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Singular integrals and rectifiability

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Certifico que aquesta memòria ha
estat realitzada per Daniel Girela
Sarrión i dirigida per mi.

Barcelona, juny de 2.016,

Xavier Tolsa i Domènech

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Contents

Introduction

The problems addressed in this dissertation live in the intersection between Harmonic Analysis and Geometric Measure Theory, and so one should say that they belong to the area of Geometric Analysis. Precisely, we have analyzed relationships between singular integral operators such as the Riesz transform with respect to general Borel measures in the Euclidean space, and metric or geometric properties of those measures or their supports.

In the next few pages we summarize the workflow we have followed in the development of this dissertation and the results we have obtained, as well as some definitions of the concepts that are needed to understand these results. The rest of pertinent definitions, auxiliary results and proofs will be found in the next chapters.

We wish to remark, as well, that the results in Chapter 1 can be found at [G1], the ones in Chapter 2 can be found at [G2] and the ones in Chapter 3 can be found at [GT], which is a collaboration with Tolsa. This does not mean that Chapter 1 and Chapter 2 have been developed independently by the author of this dissertation, as all the work presented here has been done under the guidance of Professor Tolsa.

Some definitions

A measurable function k defined in $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d \colon x = y\}$ is an n-dimensional Calderón-Zygmund kernel if there are constants $c > 0$ and $0 < \delta \leq 1$ such that

$$|k(x,y)| \leq \frac{c}{|x-y|^n} \quad \text{if} \quad x \neq y$$

and

$$|k(x,y) - k(x',y)| + |k(y,x) - k(y,x')| \leq c\frac{|x-x'|^\delta}{|x-y|^{n+\delta}} \quad \text{if} \quad |x-x'| \leq \frac{|x-y|}{2}.$$

Given a signed Radon measure ν in $\mathbb{R}^d$ and $x \in \mathbb{R}^d$, we define (at least, *formally*)

$$T\nu(x) = \int k(x,y)d\nu(y), \quad x \in \mathbb{R}^d \setminus \operatorname{supp}(\nu)$$

and we say that T is a singular integral operator with kernel k. Associated with it, one defines the truncated operators T_ε by

$$T_\varepsilon\nu(x) = \int_{|x-y|>\varepsilon} k(x,y)d\nu(y), \; x \in \mathbb{R}^d$$

for all $\varepsilon > 0$, and the maximal operator T_* by

$$T_*\nu(x) = \sup_{\varepsilon>0} |T_\varepsilon\nu(x)|, \; x \in \mathbb{R}^d.$$

If μ is a fixed positive Radon measure in $\mathbb{R}^d$ and $f \in L^1_{loc}(\mu)$, we set

$$T_\mu f = T(f\mu), \; T_{\mu,\varepsilon} f = T_\varepsilon(f\mu), \; T_{\mu,*} f = T_*(f\mu),$$

although sometimes we will omit the underlying measure μ in the subscript when there is no room for confusion.

We say that T_μ is bounded in $L^2(\mu)$ if there is a constant $C > 0$ such that, for all $\varepsilon > 0$, $||T_{\mu,\varepsilon} f||_{L^2(\mu)} \leq C||f||_{L^2(\mu)}$ for all f in $L^2(\mu)$ and, in such a case, we say that T is a Calderón-Zygmund operator. The norm of T_μ is the infimum of all those constants C.

Some important examples of this class of operators are:

- The Hilbert transform, which is defined for functions $f \in L^2(\mathbb{R})$ by

$$Hf(x) = \frac{1}{\pi}\text{p.v.} \int \frac{f(y)}{x-y} dy.$$

- The Beurling transform, which is defined for funcions $f \in L^2(\mathbb{C})$ by

$$Bf(z) = -\frac{1}{\pi^2}\text{p.v.} \int \frac{f(w)}{(w-z)^2} dw.$$

- The n-dimensional Riesz transform, which is defined for signed Radon measures ν in $\mathbb{R}^d$, at least *formally*, by

$$\mathcal{R}\nu(x) = \int \frac{x-y}{|x-y|^{n+1}} d\nu(y).$$

- The Cauchy transform, which is defined for Radon measures ν in $\mathbb{C}$, at least *formally*, by

$$\mathcal{C}\nu(z) = \int \frac{d\nu(\zeta)}{\zeta - z}.$$

Chapter 1: The Cauchy transform along a Lipschitz curve: an improvement of Cotlar's inequality and some counterexamples

In the papers [MV], [MOV] and [MOPV], Mateu, Orobitg, Pérez and Verdera showed that for certain Calderón-Zygmund operators T (in $\mathbb{R}^n$ and with respect to Lebesgue measure), the classical Cotlar's inequality

$$T_* f \lesssim M(Tf) + Mf$$

could be improved in such a way that the maximal singular integral $T_* f$ would be controlled only by the singular integral Tf. Here, M stands for the Hardy-Littlewood maximal operator, which is defined for $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy.$$

Precisely, for the Beurling transform, one has

$$B_* f \lesssim M(Bf)$$

for all $f \in L^2(\mathbb{C})$, while for the Hilbert transform, one has

$$H_* f \lesssim M^2(Hf)$$

for all $f \in L^2(\mathbb{R})$. As Verdera points out in [MV], being able to establish this type of control for other operators (say, for example, Riesz transforms with respect to general measures) could be a useful tool towards solving David-Semmes conjecture, which states that the boundedness of Riesz transforms characterizes uniform rectifiability.

In that direction, the natural first step would be to study whether an inequality like the ones above is satisfied by the Cauchy transform $\mathcal{C}$ along a Lipschitz curve Γ, since it is, modulo conjugation, the one-dimensional Riesz transform with respect to $\mathcal{H}^1\lfloor_\Gamma$ (which stands for arc-length measure along Γ) in the plane and it coincides with a constant multiple of the Hilbert transform when Γ is a straight line. However, we prove that, in general, this is not the case when Γ is the graph of a Lipschitz function.

Theorem. *Consider the Lipschitz function $A(x) = |x|$, and let $\mathcal{C}$ denote the Cauchy transform along Γ, the graph of A. Then, there exists $f \in L^2(\mathbb{R})$ such that for all $c > 0$ and all $n \geq 1$, there exists $\varepsilon > 0$ such that*

$$|\mathcal{C}_\varepsilon f(0)| > cM^n(\mathcal{C}f)(0).$$

An easy generalization of this result states that the inequality $\mathcal{C}_* f \lesssim M^n(\mathcal{C}f)$ will fail for every $n \geq 1$ at all points where Γ has an *angle*.

Our second result shows that the failure of the inequality $\mathcal{C}_* f \lesssim M^n(\mathcal{C}f)$ is not only caused by the *non-smoothness* of Γ, since, when Γ is the graph of a Lipschitz function of compact support A, it can only hold true if $A \equiv 0$, that is, if Γ is a straight line.

Theorem. *Let A be a Lipschitz function with compact support, and let $\mathcal{C}$ denote the Cauchy transform along Γ, the graph of A. Suppose A is not identically null, or, equivalently, that Γ is not a straight line. Then, there exists $x \in \mathbb{R}$ such that for all $c > 0$ there exists $f \in L^2(\mathbb{R})$ with*

$$\mathcal{C}_* f(x) > cM^n(\mathcal{C}f)(x)$$

for all $n \geq 1$.

Finally, we prove that when Γ is a sufficiently smooth Jordan curve (say, $\mathcal{C}^{1+\varepsilon}$), we have $\mathcal{C}_* f \lesssim M^2(\mathcal{C}f)$ for all $f \in L^2(\mathcal{H}^1\lfloor_\Gamma)$.

Chapter 2: Geometric conditions for the L^2-boundedness of singular integral operators with odd kernels with respect to measures with polynomial growth in $\mathbb{R}^d$

In the paper [T3], Tolsa proved that the $L^2(\mu)$-boundedness of the Cauchy transform with respect to a Radon measure μ in $\mathbb{C}$ is a sufficient condition for the $L^2(\mu)$-boundedness of all odd and sufficiently smooth 1-dimensional convolution-type singular integral operators with respect to μ. To do so, he relied on a suitable corona decomposition for measures with linear growth and finite curvature (in particular, for those measures μ for which $\mathcal{C}_\mu$ is bounded in $L^2(\mu)$) that could not easily be generalized for higher dimensions, since curvature is only available in this setting.

Using a new Corona Decomposition introduced by Azzam and Tolsa in [AT], we have proved the following result:

Theorem. *Let μ be a finite Radon measure in $\mathbb{R}^d$ with polynomial growth of degree n and such that, for all balls $B \subset \mathbb{R}^d$ with radius $r(B)$,*

$$\int_B \int_0^{r(B)} \beta^n_{\mu,2}(x,r)^2 \theta^n_\mu(x,r) \frac{dr}{r} d\mu(x) \lesssim \mu(B).$$

Then, all singular integral operators T_μ with kernels in $\mathcal{K}^n(\mathbb{R}^d)$ are bounded in $L^2(\mu)$.

Let us just remark here that $\theta^n_\mu(x,r)$ stands for the n-dimensional μ-density of the ball $B(x,r)$, i.e.,

$$\theta^n_\mu(x,r) = \frac{\mu(B(x,r)}{r^n};$$

that $\beta^n_{\mu,2}(x,r)$ stands for Jones's β_2-coefficient of the ball $B(x,r)$ with respect to μ, i.e.,

$$\beta^n_{\mu,2}(x,r) = \inf_L \left(\frac{1}{r(B)^n} \int_B \left(\frac{\operatorname{dist}(y,L)}{r(B)} \right)^2 d\mu(y) \right)^{\frac{1}{2}},$$

where the infimum is taken over all n-planes $L \subset \mathbb{R}^d$; and that $\mathcal{K}^n(\mathbb{R}^d)$ is a family of odd and sufficiently smooth n-dimensional convolution-type Calderón-Zygmund kernels.

Using this result, we obtain an interesting estimate for the Lipschitz harmonic capacity in the spirit of the comparability between the analytic capacity γ and the capacity γ_+ obtained by Tolsa in [T2], and which could serve as a first step towards characterizing those sets that are removable for Lipschitz harmonic functions in a metric-geometric way. Recall that the Lipschitz harmonic capacity of a compact set $E \subset \mathbb{R}^d$ is the natural higher-dimensional analog of analytic capacity, and is defined by

$$\kappa(E) = \sup |\langle \Delta\varphi, 1\rangle|,$$

where the supremum is taken over all Lipschitz functions $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ that are harmonic in $\mathbb{R}^d \setminus E$ and satisfy $||\nabla\varphi||_\infty \leq 1$.

The result we have obtained is the following:

Corollary. *Let E be a compact set in $\mathbb{R}^{n+1}$. Then,*

$$\kappa(E) \gtrsim \sup \mu(E),$$

where the supremum is taken over all positive Borel measures μ supported on E such that

$$\sup_{x\in\mathbb{R}^{n+1}, R>0} \left\{ \theta^n_\mu(x,R) + \int_0^\infty \beta_{\mu,2}(x,r)^2 \theta^n_\mu(x,r) \frac{dr}{r} \right\} \leq 1.$$

In fact, in order to characterize removable sets for Lipschitz harmonic functions in a metric-geometric way, one would need to have $\approx$ instead of $\gtrsim$ in the inequality above. It is worth remarking that Azzam and Tolsa have been able to obtain this type of inequality for the analytic capacity γ in [AT].

Chapter 3: The Riesz transform and quantitative rectifiability for general Radon measures

In the paper [NToV1], Nazarov, Tolsa and Volberg solved David-Semmes conjecture affirmatively in the codimension 1 case, that is, they proved that given an n-AD-regular measure in $\mathbb{R}^{n+1}$, the $L^2(\mu)$-boundedness of the n-dimensional Riesz transform implies the uniform n-rectifiability of μ. Using techniques developed in that work and in some others that are closely related, we obtain the following quantitative result that is valid for Radon measures in $\mathbb{R}^{n+1}$ with polynomial growth of degree n. To state it, denote by $\mathcal{R}$ the n-dimensional Riesz transform in $\mathbb{R}^{n+1}$; for a Radon measure μ in $\mathbb{R}^{n+1}$, $f \in L^1_{loc}(\mu)$ and $A \subset \mathbb{R}^{n+1}$ with $\mu(A) > 0$, set

$$m_{\mu,A}(f) = \frac{1}{\mu(A)} \int_A f d\mu;$$

for a ball $B \subset \mathbb{R}^{n+1}$

$$P_\mu(B) = \sum_{j=0}^\infty 2^{-j} \theta_\mu(2^j B),$$

and for a hyperplane L in $\mathbb{R}^{n+1}$

$$\beta^L_{\mu,1}(B) = \frac{1}{r(B)^n}\int_B \frac{\operatorname{dist}(x,L)}{r(B)}d\mu(x).$$

Theorem. *Let μ be a Radon measure on $\mathbb{R}^{n+1}$ and $B \subset \mathbb{R}^{n+1}$ a ball so that the following conditions hold:*

(a) For some constant $C_0 > 0$, $C_0^{-1} r(B)^n \le \mu(B) \le C_0\, r(B)^n$.

(b) $P_\mu(B) \le C_0$, and $\mu(B(x,r)) \le C_0\, r^n$ for all $x \in B$ and $0 < r \le r(B)$.

(c) There is some n-plane L passing through the centre of B such that for some $0 < \delta \ll 1$, $\beta^L_{\mu,1}(B) \le \delta$.

(d) $\mathcal{R}_{\mu\lfloor_B}$ is bounded in $L^2(\mu\lfloor_B)$ with $\|\mathcal{R}_{\mu\lfloor_B}\|_{L^2(\mu\lfloor_B)\to L^2(\mu\lfloor_B)} \le C_1$.

(e) For some constant $0 < \varepsilon \ll 1$,

$$\int_B |\mathcal{R}\mu(x) - m_{\mu,B}(\mathcal{R}\mu)|^2\, d\mu(x) \le \varepsilon\, \mu(B).$$

Then there exists some constant $\tau > 0$ such that if δ, ε are small enough (depending on C_0 and C_1), then there is a uniformly n-rectifiable set $\Gamma \subset \mathbb{R}^{n+1}$ such that

$$\mu(B \cap \Gamma) \ge \tau\, \mu(B).$$

Furthermore, the constant τ and the uniform rectifiability constants of Γ depend on all the constants above.

In particular, this result ensures the existence of some piece of positive μ-measure of $B \cap \Gamma$ where μ and the Hausdorff measure $\mathcal{H}^n$ are mutually absolutely continuous. This fact, which at first sight may appear rather surprising, is one of the main difficulties for its proof.

The main motivation for this result was the quantitative theorem by Léger on Menger curvature, and in fact one may think that this theorem is its higher-dimensional analog for Riesz transforms. Some details about this analogy are explained in Chapter 3, although we wish to remark now that the absence of a tool like Menger curvature makes the proofs be substantially different. Finally, we wish to remark that this result has turned out to be an essential tool for the solution of an old question on harmonic measure that will appear in a work by Azzam, Mourgoglou and Tolsa [AMT].

A remark about notation

As it is usual in Harmonic Analysis, a letter c (or C, or any other) will denote an absolute constant that may change its value at different occurrences. Constants with subscripts will retain their value at different occurrences, at least inside the same chapter of this dissertation. The notation $A \lesssim B$ means that there is a positive absolute constant C such that $A \le CB$, and $A \approx B$ is equivalent to $A \lesssim B \lesssim A$.

Chapter 1

The Cauchy transform along a Lipschitz curve: an improvement of Cotlar's inequality and some counterexamples

1.1 Introduction

We say that a measurable function k defined in $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d \colon x = y\}$ is an n-dimensional Calderón-Zygmund kernel if there are constants $c > 0$ and $0 < \delta \leq 1$ such that

$$|k(x,y)| \leq \frac{c}{|x-y|^n} \quad \text{if} \quad x \neq y$$

and

$$|k(x,y) - k(x',y)| + |k(y,x) - k(y,x')| \leq c\frac{|x-x'|^\delta}{|x-y|^{n+\delta}} \quad \text{if} \quad |x-x'| \leq \frac{|x-y|}{2}. \tag{1.1}$$

Given a signed Radon measure ν in $\mathbb{R}^d$ and $x \in \mathbb{R}^d$, we define

$$T\nu(x) = \int k(x,y)d\nu(y), \quad x \in \mathbb{R}^d \setminus \operatorname{supp}(\nu)$$

and we say that T is a singular integral operator with kernel k. The integral above need not be convergent for $x \in \operatorname{supp}(\nu)$, and this is why one introduces the truncated operators associated to T, which are defined, for every $\varepsilon > 0$, by

$$T_\varepsilon \nu(x) = \int_{|x-y|>\varepsilon} k(x,y)d\nu(y), \ x \in \mathbb{R}^d.$$

Notice that the integral above is absolutely convergent if, for example, $|\nu|(\mathbb{R}^d) < \infty$.

If μ is a fixed positive Radon measure in $\mathbb{R}^d$ and $f \in L^1_{loc}(\mu)$, we set

$$T_\mu f(x) = T(f\mu)(x), \ x \in \mathbb{R}^d \setminus \operatorname{supp}(\mu)$$

and, for $\varepsilon > 0$,

$$T_{\mu,\varepsilon} f(x) = T_\varepsilon(f\mu)(x), \ x \in \mathbb{R}^d.$$

We say that T_μ is bounded in $L^2(\mu)$ if there is a constant $C > 0$ such that, for all $\varepsilon > 0$, $||T_{\mu,\varepsilon} f||_{L^2(\mu)} \leq C||f||_{L^2(\mu)}$ for all f in $L^2(\mu)$ and, in such a case, we say that T is a Calderón-Zygmund operator. The norm of T_μ is the infimum of all those constants C (the same idea is used to define the boundedness of T_μ in other spaces). Some of the most important examples of this class of operators are the n-dimensional Riesz transform, given by

$$\mathcal{R}\nu(x) = \int \frac{x-y}{|x-y|^{n+1}} d\nu(y)$$

and its one-dimensional analog in $\mathbb{R}^2 \equiv \mathbb{C}$, the Cauchy transform, defined by

$$\mathcal{C}\nu(z) = \int \frac{d\nu(\zeta)}{\zeta - z}.$$

Regarding the Cauchy transform, a particularly interesting case is the one that arises when μ is the arc-length measure (or some measure comparable to this) supported on a Lipschitz graph. To be more precise, let $A\colon \mathbb{R} \to \mathbb{R}$ be a Lipschitz function with Lipschitz constant $\Lambda_1 \geq 0$, and let $\Gamma \subset \mathbb{R}^2 \equiv \mathbb{C}$ be its graph, which we parametrize by

$$z(x) = x + iA(x), \quad x \in \mathbb{R}.$$

We define a measure μ on Γ by

$$\mu(z(E)) = |E|,$$

where E is any Borel subset of $\mathbb{R}$. We will normally call $\mathcal{C}_\mu$ the Cauchy transform along Γ. Recall that, since A is Lipschitz, it is differentiable almost everywhere and, furthermore, its Lipschitz constant coincides with $||A'||_\infty$. Moreover, it is easy to check that the measure μ that we are considering is comparable to the arc-length measure on Γ.

In [C], Calderón proved that $\mathcal{C}_\mu$ is bounded in $L^2(\mu)$ when $||A'||_\infty$ is sufficiently small. Later, in [CMM], Coifman, McIntosh and Meyer proved that $\mathcal{C}_\mu$ is bounded in $L^2(\mu)$ for every Lipschitz function A. It also follows from the work of Calderón that

$$\text{p.v.}\mathcal{C}_\mu f(z) = \lim_{\varepsilon \searrow 0} \mathcal{C}_{\mu,\varepsilon} f(z)$$

exists for a.e. $z \in \operatorname{supp}(\mu)$ for all $f \in L^2(\mu)$, and, as a result, we can think of $\mathcal{C}_\mu$ to be defined as a principal value operator.

All the considerations regarding the Cauchy transform along Γ can be posed in terms of its parametrized version, which, abusing notation and language, will be again denoted by $\mathcal{C}$ and called the Cauchy transform along Γ. It is defined, for $f \in L^2(\mathbb{R})$ and $x \in \mathbb{R}$, by

$$\mathcal{C}f(x) = \text{p.v.} \int_{\mathbb{R}} \frac{f(y)}{z(y) - z(x)} dy.$$

Associated with it, we consider as well the truncated operators

$$\mathcal{C}_\varepsilon f(x) = \int_{|y-x|>\varepsilon} \frac{f(y)}{z(y) - z(x)} dy$$

and the maximal operator

$$\mathcal{C}_* f(x) = \sup_{\varepsilon > 0} |\mathcal{C}_\varepsilon f(x)|.$$

Notice that the truncated operators $\mathcal{C}_\varepsilon$ are not the exact analogues to the truncated operators $\mathcal{C}_{\mu,\varepsilon}$ defined above, which would correspond to

$$\tilde{\mathcal{C}}_\varepsilon f(x) = \int_{|z(y)-z(x)|>\varepsilon} \frac{f(y)}{z(y) - z(x)} dy.$$

We will deal with this issue later.

From the standard Calderón-Zygmund theory, we obtain that $\mathcal{C}$ is bounded in $L^p(\mathbb{R})$ for $1 < p < \infty$, it is bounded from $L^1(\mathbb{R})$ to $L^{1,\infty}(\mathbb{R})$ and from $L^\infty(\mathbb{R})$ to $BMO(\mathbb{R})$, and it satisfies the classical Cotlar's inequality, i.e., for all $f \in L^2(\mathbb{R})$ and all $x \in \mathbb{R}$,

$$\mathcal{C}_* f(x) \lesssim M(\mathcal{C}f)(x) + \mathcal{C}f(x),$$

where M is the Hardy-Littlewood maximal operator.

In the papers [MOPV], [MOV] and [MV], Mateu, Orobitg, Pérez and Verdera study the problem of controlling a maximal singular integral $T_* f$ in terms of the corresponding singular integral Tf. As it is stated in those papers, one reason to consider this problem is to gain a better understanding of David-Semmes conjecture regarding the possibility of characterizing uniform rectifiability by the boundedness of the Riesz transforms (see [DS1]).

Next we describe some of the results proved in those papers.

Definition 1.1.1. *A higher-order Riesz transform is a Calderón-Zygmund operator defined, for $f \in L^2(\mathbb{R}^n)$, by*

$$Tf(x) = p.v. \int_{\mathbb{R}^n} \frac{P(x-y)}{|x-y|^{n+d}} f(y)dy,$$

where P is a harmonic homogeneous polynomial of degree $d \geq 1$. We say that T is odd (respectively, even) if d is odd (respectively, even).

Theorem A. *Let T be a higher order Riesz transform, and let T_* be the associated maximal operator. Then,*

1. *If T is even, then for all $f \in L^2(\mathbb{R}^n)$ and all $x \in \mathbb{R}^n$,*

$$T_* f(x) \lesssim M(Tf)(x).$$

2. *If T is odd, then for all $f \in L^2(\mathbb{R}^n)$ and all $x \in \mathbb{R}^n$,*

$$T_* f(x) \lesssim M^2(Tf)(x).$$

Notice that, in particular, for the Hilbert transform we have

$$H_* f(x) \lesssim M^2(Hf)(x)$$

for all $f \in L^2(\mathbb{R})$ and all $x \in \mathbb{R}$.

Definition 1.1.2. *A* ***smooth homogeneous singular integral operator*** *is a singular integral operator which is defined, for $f \in L^2(\mathbb{R}^n)$, by*

$$Tf(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)dy,$$

where $\Omega \colon \mathbb{R}^n \to \mathbb{C}$ is a homogeneous function of degree 0 whose restriction to the unit sphere $\mathbb{S}^{n-1}$ is of class $\mathcal{C}^\infty$ and satisfies the cancellation property

$$\int_{\mathbb{S}^{n-1}} \Omega(u) d\sigma(u) = 0.$$

We will say that the operator is odd (respectively, even) if Ω is odd (respectively, even).

Theorem B. *Let T be a smooth homogeneous singular integral operator, and let T_* be the associated maximal operator. Then,*

- *If T is even, the following assertions are equivalent:*
  1. $T_*f(x) \lesssim M(Tf)(x)$ *for all* $f \in L^2(\mathbb{R}^n)$ *and all* $x \in \mathbb{R}^n$.
  2. $||T_*f||_{L^2} \lesssim ||Tf||_{L^2}$ *for all* $f \in L^2(\mathbb{R}^n)$.
- *If T is odd, the following assertions are equivalent:*
  1. $T_*f(x) \lesssim M^2(Tf)(x)$ *for all* $f \in L^2(\mathbb{R}^n)$ *and all* $x \in \mathbb{R}^n$.
  2. $||T_*f||_{L^2} \lesssim ||Tf||_{L^2}$ *for all* $f \in L^2(\mathbb{R}^n)$.

The statements in the previous two theorems concerning even operators were proved in [MOV], while those concerning odd operators were proved in [MOPV]. It is worth mentioning, as well, that Bosch-Camòs, Mateu and Orobitg extended Theorem B later in the following way in their paper [BMO1]:

Theorem C. *Let T be a smooth homogeneous singular integral operator, and let T_* be the associated maximal operator. Then, the following assertions are equivalent:*

1. $||T_*f||_{L^2} \lesssim ||Tf||_{L^2}$ *for all* $f \in L^2(\mathbb{R}^n)$.
2. *If* $1 < p < \infty$ *and* $\omega \in A_p$, *then* $||T_*f||_{L^p(\omega)} \lesssim ||Tf||_{L^p(\omega)}$ *for all* $f \in L^p(\omega)$.

Furthermore, if T is even, the conditions above are equivalent as well to

3. $||T_*f||_{L^{1,\infty}} \lesssim ||Tf||_{L^1}$ *for all* $f \in H^1(\mathbb{R}^n)$.

An interesting result concerning pointwise inequalities like the ones in Theorem A for a slightly modified version of the maximal Beurling transform and its iterates can be found in [BMO2].

Taking into account the *possible* relationship of these type of inequalities with the David-Semmes conjecture, we tried to establish some of them for the Cauchy transform along a Lipschitz curve (in fact, we only dealt with pointwise inequalities like the ones above, since the norm inequalities are almost trivial, as we will show later).

Another possible motivation to try to extend the results above for the Cauchy transform along a Lipschitz graph Γ is that it coincides with a constant multiple of the Hilbert transform when Γ is a straight line. This is a reason why one could think that the pointwise estimate $\mathcal{C}_*f \lesssim M^n(\mathcal{C}f)$ could hold for the Cauchy transform along, at least, some class of graphs Γ, for some $n \geq 1$. We will show that one cannot have a similar inequality for the Cauchy transform unless Γ is a straight line.

Theorem 1.1.1. *Consider the Lipschitz function $A(x) = |x|$, and let $\mathcal{C}$ denote the Cauchy transform along Γ, the graph of A. Then, there exists $f \in L^2(\mathbb{R})$ such that for all $c > 0$ and all $n \geq 1$, there exists $\varepsilon > 0$ such that*

$$|\mathcal{C}_\varepsilon f(0)| > cM^n(\mathcal{C}f)(0).$$

This theorem can be easily generalized to Lipschitz graphs Γ with angles, meaning with this points x where A' has a jump discontinuity, as we will show later.

After obtaining this result, one might think of establishing the inequality $\mathcal{C}_*f \lesssim M^n(\mathcal{C}f)$ imposing some restrictions on the smoothness of A. This is not the case, as the next theorem shows.

Theorem 1.1.2. *Let A be a Lipschitz function with compact support, and let $\mathcal{C}$ denote the Cauchy transform along Γ, the graph of A. Suppose A is not identically null, or, equivalently, that Γ is not a straight line. Then, there exists $x \in \mathbb{R}$ such that for all $c > 0$ there exists $f \in L^2(\mathbb{R})$ with*

$$\mathcal{C}_*f(x) > cM^n(\mathcal{C}f)(x)$$

for all $n \geq 1$.

We want to remark that the points x mentioned in this last theorem are *easy* to find. For example, when A is of class $\mathcal{C}^2$, any point x with $A''(x) \neq 0$ will do the job.

1.2 Another version of the Cauchy transform

We define a new operator which, abusing language, will also be called the Cauchy transform along Γ, by

$$Tf(x) = \frac{1}{\pi i}\text{p.v.}\int_{\mathbb{R}} \frac{f(y)}{z(y)-z(x)}dz(y),$$

where $dz(y) = z'(y)dy = (1+iA'(y))dy$. As before, associated with it, we will have the truncated operators T_ε and the maximal operator T_*. This operator is very closely related to $\mathcal{C}$. Indeed,

$$\begin{aligned} Tf(x) &= \frac{1}{\pi i}\text{p.v.}\int_{\mathbb{R}} \frac{f(y)}{z(y)-z(x)}dz(y) \\ &= \frac{1}{\pi i}\text{p.v.}\int_{\mathbb{R}} \frac{f(y)z'(y)}{z(y)-z(x)}dy = \frac{1}{\pi i}\mathcal{C}(f\cdot z')(x). \end{aligned} \tag{1.2}$$

Analogously,

$$\mathcal{C}f(x) = \pi i T\left(\frac{f}{z'}\right)(x). \tag{1.3}$$

It is clear that T satisfies the same boundedness properties that $\mathcal{C}$ satisfies (with different multiplicative constants). Moreover, by equations (1.2) and (1.3), and taking into account that $z' \in L^\infty$ and $|z'| \approx 1$, we can limit ourselves to prove Theorems 1.1.1 and 1.1.2 substituting $\mathcal{C}$ by T, $\mathcal{C}_\varepsilon$ by T_ε and $\mathcal{C}_*$ by T_*.

The main reason for using this version of the Cauchy transform is contained in the following result, which we learnt from Escauriaza ([E]).

