mathematical point of view, these models are not tackled without using computers, since numerical algorithms are designed to solve as fast as possible complex mathematical problems. In this sense, approximate solutions are achieved in much less time than exact solutions with a controlled error. Moreover, in many real models it is not pragmatically possible to obtain an exact solution, since the computational cost or the runtime may not be assumed, even being a solvable problem.

Approximate solutions have been comprehensively studied in different settings: optimization problems, variational inequalities, fixed-point theorems, Nash equilibrium problems and so forth. Hence it is natural to study approximate solutions for equilibrium problems (see, for instance, [7, 21, 22, 36, 116]). An element $\bar{x} \in S$ is said to be an approximate solution of equilibrium problem EP with error $\varepsilon \geq 0$ if

$$f(\bar{x}, x) \geq -\varepsilon \quad \forall x \in S. \tag{1.10}$$

It is denoted by $\bar{x} \in \mathrm{E}(f, S, \varepsilon)$. In the literature have been defined other concepts of approximate solutions, so that in the next chapters and for the vector-valued case we will consider a more general notion.

The notion of strict solution is well known in the area of Optimization. A point $\bar{x} \in S$ is a *strict solution* of OP if

$$g(\bar{x}) < g(x) \quad \forall x \in S \backslash \{\bar{x}\}.$$

This assertion is more restrictive than (1.2) and guarantees that $\bar{x}$ is the only solution of OP. It was also extended to vector optimization problems (see [113, 175]) and set-valued optimization problems (see [8,66]). On the other hand, it has also been defined for variational inequalities problems and applied to Economics (see for instance [47, 114]). Hence, it was natural to extend the notion of strict solution to equilibrium problems (see, for instance, [25]). An element $\bar{x} \in S$ is said to be a *strict solution of EP* if

$$f(\bar{x}, x) > 0 \quad \forall x \in S \backslash \{\bar{x}\}. \tag{1.11}$$

It is denoted by $\bar{x} \in \mathrm{S}(f, S)$. Notice that every strict solution is a solution whenever f verifies the diagonal null property. Moreover, if f is monotone (see

(1.9)) and $\bar{x}$ is a strict solution of f, then $\bar{x}$ is the unique strict solution. On the other hand, the notion of strict solution plays an important role in the Ekeland Variational Principle.

It is known that a lower semicontinuous bounded below function on a noncompact set may not attain its infimum. Variational principles allow us to add a perturbation with controllable behavior to this kind of functions so that a minimum is attained. The most known variational principal is the Ekeland Variational Principle (briefly, EVP), introduced by Ivar Ekeland [54, 55], which is a powerful tool with applications in numerous areas as Nonlinear Analysis, Convex Analysis, Optimization, Differential Geometry, Differential Equations, Fixed Point Theory, Mathematical Finance and so on. Next, let us recall it.

Theorem 1.9. Let (X, d) be a complete metric space and let $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous bounded below function. Suppose that $\varepsilon > 0$ and $x_0 \in X$ satisfy that

$$f(x_0) \leq \inf_{x \in X} f(x) + \varepsilon.$$

Then, for any $\delta > 0$ there exists $\bar{x} \in X$ such that

(*a*) $f(\bar{x}) + \dfrac{\varepsilon}{\delta} d(x_0, \bar{x}) \leq f(x_0)$,

(*b*) $d(x_0, \bar{x}) \leq \delta$,

(*c*) $f(x) + \dfrac{\varepsilon}{\delta} d(x, \bar{x}) > f(\bar{x}) \quad \forall x \in X \backslash \{\bar{x}\}$.

Notice that the perturbation function is $\dfrac{\varepsilon}{\delta} d(\cdot, \bar{x})$, and that the perturbed function $f(\cdot) + \dfrac{\varepsilon}{\delta} d(\cdot, \bar{x})$ attains a strict minimum at $\bar{x}$.

The Ekeland Variational Principle has some equivalent results in the sense that one proves each other. Some of them in the setting of Banach spaces are the Bishop-Phelps Theorem [26, 27] (in fact, this earlier result was an inspiration for Ivar Ekeland, see [56]), the Daneš' Drop Theorem [48], the Penot's Flower-Petal Theorem [153] (see also [166]); in complete metric spaces, the Kirk-Caristi's Fixed Point Theorem [119] or the Takahashi's Nonconvex Minimization Theorem [180]. Furthermore, Sullivan [178] proved that the completeness of a metric space is characterized by the EVP.

In the literature there are uncountable versions of the EVP (see, for instance, [117, Chapter 10]). Even before the equilibrium problems become popular with the work [28], Oettli and Théra [150] studied the equivalence of several results with an EVP for bifunctions and applied it to obtain an existence result for EP ([150, Theorem 6]), where the domain of the bifunction is a complete metric space. By the same approach, Bianchi, Kassay and Pini [21] also obtained several existence results in an Euclidean space, without any convexity assumption on the underlying set, whether it is closed or compact, and later they applied it to study a well-posedness notion for equilibrium problems in [24] (see also [22, 23] for the vector-valued case). In these works, the triangular inequality property on a bifunction is required, that is, $f\colon X \times X \to \mathbb{R}$ satisfies that

$$f(x,z) \leq f(x,y) + f(y,z) \quad \forall x,y,z \in X. \tag{1.12}$$

Approximate variational principles are another interesting research line. In these results, the perturbed function achieves an approximate strict solution instead of an exact strict solution under less restrictive assumptions. Combari, Marcellin and Thibault [45] obtained an approximate EVP in non complete normed linear spaces and they applied it to achieve graph convergence results for ε-Fenchel subdifferentials in a non complete setting. In Chapter 4, we will focus on exact and approximate EVPs for vector-valued bifunctions in a more general framework than the one considered until now in the literature for this kind of results, in which further information may be found. New results with significant improvements are obtained and the roles of some usual assumptions for this sort of results are clarified.

The reader may find other extensions of the EVP for bi-set-valued mappings in [12, 82, 117, 161, 183, 198] and references therein. There are more types of variational principles by depending on the perturbation function properties. The Stegall Variational Principle [177], given in a Banach space X, attains a strong minimum on $S \subset X$ and the perturbation function is a bounded linear function, but it needs stronger assumptions on S. Other important results are the smooth variational principles which are focused on the differentiability of the perturbed function when the initial function is also differentiable. This is the case of Borwein-Preiss Variational Principle [31] (see also [18, 124, 125, 137, 155]) and Deville-Godefroy-Zizler Variational Principle [50, 51], which have important applications to subdifferentiability, differentiability of convex functions, geometry of Banach Spaces, Hamilton-Jacobi equations, etc. Moreover, Finet and Quarta [61] extended the Deville-Godefroy-Zizler variational principle to bifunctions and applied it to obtain two variational principles (one of Borwein-Preiss type and the other one of Ekeland type) and also to derive existence results for equilibrium problems.

1.2 Vector equilibrium problems

In many mathematical areas, it is usual to deal with problems with multiple objectives due to their own nature. All this information must be borne in mind and a way to process it is by considering more abstract spaces which allow to work with several variables at the same time or with functions, matrices, sets and so forth as abstract elements. Hence, it seems natural to replace the real number line $\mathbb{R}$ by the Euclidean space $\mathbb{R}^n$ or, more generally, by a real linear space Y. Here, Cantor and Hausdorff works (see [33, 103]) must be highlighted since they developed the mathematical basic structure of the partially ordered linear spaces.

This extension has some technical difficulties since the real number line has very desirable properties and many mathematical tools that are useful to solve problems. On an arbitrary real linear space, an important difficulty is how to introduce a "good" order for its elements. In general, we might lose the totalness of the usual order on $\mathbb{R}$, that is, it will not be possible to compare all the elements between them. Moreover, as we are going to deal with multicriteria problems, there will be different orders to consider by depending on the nature of the problem.

Recall that a *binary relation* $\mathcal{R}$ *on a set* L is a nonempty subset $\mathcal{R} \subset L \times L$ and it is denoted $y_1 \mathcal{R} y_2$ if $(y_1, y_2) \in \mathcal{R}$. A binary relation on L, $\preceq$, is said to be

- *reflexive* if $y \preceq y \quad \forall y \in L$;
- *transitive* if $y_1 \preceq y_2$ and $y_2 \preceq y_3 \Longrightarrow y_1 \preceq y_3 \quad \forall y_1, y_2, y_3 \in L$;

- *antisymmetric* if $y_1 \preceq y_2$ and $y_2 \preceq y_1 \Longrightarrow y_1 = y_2 \quad \forall y_1, y_2 \in L$;

$\preceq$ is called a *preorder on* L if $\preceq$ is transitive, a *quasiorder on* L if $\preceq$ is reflexive and transitive, and a *partial order on* L if $\preceq$ is reflexive, transitive and antisymmetric.

A partial order $\preceq \subset L \times L$ for which every pair of elements of L are comparable (that is, $y_1 \preceq y_2$ or $y_2 \preceq y_1$ for all $y_1, y_2 \in L$) is said to be a *total order on* L.

A binary relation $\preceq$ on Y is said to be *compatible on* Y if satisfies that

- $y_1 \preceq y_2$ and $y_3 \preceq y_4 \Longrightarrow y_1 + y_3 \preceq y_2 + y_4 \quad \forall y_1, y_2, y_3, y_4 \in Y$;
- $y_1 \preceq y_2 \Longrightarrow \alpha y_1 \preceq \alpha y_2 \quad \forall y_1, y_2 \in Y, \alpha > 0$.

Partial orders on real linear spaces are closely connected to *convex cones*. A nonempty set $D \subset Y$ is said to be *convex* if verifies that

$$y_1, y_2 \in D, \alpha \in (0,1) \Rightarrow \alpha y_1 + (1-\alpha) y_2 \in D,$$

and it is said to be a *cone* if

$$y \in D, \alpha \geq 0 \Rightarrow \alpha y \in D. \tag{1.13}$$

We notice that some authors consider in this definition $\alpha > 0$ instead of $\alpha \geq 0$ and then do not require that $0 \in D$ to be a cone. A cone $D \subset Y$ is *nontrivial* or *proper* if $D \neq \{0\}$ and $D \neq Y$, and is *pointed* if

$$D \cap (-D) = \{0\}.$$

It is well known that a cone $D \subset Y$ is convex if and only if $D + D \subset D$ (see [112, Lemma 1.11]).

Throughout, we denote the binary relation

$$\leq_D := \{(y_1, y_2) \in Y \times Y : y_2 - y_1 \in D\}.$$

Next we recall an important characterization of partial orders on real linear spaces (see [117, Theorem 2.1.11]). It shows that these binary relations are defined through pointed convex cones.

Theorem 1.10. Let Y be a real linear space.

- Consider a cone $D \subset Y$. Then the binary relation $\leq_D$ is reflexive, and it is compatible on Y provided that D is convex. Moreover, D is convex if and only if $\leq_D$ is transitive, and D is pointed if and only if $\leq_D$ is antisymmetric.

- Conversely if $\preceq \subset Y \times Y$ is a reflexive relation that is compatible on Y, then

$$D := \{y \in Y : 0 \preceq y\}$$

is a cone and $\preceq = \leq_D$.

If D is a solid set (that is, $\operatorname{int} D \neq \emptyset$, where int denotes the topological interior), then the following notation is very usual in the literature:

$$y_1 <_D y_2 \Leftrightarrow y_2 - y_1 \in \operatorname{int} D.$$

This notation is extended to arbitrary nonempty sets $E \subset Y$:

$$y_1 \leq_E y_2 \Leftrightarrow y_2 - y_1 \in E,$$

and if E is solid,

$$y_1 <_E y_2 \Leftrightarrow y_2 - y_1 \in \operatorname{int} E.$$

In the algebraic framework, $<_E$ (included the case $E = D$) is usually defined through the algebraic interior (see Section 1.4) instead of the topological interior. To be precise, if E is algebraic solid (that is, $\operatorname{core} E \neq \emptyset$),

$$y_1 <_D y_2 \Leftrightarrow y_2 - y_1 \in \operatorname{core} E.$$

Remark 1.11. Let us observe that the relation $\leq_E$ is transitive and compatible with the sum of Y whenever $E + E \subset E$, i.e., under this condition, if $y_1 \leq_E y_2$, $y_2 \leq_E y_3$ and $z_1 \leq_E z_2$, then $y_1 \leq_E y_3$ and $y_1 + z_1 \leq_E y_2 + z_2$.

Notice that partial orders on $\mathbb{R}$ only can be given by $\mathbb{R}_+ := \{y \in \mathbb{R} : 0 \leq y\}$, $-\mathbb{R}_+$, and the trivial cone $\{0\}$, since there are not more pointed convex cones on $\mathbb{R}$. Clearly, $\leq = \leq_{\mathbb{R}_+}$ and is *total*. However, in general partial orders are not total orders. For instance, consider in $\mathbb{R}^2$ the usual order $\leq_{\mathbb{R}^2_+}$ given by the convex cone $\mathbb{R}^2_+ := \{(y_1, y_2) \in \mathbb{R}^2 : 0 \leq y_1, 0 \leq y_2\}$. Then $(1,0)$ and $(0,1)$ are not comparable since $(1,0) \not\leq_{\mathbb{R}^2_+} (0,1)$ and $(0,1) \not\leq_{\mathbb{R}^2_+} (1,0)$. This handicap has important consequences in vector-valued problems as, for instance, the existence of several notions of solution that coincide only in the scalar case, or the existence of infinite optimal values in many problems. Therefore, the decision maker's criterion to choose possible solutions has a very important role in vector-valued problems.

Vector optimization problems appeared at the end of 19th century in the area of Mathematical Finance, more concretely, in Utility Theory, Welfare Economics and Game Theory. Roughly speaking, the problem is to find the maximal utility when there are several customers with different criteria, even with opposing preferences. Several economists studied this kind of problems and, in particular, the contributions of Edgeworth [53] and Pareto [151] should be remarked. Indeed, we are still using the concept of Pareto optimal solution: Consider a finite dimensional vector optimization problem, VOP,

$$\min\{f(x) : x \in S\},$$

where $S \subset \mathbb{R}^m$ is nonempty, $f\colon S \to \mathbb{R}^n$, $f = (f_1, \ldots, f_n)$, and its coordinates functions are the n objectives to minimize. A point $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m) \in S$ is said to be a *Pareto optimal solution* of VOP if does not exist $x = (x_1, x_2, \ldots, x_m) \in S$ such that

$$\begin{aligned} f_i(x) &\leq f_i(\bar{x}) \quad \forall i = 1, \ldots, n, \text{ and} \\ f_j(x) &< f_j(\bar{x}) \quad \text{for some } j \in \{1, \ldots, n\}. \end{aligned}$$

That is, $\bar{x} \in S$ is a Pareto optimal solution if it is not possible to strictly improve one of the n objectives without worsening at least one of the others.

Yu [193] extended the Pareto's efficiency concept to vector optimization problems by using partial orders. In fact, a partial order is defined on $\mathbb{R}^n$ by using the Pareto's criterion: Given $x = (x_1, x_2 \ldots, x_n), y = (y_1, y_2 \ldots, y_n) \in \mathbb{R}^n$,

$$x \preceq y \Leftrightarrow x_i \leq y_i \quad \forall i = 1, \ldots, n.$$

By Theorem 1.10, the pointed convex cone that defines $\preceq$ is the nonnegative orthant of $\mathbb{R}^n$,

$$\mathbb{R}^n_+ := \{y \in \mathbb{R}^n : 0 \leq y_i \quad \forall i = 1, \ldots, n\}. \tag{1.14}$$

Hence, this cone is also called as *Pareto's cone* and $\preceq$ is denoted as $\leq_{\mathbb{R}^n_+}$. Notice that we may formulate the concept of Pareto optimality with a set equation: $\bar{x} \in S$ is a Pareto optimal solution if of VOP and only if

$$\big(f(S) - f(\bar{x})\big) \cap \big(-\mathbb{R}^n_+ \backslash \{0\}\big) = \emptyset.$$

Vector optimization problems were reintroduced independently by Borel, Von Neumann and Morgenstern (see [29, 187, 188]) in the field of Game Theory in the early 20th century. Later, Koopmans [123] introduced the notion of efficient point in the Production Theory framework, and Kuhn and Tucker [126] introduced the current mathematical formulation of the multiobjective optimization problems and the notion of proper efficient solution. Shortly afterwards, Hurwicz [111] extended the Kuhn and Tucker results to topological linear spaces.

Let us recall two notions of solution in vector optimization problems. Consider a topological linear space X, an arbitrary nonempty subset $S \subset X$ and another real topological linear space Y, which is partially ordered by a pointed convex cone D. Let the problem VOP be defined by the decision set S and the objective mapping $f\colon S \to Y$. A point $\bar{x} \in S$ is said to be an *efficient solution* (or a *nondominated solution*) *of* VOP if

$$x \in S,\ f(x) \leq_D f(\bar{x}) \Rightarrow f(x) = f(\bar{x}),$$

or equivalently,

$$\big(f(S) - f(\bar{x})\big) \cap \big(-D \backslash \{0\}\big) = \emptyset, \tag{1.15}$$

and it is denoted as $\bar{x} \in \mathrm{O}(f, S, D)$ (or simply $\bar{x} \in \mathrm{O}(f, D)$ when $S = X$). In the case that D is *solid*, then $\bar{x} \in S$ is said to be a *weak efficient solution of* VOP if

$$x \in S,\ f(x) \leq_{\operatorname{int} D \cup \{0\}} f(\bar{x}) \Rightarrow f(x) = f(\bar{x}),$$

or equivalently,

$$\big(f(S) - f(\bar{x})\big) \cap \big(-\operatorname{int} D\big) = \emptyset, \tag{1.16}$$

and it is denoted as $\bar{x} \in \mathrm{WO}(f, S, D)$ (or simply $\bar{x} \in \mathrm{WO}(f, D)$ when $S = X$). In general, $\mathrm{O}(f, S, D) \subset \mathrm{WO}(f, S, D)$. Notice that when $Y = \mathbb{R}$ and $D = \mathbb{R}_+$, then both notions coincide with the solution concept of scalar optimization problems, OP (Problem 1.1). So that both notions are considered different extensions for the vectorial case.

Vector variational inequality problems were introduced by F. Giannessi [75] in finite dimensional spaces (see also [76]). Later, Chen et al. extended them to infinite dimensional spaces (see [38, 39]), as well as vector complementary problems (see [42]). Furthermore, they have been intensively studied by Chen et al. [37,41], Lee et al. [133–136] and many other authors [131,139,174,191,194]. Several existence theorems on vector variational inequality problems have been achieved and applied to vector optimization problems and vector complementarity problems (for further details, see [40, 84] and references therein).

Let us denote by $L(X, Y)$ the set of linear mappings from X to Y and consider a mapping $T\colon S \to L(X, Y)$. In a vector variational inequality problem, VVIP, it is requested to find $\bar{x} \in S$ such that

$$\langle T(\bar{x}), x - \bar{x}\rangle \notin -D\backslash\{0\} \quad \forall x \in S. \tag{1.17}$$

A point $\bar{x} \in S$ satisfying (1.17) is denoted as $\bar{x} \in \mathrm{V}(T, S, D)$ (or simply $\bar{x} \in \mathrm{V}(T, D)$ when $S = X$). In the case that D is solid, a weak vector variational inequality problem, WVVIP, consists in finding $\bar{x} \in S$ such that

$$\langle T(\bar{x}), x - \bar{x}\rangle \notin -\operatorname{int} D \quad \forall x \in S. \tag{1.18}$$

A point $\bar{x} \in S$ satisfying (1.18) is denoted as $\bar{x} \in \mathrm{WV}(T, S, D)$ (or simply $\bar{x} \in \mathrm{WV}(T, D)$ when $S = X$). In general, $\mathrm{V}(T, S, D) \subset \mathrm{WV}(T, S, D)$. It is clear that VVIP and WVVIP coincide with the scalar case VIP (Problem 1.2).

In 1997, different researchers from the areas of vector optimization and vector variational inequalities extended the scalar equilibrium problems to the vector-valued setting in which the image space of the associated bifunction is a partially ordered linear space. The first vector equilibrium problems were introduced in the works [10], [20] and [149].

Let us consider a bifunction $f\colon S \times S \to Y$. A vector equilibrium problem, VEP, consists in finding a point $\bar{x} \in S$ such that

$$f(\bar{x}, S) \cap (-D \backslash \{0\}) = \emptyset. \tag{1.19}$$

Such $\bar{x}$ is said to be an *efficient solution of VEP* and it is denoted as $\bar{x} \in \mathrm{E}(f, S, D)$ (or simply $\bar{x} \in \mathrm{E}(f, D)$ when $S = X$). In the case that D is solid, a weak vector equilibrium problem, WVEP, consists in finding $\bar{x} \in S$ such that

$$f(\bar{x}, S) \cap (-\operatorname{int} D) = \emptyset. \tag{1.20}$$

Such $\bar{x}$ is said to be a *weak efficient solution of VEP* and it is denoted as $\bar{x} \in \mathrm{WE}(f, S, D)$ (or simply $\bar{x} \in \mathrm{WE}(f, D)$ when $S = X$). Furthermore, when Y is not endowed with any particular topology, the topological interior of D is replaced by its algebraic interior when D is algebraic solid (see Section 1.4), that is,

$$f(\bar{x}, S) \cap (-\operatorname{core} D) = \emptyset.$$

Since D is convex, both solutions coincide when D is solid.

We notice that f is not assumed to be diagonal null (i.e., $f(x, x) = 0$ for all $x \in S$) as an initial condition for a vector equilibrium problem in [10, 20, 149]. In this way, the considered problem is more general and for existence theorems, only is required conditions such as $f(x, x) \geq_D 0$ for all $x \in S$.

On the other hand, if $S = X$, f is diagonal null and for each $x \in X$ we consider the mapping $g_x : X \to Y$, $g_x(z) = f(x, z)$ for all $z \in X$, then $\bar{x}$ is an efficient (resp. weak efficient) solution of VEP if $g_{\bar{x}}(\bar{x})$ is a nondominated (resp. weak nondominated) point of the set $g_{\bar{x}}(X)$ with respect to the ordering $\leq_D$. Then, from this point of view, a vector equilibrium problem whose bifunction is diagonal null is a vector optimization problem. In chapter 4 we will analyze the behavior of several usual hypotheses, as the bifunction's diagonal null property or the triangular inequality property in Ekeland Variational Principle-type results for bifunctions.

In a similar way as in the scalar case, vector equilibrium problems reunify the vector-valued extensions of classical problems as vector optimization problems and vector variational inequalities.

Another extension considered in [10, 149] is the following: Find $\bar{x} \in S$ such that

$$f(\bar{x}, S) \subset D. \tag{1.21}$$

Such $\bar{x}$ that satisfy this condition are called *ideal efficient solutions of VEP* and will be denoted as $\bar{x} \in \mathrm{IE}(f, S, D)$ (or simply $\bar{x} \in \mathrm{IE}(f, D)$ when $S = X$). Even being another natural vector-valued extension, this kind of solutions is very restrictive and the literature about efficient solutions and weak efficient solutions is more extensive. In general, $\mathrm{IE}(f, S, D) \subset \mathrm{E}(f, S, D) \subset \mathrm{WE}(f, S, D)$.

Let us show a simple example in which these three kinds of solutions are different.

Example 1.12. Consider $X = \mathbb{R}$, $S = [0, \pi]$ and $Y = \mathbb{R}^2$ ordered by $D = \mathbb{R}^2_+$. Let $f\colon [0, \pi] \times [0, \pi] \to \mathbb{R}^2$ be given by

$$f(x, y) = \big(\sin^2 x - \sin x \sin y, \sin(x - y)\big).$$

Let us compute the sets $\mathrm{WE}\left(f, [0, \pi], \mathbb{R}^2_+\right)$, $\mathrm{E}\left(f, [0, \pi], \mathbb{R}^2_+\right)$ and $\mathrm{IE}\left(f, [0, \pi], \mathbb{R}^2_+\right)$. If we consider $\bar{x} = 0$, then

$$f(\bar{x}, x) = (0, -\sin x) \notin -\operatorname{int} \mathbb{R}^2_+ \quad \forall x \in [0, \pi],$$

so $0 \in \mathrm{WE}\left(f, [0, \pi], \mathbb{R}^2_+\right)$, and it is clear that $0 \notin \mathrm{E}\left(f, [0, \pi], \mathbb{R}^2_+\right)$.
Let $\bar{x} \in \left(0, \frac{\pi}{2}\right)$, then

$$f(\bar{x}, x) = \big(\sin \bar{x}(\sin \bar{x} - \sin x), \sin(\bar{x} - x)\big),$$

and by taking $x \in \left(\bar{x}, \frac{\pi}{2}\right)$ we have that $f(\bar{x}, x) \in -\operatorname{int} \mathbb{R}^2_+$, so $\bar{x} \notin \mathrm{WE}\left(f, [0, \pi], \mathbb{R}^2_+\right)$.
Let $\bar{x} = \frac{\pi}{2}$, then

$$f\left(\bar{x}, x\right) = (1 - \sin x, \cos x) \notin -\mathbb{R}^2_+ \backslash \{(0, 0)\} \quad \forall x \in [0, \pi],$$

so $\frac{\pi}{2} \in \mathrm{E}\left(f, [0, \pi], \mathbb{R}^2_+\right)$, and it is clear that $\frac{\pi}{2} \notin \mathrm{IE}\left(f, [0, \pi], \mathbb{R}^2_+\right)$.
If $\bar{x} \in \left(\frac{\pi}{2}, \pi\right)$, then

$$f(\bar{x}, x) = \big(\sin \bar{x}(\sin \bar{x} - \sin x), \sin(\bar{x} - x)\big) \notin -\mathbb{R}^2_+ \backslash \{(0, 0)\} \quad \forall x \in [0, \pi],$$

so $\bar{x} \in \mathrm{E}\left(f, [0,\pi], \mathbb{R}^2_+\right)$, and it is clear that $\bar{x} \notin \mathrm{IE}\left(f, [0,\pi], \mathbb{R}^2_+\right)$.
Finally, if $\bar{x} = \pi$, then

$$f(\bar{x}, x) = (0, \sin x) \in \mathbb{R}^2_+ \quad \forall x \in [0,\pi],$$

so $\pi \in \mathrm{IE}\left(f, [0,\pi], \mathbb{R}^2_+\right)$. Therefore,

$$\begin{aligned}\mathrm{IE}\left(f, [0,\pi], \mathbb{R}^2_+\right) = \{\pi\} \subsetneq \mathrm{E}\left(f, [0,\pi], \mathbb{R}^2_+\right) &= \left[\frac{\pi}{2}, \pi\right] \\ &\subsetneq \mathrm{WE}\left(f, [0,\pi], \mathbb{R}^2_+\right) = \{0\} \cup \left[\frac{\pi}{2}, \pi\right].\end{aligned}$$

This example shows that the concepts of efficient, weak efficient and ideal efficient solution, which coincide in the scalar case, may be different in the vector-valued case for the same bifunction, even on finite dimensional linear spaces, so they must be analyzed independently.

In general, solving vector-valued equilibrium problems requires a higher computational cost and the real line has important properties that makes easier dealing with scalar equilibrium problems. A recurrent technique is the scalarization, which consists in studying the solutions of a vector-valued problem through the solutions of an associated scalar problem.

A functional $\varphi\colon Y \to \mathbb{R} \cup \{\pm\infty\}$ is considered and the values of f are transferred to the extended real line by means of the composition $\varphi \circ f\colon S \times S \to \mathbb{R} \cup \{\pm\infty\}$. The scalar equilibrium problem associated to VEP is the following: find $\bar{x} \in S$ such that

$$(\varphi \circ f)(\bar{x}, x) \geq 0, \quad \forall\, x \in S, \tag{1.22}$$

More generally, a point $\bar{x} \in S$ is an approximate solution with error $\varepsilon \geq 0$ of this problem if

$$(\varphi \circ f)(\bar{x}, x) + \varepsilon \geq 0, \quad \forall\, x \in S.$$

Different characterizations of solutions of vector-valued problems are obtained depending on the properties of the selected scalarization functional φ (see, for instance, [40, 84, 112, 117, 172]). In particular, we will work with linear functionals (i.e., $\varphi \in Y'$) in Chapter 2 and an algebraic formulation of the so-called nonconvex separation functional in Chapter 3.

1.3 Objectives

This thesis project is mainly devoted to study approximate and exact solutions of vector-valued equilibrium problems on real linear spaces not necessarily endowed with any particular topology, and some related topics as vector-valued EVPs for bifunctions.

Efficiency concepts based on coradiant sets and improvement sets were introduced in vector optimization (see [91, 92, 95]), which generalize the most important approximate efficiency notions in the literature. They allow to approach efficient solutions and weak efficient solutions by taking suitable coradiant or improvement sets (see [94]).

Another interesting point of view of using coradiant or improvement sets to deal with approximate efficient solutions is that they are not topological concepts. This key fact allows to work on an algebraic framework, in the sense that the image space of the bifunction is a real linear space which is not equipped with any particular topology. The topological interior and the topological closure of a set may be replaced by algebraic counterparts as the well-known algebraic interior and the algebraic closure (see [112]) or the vector closure (see [4]). In our case, we will focus on the vector closure as a counterpart of the topological closure since it corrects some unsuitable behavior of the algebraic closure with respect to non-solid sets (both closures coincide for convex sets). Moreover, given an arbitrary set, its algebraic closure is contained in its vector closure and, on topological linear spaces, both are contained in its topological closure, so the vector closure is closer to the topological closure. For example, the algebraic closure of the set of rational numbers (which has empty interior) coincide with itself and its vector closure is the set of real numbers, as its topological closure.

Henig proper efficient solutions of vector optimization problems were generalized to equilibrium problems (see [79]). On the other hand, approximate Henig proper efficient solutions defined by coradiant sets were introduced in vector optimization very recently (see [88]). Both solution concepts were characterized by using linear scalarization techniques and generalized convexity assumptions, as the cone-convexity or an approximate counterpart of the nearly subconvexlikeness notion (see [88, 192]). A natural objective will be to extend these results to equilibrium problems on an algebraic framework and we deal with it in Chapter 2.

We notice that free-disposal sets (in particular, improvement sets) were introduced by Debreu [49] in the area of Mathematical Finance in order to approximate preference relations given by convex cones with other sort of sets. Then equilibrium problems dealing with improvement sets are a way to transfer results from Optimization to Economics. The vector closure allows to give an algebraic counterpart of the generalized nearly subconvexlikeness notion defined in [88].

Gong [81] characterized the weak efficient solutions and Henig proper efficient solutions of vector equilibrium problems by means of the so-called nonlinear scalarization functional (see [72–74, 145, 152, 168]). These results only can be applied when the ordering cone on the final space of the bifunction has nonempty topological interior, since this condition is essential to define the nonlinear scalarization functional. Recently, Qiu and He [163] extended this functional to real linear spaces ordered by (not necessarily solid) convex cones by using a kind of vector closure in a given direction.

A second objective will be to extend the nonlinear scalarization functional to real linear spaces when the ordering set is an arbitrary nonempty set and studying its main properties: monotonicity, convexity, level sets, etc. Many important properties might be obtained by using the algebraic interior and the vector closure in a given direction. This algebraic version of the nonlinear scalarization functional will be useful to characterize a very wide class of approximate solutions of vector equilibrium problems defined through free-disposal sets. Additionally, it might be also applied to achieve some results of particular problems as vector variational inequality problems and vector optimization problems. As a consequence, some approximate efficiency results for vector optimization problems on real linear spaces (see, for instance, [120]) might be improved by choosing suitable free-disposal sets. We deal with this objective in Chapter 3.

Several authors extended the Ekeland Variational Principle to vector-valued bifunctions by using linear scalarization procedures (see [22, 183]), but also was generalized by using the nonlinear scalarization functional (see, for instance, [9,11, 85,96,163,181]). In particular, an approximate EVP for set-valued mappings was obtained in [96] by using approximate strict solutions, so that the completeness of the initial metric space was not necessary.

The third objective will be to introduce the concepts of exact and approximate strict solution in vector equilibrium problems through free-disposal sets and studying its properties. Moreover, exact EVPs for bifunctions on complete metric spaces and approximate EVPs for bifunctions on (not necessarily complete) metric spaces will be obtained without any particular topology in the final space and by means of a strict fixed point theorem for set-valued mappings. The aim is to improve some EVPs of the literature. In our approach, the usual topological assumptions on the final space of the bifunction are replaced by algebraic counterparts via the vector closure in a given direction. Also, scalarization through (non necessarily continuous) linear functions and the algebraic version of the nonlinear scalarization functional defined by nonempty arbitrary ordering sets on real linear spaces will be used. As a consequence, the roles of certain common assumptions as the diagonal null property or the triangular inequality property on the bifunction will be clarified.

The vector-valued EVP for bifunctions have been applied to obtain existence results for weak efficient solutions of vector equilibrium problems (see, for instance, [9,11,22]). Topological assumptions as continuity and boundedness were required in both coordinates of the bifunction. Moreover, certain assumptions as the diagonal null property and the triangular inequality property are also assumed, but their roles have not been clarified.

As a result, the final objective will be to introduce and study "semialgebraic" semicontinuity notion, in the sense that it works when the initial space is endowed with a topology and the final space is a real linear space. This concept was motivated by a topological previous one given by Tammer [181] and considers sublevel sets and the algebraic interior in order to obtain results with weaker assumptions. Then, the role of the triangular inequality property is clarified and, by means of the algebraic version of the nonlinear scalarization functional and the semialgebraic semicontinuity, existence results for equilibrium problems are achieved under weaker assumptions. We will deal with the notions of strict solution and its approximate counterpart together with the other notions of approximate solutions and approximate proper solutions at the end of Chapter 2, and with the EVPs and the existence results for vector equilibrium problems in Chapter 4.

1.4 Preliminaries on real ordered linear spaces

Let Y be a real linear space whose algebraic dual space is denoted by Y', and $D \subset Y$ be a proper convex cone. Let $A \subset Y$ be a nonempty set. The cone generated by A is the set

$$\operatorname{cone} A := \bigcup_{\alpha \geq 0} \alpha A.$$

Some authors consider the generated cone by a set without the vertex 0, just as they define the conical sets (see (1.13)). A is said to be coradiant if $\alpha A \subset A$ for all $\alpha \geq 1$, and the notation $\operatorname{shw} A$ stands for the coradiant set generated by A (see [196]), i.e.,

$$\operatorname{shw} A := \bigcup_{\alpha \geq 1} \alpha A.$$

A is said to be a free-disposal set with respect to a convex cone $K \subset Y$ [49] if $A + K = A$ and is said to be an improvement set with respect to K [43, 95] if additionally $0 \notin A$. Let us denote

$$\begin{aligned} A^+ &:= \{\phi \in Y' : \phi(a) \geq 0, \, \forall a \in A\}, \\ A^\# &:= \{\phi \in Y' : \phi(a) > 0, \, \forall a \in A, a \neq 0\}, \\ A^{s+} &:= \bigcup_{\delta > 0} \{\phi \in Y' : \phi(a) \geq \delta, \, \forall a \in A, a \neq 0\}. \end{aligned}$$

A^+ is said to be the positive polar cone of A and $A^\#$ is said to be the quasi-interior of A^+. In general, $A^{s+} \subset A^\# \subset A^+$. For a geometrical interpretation of set A^{s+}, see Remark 4.5(ii).

The notations $\operatorname{co} A$, $\operatorname{span} A$ and $L(A) := \operatorname{span}(A - A)$ denotes the convex hull, the linear hull and the associated linear subspace of A, respectively. In addition, $\operatorname{core} A$, $\operatorname{icr} A$ and $\operatorname{vcl} A$ stand for the algebraic interior, the relative algebraic interior and the vector closure of A (see [4, 112]), i.e.,

$$\begin{aligned}
\operatorname{core} A &:= \{y \in Y \,:\, \forall v \in Y, \exists \lambda > 0 \text{ s.t. } y + [0,\lambda]v \subset A\},\\
\operatorname{icr} A &:= \{y \in Y \,:\, \forall v \in L(A), \exists \lambda > 0 \text{ s.t. } y + [0,\lambda]v \subset A\},\\
\operatorname{vcl} A &:= \{y \in Y : \exists\, v \in Y \text{ s.t. } \forall \lambda > 0\, \exists\, \lambda' \in [0,\lambda], y + \lambda' v \in A\}.
\end{aligned}$$

A is said to be algebraic solid (resp., relatively solid) if $\operatorname{core} A \neq \emptyset$ (resp., $\operatorname{icr} A \neq \emptyset$) and vectorially closed if $\operatorname{vcl} A = A$. If $0 \in \operatorname{core} A$, then A is said to be an absorbing set. $\operatorname{cl} A$ will denote the topological closure of A whenever Y is a topological linear space.