Lemma 1.2.1. *If $f \in L^p(\mathbb{R})$, $1<p<\infty$, then $T^2 f = f$.*

Proof. For $w \in \mathbb{C}$ and $\alpha > 0$, we define the upper and lower half cones with vertex at w and generatrix slope α, respectively, by

$$X^+(w,\alpha) = \{z \in \mathbb{C} \colon |\text{Re } z - \text{Re } w| < \alpha(\text{Im } z - \text{Im } w)\}$$

$$X^-(w,\alpha) = \{z \in \mathbb{C} \colon |\text{Re } z - \text{Re } w| < \alpha(\text{Im } w - \text{Im } z)\}.$$

It is immediate that for all $w \in \Gamma$ and all $0 < \alpha < \frac{1}{||A'||_\infty}$,

$$X^+(w,\alpha) \subset \{x+iy \in \mathbb{C} \colon y > A(x)\}$$

and

$$X^-(w,\alpha) \subset \{x+iy \in \mathbb{C} \colon y < A(x)\}.$$

Fix $0 < \alpha < \frac{1}{||A'||_\infty}$. Let $f \in L^p(\mathbb{R})$, and let us define, for $x \in \mathbb{R}$,

$$T_+f(x) = \lim_{\substack{w\to z(x)\\ w\in X^+(z(x),\alpha)}} \frac{1}{\pi i}\int_{\mathbb{R}} \frac{f(y)}{w-z(x)}dz(y),$$

$$T_-f(x) = \lim_{\substack{w\to z(x)\\ w\in X^-(z(x),\alpha)}} \frac{1}{\pi i}\int_{\mathbb{R}} \frac{f(y)}{w-z(x)}dz(y).$$

From Plemelj formulae (see, for example, Chapter 8 of [T6]), we obtain

$$T_+f(x) = Tf(x) + f(x);\;\; T_-f(x) = Tf(x) - f(x)$$

for a.e. $x \in \mathbb{R}$. In particular, $T = T_+ - Id$. Hence,

$$T^2 = (T_+ - Id)^2 = (T_+)^2 - 2T_+ + Id.$$

A direct application of Cauchy's integral formula gives $(T_+)^2 = 2T_+$. As a consequence, $T^2 = Id$, as desired. $\square$

As a consequence of this result, one easily gets the L^p-control of the maximal Cauchy transform in terms of the Cauchy transform, for $1 < p < \infty$.

Corollary 1.2.1. *If $f \in L^p(\mathbb{R})$, $1 < p < \infty$, then $||T_*f||_{L^p} \lesssim ||Tf||_{L^p}$.*

Proof. Indeed, taking into account the L^p-boundedness of T_* and T, and the fact that $T^2 f = f$, we get

$$||T^* f||_{L^p} \lesssim ||f||_{L^p} = ||T^2 f||_{L^p} \lesssim ||Tf||_{L^p}.$$

□

The following lemma states that T is antisymmetric with respect to dz, and its proof follows by an easy application of Fubini's Theorem.

Lemma 1.2.2. *Let $1 < p < \infty$, p' the conjugate exponent to p and $f \in L^p(\mathbb{R})$, $g \in L^{p'}(\mathbb{R})$. Then,*

$$\int_{\mathbb{R}} Tf(x) g(x) dz(x) = -\int_{\mathbb{R}} f(y) Tg(y) dz(y).$$

1.3 The proofs

We argue here as Mateu, Orobitg, Pérez and Verdera did in [MOPV], where they proved $T_*f \lesssim M^2(Tf)$, for T an odd higher order Riesz transform.

Let $f \in L^2(\mathbb{R})$, $x \in \mathbb{R}$ and $\varepsilon > 0$. We have

$$T_\varepsilon f(x) = \frac{1}{\pi i} \int_{|y-x|>\varepsilon} \frac{f(y)}{z(y) - z(x)} dz(y).$$

For $x \in \mathbb{R}$ and $\varepsilon > 0$, define

$$K_{x,\varepsilon}(y) = \frac{1}{\pi i} \frac{1}{z(y) - z(x)} \chi_{|y-x|>\varepsilon}(y),$$

so that

$$T_\varepsilon f(x) = \int_{\mathbb{R}} f(y) K_{x,\varepsilon}(y) dz(y).$$

A straightforward computation yields that $K_{x,\varepsilon} \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and

$$||K_{x,\varepsilon}||_{L^2} \leq \frac{1}{\sqrt{\varepsilon}}, \quad ||K_{x,\varepsilon}||_{L^\infty} \leq \frac{1}{\varepsilon}.$$

Now let $g_{x,\varepsilon} = T(K_{x,\varepsilon})$, so that

$$\begin{aligned} T_\varepsilon f(x) &= \int_{\mathbb{R}} f(y) K_{x,\varepsilon}(y) dz(y) = \int_{\mathbb{R}} f(y) T(T(K_{x,\varepsilon}))(y) dz(y) \\ &= -\int_{\mathbb{R}} Tf(y) T(K_{x,\varepsilon})(y) dz(y) = -\int_{\mathbb{R}} Tf(y) g_{x,\varepsilon}(y) dz(y). \end{aligned}$$

Fix $N > 0$ to be chosen later, and denote, for $a \in \mathbb{R}$ and $r > 0$,

$$I_{a,r} = (a - r, a + r).$$

Also, for a function $h \in L^1_{loc}(\mathbb{R})$ and an interval $I \subset \mathbb{R}$, denote

$$m_I h = \frac{1}{|I|} \int_I h(x) dx.$$

Then, we have,

$$\begin{aligned}
-T_\varepsilon f(x) &= \int_{\mathbb{R}} Tf(y) g_{x,\varepsilon}(y) dz(y) \\
&= \int_{|y-x|<N\varepsilon} Tf(y) g_{x,\varepsilon}(y) dz(y) + \int_{|y-x|>N\varepsilon} Tf(y) g_{x,\varepsilon}(y) dz(y) \\
&= \int_{I_{x,N\varepsilon}} Tf(y)[g_{x,\varepsilon}(y) - m_{I_{x,N\varepsilon}}(g_{x,\varepsilon})] dz(y) + m_{I_{x,N\varepsilon}}(g_{x,\varepsilon}) \int_{I_{x,N\varepsilon}} Tf(y) dz(y) \\
&\quad + \int_{|y-x|>N\varepsilon} Tf(y) g_{x,\varepsilon}(y) dz(y) = \mathsf{I} + \mathsf{II} + \mathsf{III}.
\end{aligned}$$

Let us check now that $|\mathsf{I}| \lesssim M^2(Tf)(x)$ and $|\mathsf{II}| \lesssim M(Tf)(x)$. We recall first the following results, stated in [MOPV], and whose proofs can be found in [W] and [Gr], respectively:

Lemma 1.3.1. *Let $\phi \in BMO(\mathbb{R}^n)$, ψ a measurable function in $\mathbb{R}^n$ and Q a cube in $\mathbb{R}^n$. Then,*

$$\frac{1}{|Q|} \int_Q |\phi(x) - m_Q\phi||\psi(x)| dx \leq c||\phi||_{BMO}||\psi||_{L\log L, Q},$$

where $c > 0$ only depends on n.

Lemma 1.3.2. *There exists a positive constant $c = c(n) > 0$ such that for every cube $Q \subset \mathbb{R}^n$ and every function $\psi \in L^1_{loc}(\mathbb{R}^n)$ we have*

$$||\psi||_{L\log L, Q} \leq cM^2\psi(x),$$

where $M^2 = M \circ M$ and M is the Hardy-Littlewood maximal operator.

Let us estimate $|\mathsf{I}|$ and $|\mathsf{II}|$ now. We have

$$\begin{aligned}
|\mathsf{I}| &= \left| \int_{I_{x,N\varepsilon}} Tf(y)[g_{x,\varepsilon}(y) - m_{I_{x,N\varepsilon}}(g_{x,\varepsilon})] dz(y) \right| \\
&\lesssim \int_{I_{x,N\varepsilon}} |Tf(y)||g_{x,\varepsilon}(y) - m_{I_{x,N\varepsilon}}(g_{x,\varepsilon})| dy \\
&\lesssim |I_{x,N\varepsilon}| ||g_{x,\varepsilon}||_{BMO} ||Tf||_{L(logL), I_{x,N\varepsilon}} \\
&\lesssim \varepsilon ||T(K_{x,\varepsilon})||_{BMO} M^2(Tf)(x) \\
&\lesssim \varepsilon ||K_{x,\varepsilon}||_{L^\infty} M^2(Tf)(x) \lesssim M^2(Tf)(x).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
|\mathsf{II}| &= \left| m_{I_{x,N\varepsilon}}(g_{x,\varepsilon}) \int_{I_{x,N\varepsilon}} Tf(y) dz(y) \right| \\
&\lesssim \frac{1}{|I_{x,N\varepsilon}|} \left| \int_{I_{x,N\varepsilon}} g_{x,\varepsilon}(y) dy \right| \int_{I_{x,N\varepsilon}} |Tf(y)| dy \\
&= \left| \int_{I_{x,N\varepsilon}} T(K_{x,\varepsilon})(y) dy \right| \frac{1}{|I_{x,N\varepsilon}|} \int_{I_{x,N\varepsilon}} |Tf(y)| dy \\
&\leq |I_{x,N\varepsilon}|^{\frac{1}{2}} ||T(K_{x,\varepsilon})||_{L^2} M(Tf)(x) \\
&\lesssim \varepsilon^{\frac{1}{2}} ||K_{x,\varepsilon}||_{L^2} M(Tf)(x) \lesssim M(Tf)(x),
\end{aligned}$$

as claimed.

Now, since $M(Tf) \leq M^2(Tf)$, we get

$$|\mathsf{I}| + |\mathsf{II}| \lesssim M^2(Tf)(x). \tag{1.4}$$

Let us study III now. Recall that

$$\mathsf{III} = \int_{|y-x|>N\varepsilon} Tf(y) g_{x,\varepsilon}(y) dz(y) = \int_{|y-x|>N\varepsilon} Tf(y) T(K_{x,\varepsilon}(y)) dz(y). \tag{1.5}$$

An easy contour integration argument yields the following result:

Lemma 1.3.3. *Fix $x \in \mathbb{R}$ and $\varepsilon > 0$. Then, for almost every $y \in \mathbb{R}$ with $|y-x| > \varepsilon$, we have*

$$T(K_{x,\varepsilon})(y) = \frac{1}{\pi i} \frac{1}{z(y) - z(x)} \left[B(x,\varepsilon) + G_{x,\varepsilon}(y) \right],$$

where

$$B(x,\varepsilon) = \log \frac{|z(x+\varepsilon) - z(x)|}{|z(x-\varepsilon) - z(x)|} + i\Big(\pi + \arg[z(x+\varepsilon) - z(x)] - \arg[z(x-\varepsilon) - z(x)] \Big)$$

and

$$G_{x,\varepsilon}(y) = \log \frac{|z(x-\varepsilon) - z(y)|}{|z(x+\varepsilon) - z(y)|} + i\Big(\arg[z(x-\varepsilon) - z(y)] - \arg[z(x+\varepsilon) - z(y)] \Big),$$

where, for a complex number $w \neq 0$, we consider $-\frac{\pi}{2} \leq \arg(w) < \frac{3\pi}{2}$.

Proof. Let $x \in \mathbb{R}$, $\varepsilon > 0$ and $y \in \mathbb{R}$ with $|y - x| > \varepsilon$. We will assume that $y > x$ (the case $y < x$ is treated analogously) and also that A is differentiable at y. For a set $I \subset \mathbb{R}$, denote $\Gamma(I) = \{z(t) \colon t \in I\}$, and for a complex number $w \neq 0$, let $\mathrm{Log}\,(w) = \log|w| + i \arg(w)$. Then, we have

$$\begin{aligned}
T(K_{x,\varepsilon})(y) &= \frac{1}{\pi i} \text{p.v.} \int_{\mathbb{R}} \frac{K_{x,\varepsilon}(t)}{z(t) - z(y)} dz(t) \\
&= \lim_{\substack{R\to\infty \\ \delta\to 0}} \frac{1}{\pi i} \int_{\Gamma(\{t \colon |t-x|>\varepsilon, |t-y|>\delta, |t|<R\})} \frac{dw}{(w - z(x))(w - z(y))} \\
&= \frac{1}{\pi i (z(y) - z(x))} \lim_{\substack{R\to\infty \\ \delta\to 0}} \int_{\Gamma(\{t \colon |t-x|>\varepsilon, |t-y|>\delta, |t|<R\})} \left(\frac{1}{w - z(y)} - \frac{1}{w - z(x)} \right) dw \\
&= \frac{1}{\pi i (z(y) - z(x))} \lim_{\substack{R\to\infty \\ \delta\to 0}} (\mathsf{I}_{R,\delta} + \mathsf{II}_{R,\delta} + \mathsf{III}_{R,\delta}),
\end{aligned}$$

where, for sufficiently small $\delta > 0$ and sufficiently big $R > 0$,

$$\begin{aligned}
\mathsf{I}_{R,\delta} &= \int_{\Gamma((-R, x-\varepsilon))} \left(\frac{1}{w - z(y)} - \frac{1}{w - z(x)} \right) dw \\
&= \mathrm{Log}\,[z(x-\varepsilon) - z(y)] - \mathrm{Log}\,[z(-R) - z(y)] - \mathrm{Log}\,[z(x-\varepsilon) - z(x)] + \mathrm{Log}\,[z(-R) - z(x)],
\end{aligned}$$

$$\begin{aligned}
\mathsf{II}_{R,\delta} &= \int_{\Gamma((x+\varepsilon, y-\delta))} \left(\frac{1}{w - z(y)} - \frac{1}{w - z(x)} \right) dw \\
&= \mathrm{Log}\,[z(y-\delta) - z(y)] - \mathrm{Log}\,[z(x+\varepsilon) - z(y)] - \mathrm{Log}\,[z(y-\delta) - z(x)] + \mathrm{Log}\,[z(x+\varepsilon) - z(x)]
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{III}_{R,\delta} &= \int_{\Gamma((y+\delta, R))} \left(\frac{1}{w - z(y)} - \frac{1}{w - z(x)} \right) dw \\
&= \mathrm{Log}\,[z(R) - z(y)] - \mathrm{Log}\,[z(y+\delta) - z(y)] - \mathrm{Log}\,[z(R) - z(x)] + \mathrm{Log}\,[z(y+\delta) - z(x)].
\end{aligned}$$

Gathering the previous identities, we obtain

$$\begin{aligned}
\mathrm{Re}\ (\mathsf{I}_{R,\delta} + \mathsf{II}_{R,\delta} + \mathsf{III}_{R,\delta}) = \log \frac{|z(x-\varepsilon) - z(y)||z(x+\varepsilon) - z(x)|}{|z(x+\varepsilon) - z(y)||z(x-\varepsilon) - z(x)|} \\
+ \log \frac{|z(-R) - z(x)||z(R) - z(y)|}{|z(-R) - z(y)||z(R) - z(x)|} \\
+ \log \frac{|z(y-\delta) - z(y)||z(y+\delta) - z(x)|}{|z(y+\delta) - z(y)||z(y-\delta) - z(x)|}.
\end{aligned} \tag{1.6}$$

On the other hand,

$$\begin{aligned}\text{Im }(\mathsf{I}_{R,\delta}+\mathsf{II}_{R,\delta}+\mathsf{III}_{R,\delta}) = &(\arg[z(x-\varepsilon)-z(y)]-\arg[z(x-\varepsilon)-z(x)]\\&-\arg[z(x+\varepsilon)-z(y)]+\arg[z(x+\varepsilon)-z(x)])\\&+(-\arg[z(-R)-z(y)]+\arg[z(-R)-z(x)]\\&+\arg[z(R)-z(y)]-\arg[z(R)-z(x)])\\&+(\arg[z(y-\delta)-z(y)]-\arg[z(y-\delta)-z(x)]\\&-\arg[z(y+\delta)-z(y)]+\arg[z(y+\delta)-z(x)]).\end{aligned}\tag{1.7}$$

Letting $R\to\infty$ and $\delta\to 0$ in (1.6) and (1.7), using the fact that A is differentiable at y, and adding up the results, we obtain

$$\lim_{\substack{R\to\infty\\ \delta\to 0}}(\mathsf{I}_{R,\delta}+\mathsf{II}_{R,\delta}+\mathsf{III}_{R,\delta}) = G_{x,\varepsilon}(y)+B(x,\varepsilon),$$

and so the desired conclusion follows. □

It is easy to check that the term $B(x,\varepsilon)$ satisfies the following:

Lemma 1.3.4. *Let $x\in\mathbb{R}$ and $\varepsilon>0$. Then, the following assertions are equivalent:*

1. *$B(x,\varepsilon)=0$.*
2. *Im $B(x,\varepsilon)=0$.*
3. *The points $z(x-\varepsilon)$, $z(x)$ and $z(x+\varepsilon)$ are collinear.*

On the other hand, we can prove the following decay at infinity of the term $G_{x,\varepsilon}(y)$.

Lemma 1.3.5. *Choose $N>1+4(1+||A'||_\infty)$. Then for $|y-x|>N\varepsilon$,*

$$|G_{x,\varepsilon}(y)|\lesssim\frac{\varepsilon}{|y-x|}.$$

Proof. Let

$$u_{x,\varepsilon}(y)=\text{Re }G_{x,\varepsilon}(y)=\log\frac{|z(x-\varepsilon)-z(y)|}{|z(x+\varepsilon)-z(y)|}$$

and

$$v_{x,\varepsilon}(y)=\text{Im }G_{x,\varepsilon}(y)=\arg[z(x-\varepsilon)-z(y)]-\arg[z(x+\varepsilon)-z(y)].$$

Recall that, for $w\in\mathbb{C}$, $|w|<\frac{1}{2}$,

$$|\text{Log }(1+w)|\leq 2|w|,$$

where Log is defined as in the previous lemma.

Now, for $|y-x|>N\varepsilon$, we have

$$\frac{z(x-\varepsilon)-z(y)}{z(x+\varepsilon)-z(y)}=1+\frac{z(x-\varepsilon)-z(x+\varepsilon)}{z(x+\varepsilon)-z(y)},$$

and

$$\begin{aligned}\left|\frac{z(x-\varepsilon)-z(x+\varepsilon)}{z(x+\varepsilon)-z(y)}\right| &\leq\frac{(1+\Lambda_1)2\varepsilon}{|y-(x+\varepsilon)|}\leq\frac{(1+\Lambda_1)2\varepsilon}{\frac{N-1}{N}|y-x|}\\&\leq\frac{(1+\Lambda_1)2\varepsilon}{\frac{N-1}{N}N\varepsilon}=\frac{2(1+\Lambda_1)}{N-1}\leq\frac{1}{2},\end{aligned}$$

where the last inequality holds precisely because of the choice of N. Then,

$$
\begin{aligned}
|u_{x,\varepsilon}(y)| &= \left|\log \frac{|z(x-\varepsilon)-z(y)|}{|z(x+\varepsilon)-z(y)|}\right| = \left|\log\left|1+\frac{z(x-\varepsilon)-z(x+\varepsilon)}{z(x+\varepsilon)-z(y)}\right|\right| \\
&\leq \left|\operatorname{Log}\left(1+\frac{z(x-\varepsilon)-z(x+\varepsilon)}{z(x+\varepsilon)-z(y)}\right)\right| \leq 2\left|\frac{z(x-\varepsilon)-z(x+\varepsilon)}{z(x+\varepsilon)-z(y)}\right| \\
&\leq 2\frac{(1+\Lambda_1)2\varepsilon}{\frac{N-1}{N}|y-x|} = \frac{4N(1+\Lambda_1)}{N-1}\frac{\varepsilon}{|y-x|} \lesssim \frac{\varepsilon}{|y-x|}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
|v_{x,\varepsilon}(y)| &= |\arg[z(x-\varepsilon)-z(y)] - \arg[z(x+\varepsilon)-z(y)]| \\
&\leq |\operatorname{Log}\,[z(x-\varepsilon)-z(y)] - \operatorname{Log}\,[z(x+\varepsilon)-z(y)]| \\
&= \left|\int_{\Gamma((x-\varepsilon,x+\varepsilon))}\frac{dz}{z-z(y)}\right| \leq \mathcal{H}^1(\Gamma((x-\varepsilon,x+\varepsilon)))\max_{|t-x|\leq\varepsilon}\frac{1}{|z(t)-z(y)|} \\
&\leq \frac{2(1+\Lambda_1)\varepsilon}{|y-x|} \lesssim \frac{\varepsilon}{|y-x|}.
\end{aligned}
$$

Putting all together, the lemma follows. □

As a result, going back to (1.5), and applying Lemma 1.3.3, we obtain

$$
\begin{aligned}
\mathsf{III} &= \frac{1}{\pi i}\int_{|y-x|>N\varepsilon} Tf(y)\frac{1}{z(y)-z(x)}\left[B(x,\varepsilon)+G_{x,\varepsilon}(y)\right]dz(y) \\
&= \frac{1}{\pi i}\left[B(x,\varepsilon)\int_{|y-x|>N\varepsilon} Tf(y)\frac{dz(y)}{z(y)-z(x)} + \int_{|y-x|>N\varepsilon} Tf(y)\frac{G_{x,\varepsilon}(y)dz(y)}{z(y)-z(x)}\right] \\
&= B(x,\varepsilon)T_{N\varepsilon}(Tf)(x) + \mathsf{IV}.
\end{aligned}
\tag{1.8}
$$

Now, fixing $N > 1+4(1+||A'||_\infty)$, and applying Lemma 1.3.5, we obtain

$$
\begin{aligned}
|\mathsf{IV}| &= \left|\frac{1}{\pi i}\int_{|y-x|>N\varepsilon} Tf(y)\frac{G_{x,\varepsilon}(y)dz(y)}{z(y)-z(x)}\right| \\
&\lesssim \varepsilon\int_{|y-x|>N\varepsilon}|Tf(y)|\frac{dy}{|y-x|^2} \\
&= \varepsilon\sum_{k=0}^{\infty}\int_{2^kN\varepsilon<|y-x|<2^{k+1}N\varepsilon}|Tf(y)|\frac{dy}{|y-x|^2} \\
&\leq \varepsilon\sum_{k=0}^{\infty}\frac{1}{(2^kN\varepsilon)^2}\int_{2^kN\varepsilon<|y-x|<2^{k+1}N\varepsilon}|Tf(y)|dy \\
&\leq \varepsilon\sum_{k=0}^{\infty}\frac{1}{2^{k-2}N\varepsilon}\frac{1}{2\cdot 2^{k+1}N\varepsilon}\int_{|y-x|<2^{k+1}N\varepsilon} \\
&\leq \varepsilon\left(\sum_{k=0}^{\infty}\frac{1}{2^{k-2}N\varepsilon}\right)M(Tf)(x) \lesssim M(Tf)(x) \leq M^2(Tf)(x).
\end{aligned}
\tag{1.9}
$$

As a result, gathering the estimates in (1.4), (1.8) and (1.9), we have the following:

Lemma 1.3.6. *For all $f \in L^2(\mathbb{R})$, all $x \in \mathbb{R}$ and all $\varepsilon > 0$,*

$$|T_\varepsilon f(x) + B(x,\varepsilon)T_{N\varepsilon}(Tf)(x)| \lesssim M^2(Tf)(x).$$

1.3.1 Proof of Theorem 1.1.1

Fix the Lipschitz function $A(x) = |x|$. In this case,

$$B(0,\varepsilon) = \log \frac{|z(\varepsilon) - z(0)|}{|z(-\varepsilon) - z(0)|} + i\Big(\pi + \arg[z(\varepsilon) - z(0)] - \arg[z(-\varepsilon) - z(0)]\Big) = \frac{\pi i}{2}.$$

Assume that the inequality

$$T_* f(x) \lesssim M^n(Tf)(x) \quad \text{for all } f \in L^2(\mathbb{R})$$

were true for some $n \geq 2$. Then, applying Lemma 1.3.6, this would yield

$$|B(x,\varepsilon) T_{N\varepsilon}(Tf)(x)| \lesssim M^n(Tf)(x)$$

for all $f \in L^2(\mathbb{R})$. Now, taking into account that $T^2 = Id$, and setting $x = 0$, the latter implies

$$|T_{N\varepsilon} f(0)| \lesssim M^n f(0), \tag{1.10}$$

for all $f \in L^2(\mathbb{R})$, and this is false for $f = \chi_{[0,1]}$. Indeed, $M^n f(0) \leq 1$, while for $0 < N\varepsilon < 1$,

$$\begin{aligned} T_{N\varepsilon} f(0) &= \frac{1}{\pi i} \int_{|y| > N\varepsilon} \chi_{[0,1]}(y) \frac{dz(y)}{z(y) - z(0)} \\ &= \frac{1}{\pi i} \int_{N\varepsilon}^{1} \frac{1+i}{y + iy} dy \\ &= \frac{1}{\pi i} \int_{N\varepsilon}^{1} \frac{dy}{y} = -\frac{1}{\pi i} \log(N\varepsilon), \end{aligned}$$

so

$$\lim_{\varepsilon \to 0} |T_{N\varepsilon} f(0)| = \infty,$$

yielding a contradiction with (1.10).

This counterexample can be generalized in the following way: suppose Γ has an angle at a point $z(x)$, $x \in \mathbb{R}$, meaning with this that A' has a jump discontinuity at x, i.e.,

$$\lim_{h \to 0^+} \frac{A(x+h) - A(x)}{h} = A'_+(x) \neq A'_-(x) = \lim_{h \to 0^-} \frac{A(x+h) - A(x)}{h}.$$

A straightforward computation shows now that

$$\lim_{\varepsilon \to 0} \operatorname{Im} B(x,\varepsilon) = \arctan(A'_+(x)) - \arctan(A'_-(x)) \neq 0,$$

and so $B(x,\varepsilon)$ stays away from 0 as $\varepsilon \to 0$. The same argument that was used above, substituting $\chi_{[0,1]}$ by $\chi_{[x,x+1]}$, will show that the inequality

$$T_* f(x) \lesssim M^n(Tf)(x)$$

cannot hold.

1.3.2 Proof of theorem 1.1.2

We will study now the term $T_{N\varepsilon}(Tf)(x)$ to give more light to this subject. This will lead us to prove that, when A has compact support, the inequality

$$T_* f(x) \lesssim M^n(Tf)(x)$$

can only hold when $A = 0$, i.e., when Γ is a straight line, which is a case already known since T is, essentially, the Hilbert transform.

Assume that A has compact support, say $\operatorname{supp}(A) \subset [-L, L]$, $L > 0$. Let $f \in L^2(\mathbb{R})$, and write

$$g = (Tf)\chi_{[-2L,2L]}, \quad h = (Tf)\chi_{\mathbb{R}\setminus[-2L,2L]},$$

so that $Tf = g + h$ and

$$T_{N\varepsilon}(Tf)(x) = T_{N\varepsilon}g(x) + T_{N\varepsilon}h(x).$$

Fix $x \in [-L, L]$. Observe that

$$\begin{aligned} i\pi T_{N\varepsilon}g(x) &= \int_{|y-x|>N\varepsilon} \frac{g(y)}{z(y)-z(x)} dz(y) \\ &= \sum_{k=0}^{\infty} \int_{2^k N\varepsilon<|y-x|<2^{k+1}N\varepsilon} \frac{g(y)}{z(y)-z(x)} dz(y). \end{aligned}$$

Now, taking into account that $\operatorname{supp}(g) \subset [-2L, 2L]$, one gets that, when $2^k N\varepsilon > 4L$,

$$\int_{2^k N\varepsilon<|y-x|<2^{k+1}N\varepsilon} \frac{g(y)}{z(y)-z(x)} dz(y) = 0.$$

This yields that only the first $M_{L,\varepsilon}$ terms of the sum above do not vanish, where

$$M_{L,\varepsilon} = \left\lceil \frac{\log\left(\frac{4L}{N\varepsilon}\right)}{\log 2} \right\rceil$$

(by $\lceil t \rceil$ we denote the smallest integer n such that $t \le n$).

Furthermore, for each $k \geq 0$,

$$\begin{aligned} \left| \int_{2^k N\varepsilon<|y-x|<2^{k+1}N\varepsilon} \frac{g(y)}{z(y)-z(x)} dz(y) \right| &\lesssim \int_{2^k N\varepsilon<|y-x|<2^{k+1}N\varepsilon} \frac{|g(y)|}{|y-x|} dy \\ &\lesssim \frac{1}{2^k N\varepsilon} \int_{|y-x|<2^{k+1}N\varepsilon} |g(y)| dy \\ &\lesssim Mg(x). \end{aligned}$$

Putting all together, and taking into account that $Mg \le M(Tf)$, we obtain

$$|T_{N\varepsilon}g(x)| \lesssim \left(1 + \left|\frac{\log\left(\frac{4L}{N\varepsilon}\right)}{\log 2}\right|\right) M(Tf)(x).$$

On the other hand, since $A = 0$ on $\operatorname{supp}(h)$, we get

$$i\pi T_{N\varepsilon}h(x) = \int_{|y-x|>N\varepsilon} \frac{h(y)}{z(y)-z(x)} dz(y) = \int_{|y-x|>N\varepsilon} \frac{h(y)}{y-z(x)} dy.$$

Now, for $|y - x| > N\varepsilon$,

$$\frac{1}{y-z(x)} = \frac{1}{y-x} + \left(\frac{1}{y-z(x)} - \frac{1}{y-x}\right) := \frac{1}{y-x} + D(x,y),$$

and so

$$\begin{aligned} i\pi T_{N\varepsilon}h(x) &= \int_{|y-x|>N\varepsilon} \frac{h(y)}{y-x} dy + \int_{|y-x|>N\varepsilon} h(y)D(x,y)dy \\ &:= H_{N\varepsilon}h(x) + \int_{|y-x|>N\varepsilon} h(y)D(x,y)dy. \end{aligned}$$

Observe now that, for $x \neq y$,

$$|D(x,y)| = \left| \frac{1}{y - z(x)} - \frac{1}{y - x} \right| = \left| \frac{iA(x)}{(y-x)(y-z(x))} \right| \leq \frac{|A(x)|}{|y-x|^2}.$$

Then, taking into account that $h = 0$ on $[-2L, 2L]$, and recalling that $|x| \leq L$, one gets

$$\left| \int_{|y-x|>N\varepsilon} h(y) D(x,y) dy \right| \leq |A(x)| \int_{|y-x|>L} \frac{|h(y)|}{|y-x|^2} dy.$$

Splitting the last integral into the regions $\{2^k L < |y-x| \leq 2^{k+1} L\}$, and using the fact that $M(h) \leq M(Tf)$, we get

$$\left| \int_{|y-x|>N\varepsilon} h(y) D(x,y) dy \right| \leq \frac{8}{L} |A(x)| M(Tf)(x).$$

The previous discussion shows that

$$T_{N\varepsilon}(Tf)(x) = \frac{1}{\pi i} H_{N\varepsilon} h(x) + \mathsf{V},$$

where

$$|\mathsf{V}| \leq c(x, \varepsilon, N, L) M(Tf)(x)$$

and $0 < c(x, \varepsilon, N, L) < \infty$. Recall now that, by Lemma 1.3.6, we have

$$|T_\varepsilon f(x) + B(x, \varepsilon) T_{N\varepsilon}(Tf)(x)| \lesssim M^2(Tf)(x).$$

Then, it follows that

$$\left| T_\varepsilon f(x) + \frac{1}{\pi i} B(x, \varepsilon) H_{N\varepsilon} h(x) \right| \leq c'(x, \varepsilon, N, L) M^2(Tf)(x),$$

where $0 < c'(x, \varepsilon, N, L) < \infty$.

Assume A is not identically null, and suppose that the inequality $T_* f(x) \lesssim M^n(Tf)(x)$ holds. Applying Lemma 1.3.4, we may pick $x \in [-L, L]$ and $\varepsilon > 0$ with

$$-L < x - N\varepsilon < x < x + N\varepsilon < L$$

and such that $B(x, \varepsilon) \neq 0$. Then, it follows that

$$|B(x, \varepsilon)| |H_{N\varepsilon}((Tf)\chi_{\mathbb{R} \setminus [-2L, 2L]})(x)| \leq c''(x, \varepsilon, N, L) M^n(Tf)(x),$$

with $0 < c''(x, \varepsilon, N, L) < \infty$.

Now, for each $k = 3, 4 \ldots$, pick $f_k \in L^2(\mathbb{R})$ such that $Tf_k = \chi_{[0, kL]}$, and so $(Tf_k)\chi_{\mathbb{R} \setminus [-2L, 2L]} = \chi_{(2L, kL]}$. Applying the previous inequality for each f_k, and using the fact that $M^n(Tf_k) \leq 1$, we obtain

$$|B(x, \varepsilon)| |H_{N\varepsilon}(\chi_{(2L, kL]})(x)| \leq c''(x, \varepsilon, N, L).$$

Finally, observe that

$$H_{N\varepsilon}(\chi_{(2L, kL]})(x) = \int_{2L}^{kL} \frac{dy}{y - x} = \log \frac{kL - x}{2L - x},$$

and so

$$|B(x, \varepsilon)| \log \frac{kL - x}{2L - x} \leq c''(x, \varepsilon, N, L),$$

yielding a contradiction, since the left hand side tends to ∞ as $k \to \infty$.

1.4 Further results

1.4.1 Another version of the truncated operators

Let us consider now another version of the truncated operators. Define, for $\varepsilon > 0$ and $x \in \mathbb{R}$,

$$\tilde{T}_\varepsilon f(x) = \frac{1}{\pi i} \int_{|z(y)-z(x)|>\varepsilon} \frac{f(y)}{z(y)-z(x)} dz(y)$$

and the associated maximal operator $\tilde{T}_* f(x) = \sup_{\varepsilon>0} |\tilde{T}_\varepsilon f(x)|$. This is a truncation over balls of radius ε, while the one for T_ε was a truncation over strips of width 2ε.

We consider now the same problem as before: that of giving an estimate of the form

$$\tilde{T}_* f(x) \lesssim M^n(Tf)(x),$$

and the same arguments employed before will work here. Indeed, if we define $l(x,\varepsilon) = z(x_-)$, $r(x,\varepsilon) = z(x_+)$, where

$$x_- = \sup\{t < x \colon |z(t) - z(x)| = \varepsilon\}$$

and

$$x_+ = \inf\{t > x \colon |z(t) - z(x)| = \varepsilon\},$$

then $l(x,\varepsilon)$ and $r(x,\varepsilon)$ will play the same role that $z(x-\varepsilon)$ and $z(x+\varepsilon)$ played before. Precisely, $l(x,\varepsilon)$ is the *last* point of Γ to the left of $z(x)$ that belongs to the circle centered at $z(x)$ with radius ε, and $r(x,\varepsilon)$ is the analogue of this one at the right.

Since the quantities $|y-x|$ and $|z(y)-z(x)|$ are comparable, one can repeat the arguments used before to get an analogous of Lemma 1.3.6, which will be stated now as

$$|\tilde{T}_\varepsilon f(x) + \tilde{B}(x,\varepsilon)\tilde{T}_{N\varepsilon} f(x)| \lesssim M^2(Tf)(x),$$

where

$$\tilde{B}(x,\varepsilon) = \log \frac{|r(x,\varepsilon)-z(x)|}{|l(x,\varepsilon)-z(x)|} + i\Big(\pi + \arg[r(x,\varepsilon)-z(x)] - \arg[l(x,\varepsilon)-z(x)]\Big).$$

As in Lemma 1.3.4, $\tilde{B}(x,\varepsilon) = 0$ if, and only if, $l(x,\varepsilon)$, $z(x)$ and $r(x,\varepsilon)$ are collinear.

With this tools at hand, one can prove the following results, which are the analogs to Theorems 1.1.1 and 1.1.2 in this setting.

Theorem 1.4.1. *Consider the Lipschitz function $A(x) = |x|$. Then, there exists $f \in L^2(\mathbb{R})$ such that for all $c > 0$ and all $n \geq 1$, there exists $\varepsilon > 0$ such that*

$$|\tilde{T}_\varepsilon f(0)| > cM^n(Tf)(0).$$

To prove this, one can mimic the argument in Section 1.3.1, since here we have again $\tilde{B}(0,\varepsilon) = i\frac{\pi}{2}$.

Theorem 1.4.2. *Let A be a Lipschitz function with compact support. Suppose A is not identically null, or, equivalently, that Γ is not a straight line. Then, there exists $x \in \mathbb{R}$ such that for all $c > 0$ there exists $f \in L^2(\mathbb{R})$ with*

$$\tilde{T}_* f(x) > c\tilde{T}^n(Tf)(x)$$

for all $n \geq 1$.

Again, the argument in Section 1.3.2 adapts trivially to this case, by just taking into account that, if A is not identically null, one can find $x \in \mathbb{R}$ and $\varepsilon > 0$ as small as needed such that $l(x,\varepsilon)$, $z(x)$ and $r(x,\varepsilon)$ are not collinear.

1.4.2 The case of Jordan curves

Let Γ be a Jordan curve in the plane, parametrized by a periodic function $\gamma\colon \mathbb{R} \to \mathbb{C}$. We will pose, for the moment, the following assumptions on γ:

- γ is of class $\mathcal{C}^1$.
- γ is L-periodic, $\gamma([0, L)) = \Gamma$.
- γ is injective on $[0, L)$.
- $|\gamma'(t)| = 1$ for all t.
- ω is the modulus of continuity of γ' (this means that ω is a non-negative and increasing continuous function in $[0, \infty)$ with $\omega(0) = 0$ and such that $|\gamma'(s) - \gamma'(t)| \leq \omega(|s - t|)$ for all $s, t \in \mathbb{R}$).

We denote by μ the arc-length measure on Γ. We have, for a Borel set $I \subset [0, L)$,

$$\mu(\gamma(I)) = \int_I |\gamma'(t)| dt = |I|.$$

For a point $z \in \Gamma$ and $r > 0$, denote

$$\Gamma_{z,r} = \gamma(\{t : |t - x| < r\}),$$

where $z = \gamma(x)$, $x \in \mathbb{R}$.

The Hardy-Littlewood maximal function of a function $f \in L^1(\Gamma, \mu)$ is defined, for $z \in \Gamma$, by

$$Mf(z) = \sup_{r>0} \frac{1}{\mu(\Gamma_{z,r})} \int_{\Gamma_{z,r}} |f| d\mu = \sup_{r>0} \frac{1}{2r} \int_{\Gamma_{z,r}} |f| d\mu$$

The Cauchy transform of a function $f \in L^2(\Gamma, d\mu)$ is defined, for $z \in \Gamma$, as the principal value integral

$$Tf(z) = \lim_{\varepsilon \to 0} T_\varepsilon f(z),$$

where

$$T_\varepsilon f(z) = \frac{1}{\pi i} \int_{\Gamma \setminus \Gamma_{z,\varepsilon}} \frac{f(\xi)}{\xi - z} d\xi.$$

We consider as well the maximal operator associated with T,

$$T_* f(z) = \sup_{\varepsilon > 0} |T_\varepsilon f(z)|.$$

In this section we will prove that, if γ is regular enough (we will specify later how much regularity is needed), then

$$T_* f(z) \lesssim M^2(Tf)(z) \;\text{ for all } f \in L^2(\Gamma, \mu).$$

To do so, we will follow, essentially, the same steps we have taken in Section 1.3 for the case of Lipschitz graphs. Most of the arguments there will be valid in this setting, and so we will not enter into many details. First of all, we remark that the analogs of Lemmas 1.2.1 and 1.2.2 hold now:

Lemma 1.4.1. *If $f \in L^2(\Gamma, \mu)$, $T^2 f = f$.*

Lemma 1.4.2. *If $f, g \in L^2(\Gamma, \mu)$, then*

$$\int_\Gamma Tf(z) g(z) dz = -\int_\Gamma f(z) Tg(z) dz.$$

We argue now as in Section 1.3. Fix $f \in L^2(\Gamma, \mu)$, $z \in \Gamma$ and $\varepsilon > 0$. Then, we have

$$T_\varepsilon f(z) = \frac{1}{\pi i} \int_{\Gamma \setminus \Gamma_{z,\varepsilon}} \frac{f(\xi)}{\xi - z} d\xi = \int_\Gamma f(\xi) K_{z,\varepsilon}(\xi) d\xi,$$

where

$$K_{z,\varepsilon}(\xi) = \frac{1}{\pi i(\xi - z)} \chi_{\Gamma \setminus \Gamma_{z,\varepsilon}}(\xi).$$

It is easy to check that $K_{z,\varepsilon} \in L^2(\Gamma, \mu) \cap L^\infty(\Gamma, \mu)$, and moreover

$$||K_{z,\varepsilon}||_{L^2} \lesssim \frac{1}{\sqrt{\varepsilon}}, \quad ||K_{z,\varepsilon}||_{L^\infty} \lesssim \frac{1}{\varepsilon}.$$

Since $K_{z,\varepsilon} \in L^2(\Gamma, \mu)$, we have $K_{z,\varepsilon} = T^2(K_{z,\varepsilon}) = T(g_{z,\varepsilon})$, for $g_{z,\varepsilon} = T(K_{z,\varepsilon})$. Then, we get

$$T_\varepsilon f(z) = \int_\Gamma f(\xi) K_{z,\varepsilon}(\xi) d\xi = \int_\Gamma f(\xi) T(g_{z,\varepsilon})(\xi) d\xi = -\int_\Gamma Tf(\xi) g_{z,\varepsilon}(\xi) d\xi,$$

and, as a consequence,

$$\begin{aligned}
-T_\varepsilon f(z) &= \int_\Gamma Tf(\xi) g_{z,\varepsilon}(\xi) d\xi \\
&= \int_{\Gamma_{z,2\varepsilon}} Tf(\xi) g_{z,\varepsilon}(\xi) d\xi + \int_{\Gamma \setminus \Gamma_{z,2\varepsilon}} Tf(\xi) g_{z,\varepsilon}(\xi) d\xi \\
&= \int_{\Gamma_{z,2\varepsilon}} Tf(\xi)[g_{z,\varepsilon}(\xi) - m_{\Gamma_{z,2\varepsilon}}(g_{z,\varepsilon})] d\xi + m_{\Gamma_{z,2\varepsilon}}(g_{z,\varepsilon}) \int_{\Gamma_{z,2\varepsilon}} Tf(\xi) d\xi + \int_{\Gamma \setminus \Gamma_{z,2\varepsilon}} Tf(\xi) g_{z,\varepsilon}(\xi) d\xi \\
&= \mathsf{I} + \mathsf{II} + \mathsf{III},
\end{aligned}$$

where, for a function $h \in L^1(\Gamma, \mu)$ and a Borel set $E \subset \Gamma$ with $\mu(E) > 0$,

$$m_E h = \frac{1}{\mu(E)} \int_E h d\mu.$$

Arguing essentially as in Section 1.3, one can prove that $|\mathsf{I}| \lesssim M^2(Tf)(z)$ and $|\mathsf{II}| \lesssim M(Tf)(z)$. Let us study III now.

$$\mathsf{III} = \int_{\Gamma \setminus \Gamma_{z,2\varepsilon}} Tf(\xi) g_{z,\varepsilon}(\xi) d\xi = \int_{\Gamma \setminus \Gamma_{z,2\varepsilon}} Tf(\xi) T(K_{z,\varepsilon})(\xi) d\xi.$$

A similar argument to the one used in Lemma 1.3.3 yields the following result.

Lemma 1.4.3. *For $\xi \in \Gamma \setminus \Gamma_{z,2\varepsilon}$,*

$$T(K_{z,\varepsilon})(\xi) = \frac{1}{\pi i} \frac{1}{z - \xi} [B(z, \varepsilon) + G_{z,\varepsilon}(\xi)],$$

where

$$G_{z,\varepsilon}(\xi) \lesssim \frac{\varepsilon}{|z - \xi|}$$

and

$$|B(z, \varepsilon)| \lesssim \omega(2\varepsilon).$$

Remark: The expressions of $G_{z,\varepsilon}(\xi)$ and $B(z,\varepsilon)$ are totally analogous to the ones for $G_{x,\varepsilon}(y)$ and $B(x,\varepsilon)$ in Lemma 1.3.3, for suitably chosen branches of $\arg(w - z)$ and $\arg(w - \xi)$. The estimate for $G_{z,\varepsilon}$ is proved as in Lemma 1.3.5, while the estimate for $B(z,\varepsilon)$ follows from an application of the Mean Value Theorem.

From this, it follows that

$$\begin{aligned}
\mathsf{III} &= \int_{\Gamma\setminus\Gamma_{z,2\varepsilon}} Tf(\xi)T(K_{z,\varepsilon})(\xi)d\xi \\
&= B(z,\varepsilon)\frac{1}{\pi i}\int_{\Gamma\setminus\Gamma_{z,2\varepsilon}} Tf(\xi)\frac{1}{z-\xi}d\xi + \frac{1}{\pi i}\int_{\Gamma\setminus\Gamma_{z,2\varepsilon}} Tf(\xi)\frac{G_{z,\varepsilon}(\xi)}{z-\xi}d\xi \\
&= B(z,\varepsilon)T_{2\varepsilon}(Tf)(z) + \frac{1}{\pi i}\int_{\Gamma\setminus\Gamma_{z,2\varepsilon}} Tf(\xi)\frac{G_{z,\varepsilon}(\xi)}{z-\xi}d\xi \\
&= \mathsf{III}_1 + \mathsf{III}_2.
\end{aligned}$$

On the one hand,

$$\begin{aligned}
|\mathsf{III}_2| &\le \frac{1}{\pi}\int_{\Gamma\setminus\Gamma_{z,2\varepsilon}} |Tf(\xi)|\frac{|G_{z,\varepsilon}(\xi)|}{|z-\xi|}d\mu(\xi) \\
&\lesssim \varepsilon\int_{\Gamma\setminus\Gamma_{z,2\varepsilon}} \frac{|Tf(\xi)|}{|z-\xi|^2}d\mu(\xi) \lesssim M(Tf)(z)
\end{aligned}$$

where the last inequality is shown by splitting the integral over the sets

$$\Gamma_{z,2^{k+1}\varepsilon}\setminus\Gamma_{z,2^k\varepsilon},\quad k=1,2,3\dots$$

On the other hand

$$|\mathsf{III}_1| = |B(z,\varepsilon)|\frac{1}{\pi}\int_{\Gamma\setminus\Gamma_{z,2\varepsilon}} \frac{|Tf(\xi)|}{|\xi-z|}d\mu(\xi) \lesssim \omega(2\varepsilon)\int_{\Gamma\setminus\Gamma_{z,2\varepsilon}} \frac{|Tf(\xi)|}{|z-\xi|}d\mu(\xi).$$

To estimate the last integral, we also split it over the sets

$$\Gamma_{z,2^{k+1}\varepsilon}\setminus\Gamma_{z,2^k\varepsilon},\quad k=1,2,3\dots$$

Notice that, for k big enough, $\Gamma_{z,2^k\varepsilon}=\Gamma$, and so $\Gamma_{z,2^{k+1}\varepsilon}\setminus\Gamma_{z,2^k\varepsilon}=\emptyset$. Precisely, this holds for all k such that $2^k\varepsilon>2L$, which is equivalent to

$$k > \frac{\log\frac{2L}{\varepsilon}}{\log 2}.$$

As a result, if we denote by $k_0(\varepsilon)$ the smallest integer k that satisfies the previous inequality, we have

$$\begin{aligned}
\int_{\Gamma\setminus\Gamma_{z,2\varepsilon}} \frac{|Tf(\xi)|}{|z-\xi|}d\mu(\xi) &= \sum_{k=1}^{k_0(\varepsilon)}\int_{\Gamma_{z,2^{k+1}\varepsilon}\setminus\Gamma_{z,2^k\varepsilon}} \frac{|Tf(\xi)|}{|z-\xi|}d\mu(\xi) \\
&\lesssim \sum_{k=1}^{k_0(\varepsilon)}\frac{1}{2^k\varepsilon}\int_{\Gamma_{z,2^{k+1}\varepsilon}\setminus\Gamma_{z,2^k\varepsilon}} |Tf(\xi)|d\mu(\xi) \\
&\le 4\sum_{k=1}^{k_0(\varepsilon)}\frac{1}{2\cdot 2^k\varepsilon}\int_{\Gamma_{z,2^{k+1}\varepsilon}} |Tf(\xi)|d\mu(\xi) \\
&\le 4k_0(\varepsilon)M(Tf)(z).
\end{aligned}$$

As a result,

$$|\mathsf{III}_1| \lesssim \omega(2\varepsilon)k_0(\varepsilon)M(Tf)(z) \lesssim \omega(2\varepsilon)\left|\log\frac{2L}{\varepsilon}\right|M(Tf)(z).$$

Gathering the estimates for $|\mathsf{I}|$, $|\mathsf{II}|$, $|\mathsf{III}_1|$ and $|\mathsf{III}_2|$, we have

$$|T_\varepsilon f(z)| \lesssim M^2(Tf)(z) + \omega(2\varepsilon)\left|\log\frac{2L}{\varepsilon}\right|M(Tf)(z).$$

From this, it follows that, if ω is such that $\omega(2\varepsilon)|\log\varepsilon|$ stays bounded as $\varepsilon\to 0$, then we have

$$|T_\varepsilon f(z)| \lesssim M^2(Tf)(z).$$

Thus, we have proved the following result:

Theorem 1.4.3. *With the notation established in this section, suppose γ' has a modulus of continuity ω such that $\omega(\varepsilon)|\log \varepsilon|$ stays bounded as $\varepsilon \to 0$ (this happens, for example, if $\gamma \in \mathcal{C}^{1+\delta}$ for some $\delta > 0$). Then, there exists a constant $c > 0$ such that, for all $f \in L^2(\Gamma, d\mu)$ and all $z \in \Gamma$,*

$$T_* f(z) \leq c M^2(Tf)(z).$$

We want to remark, finally, that a totally analogous result holds if one considers the truncated operators given by

$$\tilde{T}_\varepsilon f(z) = \frac{1}{\pi i} \int_{\Gamma \setminus B(z,\varepsilon)} \frac{f(\xi)}{\xi - z} d\xi.$$

Chapter 2

Geometric conditions for the L^2-boundedness of singular integral operators with odd kernels with respect to measures with polynomial growth in $\mathbb{R}^d$

2.1 Introduction

In this chapter, we study $L^2(\mu)$-boundedness of singular integral operators with sufficiently smooth convolution-type kernels. More precisely, we will consider kernels of the form $k(x, y) = K(x - y)$, where $K\colon \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ is an odd and $\mathcal{C}^2$ function that satisfies

$$|\nabla^j K(x)| \leq \frac{C(j)}{|x|^{n+j}} \text{ for all } x \neq 0 \text{ and } j \in \{0, 1, 2\}.$$

It is easy to check that the inequalities above imply that k is a Calderón-Zygmund kernel with $\delta = 1$ in (1.1). We will denote by $\mathcal{K}^n(\mathbb{R}^d)$ the class of all these kernels.

In [T3], Tolsa proved the following result[1]:

Theorem D. *Let μ be a Radon measure in $\mathbb{C}$ without atoms. If the Cauchy transform $\mathcal{C}_\mu$ is bounded in $L^2(\mu)$, then all 1-dimensional singular integral operators T_μ with kernels in $\mathcal{K}^1(\mathbb{C})$ are also bounded in $L^2(\mu)$.*

In order to prove this result, Tolsa relied on a suitable corona decomposition for measures with linear growth and finite curvature[2] and split the operator T into a sum of different operators K_R, each of which are associated to a *tree* of the corona decomposition. The operators K_R are bounded because on each tree the measure μ can be approximated by arc length on an Ahlfors-David regular curve and, moreover, the operators K_R behave in a quasiorthogonal way.

[1] Tolsa's result in [T3] is actually stated for operators with smoother kernels than the ones we consider here. However, after the publication of [T5], it is obvious that it can be generalized to obtain Theorem D

[2] We will not enter into details about curvature of measures and its relationship with the boundedness of the Cauchy transform here, but an interested reader is encouraged to read [T6, Chapters 3 and 7] for further information on this issue.

However, as that corona construction relied heavily on the relationship between the Cauchy transform and curvatures of measures, it could not be easily generalized to higher dimensions. Using a new corona decomposition that involves the β-numbers of Jones, David and Semmes instead of curvature and which is valid for all dimensions, Azzam and Tolsa [AT] have recently proved the following:

Theorem E. *Let μ be a finite Radon measure with compact support in $\mathbb{C}$ with linear growth. Then, for all $\varepsilon > 0$,*

$$||C_\varepsilon \mu||^2_{L^2(\mu)} \lesssim ||\mu|| + \iint_0^\infty \beta^n_{\mu,2}(x,r)^2 \theta^1_\mu(x,r) \frac{dr}{r} d\mu(x).$$

Some notions need to be defined here: first of all, a Borel measure μ in $\mathbb{R}^d$ is said to have polynomial growth of degree n if there is a constant $c_0 \geq 0$ such that $\mu[B(x,r)] \leq c_0 r^n$ for all $x \in \mathbb{R}^d$ and all $r > 0$ (when $n = 1$, μ is said to have linear growth). μ is said to be n-AD-regular (or just AD-regular or Ahlfors-David-regular) if there is a constant $c_0 > 0$ such that

$$c_0^{-1} r^n \leq \mu[B(x,r)] \leq c_0 r^n \text{ for all } x \in \operatorname{supp}(\mu) \text{ and all } 0 < r \leq \operatorname{diam}(\operatorname{supp}(\mu)).$$

Secondly, given a ball $B(x,r) \subset \mathbb{R}^d$, we define

$$\theta^n_\mu[B(x,r)] = \theta^n_\mu(x,r) = \frac{\mu(B(x,r))}{r^n}.$$

Finally, for $1 \leq p < \infty$, the $\beta^n_{\mu,p}$-coefficient of a ball B with radius $r(B)$ is defined by

$$\beta^n_{\mu,p}(B) = \inf_L \left(\frac{1}{r(B)^n} \int_B \left(\frac{\operatorname{dist}(y,L)}{r(B)} \right)^p d\mu(y) \right)^{\frac{1}{p}},$$

where the infimum is taken over all n-planes $L \subset \mathbb{R}^d$.

To understand the importance of these β-coefficients, recall that a set $E \subset \mathbb{R}^d$ is called n-rectifiable if there are Lipschitz maps $f_i : \mathbb{R}^n \to \mathbb{R}^d$, $i = 1, 2, \ldots$, such that

$$\mathcal{H}^n \left(E \setminus \bigcup_i f_i(\mathbb{R}^n) \right) = 0, \tag{2.1}$$

where $\mathcal{H}^n$ stands for the n-dimensional Hausdorff measure. Also, one says that a Radon measure μ on $\mathbb{R}^d$ is n-rectifiable if μ vanishes out of an n-rectifiable set $E \subset \mathbb{R}^d$ and moreover μ is absolutely continuous with respect to $\mathcal{H}^n \lfloor_E$.

With these definitions at hand, we remark now that these $\beta^n_{\mu,p}$-coefficients are a generalization of the β-numbers introduced by Jones in [J], where he used them to characterize compact subsets of the plane that are contained in a rectifiable set.

Recall as well that a measure μ in $\mathbb{R}^d$ is said to be uniformly n-rectifiable if it is n-AD-regular and there exist $\theta, M > 0$ such that for all $x \in \operatorname{supp}(\mu)$ and all $r > 0$ there is a Lipschitz mapping g from the ball $B_n(0,r)$ in $\mathbb{R}^n$ to $\mathbb{R}^d$ with $\operatorname{Lip}(g) \leq M$ such that

$$\mu(B(x,r) \cap g(B_n(0,r))) \geq \theta r^n.$$

We will refer to the constants M, θ as the UR (uniform rectifiability) constants of μ. In the particular case when $\mu = \mathcal{H}^n \lfloor_E$ for some set $E \subset \mathbb{R}^d$, we say E is uniformly is n-rectifiable if μ is uniformly n-rectifiable and we call the UR constants of μ, simply, the UR constants of E.

Another important application of the β-coefficients is, as David and Semmes proved in [DS1], that an n-AD-regular measure μ is uniformly n-rectifiable if, and only if, there is some constant $c > 0$ such that, for every ball B with centre on $\operatorname{supp}(\mu)$,

$$\int_B \int_0^{r(B)} \beta^n_{\mu,2}(x,r)^2 \frac{dr}{r} d\mu(x) \leq c\mu(B). \tag{2.2}$$

Very recently, Azzam and Tolsa (see [AT] and [T7]) have shown that a positive and finite Borel measure μ in $\mathbb{R}^d$ with

$$0 < \limsup_{r\to 0} \theta^n_\mu(x,r) < \infty \text{ for } \mu - \text{a.e. } x \in \mathbb{R}^d$$

is n-rectifiable if, and only if,

$$\int_0^1 \beta_{\mu,2}(x,r)^2 \frac{dr}{r} < \infty \tag{2.3}$$

for μ-a.e. $x \in \mathbb{R}^d$.

Using the corona decomposition from [AT], we prove the following result:

Theorem 2.1.1. *Let μ be a finite Radon measure in $\mathbb{R}^d$ with polynomial growth of degree n and such that, for all balls $B \subset \mathbb{R}^d$ with radius $r(B)$,*

$$\int_B \int_0^{r(B)} \beta^n_{\mu,2}(x,r)^2 \theta^n_\mu(x,r) \frac{dr}{r} d\mu(x) \lesssim \mu(B). \tag{2.4}$$

Then, all Calderón-Zygmund operators T_μ with kernels in $\mathcal{K}^n(\mathbb{R}^d)$ are bounded in $L^2(\mu)$.

Notice that (2.4) is a quantitative version of (2.3), just like (2.2), with no assumptions on the AD-regularity of μ. A trivial example of a measure μ that is not n-AD-regular and satisfies (2.4) is the area measure on a square (with $d = 2$ and $n = 1$). Of course, the most interesting examples with regard to this result will arise from measures that have *some n-dimensional nature* (e.g., measures supported on sets with Hausdorff dimension equal to n).