For each $q \in Y$, $\operatorname{vcl}_q A$ denotes the vector closure of A in the direction q (see [97, 99, 159, 163, 199]), defined as

$$\operatorname{vcl}_q A := \{y \in Y : \forall\, \lambda > 0\, \exists\, \lambda' \in [0,\lambda], y + \lambda' q \in A\}.$$

Also we denote by $\operatorname{ovcl}_q^{+\infty} A$ (see [97, 99]) the set of all points from which the ray with direction q is not asymptotically contained in $Y \backslash A$, i.e.,

$$\operatorname{ovcl}_q^{+\infty} A := \{y \in Y : \forall\, \lambda > 0\, \exists\, \lambda' \in [\lambda, +\infty) \text{ s.t. } y + \lambda' q \in A\}.$$

We say that A is vectorially closed by q or q-vectorially closed if $\operatorname{vcl}_q A = A$. The set $\operatorname{ovcl}_q^{+\infty} A$ is needed in order to characterize the properness of the nonconvex separation functional (see Theorem 3.2(a)). It is clear that

$$A \subset \operatorname{vcl}_q A \subset \bigcap_{\alpha > 0} (A - [0,\alpha)q) \subset \mathbb{R}q + A \quad \forall q \in Y.$$

Moreover, for each $q \in Y$ we have that

$$\begin{aligned}
\operatorname{vcl}_q(A + y) &= \operatorname{vcl}_q A + y \quad \forall y \in Y,\\
\operatorname{vcl}_q(\operatorname{vcl}_q A) &= \operatorname{vcl}_q A, && (1.23)\\
-\operatorname{ovcl}_q^{+\infty} A &= \operatorname{ovcl}_{-q}^{+\infty}(-A), && (1.24)\\
\operatorname{vcl} A &= \bigcup_{q \in Y} \operatorname{vcl}_q A. && (1.25)
\end{aligned}$$

Statement (1.23) was stated in [163, Proposition 2.4(iii)] when A is a convex cone. Observe that $\operatorname{vcl} A$ is not necessarily vectorially closed (see [4, Example 2]), even though $\operatorname{vcl}_q A$ is q-vectorially closed, for each $q \in Y$ (see (1.23)). On the other hand, if A is convex, then it is easy to check that $\operatorname{vcl}_q A$ is convex and

$$\alpha y + (1-\alpha) z \in \operatorname{ovcl}_q^{+\infty} A \quad \forall y \in \operatorname{ovcl}_q^{+\infty} A, \forall z \in \mathbb{R}q + A, \forall \alpha \in (0,1). \tag{1.26}$$

Roughly speaking, the sets $\operatorname{core} A$ and $\operatorname{icr} A$ (resp., $\operatorname{vcl} A$ and $\operatorname{vcl}_q A$) are algebraic counterparts of the topological interior (resp., topological closure) of A. If Y is a real topological linear space, we have that $\operatorname{vcl} A \subset \operatorname{cl} A$. Furthermore, if Y is endowed with the core convex topology τ (see [117, Section 6.3]) and A is convex and relatively solid, then $\operatorname{vcl} A = \operatorname{cl}_\tau A$ (see [157, Lemma 3.1]).

The conic extension of A with respect to D (resp. the open conic extension of A with respect to D whenever D is algebraic solid) is denoted by

$$\mathcal{E}^D(A) := A + D \quad (\text{resp. } \mathcal{E}_0^D(A) := A + \operatorname{core} D).$$

Let us collect some useful properties about the algebraic interior, the vector closure and the conic extensions of a set. The proofs of statements (a)-(e) may be found in [4, 91, 112, 163].

Lemma 1.13. Consider a nonempty convex set $F \subset Y$ and a proper algebraic solid convex cone $K \subset Y$. Then,

(a) $\operatorname{core} F$ is a convex set and $\operatorname{core} F = \operatorname{core}(\operatorname{core} F)$. If additionally Y is a topological linear space and $\operatorname{int} F \neq \emptyset$, then $\operatorname{core} F = \operatorname{int} F$.

(b) If F is coradiant, then $F = \mathcal{E}^{\operatorname{cone} F}(F)$.

(c) $\operatorname{icr} K = \operatorname{core} K = \mathcal{E}_0^K(\operatorname{core} K) = \mathcal{E}_0^K(K)$ and $\operatorname{core} K \cup \{0\}$ is a proper convex cone.

(d) $\operatorname{core} K = \mathcal{E}_0^K(\operatorname{vcl}_q K) = \operatorname{vcl}_q K + (0, +\infty) q$ for all $q \in \operatorname{core} K$.

(e) $\operatorname{vcl} K = \operatorname{vcl}_q K$ for all $q \in \operatorname{core} K$.

(f) $\mathcal{E}_0^K(A) = \mathcal{E}_0^K(\operatorname{vcl} A)$, for any subset $\emptyset \neq A \subset Y$.

Proof. In order to prove part (f), it is enough to show that $\mathcal{E}_0^K(\operatorname{vcl} A) \subset \mathcal{E}_0^K(A)$. Consider two arbitrary points $y \in \operatorname{vcl} A$ and $k \in \operatorname{core} K$. It follows that there exists $v \in Y$ such that $\forall \lambda > 0, \exists \lambda' \in [0, \lambda]$ such that $y + \lambda' v \in A$. Moreover, by part (a) we have that $\operatorname{core}(\operatorname{core} K) = \operatorname{core} K$, so there exists $\beta > 0$ such that $k + [0, \beta](-v) \subset \operatorname{core} K$. Take $\lambda = \beta$ and $\lambda' \in [0, \beta]$ such that $y + \lambda' v \in A$. Therefore $y + k = y + \lambda' v + k - \lambda' v \in \mathcal{E}_0^K(A)$, and the proof finishes. $\square$

Given a proper convex cone K, we have that (see [5, Proposition 2.3])

$$\operatorname{icr} K \neq \emptyset \Leftrightarrow \operatorname{icr} K^+ \neq \emptyset. \tag{1.27}$$

Moreover, if N and K are relatively solid and vectorially closed proper convex cones in Y such that $N \cap K = \{0\}$, then

$$(N \cap K)^+ = \operatorname{vcl}(N^+ + K^+)$$

(see [5, Proposition 2.4]). If additionally K is pointed, we obtain that

$$\operatorname{core} K^+ \cap (-N^+) \neq \emptyset \tag{1.28}$$

by applying [5, Theorem 2.3] with $K = -N$ and $A = \operatorname{core} K$.

For each $C \subset Y \backslash \{0\}$ and $\varepsilon \geq 0$, let us denote

$$C(\varepsilon) := \begin{cases} \varepsilon C & \text{if } \varepsilon > 0, \\ \bigcup\limits_{\varepsilon > 0} \varepsilon C & \text{if } \varepsilon = 0. \end{cases}$$

Clearly, if C is convex, then $C(\varepsilon)$ is convex for all $\varepsilon > 0$. Consider the following collections of sets:

$$\begin{aligned}
\mathcal{H} &:= \{\emptyset \neq C \subset Y \backslash \{0\} \,:\, C \cap (-D) = \emptyset\}, \\
\widehat{\mathcal{H}} &:= \{\emptyset \neq C \subset Y \backslash \{0\} \,:\, \operatorname{vcl} \operatorname{cone} C \cap (-D \backslash \{0\}) = \emptyset\}, \\
\mathcal{G} &:= \{K \subset Y \,:\, K \text{ is a proper algebraic solid convex cone such that} \\
&\qquad D \backslash \{0\} \subset \operatorname{core} K\}, \\
\mathcal{G}(C) &:= \{K \in \mathcal{G} \,:\, C \cap (-\operatorname{core} K) = \emptyset\}, \\
\mathcal{O}(C) &:= \{K \in \mathcal{G}(C) \,:\, \operatorname{core} K = K \backslash \{0\}\}.
\end{aligned}$$

Let us give some properties related to these collections and conic extensions. The proof is analogous to [88, Lemma 2.2] by using the properties of the algebraic interior of Lemma 1.13.

Lemma 1.14. Consider a nonempty set $C \subset Y\backslash\{0\}$, $\varepsilon \geq 0$ and $K \in \mathcal{G}$. Then,

(a) $\mathcal{E}_0^K(D) = \mathcal{E}_0^K(D\backslash\{0\}) = \operatorname{core} K$.

(b) If $K \in \mathcal{G}(C)$, then $\mathcal{E}_0^K(C)(\varepsilon) = \mathcal{E}_0^K\big(C(\varepsilon)\big)$.

(c) If $C \in \mathcal{H}$, then $\mathcal{E}^D(C)(\varepsilon) = \mathcal{E}^D\big(C(\varepsilon)\big)$.

It is clear that $\widehat{\mathcal{H}} \subset \mathcal{H}$, $D\backslash\{0\} \in \widehat{\mathcal{H}}$ provided that $\operatorname{vcl} D$ is pointed and $\mathcal{G}(D\backslash\{0\}) = \mathcal{G}$. For each $K \in \mathcal{G}(C)$ we have that $K \in \mathcal{G}\big(\mathcal{E}^D(C)\big)$ and $\operatorname{core} K \cup \{0\} \in \mathcal{O}(C)$. On the other hand, $\operatorname{core} K \cup \{0\} = K$ for all $K \in \mathcal{O}(C)$. Moreover, given $K \in \mathcal{G}$, let us observe that

$$K \in \mathcal{G}(C) \Leftrightarrow 0 \notin \mathcal{E}_0^K(C).$$

These collections of sets were introduced in [88] in the topological setting by using the topological closure and the topological interior. For instance, the following collection was considered instead of $\widehat{H}$:

$$\overline{\mathcal{H}} := \{\emptyset \neq C \subset Y\backslash\{0\} \,:\, \operatorname{cl}\operatorname{cone} C \cap (-D\backslash\{0\}) = \emptyset\}.$$

If $\operatorname{cone} C$ is a convex set and $\operatorname{int}(\operatorname{cone} C) \neq \emptyset$, then $\overline{\mathcal{H}} = \widehat{\mathcal{H}}$ (see Lemma 1.13(a)). In general, $\overline{\mathcal{H}} \subset \widehat{\mathcal{H}}$ since $\operatorname{vcl}\operatorname{cone} C \subset \operatorname{cl}\operatorname{cone} C$. The next example shows that this last inclusion may be strict.

Example 1.15. Let $Y = \mathbb{R}^3$, $D = \{0\} \times (-\mathbb{R}_+) \times \mathbb{R}_+$ and $C = \{(x, 1, x^2) \,:\, x \in \mathbb{R}\backslash\{0\}\}$. One may check that

$$\operatorname{vcl}\operatorname{cone} C \cap (-D\backslash\{0\}) = \emptyset,$$

$$\operatorname{cl}\operatorname{cone} C \cap (-D\backslash\{0\}) = \{0\} \times (\mathbb{R}_+\backslash\{0\}) \times \{0\},$$

so that, $\overline{\mathcal{H}} \subsetneq \widehat{\mathcal{H}}$.

Let us consider the mapping $\tau_C\colon Y' \to \mathbb{R}\cup\{-\infty\}$ given by

$$\tau_C(y') = \inf_{y\in C} y'(y) \quad \forall y' \in Y'.$$

For any nonempty sets $C_1, C_2 \subset Y\backslash\{0\}$ and $\varepsilon_1, \varepsilon_2 \in \mathbb{R}_+$ we have that

$$\tau_{C_1(\varepsilon_1)+C_2(\varepsilon_2)}(y') = \varepsilon_1\tau_{C_1}(y') + \varepsilon_2\tau_{C_2}(y')$$

for all $y' \in C_1^+ \cap C_2^+$. Let us define the collection of sets

$$\mathcal{F}_{D^+} := \{\emptyset \neq C \subset Y\backslash\{0\} : (D^+ \cap C^+)\backslash\{0\} \neq \emptyset\},$$
$$\mathcal{F}_{D^\#} := \{\emptyset \neq C \subset Y\backslash\{0\} : D^\# \cap C^+ \neq \emptyset\}.$$

We can easily check that $\mathcal{F}_{D^\#} \subset \widehat{\mathcal{H}} \cap \mathcal{F}_{D^+}$. Moreover, if Y' separates elements in Y (i.e., two different elements in Y may be separated by a hyperplane), C is a relatively solid convex set, D is pointed, vectorially closed and algebraic solid, it follows that

$$C \in \mathcal{F}_{D^\#} \Leftrightarrow C \in \widehat{\mathcal{H}}$$

by (1.28) and [112, Lemma 1.25] for $K = D$ and $N = -\operatorname{vcl}\operatorname{cone} C$. Notice that $-\operatorname{vcl}\operatorname{cone} C$ is a vectorially closed cone by applying [4, Proposition 3(iii)] and [4, Proposition 5(i)] with $A = \operatorname{cone} C$.

In the following, fix a nonempty set $E \subset Y$ such that E is algebraic solid. The following set is considered:

$$\mathcal{H}_E := \{q \in Y\backslash\{0\} : \operatorname{vcl}_q E + (0,+\infty)q = \operatorname{core} E\}.$$

Observe that for each $q \in Y\backslash\{0\}$,

$$\operatorname{vcl}_q E + (0,+\infty)q = \operatorname{core} E \iff \operatorname{vcl}_q E + (0,+\infty)q \subset \operatorname{core} E.$$

In the next proposition we show that $\mathcal{H}_E$ is nonempty whenever E is free-disposal with respect to an algebraic solid convex cone K.

It reduces to Lemma 1.13(c) and (d) by taking $E = K$. The proof of statement (a) may be found in [4, Proposition 6]).

Proposition 1.16. Suppose that E is free-disposal with respect to an algebraic solid convex cone K. Then:

(a) $E + \operatorname{core} K = \operatorname{core} E$.

(b) $\operatorname{vcl}_q E + \operatorname{core} K = \operatorname{vcl}_q E + (0, +\infty)q = \operatorname{core} E$ for all $q \in \operatorname{core} K$.

Proof. In order to show (b), let us fix an arbitrary $q \in \operatorname{core} K$ and prove the inclusions

$$\operatorname{core} E \subset \operatorname{vcl}_q E + (0, +\infty)q \subset \operatorname{vcl}_q E + \operatorname{core} K \subset \operatorname{core} E.$$

Observe that the middle one is obvious, since $q \in \operatorname{core} K$ and $\operatorname{core} K \cup \{0\}$ is a cone by Lemma 1.13(c).

First, take an arbitrary point $e \in \operatorname{core} E$. Then there exists $\lambda > 0$ such that $e - [0, \lambda]q \subset E$. In particular, $e \in \lambda q + E \subset (0, +\infty)q + \operatorname{vcl}_q E$.

Second, fix $e \in \operatorname{vcl}_q E$ and $d \in \operatorname{core} K$. Since $\operatorname{core} K = \operatorname{core}(\operatorname{core} K)$ by Lemma 1.13(a), we know that there exist $\mu > 0$ and $0 \leq \lambda \leq \mu$ such that $d - [0, \mu]q \subset \operatorname{core} K$ and $e + \lambda q \in E$. Therefore,

$$e + d = e + \lambda q + d - \lambda q \in E + \operatorname{core} K$$

and $e + d \in \operatorname{core} E$, since $E + \operatorname{core} K = \operatorname{core} E$ by part (a). $\square$

Remark 1.17. As a consequence of this result we deduce that $\operatorname{core} E \subset \mathcal{H}_E$ whenever E is an algebraic solid convex set and $E + \operatorname{cone} E \subset E$. If additionally $0 \in E$, then it is clear that $\mathcal{H}_E = \operatorname{core} E$.

Given a nonempty set X and a mapping $g \colon X \to Y$, we denote for each $y \in Y$ and a binary relation $\mathcal{R}$ on Y, the set

$$[g\mathcal{R}y] := \{x \in X : g(x)\mathcal{R}y\}.$$

Notice that this notation encompasses the level sets and the sublevel sets of g with respect to an arbitrary order on Y. A bifunction $v \colon X \times X \to Y$ is said to satisfy the triangle inequality property with respect to $\leq_D$ ($\leq_D$-t.i. property for short) if

$$v(x_1, x_3) \leq_D v(x_1, x_2) + v(x_2, x_3) \quad \forall x_1, x_2, x_3 \in X.$$

Observe that (1.12) corresponds with the $\leq_{\mathbb{R}_+}$-t.i. property. On the other hand, v is said to be diagonal null if $v(x, x) = 0$ for all $x \in X$.

In the next lemma we collect several properties of free-disposal sets, some of them in connection with the vector closure in a given direction, the binary relations $\leq_E$ and $\leq_D$ and also with the sublevel sets of bifunctions satisfying the $\leq_D$-t.i. property. Let us observe that the set E can be algebraic nonsolid (compare parts (b) and (c) with Lemma 1.13(d) and Proposition 1.16(b), where K is assumed to be algebraic solid).

Lemma 1.18. Assume that $E \subset Y$ is free-disposal with respect to D and consider a bifunction $f\colon X \times X \to Y$. Then the following statements are true:

(a) $\operatorname{vcl}_q E$ and εE are free-disposal sets with respect to D for all $q \in Y$ and $\varepsilon > 0$.

(b) $(0, +\infty)q + \operatorname{vcl}_q E \subset E$ for all $q \in D$.

(c) $[0, +\infty)q + \operatorname{vcl}_q E = \operatorname{vcl}_q E$ for all $q \in D$.

(d) $D \subset \bigcap_{q \in E} \operatorname{vcl}_q \operatorname{cone} E$.

(e) $\bigcap_{\delta > 0} (-\delta q + \operatorname{vcl}_q E) \subset \operatorname{vcl}_q E$ for all $q \in Y$.

(f) Let $y_1, y_2, y_3 \in Y$ such that $y_1 \leq_E y_2$, $y_2 \leq_D y_3$ and $z_1 \leq_D z_2$. Then, $y_1 \leq_E y_3$ and $y_1 + z_1 \leq_E y_2 + z_2$.

(g) Suppose that f satisfies the $\leq_D$-t.i. property and consider two arbitrary points $a, b \in Y$. Then, for each $x \in X$, we have

$$\left(\bigcup_{z \in [f(x,\cdot) \leq_D a]} [f(z, \cdot) \leq_E b] \right) \subset [f(x, \cdot) \leq_E a + b].$$

Proof. Let us prove parts (d) and (g), since parts (a), (b), (c), (e) and (f) follow easily from the definitions.

(d) Consider two arbitrary points $d \in D$ and $q \in E$. As E is free-disposal with respect to D we have that $q + nd \in E$ for all $n \in \mathbb{N}$. Then, $d + (1/n)q \in \operatorname{cone} E$ for all $n \in \mathbb{N}$, and it follows that $d \in \operatorname{vcl}_q \operatorname{cone} E$.

(g) Consider $x \in X$, $z \in [f(x, \cdot) \leq_D a]$ and $y \in Y$ such that $f(z, y) \leq_E b$. By part (f) and since f satisfies the $\leq_D$-t.i. property we see that

$$f(x, y) \leq_D f(x, z) + f(z, y) \leq_E a + b,$$

and as E is free-disposal w.r.t. D, we obtain that $f(x,y) \leq_E a+b$, which finishes the proof. $\square$

Remark 1.19. As a consequence of Lemma 1.18(d) we have that $E^+ \subset D^+$. Indeed, let $\phi \in E^+$, $d \in D$ and an arbitrary point $e \in E$. Then there exists a sequence $(t_n) \subset \mathbb{R}_+$, $t_n \to 0$, such that $d + t_n e \in \operatorname{cone} E$ for all n. As $\phi \in E^+$ we deduce that $\phi(d) + t_n\phi(e) \geq 0$ for all n, and taking the limit when $n \to +\infty$ it follows that $\phi(d) \geq 0$. Therefore, $\phi \in D^+$.

By dealing with an extended-real valued function, $\varphi\colon Y \to \mathbb{R} \cup \{\pm\infty\}$, we denote

$$\operatorname{dom}\varphi := \{y \in Y : \varphi(y) < +\infty\},$$
$$\operatorname{epi}\varphi := \{(y,t) \in Y \times \mathbb{R} : \varphi(y) \leq t\},$$

and we say that φ is proper if $[\varphi = -\infty] = \emptyset$ and $\operatorname{dom}\varphi \neq \emptyset$.

Moreover, φ is said to be positively homogeneous (resp. subadditive, convex) if $\varphi(\lambda y) = \lambda\varphi(y)$ for all $y \in Y$ and $\lambda > 0$ (resp. $\varphi(y_1+y_2) \leq \varphi(y_1)+\varphi(y_2)$, $\varphi(\alpha y_1 + (1-\alpha)y_2) \leq \alpha\varphi(y_1) + (1-\alpha)\varphi(y_2)$ for all $y_1, y_2 \in Y$ and $\alpha \in (0,1)$). In the previous definitions we assume the conventions $+\infty - \infty = -\infty + \infty = +\infty$.

On the other hand, given a nonempty set $C \subset Y$, we say that φ is C-nondecreasing (resp. C-increasing) if

$$y_1, y_2 \in Y, y_1 \neq y_2, y_1 - y_2 \in -C \Rightarrow \varphi(y_1) \leq \varphi(y_2) \quad (\text{resp. } \varphi(y_1) < \varphi(y_2)).$$

When C is a proper convex cone, φ is said to be nondecreasing with respect to the relation $\leq_C$ if φ is C-nondecreasing and, if C is algebraic solid, φ is said to be increasing with respect to the relation $<_C$ if φ is $\operatorname{core} C$-increasing.

In Proposition 1.20(b) we give a sufficient condition for the emptyness of the set $\operatorname{ovcl}_q^{+\infty} F$. For another sufficient condition see Proposition 1.21.

Proposition 1.20. (a) Let $y, z, q \in Y$ and $\emptyset \neq F \subset Y$. If $y \in \operatorname{ovcl}_q^{+\infty}(F+z)$ then $q \in \operatorname{vcl}_{y-z} \operatorname{cone} F$. Reciprocally, if $q \in \operatorname{vcl}_y F \backslash F$, then $y \in \operatorname{ovcl}_q^{+\infty}(\operatorname{cone} F)$.

(b) If $q \notin \operatorname{vcl}\operatorname{cone} F$, then $\operatorname{ovcl}_q^{+\infty} F = \emptyset$.

Proof. (a) Suppose that $y \in \operatorname{ovcl}_q^{+\infty}(F+z)$. Then, for each $\lambda > 0$ there exists $\lambda' \in [1/\lambda, +\infty)$ such that $y + \lambda' q \in F + z$ and so, by defining $\beta := 1/\lambda' \in (0, \lambda]$, we have

$$q + \beta(y - z) \in \beta F \subset \operatorname{cone} F.$$

As $\lambda > 0$ is arbitrary, we conclude that $q \in \operatorname{vcl}_{y-z} \operatorname{cone} F$.

Reciprocally, if $q \in \operatorname{vcl}_y F \backslash F$, then for each $\lambda > 0$ there exists $\lambda' \in [0, 1/\lambda]$ such that $q + \lambda' y \in F$. As $q \notin F$ we deduce that $\lambda' > 0$ and so $y + \beta q \in \operatorname{cone} F$, where $\beta := 1/\lambda' \in [\lambda, +\infty)$. Therefore, $y \in \operatorname{ovcl}_q^{+\infty}(\operatorname{cone} F)$ and part (a) is proved.

(b) This part is a direct consequence of the previous one and the proof is complete. □

The reciprocal result of Proposition 1.20(b) is false (see Remark 1.22). Next, we state a sufficient condition for the emptyness of the set $\operatorname{ovcl}_q^{+\infty} F$ via scalarization.

Proposition 1.21. Consider $q \in Y \backslash \{0\}$ and $\emptyset \neq F \subset Y$. Let $\varphi : Y \to \mathbb{R} \cup \{+\infty\}$ be a proper subadditive positively homogeneous functional such that $\varphi(q) < 0$. If $\inf_{z \in F} \varphi(z) > -\infty$, then $\operatorname{dom} \varphi \cap \operatorname{ovcl}_q^{+\infty} F = \emptyset$.

Proof. Assume by contradiction that there exists $y \in \operatorname{dom} \varphi \cap \operatorname{ovcl}_q^{+\infty} F$. Then there exists a sequence of positive real numbers (t_n) such that $t_n \to +\infty$ and $y + t_n q \in F$ for all n. By the properties of φ we see that

$$-\infty < \inf_{z \in F} \varphi(z) \leq \varphi(y + t_n q) \leq \varphi(y) + t_n \varphi(q), \quad \forall n,$$

which is a contradiction since $y \in \operatorname{dom} \varphi$ and $t_n \varphi(q) \to -\infty$. This finishes the proof. □

Remark 1.22. The reciprocal result of Proposition 1.21 is false. Indeed, consider $Y = \mathbb{R}^2$, $q = (-1, 0)$, $\varphi(y_1, y_2) = y_1$ for all $(y_1, y_2) \in \mathbb{R}^2$, and

$$F = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \leq 0, 0 < y_2 \leq e^{y_1}\}.$$

Then $\operatorname{ovcl}_q^{+\infty} F = \emptyset$, $q \in \operatorname{vcl} \operatorname{cone} F$ and $\inf_{z \in F} \varphi(z) = -\infty$.

Assume that X is a real linear space. $g\colon X \to Y$ is said to be D-convex on a nonempty set $S \subset X$ if S is convex and

$$g(\lambda x_1 + (1-\lambda)x_2) \in \lambda g(x_1) + (1-\lambda)g(x_2) - D,$$

for every $x_1, x_2 \in S$, $\lambda \in [0,1]$. g is said to be D-convexlike on S if $g(S) + D$ is convex. Obviously, if g is D-convex on S, then g is D-convexlike on S. In the literature, many notions concerning with generalized convexity have been defined in the algebraic framework (see, for instance, [4, 156, 202]). Next, we follow this approach and we introduce an algebraic approximate nearly subconvexlikeness concept.

In [90, Definition 2.3], a generalized convexity notion for mappings called nearly (C,ε)-subconvexlikeness was introduced on the framework of real locally convex spaces. It is an approximate counterpart of the so-called nearly subconvexlikeness notion [192, Definition 2.2], and has proven to be useful to obtain necessary conditions for different kinds of approximate proper efficient solutions of vector optimization problems via linear scalarization. Then, we will deal with an algebraic counterpart through the vector closure that will allow us to obtain necessary conditions for approximate proper efficient solutions of problem VEP by linear scalarization on an algebraic framework.

Definition 1.23. Consider $\varepsilon \geq 0$, $g\colon X \to Y$, $\emptyset \neq S \subset X$ and $\emptyset \neq C \subset Y\backslash\{0\}$. The mapping g is said to be v-nearly (C,ε)-subconvexlike (or v-nearly C-subconvexlike when $\varepsilon = 1$) on S if $\operatorname{vcl}\operatorname{cone}\big(g(S) + C(\varepsilon)\big)$ is convex.

If Y is a topological linear space and g is v-nearly (C,ε)-subconvexlike on S, then is nearly (C,ε)-subconvexlike on S (that is, $\operatorname{cl}\operatorname{cone}\big(g(S)+C(\varepsilon)\big)$ is convex). Indeed, given $\emptyset \neq F \subset Y$ such that $\operatorname{vcl} F$ is convex, we know that $\operatorname{cl}(\operatorname{vcl} F)$ is convex. Then

$$\operatorname{cl} F \subset \operatorname{cl}(\operatorname{vcl} F) \subset \operatorname{cl}(\operatorname{cl} F) = \operatorname{cl} F,$$

so that $\operatorname{cl} F = \operatorname{cl}(\operatorname{vcl} F)$ is convex and the implication follows by taking $F := \operatorname{cone}\big(g(S) + C(\varepsilon)\big)$. In the next result we give sufficient conditions for a function g to be v-nearly (C,ε)-subconvexlike on S.

Theorem 1.24. Consider $\varepsilon \geq 0$, $g\colon X \to Y$ and two nonempty sets $S \subset X$, $C \subset Y \backslash \{0\}$. It follows that:

(a) If C is convex and there exists a cone P such that g is P-convexlike on S and $\mathcal{E}^P(C) = C$ (i.e., C is free-disposal with respect to P), then g is v-nearly (C, ε)-subconvexlike on S.

(b) If C is convex and coradiant, g is $(\operatorname{cone} C)$-convexlike on S, then g is v-nearly (C, ε)-subconvexlike on S.

Proof. (a) Consider $\varepsilon > 0$. We know that $\mathcal{E}^P\big(g(S)\big)$ and $C(\varepsilon)$ are convex sets, so that $\mathcal{E}^P\big(g(S)\big) + C(\varepsilon) = g(S) + \mathcal{E}^P\big(C(\varepsilon)\big)$ is also a convex set. Since

$$\mathcal{E}^P\big(C(\varepsilon)\big) = \varepsilon \mathcal{E}^P(C) = C(\varepsilon),$$

then $g(S) + C(\varepsilon)$ is convex, which implies that g is v-nearly (C, ε)-subconvexlike on S by [4, Proposition 5(iv)].

Consider $\varepsilon = 0$ and set $C' := C(0)$. We know that C' is convex and it is easy to check that $\mathcal{E}^P(C') = C'$. By applying the previous case with $C := C'$ and $\varepsilon' = 1$, we know that g is v-nearly C'-subconvexlike on S, i.e., $\operatorname{vcl}\operatorname{cone}\big(g(S) + C(0)\big)$ is convex, so that g is v-nearly $(C, 0)$-subconvexlike on S.

(b) By Lemma 1.13(b), we have that $\mathcal{E}^{\operatorname{cone} C}(C) = C$. Then the result is a consequence of part (a). □

Finally, let us introduce two concepts of semicontinuity for vector-valued functions that work when the final space is not endowed with any topology –in this sense it could be considered "semialgebraic" semicontinuity notions–. They are inspired by a well-known notion of lower semicontinuity due to Tammer (see [181]).

Definition 1.25. Let X be a topological space, $q \in Y \backslash \{0\}$ and $\emptyset \neq H \subset Y$. A function $g\colon X \to Y$ is said to be lower semicontinuous with respect to q and H –denoted (q, H)-lsc in short form– if the sublevel sets $[g \leq_{\operatorname{vcl}_q H} rq]$ are closed for all $r \in \mathbb{R}$. If H is algebraic solid, g is said to be upper semicontinuous with respect to q and H – denoted (q, H)-usc in short form– if the sublevel sets $[g <_H rq]$ are open for all $r \in \mathbb{R}$.

Analogously, g is said to be lower semicontinuous with respect to the relation $\leq_{\mathrm{vcl}_q H}$ –denoted lsc w.r.t. $\leq_{\mathrm{vcl}_q H}$ in short form– if the sublevel sets $[g \leq_{\mathrm{vcl}_q H} y]$ are closed for all $y \in Y$.

Remark 1.26. (i) Let $\alpha \in \mathbb{R}$ and the set $H_{\alpha q} := \alpha q + D$. The $(q, H_{\alpha q})$-lsc coincides with the (q, D)-lsc, since $\mathrm{vcl}_q H_{\alpha q} = \alpha q + \mathrm{vcl}_q D$ and so

$$[g \leq_{\mathrm{vcl}_q H_{\alpha q}} rq] = [g \leq_{\mathrm{vcl}_q D} (r - \alpha)q] \quad \forall r \in \mathbb{R}.$$

On the other hand, if Y is a topological linear space and D is closed, then $\mathrm{vcl}_q D = D$ and it follows that the lsc w.r.t. $\leq_{\mathrm{vcl}_q D}$ coincides with its topological counterpart lsc notion.

(ii) The (q, D)-usc concept is weaker than the usual cone usc notion whenever the ordering cone D has nonempty topological interior. Recall that if Y is a real topological linear space, then $g \colon X \to Y$ is said to be D-upper semicontinuous if for each $x \in X$ and for each neighborhood V of $g(x)$ there exists a neighborhood U of x such that $g(z) \in V - D$ for all $z \in U$ (see, for instance, [154]). Observe that the superlevel sets $[g \geq_D y]$ are closed for all $y \in Y$ whenever g is D-upper semicontinuous and D is closed.

Consider $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $D = \mathbb{R}^2_+$ and $g \colon X \to Y$ defined as

$$g(t) = \begin{cases} (0, 0) & \text{if} \quad t < 0 \\ \left(1, -\frac{1}{t+1}\right) & \text{if} \quad t \geq 0. \end{cases}$$

g is not D-upper semicontinuous at $t = 0$. Indeed, if V is a (small enough) neighborhood of $g(0) = (1, -1)$, we may consider a point $z < 0$ in any neighborhood of 0, and hence $g(z) = (0, 0) \notin V - D$.

Moreover, take $q = (1, 1)$, then

$$[g <_D rq] = \begin{cases} \emptyset & \text{if} \quad r \leq 0 \\ (-\infty, 0) & \text{if} \quad 0 < r \leq 1 \\ \mathbb{R} & \text{if} \quad r > 1. \end{cases}$$

Then g is (q, D)-usc.

(iii) In [9, Definition 2.1(iii)] the author introduced the following upper semicontinuity concept for vector-valued functions in the spirit of Tammer's lower semicontinuity: suppose that Y is a locally convex Hausdorff topological linear space and $q \in \operatorname{int} D$. Then $g\colon X \to Y$ is said to be (q, D)-superlevel closed if the set $[g \geq_D rq]$ is closed for all $r \in \mathbb{R}$ (in [9] is also named as (q, D)-upper semicontinuous). This notion was used in [9, Theorem 4.1] to prove the existence of weak efficient solutions of vector equilibrium problems (see Remark 4.19).

Chapter 2

Algebraic notions of solution

2.1 Introduction

In the literature, there are many works that deal with optimization problems from an algebraic point of view, that is, the problems are defined on real linear spaces not endowed with any particular topology (see, for instance, books [13, 14, 110, 112], papers [1–6, 106, 107, 120, 130, 156, 157, 163, 199, 202–206], and references therein). This research line provides an alternative way to study classic problems since topological tools are not used. Some hypotheses of well-known results are reformulated or even weakened by using algebraic counterparts and, as a result, many topological components are shown to be unessential. Of course, the new achieved algebraic results remain valid on topological linear spaces, so they may generalize the previous ones.

Every real linear space may be equipped with the strongest locally convex topology, called the *core convex topology* (see [117, Section 6.3]) and some algebraic notions coincide with topological concepts in this topology. However, the algebraic counterparts only are equivalent to the topological ones under additional conditions. For example, the algebraic interior and the vector closure require convexity assumptions to coincide with the topological interior and the topological closure, respectively (see [117, Proposition 6.3.1(iii)] and [157, Lemma 3.1]). Thus, the optimality notions defined through algebraic tools are different, and their respective results may extend the previous topological ones.

On the other hand, it is interesting to consider equilibrium problems from an algebraic point of view since they encompass several important problems and we may state new results in this general setting. For example, existence theorems for saddle point problems –which are a particular case of an equilibrium problem (see Section 1.1, Problem 1.3)– have already been obtained in [6,206] in real linear spaces apart from optimization problems.

This chapter is structured as follows. In section 2.2 we focus on minimal solutions of vector equilibrium problems with respect to arbitrary ordering sets. In particular, we highlight the notion of (C,ε)-efficient solution, which was introduced by Gutiérrez, Jiménez and Novo [91, 92], and the notion of E-optimality, which was defined by Chicco, Mignanego, Pusillo and Tijs [43] in finite dimensional vector optimization problems and extended by Gutiérrez, Jiménez and Novo [95] to real locally convex spaces. We generalize these notions to equilibrium problems on real linear spaces and we study their main properties. E-weak efficient solutions for vector equilibrium problems are characterized by using a linear scalarization procedure and generalized convexity assumptions.

Section 2.3 is devoted to approximate proper efficient solutions. Henig and Benson (C,ε)-proper efficient solutions were introduced in [88, 90], respectively, for vector optimization problems in real locally convex spaces. In this section, we study them for vector equilibrium problems on real linear spaces. We establish an inclusion between them and characterize the Henig (C,ε)-proper efficient solutions by linear scaralarization under generalized convexity assumptions.

In section 2.4 we extend the notion of strict solution from vector optimization problems (see [113, 175]) to vector equilibrium problems on real linear spaces, and we introduce and study an approximate version of this concept by means of free-disposal sets. This extension is essential for the exact and approximate EVP-type results for vector-valued bifunctions obtained in Chapter 4. Some basic properties of this kind of solutions are stated.

2.2 Minimal solutions and arbitrary ordering sets

In Chapter 1, we saw that the study of approximate solutions is a relevant research line in different scalar problems and so, it is worthy to focus them from the point of view of equilibrium problems.

Gutiérrez, Jiménez and Novo introduced a new concept of approximate solution for vector optimization problems based on coradiant sets in order to unify and generalize several ε-efficiency notions of the literature (see [91, 92]), since they noticed that many of them are actually concepts of nondominated solutions with respect to certain ordering sets. Next, we will introduce it for vector equilibrium problems. Let S be an arbitrary nonempty decision set, Y be a real linear space ordered by a proper convex cone D and $f\colon S \times S \to Y$ be a bifunction.

Definition 2.1. Let $\varepsilon \geq 0$ and $C \in \mathcal{H}$. A point $\bar{x} \in S$ is a (C, ε)-efficient solution of problem VEP and it is denoted by $\bar{x} \in \mathrm{E}(f, S, C, \varepsilon)$ if

$$f(\bar{x}, S) \cap \big(- C(\varepsilon)\big) = \emptyset.$$

Remark 2.2. (i) Clearly, if $\operatorname{cone} C = D$, then $\mathrm{E}(f, S, C, 0) = \mathrm{E}(f, S, D)$.

(ii) $C \in \mathcal{H}$ is assumed in order to obtain a consistent set of approximate solutions of problem VEP. Indeed, if a nonempty set $C \subset Y \backslash \{0\}$ is such that $C \notin \mathcal{H}$, then $C(\varepsilon) \cap (-D) \neq \emptyset$ for all $\varepsilon \geq 0$, and if $\bar{x} \in S$ is a (C, ε)-efficient solution of problem VEP, we have that

$$f(\bar{x}, S) \cap \Big(\big(- C(\varepsilon)\big) \cap D \Big) = \emptyset. \tag{2.1}$$

Hence, we are removing any feasible point $\hat{x} \in S$ that does not satisfy (2.1) as possible (C, ε)-efficient solution. This fact may lead to a senseless procedure in order to deal with a suitable approximate efficient set as it was shown for vector optimization problems (and so, for equilibrium problems) in [88, Example 2.5], where the set of efficient solutions is infinite and the sets of approximate efficient solutions are empty.