When $n = d - 1$, the previous result can be applied to get an interesting estimate for the Lipschitz harmonic capacity. Recall that the Lipschitz harmonic capacity of a compact set $E \subset \mathbb{R}^d$ is defined by

$$\kappa(E) = \sup |\langle \varDelta\varphi, 1\rangle|,$$

where the supremum is taken over all Lipschitz functions $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ that are harmonic in $\mathbb{R}^d \setminus E$ and satisfy $||\nabla\varphi||_\infty \leq 1$. Here $\langle \varDelta\varphi, 1\rangle$ denotes the action of the compactly supported distributional Laplacian $\varDelta\varphi$ on the function 1. This notion was introduced by Paramonov [Pa] to study the problem of $\mathcal{C}^1$ harmonic approximation on compact subsets of $\mathbb{R}^d$ and, as it was proved by Mattila and Paramonov in [MP], serves to characterize removable sets for Lipschitz harmonic functions as those sets E with $\kappa(E) = 0$. Later, Volberg [V] proved that

$$\kappa(E) \approx \sup\{\mu(E)\colon \mu \in \varSigma_n(E), ||\mathcal{R}^n_\mu||_{L^2(\mu)\to L^2(\mu)} \leq 1\},$$

where $\varSigma_n(E)$ stands for the subset of the positive measures μ supported on E such that $\mu[B(x,r)] \leq r^n$ for all x, r and $\mathcal{R}^n_\mu$ is the n-dimensional Riesz transform with respect to μ. Using this comparability and Theorem 2.1.1, we obtain the following:

Corollary 2.1.1. *Let E be a compact set in $\mathbb{R}^{n+1}$. Then,*

$$\kappa(E) \gtrsim \sup \mu(E), \tag{2.5}$$

where the supremum is taken over all positive Borel measures μ supported on E such that

$$\sup_{x\in\mathbb{R}^{n+1}, R>0} \left\{ \theta^n_\mu(x,R) + \int_0^\infty \beta_{\mu,2}(x,r)^2 \theta^n_\mu(x,r) \frac{dr}{r} \right\} \leq 1. \tag{2.6}$$

A very interesting problem would be to show that, in fact, $\gtrsim$ may be substituted by $\approx$ in (2.5), as an analog to the comparabilty between the analytic capacity γ and the capacity γ_+ obtained by Tolsa in [T2]. This would serve to characterize removable sets for Lipschitz harmonic functions in a metric-geometric way and also to prove the bi-Lipschitz invariance of Lipschitz harmonic capacity, which is still unknown. Indeed, whenever a measure μ satisfies (2.6), it is clear that it also satisfies (2.4) and then, arguing as in Section 8 of [T4], one can prove that its image measure $\sigma = \varphi_{\#}\mu$ under a bi-Lipschitz map φ satisfies

$$\sigma(B) \leq C_\varphi r(B)^n$$

and

$$\int_B \int_0^{r(B)} \beta_{\sigma,2}^n(x,r)^2 \theta_\sigma^n(x,r) \frac{dr}{r} d\sigma(x) \lesssim C_\varphi \sigma(B),$$

for all balls B of radius $r(B)$, where C_φ is a positive constant only depending on the bi-Lipschitz constant of φ. Then, using Chebyshev's inequality, one can prove that there exists an appropriate restriction τ of σ with $||\tau|| \approx ||\sigma||$ and such that

$$\sup_{x\in\mathbb{R}^{n+1}, R>0} \left\{ \theta_\tau^n(x,R) + \int_0^\infty \beta_{\tau,2}(x,r)^2 \theta_\tau^n(x,r) \frac{dr}{r} \right\} \leq C_\varphi.$$

It is worth remarking that Azzam and Tolsa were able to obtain a comparability like the one we have described for analytic capacity in [AT]:

Theorem F. *Let $E \subset \mathbb{C}$ be compact. Then,*

$$\gamma(E) \approx \sup \mu(E),$$

where the supremum is taken over all Borel measures μ in $\mathbb{C}$ such that

$$\sup_{x\in\mathbb{R}^{n+1}, R>0} \left\{ \theta_\mu^1(x,R) + \int_0^\infty \beta_{\mu,2}(x,r)^2 \theta_\mu^1(x,r) \frac{dr}{r} \right\} \leq 1.$$

2.2 Preliminaries

2.2.1 A useful estimate

Let μ be a positive Radon measure in $\mathbb{R}^d$ such that $\mu(B(x,r)) \leq c_0 r^n$ for all $x \in \mathbb{R}^d$ and all $r > 0$. Then, for all $x \in \mathbb{R}^d$ and all $r > 0$,

$$\int_{|x-y|>r} \frac{d\mu(y)}{|x-y|^{n+1}} \leq \frac{c_0}{r}. \tag{2.7}$$

This estimate, that can be easily proved by splitting the domain of integration into annuli $\{y \in \mathbb{R}^d \colon 2^k r < |y-x| \leq 2^{k+1} r\}$, $k \geq 0$, is commonly used in Calderón-Zygmund theory, and we will also make use of it several times in this paper.

2.2.2 Notation

- If B is a ball in $\mathbb{R}^d$, we denote its radius by $r(B)$. Given $\lambda > 0$, the ball which is concentric with B and has radius $\lambda r(B)$ is denoted by λB.

- If μ is a Radon measure in $\mathbb{R}^d$ and $A \subset \mathbb{R}^d$, the restriction of μ to A is denoted $\mu\lfloor_A$ or, simply, μ_A, and it is defined by

$$\mu\lfloor_A(E) = \mu(E \cap A).$$

2.2.3 Suppressed operators

In this section, we recall the definition and most important properties of the so-called *suppressed operators*, introduced by Nazarov, Treil and Volberg in [NTV], and that may be thought of as regular truncations of a singular integral operator. All definitions and results in this section can be found in [V].

Let k be an n-dimensional antisymmetric Calderón-Zygmund kernel in $\mathbb{R}^d$. Given a non-negative and 1-Lipschitz function $\Phi\colon \mathbb{R}^d \to \mathbb{R}$, we define

$$k_\Phi(x,y) = k(x,y)\frac{1}{1+k(x,y)^2\Phi(x)^n\Phi(y)^n}.$$

Then, k_Φ is also an antisymmetric Calderón-Zygmund kernel, whose Calderón-Zygmund constants do not depend on Φ but only on those of k, such that

1. $k_\Phi(x,y) = k(x,y)$ if $\Phi(x)\Phi(y) = 0$.
2. $|k_\Phi(x,y)| \leq c(n)\min\left\{\frac{1}{\Phi(x)^n}, \frac{1}{\Phi(y)^n}\right\}$.

We denote by T_Φ the integral operator associated to the kernel k_Φ, that is, if ν is a signed Borel measure in $\mathbb{R}^d$ and $x \in \mathbb{R}^d$,

$$T_\Phi\nu(x) = \int k_\Phi(x,y)d\nu(y)$$

whenever the integral makes sense. Naturally, we can also define the associated truncated operators

$$T_{\Phi,\varepsilon}\nu(x) = \int_{|x-y|>\varepsilon} k_\Phi(x,y)d\nu(y)$$

and the maximal operator

$$T_{\Phi,*}\nu(x) = \sup_{\varepsilon>0}|T_{\Phi,\varepsilon}\nu(x)|.$$

We also introduce the Hardy-Littlewood-like maximal operator associated to Φ

$$M^r_\Phi\nu(x) = \sup_{r\geq\Phi(x)}\frac{|\nu|[B(x,r)]}{r^n}.$$

As usual, if σ is any fixed positive Borel measure in $\mathbb{R}^d$, we can make these operators act on measures of the form $f\sigma$. To simplify notation, we denote, in such a case,

$$T_{\sigma,\Phi}f = T_\Phi(f\sigma), \quad T_{\sigma,\Phi,\varepsilon}f = T_{\Phi,\varepsilon}(f\sigma), \quad M^r_{\sigma,\Phi}f = M^r_\Phi(f\sigma).$$

Lemma A. *Let ν be a signed and finite Borel measure in $\mathbb{R}^d$ and $x \in \mathbb{R}^d$.*

1. *If $\varepsilon > \Phi(x)$,*
$$|T_{\Phi,\varepsilon}\nu(x) - T_\varepsilon\nu(x)| \lesssim M^r_\Phi\nu(x).$$
2. *If $\varepsilon \leq \Phi(x)$,*
$$|T_{\Phi,\varepsilon}\nu(x) - T_{\Phi,\Phi(x)}\nu(x)| \lesssim M^r_\Phi\nu(x).$$

Finally, we state a Cotlar-type inequality that will be especially useful when dealing with suppressed operators T_Φ. To do so, we introduce a couple more of maximal operators associated to any positive Radon measure σ in $\mathbb{R}^d$: for $f \in L^1_{loc}(\sigma)$ and $x \in \mathbb{R}^d$,

$$\tilde{M}_\sigma f(x) = \sup_{r>0}\frac{1}{\sigma[B(x,3r)]}\int_{B(x,r)}|f|d\sigma, \quad \tilde{M}_{\sigma,\frac{3}{2}}f(x) = \sup_{r>0}\left(\frac{1}{\sigma[B(x,3r)]}\int_{B(x,r)}|f|^{\frac{3}{2}}d\sigma\right)^{\frac{2}{3}}.$$

Theorem G. *Let σ be a positive Radon measure in $\mathbb{R}^d$, and let, for $x \in \mathbb{R}^d$,*

$$\mathcal{R}(x) = \sup\{r > 0 \colon \sigma[B(x,r)] > C_0 r^n\},$$

where $C_0 > 0$ is some fixed constant. Let S be a singular integral operator with Calderón-Zygmund kernel s, with

$$|s(x,y)| \lesssim \min\left\{\frac{1}{\mathcal{R}(x)^n}, \frac{1}{\mathcal{R}(y)^n}\right\}.$$

and such that S_σ is bounded in $L^2(\sigma)$. Then, for all $f \in L^1_{loc}(\sigma)$ and all $x \in \mathbb{R}^d$,

$$S_*(f\sigma)(x) \lesssim \tilde{M}_\sigma(S(f\sigma))(x) + \tilde{M}_{\sigma,\frac{3}{2}} f(x).$$

2.3 The dyadic lattice of cells with small boundaries

We will use the dyadic lattice of cells with small boundaries constructed by David and Mattila in [DM, Theorem 3.2]. The properties of this dyadic lattice are summarized in the next lemma.

Lemma B (David, Mattila)**.** *Let μ be a Radon measure on $\mathbb{R}^d$, $E = supp(\mu)$, and consider two constants $K_0 > 1$ and $A_0 > 5000\, K_0$. Then, there exists a sequence $\{\mathcal{D}_k\}_{k=0}^\infty$ of families of Borel subsets of E with the following properties:*

- *For each integer $k \geq 0$, $\mathcal{D}_k$ is a partition of E, that is, the sets $Q \in \mathcal{D}_k$ are pairwise disjoint and*

$$\bigcup_{Q \in \mathcal{D}_k} Q = E.$$

- *If k, l are integers, $0 \leq k < l$, $Q \in \mathcal{D}_k$ and $R \in \mathcal{D}_l$, then either $R \subset Q$ or $Q \cap R = \emptyset$.*
- *The general position of the cells Q can be described as follows: for each $k \geq 0$ and each cell $Q \in \mathcal{D}_k$, there is a ball $B(Q) = B(z_Q, r(Q))$ such that*

$$z_Q \in E, \qquad A_0^{-k} \leq r(Q) \leq K_0\, A_0^{-k}, \qquad E \cap B(Q) \subset Q \subset E \cap 28\, B(Q),$$

  where the balls $5B(Q)$, $Q \in \mathcal{D}_k$, are pairwise disjoint.

- *The cells $Q \in \mathcal{D}_k$ have small boundaries, that is, for each $Q \in \mathcal{D}_k$ and each integer $l \geq 0$, set*

$$N_l^{ext}(Q) = \{x \in E \setminus Q : \operatorname{dist}(x, Q) < A_0^{-k-l}\},$$

$$N_l^{int}(Q) = \{x \in Q : \operatorname{dist}(x, E \setminus Q) < A_0^{-k-l}\},$$

  and

$$N_l(Q) = N_l^{ext}(Q) \cup N_l^{int}(Q).$$

  Then

$$\mu(N_l(Q)) \leq (C^{-1} K_0^{-3d-1} A_0)^{-l}\, \mu(90B(Q)). \tag{2.8}$$

- *Denote by $\mathcal{D}_k^{db}$ the family of cells $Q \in \mathcal{D}_k$ for which*

$$\mu(100B(Q)) \leq K_0\, \mu(B(Q)). \tag{2.9}$$

  Then, for all $Q \in \mathcal{D}_k \setminus \mathcal{D}_k^{db}$, we have that $r(Q) = A_0^{-k}$ and $\mu[100B(Q)] \leq K_0^{-l} \mu[100^{l+1}B(Q)]$ for all $l \geq 1$ such that $100^l \leq K_0$.

We use the notation $\mathcal{D} = \bigcup_{k\geq 0} \mathcal{D}_k$. For $Q \in \mathcal{D}$, we set $\mathcal{D}(Q) = \{P \in \mathcal{D} : P \subset Q\}$.

Remark 1. *Any two disjoint cells $Q, Q' \in \mathcal{D}$ satisfy $\frac{1}{2}B(Q) \cap \frac{1}{2}B(Q') = \emptyset$. This holds with $\frac{1}{2}$ replaced by 5 in the statements in the lemma above in case that Q, Q' belong to the same generation $\mathcal{D}_k$. If $Q \in \mathcal{D}_j$ and $Q' \in \mathcal{D}_k$ with $j \neq k$, this follows easily too. Indeed, assume $j < k$, and suppose $\frac{1}{2}B(Q) \cap \frac{1}{2}B(Q') \neq \emptyset$. Since $r(Q) << r(Q')$ (by choosing A_0 to be big enough in terms of K_0), this implies that $B(Q') \subset B(Q)$, and so*

$$B(Q') \cap E \subset B(Q) \cap E \subset Q,$$

which implies that $Q' \cap Q \neq \emptyset$ and gives a contradiction.

Given $Q \in \mathcal{D}_k$, we denote $J(Q) = k$. We set $\ell(Q) = 56\, K_0\, A_0^{-k} = \ell_k$ and we call it the side length of Q. Note that

$$\frac{1}{28}\, K_0^{-1} \ell(Q) \leq \operatorname{diam}(Q) \leq \ell(Q).$$

Observe that $r(Q) \approx \operatorname{diam}(Q) \approx \ell(Q)$. In addition, we call z_Q the center of Q, and we call the cell $Q' \in \mathcal{D}_{k-1}$ such that $Q' \supset Q$ the parent of Q. We set $B_Q = 28\, B(Q)$, so that

$$E \cap \tfrac{1}{28} B_Q \subset Q \subset B_Q.$$

We assume A_0 to be big enough so that the constant $C^{-1} K_0^{-3d-1} A_0$ in (2.8) satisfies

$$C^{-1} K_0^{-3d-1} A_0 > A_0^{1/2} > 10.$$

Then we infer that, for all $0 < \lambda \leq 1$,

$$\mu\big(\{x \in Q : \operatorname{dist}(x, E \setminus Q) \leq \lambda\, \ell(Q)\}\big) + \mu\big(\{x \in 4B_Q \setminus Q : \operatorname{dist}(x, Q) \leq \lambda\, \ell(Q)\}\big) \leq c\, \lambda^{1/2}\, \mu(3.5 B_Q). \quad (2.10)$$

We denote $\mathcal{D}^{db} = \bigcup_{k\geq 0} \mathcal{D}_k^{db}$ and $\mathcal{D}^{db}(Q) = \mathcal{D}^{db} \cap \mathcal{D}(Q)$. Note that, in particular, from (2.9) we obtain

$$\mu(100 B(Q)) \leq K_0\, \mu(Q) \qquad \text{if } Q \in \mathcal{D}^{db}.$$

For this reason we will call the cells from $\mathcal{D}^{db}$ doubling.

As it is shown in [DM, Lemma 5.28], any cell $R \in \mathcal{D}$ can be covered μ-a.e. by a family of doubling cells:

Lemma C. *Let $R \in \mathcal{D}$. Suppose that the constants A_0 and K_0 in Lemma B are chosen appropriately. Then there exists a family of doubling cells $\{Q_i\}_{i\in I} \subset \mathcal{D}^{db}$, with $Q_i \subset R$ for all i, such that their union covers μ-almost all R.*

The following result is proved in [DM, Lemma 5.31].

Lemma D. *Let $R \in \mathcal{D}$ and let $Q \subset R$ be a cell such that all the intermediate cells S, $Q \subsetneq S \subsetneq R$ are non-doubling (i.e. belong to $\mathcal{D} \setminus \mathcal{D}^{db}$). Then*

$$\mu(100 B(Q)) \leq A_0^{-10n(J(Q)-J(R)-1)} \mu(100 B(R)). \quad (2.11)$$

From the preceding lemma we infer:

Lemma E. *Let $Q, R \in \mathcal{D}$ be as in Lemma D. Then*

$$\theta_\mu(100 B(Q)) \leq K_0\, A_0^{-9n(J(Q)-J(R)-1)}\, \theta_\mu(100 B(R))$$

and

$$\sum_{S \in \mathcal{D} : Q \subset S \subset R} \theta_\mu(100 B(S)) \leq c\, \theta_\mu(100 B(R)),$$

with c depending on K_0 and A_0.

Proof. By 2.11,

$$\begin{aligned}\theta_\mu(100B(Q)) &\leq A_0^{-10n(J(Q)-J(R)-1)}\,\frac{\mu(100B(R))}{r(100B(Q))^n}\\ &= A_0^{-10n(J(Q)-J(R)-1)}\theta_\mu(100B(R))\,\frac{r(B(R))^n}{r(B(Q))^n}.\end{aligned}$$

The first inequality in the lemma follows from this estimate and the fact that

$$r(B(R)) \leq K_0\, A_0^{(J(Q)-J(R))}\, r(B(Q)).$$

The second inequality in the lemma is an immediate consequence of the first one. □

From now on we will assume that K_0 and A_0 are some big fixed constants so that the results stated in the lemmas of this section hold.

2.4 The corona decomposition

Let μ be any measure satisfying the same hypotheses as the one in Theorem 2.1.1 (e.g., the restriction of the measure μ presented there to any ball B) and construct the dyadic lattice $\mathcal{D}$ of cells with small boundaries associated to μ that is given by Lemma B. Let $R_0 \in \mathcal{D}$ be such that $\operatorname{supp}(\mu) \subset R_0$ and $\operatorname{diam}(\operatorname{supp}(\mu)) \leq \ell(R_0)$ (we can assume, without loss of generality, that $\mathcal{D}_0 = \{R_0\}$), and let Top be a family of doubling cells contained in R_0 and such that $R_0 \in \mathsf{Top}$ that we will fix below.

For every $R \in \mathsf{Top}$, denote by $\mathsf{Stop}(R)$ the family of maximal cells $Q \in \mathsf{Top}$ that are contained in R, and by $\mathsf{Tree}(R)$ the family of cells $Q \in \mathcal{D}$ that are contained in R and not contained in any $Q' \in \mathsf{Stop}(Q)$. Then, we define

$$\mathsf{Good}(R) = R \setminus \bigcup_{Q\in\mathsf{Stop}(R)} Q$$

and, for $Q \subset R$,

$$\delta_\mu(Q,R) = \int_{2B_R\setminus Q} \frac{d\mu(y)}{|y-z_Q|^n}.$$

The arguments of Azzam and Tolsa [AT, Lemma 7.2] can be easily adapted to prove the following:

Lemma F. *There exists a family $\mathsf{Top} \subset \mathcal{D}^{db}$ as above such that, for all $R \in \mathsf{Top}$, there exists a bi-Lipschitz injection $g_R\colon \mathbb{R}^n \to \mathbb{R}^d$ with the bi-Lipschitz constant bounded above by some absolute constant and with image $\Gamma_R = g(\mathbb{R}^n)$ such that*

1. *μ-almost all $\mathsf{Good}(R)$ is contained in Γ_R.*
2. *For all $Q \in \mathsf{Stop}(R)$ there exists another cell $\tilde{Q} \in \mathcal{D}(R)$ with $Q \subset \tilde{Q}$ such that $\delta_\mu(Q,\tilde{Q}) \leq c\,\theta_\mu(B_R)$ and $B_{\tilde{Q}} \cap \Gamma_R \neq \emptyset$.*
3. *For all $Q \in \mathsf{Tree}(R)$, $\theta_\mu(1.1B_Q) \leq c\,\theta_\mu(B_R)$.*

Furthermore, the cells $R \in \mathsf{Top}$ satisfy the following packing condition:

$$\sum_{R\in\mathsf{Top}} \theta_\mu(B_R)^2\mu(R) \lesssim \theta_\mu(B_{R_0})^2\mu(R_0) + \iint_0^{\ell(R_0)} \beta_{\mu,2}^n(x,r)^2\theta_\mu^n(x,r)\frac{dr}{r}d\mu(x).$$

2.5 The main lemma

For technical reasons, we will assume that the kernel k of T is not only in $\mathcal{K}^n(\mathbb{R}^d)$, but that it is also a bounded function, so that the definition of $T\mu(x)$ makes perfect sense for all $x \in \mathbb{R}^d$ if μ is a finite and compactly supported Borel measure in $\mathbb{R}^d$, which is the case we are considering. However, as all of our estimates will be independent of the L^∞-norm of k, our result can be easily extended for general Calderón-Zygmund kernels $k \in \mathcal{K}^n(\mathbb{R}^d)$ by a standard smoothing procedure (see, for example, equation (44) in [T1]).

The following sections will be devoted to proving this result:

Main Lemma 2.5.1. *Let μ be a positive Radon measure in $\mathbb{R}^d$ with compact support and polynomial growth of degree n. Then,*

$$||T\mu||^2_{L^2(\mu)} \lesssim ||\mu|| + \iint \beta^n_{\mu,2}(x,r)^2 \theta^n_\mu(x,r) \frac{dr}{r} d\mu(x).$$

Theorem 2.1.1 follows from the non-homogeneous $T(1)$ theorem [T1, Theorem 1.1 and Lemma 7.3] and the previous lemma, as it enables us to estimate $||T(\chi_B\mu)||_{L^2(\chi_B\mu)}$ for all balls $B \subset \mathbb{R}^d$. Indeed, if μ is the measure from Theorem 2.1.1, B is a ball in $\mathbb{R}^d$ and $r(B)$ is its radius, applying Lemma 2.5.1 to the measure $\chi_B\mu$, we obtain

$$||T(\chi_B\mu)||^2_{L^2(\mu)} \lesssim \mu(B) + \iint \beta^n_{\chi_B\mu,2}(x,r)^2 \theta^n_{\chi_B\mu}(x,r) \frac{dr}{r} d\mu(x) \lesssim \mu(B),$$

where the last inequality follows directly from the hypotheses of Theorem 2.1.1. Therefore, the non-homogeneous $T(1)$ theorem applies, and we obtain that T_μ is bounded in $L^2(\mu)$.

To prove the Main Lemma, we will closely follow the ideas by Tolsa in [T3], but we will use the dyadic lattice $\mathcal{D}$ associated to μ, which is introduced in Section 2.3, instead of the usual dyadic lattice of *true* cubes in $\mathbb{R}^d$. We apply Lemma F to obtain a Corona Decomposition for μ, and we decompose $T\mu$ in terms of that Corona Decomposition, since the terms that arise from it will be tractable. The main difference between our proof and Tolsa's one will be found in Section 2.8, since the fact that the cells in $\mathcal{D}$ have thin boundaries helps us to avoid going through the process of averaging over random dyadic lattices to get the estimate that is proved there.

2.6 Decomposition of $T\mu$ with respect to the corona decomposition

To estimate $||T\mu||^2_{L^2(\mu)}$ we will decompose $T\mu$ with respect to the corona decomposition from Lemma F. To do so, let ψ be a non-negative and radial $\mathcal{C}^\infty$ function such that

$$\chi_{B(0,0.001)} \leq \psi \leq \chi_{B(0,0.01)} \qquad \text{and} \qquad ||\nabla\psi|| \lesssim 1.$$

For each $k \in \mathbb{Z}$, define $\psi_k(z) = \psi(A_0^k z)$ and $\varphi_k = \psi_k - \psi_{k+1}$, so that each function φ_k is non-negative and supported on $B(0, 0.01A_0^{-k}) \setminus B(0, 0.001A_0^{-k-1})$ and, furthermore,

$$\sum_{k\in\mathbb{Z}} \varphi_k(z) = 1$$

for all $x \in \mathbb{R}^d \setminus \{0\}$.

Now observe that, for $x \in \operatorname{supp}(\mu)$ we have

$$\begin{aligned} T\mu(x) &= \int k(x,y)d\mu(y) = \int \left(\sum_{k\in\mathbb{Z}} \varphi_k(x-y) \right) k(x,y)d\mu(y) \\ &= \sum_{k\in\mathbb{Z}} \int \varphi_k(x-y)k(x,y)d\mu(y). \end{aligned}$$

Therefore, if we define

$$T_k\mu(x) = \int \varphi_k(x-y)k(x,y)d\mu(y)$$

we have

$$T\mu(x) = \sum_{k\in\mathbb{Z}} T_k\mu(x).$$

Now set $\mathcal{D}_k = \{R_0\}$ whenever $k < 0$ and $T_Q\mu = \chi_Q T_{J(Q)}\mu$ for all $Q \in \mathcal{D}$. Then,

$$\begin{aligned} T\mu &= \sum_{k\in\mathbb{Z}} T_k\mu = \sum_{k\in\mathbb{Z}} \left(\sum_{Q\in\mathcal{D}_k} \chi_Q T_k\mu \right) \\ &= \sum_{k\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_k} \chi_Q T_{J(Q)}\mu = \sum_{Q\in\mathcal{D}} T_Q\mu \\ &= \sum_{Q\in\mathcal{F}} T_Q\mu + \sum_{R\in\mathsf{Top}} \left(\sum_{Q\in\mathsf{Tree}(R)} T_Q\mu \right) \\ &= \sum_{Q\in\mathcal{F}} T_Q\mu + \sum_{R\in\mathsf{Top}} K_R\mu, \end{aligned}$$

where, for $R \in \mathsf{Top}$,

$$K_R\mu = \sum_{Q\in\mathsf{Tree}(R)} T_Q\mu$$

and $\mathcal{F}$ is a finite family of cells $Q \in \mathcal{D}$ with $\ell(Q) \approx \operatorname{diam}(\operatorname{supp}(\mu))$.

Notice that for $Q \in \mathcal{F}$, the estimate

$$||T_Q\mu||^2_{L^2(\mu)} \lesssim ||\mu||$$

holds trivially. Therefore,

$$||T\mu||^2_{L^2(\mu)} \lesssim ||\mu|| + \left|\left| \sum_{R\in\mathsf{Top}} K_R\mu \right|\right|^2_{L^2(\mu)} = \sum_{R\in\mathsf{Top}} ||K_R\mu||^2_{L^2(\mu)} + \sum_{R,R'\in\mathsf{Top}:\ R\neq R'} \langle K_R\mu, K_{R'}\mu\rangle_\mu,$$

where $\langle\cdot,\cdot\rangle_\mu$ denotes the usual pairing in $L^2(\mu)$, i.e.,

$$\langle f,g\rangle_\mu = \int fg d\mu$$

The *diagonal sum* $\sum_{R\in\mathsf{Top}} ||K_R\mu||^2_{L^2(\mu)}$ will be estimated in Section 2.7 using the fact that, on each $\mathsf{Tree}(R)$, μ can be approximated by a measure of the form $\eta\mathcal{H}^n_{\Gamma_R}$, where η is a bounded function, and $T_{\mathcal{H}^n_{\Gamma_R}}$ is bounded in $L^2(\mathcal{H}^n_{\Gamma_R})$ because Γ_R is a bi-Lipschitz image of $\mathbb{R}^n$, and thus uniformly n-rectifiable (see [T5], or the more classical reference [DS2] for the case where K is assumed to be $\mathcal{C}^\infty$ away from the origin). To deal with the *non-diagonal sum* $\sum_{R,R'\in\mathsf{Top}:\ R\neq R'} \langle K_R\mu, K_{R'}\mu\rangle_\mu$, we will use quasi-orthogonality arguments. Here, the fact that the cells from $\mathcal{D}$ have thin boundaries will be crucial.

2.7 The estimate of $\sum_{R\in\mathsf{Top}} ||K_R\mu||^2_{L^2(\mu)}$

The goal of this section is to prove the following:

Lemma 2.7.1.

$$\sum_{R\in\mathsf{Top}} ||K_R\mu||^2_{L^2(\mu)} \lesssim \sum_{R\in\mathsf{Top}} \theta_\mu(B_R)^2\mu(R).$$

2.7.1 Regularization of the stopping squares

Pick $R \in \mathsf{Top}$ and define

$$d_R(x) = \inf_{Q\in\mathsf{Tree}(R)} \{|x - z_Q| + \ell(Q)\}.$$

Notice that d_R is a 1-Lipschitz function because it is defined as the infimum of a family of 1-Lipschitz functions.

Now, we denote

$$B_0(R) = B(z_R, 29A_0^{-J(R)}), \quad W_R = \{x \in \mathbb{R}^d : d_R(x) = 0\} \tag{2.12}$$

and, for all $x \in B_0(R) \setminus W_R$, we denote by Q_x the largest cell $Q_x \in \mathcal{D}$ containing x and such that

$$\ell(Q_x) \leq \frac{1}{60} \inf_{y\in Q_x} d_R(y).$$

We define $\mathsf{Reg}(R)$ as the family of the cells $\{Q_x\}_{x\in B_0(R)\setminus W_R}$, which are pairwise disjoint. Note that

$$B_0(R) \setminus \bigcup_{Q\in\mathsf{Reg}(R)} Q = W_R \subset \mathsf{Good}(R).$$

Lemma 2.7.2. *Properties of the regularized stopping cells:*

1. *If $Q \in \mathsf{Reg}(R)$ and $x \in B(z_Q, 50\ell(Q))$, then $d_R(x) \approx \ell(Q)$.*
2. *If $Q, Q' \in \mathsf{Reg}(R)$ are such that $B(z_Q, 50\ell(Q)) \cap B(z_{Q'}, 50\ell(Q')) \neq \emptyset$, then $\ell(Q) \approx \ell(Q')$.*
3. *If $Q \in \mathsf{Reg}(R) \cap \mathcal{D}(R)$, there exists $Q' \in \mathsf{Stop}(R)$ such that $Q \subset Q'$.*
4. *If $Q \in \mathsf{Reg}(R)$, $x \in Q$ and $r > \ell(Q)$, then*

$$\mu[B(x,r) \cap B_R] \lesssim \theta_\mu(B_R) r^n.$$

Proof. 1. First, observe that by definition of $\mathsf{Reg}(R)$,

$$Q \in \mathsf{Reg}(R) \Rightarrow \ell(Q) \leq \frac{1}{60} \inf_{y\in Q} d_R(y) \leq \frac{1}{60} d_R(z_Q),$$

that is, $d_R(z_Q) \geq 60\ell(Q)$. Therefore, since d_R is 1-Lipschitz and $|x - z_Q| \leq 50\ell(Q)$,

$$d_R(x) \geq d_R(z_Q) - |x - z_Q| \geq 60\ell(Q) - 50\ell(Q) = 10\ell(Q).$$

On the other hand, again by definition of $\mathsf{Reg}(R)$, we have

$$\ell(\hat{Q}) > \frac{1}{60} \inf_{y\in\hat{Q}} d_R(y),$$

where $\hat{Q}$ is the parent of Q. Then, there exists $\hat{y} \in \hat{Q}$ such that

$$d_R(\hat{y}) < 60\ell(\hat{Q}) = 60A_0\ell(Q).$$

Now, since $x, \hat{y} \in \hat{Q}$ and $\operatorname{diam}(\hat{Q}) \leq \ell(\hat{Q}) = A_0\ell(Q)$, and taking into account once again that d_R is 1-Lipschitz, we get

$$d_R(x) \leq d_R(\hat{y}) + |x - \hat{y}| \leq 60A_0\ell(Q) + A_0\ell(Q) = 61A_0\ell(Q), \tag{2.13}$$

as desired.