(C, ε)-efficiency encompasses some of the most celebrated notions of approximate efficiency given in the literature. Assume that Y is a real topological linear space, D is a proper pointed convex cone and consider $g\colon S \to Y$.

- Set $f(x, y) := g(y) - g(x)$ in order to deal with vector optimization problems. Then it is known that (C, ε)-efficiency encompasses the ε-efficiency notions given by Kutateladze [127], Németh [148], White [189], Helbig [104], Valyi [185, 186] or Tanaka [184].

- Similar approaches to Kutateladze's ε-efficiency were introduced for vector equilibrium problems, such as the ε-equilibrium point given by Bianchi, Kassay and Pini [22], and the λ-equilibrium point given by Ansari [9]. Both extensions are clearly encompassed by the (C, ε)-efficiency notion.

On the other hand, Chicco, Mignanego, Pusillo and Tijs [43] introduced the concept of E-optimality in finite-dimensional vector optimization problems ordered by components. The idea behind this concept is similar to the (C, ε)-efficiency since the authors considered nondominated points with respect to an ordering set E that is assumed to be an improvement set with respect to the Pareto cone (see (1.14)). Gutiérrez, Jiménez and Novo [95] extended this notion to the infinite-dimensional framework by using improvement sets with respect to an arbitrary ordering convex cone. This extended E-optimality notion also unifies and generalizes several approximate ε-efficiency concepts of vector optimization problems.

Later, Lalitha and Chatterjee [132] characterized E-optimal solutions by scalarization and established stability results for these solutions. Xia, Zhang and Zhao [190] studied conic extensions of improvement sets via the quasi interior and applied the obtained result to characterize by scalarization weak efficient solutions of vector optimization problems with set-valued mappings. Recently, Gutiérrez, Huerga, Jiménez and Novo [89] obtained E-optimality results in vector optimization problems on real linear spaces.

Other more general approaches for vector optimization problems have been provided in this line by considering the usual notion of nondominated point and assuming very mild conditions on the ordering set E (see, for instance, [17, 65]).

We will consider a very general equilibrium problem given by the data f, S and E, where $E \subset Y$ is a algebraic solid set and $0 \notin \operatorname{core} E$.

Definition 2.3. A point $\bar{x} \in S$ is said to be a E-weak efficient solution (or simply, a weak efficient solution) of VEP if satisfies

$$f(\bar{x}, S) \cap (-\operatorname{core} E) = \emptyset. \tag{2.2}$$

The set of all weak efficient solutions of problem VEP is denoted by $\mathrm{WE}(f, S, E)$.

Observe that the exact case of weak efficient solutions of a vector equilibrium problem is included in the previous definition by taking $E = D$, whenever D is algebraic solid.

From now on, assume that E is free-disposal with respect to a proper convex algebraic solid cone D. Next, we give a sufficient condition for E-weak efficient solutions.

Theorem 2.4. For every $\mu \in E^+ \backslash \{0\}$, it follows that

$$\mathrm{E}\left(\mu \circ f, S, \tau_E(\mu)\right) \subset \mathrm{WE}\left(f, S, E\right).$$

Proof. Take $\bar{x} \in \mathrm{E}\left(\mu \circ f, S, \tau_E(\mu)\right)$ and suppose by contradiction that $\bar{x} \notin \mathrm{WE}(f, S, E)$. Then, by Proposition 1.16(a) we have that

$$f(\bar{x}, S) \cap \big(-E - \operatorname{core} D\big) \neq \emptyset,$$

so there exist $\bar{y} \in S, e \in E$ and $d \in \operatorname{core} D$ such that $f(\bar{x}, \bar{y}) = -e - d$. It is clear that $\mu(d) > 0$, since $\mu \in D^+ \backslash \{0\}$ (see Remark 1.19) and $d \in \operatorname{core} D$. Hence,

$$\mu \circ f(\bar{x}, \bar{y}) + \tau_E(\mu) = -\mu(e) - \mu(d) + \tau_E(\mu) < 0,$$

which is a contradiction to $\bar{x} \in \mathrm{E}\left(\mu \circ f, S, \tau_E(\mu)\right)$ (see (1.22)). $\square$

The next result provides us a necessary condition for E-weak efficient solutions under a generalized convexity assumption on the bifunction.

Theorem 2.5. Consider $\bar{x} \in S$. Assume that $f(\bar{x}, \cdot)$ is v-nearly E-subconvexlike on S, and that one of the following assumptions is satisfied

(i) $0 \in f(\bar{x}, S)$;

(ii) E is coradiant.

If $\bar{x} \in \mathrm{WE}(f, S, E)$, then there exists $\mu \in E^+ \backslash \{0\}$ such that $\bar{x} \in \mathrm{E}\left(\mu \circ f, S, \tau_E(\mu)\right)$.

Proof. Since $\bar{x} \in \mathrm{WE}(f, S, E)$, by Proposition 1.16($a$) we have that

$$f(\bar{x}, S) \cap \big(-\operatorname{core} E\big) = \emptyset.$$

By Proposition 1.16(a) it follows that

$$\big(f(\bar{x}, S) + E\big) \cap (-\operatorname{core} D) = \emptyset,$$

and then

$$\operatorname{vcl} \operatorname{cone} \big(f(\bar{x}, S) + E\big) \cap (-\operatorname{core} D) = \emptyset.$$

Since $f(\bar{x}, \cdot)$ is v-nearly E-subconvexlike on S, we know that the set $\operatorname{vcl} \operatorname{cone} \big(f(\bar{x}, S) + E\big)$ is convex. By the basic version of the separation theorem (see [112, Theorem 3.14]), there exists $\mu \in Y' \backslash \{0\}$ and $\alpha \in \mathbb{R}$ such that

$$\begin{aligned} \mu(y) \geq \alpha \geq \mu(-d) \quad & \forall y \in \operatorname{vcl} \operatorname{cone} \big(f(\bar{x}, S) + E\big), \forall d \in D, \\ \alpha > \mu(-d) \quad & \forall d \in \operatorname{core} D. \end{aligned}$$

It is easy to check that $\alpha = 0$. If (i) is satisfied, then $0 \in f(\bar{x}, S)$ and it follows that $\mu(e) \geq 0$ for all $e \in E$, so that $\mu \in E^+$. If (ii) is satisfied, then fix an arbitrary point $\bar{y} \in f(\bar{x}, S)$. We have that

$$\frac{1}{\lambda}\mu(\bar{y}) + \mu(e) = \frac{1}{\lambda}\mu(\bar{y} + \lambda e) \geq 0 \quad \forall e \in E, \forall \lambda \geq 1.$$

By taking the limit $\lambda \to +\infty$, it follows that $\mu(e) \geq 0$. Then it is clear that $\mu \in E^+$. Hence, in both cases,

$$\mu \in E^+ \backslash \{0\}.$$

It follows that

$$\mu\big(f(\bar{x}, x)\big) + \mu(e) \geq 0 \quad \forall x \in S, \forall e \in E,$$

and then

$$\mu\big(f(\bar{x}, x)\big) + \tau_E(\mu) \geq 0 \quad \forall x \in S.$$

Therefore $\bar{x} \in \mathrm{E}\left(\mu \circ f, S, \tau_E(\mu)\right)$. □

The next characterization of E-weak efficient solutions follows from Theorems 2.4 and 2.5.

Corollary 2.6. For each $x \in S$, assume that $f(x, \cdot)$ is v-nearly E-subconvexlike on S, and one of the following assumptions is satisfied

(i) $0 \in f(x, S)$;

(ii) E is coradiant.

Then

$$\mathrm{WE}(f, S, E) = \bigcup_{\mu \in E^+ \setminus \{0\}} \mathrm{E}\left(\mu \circ f, S, \tau_E(\mu)\right).$$

2.3 Approximate proper solutions

The set of efficient points of an ordered set is usually very large, so that in problems with ordered spaces it is often difficult to choose the most suitable decision. Proper efficiency was introduced in vector optimization in order to select those efficient solutions that satisfy some desirable properties as, for instance, stability, relationships with solutions of associated scalar optimization problems, etc. In this way, many authors defined and studied different proper efficiency notions as for instance, [19,30,32,71,102,105,111] (for more detailed information, see [86, 117]). To the best of our knowledge, the first concepts of approximate proper efficient solution of vector optimization problems were introduced at the end of the nineties (see [138, 141, 167]) and since the early 21st century they have also been studied on real linear spaces with algebraic tools (see [5, 107, 120, 202–206]) and have been extended to vector equilibrium problems (see, for instance, [34, 35, 77, 79–81, 83, 169, 170]).

Recently, several "proper" versions concerning with the concept of (C, ε)-efficiency has been studied (see [69, 70, 87, 88, 90, 173, 200]). In particular, we stand out the Henig (C, ε)-proper efficient solutions for vector optimization problems, whose properties have been studied in real locally convex spaces and have been characterized by means of linear scalarization under generalized convexity assumptions.

In this section, we study the Henig (C,ε)-proper efficiency for vector equilibrium problems in the framework of real linear spaces not endowed with any particular topology. By means of the vector closure, we define a generalized convexity assumption which allows us to characterize this kind of approximate proper efficient solutions by linear scalarization.

From now on in this section suppose that $\operatorname{vcl} D \cap (-D) = \{0\}$. This condition is clearly satisfied when D is pointed and vectorially closed. Next, we will give two concepts of proper efficiency for equilibrium problems on real linear spaces. The first one is based on a well-known notion introduced by Henig for vector optimization problems which was extended by Gong for vector equilibrium problems in Hausdorff locally convex spaces (named also as global efficiency or global proper efficiency, see [79, 105]).

Definition 2.7. A point $\bar{x} \in S$ is said to be a Henig (vectorial) proper efficient solution for problem VEP if there exists $K \in \mathcal{G}$ such that $\bar{x} \in \mathrm{E}(f, S, K)$. It is denoted by $\bar{x} \in \mathrm{HeV}(f, S, D)$.

Notice that it is not required the pointedness of K in the same way as Henig's definition [105] for vector optimization problems. In fact, for each $K \in \mathcal{G}$, we may set a pointed cone $K' := \big(K \backslash (-K)\big) \cup \{0\} \subset K$ and it is easy to check that $K' \in \mathcal{G}$. It follows that $\mathrm{E}(f, S, K) \subset \mathrm{E}(f, S, K')$, so that the pointedness assumption on K is not necessary. Hence, the algebraic notion of Henig proper efficiency coincides with the topological one [79, Definition 1.1] introduced by Gong in locally convex spaces which considers pointed cones.

Adán and Novo [5] defined an algebraic notion of Benson proper efficiency [19], and later Gong [79] extended it for equilibrium problems (in a topological setting). Next, we generalize the algebraic notion of Adán and Novo [5] to equilibrium problems.

Definition 2.8. A point $\bar{x} \in S$ is said to be a Benson (vectorial) proper efficient solution for problem VEP if

$$\operatorname{vcl} \operatorname{cone}(f(\bar{x}, S) + D) \cap (-D \backslash \{0\}) = \emptyset.$$

The set of all Benson proper efficient solutions of problem VEP it is denoted by $\mathrm{BeV}(f, S, D)$.

Remark 2.9. (i) By Lemma 1.13(a) and (c), it is easy to check that $\bar{x} \in S$ is a proper efficient solution in the sense of Henig for problem VEP if there exists $K \in \mathcal{G}$ such that $\bar{x} \in \mathrm{WE}(f, S, K)$.

(ii) In general,

$$\mathrm{HeV}(f, S, D) \subset \mathrm{BeV}(f, S, D) \subset \mathrm{E}(f, S, D). \tag{2.3}$$

Indeed, given $\bar{x} \in \mathrm{HeV}(f, S, D)$, there exists $K \in \mathcal{G}$ such that $f(\bar{x}, S) \cap (-\operatorname{core} K) = \emptyset$. Hence, by Lemma 1.13$(c)$,

$$(f(\bar{x}, S) + K) \cap (-\operatorname{core} K) = \emptyset$$

and, since $\operatorname{core} K \cup \{0\}$ is a cone, we have that $\operatorname{cone}(f(\bar{x}, S) + K) \cap (-\operatorname{core} K) = \emptyset$. Then it is clear that

$$\operatorname{vcl}\operatorname{cone}(f(\bar{x}, S) + K) \cap (-\operatorname{core} K) = \emptyset,$$

and since $D \backslash \{0\} \subset \operatorname{core} K$, it follows that $\bar{x} \in \mathrm{BeV}(f, S, D)$. The other inclusion is obvious.

Next, we introduce concepts of Henig and Benson (C, ε)-proper efficient solution for vector equilibrium problems on real linear spaces by means of algebraic tools.

Definition 2.10. Let $\varepsilon \geq 0$ and $C \in \widehat{\mathcal{H}}$. A point $\bar{x} \in S$ is a Henig (vectorial) (C, ε)-proper efficient solution of problem VEP, and it is denoted by $\bar{x} \in \mathrm{HeV}(f, S, C, \varepsilon)$, if there exists $K \in \mathcal{G}(C)$ such that $\bar{x} \in \mathrm{E}\left(f, S, \mathcal{E}_0^K(C), \varepsilon\right)$.

Remark 2.11. (i) The previous concept encompasses the concept of exact Henig's proper efficiency by taking $C = D \backslash \{0\}$ and an arbitrary $\varepsilon \geq 0$. Indeed, we have that $\mathcal{G}(D \backslash \{0\}) = \mathcal{G}$. Moreover, given $K \in \mathcal{G}$, it is clear by Lemma 1.14(a) that $\mathrm{WE}(f, S, K) = \mathrm{E}\left(f, S, \mathcal{E}_0^K(D \backslash \{0\}), \varepsilon\right)$ for all $\varepsilon \geq 0$, and by Remark 2.9(i), we have that

$$\mathrm{HeV}(f, S, D) = \mathrm{HeV}(f, S, D \backslash \{0\}, \varepsilon) \quad \forall \varepsilon \geq 0.$$

(ii) Suppose that $K \in \mathcal{G}$. We have that $K \in \mathcal{G}(C)$ if and only if $C \cap (-\operatorname{core} K) = \emptyset$ and, by Lemma 1.14(a) we know that $\mathcal{E}_0^K(D) = \operatorname{core} K$. Then $K \in \mathcal{G}(C)$ if and only if $\mathcal{E}_0^K(C) \cap (-D) = \emptyset$, or equivalently,

$$K \in \mathcal{G}(C) \Leftrightarrow \mathcal{E}_0^K(C) \in \mathcal{H}.$$

Thus, the condition $K \in \mathcal{G}(C)$ implies that $\mathcal{E}_0^K(C)$ is an appropiate set to deal with approximate efficient solutions of problem VEP according to Remark 2.2(ii). Moreover, condition $K \in \mathcal{G}(C)$ implies that $\operatorname{vcl}\operatorname{cone} C \cap (-\operatorname{core} K) = \emptyset$, and since $D\backslash\{0\} \subset \operatorname{core} K$, it follows that

$$\operatorname{vcl}\operatorname{cone} C \cap (-D\backslash\{0\}) = \emptyset.$$

Therefore it is natural to require the condition $C \in \widehat{\mathcal{H}}$ in Definition 2.10 for consistency. (iii) It follows by definition that

$$\begin{aligned} \operatorname{HeV}(f,S,C,\varepsilon) &= \bigcup_{K \in \mathcal{G}(C)} \operatorname{E}\left(f,S,\mathcal{E}_0^K(C),\varepsilon\right) \\ &\subset \operatorname{E}\left(f,S,C+D\backslash\{0\},\varepsilon\right) \quad \forall C \in \widehat{\mathcal{H}},\ \forall \varepsilon \geq 0. \end{aligned} \tag{2.4}$$

Definition 2.12. Let $\varepsilon \geq 0$ and $C \in \widehat{\mathcal{H}}$. A point $\bar{x} \in S$ is a Benson (vectorial) (C,ε)-proper efficient solution of problem VEP if

$$\operatorname{vcl}\operatorname{cone}\big(f(\bar{x},S) + C(\varepsilon)\big) \cap (-D) = \{0\}.$$

It is denoted by $\bar{x} \in \operatorname{BeV}(f,S,C,\varepsilon)$.

Remark 2.13. Notice that $\operatorname{BeV}(f,S,C,0) = \operatorname{BeV}(f,S,D)$ whenever $\operatorname{cone} C = D$. Indeed, it is enough to show that $\operatorname{vcl}\operatorname{cone}(A + D\backslash\{0\}) \supset \operatorname{vcl}\operatorname{cone}(A+D)$, where $A \subset Y$ is nonempty. First, it is clear that

$$\operatorname{cone}(A+D) \subset \operatorname{vcl}\operatorname{cone}(A + D\backslash\{0\}). \tag{2.5}$$

Indeed, let $y \in \operatorname{cone}(A+D)$. Then there exist $\alpha \geq 0$, $a \in A$ and $d \in D$ such that $y = \alpha(a+d)$. If $\alpha = 0$ or $d \neq 0$, then it is obvious that $y \in \operatorname{cone}(A + D\backslash\{0\})$. Otherwise, $y = \alpha a$, where $\alpha > 0$, and taking an arbitrary point $d \in D\backslash\{0\}$ it is clear that

$$y + \frac{1}{n}d = \alpha\left(a + \frac{1}{\alpha n}d\right) \in \operatorname{cone}(A + D\backslash\{0\}).$$

Hence $y \in \operatorname{vcl} \operatorname{cone}(A + D \backslash \{0\})$ and statement (2.5) holds.

Let us prove that $\operatorname{vcl} \operatorname{cone}(A + D) \subset \operatorname{vcl} \operatorname{cone}(A + D \backslash \{0\})$. Consider an arbitrary point $y \in \operatorname{vcl} \operatorname{cone}(A + D)$. Then there exist $v \in Y$ and a sequence $(\lambda_n) \subset \mathbb{R}_+$ such that $\lambda_n \to 0$ and $y + \lambda_n v \in \operatorname{cone}(A + D)$ for all n. We can assume that $\lambda_n > 0$ for all n (otherwise $y \in \operatorname{vcl} \operatorname{cone}(A + D \backslash \{0\})$ by (2.5)).

If there exists a subsequence (λ_{n_k}) such that $y + \lambda_{n_k} v \in \operatorname{cone}(A + D \backslash \{0\})$, then $y \in \operatorname{vcl} \operatorname{cone}(A + D \backslash \{0\})$ and the proof finishes. Otherwise, there exists $n_0 \in \mathbb{N}$ and sequences $(\alpha_n)_{n \geq n_0} \subset \mathbb{R}_+ \backslash \{0\}$, $(a_n)_{n \geq n_0} \subset A$ such that

$$y + \lambda_n v = \alpha_n a_n \quad \forall n \geq n_0.$$

Let $d \in D \backslash \{0\}$ be arbitrary. We have that

$$y + \lambda_n (v + d) = \alpha_n \left(a_n + \frac{\lambda_n}{\alpha_n} d \right) \in \operatorname{cone}(A + D \backslash \{0\}) \quad \forall n \geq n_0,$$

and so $y \in \operatorname{vcl} \operatorname{cone}(A + D \backslash \{0\})$, which finishes the proof.

Let us give some equivalent formulations for Henig (C, ε)-proper efficient solutions of problem VEP. This result is an algebraic version of [88, Theorem 3.3] for equilibrium problems and it can be proved in a similar way.

Theorem 2.14. Let $\varepsilon \geq 0, C \in \widehat{\mathcal{H}}$ and $\bar{x} \in S$. The following statements are equivalent:

(a) $\bar{x} \in \operatorname{HeV}(f, S, C, \varepsilon)$.

(b) There exists $K \in \mathcal{O}(C)$ such that $\bar{x} \in \operatorname{E}\left(f, S, \mathcal{E}_0^K(C), \varepsilon\right)$.

(c) There exists $K \in \mathcal{O}(C)$ such that

$$\operatorname{vcl} \operatorname{cone}\big(f(\bar{x}, S) + C(\varepsilon)\big) \cap (-\operatorname{core} K) = \emptyset.$$

(d) There exists $K \in \mathcal{O}(C)$ such that

$$\big(f(\bar{x}, S) + C(\varepsilon)\big) \cap (-\operatorname{core} K) = \emptyset.$$

(e) $\mathcal{E}^D(C) \in \widehat{\mathcal{H}}$ and $\bar{x} \in \operatorname{HeV}(f, S, \mathcal{E}^D(C), \varepsilon)$.

Remark 2.15. Consequently, for each $C \in \widehat{\mathcal{H}}$, $\varepsilon \geq 0$ such that $\mathrm{HeV}(f, S, C, \varepsilon) \neq \emptyset$,

$$\mathrm{HeV}(f, S, C, \varepsilon) = \mathrm{HeV}(f, S, \mathcal{E}^D(C), \varepsilon).$$

Hence, in Definition 2.10 we can assume without loss of generality that C is an improvement set with respect to D. Moreover, taking $C = D\backslash\{0\}$ and $\varepsilon = 1$ in Theorem 2.14, we characterize the exact Henig proper efficient solutions of problem VEP by Remark 2.11(i).

As a corollary we obtain that every Henig (C, ε)-proper efficient solution for problem VEP is also a Benson (C, ε)-proper efficient solution. This result generalizes statement (2.3) to the approximate setting.

Corollary 2.16. Let $\varepsilon \geq 0$ and $C \in \widehat{\mathcal{H}}$. It follows that

$$\mathrm{HeV}(f, S, C, \varepsilon) \subset \mathrm{BeV}(f, S, C, \varepsilon).$$

Proof. Let $\bar{x} \in \mathrm{HeV}(f, S, C, \varepsilon)$. By Theorem 2.14($c$) there exists $K \in \mathcal{O}(C)$ such that

$$\mathrm{vcl}\,\mathrm{cone}\,\big(f(\bar{x}, S) + C(\varepsilon)\big) \cap (-\,\mathrm{core}\,K) = \emptyset.$$

Since $D\backslash\{0\} \subset \mathrm{core}\,K$, it follows that

$$\mathrm{vcl}\,\mathrm{cone}\,\big(f(\bar{x}, S) + C(\varepsilon)\big) \cap (-D\backslash\{0\}) = \emptyset,$$

and so $\bar{x} \in \mathrm{BeV}(f, S, C, \varepsilon)$. □

The next result shows several properties of the set of Henig (C, ε)-proper efficient solutions of problem VEP.

Theorem 2.17. Fix $\varepsilon \geq 0$ and $C \in \widehat{\mathcal{H}}$. The following properties hold:

(a) If $\mathrm{vcl}\,C \in \widehat{\mathcal{H}}$, then $\mathcal{G}(\mathrm{vcl}\,C) = \mathcal{G}(C)$ and $\mathrm{HeV}(f, S, \mathrm{vcl}\,C, \varepsilon) = \mathrm{HeV}(f, S, C, \varepsilon)$.

(b) $\mathrm{HeV}(f, S, C', \delta) \subset \mathrm{HeV}(f, S, C, \varepsilon)$ for all $C' \in \widehat{\mathcal{H}}$ and $\delta \geq 0$ such that $C(\varepsilon) \subset \mathrm{vcl}\,C'(\delta)$.

(c) If $\mathcal{E}^D(C)$ is coradiant, then $\mathrm{HeV}(f, S, C, \delta) \subset \mathrm{HeV}(f, S, C, \varepsilon)$ for all $0 < \delta \leq \varepsilon$.

(*d*) $\mathrm{HeV}(f, S, C + C', \varepsilon) = \mathrm{HeV}(f, S, \operatorname{vcl} C + C', \varepsilon)$ for all nonempty set $C' \subset Y$ such that $\operatorname{vcl} C + C' \in \widehat{\mathcal{H}}$.

(*e*) If C is convex and coradiant, then $\mathrm{HeV}(f, S, C, \varepsilon) = \mathrm{HeV}(f, S, C + C(0), \varepsilon)$.

(*f*) If $C \subset D\backslash\{0\}$, then $\mathrm{HeV}(f, S, D) \subset \mathrm{HeV}(f, S, C, \varepsilon)$.

(*g*) If $C \subset D \subset \operatorname{vcl} C(0)$, then $\mathrm{HeV}(f, S, C, 0) = \mathrm{HeV}(f, S, D)$.

Proof. (*a*) Obviously, $\mathcal{G}(\operatorname{vcl} C) \subset \mathcal{G}(C)$. For the converse inclusion, consider $K \in \mathcal{G}(C)$. Hence $C \cap (-\operatorname{core} K) = \emptyset$ and so

$$\operatorname{vcl} C \cap (-\operatorname{core} K) = \emptyset.$$

Thus $K \in \mathcal{G}(\operatorname{vcl} C)$ and this proves that $\mathcal{G}(\operatorname{vcl} C) = \mathcal{G}(C)$. On the other hand, we know that $\mathcal{E}_0^K(\operatorname{vcl} C) = \mathcal{E}_0^K(C)$ by Lemma 1.13(*f*). Therefore

$$\mathrm{E}\left(f, S, \mathcal{E}_0^K(\operatorname{vcl} C), \varepsilon\right) = \mathrm{E}\left(f, S, \mathcal{E}_0^K(C), \varepsilon\right),$$

and so $\mathrm{HeV}(f, S, \operatorname{vcl} C, \varepsilon) = \mathrm{HeV}(f, S, C, \varepsilon)$.

(*b*) It is easy to check that $\mathcal{O}(C') \subset \mathcal{O}(C)$. Choose $\bar{x} \in \mathrm{HeV}(f, S, C', \delta)$. By Theorem 2.14(*c*) there exists $K \in \mathcal{O}(C)$ such that

$$\operatorname{vcl} \operatorname{cone}\left(f(\bar{x}, S) + C'(\delta)\right) \cap (-\operatorname{core} K) = \emptyset,$$

and by [4, Proposition 5(i)],

$$\operatorname{cone} \operatorname{vcl}\left(f(\bar{x}, S) + C'(\delta)\right) \cap (-\operatorname{core} K) = \emptyset.$$

Since $A + \operatorname{vcl} B \subset \operatorname{vcl}(A + B)$ for every nonempty sets $A, B \subset Y$, it follows that

$$\operatorname{cone}\left(f(\bar{x}, S) + \operatorname{vcl} C'(\delta)\right) \cap (-\operatorname{core} K) = \emptyset.$$

We know that $C(\varepsilon) \subset \operatorname{vcl} C'(\delta)$, then

$$\left(f(\bar{x}, S) + C(\varepsilon)\right) \cap (-\operatorname{core} K) = \emptyset,$$

and by Theorem 2.14(*d*) it follows that $\bar{x} \in \mathrm{HeV}(f, S, C, \varepsilon)$.

(*c*) Let $0 < \delta \leq \varepsilon$. Since $\mathcal{E}^D(C)$ is coradiant, it follows that $(\varepsilon/\delta)\mathcal{E}^D(C) \subset \mathcal{E}^D(C)$ and so

$$C(\varepsilon) \subset \mathcal{E}^D(C)(\varepsilon) \subset \mathcal{E}^D(C)(\delta).$$

Suppose that $\mathrm{HeV}(f, S, C, \delta) \neq \emptyset$ (otherwise the result is obvious). Hence $\mathcal{E}^D(C) \in \widehat{\mathcal{H}}$ by Theorem 2.14(*e*), and applying part (*b*) with $C' = \mathcal{E}^D(C)$ and Remark 2.15 we obtain that

$$\mathrm{HeV}(f, S, C, \delta) = \mathrm{HeV}(f, S, \mathcal{E}^D(C), \delta) \subset \mathrm{HeV}(f, S, C, \varepsilon).$$

(*d*) We have that $\mathcal{O}(C+C') = \mathcal{O}(\operatorname{vcl} C+C')$. Indeed, given $K \in \mathcal{O}(C+C')$, it follows that

$$C' \cap (-C - \operatorname{core} K) = \emptyset.$$

By applying Lemma 1.13(*f*),

$$C' \cap (-\operatorname{vcl} C - \operatorname{core} K) = \emptyset,$$

so that $(\operatorname{vcl} C + C') \cap (-\operatorname{core} K) = \emptyset$ and it follows that $K \in \mathcal{O}(\operatorname{vcl} C + C')$. Moreover, for every $K \in \mathcal{O}(C + C')$ we have that

$$\mathcal{E}_0^K(\operatorname{vcl} C + C') = \mathcal{E}_0^K(\operatorname{vcl} C) + C' = \mathcal{E}_0^K(C) + C' = \mathcal{E}_0^K(C + C').$$

Then it follows that $\mathrm{E}\left(f, S, \mathcal{E}_0^K(\operatorname{vcl} C + C'), \varepsilon\right) = \mathrm{E}\left(f, S, \mathcal{E}_0^K(C + C'), \varepsilon\right)$ for all $K \in \mathcal{O}(C+C')$, which implies that $\mathrm{HeV}(f, S, C+C', \varepsilon) = \mathrm{HeV}(f, S, \operatorname{vcl} C+C', \varepsilon)$ by Theorem 2.14(*b*).

(*e*) We have that $C \subset C + \operatorname{vcl} C(0) \subset \operatorname{vcl}\big(C + C(0)\big)$ and by [91, Lemma 3.1(v)], it follows that

$$C + C(0) \subset C \subset \operatorname{vcl}\big(C + C(0)\big). \tag{2.6}$$

Since $C \in \widehat{\mathcal{H}}$, it follows that $C + C(0) \in \widehat{\mathcal{H}}$ and $\mathcal{G}(C) = \mathcal{G}\big(C + C(0)\big)$. Moreover, by Lemma 1.13(*f*) and (2.6), we know that

$$\mathcal{E}_0^K\big(C + C(0)\big) = \mathcal{E}_0^K(C)$$

for all $K \in \mathcal{G}(C)$. Therefore

$$\mathrm{E}\left(f, S, \mathcal{E}_0^K(C), \varepsilon\right) = \mathrm{E}\Big(f, S, \mathcal{E}_0^K\big(C + C(0)\big), \varepsilon\Big)$$

for all $K \in \mathcal{G}(C)$, and so $\mathrm{HeV}(f,S,C,\varepsilon) = \mathrm{HeV}(f,S,C+C(0),\varepsilon)$.

(f) By Remark 2.11(i) and by applying part (b) with $C' = D\backslash\{0\}$ and $\delta = 1$, we obtain that

$$\mathrm{HeV}(f,S,D) = \mathrm{HeV}(f,S,D\backslash\{0\},1) \subset \mathrm{HeV}(f,S,C,\varepsilon).$$

(g) Applying part (f) with $\varepsilon = 0$, we have that

$$\mathrm{HeV}(f,S,D) \subset \mathrm{HeV}(f,S,C,0).$$

Conversely, as $(D\backslash\{0\})(1) \subset \mathrm{vcl}\, C(0)$, by Remark 2.11($i$) and part ($b$), it follows that

$$\mathrm{HeV}(f,S,C,0) \subset \mathrm{HeV}(f,S,D\backslash\{0\},1) = \mathrm{HeV}(f,S,D)$$

and the proof is complete. □

Some properties established in [88, Theorem 3.6] are proved in Theorem 2.17 for Henig vectorial (C,ε)-proper efficient solutions of problem VEP. However, in [88, Theorem 3.6(a)] one has that

$$C \in \overline{\mathcal{H}}, 0 \notin \mathrm{cl}\, C \quad \Longrightarrow \quad \mathrm{cl}\, C \in \overline{\mathcal{H}}$$

and this property is not true for the vector closure and $C \in \widehat{\mathcal{H}}$. Consider $Y = \mathbb{R}^3, D = \{0\} \times (-\mathbb{R}_+) \times \mathbb{R}_+$ and $C = \{(x,y,z) \in \mathbb{R}^3 : 0 < x, y = 1, 0 < z \leq x^2\}$. Hence $C \in \widehat{\mathcal{H}}$, $(0,0,0) \notin \mathrm{vcl}\, C$, but $\mathrm{vcl}\,\mathrm{cone}(\mathrm{vcl}\, C) \cap (-D\backslash\{0\}) = \{0\} \times \big(\mathbb{R}_+\backslash\{0\}\big) \times \{0\}$, so that $\mathrm{vcl}\, C \notin \widehat{\mathcal{H}}$. Therefore it is necessary to require that $\mathrm{vcl}\, C \in \widehat{\mathcal{H}}$ in (a).

For the rest of this section, fix $C \in \widehat{\mathcal{H}}$ and $\varepsilon \geq 0$. From statement (2.4) we know that Henig (vectorial) (C,ε)-proper efficient solutions of problem VEP are $(\mathcal{E}_0^K(C),\varepsilon)$-efficient solutions for some $K \in \mathcal{G}(C)$. Observe that the set $\mathcal{E}_0^K(C)(\varepsilon)$ is free-disposal with respect to $K \in \mathcal{G}(C)$ by Lemma 1.13(c) and Lemma 1.14(b).

Next, we provide sufficient and necessary conditions for Henig (vectorial) (C,ε)-proper efficient solutions of problem VEP through linear scalarization under a generalized convexity assumption by using Theorems 2.4 and 2.5. The proofs of these results are omitted since they may be proved in a similar way to their counterparts in vector optimization problems (see [88, Theorems 4.4, 4.5]).

Theorem 2.18. For every $\mu \in D^{\#} \cap C^{+}$ we have that

$$\mathrm{E}\left(\mu \circ f, S, \varepsilon\tau_C(\mu)\right) \subset \mathrm{HeV}(f, S, C, \varepsilon).$$

Theorem 2.19. Consider $\bar{x} \in S$. Assume that $f(\bar{x}, \cdot)$ is v-nearly (C, ε)-subconvexlike on S, and that one of the following assumptions is satisfied:

(i) $0 \in f(\bar{x}, S)$;

(ii) $C(\varepsilon)$ is coradiant.

If $\bar{x} \in \mathrm{HeV}(f, S, C, \varepsilon)$, then $C \in \mathcal{F}_{D^{\#}}$ and there exists $\mu \in D^{\#} \cap C^{+}$ such that $\bar{x} \in \mathrm{E}\left(\mu \circ f, S, \varepsilon\tau_C(\mu)\right)$.

The next corollary follows from Theorems 2.18, 2.19 and Corollary 2.16. It provides us a characterization of Henig (vectorial) (C, ε)-proper efficient solutions by linear scalarization under a generalized convexity assumption.

Corollary 2.20. It follows that

$$\bigcup_{\mu \in D^{\#} \cap C^{+}} \mathrm{E}\left(\mu \circ f, S, \varepsilon\tau_C(\mu)\right) \subset \mathrm{HeV}(f, S, C, \varepsilon) \subset \mathrm{BeV}(f, S, C, \varepsilon).$$

Moreover, for each $x \in S$ suppose that $f(x, \cdot)$ is v-nearly (C, ε)-subconvexlike on S, and one of the following assumptions is satisfied

(i) $0 \in f(x, S)$;

(ii) $C(\varepsilon)$ is coradiant.

Then, it follows that

$$\mathrm{HeV}(f, S, C, \varepsilon) = \bigcup_{\mu \in D^{\#} \cap C^{+}} \mathrm{E}\left(\mu \circ f, S, \varepsilon\tau_C(\mu)\right).$$

By Remark 2.11(i) and by applying Corollary 2.20 with $C = D\backslash\{0\}$, since $\tau_{D\backslash\{0\}}(\mu) = 0$ for all $\mu \in D^{\#}$, we obtain that

$$\mathrm{HeV}(f, S, D) = \bigcup_{\mu \in D^{\#}} \mathrm{E}(\mu \circ f, S)$$

whenever $f(x,\cdot)$ is v-nearly $\big(D\backslash\{0\},0\big)$-subconvexlike on S for all $x \in S$, i.e., whenever $\operatorname{vcl}\operatorname{cone}\big(f(x,S)+D\big)$ is convex for all $x \in S$ (see Remark 2.13). Notice that this characterization improves [79, Theorem 2.1(i)] since the v-nearly $\big(D\backslash\{0\},0\big)$-subconvexlike assumption is weaker than the D-convexlike assumption (see Theorem 1.24).

Let us provide some properties of the set of Henig (vectorial) (C,ε)-proper efficient solutions of problem VEP with respect to the coradiant hull and the cone generated by C.

Proposition 2.21. Consider $\bar{x} \in S$. Suppose that $f(\bar{x},\cdot)$ is v-nearly (C,ε)-subconvexlike on S, and that one of the following assumptions is satisfied:

(i) $0 \in f(\bar{x},S)$;

(ii) $C(\varepsilon)$ is coradiant.

It follows that

$$\bar{x} \in \operatorname{HeV}(f,S,C,\varepsilon) \Longleftrightarrow \bar{x} \in \operatorname{HeV}(f,S,\operatorname{shw} C,\varepsilon).$$

If additionally $\operatorname{co} C \in \widehat{\mathcal{H}}$, then

$$\bar{x} \in \operatorname{HeV}(f,S,C,\varepsilon) \Longleftrightarrow \bar{x} \in \operatorname{HeV}(f,S,\operatorname{co} C,\varepsilon).$$

Proof. Let us prove the first equivalence, since the second one can be obtained in a similar way. Since $\operatorname{cone}(\operatorname{shw} C) = \operatorname{cone} C$, it is clear that $\operatorname{shw} C \in \widehat{\mathcal{H}}$. If $\bar{x} \in \operatorname{HeV}(f,S,\operatorname{shw} C,\varepsilon)$, by Theorem 2.17($b$) we have that $\bar{x} \in \operatorname{HeV}(f,S,C,\varepsilon)$.