2. This follows directly from (1).

3. If such a $Q' \in \mathsf{Stop}(R)$ does not exist, we get that $Q \in \mathsf{Tree}(R)$. Then, for all $x \in Q$,

$$d_R(x) \leq \inf_{Q' \in \mathsf{Tree}(R)} [|x - z_{Q'}| + \ell(Q')] \leq |x - z_Q| + \ell(Q) \leq 2\ell(Q).$$

However, since $Q \in \mathsf{Reg}(R)$, we get

$$\ell(Q) \leq \frac{1}{60} \inf_{x \in Q} d_R(x),$$

so $d_R(x) \geq 60\ell(Q)$ for all $x \in Q$. This is a contradiction.

4. Since $x \in Q$ and $Q \in \mathsf{Reg}(R)$, by (2.13) we have $d_R(x) < 62A_0\ell(Q)$. Now, since

$$d_R(x) = \inf_{Q' \in \mathsf{Tree}(R)} [|x - z_{Q'}| + \ell(Q')]$$

we obtain that there exists $Q' \in \mathsf{Tree}(R)$ such that

$$|x - z_{Q'}| + \ell(Q') < 62A_0\ell(Q).$$

From this, we get

$$|x - z_{Q'}| < 62A_0 r \ \text{ and } \ r > \frac{1}{62A_0\ell(Q')}$$

and, therefore, we have two possibilities:

(a) There exists $Q'' \in \mathsf{Tree}(R)$ with $Q' \subset Q''$ and $\ell(Q'') \lesssim r$ such that $B(x,r) \subset 1.1B_{Q''}$. In such a case, since $Q'' \in \mathsf{Tree}(R)$, we have $\theta_\mu(1.1B_{Q''}) \lesssim \theta_\mu(B_R)$, and therefore

$$\begin{aligned}\mu[B(x,r) \cap B_R] &\leq \mu[B(x,r)] \leq \mu(1.1B_{Q''}) = \theta_\mu(1.1B_{Q''}) r(B_{Q''})^n \\ &\lesssim \theta_\mu(1.1B_{Q''}) r^n \lesssim \theta_\mu(B_R) r^n.\end{aligned}$$

(b) $B(x,r) \supset B_R$. In this case,

$$\mu[B(x,r) \cap B_R] = \mu(B_R) = \theta_\mu(B_R) r(B_R)^n \leq \theta_\mu(B_R) r^n.$$

□

2.7.2 The suppressed operators T_{Φ_R}

Fix $R \in \mathsf{Top}$ and define

$$\Phi_R(x) = \frac{1}{20A_0^2} d_R(x).$$

Lemma 2.7.3. *Properties of the suppressing function Φ_R:*

1. *If $x \in Q$ for some $Q \in \mathsf{Stop}(R)$, $\Phi_R(x) \leq \frac{1}{10A_0}\ell(Q)$.*

2. *If $x \in \mathsf{Good}(R)$, $\Phi_R(x) = 0$.*

3. *If $x \in Q$ for some $Q \in \mathsf{Reg}(R)$, then $\Phi_R(x) \gtrsim \ell(Q)$.*

4. *For all $x \in B_R$ and all $r \geq \Phi_R(x)$,*

$$\mu[B(x,r) \cap B_R] \leq C_1\, \theta_\mu(B_R) r^n. \tag{2.14}$$

Proof. 1. Let $Q \in \mathsf{Stop}(R)$ and $x \in Q$. We have

$$d_R(x) = \inf_{Q' \in \mathsf{Tree}(R)} [|x - z_{Q'}| + \ell(Q')] \leq |x - z_{\hat{Q}}| + \ell(\hat{Q}),$$

where $\hat{Q}$ is the parent of Q. Then,

$$\Phi_R(x) = \frac{1}{20A_0^2} d_R(x) \leq \frac{1}{20A_0^2} 2\ell(\hat{Q}) = \frac{1}{10A_0^2} A_0 \ell(Q) = \frac{1}{10A_0} \ell(Q).$$

2. If $x \in \mathsf{Good}(R)$, there exist arbitrarily small cells $Q \in \mathsf{Tree}(R)$ that contain x. Therefore,

$$\Phi_R(x) = \frac{1}{20A_0^2} \inf_{Q \in \mathsf{Tree}(R)} [|x - z_Q| + \ell(Q)] = 0.$$

3. This follows directly from (1) in Lemma 2.7.2.

4. First, observe that if $x \in R \setminus \bigcup_{Q \in \mathsf{Reg}(R)} Q$, then (2.14) holds for all $r > 0$, and this can be proved arguing as in (4) in Lemma 2.7.2 and taking into account that $d_R(x) = 0$. Otherwise, if $x \in Q$ for some $Q \in \mathsf{Reg}(R)$, by (1) in Lemma 2.7.2 we have that $r \gtrsim \ell(Q)$, and so (4) in Lemma 2.7.2 applies. $\square$

Lemma 2.7.4. *For $x \in R$,*

$$|K_R \mu(x)| \leq T_{\Phi_R,*}(\chi_{B_0(R)} \mu)(x) + c\theta_\mu(B_R),$$

where $B_0(R) = B(z_R, 29A_0^{-J(R)})$, which is defined in (2.12), satisfies $\theta_\mu(B_0(R)) \approx \theta_\mu(B_R)$.

Proof. The fact that $\theta_\mu(B_0(R)) \approx \theta_\mu(B_R)$ follows immediately from $R \in \mathcal{D}^{db}$.

Recall that

$$K_R \mu = \sum_{Q \in \mathsf{Tree}(R)} T_Q \mu = \sum_{Q \in \mathsf{Tree}(R)} \chi_Q T_{J(Q)} \mu.$$

Now, for $x \in R$, we have two possibilities: either $x \in Q$ for some $Q \in \mathsf{Stop}(R)$ or $x \in \mathsf{Good}(R)$.

1. Suppose $x \in Q$ for some $Q \in \mathsf{Stop}(R)$. Then,

$$\begin{aligned}
|K_R \mu(x)| &= \left| \sum_{j=J(R)}^{J(Q)-1} T_j \mu(x) \right| = \left| \int \left(\sum_{j=J(R)}^{J(Q)-1} \varphi_j(x-y) \right) k(x,y) d\mu(y) \right| \\
&= \left| \int [\psi_{J(R)}(x-y) - \psi_{J(Q)}(x-y)] k(x,y) d\mu(y) \right| \\
&= \left| \int_{|y-x| \geq 0.001 A_0^{-J(Q)-1}} [\psi_{J(R)}(x-y) - \psi_{J(Q)}(x-y)] k(x,y) \chi_{B_0(R)}(y) d\mu(y) \right| \\
&\leq |T_{2A_0^{-1}\ell(Q)}(\chi_{B_0(R)}\mu)(x)| + c\theta_\mu(B_R) \\
&\leq |T_{\Phi_R, 2A_0^{-1}\ell(Q)}(\chi_{B_0(R)}\mu)(x)| + |T_{2A_0^{-1}\ell(Q)}(\chi_{B_0(R)}\mu)(x) - T_{\Phi_R, 2A_0^{-1}\ell(Q)}(\chi_{B_0(R)}\mu)(x)| + c\theta_\mu(B_R) \\
&\leq T_{\Phi_R,*}(\chi_{B_0(R)}\mu)(x) + M^r_{\Phi_R}(\chi_{B_0(R)}\mu)(x) + c\theta_\mu(B_R) \\
&\leq T_{\Phi_R,*}(\chi_{B_0(R)}\mu)(x) + c\theta_\mu(B_R),
\end{aligned}$$

where the penultimate inequality follows from the fact that $\Phi_R(x) \le 2A_0^{-1}\ell(Q)$ and the last one from Lemma A.

2. If $x \in \mathsf{Good}(R)$, we have

$$|K_R\mu(x)| = \lim_{N\to\infty} \left| \int [\psi_{J(R)}(x-y) - \psi_N(x-y)]k(x,y)d\mu(y) \right|.$$

Then, for $N > J(R)$ we obtain, arguing as above, that

$$\begin{aligned}\left| \int [\psi_{J(R)}(x-y) - \psi_N(x-y)]k(x,y)d\mu(y) \right| &\le |T_{2\ell_{N+1}}(\chi_{B_0(R)}\mu)(x)| + c\theta_\mu(B_R)\\ &\le |T_{2\ell_{N+1}}(\chi_{B_0(R)}\mu)(x) - T_{\Phi_R,2\ell_{N+1}}(\chi_{B_0(R)}\mu)(x)|\\ &\quad + |T_{\Phi_R,2\ell_{N+1}}(\chi_{B_0(R)}\mu)(x)| + c\theta_\mu(B_R)\\ &\le M^r_{\Phi_R}(\chi_{B_0(R)}\mu)(x) + T_{\Phi_R,*}(\chi_{B_0(R)}\mu)(x) + c\theta_\mu(B_R)\\ &\le T_{\Phi_R,*}(\chi_{B_0(R)}\mu)(x) + c\theta_\mu(B_R)\end{aligned}$$

where in the penultimate inequality we used the fact that $\Phi_R(x) = 0 \le 2\ell_{N+1}$. Then, letting $N \to \infty$, we obtain

$$|K_R\mu(x)| \le T_{\Phi_R,*}(\chi_{B_0(R)}\mu)(x) + c\theta_\mu(B_R),$$

as desired.

□

2.7.3 A Cotlar-type inequality

Lemma 2.7.5. *Let $R \in \mathsf{Top}$. Then, for all $0 < s \le 1$,*

$$T_{\Phi_R,*}(f\mathcal{H}^n\lfloor_{\Gamma_R})(x) \le C_s \left[M^r_{\Phi_R}((T_*(f\mathcal{H}^n\lfloor_{\Gamma_R})^s)\mathcal{H}^n\lfloor_{\Gamma_R})(x)^{\frac{1}{s}} + M^r_{\Phi_R}(fH^n\lfloor_{\Gamma_R})(x) \right] \tag{2.15}$$

for all $x \in B_0(R)$.

Proof. Denote $\nu = f\mathcal{H}^n\lfloor_{\Gamma_R}$. We will prove that for all $x \in B_0(R)$ and all $\varepsilon > 0$,

$$T_{\Phi_R,\varepsilon}\nu(x) \le C_s \left[M^r_{\Phi_R}((T_*\nu)^s\mathcal{H}^n\lfloor_{\Gamma_R})(x)^{\frac{1}{s}} + M^r_{\Phi_R}\nu(x) \right]$$

By (2) in Lemma A, we can limit ourselves to the case $\varepsilon \ge \Phi_R(x)$. Furthermore, we can assume $\varepsilon > \varepsilon_0 := 0.9\,\mathrm{dist}(x,\Gamma_R)$ since otherwise $T_{\Phi_R,\varepsilon}\nu(x) = T_{\Phi_R,\varepsilon_0}\nu(x)$. Therefore, from now on we will assume $\varepsilon \ge \max\{\Phi_R(x), 0.9\,\mathrm{dist}(x,\Gamma_R)\}$. Notice that, in such a case, $\mathcal{H}^n(B(x,2\varepsilon)\cap\Gamma_R) \gtrsim \varepsilon^n$. We claim now that, for all $x' \in B(x,2\varepsilon)\cap\Gamma_R)$

$$|T_{\Phi_R,\varepsilon}\nu(x)| \le |T_\varepsilon\nu(x')| + CM^r_{\Phi_R}\nu(x). \tag{2.16}$$

From this, the desired result follows easily. Indeed, this implies that for all $0 < s \le 1$,

$$|T_{\Phi_R,\varepsilon}\nu(x)|^s \le T_*\nu(x')^s + CM^r_{\Phi_R}\nu(x)^s,$$

and so, taking the $\mathcal{H}^n\lfloor_{\Gamma_R}$-average for with respect to $x' \in B(x,2\varepsilon)$, we get

$$\begin{aligned}|T_{\Phi_R,\varepsilon}\nu(x)|^s &\le \frac{1}{\mathcal{H}^n[B(x,2\varepsilon)\cap\Gamma_R]}\int_{B(x,2\varepsilon)} T_*\nu(x')^s d\mathcal{H}^n\lfloor_{\Gamma_R}(x') + CM^r_{\Phi_R}\nu(x)^s\\ &\lesssim \frac{1}{\varepsilon^n}\int_{B(x,2\varepsilon)} T_*\nu(x')^s d\mathcal{H}^n\lfloor_{\Gamma_R}(x') + M^r_{\Phi_R}\nu(x)^s\\ &\lesssim M^r_{\Phi_R}((T_*\nu)^s\mathcal{H}^n\lfloor_{\Gamma_R})(x) + M^r_{\Phi_R}\nu(x)^s\end{aligned}$$

and, exponentiating by $\frac{1}{s}$, (2.15) follows.

Let us prove now (2.16). We have

$$|T_{\Phi_R,\varepsilon}\nu(x)| \leq |T_{\Phi_R,\varepsilon}\nu(x) - T_\varepsilon\nu(x)| + |T_\varepsilon\nu(x)| \lesssim |T_\varepsilon\nu(x)| + M^r_{\Phi_R}\nu(x)$$

by Lemma A, since $\varepsilon > \Phi_R(x)$. Now, for all $x' \in B(x, 2\varepsilon)$

$$\begin{aligned}|T_\varepsilon\nu(x)| &\leq |T_\varepsilon\nu(x) - T_{4\varepsilon}\nu(x)| + |T_{4\varepsilon}\nu(x)|\\
&= |T_\varepsilon\nu(x) - T_{4\varepsilon}\nu(x)| + |T(\chi_{\mathbb{R}^d\setminus B(x,4\varepsilon)}\nu)(x)|\\
&\leq |T_\varepsilon\nu(x) - T_{4\varepsilon}\nu(x)| + |T(\chi_{\mathbb{R}^d\setminus B(x,4\varepsilon)}\nu)(x) - T(\chi_{\mathbb{R}^d\setminus B(x,4\varepsilon)}\nu)(x')| + |T(\chi_{\mathbb{R}^d\setminus B(x,4\varepsilon)}\nu)(x')|\\
&\leq |T_\varepsilon\nu(x) - T_{4\varepsilon}\nu(x)| + |T(\chi_{\mathbb{R}^d\setminus B(x,4\varepsilon)}\nu)(x) - T(\chi_{\mathbb{R}^d\setminus B(x,4\varepsilon)}\nu)(x')|\\
&\quad + |T(\chi_{\mathbb{R}^d\setminus B(x,4\varepsilon)}\nu)(x') - T_\varepsilon\nu(x')| + |T_\varepsilon\nu(x')|.\end{aligned}$$

Now

$$|T_\varepsilon\nu(x) - T_{4\varepsilon}\nu(x)| = \left|\int_{\varepsilon\leq|x-y|<4\varepsilon} k(x,y)d\nu(y)\right| \lesssim \int_{\varepsilon<|x-y|\leq 4\varepsilon} \frac{d|\nu|(y)}{|x-y|^n} \lesssim \frac{|\nu|[B(x,4\varepsilon)]}{(4\varepsilon)^n} \leq M^r_{\Phi_R}\nu(x).$$

In addition

$$\begin{aligned}|T(\chi_{\mathbb{R}^d\setminus B(x,4\varepsilon)}\nu)(x) - T(\chi_{\mathbb{R}^d\setminus B(x,4\varepsilon)}\nu)(x')| &= \left|\int_{|x-y|>4\varepsilon} [k(x,y) - k(x',y)]d\nu(y)\right|\\
&\lesssim \int_{|x-y|>4\varepsilon} \frac{|x-x'|}{|x-y|^{n+1}} d|\nu|(y) \leq M^r_{\Phi_R}\nu(x),\end{aligned}$$

where the last inequality is obtained by taking into account that $|x-x'| \leq \varepsilon$ and splitting the domain of integration into annuli $\{2^k\varepsilon < |x-y| \leq 2^{k+1}\varepsilon\}$, $k = 2, 3, \dots$ Finally,

$$\begin{aligned}|T(\chi_{\mathbb{R}^d\setminus B(x,4\varepsilon)}\nu)(x') - T_\varepsilon\nu(x')| &= \left|\int_{|y-x|>4\varepsilon} k(x',y)d\nu(y) - \int_{|y-x'|>\varepsilon} k(x',y)d\nu(y)\right|\\
&= \left|\left(\int_{|y-x|>4\varepsilon, |y-x'|\leq\varepsilon} k(x',y)d\nu(y) + \int_{|y-x|>4\varepsilon, |y-x'|>\varepsilon} k(x',y)d\nu(y)\right)\right.\\
&\quad \left. - \left(\int_{|y-x'|>\varepsilon, |y-x|>4\varepsilon} k(x',y)d\nu(y) + \int_{|y-x'|>\varepsilon, |y-x|\leq 4\varepsilon} k(x',y)d\nu(y)\right)\right|\\
&= \left|\int_{|y-x|>4\varepsilon, |y-x'|\leq\varepsilon} k(x',y)d\nu(y) - \int_{|y-x'|>\varepsilon, |y-x|\leq 4\varepsilon} k(x',y)d\nu(y)\right|\end{aligned}$$

Here, the first integral vanishes, since $|x-x'| < 2\varepsilon$ and $|y-x| \leq \varepsilon$ imply that $|y-x| < 3\varepsilon$. Therefore,

$$\begin{aligned}|T(\chi_{\mathbb{R}^d\setminus B(x,4\varepsilon)}\nu)(x') - T_\varepsilon\nu(x')| &\leq \left|\int_{|y-x'|>\varepsilon, |y-x|\leq 4\varepsilon} k(x',y)d\nu(y)\right|\\
&\lesssim \int_{|y-x'|>\varepsilon, |y-x|\leq 4\varepsilon} \frac{d|\nu|(y)}{|x'-y|^n}\\
&\leq \frac{|\nu|[B(x,4\varepsilon)]}{\varepsilon^n} \lesssim M^r_{\Phi_R}\nu(x).\end{aligned}$$

This completes the proof of (2.16) and, hence, of the lemma.

□

2.7.4 L^2-boundedness of T_{μ,Φ_R}

Lemma 2.7.6. *Let $R \in \mathsf{Top}$ and consider the measure $\sigma_R = \theta_\mu(B_R)\mathcal{H}^n\lfloor_{\Gamma_R}$. Then, for $1 < p < \infty$, T_{σ_R,Φ_R} is bounded from $L^p(\sigma_R)$ to $L^p(\chi_{B_0(R)}\mu)$, with norm bounded by $C_p\theta_\mu(B_R)$. Furthermore, T_{σ_R,Φ_R} is bounded from $L^1(\sigma_R)$ to $L^{1,\infty}(\chi_{B_0(R)}\mu)$, with norm bounded by $C\theta_\mu(B_R)$.*

Proof. First of all, we observe that the maximal operator $M^r_{\sigma_R,\Phi_R}$ is bounded from $L^\infty(\sigma_R)$ to $L^\infty(\chi_{B_0(R)}\mu)$ with norm bounded by $C\theta_\mu(B_R)$. Indeed, if $f \in L^\infty(\sigma_R)$, and $x \in B_0(R)$

$$\begin{aligned} M^r_{\sigma_R,\Phi_R}f(x) &= \sup_{r\geq\Phi_R(x)} \frac{1}{r^n}\int_{B(x,r)\cap B_0(R)} |f|d\mu \leq ||f||_{L^\infty(\sigma_R)} \sup_{r\geq\Phi_R(x)} \frac{\mu[B(x,r)\cap B_0(R)]}{r^n} \\ &\lesssim \theta_\mu(B_R)||f||_{L^\infty(\sigma_R)}, \end{aligned}$$

by (4) in Lemma 2.7.3. Therefore,

$$||M^r_{\sigma_R,\Phi_R}f||_{L^\infty(\chi_{B_0(R)}\mu)} \lesssim \theta_\mu(B_R)||f||_{L^\infty(\sigma_R)},$$

as claimed.

Now, let us check that $M^r_{\sigma_R,\Phi_R}$ is bounded from $L^1(\sigma_R)$ to $L^{1,\infty}(\chi_{B_0(R)}\mu)$ with norm bounded by $C\theta_\mu(B_R)$. In fact, we will prove a slightly stronger result, as we will deal with a non-centered version of $M^r_{\sigma_R,\Phi_R}$, which will be useful for technical reasons. Define, for $f \in L^1(\sigma_R)$ and $x \in \mathbb{R}^d$,

$$N^r_{\sigma_R,\Phi_R}f(x) = \sup \frac{1}{r(B)^n}\int_B |f|d\sigma_R,$$

where the supremum is taken over all balls B with $x \in B$ and such that $\mu(5B) \leq C_1\theta_\mu(B_R)(5r(B))^n$, where C_1 is the same constant that appears in (4) of Lemma 2.7.3. Clearly,

$$M^r_{\sigma_R,\Phi_R}f(x) \leq N^r_{\sigma_R,\Phi_R}f(x),$$

so the weak $(1,1)$ inequality for $M^r_{\sigma_R,\Phi_R}$ will follow from that for $N^r_{\sigma_R,\Phi_R}$.

Let $f \in L^1(\sigma_R)$, $\lambda > 0$, and consider

$$\Omega_\lambda = \{x \in B_0(R)\colon N^r_{\sigma_R,\Phi_R}f(x) > \lambda\}$$

By definition of $N^r_{\sigma_R,\Phi_R}$, for every $x \in \Omega_\lambda$, there exists a ball B_x containing x with $\mu(5B_x) \leq C_1\theta_\mu(B_R)(5r(B))^n$ and such that

$$\frac{1}{r(B_x)^n}\int_{B_x} |f|d\sigma_R > \lambda,$$

which is equivalent to

$$r(B_x)^n < \frac{1}{\lambda}\int_{B_x} |f|d\sigma_R. \tag{2.17}$$

Now, applying the $5r$-covering theorem, we may extract a countable and disjoint subfamily $\{B_i\}$ of $\{B_x\}_{x\in\Omega_\lambda}$ such that the balls $\{5B_i\}$ cover Ω_λ. Then, we have

$$\begin{aligned} \mu(\Omega_\lambda) &\leq \sum_i \mu(5B_i) \leq \sum_i C_1\theta_\mu(B_R)(5r(B_i))^n \lesssim \theta_\mu(B_R)\sum_i r(B_i)^n \\ &\leq \theta_\mu(B_R)\sum_i \frac{1}{\lambda}\int_{B_i} |f|d\sigma_R \leq \frac{\theta_\mu(B_R)}{\lambda}\int_{\Omega_\lambda} |f|d\sigma_R \leq \frac{\theta_\mu(B_R)}{\lambda}||f||_{L^1(\sigma_R)}, \end{aligned} \tag{2.18}$$

which proves that $N^r_{\sigma_R,\Phi_R}$ (and also $M^r_{\sigma_R,\Phi_R}$) is bounded from $L^1(\sigma_R)$ to $L^{1,\infty}(\chi_{B_0(R)}\mu)$ with norm bounded by $C\theta_\mu(B_R)$. Then, Marcinkiewicz's Interpolation Theorem applies and so, for $1 < p < \infty$ $M^r_{\sigma_R,\Phi_R}$ is bounded from $L^p(\sigma_R)$ to $L^p(\chi_{B_0(R)}\mu)$ with norm bounded by $C_p\theta_\mu(B_R)$

Notice that (2.15) in Lemma 2.7.5 can be restated as

$$T_{\sigma_R,\Phi_R,*}f(x) \leq C_s[M^r_{\sigma_R,\Phi_R}((T_{\mathcal{H}^n\lfloor_{\Gamma_R}}f)^s)(x)^{\frac{1}{s}} + M^r_{\sigma_R,\Phi_R}f(x)]. \tag{2.19}$$

Then, taking $s = 1$ and using the $L^p(\sigma_R) \to L^p(\chi_{B_0(R)}\mu)$-boundedness of $M^r_{\sigma_R,\Phi_R}$, we obtain that $T_{\sigma_R,\Phi_R,*}$ is bounded from $L^p(\sigma_R)$ to $L^p(\chi_{B_0(R)}\mu)$ with norm bounded by $C_p\theta_\mu(B_R)$.

To deal with the weak $(1,1)$ case, we will need to work a little harder. Going back to (2.19), with $s = \frac{1}{2}$, we get that for $f \in L^1(\sigma_R)$,

$$T_{\sigma_R,\Phi_R,*}f(x) \leq C[M^r_{\sigma_R,\Phi_R}((T_{\mathcal{H}^n\lfloor_{\Gamma_R}}f)^{\frac{1}{2}})(x)^2 + M^r_{\sigma_R,\Phi_R}f(x)]$$

and so, for $\lambda > 0$,

$$\begin{aligned}
\mu(\{x \in B_0(R)\colon T_{\sigma_R,\Phi_R,*}f(x) > \lambda\}) &\leq \mu\left(\left\{x \in B_0(R)\colon M^r_{\sigma_R,\Phi_R}((T_{\mathcal{H}^n\lfloor_{\Gamma_R}}f)^{\frac{1}{2}})(x)^2 > \frac{\lambda}{2C}\right\}\right) \\
&\quad + \mu\left(\left\{x \in B_0(R)\colon M^r_{\sigma_R,\Phi_R}f(x) > \frac{\lambda}{2C}\right\}\right) \\
&\leq \mu\left(\left\{x \in B_0(R)\colon M^r_{\sigma_R,\Phi_R}((T_{\sigma_R}f)^{\frac{1}{2}})(x) > \left(\frac{\lambda}{2C}\right)^{\frac{1}{2}}\theta_\mu(B_R)^{\frac{1}{2}}\right\}\right) \\
&\quad + \mu\left(\left\{x \in B_0(R)\colon M^r_{\sigma_R,\Phi_R}f(x) > \frac{\lambda}{2C}\right\}\right)
\end{aligned}$$

Here, the second term is bounded by $C\frac{\theta_\mu(B_R)}{\lambda}||f||_{L^1(\sigma_R)}$ because of the weak $(1,1)$-inequality for $M^r_{\sigma_R,\Phi_R}$. To deal with the first term, we will use the weak $(1,1)$-inequality (2.18) for $N^r_{\sigma_R,\Phi_R}$. Denote

$$\Omega = \left\{x \in B_0(R)\colon N^r_{\sigma_R,\Phi_R}((T_{\sigma_R}f)^{\frac{1}{2}})(x) > \left(\frac{\lambda}{2C}\right)^{\frac{1}{2}}\theta_\mu(B_R)^{\frac{1}{2}}\right\}$$

so that

$$\begin{aligned}
\mu\left(\left\{x \in B_0(R)\colon M^r_{\sigma_R,\Phi_R}((T_{\sigma_R}f)^{\frac{1}{2}})(x) > \left(\frac{\lambda}{2C}\right)^{\frac{1}{2}}\theta_\mu(B_R)^{\frac{1}{2}}\right\}\right) &\leq \mu(\Omega) \lesssim \frac{\theta_\mu(B_R)}{\lambda^{\frac{1}{2}}\theta_\mu(B_R)^{\frac{1}{2}}}\int_\Omega |T_{\sigma_R}f|^{\frac{1}{2}}d\mu \\
&\lesssim \frac{\theta_\mu(B_R)^{\frac{1}{2}}}{\lambda^{\frac{1}{2}}}\mu(\Omega)^{\frac{1}{2}}||T_{\sigma_R}f||^{\frac{1}{2}}_{L^{1,\infty}(\mu)} \\
&= \mu(\Omega)^{\frac{1}{2}}\frac{1}{\lambda^{\frac{1}{2}}}||T_{\sigma_R}f||^{\frac{1}{2}}_{L^{1,\infty}(\sigma_R)},
\end{aligned}$$

which implies that $\mu(\Omega) \lesssim \frac{1}{\lambda}||T_{\sigma_R}f||_{L^{1,\infty}(\sigma_R)}$, and therefore

$$\mu\left(\left\{x \in B_0(R)\colon M^r_{\sigma_R,\Phi_R}((T_{\sigma_R}f)^{\frac{1}{2}})(x) > \frac{\lambda^{\frac{1}{2}}}{\sqrt{2C}}\theta_\mu(B_R)\right\}\right) \lesssim \frac{1}{\lambda}||T_{\sigma_R}f||_{L^{1,\infty}(\sigma_R)} \lesssim \frac{\theta_\mu(B_R)}{\lambda}||f||_{L^1(\sigma_R)},$$

where we used the fact that T_{σ_R} is bounded from $L^1(\sigma_R)$ to $L^{1,\infty}(\sigma_R)$ with norm bounded by $C\theta_\mu(B_R)$. This completes the proof of the lemma.

□

We recall here a lemma that is also used at [T3] that will be useful. Its proof is based on the combined use of both Marcinkiewicz's and Riesz-Thorin's Interpolation Theorems.

Lemma 2.7.7. *Let τ be a Radon measure in $\mathbb{R}^d$ and let T be a linear operator that is bounded in $L^2(\tau)$ with norm N_2. Suppose further that both T and its adjoint T^* are bounded from $L^1(\tau)$ to $L^{1,\infty}(\tau)$ with norm bounded by N_1. Then $N_2 \leq cN_1$, where c is an absolute constant.*

Lemma 2.7.8. *T_{μ,Φ_R} is bounded on $L^2(\chi_{B_0(R)}\mu)$ with norm bounded by $C\theta_\mu(B_R)$.*

Proof. Since T_{μ,Φ_R} is antisymmetric, by the previous lemma, we can limit ourselves to prove that it is bounded from $L^1(\chi_{B_0(R)}\mu)$ to $L^{1,\infty}(\chi_{B_0(R)}\mu)$ with norm bounded by $C\theta_\mu(B_R)$.

Let $f \in L^1(\chi_{B_0(R)})$ and denote $\mathsf{Reg}(R) = \{Q_i\}_{i=1}^\infty$, where we assume that the side-lengths $\ell(Q_i)$ are non-increasing. Arguing as in (4) of Lemma 2.7.2, it is easy to check that every cell Q_i is contained in a cell Q_i' such that $\theta_\mu(Q_i') \lesssim \theta_\mu(B_R)$, $\delta_\mu(Q_i, Q_i') \lesssim \theta_\mu(B_R)$, $Q_i' \cap \Gamma_R \neq \emptyset$ and $\mathcal{H}^n(Q_i' \cap \Gamma_R) \approx \ell(Q_i')^n$.

Set

$$g = f\chi_{B_0(R)\setminus\bigcup_i Q_i}, \quad b = \sum_i f\chi_{Q_i}$$

so that $f = g + b$. Since $B_0(R) \setminus \bigcup_i Q_i \subset \mathsf{Good}(R)$ and this is contained in Γ_R (up to a set of μ-measure zero), by the Radon-Nikodym theorem we obtain that

$$\mu\lfloor_{B_0(R)\setminus\bigcup_i Q_i} = \eta\mathcal{H}^n_{\Gamma_R},$$

where η is some function with $0 \le \eta \le C\theta_\mu[B_0(R)] \lesssim \theta_\mu(B_R)$. Then, by Lemma 2.7.6, we have that, for $\lambda > 0$,

$$\begin{aligned}
\mu(\{x \in B_0(R)\colon |T_{\mu,\Phi_R} g(x)| > \lambda\}) &= \mu(\{x \in B_0(R)\colon |T_{\mathcal{H}^n_{\Gamma_R},\Phi_R}(g\eta)(x)| > \lambda\}) \\
&= \mu\left(\{x \in B_0(R)\colon |T_{\sigma_R,\Phi_R}(g\eta)(x)| > \theta_\mu(B_R)\lambda\}\right) \\
&\lesssim \frac{1}{\lambda}||g\eta||_{L^1(\sigma_R)} = \frac{\theta_\mu(B_R)}{\lambda}||g\eta||_{L^1(\mathcal{H}^n\lfloor_{\Gamma_R})} = \frac{\theta_\mu(B_R)}{\lambda}||f||_{L^1(\mu)}
\end{aligned} \tag{2.20}$$

Now, to deal with $T_{\mu,\Phi_R} b$, we define, for every $i \ge 1$

$$\gamma_i(x) = \left(\frac{1}{\mathcal{H}^n(B_{Q_i'} \cap \Gamma_R)} \int_{Q_i} f d\mu\right) \chi_{B_{Q_i'}\cap\Gamma_R}(x), \quad \nu_i = (f\chi_{Q_i})\mu - \gamma_i \mathcal{H}^n_{\Gamma_R},$$

so that ν_i is supported on $B_{Q_i'}$ and satisfies $\int d\nu_i = 0$, and we write

$$b\mu = \sum_i \nu_i + \sum_i \gamma_i \mathcal{H}^n_{\Gamma_R}$$

so that

$$T_{\mu,\Phi_R} b = T_{\Phi_R}(b\mu) = T_{\Phi_R}\left(\sum_i \nu_i\right) + T_{\Phi_R}\left(\sum_i \gamma_i \mathcal{H}^n_{\Gamma_R}\right).$$

Now, again by Lemma 2.7.6, we get

$$\begin{aligned}
\mu\left(\left\{x \in B_0(R)\colon \left|T_{\Phi_R}\left(\sum_i \gamma_i \mathcal{H}^n_{\Gamma_R}\right)(x)\right| > \lambda\right\}\right) &= \mu\left(\left\{x \in B_0(R)\colon \left|T_{\Phi_R,\sigma_R}\left(\sum_i \gamma_i\right)(x)\right| > \theta_\mu(B_R)\lambda\right\}\right) \\
&\lesssim \frac{1}{\lambda}\left|\left|\sum_i \gamma_i\right|\right|_{L^1(\sigma_R)} \le \frac{\theta_\mu(B_R)}{\lambda}\sum_i \int |\gamma_i| d\mathcal{H}^n_{\Gamma_R} \\
&\le \frac{\theta_\mu(B_R)}{\lambda}||f||_{L^1(\mu)}.
\end{aligned} \tag{2.21}$$

Finally, to deal with the term $T_{\Phi_R}\left(\sum_i \nu_i\right)$, we apply Chebyshev's inequality to get

$$\begin{aligned}
\mu\left(\left\{x \in B_0(R)\colon \left|T_{\Phi_R}\left(\sum_i \nu_i\right)(x)\right| > \lambda\right\}\right) &\le \frac{1}{\lambda}\int_{B_0(R)} \left|T_{\Phi_R}\left(\sum_i \nu_i\right)\right| d\mu \\
&= \frac{1}{\lambda}\left(\sum_i \int_{2B_{Q_i'}} |T_{\Phi_R}\nu_i| d\mu + \int_{B_0(R)\setminus 2B_{Q_i'}} |T_{\Phi_R}\nu_i| d\mu\right)
\end{aligned} \tag{2.22}$$

Now, since $\int d\nu_i = 0$, for $x \notin 2B_{Q_i'}$ we have

$$
\begin{aligned}
|T_{\Phi_R}\nu_i(x)| &= \left|\int_{B_{Q_i'}} k_{\Phi_R}(x,y)d\nu_i(y)\right| = \left|\int_{B_{Q_i'}} [k_{\Phi_R}(x,y) - k_{\Phi_R}(x,z_{Q_i'})]d\nu_i(y)\right| \\
&\lesssim \int_{B_{Q_i'}} \frac{|y-z_{Q_i'}|}{|x-z_{Q_i'}|^{n+1}} d|\nu_i|(y) \lesssim \frac{\ell(Q_i')||\nu_i||}{|x-z_{Q_i'}|^{n+1}}
\end{aligned}
$$

and so

$$
\int_{\mathbb{R}^d \setminus 2B_{Q_i'}} |T_{\Phi_R}\nu_i|d\mu \lesssim \int_{B_0(R)\setminus 2B_{Q_i'}} \frac{\ell(Q_i')||\nu_i||}{|x-z_{Q_i'}|^{n+1}} d\mu \lesssim \theta_\mu(B_R)||\nu_i|| \lesssim \theta_\mu(B_R)\int_{Q_i}|f|d\mu. \tag{2.23}
$$

On the other hand,

$$
\begin{aligned}
\int_{2B_{Q_i'}} |T_{\Phi_R}\nu_i|d\mu &\le \int_{2B_{Q_i'}} |T_{\Phi_R}((f\chi_{Q_i})\mu)|d\mu + \int_{2B_{Q_i'}} |T_{\Phi_R}(\gamma_i \mathcal{H}^n_{\Gamma_R})|d\mu \\
&\le \int_{Q_i} |T_{\Phi_R}((f\chi_{Q_i})\mu)|d\mu + \int_{2B_{Q_i'}\setminus Q_i} |T_{\Phi_R}((f\chi_{Q_i})\mu)|d\mu + \int_{2B_{Q_i'}} |T_{\Phi_R}(\gamma_i \mathcal{H}^n_{\Gamma_R})|d\mu \\
&= \mathsf{I}_1 + \mathsf{I}_2 + \mathsf{I}_3.
\end{aligned}
$$

Now, to bound I_1 we use the fact that for all $x \in Q_i$, $\Phi_R(x) \ge \ell(Q_i)$, by (3) in Lemma 2.7.3, and so $|k_{\Phi_R}(x,y)| \lesssim \ell(Q_i)$ for all $x, y \in Q_i$. Hence,

$$
|T_{\Phi_R}((f\chi_{Q_i})\mu)(x)| \lesssim \frac{1}{\ell(Q_i)^n}\int_{Q_i}|f|d\mu
$$

and so

$$
\mathsf{I}_1 \lesssim \frac{\mu(Q_i)}{\ell(Q_i)^n}\int_{Q_i}|f|d\mu \lesssim \theta_\mu(B_R)\int_{Q_i}|f|d\mu,
$$

by (4) in Lemma 2.7.3.

To bound I_2, we observe that for $x \in 2B_{Q_i'} \setminus Q_i$,

$$
|T_{\Phi_R}((\chi_{Q_i}f)\mu)(x)| = \left|\int_{Q_i} k_{\Phi_R}(x,y)f(y)d\mu(y)\right| \lesssim \frac{1}{|x-z_{Q_i}|^n}\int_{Q_i}|f|d\mu
$$

and so

$$
\begin{aligned}
\mathsf{I}_2 &= \int_{2B_{Q_i'}\setminus Q_i} |T_{\Phi_R}((f\chi_{Q_i})\mu)|d\mu \lesssim \int_{Q_i}|f|d\mu \int_{2B_{Q_i'}\setminus Q_i} \frac{1}{|x-z_{Q_i}|^n}d\mu(x) \\
&= \delta_\mu(Q_i, Q_i')\int_{Q_i}|f|d\mu \lesssim \theta_\mu(B_R)\int_{Q_i}|f|d\mu.
\end{aligned}
$$

Finally, by Lemma 2.7.6

$$
\begin{aligned}
\mathsf{I}_3 &= \int_{2B_{Q_i'}} |T_{\Phi_R}(\gamma_i \mathcal{H}^n_{\Gamma_R})|d\mu \le \mu(2B_{Q_i'})^{\frac{1}{2}} \left(\int_{2B_{Q_i'}} |T_{\Phi_R}(\gamma_i \mathcal{H}^n_{\Gamma_R})|^2 d\mu\right)^{\frac{1}{2}} \\
&\le \frac{1}{\theta_\mu(B_R)}\mu(2B_{Q_i'})^{\frac{1}{2}}\left(\int |T_{\Phi_R}(\gamma_i\sigma_R)|^2 d\mu\right)^{\frac{1}{2}} \lesssim \mu(2B_{Q_i'})^{\frac{1}{2}}||\gamma_i||_{L^2(\sigma_R)} \\
&\le \mu(Q_i')^{\frac{1}{2}}\theta_\mu(B_R)^{\frac{1}{2}}\frac{1}{\mathcal{H}^n(Q_i'\cap\Gamma_R)}\int_{Q_i}|f|d\mu \lesssim \theta_\mu(B_R)\int_{Q_i}|f|d\mu.
\end{aligned}
$$

Gathering the estimates for I_1, I_2 and I_3, we obtain

$$
\int_{2B_{Q_i'}} |T_{\Phi_R}\nu_i|d\mu \lesssim \theta_\mu(B_R)\int_{Q_i}|f|d\mu,
$$

and so, going back to (2.22) and also taking into account (2.23), we obtain

$$\mu\left(\left\{x\in B_0(R)\colon \left|T_{\Phi_R}\left(\sum_i \nu_i\right)(x)\right|>\lambda\right\}\right)\lesssim\frac{1}{\lambda}\int|f|d\mu$$

This, together with (2.20) and (2.21), imply the weak $(1,1)$ inequality

$$\mu\left(\{x\in B_0(R)\colon |T_{\mu,\Phi_R}f(x)|>\lambda\}\right)\lesssim\frac{\theta_\mu(B_R)}{\lambda}||f||_{L^1(\mu)}$$

that we were looking for. □

2.7.5 L^2-boundedness of $T_{\Phi_R,\mu,*}$

Lemma 2.7.9. *For $R\in\mathsf{Top}$, $T_{\Phi_R,\mu,*}$ is bounded in $L^2(\chi_{B_0(R)}\mu)$ with norm bounded by $c\theta_\mu(B_R)$.*

Proof. This is a direct consecuence of Theorem G and Lemma 2.7.8, taking $S=T_{\Phi_R}$, $\sigma=\chi_{B_0(R)}\mu$ and $C_0\approx\theta_\mu(B_R)$. □

With all these tools at hand, we can prove Lemma 2.7.1. Indeed, given $R\in\mathsf{Top}$, by Lemmas 2.7.4 and 2.7.9 we have

$$||K_R\mu||_{L^2(\mu)}\le||T_{\Phi_R,*}(\chi_{B_0(R)}\mu)||_{L^2(\chi_R\mu)}+c\theta_\mu(B_R)\mu(R)^{\frac{1}{2}}\lesssim\theta_\mu(B_R)\mu(R)^{\frac{1}{2}},$$

and the desired conclusion follows after squaring both sides and summing over $R\in\mathsf{Top}$.

2.8 The estimate of $\sum_{R,R'\in\mathsf{Top},R\neq R'}\langle K_R\mu,K_{R'}\mu\rangle_\mu$

Given $R,R'\in\mathsf{Top}$, $R\neq R'$, $\langle K_R\mu,K_{R'}\mu\rangle_\mu=0$ unless $R\cap R'\neq\emptyset$. Then,

$$\sum_{R,R'\in\mathsf{Top},R\neq R'}\langle K_R\mu,K_{R'}\mu\rangle_\mu=2\sum_{Q,R\in\mathsf{Top},Q\subsetneq R}\langle K_Q\mu,K_R\mu\rangle_\mu$$

Arguing as in [T3], we can guess that bounding this sum would be relatively easy if

$$\int_Q K_Q\mu=0,$$

but this is, in general, not the case. Indeed,

$$K_Q\mu=\sum_{M\in\mathsf{Tree}(R)}T_M\mu=\sum_{M\in\mathsf{Tree}(R)}\chi_M T_{J(M)}\mu,$$

and while it is true that for all $M\in\mathsf{Tree}(R)$

$$\int_M T_{J(M)}(\chi_M\mu)d\mu=0$$

by antisimmetry, this does not imply that

$$\int_M T_{J(M)}\mu=0$$

and so

$$\int_Q K_Q\mu d\mu=0$$

will not be true in general. Still, the fact that

$$\int_M T_i(\chi_M\mu)d\mu=0$$

for all $i \geq 0$ and all $M \in \mathcal{D}$ will be useful, as we will see in the proof of Lemma 2.8.1.

We have

$$\begin{aligned}
\sum_{Q,R\in\mathsf{Top},Q\subsetneq R} \langle K_Q\mu, K_R\mu\rangle_\mu &= \sum_{R\in\mathsf{Top}} \sum_{P\in\mathsf{Stop}(R)} \sum_{Q\in\mathsf{Top},Q\subset P} \langle K_Q\mu, K_R\mu\rangle_\mu \\
&= \sum_{R\in\mathsf{Top}} \sum_{P\in\mathsf{Stop}(R)} \sum_{Q\in\mathsf{Top},Q\subset P} \sum_{Q'\in\mathsf{Tree}(Q)} \langle T_{Q'}\mu, K_R\mu\rangle_\mu \\
&= \sum_{R\in\mathsf{Top}} \sum_{P\in\mathsf{Stop}(R)} \sum_{Q\in\mathcal{D}(P)} \langle T_Q\mu, K_R\mu\rangle_\mu \\
&= \sum_{R\in\mathsf{Top}} \sum_{P\in\mathsf{Stop}(R)} \sum_{i=J(P)}^{\infty} \sum_{Q\in\mathcal{D}_i(P)} \langle \chi_Q T_i\mu, K_R\mu\rangle_\mu \\
&= \sum_{R\in\mathsf{Top}} \sum_{P\in\mathsf{Stop}(R)} \sum_{i=J(P)}^{\infty} \langle \chi_P T_i\mu, K_R\mu\rangle_\mu
\end{aligned}$$

Now, fixed $R \in \mathsf{Top}$, $P \in \mathsf{Stop}(R)$ and $i \geq J(P)$, we define $m(J(P), i)$ as some intermediate number between $J(P)$ and i (for example, the integer part of the arithmetic mean of $J(P)$ and i), and we decompose

$$P = \bigcup_{S\in\mathcal{D}_{m(J(P),i)}:\, S\subset P} S$$

so that

$$\begin{aligned}
\sum_{Q,R\in\mathsf{Top},Q\subsetneq R} \langle K_Q\mu, K_R\mu\rangle_\mu &= \sum_{R\in\mathsf{Top}} \sum_{P\in\mathsf{Stop}(R)} \sum_{i=J(P)}^{\infty} \langle \chi_P T_i\mu, K_R\mu\rangle_\mu \\
&= \sum_{R\in\mathsf{Top}} \sum_{P\in\mathsf{Stop}(R)} \sum_{i=J(P)}^{\infty} \sum_{S\in\mathcal{D}_{m(J(P),i)}} \langle \chi_S T_i\mu, K_R\mu\rangle_\mu \\
&= \sum_{R\in\mathsf{Top}} \sum_{P\in\mathsf{Stop}(R)} \sum_{i=J(P)}^{\infty} \sum_{S\in\mathcal{D}_{m(J(P),i)}} \langle \chi_S T_i(\chi_S\mu), K_R\mu\rangle_\mu \\
&\quad + \sum_{R\in\mathsf{Top}} \sum_{P\in\mathsf{Stop}(R)} \sum_{i=J(P)}^{\infty} \sum_{S\in\mathcal{D}_{m(J(P),i)}} \langle \chi_S T_i(\chi_{\mathbb{R}^d\setminus S}\mu), K_R\mu\rangle_\mu := \mathsf{ND}_1 + \mathsf{ND}_2
\end{aligned}$$

2.8.1 The estimate of ND_1

Lemma 2.8.1.

$$\mathsf{ND}_1 \lesssim \sum_{R\in\mathsf{Top}} \theta_\mu(B_R)^2\mu(R)$$

Proof. Recall that

$$\mathsf{ND}_1 = \sum_{R\in\mathsf{Top}} \sum_{P\in\mathsf{Stop}(R)} \sum_{i=J(P)}^{\infty} \sum_{S\in\mathcal{D}_{m(J(P),i)}} \langle \chi_S T_i(\chi_S\mu), K_R\mu\rangle_\mu$$

Fix $R \in \mathsf{Top}$, $P \in \mathsf{Stop}(R)$, $i \geq J(P)$ and $S \in \mathcal{D}_m(J(P), i)$. Since

$$\int_S T_i(\chi_S\mu)d\mu = 0,$$

we have

$$\langle \chi_S T_i(\chi_S\mu), K_R\mu\rangle_\mu = \int_S T_i(\chi_S\mu)K_R\mu d\mu = \int_S T_i(X_S\mu)[K_R\mu - K_R\mu(z_S)]d\mu.$$

Now, given $x \in S$, since $S \subset P$ and $P \in \mathsf{Stop}(R)$, we have that the cells from $\mathsf{Tree}(R)$ that contain x are the chain in $\mathcal{D}$ that starts in the parent of P and ends in R. Therefore,

$$
\begin{aligned}
K_R\mu(x) &= \sum_{Q\in\mathsf{Tree}(R):\, x\in Q} T_Q\mu(x)\\
&= \sum_{j\in J(R)}^{J(P)-1} T_j\mu(x)\\
&= \int \left(\sum_{j\in J(R)}^{J(P)-1} \varphi_j(x-y)\right) k(x,y)d\mu(y)\\
&= \int \left[\psi_{J(R)}(x-y) - \psi_{J(P)}(x-y)\right] k(x,y)d\mu(y)
\end{aligned}
$$

If we denote

$$
\zeta_{R,P}(x,y) = \left[\psi_{J(R)}(x-y) - \psi_{J(P)}(x-y)\right] k(x,y)
$$

it is easy to check that for $x, x' \in S$ we have

$$
|\zeta_{R,P}(x,y) - \zeta_{R,P}(x',y)| \lesssim \frac{|x-x'|}{(\ell(P)+|x-y|)^{n+1}}.
$$

Therefore, for $x \in S$,

$$
\begin{aligned}
|K_R\mu(x) - K_R\mu(z_S)| &\lesssim \int_{\operatorname{dist}(y,P)\le 0.01A_0^{-J(R)}} \frac{|x-z_S|}{(\ell(P)+|x-y|)^{n+1}} d\mu(y)\\
&\lesssim \frac{\ell(S)}{\ell(P)}\theta_\mu(B_R),
\end{aligned}
$$

where the last inequality follows from (2.7), and so

$$
\begin{aligned}
|\langle \chi_S T_i(\chi_S\mu), K_R\mu\rangle_\mu| &\lesssim \frac{\ell(S)}{\ell(P)}\theta_\mu(B_R)\int_S |T_i(\chi_S\mu)|d\mu\\
&= \frac{\ell_{m(J(P),i)}}{\ell_{J(P)}}\theta_\mu(B_R)\int_S |T_i(\chi_S\mu)|d\mu\\
&\approx A_0^{\frac{J(P)-i}{2}}\theta_\mu(B_R)\int_S |T_i(\chi_S\mu)|d\mu.
\end{aligned}
$$

Now, for $x \in S$,

$$
\begin{aligned}
|T_i(\chi_S\mu)(x)| &= \left|\int_{y\in S} \varphi_i(x-y)k(x,y)d\mu(y)\right|\\
&= \left|\int_{y\in S,\, 0.001A_0^{-i-1}<|x-y|<0.01A_0^{-i}} \varphi_i(x-y)k(x,y)d\mu(y)\right|\\
&\lesssim \int_{y\in S,\, 0.001A_0^{-i-1}<|x-y|<0.01A_0^{-i}} \frac{d\mu(y)}{|x-y|^n}\\
&\lesssim \frac{\mu[B(x,0.01A_0^{-i})]}{A_0^{-ni}} := \theta_{\mu,i}(x)
\end{aligned}
$$

and so

$$
|\langle \chi_S T_i(\chi_S\mu), K_R\mu\rangle_\mu| \lesssim A_0^{\frac{J(P)-i}{2}}\theta_\mu(B_R)\int_S \theta_{\mu,i}(x)d\mu(x).
$$

Therefore,

$$
\begin{aligned}
\mathsf{ND}_1 &\leq \sum_{R\in\mathsf{Top}}\sum_{P\in\mathsf{Stop}(R)}\sum_{i=J(P)}^{\infty}\sum_{S\in\mathcal{D}_{m(J(P),i)}} |\langle\chi_S T_i(\chi_S\mu), K_R\mu\rangle_\mu| \\
&\lesssim \sum_{R\in\mathsf{Top}}\sum_{P\in\mathsf{Stop}(R)}\sum_{i=J(P)}^{\infty}\sum_{S\in\mathcal{D}_{m(J(P),i)}} A_0^{\frac{J(P)-i}{2}}\theta_\mu(B_R)\int_S \theta_{\mu,i}(x)d\mu(x) \\
&\lesssim \sum_{R\in\mathsf{Top}}\theta_\mu(B_R)\sum_{P\in\mathsf{Stop}(R)} A_0^{\frac{J(P)}{2}}\sum_{i=J(P)}^{\infty} A_0^{-\frac{i}{2}}\int_P \theta_{\mu,i}(x)d\mu(x) \\
&= \sum_{R\in\mathsf{Top}}\theta_\mu(B_R)\sum_{P\in\mathsf{Stop}(R)} A_0^{\frac{J(P)}{2}}\sum_{i=J(P)}^{\infty} A_0^{-\frac{i}{2}}\sum_{P'\in\mathcal{D}_i\colon P'\subset P}\int_{P'} \theta_{\mu,i}(x)d\mu(x) \\
&\lesssim \sum_{R\in\mathsf{Top}}\theta_\mu(B_R)\sum_{P\in\mathsf{Stop}(R)} A_0^{\frac{J(P)}{2}}\sum_{P'\in\mathcal{D}(P)} A_0^{-\frac{J(P')}{2}}\theta_\mu[1.01B_{P'}]\mu(P'),
\end{aligned}
$$

We reorganize the previous sum, to obtain

$$
\mathsf{ND}_1 \lesssim \sum_{R\in\mathsf{Top}}\theta_\mu(B_R)\sum_{P\in\mathsf{Stop}(R)} A_0^{\frac{J(P)}{2}}\sum_{P''\in\mathsf{Top}\colon P''\subset P}\ \sum_{P'\in\mathsf{Tree}(P'')} A_0^{-\frac{J(P')}{2}}\theta_\mu[1.01B_{P'}]\mu(P') \tag{2.24}
$$

and from the fact that $P'\in\mathsf{Tree}(P'')$, we obtain that $\theta_\mu(1.01B_{P'})\lesssim\theta_\mu(B_{P''})$, so

$$
\begin{aligned}
\mathsf{ND}_1 &\lesssim \sum_{R\in\mathsf{Top}}\theta_\mu(B_R)\sum_{P\in\mathsf{Stop}(R)} A_0^{\frac{J(P)}{2}}\sum_{P''\in\mathsf{Top}\colon P''\subset P}\theta_\mu(B_{P''})\sum_{P'\in\mathsf{Tree}(P'')} A_0^{-\frac{J(P')}{2}}\mu(P') \\
&\lesssim \sum_{R\in\mathsf{Top}}\theta_\mu(B_R)\sum_{P\in\mathsf{Stop}(R)} A_0^{\frac{J(P)}{2}}\sum_{P''\in\mathsf{Top}\colon P''\subset P}\theta_\mu(B_{P''})A_0^{-\frac{J(P'')}{2}}\mu(P'') \qquad (2.25) \\
&= \sum_{R\in\mathsf{Top}}\theta_\mu(B_R)\sum_{P''\in\mathsf{Top}\colon P''\subsetneq R} A_0^{\frac{J(R_{P''})-J(P'')}{2}}\theta_\mu(B_{P''})\mu(P'')
\end{aligned}
$$

where, given $R, P''\in\mathsf{Top}$ with $P''\subsetneq R$, $R_{P''}$ is the cell from $\mathsf{Stop}(R)$ that contains P''. To deal with this sum, we need to organize it in trees. To do so, define $\mathsf{Stop}^1(R)=\mathsf{Stop}(R)$ and, for $k>1$,

$$
\mathsf{Stop}^k(R)=\{Q\in\mathcal{D}(R)\colon \text{there exists } Q'\in\mathsf{Stop}^{k-1}(R) \text{ with } Q\in\mathsf{Stop}(Q')\}
$$

so that

$$
\{P\in\mathsf{Top}\colon P\subsetneq R\}=\bigcup_{k=1}^{\infty}\mathsf{Stop}^k(R).
$$

This way, renaming P'' as P in (2.25), we have

$$
\begin{aligned}
\mathsf{ND}_1 &\lesssim \sum_{R\in\mathsf{Top}}\theta_\mu(B_R)\sum_{P\in\mathsf{Top}\colon P\subsetneq R} A_0^{\frac{J(R_P)-J(P)}{2}}\theta_\mu(B_P)\mu(P) \\
&= \sum_{R\in\mathsf{Top}}\theta_\mu(B_R)\sum_{k=1}^{\infty}\sum_{P\in\mathsf{Stop}^k(R)} A_0^{\frac{J(R_P)-J(P)}{2}}\theta_\mu(B_P)\mu(P) \\
&\lesssim \sum_{R\in\mathsf{Top}}\theta_\mu(B_R)\sum_{k=1}^{\infty} A_0^{-\frac{k}{2}}\sum_{P\in\mathsf{Stop}^k(R)}\theta_\mu(B_P)\mu(P)^{\frac12}\mu(P)^{\frac12},
\end{aligned}
$$

because $P \in \mathsf{Stop}^k(R) \Rightarrow J(P) - J(R_P) \geq k-1$. Then, using Cauchy-Schwarz's inequality twice, we get

$$\begin{aligned}
\mathsf{ND}_1 &\lesssim \sum_{R\in\mathsf{Top}} \theta_\mu(B_R) \sum_{k=1}^{\infty} A_0^{-\frac{k}{2}} \left(\sum_{P\in\mathsf{Stop}^k(R)} \theta_\mu(B_P)^2 \mu(P) \right)^{\frac{1}{2}} \left(\sum_{P\in\mathsf{Stop}^k(R)} \mu(P) \right)^{\frac{1}{2}} \\
&= \sum_{k=1}^{\infty} A_0^{-\frac{k}{2}} \sum_{R\in\mathsf{Top}} \theta_\mu(B_R)\mu(R)^{\frac{1}{2}} \left(\sum_{Q\in\mathsf{Stop}^k(R)} \theta_\mu(B_P)^2 \mu(P) \right)^{\frac{1}{2}} \\
&\leq \sum_{k=1}^{\infty} A_0^{-\frac{k}{2}} \left(\sum_{R\in\mathsf{Top}} \theta_\mu(B_R)^2 \mu(R) \right)^{\frac{1}{2}} \left(\sum_{R\in\mathsf{Top}} \sum_{P\in\mathsf{Stop}^k(R)} \theta_\mu(B_P)^2 \mu(P) \right)^{\frac{1}{2}} \\
&\lesssim \sum_{R\in\mathsf{Top}} \theta_\mu(B_R)^2 \mu(R),
\end{aligned}$$

as desired. □

2.8.2 The estimate of ND_2

Lemma 2.8.2.