For the converse implication, if $\bar{x} \in \operatorname{HeV}(f,S,C,\varepsilon)$, by Theorem 2.19 there exists $\mu \in D^{\#} \cap C^{+}$ such that $\bar{x} \in \operatorname{E}\big(\mu \circ f, S, \varepsilon\tau_C(\mu)\big)$. It is easy to check that $\tau_C(\mu) = \tau_{\operatorname{shw} C}(\mu)$. Therefore $\mu \in (\operatorname{shw} C)^{+}$ and by Theorem 2.18 we have that $\bar{x} \in \operatorname{HeV}(f,S,\operatorname{shw} C,\varepsilon)$, which finishes the proof. $\square$

2.4 Strict solutions

The concept of strict efficient solution was studied for vector optimization problems in finite dimensional spaces by Smale [175, p. 220]. It was extended by Jiménez to the framework of normed spaces (see [113, Definition 3.2]) and an approximate counterpart based on coradiant sets was introduced in [96]. In the following, we introduce a version of this concept for vector equilibrium problems by using free-disposal sets and, in Chapter 4, we will use it to obtain EVPs for vector-valued bifunctions whose space is not endowed with any particular topology.

Definition 2.22. A point $\bar{x} \in S$ is a strict efficient solution of VEP, denoted $\bar{x} \in \mathrm{S}(f, S, D)$ (and $\bar{x} \in \mathrm{S}(f, D)$ if $S = X$), if

$$f(\bar{x}, x) \notin -D \quad \forall x \in S \backslash \{\bar{x}\}.$$

This concept reduces to the notion of strict solution of a scalar equilibrium problem (see [24]) by considering $Y = \mathbb{R}$ and $D = \mathbb{R}_+$. Moreover, it is clear that $\mathrm{S}(f, S, D) \subset \mathrm{E}(f, S, D)$ whenever f is diagonal null, and this inclusion may be strict. For example, consider $X = Y = \mathbb{R}$, $S = [-1, 1]$, $D = \mathbb{R}_+$ and $f\colon [-1, 1] \times [-1, 1] \to \mathbb{R}$ such that

$$f(x, y) = \begin{cases} \max\left\{-x, \frac{1}{2}\right\}, & \text{if } x \leq 0 \text{ and } x \neq y, \\ 0, & \text{if } x = y \text{ or } x \in \left(\frac{1}{2}, \frac{3}{4}\right), \\ -1, & \text{otherwise.} \end{cases}$$

Then $\mathrm{S}(f, S, D) = [-1, 0] \subsetneq \mathrm{E}(f, S, D) = [-1, 0] \cup \left(\frac{1}{2}, \frac{3}{4}\right)$.

On the other hand, if we consider $f(x, y) = g(y) - g(x)$, where $g\colon S \to Y$ is a vector-valued mapping, we have that $\bar{x} \in S$ is a strict solution of VEP if and only if $g(\bar{x})$ is a nondominated point of the image set $g(S)$ (i.e., $g(x) \notin g(\bar{x}) - D \backslash \{0\}$ for all $x \in S$) and

$$x \in S, \quad g(x) = g(\bar{x}) \Rightarrow x = \bar{x},$$

so that $\bar{x}$ is a strict solution of problem VOP with objective function g. Conversely, if f is diagonal null and for each $x \in X$ we consider the mapping $g_x\colon X \to Y$, $g_x(z) = f(x, z)$ for all $z \in X$, then it is clear that $\bar{x}$ is a strict efficient solution of VEP if it is a strict efficient solution of VOP with objective function $g_{\bar{x}}$. Then, from this point of view, the strict efficient solutions of a vector equilibrium problem whose bifunction is diagonal null coincide with the strict efficient solutions of a vector optimization problem.

In the rest of the section, it is assumed that $f\colon S\times S\to Y$ and E is a fixed free-disposal set with respect to D. Now let us introduce a concept of approximate strict solution for equilibrium problems based on free-disposal sets.

Definition 2.23. A point $\bar{x}\in S$ is an E-strict efficient solution of VEP, denoted $\bar{x}\in \mathrm{S}(f,S,E)$ (and $\bar{x}\in \mathrm{S}(f,E)$ if $S=X$), if

$$f(\bar{x},x)\notin -E \quad \forall x\in S\backslash\{\bar{x}\}.$$

Equivalently, $\bar{x}\in \mathrm{S}(f,S,E)$ if and only if $[f(\bar{x},\cdot)\leq_E 0]\subset\{\bar{x}\}$. Thus, if f is diagonal null, then $\bar{x}\in \mathrm{S}(f,S,E)$ if and only if

$$[f(\bar{x},\cdot)\leq_E 0]=\begin{cases}\{\bar{x}\} & \text{if } 0\in E,\\ \emptyset & \text{otherwise.}\end{cases} \tag{2.7}$$

In particular, if E is an improvement set and f is diagonal null, then $\bar{x}$ is an E-strict efficient solution of VEP if it is an approximate nondominated solution of VOP with objective function $g_{\bar{x}}$ (see [95]). Then, the E-strict efficient solutions of a vector equilibrium problem whose bifunction is diagonal null and E is an improvement set coincide with approximate efficient solutions of a vector optimization problem.

Next we show some basic properties of the sets of E-strict efficient solutions of problem VEP.

Proposition 2.24. (*a*) If $E\subset D$, then $\mathrm{S}(f,S,D)\subset \bigcap\limits_{\varepsilon\geq 0}\mathrm{S}(f,S,\varepsilon E)$. On the other hand, if E is coradiant, then $\mathrm{S}(f,S,\varepsilon_1 E)\subset \mathrm{S}(f,S,\varepsilon_2 E)$ for all $\varepsilon_1,\varepsilon_2\in\mathbb{R}_+$, $0<\varepsilon_1<\varepsilon_2$.

(*b*) Assume that $\operatorname{cone} E$ is vectorially closed by $q\in E$. Then it follows that $\bigcap\limits_{\varepsilon\geq 0}\mathrm{S}(f,S,\varepsilon E)\subset \mathrm{S}(f,S,D)$. If additionally we have that $f(x,y)=0$ only if $x=y$, then $\bigcap\limits_{\varepsilon>0}\mathrm{S}(f,S,\varepsilon E)\subset \mathrm{S}(f,S,D)$.

(*c*) Assume that E is an improvement set and f is diagonal null and satisfies the $\leq_D$-t.i. property. Then

$$\left(\bigcup_{\bar{x}\in \mathrm{S}(f,S,E)}[f(\bar{x},\cdot)\leq_D 0]\right)\subset \mathrm{S}(f,S,E).$$

Proof. (a) The first part is obvious. Let $\varepsilon_1, \varepsilon_2 \in \mathbb{R}_+$, $0 < \varepsilon_1 < \varepsilon_2$. Since E is coradiant, we have that $(\varepsilon_2/\varepsilon_1)E \subset E$. Then $\varepsilon_2 E \subset \varepsilon_1 E$ and the result follows immediately.

(b) Let us prove the first part, since the second one is similar. Take $\bar{x} \in \bigcap_{\varepsilon \geq 0} \mathrm{S}(f, S, \varepsilon E)$. Then, for each $\varepsilon \geq 0$, we have that

$$f(\bar{x}, x) \notin -\varepsilon E \quad \forall x \in S \backslash \{\bar{x}\}. \tag{2.8}$$

By contradiction, suppose that there exists $y \in S\backslash\{\bar{x}\}$ such that $f(\bar{x}, y) \in -D$. By Lemma 1.18(d) we know that

$$f(\bar{x}, y) \in -\bigcap_{e \in E} \mathrm{vcl}_e \operatorname{cone} E \subset -\mathrm{vcl}_q \operatorname{cone} E = -\operatorname{cone} E.$$

Hence, there exists $\alpha \geq 0$ such that $f(\bar{x}, y) \in -\alpha E$, which is a contradiction to (2.8). Therefore, $\bar{x} \in \mathrm{S}(f, S, D)$ and the proof of part (b) is completed.

(c) Consider $\bar{x} \in \mathrm{S}(f, S, E)$ and $y \in Y$ such that $f(\bar{x}, y) \leq_D 0$. By Lemma 1.18(g) and statement (2.7) we deduce that

$$[f(y, \cdot) \leq_E 0] \subset [f(\bar{x}, \cdot) \leq_E 0] = \emptyset$$

and the result follows. □

The next example illustrates Proposition 2.24.

Example 2.25. Let $X = \mathbb{R}$, $S = \mathbb{R}_+, Y = \mathbb{R}^2, D = \mathbb{R}^2_+$ and $f \colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^2$ such that

$$f(x, y) = (1 - e^{x-y}, y^2 - x^2).$$

Observe that $f(x, y) = (0, 0)$ if and only if $x = y$.

We have that $f(\bar{x}, x) \in -D$ if and only if $x \leq \bar{x}$. Thus, $\mathrm{S}(f, S, D) = \{0\}$.

On the other hand, consider $E = \{(x, y) \in \mathbb{R}^2_+ : x + y \geq 1\}$. We have that $f(\bar{x}, x) \in -E$ if and only if $x \leq \bar{x}$ and $x^2 - \bar{x}^2 - e^{\bar{x}-x} \leq -2$. Define $g \colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ such that $g(x, y) = x^2 - y^2 - e^{y-x}$. Since $g(\cdot, \bar{x})$ is a strictly increasing real function, then $g(0, \bar{x}) = -\bar{x}^2 - e^{\bar{x}}$ is the minimum value of $g(\cdot, \bar{x})$ for each $\bar{x} \in \mathbb{R}_+$. Let $x_1 \in \mathbb{R}_+$ be such that $x_1^2 + e^{x_1} = 2$ ($x_1 \approx 0.537274$). Since $g(0, \cdot)$ is a strictly decreasing real function, we obtain that $\mathrm{S}(f, S, E) = [0, x_1)$.

Moreover, for $\varepsilon > 0$ we have that $f(\bar{x}, x) \in -\varepsilon E$ if and only if $x \leq \bar{x}$ and $x^2 - \bar{x}^2 - e^{\bar{x}-x} \leq -(1+\varepsilon)$. Set $x_\varepsilon > 0$ such that $x_\varepsilon^2 + e^{x_\varepsilon} = 1 + \varepsilon$. Hence $\mathrm{S}(f, S, \varepsilon E) = [0, x_\varepsilon)$.

It is easy to check that $0 < x_{\varepsilon_1} < x_{\varepsilon_2}$ for $0 < \varepsilon_1 < \varepsilon_2$ and that $x_\varepsilon \to 0$ when $\varepsilon \to 0$. Thus, we may verify that $\bigcap_{\varepsilon>0} \mathrm{S}(f, S, \varepsilon E) = \{0\} = \mathrm{S}(f, S, D)$. In addition, we may also check this fact by using parts (a) and (b) of Proposition 2.24, since cone E is topologically closed (and so it is vectorially closed by q for all $q \in \mathbb{R}^2 \backslash \{(0,0)\}$) and $E \subset D$.

Chapter 3

Nonconvex separation functional in linear spaces

3.1 Introduction

In Section 2.1 we cited many works of the literature dealing with vector optimization problems where the image space is not endowed with any particular topology. Except [163, 199], all those papers focus on convex problems, in the sense that their results are obtained by assuming generalized convexity assumptions on the data of the problem. In consequence, their main mathematical tools are generalized convexity concepts, convex separation theorems, alternative theorems and algebraic counterparts of some usual topological concepts, as the vector closure, the algebraic interior and the relative algebraic interior.

In this way, several important concepts and results of vector optimization have been extended to this algebraic setting. Let us underline several algebraic reformulations of solution concepts and their characterization by linear scalarization (see [4, 5, 106, 107, 157, 202–204]), Lagrangian optimality conditions, saddle point theorems and duality assertions (see [4–6, 106, 107, 204–206]), and several relations and properties on cone convexity and cone quasiconvexity concepts of vector-valued and set-valued mappings (see [4, 129, 130, 156, 202]).

However, to the best of our knowledge, in the literature only a few papers focus on nonconvex vector optimization problems whose image space has not any particular topology (see [120, 159, 163]). On the other hand, recently La Torre, Popovici and Rocca (see [128–130]) suggested to deal with the well-known nonconvex separation functional –in these papers it is called Gerstewitz's scalarization function and also smallest strictly monotonic function– in the framework of a real linear space not necessarily endowed with a topology.

The main objective of this chapter is to study fundamental properties of this functional when it takes values just in a real linear space, and to use them to characterize via scalarization several kinds of weak solution of vector equilibrium problems whose image space is not equipped with any particular topology. Some preliminary results in this line have been obtained in [120, 128, 159, 163] for set-valued mappings and vector optimization problems.

In Section 3.2, the main properties of the nonconvex separation functional are extended from the topological framework to the linear setting via suitable algebraic counterparts. The obtained results are compared with the previous ones in the literature and the main improvements are shown.

Through this functional, in Section 3.3 we characterize by scalarization weak efficient solutions of vector equilibrium problems. These characterizations are stated via a general class of algebraic solid ordering sets, in such a way that the obtained results generalize several others of the recent literature.

Finally, we apply the obtained characterizations to scalarize weak efficient solutions of both vector variational inequalities and vector optimization problems. Recall that both problems are particular cases of a vector equilibrium problem.

3.2 Algebraic formulation of the nonconvex separation functional

Throughout this chapter, a nonempty set $K \subset Y$ is said to be a cone if $\alpha K \subset K$ for all $\alpha > 0$. In other words, we consider that a cone that may not contain the vertex 0.

Let $q \in Y\backslash\{0\}$ and $\emptyset \neq E \subset Y$. The so-called nonconvex separation functional $\varphi^q_E : Y \to \mathbb{R} \cup \{\pm\infty\}$ is defined as follows:

$$\varphi^q_E(y) := \begin{cases} +\infty & \text{if } y \notin \mathbb{R}q - E, \\ \inf\{t \in \mathbb{R} : y \in tq - E\} & \text{otherwise.} \end{cases} \tag{3.1}$$

It was independently introduced in [73, 145, 152, 168] and it is called by different names: Gerstewitz's function, nonlinear scalarization function, smallest strictly monotonic function [145], shortage function [146] and so on. For its main properties, see [40, 63, 64, 74, 84, 95, 117, 163, 182] and the references therein.

In [120, 128, 159, 163] some preliminary properties of this functional were proved in the setting of a real linear space. To be precise, it was proved (see [128, Remark 2.3]) that φ^q_E is finite whenever E is a vectorially closed and algebraic solid proper convex cone and $q \in \operatorname{core} E$. Moreover, by assuming these hypotheses, in [120, Proposition 4.9] the following dual reformulation was stated:

$$\varphi^q_E(y) = \sup\{\xi(y) : \xi \in E^+, \xi(q) = 1\} \quad \forall y \in Y.$$

On the other hand, the following properties were obtained in [163, Section 2] by assuming that E is a convex cone containing 0 and $q \in E\backslash(-E)$: φ^q_E is subbaditive, positively homogeneous and E-nondecreasing and satisfies

$$[\varphi^q_E = -\infty] = \emptyset \iff q \notin -\operatorname{vcl} E, \tag{3.2}$$

$$\varphi^q_E(y + rq) = \varphi^q_E(y) + r \quad \forall y \in Y, r \in \mathbb{R},$$

$$[\varphi^q_E \leq 0] = -\operatorname{vcl}_q E, \tag{3.3}$$

$$[\varphi^q_E < 0] = (-\infty, 0)q - E. \tag{3.4}$$

Next, additional fundamental properties of the nonconvex separation functional φ^q_E are generalized to a real linear space via algebraic concepts. Let us observe that E is an arbitrary nonempty set and q is an arbitrary direction, so that we are not assuming any hypothesis on E and q.

Lemma 3.1. Let $q \in Y\backslash\{0\}$, $\emptyset \neq E \subset Y$ and $y \in Y$.

(a) If $y \in tq - \operatorname{vcl}_q E$ for some $t \in \mathbb{R}$, then $\varphi^q_E(y) \leq t$.

(b) Suppose that $\varphi^q_E(y) \in \mathbb{R}$. Then $y \in \varphi^q_E(y)q - \operatorname{vcl}_q E$.

(c) $\varphi^q_E = \varphi^q_{\mathrm{vcl}_q E}$.

Proof. (a) Assume that $y \in tq - \mathrm{vcl}_q E$ for some $t \in \mathbb{R}$. Then

$$\forall \lambda > 0, \exists \lambda' \in [0, \lambda] \text{ s.t. } (t + \lambda')q - y \in E.$$

Thus,

$$\varphi^q_E(y) \leq t + \lambda' \leq t + \lambda \quad \forall \lambda > 0,$$

and we obtain the result by taking $\lambda \to 0$.

(b) Suppose that $\varphi^q_E(y) \in \mathbb{R}$. Then, for all $\lambda > 0$ there exists $\lambda' \in [0, \lambda)$ such that $y \in (\varphi^q_E(y) + \lambda')q - E$, or equivalently, $(\varphi^q_E(y)q - y) + \lambda' q \in E$. Hence $\varphi^q_E(y)q - y \in \mathrm{vcl}_q E$, and the result follows.

(c) Since $E \subset \mathrm{vcl}_q E$, we have that $\varphi^q_E(y) \geq \varphi^q_{\mathrm{vcl}_q E}(y)$ for all $y \in Y$. In order to prove the reciprocal inequality we distinguish two cases, since in the case of $\varphi^q_{\mathrm{vcl}_q E}(y) = +\infty$, there is nothing to prove.

If $\varphi^q_{\mathrm{vcl}_q E}(y) \in \mathbb{R}$, then by applying part (b) for the set $\mathrm{vcl}_q E$ and (1.23), we have that

$$y \in \varphi^q_{\mathrm{vcl}_q E}(y)q - \mathrm{vcl}_q(\mathrm{vcl}_q E) = \varphi^q_{\mathrm{vcl}_q E}(y)q - \mathrm{vcl}_q E,$$

and this implies by part (a) that $\varphi^q_E(y) \leq \varphi^q_{\mathrm{vcl}_q E}(y)$.

Otherwise, if $\varphi^q_{\mathrm{vcl}_q E}(y) = -\infty$, then for each $t \in \mathbb{R}$ there exists a real number $t' < t$ such that $y \in t'q - \mathrm{vcl}_q E$ and, by part (a), $\varphi^q_E(y) \leq t'$. Taking $t \to -\infty$ we obtain $\varphi^q_E(y) = -\infty$ and the proof finishes. $\square$

Theorem 3.2. Consider $q \in Y \backslash \{0\}$ and $\emptyset \neq E \subset Y$. We have the following properties:

(a) $\mathrm{dom}\, \varphi^q_E = \mathbb{R}q - E$ and $[\varphi^q_E = -\infty] = \mathrm{ovcl}^{+\infty}_q(-E)$.

(b) φ^q_E is proper if and only if $\mathrm{ovcl}^{+\infty}_q(-E) = \emptyset$. φ^q_E is finite if and only if $\mathrm{ovcl}^{+\infty}_q(-E) = \emptyset$ and $Y = \mathbb{R}q - E$.

(c) $\varphi^q_E(y + rq) = \varphi^q_E(y) + r$ for all $y \in Y$ and for all $r \in \mathbb{R}$.

(d) $[\varphi^q_E \mathcal{R} r] = [\varphi^q_E \mathcal{R} 0] + rq$ for all $\mathcal{R} \in \{\leq, <, =, \geq, >\}$ and for all $r \in \mathbb{R}$.

(e) $[\varphi^q_E \leq 0] = (-\infty, 0]q - \mathrm{vcl}_q E$.

(f) $[\varphi^q_E < 0] = (-\infty, 0)q - \operatorname{vcl}_q E$.

(g) $[\varphi^q_E = 0] = (-\operatorname{vcl}_q E)\backslash((-\infty, 0)q - \operatorname{vcl}_q E)$.

(h) $[\varphi^q_E \geq 0] = Y\backslash((-\infty, 0)q - \operatorname{vcl}_q E)$.

Proof. (a) Let us prove $[\varphi^q_E = -\infty] = \operatorname{ovcl}^{+\infty}_q(-E)$, since the first part is obvious. Then, a point $y \in [\varphi^q_E = -\infty]$ if and only if

$$\forall t < 0, \exists t' \in (-\infty, t) \text{ such that } y \in t'q - E,$$

or equivalently,

$$\forall s > 0, \exists s' \in (s, +\infty) \text{ such that } y + s'q \in -E,$$

i.e., $y \in \operatorname{ovcl}^{+\infty}_q(-E)$.

(b) This part is an obvious consequence of part (a).

(c) Fix $y \in Y$ and $r \in \mathbb{R}$. It is clear that $y \in \mathbb{R}q - E$ if and only if $y + rq \in \mathbb{R}q - E$, and so $\varphi^q_E(y) + r = +\infty$ if and only if $\varphi^q_E(y + rq) = +\infty$. Otherwise,

$$\begin{aligned}\varphi^q_E(y) + r &= \inf\{t + r \,:\, t \in \mathbb{R}, y \in tq - E\} \\ &= \inf\{s \in \mathbb{R} \,:\, y \in (s - r)q - E\} \\ &= \varphi^q_E(y + rq)\end{aligned}$$

and part (c) is proved.

(d) This part follows directly from part (c).

(e) By definition, $y \in [\varphi^q_E \leq 0]$ if and only if $\varphi^q_E(y) \in [-\infty, 0]$. If $\varphi^q_E(y) \in (-\infty, 0]$, by Lemma 3.1$(b)$, we deduce that

$$y \in \varphi^q_E(y)q - \operatorname{vcl}_q E \subset (-\infty, 0]q - \operatorname{vcl}_q E.$$

On the other hand, it is obvious that

$$\operatorname{ovcl}^{+\infty}_q(-E) \subset (-\infty, t]q - \operatorname{vcl}_q E \quad \forall\, t < 0,$$

and then, by part (a) we see that $y \in (-\infty, 0]q - \operatorname{vcl}_q E$ whenever $\varphi^q_E(y) = -\infty$.

Conversely, if $y \in (-\infty, 0]q - \operatorname{vcl}_q E$, then $\varphi^q_{\operatorname{vcl}_q E}(y) \leq 0$ and by Lemma 3.1(c) it follows that $\varphi^q_E(y) \in [-\infty, 0]$, and the proof of part (e) is complete.

(f) The proof of this part is analogous to the proof of part (e).

(g) Take an arbitrary $y \in [\varphi^q_E = 0]$. By Lemma 3.1(b), $y \in 0q - \operatorname{vcl}_q E = -\operatorname{vcl}_q E$, and by part ($f$) it is clear that $y \notin (-\infty, 0)q - \operatorname{vcl}_q E$.

Reciprocally, consider an arbitrary $y \in (-\operatorname{vcl}_q E)\backslash((-\infty, 0)q - \operatorname{vcl}_q E)$. By part ($f$) we obtain $\varphi^q_E(y) \geq 0$ and since $y \in -\operatorname{vcl}_q E$, by Lemma 3.1($c$) it is clear that $\varphi^q_E(y) = \varphi^q_{\operatorname{vcl}_q E}(y) \leq 0$. Thus, $\varphi^q_E(y) = 0$.

(h) This part is a direct consequence of part (f) and the proof finishes. $\square$

In order to apply parts (a) and (b), let us observe that $\mathbb{R}q - E = \mathbb{R}q - \operatorname{vcl}_q E$.

Remark 3.3. By combining Theorem 3.2, Proposition 1.16(b) and Lemma 1.18(c) one obtains more tractable statements on the sublevel sets of the mapping φ^q_E whenever certain conditions are fulfilled. For example, if E is free-disposal with respect to an algebraic solid convex cone K, then

$$
\begin{aligned}
[\varphi^q_E < 0] &= -\operatorname{core} E \quad \forall q \in \operatorname{core} K, \\
[\varphi^q_E \leq 0] &= -\operatorname{vcl}_q E \quad \forall q \in K.
\end{aligned}
$$

In the following example we illustrate part (a) of Theorem 3.2.

Example 3.4. Let $Y = \mathbb{R}^2$, $q = (1, 1)$ and

$$E = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 = y_2 \leq 0\} \cup \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \geq 0, y_2 = 0\}.$$

It is easy to check that

$$
\varphi^q_E(y_1, y_2) = \begin{cases} +\infty & \text{if } \; y_1 > y_2, \\ y_2 & \text{if } \; y_1 < y_2, \\ -\infty & \text{if } \; y_1 = y_2. \end{cases}
$$

Then by Theorem 3.2(a) we see that

$$
\begin{aligned}
\mathbb{R}q - E &= \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \leq y_2\}, \\
\operatorname{ovcl}^{+\infty}_q(-E) &= \{(y_1, y_2) \in \mathbb{R}^2 : y_1 = y_2\}.
\end{aligned}
$$

The next example illustrates part (g) of Theorem 3.2. Observe that E is free-disposal with respect to $K = [0, +\infty)q$, but $\operatorname{vcl}_q E + (0, +\infty)q$ is strictly contained in $\operatorname{vcl}_q E$.

Example 3.5. Consider $Y = \mathbb{R}$, $E = (-\infty, 1)$, $q = -1$ and $p = 1$. We have that $\operatorname{vcl}_q E = (-\infty, 1]$ and $\operatorname{vcl}_p E = E$.

Since $(-\infty, 0)q - \operatorname{vcl}_q E = (-1, +\infty)$ and $(-\infty, 0)p - \operatorname{vcl}_p E = \mathbb{R}$, by Theorem 3.2$(g)$, we know that $[\varphi_E^q = 0] = \{-1\}$ and $[\varphi_E^p = 0] = \emptyset$. These results are easily checked since $\varphi_E^q(y) = -(1 + y)$ and $\varphi_E^p(y) = -\infty$ for all $y \in \mathbb{R}$.

Remark 3.6. Lemma 3.1 and Theorem 3.2 state well-known properties of the nonconvex separation functional φ_E^q without assuming any assumption. As a consequence, they generalize several recent results of the literature, where these properties are obtained under certain hypotheses (see [60,63,64,117,128,163,176]). Let us show some examples.

(a) In [163, Lemma 2.6] (see also (3.2)) it was proved that

$$[\varphi_K^q = -\infty] \neq \emptyset \iff q \in -\operatorname{vcl} K \tag{3.5}$$

whenever K is a convex cone containing 0 and $q \in K \backslash (-K)$. This result is a particular case of Proposition 1.20(a) and Theorem 3.2(a). Indeed, by Theorem 3.2(a) with $E = K$ we see that $[\varphi_K^q = -\infty] \neq \emptyset$ if and only if $\operatorname{ovcl}_q^{+\infty}(-K) \neq \emptyset$, and by Proposition 1.20(a) and statements (1.24) and (1.25) we obtain that

$$\operatorname{ovcl}_q^{+\infty}(-K) \neq \emptyset \iff q \in -\operatorname{vcl} K.$$

Let us observe that the convexity of the cone K is not needed.

On the other hand, if K is an algebraic solid vectorially closed convex cone and $q \in \operatorname{core} K$, then it was stated in [128, Lemma 2.2] that $[\varphi_K^q = -\infty] = \emptyset$ whenever $K \neq Y$. This result is a particular case of part (a) of Theorem 3.2. Indeed, reasoning by contradiction, if $[\varphi_K^q = -\infty] \neq \emptyset$ then by (3.5) we obtain that $q \in -K$, since K is vectorially closed. Then $0 = q - q \in \operatorname{core} K + K = \operatorname{core} K$ by Lemma 1.13(c), and so $K = Y$ since K is a cone, which is a contradiction.

(b) It is not hard to check that for each $q \in Y$ and $K \subset Y$ we have that

$$(0, +\infty)q + K = (0, +\infty)q + \operatorname{vcl}_q K.$$

Moreover, if K is a convex cone and $q \in K$, then

$$[0, +\infty)q + \operatorname{vcl}_q K = \operatorname{vcl}_q K.$$

Thus, Lemma 2.9 and Lemma 2.10 of [163] are particular cases of Theorem 3.2 (see also (3.3) and (3.4)).

(c) Consider $Y = \mathbb{R}^2$, $q = (1, 1)$ and

$$E = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \in \mathbb{Q} \cap (0, +\infty), y_2 = y_1\}.$$

It is obvious that

$$\begin{aligned} \operatorname{vcl}_q E &= \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \geq 0, y_2 = y_1\}, \\ [0, +\infty)q + \operatorname{vcl}_q E &= \operatorname{vcl}_q E, \\ (0, +\infty)q + \operatorname{vcl}_q E &= \operatorname{vcl}_q E \backslash \{(0, 0)\} \end{aligned}$$

and then, by Lemma 3.1 and Theorem 3.2 we obtain that

$$\varphi_E^q(y_1, y_2) = \varphi_{\operatorname{vcl}_q E}^q(y_1, y_2) = \begin{cases} +\infty & \text{if } y_1 \neq y_2 \\ y_1 & \text{otherwise,} \end{cases}$$

$$[\varphi_E^q \leq 0] = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \leq 0, y_2 = y_1\}, \tag{3.6}$$

$$[\varphi_E^q < 0] = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 < 0, y_2 = y_1\},$$

$$[\varphi_E^q = 0] = \{(0, 0)\}. \tag{3.7}$$

Formulas (3.6) and (3.7) cannot be obtained through [63, Proposition 4.1], and parts (ii) and (iii) of [60, Proposition 2.1], parts (a) and (c) of [63, Corollary 4.1], parts (b) and (c) of [64, Proposition 3.2], [64, Corollaries 3.3(a) and 3.4(a)], [176, Theorem 3.3] and statements (5.6) and (5.12) of [117] cannot be applied. Let us observe that, in this particular example, $\operatorname{vcl}_q E$ coincides with the topological closure of E, and so the conclusion of [63, Corollary 4.1(a)] is satisfied.

In the next two results, we characterize when φ_E^q is C-nondecreasing and C-increasing.

Theorem 3.7. Let C and E be nonempty subsets of Y and $q \in Y \backslash \{0\}$. The following statements are equivalent:

(a) φ_E^q is C-nondecreasing.

(b) $\operatorname{vcl}_q E + C \subset [0,+\infty)q + \operatorname{vcl}_q E$.

(c) $E + C \subset [0,+\infty)q + \operatorname{vcl}_q E$.

Proof. First we prove $(a) \Rightarrow (b)$. Suppose that φ_E^q is C-nondecreasing and fix $e \in \operatorname{vcl}_q E$ and $c \in C$. Since $-c = -c - e - (-e) \in -C$, by Lemma 3.1$(c)$ we have that

$$\varphi_E^q(-c-e) \leq \varphi_E^q(-e) = \varphi_{\operatorname{vcl}_q E}^q(-e) \leq 0.$$

Hence, by Theorem 3.2(e) we see that

$$-c - e \in [\varphi_E^q \leq 0] = (-\infty, 0]q - \operatorname{vcl}_q E$$

and the implication is proved.

It is obvious that $(b) \Rightarrow (c)$. Then let us prove $(c) \Rightarrow (a)$. Suppose that $E + C \subset [0,+\infty)q + \operatorname{vcl}_q E$ and take $y_1, y_2 \in Y$, $y_1 \neq y_2$, such that $y_1 - y_2 \in -C$. It is clear that $\varphi_E^q(y_1) \leq \varphi_E^q(y_2)$ whenever $\varphi_E^q(y_2) = +\infty$. Thus suppose that $\varphi_E^q(y_2) < +\infty$ and consider an arbitrary $t \in \mathbb{R}$ such that $y_2 \in tq - E$. Then,

$$y_1 \in y_2 - C \subset tq - E - C \subset tq + (-\infty, 0]q - \operatorname{vcl}_q E$$

and by Lemma 3.1(c) it follows that

$$\varphi_E^q(y_1) = \varphi_{\operatorname{vcl}_q E}^q(y_1) \leq t.$$

Then $\varphi_E^q(y_1) \leq \varphi_E^q(y_2)$ and the proof finishes. □

Let us observe that if the nonconvex separation functional φ_E^q is C-nondecreasing, then

$$[\varphi_E^q = -\infty] - C\backslash\{0\} \subset [\varphi_E^q = -\infty],$$
$$[\varphi_E^q = +\infty] + C\backslash\{0\} \subset [\varphi_E^q = +\infty].$$

Thus, if φ_E^q is not finite, then it cannot be C-increasing.

Theorem 3.8. Let C and E be nonempty subsets of Y and $q \in Y\backslash\{0\}$. Suppose that φ_E^q is finite. Then the following statements are equivalent:

(a) φ_E^q is C-increasing.

(b) $\operatorname{vcl}_q E + C\backslash\{0\} \subset (0,+\infty)q + \operatorname{vcl}_q E$.

Proof. Let us only prove $(b) \Rightarrow (a)$, since the reciprocal implication follows the same proof as in Theorem 3.7. Assume that statement (b) is fulfilled and consider $y_1, y_2 \in Y$ such that $y_1 \in y_2 - C\backslash\{0\}$. We have that $\varphi_E^q(y_2) \in \mathbb{R}$ since φ_E^q is finite. Then we can apply Lemma 3.1(b) and we deduce that $y_2 \in \varphi_E^q(y_2)q - \operatorname{vcl}_q E$, and so

$$y_1 \in \varphi_E^q(y_2)q - \operatorname{vcl}_q E - C\backslash\{0\} \subset \varphi_E^q(y_2)q - (0,+\infty)q - \operatorname{vcl}_q E.$$

Hence, by Lemma 3.1(c) we obtain

$$\varphi_E^q(y_1) = \varphi_{\operatorname{vcl}_q E}^q(y_1) < \varphi_E^q(y_2)$$

and the proof is completed. □

By Proposition 1.16(b) and Theorems 3.7 and 3.8, if E is free-disposal with respect to an algebraic solid convex cone K, $q \in \operatorname{core} K$ and $\varphi_E^q > -\infty$, then we have that φ_E^q is nondecreasing w.r.t. $\leq_K$, and it is also increasing w.r.t. $<_K$.

Remark 3.9. The previous theorems generalize some similar results in the literature obtained in the topological setting. Let us show some examples.

(a) It is obvious that statement (c) of Theorem 3.7 is fulfilled whenever E is a convex cone and $C = E$. Then the sufficient condition stated in [163, Lemma 2.11] to guaranty that φ_E^q is E-nondecreasing is a particular case of Theorem 3.7.

(b) In [117, Theorem 5.2.3(d)] (see also [117, Remark 5.2.2]), under the assumptions $E + [0,\infty)q = E$ and E topologically closed, it is obtained that φ_E^q is C-nondecreasing if and only if $E + C \subset E$. Then, this last condition is equivalent to statement (c) of Theorem 3.7. Indeed, as E is topologically closed we have that $E = \operatorname{vcl}_q E$. Thus, if $E + C \subset E$, we have that

$$E + C \subset E \subset [0,+\infty)q + \operatorname{vcl}_q E.$$

Reciprocally, if statement (c) of Theorem 3.7 is true, then

$$E + C \subset [0,+\infty)q + \operatorname{vcl}_q E = [0,+\infty)q + E = E.$$

As a consequence, it is clear that Theorem 3.7 improves [117, Theorem 5.2.3(d)], since Theorem 3.7 does not assume any hypothesis. The same conclusion is derived by comparing Theorem 3.7 with part 1 of [176, Theorem 3.8], where stronger hypotheses than [117, Theorem 5.2.3(d)] are assumed.

(c) In [117, Theorem 5.2.6(g)] (see also [117, Remark 5.2.2]), by assuming that E is closed, solid and such that $E+(0,+\infty)q = \operatorname{int} E$, it is proved that φ_E^q is C-increasing if and only if $E+C\backslash\{0\} \subset \operatorname{int} E$, i.e., if and only if $\operatorname{vcl}_q E+C\backslash\{0\} \subset (0,+\infty)q+\operatorname{vcl}_q E$, that coincides with the equivalent condition stated in part (b) of Theorem 3.8.

Thus, Theorem 3.8 improves [117, Theorem 5.2.6(g)], since it does not assume any hypothesis. The same conclusion is derived by comparing Theorem 3.8 with part 4 of [176, Theorem 3.8] (in this result it is also imposed that zero belongs to the topological boundary of the set E).

(d) The characterizations proved in Theorem 3.7 and Theorem 3.8 extend parts (a) and (b) of [64, Lemma 3.7]. For it, consider $E = B - A$ and $C = B$, $C = \operatorname{int} B$, respectively. On the other hand, let us observe that [64, Lemma 3.7(b)] is not always true if the nonconvex separation functional is not finite. For example, if $Y = \mathbb{R}^2$, $A = \{(0,0)\}$, $q = (-1,0)$ and

$$B = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \geq 0\},$$

then it is clear that $\varphi_B^q(0,0) = \varphi_B^q(-1,0) = -\infty$ and $(-1,0) \in (0,0) - \operatorname{int} B$.

Example 3.10. Let $Y = \mathbb{R}^2$, $q = (1,1)$, $C_1 = \mathbb{R}^2_+$ and

$$E = C_2 = \{(y_1, y_2) \in Y : y_1 > 0, y_2 > 0\}.$$

It is clear that $\operatorname{vcl}_q E = C_1$. Then,

$$\begin{aligned} E + C_1 &= C_2, \\ E + C_2 &= C_2, \\ [0,+\infty)q + \operatorname{vcl}_q E &= C_1, \\ \operatorname{vcl}_q E + C_1\backslash\{0\} &= C_1\backslash\{0\}, \\ \operatorname{vcl}_q E + C_2\backslash\{0\} &= C_2, \\ (0,+\infty)q + \operatorname{vcl}_q E &= C_2, \end{aligned}$$

and by Theorem 3.7 and Theorem 3.8 we deduce that φ_E^q is C_1-nondecreasing and C_2-increasing. However, it is not C_1-increasing. Observe that $E + C_1\backslash\{0\} \subset (0, +\infty)q + \operatorname{vcl}_q E$, and so this condition is not sufficient to guaranty that φ_E^q is C_1-increasing.

In the last result of this section, Theorem 3.13, we characterize when φ_E^q is convex or positively homogeneous. For this aim, the following two lemmas are needed. The first one is an algebraic reformulation of some properties considered in the proof of [84, Theorem 2.3.1(a)]. The second one shows that the (possibly non proper) functional φ_E^q is convex (resp. positively homogeneous) if and only if $\operatorname{epi} \varphi_E^q$ is convex (resp. a cone).

Lemma 3.11. Let $E \subset Y$ be a nonempty set, $q \in Y\backslash\{0\}$ be a direction such that

$$\operatorname{vcl}_q E + (0, +\infty)q \subset \operatorname{vcl}_q E$$

and the linear functional $T : Y \times \mathbb{R} \to Y$ defined by $T(y, t) = tq - y$ for all $(y, t) \in Y \times \mathbb{R}$. Then $T^{-1}(\operatorname{vcl}_q E) = \operatorname{epi} \varphi_E^q$ and $\operatorname{vcl}_q E = T(\operatorname{epi} \varphi_E^q)$.

Proof. Clearly, T is surjective and

$$T^{-1}(\operatorname{vcl}_q E) = \{(y, t) \in Y \times \mathbb{R} \,:\, y \in tq - \operatorname{vcl}_q E\}. \tag{3.8}$$

Let us prove that $T^{-1}(\operatorname{vcl}_q E) = \operatorname{epi} \varphi_E^q$. Indeed, if $(y, t) \in T^{-1}(\operatorname{vcl}_q E)$, then by (3.8) and Lemma 3.1(a) we have that $\varphi_E^q(y) \leq t$ and $(y, t) \in \operatorname{epi} \varphi_E^q$.

Reciprocally, take an arbitrary $(y, t) \in \operatorname{epi} \varphi_E^q$. If $\varphi_E^q(y) \in \mathbb{R}$, then $y \in \varphi_E^q(y)q - \operatorname{vcl}_q E$ by Lemma 3.1(b), and so

$$y \in tq - \operatorname{vcl}_q E - \big(t - \varphi_E^q(y)\big)q \subset tq - \operatorname{vcl}_q E.$$

Analogously, if $\varphi_E^q(y) = -\infty$, then there exists $t' \in (-\infty, t]$ such that $y \in t'q - \operatorname{vcl}_q E$. Hence, $y \in tq - \operatorname{vcl}_q E - (t - t')q \subset tq - \operatorname{vcl}_q E$, and in both cases, by (3.8), we deduce that $(y, t) \in T^{-1}(\operatorname{vcl}_q E)$.

Thus, we have that $\operatorname{epi} \varphi_E^q = T^{-1}(\operatorname{vcl}_q E)$, and since T is surjective, we conclude that $T(\operatorname{epi} \varphi_E^q) = \operatorname{vcl}_q E$, and the proof finishes. $\square$

Let us observe that

$$E+(0,+\infty)q\subset \operatorname{vcl}_q E \iff \operatorname{vcl}_q E+(0,+\infty)q\subset \operatorname{vcl}_q E \tag{3.9}$$

$$\iff \operatorname{vcl}_q E+[0,+\infty)q= \operatorname{vcl}_q E \tag{3.10}$$

$$\iff [\varphi_E^q\leq 0]=-\operatorname{vcl}_q E. \tag{3.11}$$

Lemma 3.12. Let $E\subset Y$ be a nonempty set and $q\in Y\backslash\{0\}$. The following statements are true:

(a) φ_E^q is positively homogeneous if and only if $\operatorname{epi}\varphi_E^q$ is a cone.

(b) φ_E^q is convex if and only if $\operatorname{epi}\varphi_E^q$ is convex.

Proof. Both necessary conditions are well known for an arbitrary proper functional $\varphi: Y\to\mathbb{R}\cup\{+\infty\}$, and they are easy to check in a similar way for any (possibly no proper) functional $\varphi: Y\to\mathbb{R}\cup\{\pm\infty\}$, $\varphi\not\equiv+\infty$. Then we prove the sufficient condition of part (b), since the proof of the sufficient condition of part (a) is similar.

Consider $y_1,y_2\in Y$ and $\alpha\in(0,1)$. The inequality

$$\varphi_E^q(\alpha y_1+(1-\alpha)y_2)\leq\alpha\varphi_E^q(y_1)+(1-\alpha)\varphi_E^q(y_2) \tag{3.12}$$

is obvious if $y_1\notin\operatorname{dom}\varphi_E^q$ or $y_2\notin\operatorname{dom}\varphi_E^q$ due to the conventions $-\infty+\infty=+\infty-\infty=+\infty$. Then suppose that $y_1,y_2\in\operatorname{dom}\varphi_E^q$. If $\varphi_E^q(y_1)=-\infty$ or $\varphi_E^q(y_2)=-\infty$, then $\varphi_E^q(\alpha y_1+(1-\alpha)y_2)=-\infty$ and statement (3.12) is fulfilled. Indeed, assume that $\varphi_E^q(y_1)=-\infty$ (the reasoning in the case $\varphi_E^q(y_2)=-\infty$ is equivalent). Then $(y_1,t)\in\operatorname{epi}\varphi_E^q$ for all $t\in\mathbb{R}$, and since $y_2\in\operatorname{dom}\varphi_E^q$, there exists $s\in\mathbb{R}$ such that $(y_2,s)\in\operatorname{epi}\varphi_E^q$. As $\operatorname{epi}\varphi_E^q$ is convex, we deduce that $(\alpha y_1+(1-\alpha)y_2,\alpha t+(1-\alpha)s)\in\operatorname{epi}\varphi_E^q$ for all $t\in\mathbb{R}$, i.e.,

$$\varphi_E^q(\alpha y_1+(1-\alpha)y_2)\leq\alpha t+(1-\alpha)s\quad\forall t\in\mathbb{R},$$

and so we have $\varphi_E^q(\alpha y_1+(1-\alpha)y_2)=-\infty$.

Finally, if $\varphi_E^q(y_1)\in\mathbb{R}$ and $\varphi_E^q(y_2)\in\mathbb{R}$, then

$$(y_1,\varphi_E^q(y_1)),(y_2,\varphi_E^q(y_2))\in\operatorname{epi}\varphi_E^q$$

and (3.12) holds since $\operatorname{epi}\varphi_E^q$ is convex, which finishes the proof. □

Observe from Lemma 3.12(b) that the set $[\varphi_E^q = -\infty]$ is convex whenever $\operatorname{epi}\varphi_E^q$ is convex.

Theorem 3.13. Let $E \subset Y$ be a nonempty set and consider $q \in Y\backslash\{0\}$. Then,

(a) If $\operatorname{vcl}_q E$ is a cone, then φ_E^q is positively homogeneous.

(b) If $\operatorname{vcl}_q E$ is convex, then φ_E^q is convex.

If additionally we have that

$$\operatorname{vcl}_q E + (0, +\infty)q \subset \operatorname{vcl}_q E, \tag{3.13}$$

then the following characterizations are true:

(c) φ_E^q is positively homogeneous if and only if $\operatorname{vcl}_q E$ is a cone.

(d) φ_E^q is convex if and only if $\operatorname{vcl}_q E$ is convex.

Proof. (a) Let $y \in Y$ and $\alpha > 0$. If $y \in \operatorname{dom}\varphi_E^q$ and $t \in \mathbb{R}$ satisfies $y \in tq - \operatorname{vcl}_q E$, then $\alpha y \in \alpha tq - \operatorname{vcl}_q E$, since $\operatorname{vcl}_q E$ is a cone, and by Lemma 3.1(a) we see that $\varphi_E^q(\alpha y) \leq \alpha t$. Then,

$$\varphi_E^q(\alpha y) \leq \alpha\varphi_E^q(y).$$

Obviously this inequality is also true whenever $y \notin \operatorname{dom}\varphi_E^q$, and by applying it to $(1/\alpha)$ and αy instead of α and y we obtain

$$\varphi_E^q(y) = \varphi_E^q((1/\alpha)(\alpha y)) \leq (1/\alpha)\varphi_E^q(\alpha y)$$

and the proof of part (a) finishes.

(b) Observe by (1.26), Lemma 3.1(c) and Theorem 3.2(a) that for each $\alpha \in (0, 1)$,

$$\alpha y_1 + (1-\alpha)y_2 \in [\varphi_E^q = -\infty] \quad \forall\, y_1 \in [\varphi_E^q = -\infty], \forall\, y_2 \in \operatorname{dom}\varphi_E^q. \tag{3.14}$$

Thus, the proof of part (b) is complete if inequality (3.12) is stated for $y_1, y_2 \in Y$ such that $\varphi_E^q(y_1) \in \mathbb{R}$ and $\varphi_E^q(y_2) \in \mathbb{R}$. Let us check this case. By Lemma 3.1(b) we have that $y_i \in \varphi_E^q(y_i)q - \operatorname{vcl}_q E$, $i = 1, 2$, and since $\operatorname{vcl}_q E$ is convex we obtain $\alpha y_1 + (1-\alpha)y_2 \in (\alpha\varphi_E^q(y_1) + (1-\alpha)\varphi_E^q(y_2))q - \operatorname{vcl}_q E$. Hence, by Lemma 3.1($a$)

we deduce that $\varphi_E^q(\alpha y_1 + (1-\alpha)y_2) \leq \alpha\varphi_E^q(y_1) + (1-\alpha)\varphi_E^q(y_2)$ and the proof of part (b) is complete.

Finally, by Lemma 3.11 it is clear that $\mathrm{vcl}_q\, E$ is convex (resp. a cone) if and only if $\mathrm{epi}\, \varphi_E^q$ is convex (resp. a cone). Then parts (c) and (d) follow by applying Lemma 3.12. □

Notice that hypothesis (3.13) has different equivalent formulations (see (3.9)-(3.11)). The necessary conditions of parts (c) and (d) of Theorem 3.13 are not true without this hypothesis, as is shown in the following example.

Example 3.14. Let $Y = \mathbb{R}$, $E = [0,1] \cup [2,+\infty)$ and $q = 1$. For each $y \in Y$ we have that

$$\{t \in \mathbb{R} : y \in tq - E\} = [y, y+1] \cup [y+2, +\infty)$$

and so $\varphi_E^q(y) = y$ for all $y \in \mathbb{R}$. Therefore, φ_E^q is positively homogeneous and convex, but $\mathrm{vcl}_q\, E$ is neither a cone nor a convex set. Notice that hypothesis (3.13) is not fulfilled.

If E is convex (resp. a cone), then it is easy to prove that $\mathrm{vcl}_q\, E$ is also convex (resp. a cone). Thus, if E is a convex cone, then assumption (3.13) is fulfilled whenever $q \in E$ and [163, Lemma 2.8] is a particular case of the sufficient conditions stated in Theorem 3.13.

Notice that if E is a topologically closed set, then condition (3.13) reduces to $E + [0,+\infty)q = E$, since $\mathrm{vcl}_q\, E = E$. Thus, Theorem 3.13 extends [117, Theorem 5.2.3(a)] to a not necessarily topologically closed set E. Analogously, Theorem 3.13 generalizes [176, Theorems 3.5 and 3.7], where E is (topologically) closed and solid, zero belongs to the topological boundary of the set E and a stronger condition than (3.13) is required, and also [64, Corollary 3.4(b)], where only the sufficient condition is stated under the following assumption: $\mathrm{cl}\, E + (0,+\infty)q \subset E$. Notice that this assumption implies that $\mathrm{cl}\, E = \mathrm{vcl}_q\, E$ and so it is stronger than (3.13).

Moreover, observe that [176, Theorem 3.7] is not correct, since the convexity of φ_E^q does not imply that E is a cone. Indeed, it is easy to check that the data

$Y := \mathbb{R}^2$, $q := (1,1)$ and

$$E := \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \geq -1, y_2 \geq -1, y_1 + y_2 \geq 0\}$$

satisfy the assumptions of [176, Theorem 3.7], E is not a cone and

$$\varphi_E^q(y_1, y_2) = \max\left\{y_1 - 1, y_2 - 1, \frac{y_1 + y_2}{2}\right\} \quad \forall (y_1, y_2) \in \mathbb{R}^2,$$

which is convex.

Let us show an example to illustrate Theorem 3.2, Theorem 3.7 and Theorem 3.8.

Example 3.15. Consider $Y = \mathbb{R}^2$, $q = (1,1)$, $p = (0,1)$ and $E = E_1 \cup E_2$, where $E_1 = \{(y_1, y_2) \in \mathbb{R}^2 \,:\, y_1 - 1 < y_2 < y_1 + 1\}$ and $E_2 = \{(y_1, y_2) \in \mathbb{R}^2 \,:\, -1 - y_1 < y_2 < 1 - y_1\}$. It is easy to obtain that $\operatorname{vcl}_q E_2 = \{(y_1, y_2) \in \mathbb{R}^2 \,:\, -1 - y_1 \leq y_2 < 1 - y_1\}$, $\operatorname{vcl}_q E = E_1 \cup \operatorname{vcl}_q E_2$ and

$$\varphi_E^q(y_1, y_2) = \begin{cases} \dfrac{y_1 + y_2 - 1}{2}, & \text{if } (y_1, y_2) \notin E_1, \\ -\infty, & \text{otherwise.} \end{cases}$$

By applying Theorem 3.2 it follows that:

$$\begin{aligned}
\operatorname{dom} \varphi_E^q &= \mathbb{R}q - E = \mathbb{R}^2, \\
[\varphi_E^q = -\infty] &= \operatorname{ovcl}_q^{+\infty}(-E) = E_1, \\
[\varphi_E^q \leq 0] &= (-\infty, 0]q - \operatorname{vcl}_q E = E_1 \cup \{(y_1, y_2) \in \mathbb{R}^2 \,:\, y_2 \leq 1 - y_1\}, \\
[\varphi_E^q < 0] &= (-\infty, 0)q - \operatorname{vcl}_q E = E_1 \cup \{(y_1, y_2) \in \mathbb{R}^2 \,:\, y_2 < 1 - y_1\}, \\
[\varphi_E^q = 0] &= (-\operatorname{vcl}_q E) \backslash ((-\infty, 0)q - \operatorname{vcl}_q E) \\
&= \{(y_1, y_2) \in \mathbb{R}^2 \,:\, y_2 = 1 - y_1, y_1 \in (-\infty, 0] \cup [1, +\infty)\}.
\end{aligned}$$

Moreover, observe that given an arbitrary vector $d \in \mathbb{R}^2$ we have

$$E + d \subset [0, +\infty)q + \operatorname{vcl}_q E \iff d \in [0, +\infty)q.$$

Hence, by applying Theorem 3.7 we deduce that φ_E^q is C-nondecreasing if and only if $C \subset [0, +\infty)q$.

On the other hand, it is not hard to check that $\varphi_E^p(y_1, y_2) = -|y_1| + y_2 - 1$ for all $(y_1, y_2) \in \mathbb{R}^2$. Then, by Theorem 3.2 it follows that

$$(0, +\infty)p + \mathrm{vcl}_p\, E = -[\varphi_E^p < 0] = \{(y_1, y_2) \in \mathbb{R}^2 : -|y_1| - 1 < y_2\}$$

and by Theorem 3.8 we obtain that φ_E^p is C-increasing if and only if

$$\mathrm{vcl}_p\, E + d \subset \{(y_1, y_2) \in \mathbb{R}^2 : -|y_1| - 1 < y_2\} \quad \forall d \in C\backslash\{(0, 0)\},$$

i.e., if and only if $|d_1| < d_2$ for all $(d_1, d_2) \in C\backslash\{(0, 0)\}$.

3.3 Characterization by scalarization of weak efficient solutions

For this section, we will consider an algebraic solid set $E \subset Y$ such that $0 \notin \mathrm{core}\, E$, and we will characterize the E-weak efficient solutions of problem VEP (see Definition 2.3) by means of the just studied algebraic formulation of the nonconvex separation functional.

Theorem 3.16. It follows that

$$\mathrm{WE}(f, S, E) = \mathrm{E}(\varphi_E^q \circ f, S) \quad \forall q \in \mathcal{H}_E.$$

Proof. Fix $q \in \mathcal{H}_E$ and $\bar{x} \in S$. Then $\bar{x} \in \mathrm{WE}(f, S, E)$ if and only if

$$f(\bar{x}, S) \cap (-\,\mathrm{core}\, E) = \emptyset. \tag{3.15}$$

By the definition of the set $\mathcal{H}_E$ and Theorem 3.2(h) it follows that (3.15) is equivalent to $\varphi_E^q(f(\bar{x}, x)) \geq 0$ for all $x \in S$, i.e., $\bar{x} \in \mathrm{E}(\varphi_E^q \circ f, S)$, and the proof is complete. □

We recall that weak E-optimality encompasses the so-called approximate weak efficient solutions of problem VEP with $E = \varepsilon q + K$, where $q \in \mathrm{core}\, K$ and $\varepsilon \geq 0$ (see [9, 22]). Next we characterize them.

Corollary 3.17. Suppose that K is a proper algebraic solid convex cone. Then, we have that

$$\mathrm{WE}(f, S, \varepsilon q + K) = \mathrm{E}(\varphi_K^q \circ f, S, \varepsilon) \quad \forall q \in \mathrm{core}\, K, \forall \varepsilon \geq 0.$$

Proof. Let us consider $q \in \operatorname{core} K$, $\varepsilon \geq 0$ and define $E := K$ and the mapping $f_{\varepsilon q} := f + \varepsilon q$, i.e., $f_{\varepsilon q}(x, z) := f(x, z) + \varepsilon q$ for all $x, z \in S$. By Remark 1.17 we see that $q \in \mathcal{H}_E$ and then, by applying Theorem 3.16 we deduce that $\mathrm{WE}(f_{\varepsilon q}, S, E) = \mathrm{E}(\varphi_E^q \circ f_{\varepsilon q}, S)$.

It is obvious that $\mathrm{WE}(f_{\varepsilon q}, S, E) = \mathrm{WE}(f, S, \varepsilon q + K)$. Moreover, by Theorem 3.2(c) it follows that

$$(\varphi_E^q \circ f_{\varepsilon q})(x, z) = \varphi_E^q(f(x, z) + \varepsilon q) = (\varphi_E^q \circ f)(x, z) + \varepsilon \quad \forall x, z \in S,$$

and then $\mathrm{E}(\varphi_E^q \circ f_{\varepsilon q}, S) = \mathrm{E}(\varphi_K^q \circ f, S, \varepsilon)$, which finishes the proof. $\square$

Corrollary 3.17 extends [165, Theorem 3.1] from the topological setting to the algebraic setting. Moreover, let us observe that condition $f(x, x) = 0$ for all $x \in S$ (i.e., f is diagonal null in S) is not needed.

When E is a convex cone, we have the following corollary.

Corollary 3.18. Suppose that K is a proper algebraic solid convex cone. Then, the following statements are equivalent:

(a) $\bar{x} \in \mathrm{WE}(f, S, K)$.

(b) $\bar{x} \in \mathrm{E}(\varphi_K^q \circ f, S)$ for all $q \in \operatorname{core} K$.

(c) There exists a function $\varphi : Y \to \mathbb{R}$ positively homogeneous, subadditive and $\operatorname{core} K$-increasing such that $\bar{x} \in \mathrm{E}(\varphi \circ f, S)$.

Proof. By applying Corollary 3.17 with $\varepsilon = 0$ we deduce that part (a) implies part (b).

Next, let us consider an arbitrary $q \in \operatorname{core} K$. It is easy to check that $\operatorname{vcl}_q K$ is a convex cone. Then, by Theorem 3.13 we have that φ_K^q is positively homogeneous and subadditive.

On the other hand, it is obvious that $\mathbb{R}q - K = Y$. Moreover, as $\operatorname{core} K \neq \emptyset$, by [4, Propositions 3(iv) and 5(i)] we see that $\operatorname{vcl} K$ is a convex cone, and by Lemma 1.13(d) and (e) we deduce that

$$\operatorname{vcl} K + \operatorname{core} K = \operatorname{core} K.$$

Therefore, $q \notin -\operatorname{vcl} K$ (otherwise, $0 = -q + q \in \operatorname{vcl} K + \operatorname{core} K = \operatorname{core} K$, that is a contradiction since K is proper) and by Remark 3.6(a) and Theorem 3.2(b) we deduce that φ_K^q is finite. Analogously, by applying Proposition 1.16(b) to $E = K$ we see that

$$\operatorname{vcl}_q K + \operatorname{core} K = (0, +\infty)q + \operatorname{vcl}_q K \quad \forall q \in \operatorname{core} K,$$

and by Theorem 3.8 we have that φ_K^q is $\operatorname{core} K$-increasing for all $q \in \operatorname{core} K$.

Therefore, since φ_K^q is finite, positively homogeneous, subadditive and $\operatorname{core} K$-increasing for all $q \in \operatorname{core} K$, then we deduce part (c) from part (b) by taking $\varphi := \varphi_K^q$.

Next we prove that part (c) implies part (a), which completes the proof. Indeed, assume that $\bar{x} \in \mathrm{E}(\varphi \circ f, S)$, where $\varphi : Y \to \mathbb{R}$ is positively homogeneous, subadditive and $\operatorname{core} K$-increasing, and reasoning by contradiction suppose that there exists $x \in S$ such that $f(\bar{x}, x) \in -\operatorname{core} K$. Then,

$$\varphi(f(\bar{x}, x)) < \varphi(0) = 0,$$

which is a contradiction since $\bar{x} \in \mathrm{E}(\varphi \circ f, S)$, and the proof finishes. □

Corollary 3.18 encompasses [81, Theorem 3.1] by assuming that Y is a real locally convex Hausdorff topological linear space and K is the ordering cone. Indeed, by Lemma 1.13(a) we see that $\operatorname{int} K = \operatorname{core} K$ whenever K is topological solid, and so [81, Theorem 3.1] is a particular case of Corollary 3.18. Let us observe that the continuity of functional φ in [81, Theorem 3.1] cannot be obtained in the algebraic framework. Moreover, the pointedness assumption on the ordering cone (i.e., condition $K \cap (-K) = \emptyset$) is not needed. In this sense, Corollary 3.18 is more general than [81, Theorem 3.1] in the topological setting too. For example, by considering $Y = \mathbb{R}^2$ and the (not pointed) ordering cone $K = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \geq 0\}$, it is clear that [81, Theorem 3.1] cannot be applied. However, the assumptions of Corollary 3.18 are fulfilled.

By Theorem 3.16 we can obtain scalarization results for those problems that can be reformulated as an equilibrium problem. Let us show this fact with problems VVIP and VOP (see (1.18) and (1.16)).

First, we characterize the solutions of an extended formulation of WVVIP in linear spaces, where the ordering set E is free-disposal with respect to a proper algebraic solid convex cone K. To be precise, the problem consists in finding $\bar{x} \in S$ such that

$$\langle T(\bar{x}), x - \bar{x} \rangle \notin -\operatorname{core} E \quad \forall x \in S, \tag{3.16}$$

where $T : S \to L(X, Y)$. We denote the set of all solutions of (3.16) by $\mathrm{WV}(T, S, E)$.

By applying Theorem 3.16 we characterize the solutions of this generalized vector variational inequality problem via solutions of a scalar equilibrium problem.

Corollary 3.19. Let E be free-disposal with respect to a proper algebraic solid convex cone K. For every $q \in \operatorname{core} K$, it follows that

$$\bar{x} \in \mathrm{WV}(T, S, E) \Leftrightarrow \varphi_E^q\big(\langle T(\bar{x}), x - \bar{x} \rangle\big) \geq 0 \quad \forall x \in S.$$

Proof. It is easy to check that $\mathrm{WV}(T, S, E) = \mathrm{WE}(f, S, E)$, where $f : S \times S \to Y$, $f(x_1, x_2) = \langle T(x_1), x_2 - x_1 \rangle$ for all $x_1, x_2 \in S$. Then the result is a direct consequence of Proposition 1.16(b) and Theorem 3.16. □

Next, we will deal with problem VOP (see page 15), where a vector-valued mapping $g : S \to Y$ is minimized with respect to the preference relation $\leq_D$ defined in Y through a proper algebraic solid convex cone $D \subset Y$. We will characterize the approximate weak efficient solutions of these problems given by the well-known concept of ε-efficient solution introduced by Kutateladze [127]. To be precise, consider $q \in \operatorname{core} D$ and $\varepsilon \geq 0$. A point $\bar{x} \in S$ is said to be a weak εq-efficient solution of the vector optimization problem VOP

$$\mathrm{Min}_D\{g(x) : x \in S\},$$

denoted by $\bar{x} \in \mathrm{WO}(g, S, D, \varepsilon q)$, if there is not a point $x \in S$ such that $g(x) \in g(\bar{x}) - \varepsilon q - \operatorname{core} D$. Next, these approximate solutions are characterized by suboptimal solutions of the scalar optimization problem OP

$$\mathrm{Min}\{(\varphi_D^q \circ g)(x) : x \in S\}.$$

Then the following sets are considered:

$$\varepsilon\text{-argmin}(\varphi_D^q \circ g, S) := \{\bar{x} \in S : (\varphi_D^q \circ g)(\bar{x}) - \varepsilon \leq (\varphi_D^q \circ g)(x) \quad \forall x \in S\}.$$

Moreover, for each $x \in S$, we denote the mapping $g_x : S \to Y$, $g_x(z) := g(z)-g(x)$ for all $z \in S$.

Corollary 3.20. Suppose that D is a proper algebraic solid convex cone. Then, for each $q \in \operatorname{core} D$ and $\varepsilon \geq 0$ we have that

$$\bar{x} \in \mathrm{WO}(g, S, D, \varepsilon q) \iff \bar{x} \in \varepsilon\text{-argmin}(\varphi_D^q \circ g_{\bar{x}}, S), \tag{3.17}$$

$$\mathrm{WO}(g, S, D, \varepsilon q) = \bigcup_{x \in S} \varepsilon\text{-argmin}(\varphi_D^q \circ g_x, S). \tag{3.18}$$

Proof. Let $f : S \times S \to Y$ be the mapping $f(x, z) := g_x(z)$ for all $x, z \in S$. Then statement (3.17) follows by applying Corollary 3.17, since

$$\mathrm{WO}(g, S, D, \varepsilon q) = \mathrm{WE}(f, S, \varepsilon q + D)$$

and

$$\bar{x} \in \mathrm{E}(\varphi_D^q \circ f, S, \varepsilon) \iff \bar{x} \in \varepsilon\text{-argmin}(\varphi_D^q \circ g_{\bar{x}}, S),$$

since $\varphi_D^q(0) = 0$. In order to state (3.18), we need to prove the inclusion

$$\varepsilon\text{-argmin}(\varphi_D^q \circ g_x, S) \subset \mathrm{WO}(g, S, D, \varepsilon q) \quad \forall x \in S.$$

Let us prove the following slightly more general relation:

$$\varepsilon\text{-argmin}(\varphi_D^q \circ (g - y), S) \subset \mathrm{WO}(g, S, D, \varepsilon q) \quad \forall y \in Y,$$

where $g - y : S \to Y$, $(g - y)(x) = g(x) - y$ for all $x \in S$. Indeed, let $z \in \varepsilon\text{-argmin}(\varphi_D^q \circ (g - y), S)$ and suppose that $z \notin \mathrm{WO}(g, S, D, \varepsilon q)$. Then there exists $z' \in S$ such that $g(z') \in g(z) - \varepsilon q - \operatorname{core} D$, or equivalently,

$$g(z') - y \in g(z) - y - \varepsilon q - \operatorname{core} D. \tag{3.19}$$

As φ_D^q is $\operatorname{core} D$-increasing (see the proof of Corollary 3.18), from (3.19) and Theorem 3.2(c) we obtain

$$\varphi_D^q(g(z') - y) < \varphi_D^q(g(z) - y - \varepsilon q) = \varphi_D^q(g(z) - y) - \varepsilon,$$

which is a contradiction, since $z \in \varepsilon\text{-argmin}(\varphi_D^q \circ (g - y), S)$, and the proof finishes. $\square$

The above corollary improves Theorem 4.12 of [120], where it is proved that

$$\bar{x} \in \mathrm{WO}(g, S, D, \varepsilon q) \Rightarrow \bar{x} \in \varepsilon\text{-argmin}(\varphi_D^q \circ (g_{\bar{x}} + \alpha q), S) \quad \forall \alpha \in \mathbb{R},$$

by assuming the additional assumptions $\varepsilon > 0$, D pointed and vectorially closed and also that the set $F := \operatorname{cone}(f(S) - f(\bar{x}) + D + \varepsilon q)$ is convex and algebraic solid. By characterization (3.17) we see that implication above is also true for $\varepsilon = 0$ and the additional assumptions on the cones D and F are not needed. Also, let us observe that parameter α is superfluous, since by Theorem 3.2(c), for each $\alpha \in \mathbb{R}$, we have that

$$\varphi_D^q \circ (g_{\bar{x}} + \alpha q)(x) = (\varphi_D^q \circ g_{\bar{x}})(x) + \alpha \quad \forall x \in S,$$

and then to minimize the function $\varphi_D^q \circ (g_{\bar{x}} + \alpha q)$ is the same as to minimize the function $\varphi_D^q \circ g_{\bar{x}}$.

Chapter 4

Ekeland variational principles and existence of solutions

4.1 Introduction

In Chapter 1 we have seen that the Ekeland variational principle is one of the most important mathematical tools in Nonlinear Analysis and Optimization. As a consequence, during the last decade it has been extended to different settings and, in particular, to vector-valued or set-valued functions and bifunctions (see, for instance, [9, 11, 22, 61, 93, 118, 140, 142, 158, 160, 161, 183, 198]).

In 1993, Oettli and Théra [150] studied an EVP and several equivalent results for scalar bifunctions. They obtained a set of implications which included a new Takahashi's nonconvex minimization principle. In 2005, Bianchi, Kassay and Pini [21] also obtained an EVP for scalar bifunctions in order to achieve existence results for equilibrium problems without convexity assumptions. Shortly after, in 2007-2008, Bianchi, Kassay and Pini [22], Ansari [9], Araya, Kimura and Tanaka [11], and Finet and Quarta [61] generalized it for vector-valued bifunctions, obtaining different versions. Recently, Qiu [160] also obtained a new version in a very general framework. Moreover, it has been extended to set-valued bifunctions (for instance, by Zeng and Li [198], Tammer and Zălinescu [183], Gong [82] and Qiu [161]). All these works deal with EVPs whose perturbed function achieves an exact solution.

A priori, one may think that an EVP whose associated perturbed function attains an approximate solution requires weaker hypotheses. In fact, in the context of optimization problems, Combari, Marcellin and Thibault [45] and Gutiérrez, Jiménez, Novo and Thibault [96] obtained approximate EVPs without assuming hypotheses concerning semicontinuity on the function. In both results, the notion of approximate strict solution played a key role. In Section 2.4 we studied a new concept of strict solution for vector equilibrium problems and an approximate counterpart through free-disposal sets which will also play a key role to derive exact and approximate EVPs in the equilibria context.

For this aim, in Section 4.2 we state a strict fixed point-type theorem, inspired by results of [140] and [159]. Then, in Section 4.3, we will obtain from it exact and approximate EVPs for vector equilibrium problems without assuming any topology on the final space of the associated bifunction. Additionally, some non-topological mathematical tools are needed, as the algebraic formulation of the so-called nonlinear scalarization functional (see Section 3.2), and a semialgebraic counterpart of a well-known lower semicontinuity concept due to Tammer [181] that was introduced in Section 1.4. The improvements of the main obtained results with respect to corresponding ones of the literature are shown, and the performance of certain assumptions usually required in proving them is clarified.

On the other hand, the first works on equilibrium problems [10, 20, 28, 149] (see also [76]) and many subsequent ones provide existence theorems under convexity assumptions on the bifunction and via topological tools. Moreover, many of the previously cited EVPs (for instance, [9, 11, 22]) were applied to obtain existence results for weak efficient solutions without convexity assumptions but under (topological) semicontinuity assumptions.

Following this research line, in Section 4.4, we will obtain an existence result for weak efficient solutions of vector equilibrium problems in the non-topological setting –in the sense that the image space of the bifunction is not endowed with any particular topology– without assuming convexity assumptions. For this aim, the main mathematical tools are the "semialgebraic" upper semicontinuity notion, which was defined in Section 1.4, and the algebraic formulation of the nonconvex separation functional of Section 3.2. As a result, the new existence result requires weaker hypotheses than the previous ones of the literature obtained via EVPs for bifunctions. Moreover, the roles of usual assumptions in that kind of results are analyzed.

4.2 A strict fixed point theorem for set-valued mappings

The mathematical tool that we are going to obtain in this section is an existence result for a kind of strict fixed point of a set-valued mapping. It is inspired by [140, Theorem 3.1] and [159, Theorem 2.1]. Recall that, $\bar{x} \in X$ is a strict fixed point of a set-valued mapping $S\colon X \rightrightarrows X$ if it satisfies that $S(\bar{x}) = \{\bar{x}\}$.

Theorem 4.1. Let (X, d) be a metric space. Consider a set-valued mapping $S : X \rightrightarrows X$, a function $m : X \to \mathbb{R} \cup \{\pm\infty\}$ and $x_0 \in X$ such that $S : S(x_0) \rightrightarrows S(x_0)$, $S(x_0)\backslash\{x_0\} \neq \emptyset$ and $m : S(x_0)\backslash\{x_0\} \to [0, +\infty]$. Assume the following conditions:

(*i*) If $x \in S(x_0)$ and $y \in S(x)$, then $S(y) \subset S(x)$.

(*ii*) There exists $x_1 \in S(x_0)\backslash\{x_0\}$ such that $m(x_1) < +\infty$.

(*iii*) There exists $\delta \geq 0$ such that for each $x \in S(x_0)$, $m(x) < +\infty$, and $y \in S(x)$, $y \neq x$, then

$$m(y) + \delta < m(x).$$

Then there exists a point $\bar{x} \in S(x_0)$ such that $S(\bar{x}) \subset \{\bar{x}\}$ whenever one of the following statements is true:

(*a*) $\delta > 0$, or

(*b*) $\delta = 0$, (X, d) is complete, the sets $S(x)$ are closed for all $x \in S(x_0)$, and there exists $\alpha > 0$ such that for each $x \in S(x_0)$ and $y \in S(x)$ such that $m(x), m(y) \in \mathbb{R}$, it follows that

$$\alpha d(x, y) \leq m(x) - m(y). \tag{4.1}$$

Proof. Suppose that $S(x_1) \not\subset \{x_1\}$ (otherwise the result is proved by considering $\bar{x} := x_1$). Let $x_2 \in S(x_1)$, $x_2 \neq x_1$. By hypotheses (i) and (iii) we see that $S(x_2) \subset S(x_1)$ and

$$m(x_2) + \delta < m(x_1). \tag{4.2}$$

If $S(x_2) \subset \{x_2\}$, then we define $\bar{x} := x_2$ and the proof finishes. Otherwise there exists $x_3 \in S(x_2)$, $x_3 \neq x_2$, and by hypotheses (i) and (iii) and statement (4.2) it follows that $S(x_3) \subset S(x_2) \subset S(x_1)$ and

$$m(x_3) + 2\delta < m(x_2) + \delta < m(x_1).$$

In this way we obtain a sequence $(x_n)_{n=1}^k \subset S(x_0)$, $k \in \mathbb{N} \cup \{+\infty\}$ such that $S(x_{n+1}) \subset S(x_n)$ and

$$m(x_{n+1}) + n\delta < m(x_1), \quad 1 \leq n \leq k-1. \tag{4.3}$$

Let us show that $(m(x_n))_{n=1}^k$ is a sequence of nonnegative real numbers. If $x_n \in S(x_0)\backslash\{x_0\}$ for all $2 \leq n \leq k$, there is nothing to prove. Otherwise there exists $n \geq 2$ such that $x_n = x_0$, and then $x_0 \in S(x_0)$ and $\delta \leq m(x_0) < +\infty$. Indeed, by (4.3) it is clear that $m(x_0) < +\infty$. On the other hand, by (ii) $m(x_1) \geq 0$ and by applying assumption (iii) with $x = x_0$ and $y = x_1$ we have that

$$\delta \leq m(x_1) + \delta < m(x_0) = m(x_n).$$

Therefore, $(m(x_n))_{n=1}^k$ is a sequence of nonnegative real numbers and

$$\inf\{m(x) : x \in S(x_n)\} \geq 0, \quad 1 \leq n \leq k.$$

Case (a) $(\delta > 0)$. Relation (4.3) implies that $\delta < \dfrac{1}{n} m(x_1)$, for $1 \leq n \leq k-1$. Since by (ii) $m(x_1) < +\infty$, we conclude that $k < +\infty$. Otherwise, by taking the limit as $n \to \infty$ in the relation above, we obtain that $\delta = 0$, which is a contradiction. Therefore, there exists $k_0 \in \mathbb{N}$, $k_0 \geq 1$, such that $S(x_{k_0}) \subset \{x_{k_0}\}$ and the proof of part (a) is completed by defining $\bar{x} := x_{k_0}$.