$$\mathsf{ND}_2 \lesssim \sum_{R\in\mathsf{Top}} \theta_\mu(B_R)^2 \mu(R).$$

Proof. Recall that

$$\mathsf{ND}_2 = \sum_{R\in\mathsf{Top}} \sum_{P\in\mathsf{Stop}(R)} \sum_{i=J(P)}^{\infty} \sum_{S\in\mathcal{D}_{m(J(P),i)}} \langle \chi_S T_i(\chi_{\mathbb{R}^d\setminus S}\mu), K_R\mu\rangle_\mu.$$

Fix $R \in \mathsf{Top}$, $P \in \mathsf{Stop}(R)$, $i \geq J(P)$ and $S \in \mathcal{D}_{m(J(P),i)}$. We have

$$\langle \chi_S T_i(\chi_{\mathbb{R}^d\setminus S}\mu), K_R\mu\rangle_\mu = \int_S T_i(\chi_{\mathbb{R}^d\setminus S}\mu) K_R\mu \, d\mu.$$

Now, if $x \in S$,

$$T_i(\chi_{\mathbb{R}^d\setminus S}\mu)(x) = \int_{\mathbb{R}^d\setminus S} \varphi_i(x-y)k(x,y)d\mu(y) = \int_{y\notin S,\, 0.001A_0^{-i-1}<|x-y|<0.01A_0^i} \varphi_i(x-y)k(x,y)d\mu(y),$$

so $T_i(\chi_{\mathbb{R}^d\setminus S}\mu)(x) = 0$ unless $\operatorname{dist}(x, E\setminus S) < 0.01A_0^{-i}$ (where, as we stated earlier, $E = \operatorname{supp}(\mu)$). Thus, if we denote

$$\partial_i S = \{x \in S\colon \operatorname{dist}(x, E\setminus S) \leq 0.01A_0^{-i}\}$$

we have that

$$\operatorname{supp}(\chi_S T_i(\chi_{\mathbb{R}^d\setminus S}\mu)) \subset \partial_i S.$$

Then,

$$\langle \chi_S T_i(\chi_{\mathbb{R}^d\setminus S_\mu}), K_R\mu\rangle_\mu = \int_{\partial_i S} T_i(\chi_{\mathbb{R}^d\setminus S}\mu) K_R\mu \, d\mu = \sum_{M\in\mathcal{D}_i\colon M\subset S} \int_{\partial_i S\cap M} T_i(\chi_{\mathbb{R}^d\setminus S}\mu) K_R\mu \, d\mu.$$

Now, for $M \in \mathcal{D}_i$ with $M \subset S$ and $x \in \partial_i S \cap M$, we have

$$\begin{aligned}
|T_i(\chi_{\mathbb{R}^d\setminus S}\mu)(x)| &= \left| \int_{y\notin S,\, 0.001A_0^{-i-1}<|x-y|<0.01A_0^i} \varphi_i(x-y)k(x,y)d\mu(y) \right| \\
&\lesssim \int_{0.001A_0^{-i-1}<|x-y|<0.01A_0^i} \frac{d\mu(y)}{|x-y|^n} \leq \frac{\mu[B(x, 0.01A_0^{-i})]}{A_0^{-ni}} \lesssim \theta_\mu[1.01B_M].
\end{aligned}$$

Therefore,

$$
\begin{aligned}
|\langle \chi_S T_i(\chi_{\mathbb{R}^d\setminus S}\mu), K_R\mu\rangle_\mu| &\leq \sum_{M\in\mathcal{D}_i:\, M\subset S} \left| \int_{\partial_i S\cap M} T_i(\chi_{\mathbb{R}^d\setminus S}\mu) K_R\mu d\mu \right| \\
&\lesssim \sum_{M\in\mathcal{D}_i:\, M\subset S} \theta_\mu[1.01B_M] \int_{\partial_i S\cap M} |K_R\mu| d\mu.
\end{aligned}
$$

Then, if we denote

$$
\partial_i \mathcal{D}_{m(J(P),i)} = \bigcup_{S\in\mathcal{D}_m(J(P),i)} \partial_i S
$$

we have

$$
\begin{aligned}
\left| \sum_{i=J(P)}^{\infty} \sum_{S\in\mathcal{D}_{m(J(P),i)}} \langle \chi_S T_i(\chi_{\mathbb{R}^d\setminus S}\mu), K_R\mu\rangle_\mu \right| &\lesssim \sum_{i=J(P)}^{\infty} \sum_{S\in\mathcal{D}_{m(J(P),i)}} \sum_{M\in\mathcal{D}_i:\, M\subset S} \theta_\mu[1.01B_M] \int_{\partial_i S\cap M} |K_R\mu| d\mu \\
&\lesssim \sum_{i=J(P)}^{\infty} \sum_{M\in\mathcal{D}_i:\, M\subset P} \theta_\mu[1.01B_M] \int_{\partial_{J(M)}\mathcal{D}_{m(J(P),J(M))}\cap M} |K_R\mu| d\mu \\
&= \sum_{P'\in\mathsf{Top}:\, P'\subset P} \sum_{M\in\mathsf{Tree}(P')} \theta_\mu[1.01B_M] \int_{\partial_{J(M)}\mathcal{D}_{m(J(P),J(M))}\cap M} |K_R\mu| d\mu.
\end{aligned}
$$

Here we have that $\theta_\mu[1.01B_M] \lesssim \theta_\mu(B_{P'})$ for $M \in \mathsf{Tree}(P')$, and therefore

$$
\begin{aligned}
\left| \sum_{i=J(P)}^{\infty} \sum_{S\in\mathcal{D}_{m(J(P),i)}} \langle \chi_S T_i(\chi_{\mathbb{R}^d\setminus S}\mu), K_R\mu\rangle_\mu \right| &\lesssim \sum_{P'\in\mathsf{Top}:\, P'\subset P} \theta_\mu(B_{P'}) \sum_{M\in\mathsf{Tree}(P')} \int_{\partial_{J(M)}\mathcal{D}_{m(J(P),J(M))}\cap M} |K_R\mu| d\mu \\
&\lesssim \sum_{P'\in\mathsf{Top}:\, P'\subset P} \theta_\mu(B_{P'}) \sum_{i=J(P')}^{\infty} \int_{\partial_i\mathcal{D}_{m(J(P),i)}\cap P'} |K_R\mu| d\mu.
\end{aligned}
$$

Here we use Cauchy-Schwarz's inequality to get

$$
\left| \sum_{i=J(P)}^{\infty} \sum_{S\in\mathcal{D}_{m(J(P),i)}} \langle \chi_S T_i(\chi_{\mathbb{R}^d\setminus S}\mu), K_R\mu\rangle_\mu \right| \lesssim \sum_{P'\in\mathsf{Top}:\, P'\subset P} \theta_\mu(B_{P'}) \sum_{i=J(P')}^{\infty} ||K_R\mu||_{L^2(\chi_{P'}\mu)} \mu[(\partial_i\mathcal{D}_{m(J(P),i)})\cap P']^{\frac{1}{2}}
$$

Now, given $R, P' \in \mathsf{Top}$ with $P' \subsetneq R$, we set

$$
\mu_{R,P'} = \left(\sum_{i=J(P')}^{\infty} \mu[(\partial_i\mathcal{D}_{m(J(R_{P'}),i)}) \cap P']^{\frac{1}{2}} \right)^2
$$

so that

$$
\begin{aligned}
\mathsf{ND}_2 &\lesssim \sum_{R\in\mathsf{Top}} \sum_{P'\in\mathsf{Top}:\, P'\subsetneq R} \theta_\mu(B_{P'}) ||K_R\mu||_{L^2(\chi_{P'}\mu)} \mu_{R,P'}^{\frac{1}{2}} \\
&= \sum_{k=1}^{\infty} \sum_{R\in\mathsf{Top}} \sum_{Q\in\mathsf{Stop}^k(R)} \theta_\mu(B_Q) ||K_R\mu||_{L^2(\chi_Q\mu)} \mu_{R,Q}^{\frac{1}{2}},
\end{aligned}
$$

and here, we use Cauchy-Schwarz's inequality twice again to get

$$\begin{aligned}
\mathsf{ND}_2 &\lesssim \sum_{k=1}^{\infty} \sum_{R \in \mathsf{Top}} \left(\sum_{Q \in \mathsf{Stop}^k(R)} ||K_R \mu||_{L^2(\chi_Q \mu)}^2 \right)^{\frac{1}{2}} \left(\sum_{Q \in \mathsf{Stop}^k(R)} \theta_\mu(B_Q)^2 \mu_{R,Q} \right)^{\frac{1}{2}} \\
&\leq \sum_{k=1}^{\infty} \sum_{R \in \mathsf{Top}} ||K_R \mu||_{L^2(\mu)} \left(\sum_{Q \in \mathsf{Stop}^k(R)} \theta_\mu(B_Q)^2 \mu_{R,Q} \right)^{\frac{1}{2}} \\
&\leq \sum_{k=1}^{\infty} \left(\sum_{R \in \mathsf{Top}} ||K_R \mu||_{L^2(\mu)}^2 \right)^{\frac{1}{2}} \left(\sum_{R \in \mathsf{Top}} \sum_{Q \in \mathsf{Stop}^k(R)} \theta_\mu(B_Q)^2 \mu_{R,Q} \right)^{\frac{1}{2}} \\
&\lesssim \left(\sum_{R \in \mathsf{Top}} \theta_\mu(B_R)^2 \mu(R) \right)^{\frac{1}{2}} \sum_{k=1}^{\infty} \left(\sum_{R \in \mathsf{Top}} \sum_{Q \in \mathsf{Stop}^k(R)} \theta_\mu(B_Q)^2 \mu_{R,Q} \right)^{\frac{1}{2}},
\end{aligned}$$

where the last inequality follows from Lemma 2.7.1. Therefore, if we prove that

$$\sum_{k=1}^{\infty} \left(\sum_{R \in \mathsf{Top}} \sum_{Q \in \mathsf{Stop}^k(R)} \theta_\mu(B_Q)^2 \mu_{R,Q} \right)^{\frac{1}{2}} \lesssim \left(\sum_{R \in \mathsf{Top}} \theta_\mu(B_R)^2 \mu(B_R) \right)^{\frac{1}{2}},$$

we will reach the desired conclusion. To do so, recall that for fixed $k \geq 1$, $R \in \mathsf{Top}$ and $Q \in \mathsf{Stop}^k(R)$

$$\mu_{R,Q} = \left(\sum_{i=J(Q)}^{\infty} \mu[(\partial_i \mathcal{D}_{m(J(R_Q),i)}) \cap Q]^{\frac{1}{2}} \right)^2.$$

Now, recalling that $\ell_s = 56 C_0 A_0^{-s} = \ell(S)$ whenever $S \in \mathcal{D}_s$, we have, for all $i \geq J(Q)$,

$$\begin{aligned}
\mu[(\partial_i \mathcal{D}_{m(J(R_Q),i)}) \cap Q] &= \sum_{S \in \mathcal{D}_{m(J(R_Q),i)} \colon S \subset Q} \mu(\partial_i S) \\
&= \sum_{S \in \mathcal{D}_{m(J(R_Q),i)} \colon S \subset Q} \mu\left(\left\{ x \in S \colon \operatorname{dist}(x, \mathbb{R}^d \setminus S) < \frac{0.01 A_0^{-i}}{\ell(S)} \ell(S) \right\} \right) \\
&\lesssim \sum_{S \in \mathcal{D}_{m(J(R_Q),i)} \colon S \subset Q} \left(\frac{\ell_i}{\ell_m} \right)^{\frac{1}{2}} \mu(3.5 B_S) \\
&\lesssim A_0^{\frac{J(R_Q)-i}{2}} \mu(B_Q),
\end{aligned}$$

where the penultimate inequality follows from (2.10). Therefore,

$$\mu_{R,Q} \lesssim \left(\sum_{i=J(Q)}^{\infty} \left(A_0^{\frac{J(R_Q)-i}{2}} \mu(B_Q) \right)^{\frac{1}{2}} \right)^2 = \mu(B_Q) \left(\sum_{i=J(Q)}^{\infty} A_0^{\frac{J(R_Q)-i}{4}} \right)^2 \lesssim \mu(B_Q) A_0^{\frac{J(R_Q)-J(Q)}{2}}$$

and so

$$\sum_{k=1}^{\infty}\left(\sum_{R\in\mathsf{Top}}\sum_{Q\in\mathsf{Stop}^k(R)}\theta_\mu(B_Q)^2\mu_{R,Q}\right)^{\frac{1}{2}} \lesssim \sum_{k=1}^{\infty}\left(\sum_{R\in\mathsf{Top}}\sum_{Q\in\mathsf{Stop}^k(R)}\theta_\mu(B_Q)^2\mu(B_Q)A_0^{\frac{J(R_Q)-J(Q)}{2}}\right)^{\frac{1}{2}}$$

$$\lesssim \sum_{k=1}^{\infty}A_0^{-\frac{k}{4}}\left(\sum_{R\in\mathsf{Top}}\sum_{Q\in\mathsf{Stop}^k(R)}\theta_\mu(B_Q)^2\mu(B_Q)\right)^{\frac{1}{2}}$$

$$\lesssim \left(\sum_{R\in\mathsf{Top}}\theta_\mu(B_R)^2\mu(B_R)\right)^{\frac{1}{2}}$$

$$\lesssim \left(\sum_{R\in\mathsf{Top}}\theta_\mu(B_R)^2\mu(R)\right)^{\frac{1}{2}},$$

as desired. □

2.9 The proof of the main lemma 2.5.1

This is a straightforward consequence of Lemmas 2.7.1, 2.8.1, 2.8.2 and F. Indeed, going back to Section 2.6,

$$||T\mu||^2_{L^2(\mu)} = \sum_{R\in\mathsf{Top}}||K_R\mu||^2_{L^2(\mu)} + \sum_{R,R'\in\mathsf{Top}}\langle K_R\mu, K_{R'}\mu\rangle_\mu.$$

Now, by Lemma 2.7.1,

$$\sum_{R\in\mathsf{Top}}||K_R\mu||^2_{L^2(\mu)} \lesssim \sum_{R\in\mathsf{Top}}\theta_\mu(B_R)^2\mu(R),$$

and by Lemmas 2.8.1 and 2.8.2

$$\left|\sum_{R,R'\in\mathsf{Top}}\langle K_R\mu, K_{R'}\mu\rangle_\mu\right| \lesssim \sum_{R\in\mathsf{Top}}\theta_\mu(B_R)^2\mu(R),$$

so

$$||T\mu||^2_{L^2(\mu)} \lesssim \sum_{R\in\mathsf{Top}}\theta_\mu(B_R)^2\mu(R) \lesssim ||\mu|| + \iint_0^1 \beta_{\mu,2}(x,r)\theta_\mu[B(x,r)]\frac{dr}{r}d\mu(x),$$

as desired.

2.10 The proof of Corollary 2.1.1

The key idea behind the proof is to use Volberg's characterization of Lipschitz harmonic capacity [V, Lemma 5.15], which states that

$$\kappa(E) \approx \sup\mu(E),$$

where the supremum is taken over all positive Borel measures μ supported on E such that $\mu[B(x,r)] \le r^n$ for all $x \in \mathbb{R}^{n+1}$ and all $r > 0$ and such that the n-dimensional Riesz transform $\mathcal{R}$ with respect to μ is bounded in $L^2(\mu)$ with norm ≤ 1.

Then, to prove Corollary 2.1.1, let μ be a positive Borel measure supported on E satisfying (2.6). Then, clearly $\mu[B(x,r)] \le r^n$ for all $x \in \mathbb{R}^{n+1}$ and all $r > 0$, and furthermore, applying Theorem 2.1.1, we get that $\mathcal{R}_\mu$ is bounded in $L^2(\mu)$ and its norm is bounded by some absolute constant. Therefore, for an appropriate multiple ν of μ we have that $\nu[B(x,r)] \le r^n$ and $||\mathcal{R}_\nu||_{L^2(\nu)\to L^2(\nu)} \le 1$, and so $\mu(E) \lesssim \nu(E) \lesssim \kappa(E)$, as desired.

Chapter 3

The Riesz transform and quantitative rectifiability for general Radon measures

3.1 Introduction

In the work [NToV1] it was shown that, given an n-AD-regular measure μ in $\mathbb{R}^{n+1}$, the $L^2(\mu)$-boundedness of the n-dimensional Riesz transform implies the uniform n-rectifiability of μ. In the codimension 1 case, this result solved a long standing problem raised by David and Semmes [DS1]. In the present chapter we obtain a related quantitative result which is valid for general Radon measures in $\mathbb{R}^{n+1}$ with polynomial growth of order n. Our result turns out to be an essential tool for the solution of an old question on harmonic measure which has appeared in a work by Azzam, Mourgoglou and Tolsa [AMT].

To state our main theorem in detail we need to introduce some notation and terminology. Let μ be a Radon measure in $\mathbb{R}^{n+1}$. For $f \in L^1_{loc}(\mu)$ and $A \subset \mathbb{R}^{n+1}$ with $\mu(A) > 0$, we consider the μ-mean of f over A

$$m_{\mu,A}(f) = \fint_A f\, d\mu = \frac{1}{\mu(A)} \int_A f\, d\mu.$$

Also, given a ball $B \subset \mathbb{R}^{n+1}$ and an n-plane L in $\mathbb{R}^{n+1}$, we denote

$$\beta^L_{\mu,1}(B) = \frac{1}{r(B)^n} \int_B \frac{\operatorname{dist}(x, L)}{r(B)}\, d\mu(x),$$

where $r(B)$ stands for the radius of B. In a sense, this coefficient measures how close the points from $\operatorname{supp}(\mu)$ are to the n-plane L in the ball B. We also set

$$P_\mu(B) = \sum_{j \geq 0} 2^{-j}\, \theta_\mu(2^j B),$$

so $P_\mu(B)$ is some kind of smoothened version of the usual n-dimensional density of μ on B. Finally, denote by K the n-dimensional Riesz kernel in $\mathbb{R}^{n+1}$, that is,

$$K(x, y) = K(x - y) = \frac{x - y}{|x - y|^{n+1}}$$

and by $\mathcal{R}$ the associated Riesz transform.

Our main theorem is the following:

Theorem 3.1.1. *Let μ be a Radon measure on $\mathbb{R}^{n+1}$ and $B \subset \mathbb{R}^{n+1}$ a ball so that the following conditions hold:*

(a) For some constant $C_0 > 0$, $C_0^{-1} r(B)^n \le \mu(B) \le C_0\, r(B)^n$.

(b) $P_\mu(B) \le C_0$, and $\mu(B(x,r)) \le C_0\, r^n$ for all $x \in B$ and $0 < r \le r(B)$.

(c) There is some n-plane L passing through the centre of B such that for some $0 < \delta \ll 1$, it holds $\beta^L_{\mu,1}(B) \le \delta$.

(d) $\mathcal{R}_{\mu\lfloor_B}$ is bounded in $L^2(\mu\lfloor_B)$ with $\|\mathcal{R}_{\mu\lfloor_B}\|_{L^2(\mu\lfloor_B)\to L^2(\mu\lfloor_B)} \le C_1$.

(e) For some constant $0 < \varepsilon \ll 1$,

$$\int_B |\mathcal{R}\mu(x) - m_{\mu,B}(\mathcal{R}\mu)|^2\, d\mu(x) \le \varepsilon\, \mu(B).$$

Then there exists some constant $\tau > 0$ such that if δ, ε are small enough (depending on C_0 and C_1), then there is a uniformly n-rectifiable set $\Gamma \subset \mathbb{R}^{n+1}$ such that

$$\mu(B \cap \Gamma) \ge \tau\, \mu(B).$$

The constant τ and the UR constants of Γ depend on all the constants above.

We remark that it is immediate to check that the condition (b) above holds, for example, if μ has polynomial growth of order n (with constant C_0). The statement in (b) which involves $P_\mu(B)$ is a little more general and it is more convenient for applications. Finally, we warn the reader that in the case that μ is not a finite measure, the statement (e) should be understood in the BMO sense: the fact that $P_\mu(B) < \infty$ guarantees that $\mathcal{R}\mu(x) - m_{\mu,B}(\mathcal{R}\mu)$ is correctly defined.

Note that, in particular, the theorem above ensures the existence of some piece of positive μ-measure of $B \cap \Gamma$ where μ and the Hausdorff measure $\mathcal{H}^n$ are mutually absolutely continuous. This fact, which at first sight may appear rather surprising, is one of the main difficulties for the proof of this result.

It is worth comparing Theorem 3.1.1 to Léger's theorem on Menger curvature. Given three points $x, y, z \in \mathbb{R}^2$, their Menger curvature is

$$c(x,y,z) = \frac{1}{R(x,y,z)},$$

where $R(x,y,z)$ is the radius of the circumference passing through x, y, z if they are pairwise different, and $c(x,y,z) = 0$ otherwise. The curvature of μ is defined by

$$c^2(\mu) = \iiint c(x,y,z)^2\, d\mu(x)\, d\mu(y)\, d\mu(z).$$

This notion was first introduced by Melnikov [M] when studying analytic capacity and, modulo an "error term", is comparable to the squared $L^2(\mu)$-norm of the Cauchy transform of μ (see [MeV]). One of the main ingredients of the proof of Vitushkin's conjecture for removable singularities for bounded analytic functions by David [D2] is Léger's theorem [L] (sometimes called also David-Léger theorem). The quantitative version of this theorem asserts that if μ is a Radon measure in $\mathbb{R}^2$ with linear growth and B is a ball such that $\mu(B) \approx r(B)$ and furthermore $c^2(\mu\lfloor_B) \le \varepsilon\, \mu(B)$ for some small enough $\varepsilon > 0$, then there exists some (possibly rotated) Lipschitz graph $\Gamma \subset \mathbb{R}^2$ such that $\mu(B \cap \Gamma) \ge \frac{9}{10}\mu(B)$. In particular, as in Theorem 3.1.1, it follows that a big piece of $\mu\lfloor_B$ is mutually absolutely continuous with respect to $\mathcal{H}^1$ on some subset of Γ.

In a sense, one can think that Theorem 3.1.1 is an analog for Riesz transforms of the quantitative Léger theorem for Menger curvature. Indeed, the role of the assumption (e) in Theorem 3.1.1 is played by the condition $c^2(\mu\lfloor_B) \leq \varepsilon\,\mu(B)$. Furthermore, it is not difficult to check that this condition implies that there exists some line L such that $\beta^L_{\mu,1}(B) \leq \delta\,\mu(B)$, with $\delta = \delta(\varepsilon) \to 0$ as $\varepsilon \to 0$, as an analog to the assumption (c) of Theorem 3.1.1.

On the other hand, from the theorem of Léger described above, it follows easily that if $\mathcal{H}^1(E) < \infty$ and $c^2(\mathcal{H}^1\lfloor_E) < \infty$, then E is 1-rectifiable. The analogous result in the codimension 1 case in $\mathbb{R}^{n+1}$ (proved in [NToV2]) asserts if $E \subset \mathbb{R}^{n+1}$, $\mathcal{H}^n(E) < \infty$, $\mathcal{H}^n\lfloor_E$ has growth of order n, and $\|\mathcal{R}(\mathcal{H}^n\lfloor_E)\|_{L^2(\mathcal{H}^n\lfloor_E)} < \infty$, then E is n-rectifiable. However, as far as we know, this cannot be proved easily using Theorem 3.1.1.

The proof of Theorem 3.1.1 is substantially different from the one of Léger's theorem: when estimating the $L^2(\mu)$-norm of $\mathcal{R}\mu$ we are dealing with a singular integral and there may be cancellations among different scales. Therefore, the situation is more delicate than in the case of the curvature $c^2(\mu)$, which is defined by a non-negative integrand (namely, the squared Menger curvature of three points).

To prove Theorem 3.1.1 we will apply some of the techniques developed in [ENV] and [NToV1]. In particular, by using a variational argument, we will estimate from below the $L^2(\mu)$-norm of the Riesz transform of a suitable periodization of a smoothened version of the measure μ restricted to some appropriate cube Q_0. The assumption that $\beta^L_{\mu,1}(B) \leq \delta$ in (c) is necessary for technical reasons, and we do not know if the theorem holds without this condition.

Finally, we are going to describe the aforementioned result on harmonic measure from [AMT] whose proof uses Theorem 3.1.1 as an essential tool. For simplicity, we will only state it for domains $\Omega_1, \Omega_2 \subset \mathbb{R}^{n+1}$ satisfying the condition

$$\mathcal{H}^s_\infty((\mathbb{R}^{n+1} \setminus \Omega_i) \cap B(x,r)) \approx r^s \quad \text{for all } x \in \partial\Omega_i \text{ and } 0 < r \leq r_0, \tag{3.1}$$

for some fixed $s \in (n-1, n+1]$ and $r_0 > 0$, where $\mathcal{H}^s_\infty$ stands for the s-dimensional Hausdorff content. For example, the so-called NTA domains introduced by Jerison and Kenig [JK] satisfy this condition, and also the simply connected domains in the plane.

Theorem 3.1.2 (Azzam, Mourgoglou, Tolsa)**.** *Let $\Omega_1, \Omega_2 \subseteq \mathbb{R}^{n+1}$, $n \geq 2$, be two disjoint connected domains with $\partial\Omega_1 = \partial\Omega_2$ so that (3.1) holds. For $i = 1, 2$, let $\omega^i = \omega^{x_i}_{\Omega_i}$ be the respective harmonic measures with poles at $x_i \in \Omega_i$, and let $E \subset \partial\Omega_1$ be a Borel set. If $\omega^1 \ll \omega^2 \ll \omega^1$ on E, then E contains an n-rectifiable subset F with $\omega^1(E \setminus F) = \omega^2(E \setminus F) = 0$ where ω^1 and ω^2 are mutually absolutely continuous with respect to $\mathcal{H}^n$.*

Up to now, this result was known only in the case when Ω_1, Ω_2 are planar domains, by results of Bishop, Carleson, Garnett and Jones [BCGJ] and Bishop [B1], and it was an open problem to extend it to higher dimensions (see Conjecture 8 in [B2]). For a partial result in the higher dimensional case, see the nice work [KPT] by Kenig, Preiss, and Toro.

3.2 The Main Lemma

3.2.1 Preliminaries and statement of the Main Lemma

Given two Radon measures μ and σ and a cube $Q \subset \mathbb{R}^{n+1}$, we set

$$d_Q(\mu, \sigma) = \sup_f \int f\, d(\mu - \sigma),$$

where the supremum is taken over all 1-Lipschitz functions supported on Q. Given an n-plane L in $\mathbb{R}^{n+1}$, we denote

$$\alpha_\mu^L(Q) = \frac{1}{\ell(Q)^{n+1}} \inf_{c\geq 0} d_Q(\mu, c\,\mathcal{H}^n\lfloor_L).$$

We say that Q has t-thin boundary with respect to μ if, for some constant $t > 0$,

$$\mu\left(\{x \in 2Q : \mathrm{dist}(x, \partial Q) \leq \lambda\,\ell(Q)\}\right) \leq t\,\lambda\,\mu(2Q) \qquad \text{for all } \lambda > 0.$$

It is well known that for any given cube $Q_0 \subset \mathbb{R}^{n+1}$ and any constant $a > 1$, there exists another cube Q with t-thin boundary such that $Q_0 \subset Q \subset aQ_0$, with t depending just on n and a. For the proof, we refer the reader to Lemma 9.43 of [T6], for example.

Main Lemma 3.2.1. *Let $n \geq 1$ and let $C_0, C_1 > 0$ be two arbitrary constants. Then, there exist constants $A = A(C_0, C_1, n) > 10$ big enough and $\varepsilon = \varepsilon(C_0, C_1, n) > 0$ small enough such that if $\delta = \delta(A, C_0, C_1, n) > 0$ is small enough, the following holds: let μ be a Radon measure in $\mathbb{R}^{n+1}$ and $Q_0 \subset \mathbb{R}^{n+1}$ a cube centered at the origin satisfying the following properties:*

(a) $\mu(Q_0) = \ell(Q_0)^n$.

(b) $P_\mu(AQ_0) \leq C_0$.

(c) For all $x \in 2Q_0$ *and* $0 < r \leq \ell(Q_0)$, $\theta_\mu(B(x,r)) \leq C_0$.

(d) Q_0 *has* C_0*-thin boundary.*

(e) $\alpha_\mu^H(3AQ_0) \leq \delta$, *where* $H = \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$.

(f) $\mathcal{R}_{\mu\lfloor_{2Q_0}}$ *is bounded in* $L^2(\mu\lfloor_{2Q_0})$ *with* $\|\mathcal{R}_{\mu\lfloor_{2Q_0}}\|_{L^2(\mu\lfloor_{2Q_0})\to L^2(\mu\lfloor_{2Q_0})} \leq C_1$.

(g) $\displaystyle\int_{Q_0} |\mathcal{R}\mu(x) - m_{\mu,Q_0}(\mathcal{R}\mu)|^2\, d\mu(x) \leq \varepsilon\,\mu(Q_0)$.

Then, there exists some constant $\tau > 0$ and a uniformly n-rectifiable set $\Gamma \subset \mathbb{R}^{n+1}$ such that

$$\mu(Q_0 \cap \Gamma) \geq \tau\,\mu(Q_0).$$

Furthermore, the constant τ and the UR constants of Γ depend on all the constants above.