Case (b) $(\delta = 0)$. Let us suppose a point $x \in S(x_0)$ such that $S(x) \not\subset \{x\}, m(x) \in \mathbb{R}$ and $\inf_{y \in S(x)} m(y) > -\infty$. By condition (iii) we deduce that there exists $y_x \in S(x)$, such that

$$m(y_x) \leq \inf_{y \in S(x)} m(y) + (1/2)(m(x) - \inf_{y \in S(x)} m(y)). \tag{4.4}$$

Thus, via the same iterative process as the previous one but applying condition (4.4) to choose a suitable point in each step –observe that these points fulfill the assumptions required to state (4.4)– we obtain a sequence $(x_n)_{n=0}^k$ such that $S(x_{n+1}) \subset S(x_n)$ and $x_{n+1} := y_{x_n}$ for all $n \in \mathbb{N}$, where $k < +\infty$ and $S(x_k) \subset \{x_k\}$ or $k = +\infty$.

The proof finishes in the first case by defining $\bar{x} := x_k$. Then suppose that $k = +\infty$. By hypothesis (iii) we see that $(m(x_n))$ is a decreasing sequence. As $m(x_n) \geq 0$ for all n, we deduce that there exists $c \geq 0$ such that $m(x_n) \to c$. On the other hand, by condition (4.1) it follows that

$$\begin{aligned} \alpha d(x_{n+j}, x_n) &\leq \sum_{i=n+1}^{n+j} \alpha d(x_i, x_{i-1}) \\ &\leq \sum_{i=n+1}^{n+j} (m(x_{i-1}) - m(x_i)) \\ &= m(x_n) - m(x_{n+j}) \quad \forall n, j \in \mathbb{N}. \end{aligned}$$

Thus, (x_n) is a Cauchy sequence and as (X, d) is a complete metric space there exists $\bar{x} \in X$ such that $x_n \to \bar{x}$. For each $n \in \mathbb{N}$ it is clear that $\bar{x} \in S(x_n)$ since $(x_{n+j})_{j=1}^{+\infty} \subset S(x_n)$ and the sets $S(x_n)$ are closed. Then, by assumption (iii) we have $m(\bar{x}) \leq m(x_n)$ and taking the limit when $n \to \infty$ we see that $m(\bar{x}) \leq c$.

Suppose that $S(\bar{x}) \not\subset \{\bar{x}\}$. Then there exists $x' \in S(\bar{x})$, $x' \neq \bar{x}$. By hypothesis (iii) we have that $m(x') < m(\bar{x})$. As $x_{n+1} = y_{x_n}$ and $\bar{x} \in S(x_n)$, by hypothesis (i) it follows that

$$2m(x_{n+1}) - m(x_n) \leq \inf_{y \in S(x_n)} m(y) \leq m(x') \quad \forall n \in \mathbb{N}.$$

By taking the limit when $n \to +\infty$ we obtain $c \leq m(x')$ and so $c < m(\bar{x})$, which is a contradiction. Therefore $S(\bar{x}) \subset \{\bar{x}\}$ and the proof finishes. □

Remark 4.2. (i) Theorem 4.1 states the existence of a strict fixed point whenever the set-valued mapping S satisfies $x \in S(x)$ for all $x \in S(x_0)$. Thus it is a sort of endpoint (or stationary point) theorem.

(ii) Part (a) of Theorem 4.1 works even though X is not a metric space.

(*iii*) It is easy to check that Theorem 4.1 reduces to [96, Theorem 4.1] by considering the following data (we use the notations of [96]):

$$\begin{aligned} S(x) &:= L(\preceq_{\varepsilon C}, P(\cdot, x), P(x,x)) \quad \forall x \in X, \\ m(x) &:= \varphi(P(x,x)) - \inf\{\varphi(P(y,y)) : y \in S(x_0)\backslash\{x_0\}\} \quad \forall x \in X, \\ \delta &:= c/2. \end{aligned}$$

(*iv*) Ekeland Variational Principle is a direct consequence of Theorem 4.1(*b*). Indeed, when the metric space (X, d) is complete and $h : X \to \mathbb{R}\cup\{+\infty\}$ is a lower bounded and lower semicontinuous function, then for each $x_0 \in X$, $h(x_0) < +\infty$, Theorem 4.1(*b*) can be applied to the following data:

$$\begin{aligned} S(x) &:= \{y \in X : h(y) + d(y, x) \le h(x)\} \quad \forall x \in X, \\ m(x) &:= h(x) - \inf\{h(y) : y \in S(x_0)\} \quad \forall x \in X, \\ \delta &:= 0. \end{aligned}$$

As a consequence, there exists $\bar{x} \in S(x_0)$ such that $S(\bar{x}) = \{\bar{x}\}$, i.e.,

$$\begin{aligned} h(\bar{x}) + d(\bar{x}, x_0) &\le h(x_0), \\ h(x) + d(x, \bar{x}) &> h(\bar{x}) \quad \forall x \in X\backslash\{\bar{x}\}, \end{aligned}$$

and so the conclusions of the Ekeland Variational Principle are obtained.

Let us illustrate the usefulness of Theorem 4.1 to derive EVPs. Next result was stated in [11, Theorem 2.1] for diagonal null bifunctions and generalizes some similar previous ones.

Theorem 4.3. Consider that (X, d) is a complete metric space, Y is a topological linear space, D has nonempty topological interior, $q \in \operatorname{int} D$ and $f\colon X \times X \to Y$. Suppose that f satisfies the $\le_D$-t.i. property and for each $x \in X$ there exists $y \in Y$ such that $f(x, X) \cap (y - \operatorname{int} D) = \emptyset$ and the following set is closed:

$$\{z \in X : f(x, z) + d(x, z)q \le_D 0\}.$$

Then, for every $x_0 \in X$ there exists $\bar{x} \in X$ such that

(*a*) $f(x_0, \bar{x}) + d(x_0, \bar{x})q \le_D 0$ or $\bar{x} = x_0$,

(b) $f(\bar{x},x)+d(x,\bar{x})q \not\leq_D 0$ for all $x \in X\backslash\{\bar{x}\}$.

Proof. Since $q \in \operatorname{int} D$, it is easy to check that φ_D^q is finite, nondecreasing w.r.t. $\leq_D$ and $[\varphi_D^q < 0] = -\operatorname{int} D$ (see Theorems 3.2 and 3.7 and Lemma 1.13). Moreover, for each $y \in Y$ there exists $t \in \mathbb{R}$ such that $tq \leq_D y$. So, for an arbitrary point $x_0 \in X$ there exists $c \in \mathbb{R}$ such that $f(x_0, X) \cap (cq - \operatorname{int} D) = \emptyset$ and then by Theorem 3.2(c) we have that $\varphi_D^q(f(x_0,z)) \geq c$ for all $z \in X$.

Let us define for each $x \in X$ the mappings

$$\begin{aligned} S(x) &:= \{z \in X : f(x,z) + d(x,z)q \leq_D 0\}, \\ m(x) &:= \varphi_D^q(f(x_0,x)) - c. \end{aligned}$$

If $S(x_0) \subset \{x_0\}$, then the result follows by considering $\bar{x} = x_0$.

Suppose that $S(x_0) \not\subset \{x_0\}$. Assumption (i) of Theorem 4.1 is a direct consequence of the $\leq_D$-t.i. property of f and assumption (ii) is clear. On the other hand, if $y \in S(x)$, $y \neq x$, then

$$f(x_0,y) + d(x,y)q \leq_D f(x_0,x) + f(x,y) + d(x,y)q \leq_D f(x_0,x).$$

As φ_D^q is nondecreasing w.r.t. $\leq_D$, by Theorem 3.2(c) we have that

$$\begin{aligned} m(y) &< m(y) + d(x,y) \\ &= \varphi_D^q(f(x_0,y) + d(x,y)q) - c \\ &\leq \varphi_D^q(f(x_0,x)) - c \\ &= m(x). \end{aligned}$$

In particular, $d(x,y) \leq m(x) - m(y)$ and assumptions (iii) and (b) are fulfilled. Then the result follows by applying Theorem 4.1. □

Let us observe that part (a) of Theorem 4.3 is equivalent to the statement $f(x_0,\bar{x}) + d(x_0,\bar{x})q \leq_D 0$ whenever $f(x_0,x_0) \in -D$ (in particular, if f is diagonal null).

4.3 Ekeland Variational Principles

It is known that the EVP relies on the completeness hypothesis of the initial metric space of the function (see [178]). Nevertheless, an approximate version of

EVP was obtained in [45] for extended real valued functions without using any completeness hypothesis. The main difference is that the perturbed function achieves an approximate minimum with an arbitrary error $\delta > 0$ instead of an exact minimum. In [96], this approach was generalized to set-valued optimization problems via a sort of approximate strict solutions based on a certain class of coradiant sets.

Next, we obtain approximate and exact EVPs related to problem VEP without considering any particular topology on the real linear space Y. In this section, assume that the decision set is $S = X$, (X, d) is a metric space and $E \subset Y$ is a free-disposal set with respect to D.

Theorem 4.4. Let $x_0 \in X$, $q \in D\backslash\{0\}$ and $\phi \in E^+\backslash\{0\}$.

Suppose that one of the following conditions is true:

(i) $0 \notin E$, $E + E \subset E$, $\phi \in E^{s+}$ and

$$c_1 := \inf\{\phi(f(x_0, x)) + \phi(q)d(x_0, x) : x \in [f(x_0, \cdot) + d(x_0, \cdot)q \leq_E f(x_0, x_0)]\} > -\infty. \quad (4.5)$$

(ii) (X, d) is complete, $E = \mathrm{vcl}_q D$, $\phi(q) > 0$,

$$c_2 := \inf\{\phi(f(x_0, x)) : x \in [f(x_0, \cdot) + d(x_0, \cdot)q \leq_{\mathrm{vcl}_q D} f(x_0, x_0)]\} > -\infty,$$

and the sets

$$[f(x_0, \cdot) + d(x, \cdot)q \leq_{\mathrm{vcl}_q D} f(x_0, x)], \quad x \in [f(x_0, \cdot) + d(x_0, \cdot)q \leq_{\mathrm{vcl}_q D} f(x_0, x_0)],$$

are closed.

Then there exists $\bar{x} \in X$ such that

(a) $f(x_0, \bar{x}) + d(x_0, \bar{x})q \leq_E f(x_0, x_0)$ or $\bar{x} = x_0$,

(b) $f(x_0, x) + d(\bar{x}, x)q \not\leq_E f(x_0, \bar{x}) \quad \forall x \in X\backslash\{\bar{x}\}$.

Proof. (i) In order to apply Theorem 4.1(a), let us define

$$S(x) := \{y \in X : f(x_0, y) + d(x, y)q \leq_E f(x_0, x)\} \quad \forall x \in X,$$
$$m(x) := \phi(f(x_0, x)) + \phi(q)d(x_0, x) - c_1 \quad \forall x \in X.$$

If $S(x_0)\backslash\{x_0\} = \emptyset$, then the result follows by defining $\bar{x} := x_0$. Otherwise, assumptions (i)-(iii) and (a) of Theorem 4.1 are true. Indeed, consider $x \in X$, $y \in S(x)$ and $z \in S(y)$. Then,

$$\begin{aligned} f(x_0, y) + d(x, y)q &\leq_E f(x_0, x), \\ f(x_0, z) + d(y, z)q &\leq_E f(x_0, y). \end{aligned}$$

As $E + E \subset E$ and $q \in D$, by Remark 1.11 we have

$$f(x_0, z) + d(x, z)q \leq_D f(x_0, z) + (d(x, y) + d(y, z))q \leq_E f(x_0, x).$$

As E is free-disposal with respect to D, by Lemma 1.18(f) we deduce that

$$f(x_0, z) + d(x, z)q \leq_E f(x_0, x)$$

and so $z \in S(x)$. Thus, $S(y) \subset S(x)$ and assumption (i) of Theorem 4.1 holds. We also see that $S : S(x_0) \rightrightarrows S(x_0)$.

For each $x \in S(x_0)$ it is clear that

$$\begin{aligned} &\phi(f(x_0, x)) + \phi(q)d(x_0, x) \\ &\geq \inf\{\phi(f(x_0, z)) + \phi(q)d(x_0, z) : z \in S(x_0)\} = c_1 \end{aligned}$$

and so m is well defined and assumption (ii) is fulfilled. Finally, consider $x \in S(x_0)$ and $y \in S(x)$, $y \neq x$. Then, $f(x_0, y) + d(x, y)q \leq_E f(x_0, x)$. As $\phi \in E^{s+}$, there exists $t > 0$ such that

$$\phi(f(x_0, y)) + \phi(q)d(x, y) + t \leq \phi(f(x_0, x)).$$

Moreover, by Remark 1.19 we have that $\phi(q) \geq 0$ and so

$$\begin{aligned} m(y) + t &= \phi(f(x_0, y)) + \phi(q)d(x_0, y) - c_1 + t \\ &\leq \phi(f(x_0, y)) + \phi(q)d(x_0, x) + \phi(q)d(x, y) - c_1 + t \\ &\leq \phi(f(x_0, x)) + \phi(q)d(x_0, x) - c_1 \\ &= m(x). \end{aligned}$$

Therefore, by considering $\delta := t/2$ we see that assumption (iii) is true. Then, by Theorem 4.1(a) we have that there exists $\bar{x} \in S(x_0)$ such that $S(\bar{x})\backslash\{\bar{x}\} = \emptyset$.

Thus,

$$f(x_0,\bar{x}) + d(x_0,\bar{x})q \leq_E f(x_0,x_0),$$
$$f(x_0,x) + d(\bar{x},x)q \not\leq_E f(x_0,\bar{x}) \quad \forall x \in X\backslash\{\bar{x}\},$$

and the proof of part (i) finishes.

(ii) Consider the same mapping $S : X \rightrightarrows X$ as in part (i) and

$$m(x) := \phi(f(x_0,x)) - c_2 \quad \forall x \in X.$$

If $S(x_0)\backslash\{x_0\} = \emptyset$, then the result follows by defining $\bar{x} := x_0$. Otherwise, assumptions (i)-(iii) and (b) of Theorem 4.1 are fulfilled. Indeed, since $\operatorname{vcl}_q D$ is a free-disposal –with respect to D, see Lemma 1.18(a)– convex cone, assumptions (i) and (ii) of Theorem 4.1 can be checked as in the proof of part (i). In particular, it follows that $S : S(x_0) \rightrightarrows S(x_0)$ and m is well defined.

On the other hand, consider $x \in S(x_0)$ and $y \in S(x)$, $y \neq x$. Then, $f(x_0,y) + d(x,y)q \leq_{\operatorname{vcl}_q D} f(x_0,x)$. As $\phi \in \operatorname{vcl}_q(D)^+ = D^+$ we have that

$$\phi(f(x_0,y)) + \phi(q)d(x,y) \leq \phi(f(x_0,x)) \tag{4.6}$$

and so

$$\begin{aligned} m(y) &= \phi(f(x_0,y)) - c_2 \\ &< \phi(f(x_0,y)) + \phi(q)d(x,y) - c_2 \\ &\leq \phi(f(x_0,x)) - c_2 \\ &= m(x). \end{aligned}$$

Therefore, assumption (iii) is fulfilled with $\delta = 0$. Finally, by (4.6) we see that

$$\phi(q)d(x,y) \leq (\phi(f(x_0,x)) - c_2) - (\phi(f(x_0,y)) - c_2) = m(x) - m(y)$$

and so inequality (4.1) is true. From here the proof finishes as in part (i). □

Remark 4.5. (i) A simple condition that implies assumption (4.5) is the following one:

$$\inf\{\phi(f(x_0,x)) : x \in [f(x_0,\cdot) \leq_E f(x_0,x_0)]\} > -\infty.$$

Indeed, if $q \in D$ then by Lemma 1.18(*f*) it is clear that

$$[f(x_0,\cdot)+d(x_0,\cdot)q \leq_E f(x_0,x_0)] \subset [f(x_0,\cdot) \leq_E f(x_0,x_0)]$$

and then for each $\phi \in E^+$ it follows that

$$\begin{aligned} c_1 &\geq \inf\{\phi(f(x_0,x)) : x \in [f(x_0,\cdot)+d(x_0,\cdot)q \leq_E f(x_0,x_0)]\} \\ &\geq \inf\{\phi(f(x_0,x)) : x \in [f(x_0,\cdot) \leq_E f(x_0,x_0)]\}. \end{aligned}$$

(*ii*) When E is convex, the existence of linear functionals $\phi \in E^{s+}$ is characterized by [110, Theorem 4C]. To be precise, if E is convex, then $E^{s+} \neq \emptyset$ if and only if there exists a convex absorbing set $V \subset Y$ such that $V \cap E = \emptyset$. In particular, observe that E is not a cone whenever $E^{s+} \neq \emptyset$.

On the other hand, we have that $E^{s+} \neq \emptyset$ whenever E is convex, algebraic solid and $0 \notin \operatorname{vcl} E$. Indeed, consider $v \in \operatorname{core} E$. Since $0 \notin \operatorname{vcl} E$, there exists $\lambda > 0$ such that $[0,\lambda]v \cap E = \emptyset$. The result follows by applying [112, Theorem 3.14] with $S = E$ and $T = [0,\lambda]v$.

When Y is a real locally convex Hausdorff topological linear space, then the existence of a convex absorbing set $V \subset Y$ such that $V \cap E = \emptyset$ is equivalent to the condition $0 \notin \operatorname{cl} E$ (see [110, Theorem 11F]). This condition is natural to deal with approximate efficient points of sets in Y. Indeed, these points can be dominated only by others near of them, and a way to state this property is to consider nondominated points with respect to ordering sets E such that $0 \notin \operatorname{cl} E$.

(*iii*) Consider a nonempty set $H \subset Y$. The following easy relationships are useful to check the assumptions of Theorem 4.4:

- If H is convex, then $H + H \subset H$ if and only if H is coradiant.
- If H is convex and coradiant, then H is free-disposal w.r.t. $\operatorname{cone} H$.
- If H is convex, coradiant and $D \subset \operatorname{cone} H$, then H is free-disposal w.r.t. D.
- If H is convex, coradiant and $\operatorname{cone} H$ is vectorially closed by q, for some $q \in H$, then H is free-disposal w.r.t. D if and only if $D \subset \operatorname{cone} H$.

(*iv*) In Theorem 4.4 we did not use the symmetric property of d, so that the result is also valid in quasi-metric spaces.

Next we show that the lower semicontinuity of f w.r.t. $\leq_{\mathrm{vcl}_q D}$ implies that the sets

$$[f(x_0,\cdot)+d(x,\cdot)q \leq_{\mathrm{vcl}_q D} f(x_0,x)] \tag{4.7}$$

are closed (compare with Proposition 4.13).

Proposition 4.6. Let $x_0, x \in X$ and $q \in D\backslash\{0\}$. If $f(x_0,\cdot): X \to Y$ is lsc w.r.t. $\leq_{\mathrm{vcl}_q D}$, then (4.7) is a closed set.

Proof. Consider $z \in X$ and a sequence $(x_n) \subset X$ such that $x_n \to z$ and

$$f(x_0,x_n)+d(x,x_n)q \leq_{\mathrm{vcl}_q D} f(x_0,x) \quad \forall n.$$

Let us fix an arbitrary $\delta > 0$. As $d(x,x_n) \to d(x,z)$ there exists $n_0 \in \mathbb{N}$ such that $-d(x,x_n)q \in -d(x,z)q+\delta q - D$ for all $n \geq n_0$, and by Lemma 1.18(a) we have that

$$\begin{aligned} f(x_0,x_n) &= (f(x_0,x_n)+d(x,x_n)q)-d(x,x_n)q \\ &\in f(x_0,x)-d(x,z)q+\delta q-\mathrm{vcl}_q D. \end{aligned}$$

Since $f(x_0,\cdot)$ is lsc w.r.t. $\leq_{\mathrm{vcl}_q D}$, we deduce that

$$f(x_0,z) \in f(x_0,x)-d(x,z)q+\delta q-\mathrm{vcl}_q D.$$

Therefore, by Lemma 1.18(e)

$$f(x_0,z)+d(x,z)q-f(x_0,x) \in \bigcap_{\delta>0}(\delta q-\mathrm{vcl}_q D) \subset -\mathrm{vcl}_q D$$

and the proof is finished. ☐

In the following corollary, we show the roles of the diagonal null and $\leq_D$-t.i. properties in the EVPs for bifunctions of the literature.

Corollary 4.7. Assume the hypotheses of Theorem 4.4 and, in addition, suppose that f is diagonal null and satisfies the $\leq_D$-t.i. property. Then there exists $\bar{x} \in X$ such that

(a) $f(x_0,\bar{x})+d(x_0,\bar{x})q \leq_E 0$ or $\bar{x}=x_0$,

(b) $f(\bar{x}, x) + d(\bar{x}, x)q \not\leq_E 0 \quad \forall x \in X \backslash \{\bar{x}\}$.

Proof. By applying Theorem 4.4 we see that there exists $\bar{x} \in X$ such that

$$\begin{aligned} f(x_0, \bar{x}) + d(x_0, \bar{x})q &\leq_E f(x_0, x_0) \text{ or } \bar{x} = x_0, \\ f(x_0, x) + d(\bar{x}, x)q &\not\leq_E f(x_0, \bar{x}) \quad \forall x \in X \backslash \{\bar{x}\}, \end{aligned} \tag{4.8}$$

and part (a) is proved since f is diagonal null. In order to state part (b), suppose by contradiction that there exists $z \in X \backslash \{\bar{x}\}$ such that $f(\bar{x}, z) + d(\bar{x}, z)q \leq_E 0$. By the $\leq_D$-t.i. property we have that

$$f(x_0, z) + d(\bar{x}, z)q \leq_D f(x_0, \bar{x}) + f(\bar{x}, z) + d(\bar{x}, z)q \leq_E f(x_0, \bar{x})$$

and by Lemma 1.18(f) we obtain that $f(x_0, z) + d(\bar{x}, z)q \leq_E f(x_0, \bar{x})$, which is a contradiction with (4.8). Therefore, statement (b) follows and the proof finishes. □

Corollary 4.7 reduces to the single-valued version of [96, Theorem 5.1]. Let us underline that our framework here is more general since Y is a real linear space and E is not necessarily contained in D. For instance, it can be applied for sets as $E_q = q + K$, where $q \in D \backslash (-K)$ and K is a convex cone such that $D \subset K$, $D \neq K$. In the first statement of [96, Theorem 5.1], the case $\bar{x} = x_0$ should be separated since $\leq_E$ does not verify the reflexive property.

On the other hand, Corollary 4.7 via the sufficient condition of Proposition 4.6 generalizes several exact EVPs for bifunctions of the literature. Indeed, it reduces to [22, Theorem 1] and [140, Corollary 4.3] by considering the following data and additional assumptions (we use the notations of [22, Theorem 1]): Y is a locally convex topological linear space, D is closed, $q := e$ and $\phi := e^*$. Let us observe that $\operatorname{vcl}_q D = D$, since D is closed, and Corollary 4.7 works with the assumption $f(x, x) \leq_D 0$ of [140, Corollary 4.3] (weaker than the diagonal null condition).

Analogously it reduces to the single-valued version of [198, Theorem 3.1], where additionally is assumed that D has nonempty topological interior and $f(\cdot, \cdot)$ is order bounded from below (i.e., there exists $y \in Y$ such that $y \leq_D f(x, z)$ for all $x, z \in X$).

Next, we focus on the particular case $E_q = q + D$, with $q \in D\backslash\{0\}$. By means of Theorem 4.1 and the algebraic version of the Gerstewitz's separation functional, we obtain for this case an exact and approximate EVP.

Theorem 4.8. Let $q \in D\backslash\{0\}$, $\alpha > 1$ and $x_0 \in X$ such that $f(x_0, x_0) \in -D$. If $x_0 \in \mathrm{S}(f, \alpha E_q)$, then there exists $\bar{x} \in X$ such that

(a) $f(x_0, \bar{x}) + d(x_0, \bar{x})q \leq_{E_q} f(x_0, x_0)$ or $\bar{x} = x_0$,

(b) $d(x_0, \bar{x}) < \alpha - 1$,

(c) $f(x_0, x) + d(\bar{x}, x)q \not\leq_{E_q} f(x_0, \bar{x})$ for all $x \in X\backslash\{\bar{x}\}$.

Proof. Let us define $S : X \rightrightarrows X$ and $m : X \to \mathbb{R} \cup \{\pm\infty\}$ as follows:

$$S(x) := \{y \in X : f(x_0, y) + d(x, y)q \leq_{E_q} f(x_0, x)\} \quad \forall x \in X,$$
$$m(x) := \varphi^q_{E_q}(f(x_0, x)) + d(x_0, x) + \alpha - 1 \quad \forall x \in X.$$

If $S(x_0)\backslash\{x_0\} = \emptyset$, then the result is proved. Otherwise, the assumptions of Theorem 4.1(a) are fulfilled. Indeed, assumption (i) can be checked as in the proof of Theorem 4.4, since E_q is free-disposal with respect to D and $E_q + E_q \subset E_q$. In particular, we see that $S : S(x_0) \rightrightarrows S(x_0)$.

On the other hand, for each $x \in S(x_0)\backslash\{x_0\}$ we have that $f(x_0, x) \not\leq_{\alpha E_q} 0$, since $x_0 \in \mathrm{S}(f, \alpha E_q)$ (see statement (2.7)). Thus,

$$f(x_0, x) + (\alpha - 1)q \not\leq_{E_q} 0 \tag{4.9}$$

and by Theorem 3.2$(c)(f)$ and Lemma 1.18(b) we obtain that

$$m(x) \geq \varphi^q_{E_q}(f(x_0, x)) + \alpha - 1 = \varphi^q_{E_q}(f(x_0, x) + (\alpha - 1)q) \geq 0.$$

Moreover,

$$f(x_0, x) + d(x_0, x)q \leq_{E_q} f(x_0, x_0) \leq_D 0$$

and by Lemma 1.18(f) it follows that

$$f(x_0, x) + d(x_0, x)q \leq_{E_q} 0. \tag{4.10}$$

Therefore,

$$m(x) = \varphi^q_{E_q}(f(x_0, x) + d(x_0, x)q) + \alpha - 1 \le \alpha - 1$$

and so $m : S(x_0)\backslash\{x_0\} \to [0, +\infty]$ and assumption (ii) is hold. Observe by (4.9) and (4.10) that $d(x_0, x) < \alpha - 1$.

Finally, let us check assumption (iii). Consider $x \in S(x_0)$ and $y \in S(x)$. Then, $f(x_0, y) + d(x, y)q \le_{E_q} f(x_0, x)$ and so $f(x_0, y) + (d(x, y) + 1)q \le_D f(x_0, x)$. Therefore, by Theorem 3.2(c) and Theorem 3.7 we have that

$$\begin{aligned} m(y) + 1 &= \varphi^q_{E_q}(f(x_0, y)) + d(x_0, y) + \alpha \\ &\le \varphi^q_{E_q}(f(x_0, y) + (d(x, y) + 1)q) + d(x_0, x) + \alpha - 1 \\ &\le \varphi^q_{E_q}(f(x_0, x)) + d(x_0, x) + \alpha - 1 \\ &= m(x), \end{aligned}$$

and assumption (iii) is fulfilled by considering $\delta = 1/2$.

By applying Theorem 4.1(a) we deduce that there exists a point $\bar{x} \in S(x_0)$ such that $S(\bar{x}) \subset \{\bar{x}\}$, which finishes the proof. □

As in Corollary 4.7, the following result shows the role of $\le_D$-t.i. property in the main EVPs for bifunctions of the literature.

Corollary 4.9. Let $q \in D\backslash\{0\}$, $\alpha > 1$, $x_0 \in X$ such that $f(x_0, x_0) \in -D$ and suppose that f satisfies the $\le_D$-t.i. property. If $x_0 \in \mathrm{S}(f, \alpha E_q)$, then there exists $\bar{x} \in X$ such that

(a) $f(x_0, \bar{x}) + d(x_0, \bar{x})q \le_{E_q} f(x_0, x_0)$ or $\bar{x} = x_0$,

(b) $d(x_0, \bar{x}) < \alpha - 1$,

(c) $f(\bar{x}, x) + d(\bar{x}, x)q \not\le_{E_q} 0$ for all $x \in X\backslash\{\bar{x}\}$.

By applying Corollary 4.9 to $f : X \times X \to Y$ such that $f(x, y) = g(y) - g(x)$ for a mapping $g : X \to Y$, we retrieve the single-valued version of [96, Theorem 5.2] without assuming the solidness of the ordering cone ($\operatorname{int} D \neq \emptyset$). To be precise, Corollary 4.9 reduces to the single-valued version of [96, Theorem 5.2] by assuming that Y is a locally convex Hausdorff topological linear space and by considering the following data (we use the notations of [96, Theorem 5.2]): $\alpha = (\varepsilon/\delta) + 1$, δq instead of q and the distance $(\varepsilon/(\lambda\delta))d(\cdot,\cdot)$. If additionally $Y = \mathbb{R}$, $D = \mathbb{R}_+$ and $q = 1$, then Corollary 4.9 reduces to [45, Proposition 2.5] whenever the approximate solution of the scalar optimization problem is strict.

Next we obtain two exact versions of the EVP for bifunctions. The following two lemmas are needed.

Lemma 4.10. Consider $y_1, y_2 \in Y$ and $s \in \mathbb{R}$ such that $\varphi_E^q(y_1) + s \not\leq \varphi_E^q(y_2)$. Then, $y_1 + sq \not\leq_D y_2$.

Proof. Suppose reasoning by contradiction that $y_1+sq \leq_D y_2$. As the mapping φ_E^q is nondecreasing w.r.t. $\leq_D$ (see Theorem 3.7) we see that $\varphi_E^q(y_1 + sq) \leq \varphi_E^q(y_2)$. Then, by Theorem 3.2$(c)$ we conclude that $\varphi_E^q(y_1) + s \leq \varphi_E^q(y_2)$, which is a contradiction. Thus, $y_1 + sq \not\leq_D y_2$ and the proof finishes. $\square$

Next we denote

$$\mathcal{F}(q, E) := (\mathbb{R}q - \operatorname{vcl}_q E) \cap (Y \backslash \operatorname{ovcl}_q^{+\infty}(-E))$$

and for all $x_0, x \in X$ such that $f(x_0, x) \in \mathcal{F}(q, E)$,

$$H_q(x) := \{z \in X : f(x_0, z) + d(x, z)q \leq_{\operatorname{vcl}_q E} \varphi_E^q(f(x_0, x))q\}.$$

By Theorem 3.2(b) it follows that $\varphi_E^q(y) \in \mathbb{R}$ if and only if $y \in \mathcal{F}(q, E)$.

Lemma 4.11. Suppose that $x_0 \in \mathrm{S}(f, E)$ and $q \in D\backslash\{0\}$. Then, for each $x \in X\backslash\{x_0\}$ we have $\varphi_E^q(f(x_0, x)) \geq 0$, and $f(x_0, x) \in \mathcal{F}(q, E)$ if and only if $f(x_0, x) \in \mathbb{R}q - \operatorname{vcl}_q E$. If additionally $f(x_0, x_0) \in \mathcal{F}(q, E)$, then $f(x_0, x) \in \mathcal{F}(q, E)$ for all $x \in H_q(x_0)$.

Proof. As $x_0 \in \mathrm{S}(f, E)$ we have that

$$f(x_0, x) \notin -E \quad \forall x \in X\backslash\{x_0\}, \tag{4.11}$$

and by Lemma 1.18(b) we deduce that

$$f(x_0, x) \notin (-\infty, 0)q - \operatorname{vcl}_q E \quad \forall x \in X\backslash\{x_0\}.$$

Then, by Theorem 3.2(f) it follows that

$$\varphi_E^q(f(x_0, x)) \geq 0 \quad \forall x \in X\backslash\{x_0\}.$$

From here the first part of the result follows by applying Theorem 3.2(a).

Suppose that $f(x_0, x_0) \in \mathcal{F}(q, E)$. By applying Theorem 3.2(c)(e) and Lemma 1.18(c) it is not hard to check that

$$H_q(x_0) = \{x \in X : \varphi_E^q(f(x_0, x)) + d(x_0, x) \leq \varphi_E^q(f(x_0, x_0))\} \tag{4.12}$$

and so $\varphi_E^q(f(x_0, x)) < +\infty$ for all $x \in H_q(x_0)$, which finishes the proof. □

Theorem 4.12. Assume that (X, d) is complete and consider $q \in D\backslash\{0\}$ and $x_0 \in X$ such that $f(x_0, x_0) \in \mathcal{F}(q, E)$. If $x_0 \in \mathrm{S}(f, E)$ and $H_q(x)$ is closed for all $x \in H_q(x_0)$, then there exists $\bar{x} \in X$ such that

(a) $f(x_0, \bar{x}) + d(\bar{x}, x_0)q \leq_{\mathrm{vcl}_q E} \varphi_E^q(f(x_0, x_0))q$,

(b) $d(\bar{x}, x_0) \leq \varphi_E^q(f(x_0, x_0))$ or $\bar{x} = x_0$,

(c) $f(x_0, x) + d(\bar{x}, x)q \not\leq_D f(x_0, \bar{x})$ for all $x \in X\backslash\{\bar{x}\}$.

Proof. First, let us observe by Lemma 4.11 and (4.12) that the sets $H_q(x)$ are well defined for all $x \in H_q(x_0)$ and $x_0 \in H_q(x_0)$. Thus, $f(x_0, x_0) \leq_{\mathrm{vcl}_q E} \varphi_E^q(f(x_0, x_0))q$.

Consider the following mappings $S : X \rightrightarrows X$ and $m : X \to \mathbb{R} \cup \{\pm\infty\}$: $S(x) := H_q(x)$ for all $x \in X$ such that $f(x_0, x) \in \mathcal{F}(q, E)$ and $S(x) := X$ otherwise, and $m(x) = \varphi_E^q(f(x_0, x))$ for all $x \in X$. By Lemma 4.11 we see that $S(x) = H_q(x)$ for all $x \in S(x_0)$, and $m : S(x_0)\backslash\{x_0\} \to [0, +\infty]$.

On the other hand, by using the same reasoning as in the proof of Lemma 4.11 to obtain statement (4.12), we have that

$$S(x) = \{z \in X : \varphi_E^q(f(x_0, z)) + d(x, z) \leq \varphi_E^q(f(x_0, x))\} \quad \forall x \in H_q(x_0). \tag{4.13}$$

Observe that $x \in S(x)$ and so $f(x_0, x) \leq_{\mathrm{vcl}_q E} \varphi_E^q(f(x_0, x))q$ for all $x \in H_q(x_0)$ (see Lemma 3.1(b)).

If $S(x_0)\backslash\{x_0\} = \emptyset$, then the result follows by defining $\bar{x} = x_0$. Indeed, parts (a) and (b) are clear and part (c) is a consequence of Lemma 4.10. Then, let us assume that $S(x_0)\backslash\{x_0\} \neq \emptyset$.

In order to check assumption (i) of Theorem 4.1, let us consider $x \in S(x_0)$, $y \in S(x)$ and $z \in S(y)$. As $S(x) = H_q(x)$ it follows that $f(x_0, y) \in \mathcal{F}(q, E)$ and then $S(y) = H_q(y)$. Thus, by (4.13) we have that

$$\begin{aligned}\varphi_E^q(f(x_0, z)) + d(x, z) &\le \varphi_E^q(f(x_0, z)) + d(x, y) + d(y, z)\\ &\le \varphi_E^q(f(x_0, y)) + d(x, y)\\ &\le \varphi_E^q(f(x_0, x)).\end{aligned}$$

Thus, $z \in S(x)$ and assumption (i) of Theorem 4.1 holds. In particular, $S : S(x_0) \rightrightarrows S(x_0)$.

As $f(x_0, x_0) \in \mathcal{F}(q, E)$ it is clear that

$$m(x) \le \varphi_E^q(f(x_0, x)) + d(x_0, x) \le \varphi_E^q(f(x_0, x_0)) < +\infty \quad \forall x \in S(x_0)$$

and then assumption (ii) of Theorem 4.1 is fulfilled.

On the other hand, consider $x \in S(x_0)$ and $y \in S(x)$, $y \neq x$. Then, $\varphi_E^q(f(x_0, y)) \in \mathbb{R}$ and so

$$m(y) < \varphi_E^q(f(x_0, y)) + d(x, y) \le \varphi_E^q(f(x_0, x)) = m(x).$$

Therefore, assumption (iii) of Theorem 4.1 is fulfilled with $\delta = 0$. Finally, for each $x \in S(x_0)$ the set $S(x)$ is supposed to be closed and we have that

$$d(x, y) \le \varphi_E^q(f(x_0, x)) - \varphi_E^q(f(x_0, y)) = m(x) - m(y) \quad \forall y \in S(x).$$

Thus, all assumptions of Theorem 4.1 are fulfilled and so there exists $\bar{x} \in S(x_0)$ such that $S(\bar{x}) = \{\bar{x}\}$. Let us check that statements (a)-(c) are true.

Part (a) is obvious and statement (c) is a direct consequence of Lemma 4.10. In order to state part (b) suppose that $\bar{x} \neq x_0$ and observe by part (a) and statement (4.11) that $f(x_0, \bar{x}) \notin -E$ and $f(x_0, \bar{x}) + d(\bar{x}, x_0)q \le_{\mathrm{vcl}_q E} \varphi_E^q(f(x_0, x_0))q$. If $d(\bar{x}, x_0) - \varphi_E^q(f(x_0, x_0)) > 0$, then by Lemma 1.18$(b)$ we have that

$$f(x_0, \bar{x}) \in -(d(\bar{x}, x_0) - \varphi_E^q(f(x_0, x_0)))q - \mathrm{vcl}_q E \subset -E,$$

which is a contradiction. Thus, $d(\bar{x}, x_0) \le \varphi_E^q(f(x_0, x_0)$ and the proof finishes. $\square$

Next we prove that the closedness of the sets $H_q(x)$ is fulfilled whenever $f(x_0,\cdot): X \to Y$ is (q,E)-lsc.

Proposition 4.13. Let $x_0, x \in X$ and $q \in D\backslash\{0\}$ such that $\varphi_E^q(f(x_0,x)) \in \mathbb{R}$. If $f(x_0,\cdot): X \to Y$ is (q,E)-lsc, then $H_q(x)$ is closed.

Proof. Consider a sequence $(x_n) \subset H_q(x)$ and suppose that $x_n \to z$. Thus, $d(x_n,x) \to d(z,x)$ and

$$f(x_0,x_n) + d(x,x_n)q \leq_{\mathrm{vcl}_q E} \varphi_E^q(f(x_0,x))q \quad \forall n.$$

Let us fix an arbitrary $\delta > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $-d(x,x_n)q \in -d(x,z)q + \delta q - D$ for all $n \geq n_0$, and by Lemma 1.18(a) we have that

$$\begin{aligned} f(x_0,x_n) &= (f(x_0,x_n) + d(x,x_n)q) - d(x,x_n)q \\ &\in (\varphi_E^q(f(x_0,x)) - d(x,z) + \delta)q - \mathrm{vcl}_q E. \end{aligned}$$

Since $f(x_0,\cdot)$ is (q,E)-lsc, we deduce that

$$f(x_0,z) \in (\varphi_E^q(f(x_0,x)) - d(x,z) + \delta)q - \mathrm{vcl}_q E.$$

Therefore, by Lemma 1.18(e)

$$f(x_0,z) + (d(x,z) - \varphi_E^q(f(x_0,x)))q \in \bigcap_{\delta>0}(\delta q - \mathrm{vcl}_q E) \subset -\mathrm{vcl}_q E$$

and the proof is finished. □

Corollary 4.14. Assume that (X,d) is complete, E is an improvement set and f is diagonal null and satisfies the $\leq_D$-t.i. property. Let $q \in D\backslash\{0\}$ and suppose that $\mathbb{R}q \cap \mathrm{vcl}_q E \neq \emptyset$. If $x_0 \in \mathrm{S}(f,E)$ and $H_q(x)$ is closed for all $x \in H_q(x_0)$, then there exists $\bar{x} \in X$ such that

(a) $f(x_0,\bar{x}) + d(\bar{x},x_0)q \leq_{\mathrm{vcl}_q E} \varphi_E^q(0)q$,

(b) $d(\bar{x},x_0) \leq \varphi_E^q(0)$,

(c) $f(\bar{x},x) + d(x,\bar{x})q \not\leq_D 0 \ \forall x \in X\backslash\{\bar{x}\}$.

Proof. Let us check that $f(x_0, x_0) \in \mathcal{F}(q, E)$. As f is diagonal null, we have to prove that $\varphi_E^q(0) \in \mathbb{R}$. It is clear that $\varphi_E^q(0) < +\infty$, since $\mathbb{R}q \cap \operatorname{vcl}_q E \neq \emptyset$ (see Theorem 3.2(a)). On the other hand, $0 \notin E$ and so by Theorem 3.2(f) and Lemma 1.18(b) we deduce that $\varphi_E^q(0) \geq 0$.

By applying Theorem 4.12 we deduce that statements (a) and (b) are true and additionally the following condition is also fulfilled:

$$f(x_0, x) + d(\bar{x}, x)q \not\leq_D f(x_0, \bar{x}) \quad \forall x \in X \backslash \{\bar{x}\}. \tag{4.14}$$

Let us suppose, reasoning by contradiction, that there exists a point $u \in X$, $u \neq \bar{x}$, such that

$$f(\bar{x}, u) + d(u, \bar{x})q \leq_D 0. \tag{4.15}$$

Then, by the $\leq_D$-t.i. property and (4.15) we deduce that

$$f(x_0, u) + d(\bar{x}, u)q \leq_D f(x_0, \bar{x}) + f(\bar{x}, u) + d(u, \bar{x})q \leq_D f(x_0, \bar{x}),$$

which is contrary to (4.14). Therefore, statement (c) is true and the proof finishes. □

Corollary 4.14 reduces to the EVP for vector-valued bifunctions given in [22, Theorem 2] by considering the improvement set $E = \varepsilon e + K$, the distance $(\varepsilon/\lambda)d(\cdot, \cdot)$ and $q = e$ (we use the notations of [22, Theorem 2]). In particular, let us observe that $\varphi_E^q(0) = \varepsilon$ and $\operatorname{vcl}_q E = E$ –since K is closed– and so, since f is diagonal null and $\leq_K$-t.i. property is fulfilled, it follows that

$$\begin{aligned}
& f(x_0, \bar{x}) + (\varepsilon/\lambda)d(\bar{x}, x_0)q \leq_{\operatorname{vcl}_q E} \varphi_E^q(0)q \\
\Rightarrow \quad & f(x_0, \bar{x}) + (\varepsilon/\lambda)d(\bar{x}, x_0)q \leq_K 0 \\
\Rightarrow \quad & 0 \leq_K -f(x_0, \bar{x}) \leq_K f(\bar{x}, x_0) \\
\Rightarrow \quad & f(\bar{x}, x_0) \in K.
\end{aligned}$$

Moreover, [22, Theorem 2] considers additional and stronger assumptions. To be precise, it is assumed that Y is a locally convex Hausdorff topological linear space, the ordering cone $D = K$ is closed, $f(z, \cdot)$ is quasi lower semicontinuous for all $z \in X$ (i.e., the set $[f(z, \cdot) \leq_D b]$ is closed for all $z \in X$ and $b \in Y$) and there exists $e \in D\backslash(-D)$ and $\phi \in D^+$ such that $\phi(e) = 1$ and $\phi \circ f(z, \cdot)$ is bounded from below for all $z \in X$.

Analogously, Corollary 4.14 reduces to [9, Theorem 3.1] when is applied to a complete metric space instead of a quasi-metric space, and the perturbation function is defined by the distance instead of a W-distance. Indeed, this result supposes that Y is a locally convex Hausdorff topological linear space, the ordering cone D is solid and the function $f(x, \cdot) : X \to Y$ is order bounded from below for all $x \in X$ (i.e., there exists a point $y \in Y$ such that $y \leq_D f(x, z)$ for all $z \in X$). Let us check these assumptions imply for each point $x_0 \in X$ and $q \in \operatorname{int} D$ that there exists $\varepsilon > 0$ such that $x_0 \in \mathrm{S}(f, E_{\varepsilon q})$.

Let $q \in \operatorname{int} D$ and $x_0 \in X$. It is easy to see that for all $y \in Y$ there exists $t > 0$ such that $y \in -tq + \operatorname{int} D$. Then, as $f(x_0, \cdot)$ is order bounded from below, there exists $t_0 > 0$ such that $-t_0 q <_D f(x_0, x)$ for all $x \in X$. From here it follows that $x_0 \in \mathrm{S}(f, E_{t_0 q})$. Otherwise there exists $x' \in X\backslash\{x_0\}$ such that $f(x_0, x') \in (-t_0 q + \operatorname{int} D) \cap (-t_0 q - D)$ and so

$$0 = f(x_0, x') - f(x_0, x') \in \operatorname{int} D + D = \operatorname{int} D,$$

which is a contradiction since D is proper.

Thus, Corollary 4.14 can be applied for each $x_0 \in X$ and the set $E = E_{\varepsilon q}$ via Proposition 4.13 and Remark 1.26(i) and so the conclusions of [9, Theorem 3.1] are obtained. Let us observe that in this case the statement

$$f(x_0, \bar{x}) + d(\bar{x}, x_0)q \leq_{\operatorname{vcl}_q E_{\varepsilon q}} \varphi^q_{E_{\varepsilon q}}(0)q,$$

is equivalent to $f(x_0, \bar{x}) + d(\bar{x}, x_0)q \leq_D 0$, since $\operatorname{vcl}_q E_{\varepsilon q} = \varepsilon q + \operatorname{vcl}_q D = \varepsilon q + D$ –in [9, Theorem 3.1] the ordering cone is assumed to be closed– and $\varphi^q_{E_{\varepsilon q}}(0) = \varepsilon$.

4.4 Weierstrass Theorem for bifunctions

In the literature, there are many existence results for weak efficient solutions of problem VEP derived by different EVPs for bifunctions without assuming any convexity assumption. Some of them require the $\leq_D$-t.i. property. Next we obtain a new existence result that improves the previous ones since it requires weaker assumptions and the image space of the bifunction is a real linear space. Moreover, the role of the $\leq_D$-t.i. property is clarified.

In the following we assume that X is a topological space, the decision set is the whole space $S = X$, and D is algebraic solid. The following lemma is needed.

Lemma 4.15. Let $x_0, \bar{x} \in X$ be such that $f(\bar{x}, x_0) \notin f(x, x_0) - \operatorname{core} D$ for all $x \in X$. If f satisfies the $\leq_D$-t.i. property, then $\bar{x} \in \mathrm{WE}(f, D)$.

Proof. Suppose, reasoning by contradiction, that $\bar{x} \notin \mathrm{WE}(f, D)$. Then there exists $x \in X$ such that $f(\bar{x}, x) \in -\operatorname{core} D$, i.e., $f(\bar{x}, x) <_D 0$. As f satisfies the $\leq_D$-t.i. property, we see that

$$f(\bar{x}, x_0) \leq_D f(\bar{x}, x) + f(x, x_0) <_D f(x, x_0).$$

Since $D + \operatorname{core} D = \operatorname{core} D$ we have that $f(\bar{x}, x_0) <_D f(x, x_0)$ and a contradiction is obtained. □

From the previous lemma we see that if $f(\bar{x}, x_0)$ is a weak maximal point of the set $f(X, x_0)$ with respect to the relation $\leq_D$ (i.e., $f(\bar{x}, x_0) \not<_D f(x, x_0)$ for all $x \in X$), then $f(\bar{x}, \bar{x})$ is a weak minimal point of the set $f(\bar{x}, X)$ with respect to the same relation (i.e., $f(\bar{x}, x) \not<_D f(\bar{x}, \bar{x})$ for all $x \in X$), whenever the bifunction f satisfies the $\leq_D$-t.i. property and f is diagonal null.

Theorem 4.16. Assume that X is compact, f satisfies the $\leq_D$-t.i. property and there exist $x \in X$ and $q \in \operatorname{core} D$ such that $\varphi_D^q \circ f(\cdot, x) \colon X \to \mathbb{R}$ is usc. Then $\mathrm{WE}(f, D) \neq \emptyset$.

Proof. Let $q \in \operatorname{core} D$ and $x \in X$ be such that $\varphi_D^q \circ f(\cdot, x)$ is usc. By Theorem 3.2(b), we know that $\varphi_D^q \circ f(\cdot, x)$ is finite, and by applying the Weierstrass theorem we deduce that there exists a point $\bar{x} \in X$ such that

$$\varphi_D^q \circ f(\bar{x}, x) \geq \varphi_D^q \circ f(z, x) \quad \forall z \in X. \tag{4.16}$$

By Theorem 3.8 we know that φ_D^q is increasing w.r.t. $<_D$, and so by (4.16) we obtain $f(\bar{x}, x) \notin f(z, x) - \operatorname{core} D$ for all $z \in X$. Then, by applying Lemma 4.15 we have that $\bar{x} \in \mathrm{WE}(f, D)$ and the proof finishes. □

Proposition 4.17. Let $x \in X$ and $q \in \operatorname{core} D$. If $f(\cdot, x)\colon X \to Y$ is (q, D)-usc, then $\varphi_D^q \circ f(\cdot, x) : X \to \mathbb{R}$ is usc.

Proof. Consider $c \in \mathbb{R}$, $z \in X$ and a net $(z_i) \subset X$ such that $z_i \to z$ and $\varphi_D^q \circ f(z_i, x) \geq c$ for all i. By Remark 3.3 we have that

$$[\varphi_D^q < 0] = -\operatorname{core} D \tag{4.17}$$

and so, by Theorem 3.2(c) it follows that $z_i \notin [f(\cdot, x) <_D cq]$ for all i.

Since $f(\cdot, x)$ is (q, D)-usc, it follows that $[f(\cdot, x) <_D cq]$ is an open set, and so $z \notin [f(\cdot, x) <_D cq]$. Thus, $f(z, x) - cq \notin -\operatorname{core} D$, and by Theorem 3.2$(c)$ and (4.17), we have that $\varphi_D^q(f(z, x)) \geq c$. Therefore, the set $[\varphi_D^q \circ f(\cdot, x) \geq c]$ is closed for all $c \in \mathbb{R}$, and the result is proved. □

The next corollary is a direct consequence of Proposition 4.17 and Theorem 4.16.

Corollary 4.18. Assume that X is compact, f satisfies the $\leq_D$-t.i. property and there exist $x \in X$ and $q \in \operatorname{core} D$ such that $f(\cdot, x) : X \to Y$ is (q, D)-usc. Then $\mathrm{WE}(f, D) \neq \emptyset$.

Remark 4.19. Corollary 4.18 improves several existence results of weak efficient solutions of vector equilibrium problems stated in the literature, since it considers weaker assumptions. For example, in [9, Theorem 4.1] the following additional assumptions are required: Y is a locally convex Hausdorff topological linear space, D is closed, f is diagonal null, $f(z, \cdot)$ is (q, D)-lsc for all $z \in X$ (i.e., $[f(z, \cdot) \leq_D rq]$ is closed for all $z \in X$ and $r \in \mathbb{R}$) and $f(z, \cdot)$ is order bounded from below for all $z \in X$ (i.e., for each $z \in X$ there exists a point $y \in Y$ such that $y \leq_D f(z, u)$ for all $u \in X$).

Moreover, the (q, D)-superlevel closedness notion is required (see Remark 1.26(iii)) through the following property (see [9, Lemma 2.2]): if $g\colon X \to Y$ is (q, D)-superlevel closed, then $\varphi_D^q \circ g : X \to \mathbb{R}$ is upper semicontinuous. For its proof, the author cites [181], where a similar result is given for (q, D)-lower semicontinuous functions.

Although the corresponding result [181, Lemma 4.1] is correct, this extension for (q,D)-superlevel closed functions is not true. Indeed, let $X=\mathbb{R}$, $Y=\mathbb{R}^2$, $D=\mathbb{R}^2_+$, $q=(1,1)$ and $g:X\to Y$ defined as

$$g(t)=\begin{cases}(t,1) & \text{if} \quad t<0\\ (t,0) & \text{if} \quad t\geq 0.\end{cases}$$

Clearly,

$$[g\geq_D rq]=\begin{cases}[r,+\infty) & \text{if} \quad r\leq 0\\ \emptyset & \text{if} \quad r>0.\end{cases}$$

Then g is (q,D)-superlevel closed. Moreover,

$$(\varphi_D^q\circ g)(t)=\begin{cases}1 & \text{if} \quad t<0\\ t & \text{if} \quad t\geq 0.\end{cases}$$

Thus, $[\varphi_D^q\circ g\geq 1/2]=(-\infty,0)\cup[1/2,+\infty)$ and $\varphi_D^q\circ g$ is not upper semicontinuous.

It is proved in Proposition 4.17 that $\varphi_D^q\circ g$ is upper semicontinuous whenever g fulfills the (q,D)-usc concept introduced in Definition 1.25.

Analogously, in [22, Theorem 3], the next additional assumptions are supposed: Y is a locally convex Hausdorff topological linear space, D is closed, f is diagonal null, $f(z,\cdot)$ is quasi lower semicontinuous for all $z\in X$ (i.e., the set $[f(z,\cdot)\leq_D y]$ is closed for all $z\in X$ and $y\in Y$) and there exist $e\in D\backslash(-D)$ and a functional $\phi\in D^\circ$ such that $\phi(e)=1$ and $\phi\circ f(z,\cdot)$ is bounded from below for all $z\in X$. Moreover, a stronger upper semicontinuous notion is required (see Remark 1.26(*ii*)).

Finally, in the single-valued version of [198, Theorem 4.1], it is assumed that Y is a locally convex Hausdorff topological linear space, D is closed, f is diagonal null, $f(z,\cdot)$ is quasi lower semicontinuous for all $z\in X$, $f(\cdot,\cdot)$ is order bounded from below (i.e., there exists a point $y\in Y$ such that $y\leq_D f(z,u)$ for all $z,u\in X$). Moreover, the continuity of the function $f(\cdot,z)$ is required for all $z\in X$.

Let us illustrate Corollary 4.18 with an example in which the weakening of the assumptions plays a key role.

Example 4.20. Consider $X = [-4, 4]$, $Y = \mathbb{R}^2$, $D = \mathbb{R}^2_+$ and $g \colon X \to Y$ defined as

$$g(t) = \begin{cases} (-t, -t) & \text{if} \quad t < 0 \\ (t-2, -2) & \text{if} \quad 0 \leq t < 2 \\ (-2, t) & \text{if} \quad t \geq 2. \end{cases}$$

Fix $q = (1, 1) \in \operatorname{core} D$ and $x = -1$. We define the bifunction $f \colon X \times X \to Y$ as $f(s, t) = g(t) - g(s)$. Clearly, f satisfies the $\leq_D$-t.i. property. Let us check that $f(\cdot, x)$ is (q, D)-usc, that is,

$$[f(\cdot, x) <_D rq] = [g >_D (1 - r, 1 - r)] \tag{4.18}$$

is open for all $r \in \mathbb{R}$. Since

$$[g >_D (s, s)] = \begin{cases} [-4, 4] & \text{if} \quad s < -2 \\ [-4, 0) & \text{if} \quad -2 \leq s \leq 0 \\ [-4, -s) & \text{if} \quad 0 < s < 4 \\ \emptyset & \text{if} \quad s \geq 4 \end{cases},$$

then the sublevel sets of (4.18) are open in $[-4, 4]$ for all $r \in \mathbb{R}$. Hence, by Corollary 4.18, we have that $\operatorname{WE}(f, D) \neq \emptyset$ (in fact, one may compute that $\operatorname{WE}(f, D) = [0, 4]$). In addition, notice that $f(\cdot, x)$ is not (e, D)-superlevel closed, for any $e \in \operatorname{int} D$. Indeed, by taking $r = 0$ we have

$$[f(\cdot, x) \geq_D re] = [f(\cdot, x) \geq_D (0, 0)] = [g \leq_D (1, 1)] = [-1, 2)$$

which is not closed in $[-4, 4]$.

As a result, it follows that $f(x, \cdot)$ is not (e, D)-lsc for any $e \in \operatorname{int} D$, since

$$[f(x, \cdot) \leq_D (0, 0)] = [f(\cdot, x) \geq_D (0, 0)].$$

Therefore, [9, Theorem 4.1], [22, Theorem 3] and [198, Theorem 4.1] cannot be applied.

Chapter 5

Conclusions and future lines of development

In this last chapter, we summarize the most relevant results of this memory, in order to provide a global vision of the work done for the readers. In addition, some future lines of development that have emerged during the elaboration of this memory are exposed.

5.1 Conclusions

The main objective of this doctoral thesis has been to analyze the solutions of vector equilibrium problems. Just like it was indicated in the introduction, the most important feature of this kind of problems is the generalization of several classic mathematical problems, in such a way we may deal with all of them at once. Following this unifying approach, we generalized well-known concepts of solution from vector optimization problems to vector equilibrium problems. Furthermore, we consider an algebraic framework since it often allows to weaken certain topological assumptions by using certain algebraic counterparts.

Chapter 2 is devoted to introducing and studying certain algebraic notions of solution of vector equilibria in connection with arbitrary ordering sets. First, in Section 2.2, the concept of (C, ε)-efficiency introduced by Gutiérrez, Jiménez and Novo [91, 92] in vector optimization problems is generalized to vector equilibrium problems. In this notion, an arbitrary coradiant set (see Section 1.4) plays the role of ordering set, so that it encompasses many ε-efficiency notions for vector optimization problems (for instance, [104, 127, 148, 184–186]) and other approximate efficiency concepts for vector equilibrium problems, such as the ε-equilibrium and λ-equilibrium points defined by Bianchi, Kassay and Pini [22] and Ansari [9], respectively.

Secondly, the notion of E-weak efficient solution, which was introduced by Chicco, Mignanego, Pusillo and Tijs [43] for vector optimization problems in finite-dimensional spaces and extended later by Gutiérrez, Jiménez and Novo [95] to locally convex spaces, is generalized to vector equilibrium problems on real linear spaces. In this notion, an algebraic solid set E is considered as the ordering set. We characterize this kind of solutions when E is free disposal with respect to a proper algebraic solid convex cone D under a generalized convexity assumption by a linear scalarization procedure.

In Section 2.3 we deal with (C, ε)-proper efficiency notions. Benson [19] and Henig [105] proposed (exact) proper efficiency notions that nowadays are well known. Adán and Novo [5] extended the Benson's one to real linear spaces by means of the vector closure [4], Gong [78,79] generalized both notions to vector equilibrium problems, and Zhou and Peng [202] introduced the first approximate counterparts of both notions in real linear spaces.

Recently, Gutiérrez, Huerga, Jiménez and Novo [88,90] have defined notions of Henig and Benson (C, ε)-proper efficiency in vector optimization that improve the previous ones with respect to the limit behaviour of their solutions when the error ε tends to zero. In this section, both notions are generalized to vector equilibrium problems on real linear spaces and their main properties and relationships are studied. Moreover, since Henig (C, ε)-proper efficient solutions can be formulated as $(\mathcal{E}_0^K(C), \varepsilon)$-efficient solutions for some $K \in \mathcal{G}(C)$ and $\big(\mathcal{E}_0^K(C)\big)(\varepsilon)$ is free-disposal with respect to K for every $\varepsilon \geq 0$, we applied the obtained results in the previous section to characterize the Henig (C, ε)-proper efficient solutions under a generalized convexity assumption.

On the other hand, the concept of strict solution is well known in scalar and vector optimization problems (see [66, 113, 175]), and also in scalar equilibrium problems (see [25]). Strict solutions for vector equilibrium problems are introduced in Section 2.4 in order to generalize all of them.

An approximate counterpart based on coradiant sets was defined by Gutiérrez, Jiménez, Novo and Thibault [96] with the aim of stating approximate EVPs in set-valued optimization problems. Following this approach, we introduce the concept of E-strict solution, where E is a free-disposal set w.r.t. a convex cone, and we studied its main properties in order to obtain approximate EVPs for vector-valued bifunctions.

Chapter 3 is devoted to the algebraic study of the nonconvex separation functional φ_A^q (see [117, Chapter 5] for detailed information). This research line has been proposed in the literature by different authors. Indeed, La Torre, Popovici and Rocca [128–130] suggested studying this functional in the framework of a real linear space Y not endowed with any particular topology, and Qiu and He [163] dealt with φ_D^q, where D is a convex cone and $q \in D\backslash(-D)$ it in this setting.

In Section 3.2, we study functional φ_A^q in the same setting as above for an arbitrary nonempty set $A \subset Y$ and a vector $q \in Y\backslash\{0\}$. By using algebraic counterparts of topological concepts such as the vector closure in a given direction and the algebraic interior, we obtain important properties of φ_A^q.

To be precise, in Lemma 3.1 and Theorem 3.2 we characterize its level sets, its effective domain, and other well-known properties without assuming any assumption. Furthermore, in Theorems 3.7 and 3.8 we characterize the monotonicity of φ_A^q with respect to an arbitrary ordering set $C \subset Y$. In Theorem 3.13, we give sufficient conditions for the positive homogeneity and the convexity of φ_A^q, and under condition (3.13), both properties are characterized. These results generalize and improve several recent ones in the literature (see [60, 63, 64, 117, 128, 163, 176]).

In Section 3.3, the algebraic formulation of the nonconvex separation functional is applied to scalarize E-weak efficient solutions of vector equilibrium problems on real linear spaces (Theorem 3.16). This result encompasses and improves many exact and approximate scalarization results of the literature. For example, [81, Theorem 3.1] is extended via Corollary 3.18 and [165, Theorem 3.1] is generalized by taking $E = \varepsilon q + K$ in Corollary 3.17, where $q \in \operatorname{core} K$ and $\varepsilon \geq 0$. Both results require less and weaker assumptions.

In addition, we applied Theorem 3.16 to characterize by scalarization the weak solutions of a wide class of vector variational inequality problems (Corollary 3.19) and vector optimization problems (Corollary 3.20). As an illustration of the power of the obtained results observe for instance that [120, Theorem 4.12] is improved by Corollary 3.20, since the first one requires additional assumptions such as pointedness and vectorially closedness on the ordering cone, a convexity assumption related with the data of the problem, and so on.

Chapter 4 focuses on Ekeland variational principles and the existence of weak solutions for vector equilibrium problems. In Section 4.2, a powerful tool to derive EVPs and approximate EVPs (Theorem 4.1) is obtained. It is an existence result for a kind of strict fixed point of a set-valued mapping inspired by [140, Theorem 3.1] and [159, Theorem 2.1]. For instance, the original EVP and [11, Theorem 2.1] may be derived from it (see Remark 4.2(*iv*) and Theorem 4.3, respectively).

In Section 4.3, we use Theorem 4.1 to derive several EVPs for vector-valued bifunctions (see Theorems 4.4, 4.8 and 4.12). Let us also highlight the EVP derived in Corollary 4.7, which shows the roles of the diagonal null and $\leq_D$-t.i. properties in the main EVPs for bifunctions of the literature, reduces to [22, Theorem 1] and [140, Corollary 4.3] by considering additional assumptions, and also encompasses the single-valued versions of [96, Theorem 5.1] and [198, Theorem 3.1].

Theorem 4.8 focuses on EVPs for the particular case $E_q = q + D$, with $q \in D \backslash \{0\}$. The algebraic formulation of nonconvex separation functional (see Chapter 3) plays a key role in proving this result. From it, we obtained Corollary 4.9 that shows, as well as Corollary 4.7, the role of $\leq_D$-t.i. property in the main EVPs for bifunctions of the literature. It reduces to the single-valued version of [96, Theorem 5.2] without assuming the solidness of the ordering cone, and also to [45, Proposition 2.5] whenever the approximate solution of the scalar optimization problem is strict.

Theorem 4.12 is an exact EVP for vector-valued bifunction that reduces Corollary 4.14 to the EVPs given in [22, Theorem 2] and [9, Theorem 3.1], which require additional and stronger assumptions on the ordering cone and the bifunction.

In Section 4.4, we clarified the role of the $\leq_D$-t.i. property to prove existence results for weak efficient solutions of vector equilibrium problems. Roughly speaking, we proved that those points whose values are weak maximal points with respect to the first variable of the bifunction are also weak minimal points with respect to the second variable if the bifunction satisfies the $\leq_D$-t.i. property and is diagonal null. This fact provides a new point of view to obtain existence results in vector equilibrium problems.

Following this approach, and via the algebraic formulation of the nonconvex functional (see Chapter 3) we obtained a Weierstrass theorem for weak efficient solutions of vector equilibrium problems in an algebraic framework (Corollary 4.18), which generalizes and significantly improves some corresponding results of the literature (see [9, 22, 198]). In particular, some lower semicontinuity and boundedness assumptions have been removed, and only it is required the (q, D)-usc at one fixed point in the second variable of the bifunction (see Section 1.4).

As a result of this doctoral thesis, three works [97–99] were published in two high impact journals from the Journal Citation Reports (JCR). Chapter 3 is based on [99] and Chapter 4 on [97] (Sections 4.2 and 4.3) and [98] (Section 4.4).

5.2 Future lines of development

In this section, potential future lines of development that have emerged during the present work are enumerated. On the one hand, specific questions are proposed with the aim to continue and complete the results of this memory. On the other hand, other general related questions are posed with the goal to develop them in a more independent way.

Line 1. Zhao, Chen and Yang [200, Section 6] suggested to study nonlinear scalarization results for approximate proper efficient solutions of vector optimization problems. In fact, we saw in Chapter 2 that the bibliography about linear scalarization under convexity assumptions of approximate efficiency and approximate proper efficiency is wide, but the bibliography about nonlinear scalarization results of approximate proper efficiency in vector equilibria without assuming any convexity hypothesis is brief.

Gong [81] obtained in the topological framework some characterizations of several type of solutions of vector equilibrium problems through the nonconvex separation functional, in particular, for (exact) proper efficient solutions in the sense of Henig.

The algebraic formulation of the nonconvex separation functional and its properties presented in Chapter 3 may work very properly to characterize by scalarization the (C,ε)-Henig proper efficient solutions of equilibrium problems in the algebraic framework and without supposing any convexity condition.

Observe that Gong required some topological assumptions such as the solidness of the ordering cone and semicontinuity properties on the scalarizing functional, and we may replace the ordering cone with a generalized ordering set such as a coradiant set, which might be algebraic solid. Analogously, the semicontinuity assumption could be replaced with some semialgebraic counterpart, as the ones introduced in Section 1.4.

Line 2. Inspired in our paper [99], Qiu [162] has recently defined an extension of the nonconvex separation functional (see Section 3.2). To be precise, the next functional is introduced,

$$\varphi_E^Q(y) := \begin{cases} +\infty & \text{if } y \notin \mathbb{R}Q - E, \\ \inf\{t \in \mathbb{R} : y \in tQ - E\} & \text{otherwise,} \end{cases}$$

where the point $q \in Y\backslash\{0\}$ in (3.1) is replaced with a set $Q \subset Y\backslash\{0\}$.

Qiu has obtained important properties of this functional under certain assumptions and he has applied them to get EVPs for set-valued functionals, where the perturbation consists of a subset of the ordering cone multiplied by the distance function. Some of these results have been stated with respect to approximate efficient solutions in the sense of Németh [148] of set-valued vector optimization problems.

It could be possible to go deeper in the study of functional φ_E^Q and to obtain alternative properties or to improve the existing ones. Moreover, it may be applied to set-valued vector equilibrium problems (such as [10,12,82,161,198]) defined by an arbitrary ordering set in order to characterize by scalarization their exact and approximate weak efficient solutions and also to obtain EVPs, in such a way that the results given in Sections 3.2, 3.3 and 4.3 may be extended.

Line 3. Hiriart-Urruty [108, 109] introduced the well-known oriented distance function (also called signed distance function). This function $\Delta_A\colon Y \to \mathbb{R} \cup \{\pm\infty\}$ is defined by

$$\Delta_A(y) := d(y, A) - d(y, Y \backslash A) \quad \forall y \in Y,$$

where A is a subset of a normed space Y, $d(y, A) := \inf\{\|a - y\| \,:\, a \in A\}$ and $d(y, \emptyset) = +\infty$. It is known that Δ_A has very good properties such as Lipschitz continuity, convexity, positive homogeneity or monotonicity by depending on the properties of A (see [100, 101, 117, 143, 195] for further information). In fact, Zaffaroni [195] applied this functional to obtain characterizations for several types of solution of a vector optimization problem by means of a scalarization procedure, and Ha [100] also applied it to characterize the so-called Q-minimal points, which encompass many notions of efficiency and proper efficiency of vector optimization problems. The main advantage of the oriented distance function with respect to other nonlinear scalarization functions is that its properties do not require the solidness of the ordering set to be fulfilled.

The oriented distance function has been applied also to deal with approximate efficient solutions of single-valued and set-valued vector optimization problems (see [68, 69, 179]), and Zhao et al. [201] obtained some results about E-optimality in vector optimization problems, being E an improvement set. In contrast, the literature concerning with the oriented distance function applied to vector equilibrium problems in some way is brief to the best of our knowledge (see, for instance, [52, 171]). In Section 3.3 we characterized E-weak efficient solutions of vector equilibrium problems by means of the algebraic formulation of the nonconvex separation functional. Then, a new research line would be to characterize by means of the oriented distance function E-efficient solutions of vector equilibrium problems with respect to a possible nonsolid ordering set E.

Line 4. Gong [81] characterized the weak efficient solutions and the Henig proper efficient solutions of vector equilibrium problems by means of the nonconvex separation functional. By combining these characterizations with generalized differential calculus, he obtained necessary and sufficient stationary point conditions for weak efficient solutions and Henig proper efficient solutions of nonsmooth vector equilibrium problems. However, these results only may be applied when the ordering cone has nonempty topological interior, since this condition is essential to the nonconvex separation functional satisfies suitable properties.

The algebraic formulation of the nonconvex separation functional and its properties presented in Section 3.2 allows us to consider nonsolid ordering cones, so it may be applied to generalize Gong's results to other kind of solutions of vector equilibrium problems as Benson proper efficient solutions, whose definition does not require the solidness of the ordering set.

Line 5. By means of linear scalarization, necessary and sufficient Lagrangian conditions were obtained for weak efficient solutions, Henig proper efficient solutions and superefficient solutions of vector equilibrium problems (see [80, 144, 164]). In these papers, the feasible set is given by a cone constraint and satisfies the so-called Slater's constraint qualification, and the functions that define the problem fulfill certain generalized convexity conditions (see Section 1.4). On the other hand, several authors studied this kind of results for vector optimization problems via algebraic notions (see [4–6, 106, 107, 120, 157, 204–206]).

It would be interesting to obtain necessary and sufficient Lagrangian conditions for approximate proper efficient solutions of vector equilibrium problems in the algebraic framework (see Section 2.3) by assuming weaker generalized convexity assumptions and more general constraint qualifications (see, for instance, [67, 89, 197]).

Line 6. Ekeland Variational Principle is well known for its multiple applications. For instance, existence results for vector equilibrium problems and fixed-point theorems for set-valued mappings have been obtained from EVPs for bifunctions (see, for instance, [9, 11, 22, 150]). As a result, the EVPs obtained in Section 4.3 may be applied to generalize or improve that results.

Line 7. Theorem 4.1 is a kind of strict fixed point theorem for set-valued mappings and it is inspired in two previous results by Lin et al. [140, Theorem 3.1] and Qiu [159, Theorem 2.1]. As a consequence of these two results, the authors derived EVPs for set-valued optimization problems. Then, Theorem 4.1 may be also applied to obtain EVPs for set-valued bifunctions (see, for instance, [12, 82, 117, 161, 183, 198]).

Observe that this line is directly related to Line 2, since EVPs for set-valued bifunctions may be provided from Theorem 4.1 together with the functional φ_E^Q.

Line 8. Finet et al. [62], Berdnarczuk and Zagrodny [18], and Kruger et al. [125] extended the well-known Borwein-Preiss Variational Principle [31, 124, 137] to vector-valued functions. On the other hand, Plubtieng and Seangwattana [155] extended it to a system of equilibrium problems. Furthermore, Finet and Quarta [61] generalized the Deville-Godefroy-Zizler Variational Principle [50, 51] to vector equilibrium problems, and from this result, they derived an Ekeland Variational Principle [54, 55] and a Borwein-Preiss Variational Principle for vector equilibrium problems.

In their extension, Finet and Quarta [61] introduced a lower semicontinuity notion, the coordinate free lower semicontinuity, which is assumed on the bifunction, as well as other usual assumptions as the diagonal null condition and the triangle inequality property (see Section 1.4). Then, it would be interesting to study if the (q, H)-semicontinuity concept defined in Section 1.4 may replace the coordinate free lower semicontinuity and also to analyze the roles of both diagonal null and triangle inequality properties in a similar way as we did in Section 4.3 with the Ekeland Variational Principle for bifunctions. Furthermore, an approximate version of Deville-Godefroy-Zizler Variational Principle to vector equilibrium problems via arbitrary ordering sets such as free-disposal sets have not been studied yet.

Line 9. Bao and Mordukhovich [15, 16] obtained subdifferential versions of the Ekeland Variational Principle for approximate minimizers of set-valued functions defined on Asplund spaces. Moreover, Ha [101] formulated two versions of the Ekeland Variational Principle for Henig proper minimizers and super minimizers of set-valued functions involving coderivatives in the sense of Ioffe, Clarke and Mordukhovich.

On the other hand, in Section 2.3 the concepts of (C, ε)-Henig proper efficiency and (C, ε)-Benson proper efficiency were extended from vector optimization problems to vector equilibrium problems. Then, it would be interesting to derive subdifferential versions of the Ekeland Variational Principle for set-valued bifunctions and set-valued perturbations based on the previously mentioned coderivates.

Bibliography

[1] M. Adán, V. Novo, *Partial and generalized subconvexity in vector optimization problems*, J. Convex Anal., 8 (2001), pp. 583–594.

[2] M. Adán, V. Novo, *Optimality conditions for vector optimization problems with generalized convexity in real linear spaces*, Optimization, 51 (2002), pp. 73–91.

[3] M. Adán, V. Novo, *Efficient and weak efficient points in vector optimization with generalized cone convexity*, Appl. Math. Lett., 16 (2003), pp. 221–225.

[4] M. Adán, V. Novo, *Weak efficiency in vector optimization using a closure of algebraic type under cone-convexlikeness*, European J. Oper. Res., 149 (2003), pp. 641–653.

[5] M. Adán, V. Novo, *Proper efficiency in vector optimization on real linear spaces*, J. Optim. Theory Appl., 121 (2004), pp. 515–540.

[6] M. Adán, V. Novo, *Duality and saddle-points for convex-like vector optimization problems on real linear spaces*, TOP, 13 (2005), pp. 343–357.

[7] B. Alleche, V.D. Rădulescu, *Solutions and approximate solutions of quasi-equilibrium problems in Banach spaces*, J. Optim. Theory Appl., 170 (2016), pp. 629–649.

[8] M. Alonso, L. Rodríguez-Marín, *Set-relations and optimality conditions in set-valued maps*, Nonlinear Anal., 63 (2005), pp. 1167–1179.

[9] Q.H. Ansari, *Vectorial form of Ekeland-type variational principle with applications to vector equilibrium problems and fixed point theory*, J. Math. Anal. Appl., 334 (2007), pp. 561–575.

[10] Q.H. Ansari, W. Oettli, D. Schläger, *A generalization of vectorial equilibria*, Math. Methods Oper. Res., 46 (1997), pp. 147–152.

[11] Y. Araya, K. Kimura, T. Tanaka, *Existence of vector equilibria via Ekeland's variational principle*, Taiwanese J. Math., 12 (2008), pp. 1991–2000.

[12] J.S. Bae, S.H. Cho, *Fixed points and variational principle with applications to equilibrium problems on cone metric spaces*, J. Korean Math. Soc., 50 (2013), pp. 95–109.

[13] J. Bair, R. Fourneau, *Etude Géométrique des Espaces Vectoriels I. Une Introduction*, Springer, Berlin, 1975.

[14] J. Bair, R. Fourneau, *Etude Géométrique des Espaces Vectoriels II. Polyèdres et Polytopes Convexes*, Springer, Berlin, 1980.

[15] T.Q. Bao, B.S. Mordukhovich, *Variational principles for set-valued mappings with applications to multiobjective optimization*, Control Cybernet., 36 (2007), pp. 531–562.

[16] T.Q. Bao, B.S. Mordukhovich, *Relative Pareto minimizers for multiobjective problems: existence and optimality conditions*, Math. Program., 122 (2010), pp. 301–347.

[17] T.Q. Bao, B.S. Mordukhovich, *Extended Pareto optimality in multiobjective problems*, in Recent Developments in Vector Optimization (Q.H. Ansari, J.C. Yao eds.), Springer-Verlag, Berlin, 2012, pp. 467–515.

[18] E.M. Bednarczuk, D. Zagrodny, *A smooth vector variational principle*, SIAM J. Control Optim., 48 (2010), pp. 3735–3745.

[19] H.P. Benson, *An improved definition of proper efficiency for vector maximization with respect to cones*, J. Math. Anal. Appl., 71 (1979), pp. 232–241.

[20] M. Bianchi, N. Hadjisavvas, S. Schaible, *Vector equilibrium problems with generalized monotone bifunctions*, J. Optim. Theory Appl., 92 (1997), pp. 527–542.

[21] M. Bianchi, G. Kassay, R. Pini, *Existence of equilibria via Ekeland's principle*, J. Math. Anal. Appl., 305 (2005), pp. 502–512.

[22] M. Bianchi, G. Kassay, R. Pini, *Ekeland's principle for vector equilibrium problems*, Nonlinear Anal., 66 (2007), pp. 1454–1464.

[23] M. Bianchi, G. Kassay, R. Pini, *Well-posedness for vector equilibrium problems*, Math. Methods Oper. Res., 70 (2009), pp. 171–182.

[24] M. Bianchi, G. Kassay, R. Pini, *Well-posed equilibrium problems*, Nonlinear Anal., 72 (2010), pp. 460–468.

[25] M. Bianchi, I.V. Konnov, R. Pini, *Lexicographic and sequential equilibrium problems*, J. Global Optim., 46 (2010), pp. 551–560.

[26] E. Bishop, R.R. Phelps, *A proof that every Banach space is subreflexive*, Bull. Amer. Math. Soc., 67 (1961), pp. 97–98.

[27] E. Bishop, R.R. Phelps, *The support functionals of a convex set*, In Proc. Sympos. Pure Math. VII, Amer. Math. Soc. (V.L. Klee ed.), Providence, Rhode Island, 1963, pp. 27–35.

[28] E. Blum, W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student, 63 (1994), pp. 123–145.

[29] E. Borel, *La théorie du jeu et les équations intégrales à noyau symétrique gauche*, C. R. Math. Acad. Sci. Paris, 173 (1921), pp. 1304–1308.

[30] J.M. Borwein, *Proper efficient points for maximizations with respect to cones*, SIAM J. Control Optim., 15 (1977), pp. 57–63.

[31] J.M. Borwein, D. Preiss, *A smooth variational principle with applications to subdifferentiability and to differentiability of convex functions*, Trans. Amer. Math. Soc., 303 (1987), pp. 517–527.

[32] J.M. Borwein, D. Zhuang, *Super efficiency in vector optimization*, Trans. Am. Math. Soc. 338 (1993), pp. 105–122.

[33] G. Cantor, *Beiträge zur Begründung der transfiniten Mengenlehre*, Math. Ann., 49 (1897), pp. 207–246.

[34] A. Capătă, *Existence results for proper efficient solutions of vector equilibrium problems and applications*, J. Global Optim., 51 (2011), pp. 657–675.

[35] A. Capătă, *Existence results for globally efficient solutions of vector equilibrium problems via a generalized KKM principle*, Acta Math. Sci. Ser. B Engl. Ed., 37 (2017), pp. 463–476.

[36] M. Castellani, M. Giuli, *Approximate solutions of quasiequilibrium problems in Banach spaces*, J. Global Optim., 64 (2016), pp. 615–620.

[37] G.Y. Chen, *Existence of solutions for a vector variational inequality: an extension of the Hartmann-Stampacchia theorem*, J. Optim. Theory Appl., 74 (1992), pp. 445–456.

[38] G.Y. Chen, G.M. Cheng, *Vector variational inequality and vector optimization problem*, in Lecture Notes in Economics and Mathematical Systems 285, Springer, Berlin, 1987, pp. 408–416.

[39] G.Y. Chen, B.D. Craven, *A vector variational inequality and optimization over an efficient set*, Z. Oper. Res., 34 (1990), pp. 1–12.

[40] G.Y. Chen, X.X. Huang, X. Yang, *Vector Optimization. Set-Valued and Variational Analysis*, Lecture Notes in Economics and Mathematical Systems 541, Springer, Berlin, 2005.

[41] G.Y. Chen, S.J. Li, *Existence of solutions for generalized vector quasivariational inequality*, J. Optim. Theory Appl., 90 (1996), pp. 321–334.

[42] G.Y. Chen, X.Q. Yang, *The vector complementary problem and its equivalences with the weak minimal element in ordered spaces*, J. Math. Anal. Appl., 153 (1990), pp. 136–158.

[43] M. Chicco, F. Mignanego, L. Pusillo, S. Tijs, *Vector optimization problems via improvement sets*, J. Optim. Theory. Appl., 150 (2011), pp. 516–529.

[44] F.H. Clarke, *Optimization and Nonsmooth Analysis*, John Wiley & Sons, New York, 1983.

[45] C. Combari, S. Marcellin, L. Thibault, *On the graph convergence of ε-Fenchel subdifferentials of convex functions*, J. Nonlinear Convex Anal., 4 (2003), pp. 309–324.

[46] B.D. Craven, *Control and Optimization*, Chapman & Hall, London, 1995.

[47] G.P. Crespi, A. Guerraggio, M. Rocca, *Minty variational inequality and optimization: scalar and vector case*, in Generalized Convexity, Generalized Monotonicity and Applications, (A. Eberhard, N. Hadjisavvas, D.T. Luc eds.), Springer, Boston, Massachusetts, 2005, pp. 193–211.

[48] J. Daneš, *A geometric theorem useful in nonlinear functional analysis*, Boll. Un. Mat. Ital., 6 (1972), pp. 369–375.

[49] G. Debreu, *Theory of Value: An Axiomatic Analysis of Economic Equilibrium*, John Wiley & Sons, New York, 1959.

[50] R. Deville, G. Godefroy, V. Zizler, *Un principe variationnel utilisant des fonctions bosses*, C. R. Acad. Sci. Paris Sér. I Math., 312 (1991), pp. 281–286.

[51] R. Deville, G. Godefroy, V. Zizler, *A smooth variational principle with applications to Hamilton-Jacobi equations in infinite dimensions*, J. Funct. Anal., 111 (1993), pp. 197–212.

[52] M. Durea, J. Dutta, C. Tammer, *Lagrange multipliers for ε-Pareto solutions in vector optimization with nonsolid cones in Banach spaces*, J. Optim. Theory Appl., 145 (2010), pp. 196–211.

[53] F.Y. Edgeworth, *Mathematical Psychics. An Essay on the Application of Mathematics to the Moral Sciences*, C. Kegan Paul & Co, London, 1881.

[54] I. Ekeland, *Sur les problèmes variationnels*, C. R. Acad. Sci. Paris Sér. A-B, 275 (1972), pp. 1057–1059.

[55] I. Ekeland, *On the variational principle*, J. Math. Anal. Appl., 47 (1974), pp. 324–353.

[56] I. Ekeland, *Nonconvex minimization problems*, Bull. Amer. Math. Soc. (N.S.), 1 (1979), pp. 443–474.

[57] F. Facchinei, J.S. Pang, *Finite-Dimensional Variational Inequalities and Complementary Problems I*, Springer, New York, 2003.

[58] K. Fan, *A generalization of Tychonoff's fixed point theorem*, Math. Ann., 142 (1961), pp. 305–310.

[59] K. Fan, *A minimax inequality and applications*, in Inequalities III, Proc. Third Sympos., Univ. California, Los Angeles, Calif., 1969 (O. Shisha ed.), pp. 103–113, Academic Press, New York, 1972.

[60] A. Farajzadeh, B.S. Lee, S. Plubtieng, *On generalized quasi-vector equilibrium problems via scalarization method*, J. Optim. Theory Appl., 168 (2016), pp. 584–599.

[61] C. Finet, L. Quarta, *Vector-valued perturbed equilibrium problems*, J. Math. Anal. Appl., 343 (2008), pp. 531–545.

[62] C. Finet, L. Quarta, C. Troestler, *Vector-valued variational principles*, Nonlinear Anal., 52 (2003), pp. 197–218.

[63] F. Flores-Bazán, F. Flores-Bazán, S. Laengle, *Characterizing efficiency on infinite-dimensional commodity spaces with ordering cones having possibly empty interior*, J. Optim. Theory Appl., 164 (2015), pp. 455–478.

[64] F. Flores-Bazán, E. Hernández, *A unified vector optimization problem: complete scalarizations and applications*, Optimization, 60 (2011), pp. 1399–1419.

[65] F. Flores-Bazán, E. Hernández, *Optimality conditions for a unified vector optimization problem with not necessarily preordering relations*, J. Global Optim., 56 (2013), pp. 299–315.

[66] F. Flores-Bazán, B. Jiménez, *Strict efficiency in set-valued optimization*, SIAM J. Control Optim., 48 (2009), pp. 881–908.

[67] J.B.G. Frenk, G. Kassay, *On classes of generalized convex functions, Gordan-Farkas type theorems, and Lagrangian duality*, J. Optim. Theory Appl., 102 (1999), pp. 315–343.

[68] Y. Gao, S.H. Hou, X.M. Yang, *Existence and optimality conditions for approximate solutions to vector optimization problems*, J. Optim. Theory Appl., 152 (2012), pp. 97–120.

[69] Y. Gao, X.M. Yang, K.L. Teo, *Optimality conditions for approximate solutions of vector optimization problems*, J. Ind. Manag. Optim., 7 (2011), pp. 483–496.

[70] Y. Gao, X.M. Yang, J. Yang, H. Yan, *Scalarizations and Lagrange multipliers for approximate solutions in the vector optimization problems with set-valued maps*, J. Ind. Manag. Optim., 11 (2015), pp. 673–683.

[71] A.M. Geoffrion, *Proper efficiency and the theory of vector maximization*, J. Math. Anal. Appl., 22 (1968), pp. 618–630.

[72] C. Gerstewitz, *Nichtkonvexe dualität in der vektoroptimierung*, Wiss. Z. Tech. Hochsch. Leuna-Merseburg, 25 (1983), pp. 357–364.

[73] C. Gerstewitz, E. Iwanow, *Dualität für nichtkonvexe vektoroptimierungsprobleme*, Wiss. Z. Tech. Hochsch. Ilmenau, 31 (1985), pp. 61–81.

[74] C. Gerth, P. Weidner, *Nonconvex separation theorems and some applications in vector optimization*, J. Optim. Theory Appl., 67 (1990), pp. 297–320.

[75] F. Giannessi, *Theorems of alternative, quadratic programs and complementarity problems*, in Variational Inequalities and Complementarity Problems (R.W. Cottle, F. Giannessi, J.L. Lions eds.), John Wiley & Sons, Chichester, 1980, pp. 151–186.

[76] F. Giannessi (ed.), *Vector Variational Inequalities and Vector Equilibria. Mathematical Theories*, Nonconvex Optimization and its Applications 38, Kluwer Academic Publishers, Dordrecht, 2000.

[77] X.H. Gong, *Efficiency and Henig efficiency for vector equilibrium problems*, J. Optim. Theory Appl., 108 (2001), pp. 139–154.

[78] X.H. Gong, *Optimality conditions for Henig and globally proper efficient solutions with ordering cone has empty interior*, J. Math. Anal. Appl., 307 (2005), pp. 12–31.

[79] X.H. Gong, *Connectedness of the solution sets and scalarization for vector equilibrium problems*, J. Optim. Theory Appl., 133 (2007), pp. 151–161.

[80] X.H. Gong, *Optimality conditions for vector equilibrium problems*, J. Math. Anal. Appl., 342 (2008), pp. 1455–1466.

[81] X.H. Gong, *Scalarization and optimality conditions for vector equilibrium problems*, Nonlinear Anal., 73 (2010), pp. 3598–3612.

[82] X.H. Gong, *Ekeland's principle for set-valued vector equilibrium problems*, Acta Math. Sci. Ser. B Engl. Ed., 34 (2014), pp. 1179–1192.

[83] X.H. Gong, W.T. Fu, W. Liu, *Super efficiency for a vector equilibrium in locally convex topological vector spaces*, in Vector Variational Inequalities and Vector Equilibria: Mathematical Theories (F. Giannessi ed.). Nonconvex Optimization and Its Applications 38, Kluwer, Dordrecht, 2000, pp. 233–252.

[84] A. Göpfert, H. Rihai, C. Tammer, C. Zălinescu, *Variational Methods in Partially Ordered Spaces*, Springer, Berlin, 2003.

[85] A. Göpfert, C. Tammer, C. Zălinescu, *On the vectorial Ekeland's variational principle and minimal points in product spaces*, Nonlinear Anal., 39 (2000), pp. 909–922.

[86] A. Guerraggio, E. Molho, A. Zaffaroni, *On the notion of proper efficiency in vector optimization*, J. Optim. Theory Appl. 82 (1994), pp. 1–21.

[87] C. Gutiérrez, L. Huerga, B. Jiménez, V. Novo, *Proper approximate solutions and ε-subdifferentials in vector optimization: Basic properties and limit behaviour*, Nonlinear Anal., 79 (2013), pp. 52–67.

[88] C. Gutiérrez, L. Huerga, B. Jiménez, V. Novo, *Henig approximate proper efficiency and optimization problems with difference of vector mappings*, J. Convex Anal., 23 (2016), pp. 661–690.

[89] C. Gutiérrez, L. Huerga, B. Jiménez, V. Novo, *Approximate solutions of vector optimization problems via improvement sets in real linear spaces*, J. Global Optim., online (2017), DOI 10.1007/s10898-017- 0593-y.

[90] C. Gutiérrez, L. Huerga, V. Novo, *Scalarization and saddle points of approximate proper solutions in nearly subconvexlike vector optimization problems*, J. Math. Anal. Appl., 389 (2012), pp. 1046–1058.

[91] C. Gutiérrez, B. Jiménez, V. Novo, *On approximate efficiency in multiobjective programming*, Math. Methods Oper. Res., 64 (2006), pp. 165–185.

[92] C. Gutiérrez, B. Jiménez, V. Novo, *A unified approach and optimality conditions for approximate solutions of vector optimization problems*, SIAM J. Optim., 17 (2006), pp. 688–710.

[93] C. Gutiérrez, B. Jiménez, V. Novo, *A set-valued Ekeland's variational principle in vector optimization*, SIAM J. Control Optim., 47 (2008), pp. 883–903.

[94] C. Gutiérrez, B. Jiménez, V. Novo, *A generic approach to approximate efficiency and applications to vector optimization with set-valued maps*, J. Global Optim., 49 (2011), pp. 313–342.

[95] C. Gutiérrez, B. Jiménez, V. Novo, *Improvement sets and vector optimization*, European J. Oper. Res., 223 (2012), pp. 304–311.

[96] C. Gutiérrez, B. Jiménez, V. Novo, L. Thibault, *Strict approximate solutions in set-valued optimization with applications to the approximate Ekeland variational principle*, Nonlinear Anal., 73 (2010), pp. 3842–3855.

[97] C. Gutiérrez, G. Kassay, V. Novo, J.L. Ródenas-Pedregosa, *Ekeland variational principles in vector equilibrium problems*, SIAM J. Optim., 27 (2017), pp. 2405–2425.

[98] C. Gutiérrez, V. Novo, J.L. Ródenas-Pedregosa, *A note on existence of weak efficient solutions for vector equilibrium problems*, Optim. Lett., online (2018), DOI 10.1007/s11590-018-1242-1.

[99] C. Gutiérrez, V. Novo, J.L. Ródenas-Pedregosa, T. Tanaka, *Nonconvex separation functional in linear spaces with applications to vector equilibria*, SIAM J. Optim., 26 (2016), pp. 2677–2695.

[100] T.X.D. Ha, *Optimality conditions for several types of efficient solutions of set-valued optimization problems*, in Nonlinear Analysis and Variational Problems, Springer Optimization and Its Applications 35, pp. 305–324. Springer, New York, 2010.

[101] T.X.D. HA, *The Ekeland variational principle for Henig proper minimizers and super minimizers*, J. Math. Anal. Appl., 364 (2010), pp. 156–170.

[102] R. HARTLEY, *On cone-efficiency, cone-convexity, and cone-compactness*, SIAM J. Appl. Math., 34 (1978), pp. 211–222.

[103] F. HAUSDORFF, *Untersuchungen über Ordnungstypen*, Berichte über die Verhandlungen der Königlich-Sächsischen Gesellschaft der Wissenschaften zu Leipzig. Mathematisch-Physische Klasse 58 (1906), pp. 106–169.

[104] S. HELBIG, *On a new concept for ε-efficiency*, talk at "Optimization Days 1992", Montreal, 1992.

[105] M.I. HENIG, *Proper efficiency with respect to cones*, J. Optim. Theory Appl., 36 (1982), pp. 387–407.

[106] E. HERNÁNDEZ, B. JIMÉNEZ, V. NOVO, *Benson proper efficiency in set-valued optimization on real linear spaces*, in Lecture Notes in Economics and Mathematical Systems 563, Springer, Berlin, 2006, pp. 45–59.

[107] E. HERNÁNDEZ, B. JIMÉNEZ, V. NOVO, *Weak and proper efficiency in set-valued optimization on real linear spaces*, J. Convex Anal., 14 (2007), pp. 275–296.

[108] J.B. HIRIART-URRUTY, *New concepts in nondifferentiable programming*, Bull. Soc. Math. France, 60 (1979), pp. 57–85.

[109] J.B. HIRIART-URRUTY, *Tangent cones, generalized gradients and mathematical programming in Banach spaces*, Math. Oper. Res., 4 (1979), pp. 79–97.

[110] R.B. HOLMES, *Geometric Functional Analysis and its Applications*, Springer, Berlin, 1975.

[111] L. HURWICZ, *Programming in linear spaces*, in Studies in Linear and Non-Linear Programming (K.J. Arrow, L. Hurwicz, H. Uzawa eds.), Stanford University press, Stanford, California, 1958.

[112] J. Jahn, *Vector Optimization. Theory, Applications, and Extensions*, Springer, Berlin, 2011.

[113] B. Jiménez, *Strict efficiency in vector optimization*, J. Math. Anal. Appl., 265 (2002), pp. 264–284.

[114] R. John, *Variational inequalities and pseudomonotone functions: some characterizations*, in Generalized Convexity, Generalized Monotonicity: Recent Results (J.P. Crouzeix, J.E. Martínez-Legaz, M. Volle eds.), Kluwer, Dordrecht, 1998, pp. 291–301.

[115] E.M. Kalmoun, H. Riahi, *Topological KKM theorems and generalized vector equilibria on G-convex spaces with applications*, Proc. Amer. Math. Soc., 129 (2001), pp. 1335–1348.

[116] P. Kas, G. Kassay, Z.B. Sensoy, *On generalized equilibrium points*, J. Math. Anal. Appl., 296 (2004), pp. 619–633.

[117] A.A. Khan, C. Tammer, C. Zălinescu, *Set-valued Optimization. An Introduction with Applications*, Springer, Berlin, 2015.

[118] P.Q. Khanh, D.N. Quy, *A generalized distance and enhanced Ekeland's variational principle for vector functions*, Nonlinear Anal., 73 (2010), pp. 2245–2259.

[119] W.A. Kirk, J. Caristi, *Mappings theorems in metric and Banach spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., 23 (1975), pp. 891–894.

[120] E. Kiyani, M. Soleimani-damaneh, *Approximate proper efficiency on real linear vector spaces*, Pac. J. Optim., 10 (2014), pp. 715–734.

[121] B. Knaster, C. Kuratowski, S. Mazurkiewicz, *Ein beweis des fixpunktsatzes für n-dimensionale simplexe*, Fund. Math., 14 (1929), pp. 132–137.

[122] I.V. KONNOV, *Equilibrium Models and Variational Inequalities*, Elsevier, Amsterdam, 2007.

[123] T.C. KOOPMANS, *Analysis of production as an efficient combination of activities*, in Activity Analysis of Production and Allocation (T.C. Koopmans ed.), John Wiley & Sons, New York, 1951, pp. 33–97.

[124] A.Y. KRUGER, S. PLUBTIENG, T. SEANGWATTANA, *Borwein-Preiss variational principle revisited*, J. Math. Anal. Appl., 435 (2016), pp. 1183–1193.

[125] A.Y. KRUGER, S. PLUBTIENG, T. SEANGWATTANA, *Borwein-Preiss vector variational principle*, Positivity, 21 (2017), pp. 1273–1292.

[126] H.W. KUHN, A.W. TUCKER, *Nonlinear programming*, in Proc. of the Second Berkeley Symposium on Mathematical Statistics and Probability (J. Neyman ed.), pp. 481–492, University of California Press, Berkeley, California, 1951.

[127] S.S. KUTATELADZE, *Convex ε-programming*, Soviet Math. Dokl., 20 (1979), pp. 391–393.

[128] D. LA TORRE, N. POPOVICI, M. ROCCA, *Scalar characterization of explicitly quaxiconvex set-valued maps*, Working paper 2008-01, Department of Economics, Management and Quantitative Methods at Università degli Studi di Milano, Milano, (2008): `http://wp.demm.unimi.it/tl_files/wp/2008/DEMM-2008_001wp.pdf`

[129] D. LA TORRE, N. POPOVICI, M. ROCCA, *Scalar characterizations of weakly cone-convex and weakly cone-quasiconvex functions*, Nonlinear Anal., 72 (2010), pp. 1909–1915.

[130] D. LA TORRE, N. POPOVICI, M. ROCCA, *A note on explicitly quasiconvex set-valued maps*, J. Nonlinear Convex Anal., 12 (2011), pp. 113–118.

[131] T.C. LAI, J.C. YAO, *Existence results for VVIP*, Appl. Math. Lett., 9 (1996), pp. 17–19.

[132] C.S. Lalitha, P. Chatterjee, *Stability and scalarization in vector optimization using improvement sets*, J. Optim. Theory Appl., 166 (2015), pp. 825–843.

[133] G.M. Lee, D.S. Kim, B.S. Lee, *Generalized vector variational inequality*, Appl. Math. Lett., 9 (1996), pp. 39–42.

[134] G.M. Lee, D.S. Kim, B.S. Lee, S.J. Cho, *Generalized vector variational inequality and fuzzy extension*, Appl. Math. Lett., 6 (1993), pp. 47–51.

[135] G.M. Lee, D.S. Kim, B.S. Lee, N.D. Yen, *Vector variational inequality as a tool for studying vector optimization problems*, Nonlinear Anal., 34 (1998), pp. 745–765.

[136] G.M. Lee, B.S. Lee, S.S. Chang, *On vector quasivariational inequalities*, J. Math. Anal. Appl., 203 (1996), pp. 626–638.

[137] Y.X. Li, S.Z. Shi, *A generalization of Ekeland's ε-variational principle and its Borwein-Preiss smooth variant*, J. Math. Anal. Appl., 246 (2000), pp. 308–319.

[138] Z. Li, S. Wang, *ε-approximate solutions in multiobjective optimization*, Optimization, 44 (1998), pp. 161–174.

[139] L.J. Lin, *Pre-vector variational inequalities*, Bull. Aust. Math. Soc., 53 (1996), pp. 63–70.

[140] L.J. Lin, M. Balaj, Y.C. Ye, *Quasi-variational relation problems and generalized Ekeland's variational principle with applications*, Optimization, 63 (2014), pp. 1353–1365.

[141] J.C. Liu, *ε-properly efficient solutions to nondifferentiable multiobjective programming problems*, Appl. Math. Lett., 12 (1999), pp. 109–113.

[142] C.G. Liu, K.F. Ng, *Ekeland's variational principle for set-valued functions*, SIAM J. Optim., 21 (2011), pp. 41–56.

[143] C.G. Liu, K.F. Ng, W.H. Yang, *Merit functions in vector optimization*, Math. Program. Ser. A, 119 (2009), pp. 215–237.

[144] X.J. Long, Y.Q. Huang, Z.Y. Peng, *Optimality conditions for the Henig efficient solution of vector equilibrium problems with constraints*, Optim. Lett., 5 (2011), pp. 717–728.

[145] D.T. Luc, *Theory of Vector Optimization*, in Lecture Notes in Economics and Mathematical Systems 319, Springer, Berlin, 1989.

[146] D.G. Luenberger, *New optimality principles for economic efficiency and equilibrium*, J. Optim. Theory Appl., 75 (1992), pp. 221–264.

[147] L.D. Muu, W. Oettli, *Convergence of an adaptive penalty scheme for finding constrained equilibria*, Nonlinear Anal., 18 (1992), pp. 1159–1166.

[148] A.B. Németh, *A nonconvex vector minimization problem*, Nonlinear Anal., 10 (1986), pp. 669–678.

[149] W. Oettli, *A remark on vector-valued equilibria and generalized monotonicity*, Acta Math. Vietnam., 22 (1997), pp. 213–221.

[150] W. Oettli, M. Théra, *Equivalents of Ekeland's principle*, Bull. Aust. Math. Soc., 48 (1993), pp. 385–392.

[151] V. Pareto, *Manuale di Economia Politica*, Società Editrice Libraria, Milano, 1906.

[152] A. Pascoletti, P. Serafini, *Scalarizing vector optimization problems*, J. Optim. Theory Appl., 42 (1984), pp. 499–524.

[153] J.P. Penot, *The drop theorem, the petal theorem and Ekeland's variational principle*, Nonlinear Anal., 10 (1986), pp. 813–822.

[154] J.P. Penot, M., Thera, *Semi-continuous mappings in general topology*, Arch. Math., 38 (1982), pp. 158–166.

[155] S. Plubtieng, T. Seangwattana, *The Borwein-Preiss variational principle for nonconvex countable systems of equilibrium problems*, J. Nonlinear Sci. Appl., 9 (2016), pp. 2224–2232

[156] N. Popovici, *Explicitly quasiconvex set-valued optimization*, J. Global Optim., 38 (2007), pp. 103–118.

[157] J.H. Qiu, *Dual characterization and scalarization for Benson proper efficiency*, SIAM J. Optim., 19 (2008), pp. 144–162.

[158] J.H. Qiu, *Set-valued quasi-metrics and a general Ekeland's variational principle in vector optimization*, SIAM J. Control Optim., 51 (2013), pp. 1350–1371.

[159] J.H. Qiu, *A pre-order principle and set-valued Ekeland variational principle*, J. Math. Anal. Appl., 419 (2014), pp. 904–937.

[160] J.H. Qiu, *An equilibrium version of vectorial Ekeland variational principle and its applications to equilibrium problems*, Nonlinear Anal. Real World Appl. 27 (2016), pp. 26–42.

[161] J.H. Qiu, *An equilibrium version of set-valued Ekeland variational principle and its applications to set-valued vector equilibrium problems*, Acta Math. Sin. (Engl. Ser.), 33 (2017), pp. 210–234.

[162] J.H. Qiu, *Ekeland variational principles with set-valued objective functions and set-valued perturbations*, preprint, (2017): `http://arxiv.org/abs/1708.05126v1`

[163] J.H. Qiu, F. He, *A general vectorial Ekeland's variational principle with a P-distance*, Acta Math. Sin. (Engl. Ser.), 29 (2013), pp. 1655–1678.

[164] Q.S. Qiu, *Optimality conditions for vector equilibrium problems with constraints*, J. Ind. Manag. Optim., 5 (2009), pp. 783–790.

[165] Q.S. Qiu, X.M. Yang, *Scalarization of approximate solution for vector equilibrium problems*, J. Ind. Manag. Optim., 9 (2013), pp. 143–151.

[166] S. Rolewicz, *On drop property*, Studia Math., 85 (1986), pp. 27–35.

[167] W.D. Rong, *Proper ε-efficiency in vector optimization problems with cone-subconvexlikeness*, Acta Sci. Natur. Univ. NeiMongol, 28 (1997), pp. 609–613.

[168] A.M. Rubinov, *Sublinear operators and their applications*, Russian Math. Surveys, 32 (1977), pp. 115–175.

[169] P.H. Sach, *Henig proper generalized vector quasiequilibrium problems*, Optim. Lett., 7 (2013), pp. 173–184.

[170] P.H. Sach, N.B. Minh, *New results on Henig proper generalized vector quasiequilibrium problems*, Numer. Funct. Anal. Optim. 36 (2015), 387–403.

[171] P.H. Sach, L.A. Tuan, *Lower semicontinuity results in parametric multivalued weak vector equilibrium problems and applications*, Numer. Funct. Anal. Optim. 37 (2016), 753–785.

[172] Y. Sawaragi, H. Nakayama, T. Tanino, *Theory of multiobjective optimization.* Academic Press, Orlando, Florida, 1985.

[173] A. Sayadi-bander, L. Pourkarimi, R. Kasimbeyli, H. Basirzadeh, *Coradiant sets and ε-efficiency in multiobjective optimization*, J. Global Optim., 68 (2017), pp. 587–600.

[174] A.H. Siddiqi, Q.H. Ansari, A. Khaliq, *On vector variational inequalities*, J. Optim. Theory Appl., 84 (1995), pp. 171–180.

[175] S. Smale, *Global analysis and economics, V. Pareto theory with constraints*, J. Math. Econom., 1 (1974), pp. 213–221.

[176] B. Soleimani, *Characterization of approximate solutions of vector optimization problems with a variable order structure*, J. Optim. Theory Appl., 162 (2014), pp. 605–632.

[177] C. Stegall, *Optimization of functions on certain subsets of Banach spaces*, Math. Ann., 236 (1978), pp. 171–176.

[178] F. Sullivan, *A characterization of complete metric spaces*, Proc. Amer. Math. Soc., 83 (1981), pp. 345–346.

[179] A. Taa, *ε-Subdifferentials of set-valued maps and ε-weak Pareto optimality for multiobjective optimization*, Math. Methods Oper. Res., 62 (2005), pp. 187–209.

[180] W. Takahashi, *Existence theorems generalizing fixed point theorems for multivalued mappings*, in Fixed Point Theory and Applications (J.B. Baillon, M. Théra eds.), Pitman Research Notes in Mathematics 252, Longman (Harlow), 1991, pp. 397–406.

[181] C. Tammer, *A generalization of Ekeland's variational principle*, Optimization, 25 (1992), pp. 129–141.

[182] C. Tammer, C. Zălinescu, *Lipschitz properties of the scalarization function and applications*, Optimization, 59 (2010), pp. 305–319.

[183] C. Tammer, C. Zălinescu, *Vector variational principles for set-valued functions*, Optimization, 60 (2011), pp. 839–857.

[184] T. Tanaka, *A new approach to approximation of solutions in vector optimization problems*, in Proc. of APORS (M. Fushimi, K. Tone eds.), World Scientific Publishing, Singapore, 1995, pp. 497–504.

[185] I. Vályi, *Approximate solutions of vector optimization problems*, in Systems Analysis and Simulation (A. Sydow, M. Thoma, R.Vichnevetsky eds.) Akademie-Verlag, Berlin, 1985, pp. 246–250.

[186] I. Vályi, *Approximate saddle-point theorems in vector optimization*, J. Optim. Theory Appl., 55 (1987), pp. 435–448.

[187] J. von Neumann, *Zur theorie der gesellschaftsspiele*, Math. Ann., 100 (1928), pp. 295–320.

[188] J. von Neumann, O. Morgenstern, *Theory of Games and Economic Behavior*, Princeton University Press, Princeton, 1943.

[189] D.J. White, *Epsilon efficiency*, J. Optim. Theory Appl., 49 (1986), pp. 319–337.

[190] Y.M. Xia, W.L. Zhang, K.Q. Zhao, *Characterizations of improvement sets via quasi interior and applications in vector optimization*, Optim. Lett., 10 (2016), pp. 769–780.

[191] X.Q. Yang, *Vector variational inequality and its duality*, Nonlinear Anal., 21 (1993), pp. 869–877.

[192] X.M. Yang, D. Li, S.Y. Wang, *Near-subconvexlikeness in vector optimization with set-valued functions*, J. Optim. Theory Appl., 110 (2001), pp. 413–427.

[193] P.L. Yu, *Cone convexity, cone extreme points and nondominated solutions in decision problems with multiple objectives*, J. Optim. Theory Appl., 14 (1974), pp. 319–377.

[194] S.J. Yu, J.C. Yao, *On vector variational inequalities*, J. Optim. Theory Appl., 89 (1996), pp. 749–769.

[195] A. Zaffaroni, *Degrees of efficiency and degrees of minimality*, SIAM J. Control Optim., 42 (2003), pp. 1071–1086.

[196] A. Zaffaroni, *Superlinear separation for radiant and coradiant sets*, Optimization, 56 (2007), pp. 267–285.

[197] C. Zălinescu, *Convex Analysis in General Vector Spaces*, World Scientific, New Jersey, 2002.

[198] J. Zeng, S.J. Li, *An Ekeland's variational principle for set-valued mappings with applications*, J. Comput. Appl. Math., 230 (2009), pp. 477–484.

[199] J. Zhang, J.H. Qiu, *Vectorial variational principle with variable set-valued perturbation*, Acta Math. Sin. (Engl. Ser.), 31 (2015), pp. 595–614.

[200] K.Q. Zhao, G.Y. Chen, X.M. Yang, *Approximate proper efficiency in vector optimization*, Optimization, 64 (2015), pp. 1777–1793.

[201] K.Q. Zhao, Y.M. Xia, X.M. Yang, *Nonlinear scalarization characterizations of E-efficiency in vector optimization*, Taiwanese J. Math., 19 (2015), pp. 455–466.

[202] Z.A. Zhou, J.W. Peng, *Scalarization of set-valued optimization problems with generalized cone subconvexlikeness in real ordered linear spaces*, J. Optim. Theory Appl., 154 (2012), pp. 830–841.

[203] Z.A. Zhou, X.M. Yang, *Scalarization of ε-super efficient solutions of set-valued optimization problems in real ordered linear spaces*, J. Optim. Theory Appl., 162 (2014), pp. 680–693.

[204] Z.A. Zhou, X.M. Yang, J.W. Peng, *ε-Henig proper efficiency of set-valued optimization problems in real ordered linear spaces*, Optim. Lett., 8 (2014), pp. 1813–1827.

[205] Z.A. Zhou, X.M. Yang, J.W. Peng, *ε-Optimality conditions of vector optimization problems with set-valued maps based on the algebraic interior in real linear spaces*, Optim. Lett., 8 (2014), pp. 1047–1061.

[206] Z.A. Zhou, X.M. Yang, J.W. Peng, *Optimality conditions of set-valued optimization problem involving relative algebraic interior in ordered linear spaces*, Optimization, 63 (2014), pp. 433–446.

Symbols list