Note that condition (c) in the Main Lemma implies that $\mu(2Q_0) \lesssim C_0\,\mu(Q_0)$.

3.2.2 Reduction of Theorem 3.1.1 to the Main Lemma

Assume that the Main Lemma is proved. Then, in order to prove Theorem 3.1.1 it is enough to show the following:

Lemma 3.2.2. *Let μ and $B \subset \mathbb{R}^{n+1}$ satisfy the assumptions of Theorem 3.1.1 with constants C_0, C_1, δ, and ε. For all $A' \geq 10$ and all $\delta', \varepsilon' > 0$, if δ and ε are small enough, there exists a cube Q_0 satisfying:*

(a) $A'Q_0 \subset B$ *and* $\mathrm{dist}(A'Q_0, \partial B) \geq C_0'^{-1}\, r(B)$, *with* C_0' *depending only on* C_0 *and* n.

(b) For some constant $\gamma = \gamma(\delta') > 0$, $\gamma\, r(B) \leq \ell(Q_0) \leq A'^{-1}\, r(B)$.

(c) $\mu(Q_0) \geq C_0'^{-1} \ell(Q_0)^n$.

(d) Q_0 *has* C_0'*-thin boundary.*

(e) $\alpha_\mu^L(3A'Q_0) \leq \delta'$, *where* L *is some* n*-plane that passes through the centre of* Q_0 *and is parallel to one of its faces.*

(f) $\displaystyle\int_{Q_0} |\mathcal{R}\mu(x) - m_{\mu,Q_0}(\mathcal{R}\mu)|^2\, d\mu(x) \leq \varepsilon'\, \mu(Q_0).$

Before proving this lemma, we show how it is used to reduce Theorem 3.1.1 to the Main Lemma 3.2.1.

Proof of Theorem 3.1.1 using Lemma 3.2.2 and the Main Lemma 3.2.1. Let $B \subset \mathbb{R}^{n+1}$ be a ball satisfying the assumptions of Theorem 3.1.1 with constants C_0, C_1, δ, and ε. Let Q_0 be the cube given by Lemma 3.2.2, for some constants $A' \geq 10$ and $\delta', \varepsilon' > 0$ to be fixed below. We just have to check that the assumptions (a)-(g) of the Main Lemma are satisfied by the measure

$$\widetilde{\mu} = \frac{\ell(Q_0)^n}{\mu(Q_0)}\, \mu$$

if A' is big enough and δ', ε' are small enough.

Obviously, the assumption (a) from the Main Lemma is satisfied by $\widetilde{\mu}$. To show that (b) holds (with a constant that may differ from C_0), note first that

$$\frac{\ell(Q_0)^n}{\mu(Q_0)} \approx 1, \tag{3.2}$$

with the implicit constant depending on C_0 and C_0'. Indeed, from the assumption (c) in Theorem 3.1.1, $\mu(Q_0) \lesssim C_0 \ell(Q_0)^n$, and by (c) in Lemma 3.2.2, $\mu(Q_0) \geq C_0'^{-1}\ell(Q_0)^n$. Then, we have

$$P_{\widetilde{\mu}}(A'Q_0) \lesssim P_\mu(A'Q_0) = \sum_{j\geq 0: 2^j A'Q_0 \subset B:} 2^{-j}\theta_\mu(2^j A'Q_0) + \sum_{j\geq 0: 2^j A'Q_0 \not\subset B:} 2^{-j}\theta_\mu(2^j A'Q_0).$$

The first sum on the right hand side does not exceed $C\, C_0'$ because $\theta_\mu(2^j A'Q_0) \lesssim C_0$ for all cubes $2^j A'Q_0$ contained in B. Also, one can check that the last sum is bounded by $C\, P_\mu(B)$ because $\ell(2A'^j Q_0) \gtrsim r(B)$ for all j's in this sum, taking into account that $\operatorname{dist}(A'Q_0, \partial B) \geq C_0'^{-1}\, r(B)$.

The assumptions (d)-(e) in the Main Lemma are obviously satisfied too because of (3.2) and the analogous conditions in Lemma 3.2.2, with some different constants $C_0'', \delta'', \varepsilon''$ replacing C_0, δ, ε. $\square$

3.2.3 The proof of Lemma 3.2.2

We identify $\mathbb{R}^n$ with the horizontal n-plane $H = \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$ below. Then, given a measure σ in $\mathbb{R}^n$ and a cube $Q \in \mathbb{R}^n$, we denote

$$\alpha_\sigma^{\mathbb{R}^n}(Q) = \frac{1}{\ell(Q)^{n+1}} \inf_{c\geq 0} d_Q(\sigma, c\,\mathcal{H}^n|_{\mathbb{R}^n}), \tag{3.3}$$

where the infimum is taken over all constants $c > 0$. Note that

$$\alpha_\sigma^{\mathbb{R}^n}(Q) \approx \alpha_\sigma^H(\widehat{Q}),$$

where $\widehat{Q} = Q \times [-\ell(Q)/2, \ell(Q)/2]$. This follows easily from the fact that any 1-Lipschitz function in Q can be extended to a C-Lipschitz function on $\widehat{Q}$, with $C \lesssim 1$.

We need a couple of auxiliary results. The first one is the following:

Lemma 3.2.3. *Suppose that* σ *is some finite measure supported on* $\mathbb{R}^n$ *such that* $d\sigma(x) = \rho(x)\, dx$, *with* $\|\rho\|_\infty < \infty$. *Then, for every* $R \in \mathcal{D}(\mathbb{R}^n)$ *we have*

$$\sum_{Q\in\mathcal{D}(\mathbb{R}^n): Q\subset R} \alpha_\sigma^{\mathbb{R}^n}(3Q)^2 \ell(Q)^n \lesssim \|\rho\|_\infty^2 \ell(R)^n,$$

where $\mathcal{D}(\mathbb{R}^n)$ *stands for the family of the usual dyadic cubes in* $\mathbb{R}^n$.

This lemma can be proved by arguments that are similar to the ones used in [T5] to show that the analogous estimate holds for Lipschitz graphs. Alternatively, it can be thought of as a corollary of that result for the case where the Lipschitz graph is just a horizontal n-plane, using the auxiliary AD-regular measure $\widetilde{\sigma} = 2\|\rho\|_\infty \mathcal{H}^n|_{\mathbb{R}^n} + \sigma$ and taking into account that $\alpha_\sigma^{\mathbb{R}^n}(Q) = \alpha_{\widetilde{\sigma}}^{\mathbb{R}^n}(Q)$ for any cube $Q \subset \mathbb{R}^n$.

The second auxiliary result we need is the next one:

Lemma 3.2.4. *Let σ be some finite measure in $\mathbb{R}^n$ and $R \in \mathcal{D}(\mathbb{R}^n)$ such that*

$$\sigma(Q) \leq C_2 \ell(Q)^n$$

for all cubes $Q \in \mathcal{D}(\mathbb{R}^n)$ with $\ell(Q) \geq \ell_0$. Then, for every $R \in \mathcal{D}(\mathbb{R}^n)$ we have

$$\sum_{\substack{Q \in \mathcal{D}(\mathbb{R}^n): Q \subset R \\ \ell(Q) \geq \ell_0}} \alpha_\sigma^{\mathbb{R}^n}(3Q)^2 \ell(Q)^n \lesssim C_2^2 \ell(R)^n.$$

Proof. Let $\varphi(x) = m(B(0,\ell_0))^{-1} \chi_{B(0,\ell_0)}(x)$. Consider the function $\rho = \varphi * \sigma$ and the measure $d\nu = \rho\, dx$. We have $\|\rho\|_\infty \lesssim C_2$, since for all $x \in \mathbb{R}^n$

$$\rho(x) = \frac{1}{m(B(0,\ell_0))} \int \varphi(x-y)\, d\sigma(y) = \frac{\sigma(B(x,\ell_0))}{m(B(x,\ell_0))} \lesssim C_2.$$

Let us check that

$$\operatorname{dist}_{3Q}(\nu, \sigma) \lesssim C_2 \ell_0 \ell(Q)^n \qquad \text{for any cube } Q \text{ with } \ell(Q) \geq \ell_0. \tag{3.4}$$

For any 1-Lipschitz function f supported on $3Q$, we have

$$\left| \int f\, d\nu - \int f\, d\sigma \right| = \left| \int f\, (\varphi * \sigma)\, dx - \int f\, d\sigma \right| = \left| \int f * \varphi\, d\sigma - \int f\, d\sigma \right|.$$

Since f is 1-Lipschitz we have

$$|f(x) - f * \varphi(x)| = \left| \int_{y \in B(x,\ell_0)} \big(f(x) - f(y)\big) \varphi(x-y)\, dy \right| \leq \int \ell_0\, \varphi(x-y)\, dy = \ell_0.$$

Thus,

$$\left| \int f\, d\nu - \int f\, d\sigma \right| \lesssim \ell_0 \sigma(6Q) \lesssim C_2 \ell_0 \ell(Q)^n,$$

since $\operatorname{supp}(f) \cup \operatorname{supp}(f * \varphi) \subset 6Q$, and so (3.4) holds.

From (3.4) we infer that

$$\alpha_\sigma^{\mathbb{R}^n}(3Q) \lesssim \alpha_\nu^{R^n}(Q) + C_2 \frac{\ell_0}{\ell(Q)},$$

and by Lemma 3.2.3,

$$\sum_{\substack{Q \in \mathcal{D}(\mathbb{R}^n): Q \subset R \\ \ell(Q) \geq \ell_0}} \alpha_\sigma^{\mathbb{R}^n}(3Q)^2 \ell(Q)^n \lesssim \sum_{\substack{Q \in \mathcal{D}(\mathbb{R}^n): Q \subset R \\ \ell(Q) \geq \ell_0}} \Big(\alpha_\nu^{\mathbb{R}^n}(3Q)^2 + C_2^2 \frac{\ell_0^2}{\ell(Q)^2} \Big) \ell(Q)^n \lesssim C_2^2\, \ell(R)^n.$$

$\square$

Proof of Lemma 3.2.2. Let μ and B be as in Theorem 3.1.1. By a suitable translation and rotation we may assume that the n-plane L from Theorem 3.1.1 coincides with the horizontal n-plane $H = \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$ and that $B = B(0, r_0)$. Our first objective consists in finding an auxiliary cube R_0 contained in B, centered in H, and far from ∂B, so that $\mu(R_0) \approx \mu(B)$. The cube Q_0, to be chosen later, will be an appropriate cube contained in R_0.

To find R_0, for some constant $0 < d < 1/10$ to be fixed below, we consider a grid $\mathcal{Q}$ of n-dimensional cubes with side length $2d\,r_0$ in H, so that they cover H and have disjoint interiors. We also consider the family of $(n+1)$-dimensional cubes

$$\widehat{\mathcal{Q}} = \{Q \times [-d\,r_0, d\,r_0] : Q \in \mathcal{Q}\},$$

so that the union of the cubes from $\widehat{\mathcal{Q}}$ conforms the strip

$$V = \{x \in \mathbb{R}^{n+1} : \mathrm{dist}(x, H) \le d\,r_0\}.$$

For any constant $0 < a < 1$ we have

$$\mu\big(B \setminus (aB \cap V)\big) \le \sum_{\substack{P \in \widehat{\mathcal{Q}}: \\ P \cap (B \setminus aB) \neq \varnothing}} \mu(P \cap B) + \mu\Bigg(B \setminus \bigcup_{P \in \widehat{\mathcal{Q}}} P\Bigg) := \mathsf{S}_1 + \mathsf{S}_2.$$

To bound S_1 we use the growth condition of order n of $\mu\lfloor_B$:

$$\mathsf{S}_1 \lesssim_{C_0} \sum_{\substack{P \in \widehat{\mathcal{Q}}: \\ P \cap (B \setminus aB) \neq \varnothing}} \ell(P)^n \lesssim \mathcal{H}^n\big(H \cap A\big(0, (a - n^{1/2}2d)r_0, (1 + n^{1/2}2d)r_0\big)\big) \lesssim_{C_0} (d + 1 - a)r_0^n.$$

To estimate S_2 we use the fact that the distance from the points $x \in B \setminus \bigcup_{P \in \widehat{\mathcal{Q}}} P$ to H is larger than dr_0 and apply Chebyshev's inequality:

$$\mathsf{S}_2 \le \int_B \frac{\mathrm{dist}(x, H)}{d\,r_0}\, d\mu(x) = \frac{1}{d}\, \beta^H_{\mu,1}(B)\, r_0^n.$$

Then, we obtain

$$\mu\big(B \setminus (aB \cap V)\big) \le C(C_0)\bigg((d + 1 - a) + \frac{1}{d}\, \beta^H_{\mu,1}(B)\bigg)\, \mu(B).$$

We take now d and a so that

$$10(n+1)^{1/2}d = (1 - a) = \frac{1}{10C(C_0)},$$

and we assume

$$\beta^H_{\mu,1}(B) \le \delta \le \frac{d}{10C(C_0)} = \frac{1}{10(n+1)^{1/2}(10C(C_0))^2},$$

so that $\mu\big(B \setminus (aB \cap V)\big) \le \frac{3}{10}\, \mu(B)$. Now we choose R_0 to be a cube from $\widehat{\mathcal{Q}}$ which intersects $aB \cap V$ and has maximal μ-measure. Obviously,

$$\mu(R_0) \approx_{C_0} \mu(aB \cap V) \approx_{C_0} \mu(B),$$

and since $\mathrm{diam}(R_0) = 2(n+1)^{1/2}d\,r_0 = \frac{1-a}{5}\, r_0$, it follows that

$$\mathrm{dist}(2R_0, \partial B) \approx_{C_0} (1 - a)r_0 \approx_{C_0} r_0.$$

The cube Q_0 we are looking for will be an appropriate cube contained in R_0. To find it, first we consider the thin strip

$$V_\delta = \big\{x \in \mathbb{R}^{n+1} : \operatorname{dist}(x, H) \leq \delta^{1/2} r_0\big\}.$$

Observe that

$$\mu(B \setminus V_\delta) \leq \int_B \frac{\operatorname{dist}(x,H)}{\delta^{1/2} r_0}\, d\mu(x) = \frac{\beta^H_{\mu,1}(B)}{\delta^{1/2}}\, r_0^n \lesssim_{C_0} \delta^{1/2}\, \mu(B). \tag{3.5}$$

Denote by $\varPi$ the orthogonal projection on H and consider the measure $\sigma = \varPi_\#(\mu|_{V_\delta})$. Since V_δ has width $2\delta^{1/2}$, from the growth condition (b) in Theorem 3.1.1, it follows that $\sigma(Q) \lesssim_{C_0} \ell(Q)^n$ for any cube Q centered on H with $\ell(Q) \geq \delta^{1/2}\, r_0$.

Assume without loss of generality that R_0 is a dyadic cube. Then, by Lemma 3.2.4,

$$\sum_{\substack{Q \in \mathcal{D}(\mathbb{R}^n, R_0)\\ \ell(Q) \geq \delta^{1/2} r_0}} \alpha_\sigma^{\mathbb{R}^n}(3Q)^2 \ell(Q)^n \lesssim_{C_0} \ell(R_0)^n,$$

where $\mathcal{D}(\mathbb{R}^n, R_0)$ stands for the family of dyadic cubes in $\mathbb{R}^n$ contained in R_0. From this inequality, it easily follows that, for any constant $A' > 10$,

$$\sum_{\substack{Q \in \mathcal{D}(\mathbb{R}^n, R_0): \ell(A'Q) \leq \ell(R_0)\\ \ell(Q) \geq \delta^{1/(4n+1)} r_0}} \alpha_\sigma^{\mathbb{R}^n}(4A'Q)^2 \ell(Q)^n \lesssim_{C_0, A'} \ell(R_0)^n.$$

Note that we have used the fact that $\delta^{1/(4n+1)} > \delta^{1/2}$.

Since the number of dyadic generations between the largest cubes $Q \in \mathcal{D}(\mathbb{R}^n)$ with $\ell(A'Q) \leq \ell(R_0)$ and the smallest ones with side length $\ell(Q) \geq \delta^{1/(4n+1)} r_0$ is comparable to

$$\log_2 \frac{C(A')\ell(R_0)}{\delta^{1/(4n+1)} r_0} \approx \log_2 \frac{C(A', C_0)}{\delta^{1/(4n+1)}},$$

we infer that there exists some intermediate generation j such that

$$\sum_{Q \in \mathcal{D}_j(\mathbb{R}^n, R_0): \ell(A'Q) \leq \ell(R_0)} \alpha_\sigma^{\mathbb{R}^n}(4A'Q)^2 \ell(Q)^n \lesssim_{C_0, A'} \frac{1}{\log_2 \frac{C(A',C_0)}{\delta^{1/(4n+1)}}} \ell(R_0)^n.$$

Thus, for any $\delta' > 0$, if δ is small enough, we derive

$$\sum_{\substack{Q \in \mathcal{D}_j(\mathbb{R}^n, R_0):\\ \ell(A'Q) \leq \ell(R_0)}} \alpha_\sigma^{\mathbb{R}^n}(4A'Q)^2\, \sigma(Q) \leq C(C_0) \sum_{\substack{Q \in \mathcal{D}_j(\mathbb{R}^n, R_0):\\ \ell(A'Q) \leq \ell(R_0)}} \alpha_\sigma^{\mathbb{R}^n}(4A'Q)^2 \ell(Q)^n \leq \frac{\delta'^2}{50} \sigma(R_0).$$

Denote by $\mathcal{G}$ the subfamily of cubes from $\mathcal{D}_j(\mathbb{R}^n, R_0)$ such that $\theta_\sigma(Q) \geq \frac{1}{2}\,\theta_\sigma(R_0)$. Observe that

$$\sum_{Q \in \mathcal{D}_j(\mathbb{R}^n, R_0) \setminus \mathcal{G}} \sigma(Q) \leq \frac{1}{2}\,\theta_\sigma(R_0) \sum_{Q \in \mathcal{D}_j(\mathbb{R}^n, R_0) \setminus \mathcal{G}} \ell(Q)^n \leq \frac{1}{2}\,\theta_\sigma(R_0)\, \ell(R_0)^n = \frac{1}{2}\,\sigma(R_0).$$

Hence, $\sum_{Q \in \mathcal{G}} \sigma(Q) \geq \frac{1}{2}\,\sigma(R_0)$, and so

$$\sum_{Q \in \mathcal{G}} \alpha_\sigma^{\mathbb{R}^n}(4A'Q)^2\, \sigma(Q) \leq \frac{\delta'^2}{50} \sigma(R_0) \leq \frac{\delta'^2}{25}\, \sigma\bigg(\bigcup_{Q \in \mathcal{G}} Q\bigg).$$

Therefore, we obtain that there exists some cube $Q \in \mathcal{G}$ such that $\alpha_\sigma^{\mathbb{R}^n}(4A'Q) \leq \frac{\delta'}{5}$.

Denote $\widehat{Q} = Q \times [-\ell(Q)/2, \ell(Q)/2]$. Now we wish to bound $\alpha_\mu^H(4A'\widehat{Q})$ in terms of $\alpha_\sigma^{\mathbb{R}^n}(4A'Q)$. Let c_H be the constant that minimizes the infimum in the definition of $\alpha_\sigma^{\mathbb{R}^n}(4A'Q)$ in (3.3). Given any 1-Lipschitz function f supported on $4A'\widehat{Q}$ we have

$$\left|\int f\, d(\mu - c_H\, \mathcal{H}^n|_H)\right| \leq \int_{4A'\widehat{Q} \setminus V_\delta} |f|\, d\mu + \left|\int f\, d(\mu|_{V_\delta} - \sigma)\right| + \left|\int f\, d(\sigma - c_H\, \mathcal{H}^n|_H)\right| =: \mathsf{I}_1 + \mathsf{I}_2 + \mathsf{I}_3.$$

By (3.5), and using also the fact that $\ell(Q) \geq \delta^{1/(4n+4)} r_0$, we have

$$\mathsf{I}_1 \leq \|f\|_\infty \, \mu(B \setminus V_\delta) \lesssim_{C_0} \delta^{1/2} \, \ell(4A'Q) \, \mu(B) \lesssim_{C_0,A'} \delta^{1/2} \, \ell(Q) \, r_0^n \lesssim_{C_0,A'} \delta^{1/4} \, \ell(Q)^{n+1}.$$

We deal with I_2 now: by the definition of σ and the Lipschitz condition on f, we get

$$\begin{aligned}\mathsf{I}_2 &= \left| \int_{4A'\widehat{Q}} f(x) - f(\Pi(x)) \, d\mu|_{V_\delta}(x) \right| \\ &\leq \int_{4A'\widehat{Q}} \operatorname{dist}(x, H) \, d\mu|_{V_\delta}(x) \leq \beta_{\mu,1}(B) \, r_0^{n+1} \leq \delta \, r_0^{n+1} \leq \delta^{3/4} \, \ell(Q)^{n+1}.\end{aligned}$$

Finally, concerning I_3, we have

$$\mathsf{I}_3 \leq \alpha_\sigma^{\mathbb{R}^n}(4A'Q) \, \ell(4A'Q)^{n+1} \leq \frac{\delta'}{5} \, \ell(4A'Q)^{n+1}.$$

Gathering the estimates obtained for I_1, I_2, I_3 and choosing δ small enough we obtain

$$\left| \int f \, d(\mu - c_H \, \mathcal{H}^n|_H) \right| \leq \frac{\delta'}{2} \, \ell(4A'Q)^{n+1},$$

and thus $\alpha_\mu^H(4A'\widehat{Q}) \leq \frac{\delta'}{2}$.

Finally, we choose Q_0 to be a cube with thin boundary such that $\widehat{Q} \subset Q_0 \subset 1.1\widehat{Q}$. Since $3A'Q_0 \subset 4A'\widehat{Q}$ and $\ell(3A'Q_0) \approx \ell(4A'\widehat{Q})$, we get that $\alpha_\mu^H(3A'Q_0) \lesssim \alpha_\mu^H(4A'\widehat{Q}) \lesssim \delta'$. Then, it is easy to check that Q_0 satisfies all the properties (a)-(e) by construction, while regarding (f), we have

$$\begin{aligned}\int_{Q_0} |\mathcal{R}\mu(x) - m_{\mu,Q_0}(\mathcal{R}\mu)|^2 \, d\mu(x) &\leq 2 \int_{Q_0} |\mathcal{R}\mu(x) - m_{\mu,B}(\mathcal{R}\mu)|^2 \, d\mu(x) \\ &\leq 2 \, \varepsilon \, \mu(B) \approx_{C_0,\delta} \varepsilon \, \mu(Q_0).\end{aligned}$$

Thus, if ε is small enough, (f) holds. $\square$

3.3 The Localization Lemma

We assume that the hypotheses of the Main Lemma 3.2.1 hold. Below we allow all the constants denoted by C and all the implicit constants in the relations $\lesssim$ and $\approx$ to depend on the constants C_0 and C_1 in the Main Lemma (but not on A, δ or ε).

Recall that we denote by H the horizontal hyperplane $\{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$. Also we let c_H be some constant that minimizes the infimum in the definition of $\alpha_\mu^H(3AQ_0)$ and we denote $\mathcal{L}_H = c_H \, \mathcal{H}^n\lfloor_H$.

Lemma 3.3.1. *If δ is small enough (depending on A), then we have $c_H \approx 1$ and $\mu(AQ_0) \lesssim A^n \mu(Q_0)$.*

Proof. Let φ be a non-negative $\mathcal{C}^1$ function supported on $2Q_0$ which equals 1 on Q_0 and satisfies $\|\nabla\varphi\|_\infty \lesssim 1/\ell(Q_0)$. Then, we have

$$\left| \int \varphi \, d(\mu - \mathcal{L}_H) \right| \leq \|\nabla\varphi\|_\infty \, \ell(3AQ_0)^{n+1} \, \alpha_\mu^H(3AQ_0) \lesssim A^{n+1} \, \delta \, \ell(Q_0)^n. \tag{3.6}$$

Note that the left hand side above equals

$$\left| \int \varphi \, d\mu - c_H \int_H \varphi \, d\mathcal{H}^n \right| = |c_1 - c_H| \int_H \varphi \, d\mathcal{H}^n,$$

with

$$c_1 = \frac{\int \varphi\, d\mu}{\int_H \varphi\, d\mathcal{H}^n} \approx 1,$$

since

$$\mu(Q_0) \le \int \varphi\, d\mu \le \mu(2Q_0) \lesssim C_0\, \ell(2Q_0)^n \lesssim C_0\, \mu(Q_0),$$

and trivially

$$\int_H \varphi d\mathcal{H}^n \approx \ell(Q_0)^n.$$

Then, from (3.6) we obtain that

$$|c_1 - c_H| \lesssim A^{n+1}\, \delta\, \frac{\ell(Q_0)^n}{\int_H \varphi\, d\mathcal{H}^n} \lesssim A^{n+1}\, \delta.$$

The right hand side is $\ll 1 \approx c_1$ if δ is small enough (depending on A), and so we infer that

$$c_H \approx c_1 \approx 1.$$

In order to estimate $\mu(AQ_0)$, we take another auxiliary non-negative $\mathcal{C}^1$ function $\widetilde{\varphi}$ supported on $3AQ_0$ which equals 1 on AQ_0 and satisfies $\|\nabla\widetilde{\varphi}\|_\infty \lesssim 1/\ell(AQ_0)$. Then, we have

$$\begin{aligned}
\mu(AQ_0) &\le \int \widetilde{\varphi}\, d\mu \\
&\le \left| \int \widetilde{\varphi}\, d(\mu - \mathcal{L}_H) \right| + \int \widetilde{\varphi}\, d\mathcal{L}_H \\
&\lesssim \|\nabla\widetilde{\varphi}\|_\infty\, \ell(3AQ_0)^{n+1}\, \alpha_\mu^H(3AQ_0) + c_H \ell(3AQ_0)^n \\
&\lesssim A^n\, \delta\, \ell(Q_0)^n + \ell(3AQ_0)^n \lesssim A^n\, \ell(Q_0)^n.
\end{aligned}$$

□

Lemma 3.3.2 (Localization Lemma)**.** *If δ is small enough (depending on A), then we have*

$$\int_{Q_0} |\mathcal{R}_\mu \chi_{AQ_0}|^2\, d\mu \lesssim \left(\varepsilon + \frac{1}{A^2} + A^{2n+1} \delta^{1/(8n+8)} \right) \mu(Q_0).$$

Proof. Note first that, by standard estimates, for $x, y \in Q_0$, we have

$$\begin{aligned}
\left| \mathcal{R}_\mu \chi_{(AQ_0)^c}(x) - \mathcal{R}_\mu \chi_{(AQ_0)^c}(y) \right| &\lesssim \int_{(AQ_0)^c} \frac{|x-y|}{|x-z|^{n+1}}\, d\mu(z) \\
&\lesssim \frac{|x-y|}{\ell(AQ_0)}\, P_\mu(AQ_0) \lesssim \frac{1}{A}\, P_\mu(AQ_0) \lesssim \frac{1}{A},
\end{aligned}$$

taking into account the assumption (b) of the Main Lemma 3.2.1 for the last inequality. As a consequence,

$$\left| \mathcal{R}_\mu \chi_{(AQ_0)^c}(x) - m_{\mu,Q_0}(\mathcal{R}_\mu \chi_{(AQ_0)^c}) \right| \lesssim \frac{1}{A},$$

and so

$$\int_{Q_0} \left| \mathcal{R}_\mu \chi_{(AQ_0)^c}(x) - m_{\mu,Q_0}(\mathcal{R}_\mu \chi_{(AQ_0)^c}) \right|^2 d\mu(x) \lesssim \frac{1}{A^2}\, \mu(Q_0).$$

Together with the assumption (g) in the Main Lemma 3.2.1, this gives

$$\begin{aligned}
\int_{Q_0} |\mathcal{R}_\mu \chi_{AQ_0} - m_{\mu,Q_0}(\mathcal{R}_\mu \chi_{AQ_0})|^2\, d\mu &\le 2 \int_{Q_0} \left| \mathcal{R}\mu - m_{\mu,Q_0}(\mathcal{R}\mu) \right|^2 d\mu \\
&\quad + 2 \int_{Q_0} \left| \mathcal{R}_\mu \chi_{(AQ_0)^c} - m_{\mu,Q_0}(\mathcal{R}_\mu \chi_{(AQ_0)^c}) \right|^2 d\mu \\
&\lesssim \varepsilon\, \mu(Q_0) + \frac{1}{A^2}\, \mu(Q_0).
\end{aligned}$$

Hence, to conclude the proof of the lemma it suffices to show that

$$\left|m_{\mu,Q_0}(\mathcal{R}_\mu \chi_{AQ_0})\right| \lesssim C(A)\delta^{1/4(n+1)^2},$$

which is equivalent to

$$\left|m_{\mu,Q_0}(\mathcal{R}_\mu \chi_{AQ_0\setminus Q_0})\right| \le C(A)\delta^{1/4(n+1)^2}. \tag{3.7}$$

since, by the antisymmetry of the Riesz kernel, we have $m_{\mu,Q_0}(\mathcal{R}_\mu \chi_{Q_0}) = 0$.

To prove (3.7), we take first some small constant $0 < \kappa < 1/10$ to be fixed below. We let φ be some $\mathcal{C}^1$ function which equals 1 on $(1-\kappa)AQ_0 \setminus (1+\kappa)Q_0$ and vanishes out of $AQ_0 \setminus (1+\frac{\kappa}{2})Q_0$, so that φ is even and, furthermore $\|\nabla\varphi\|_\infty \lesssim (\kappa\,\ell(Q_0))^{-1}$. Then we split

$$\left|\int_{Q_0} \mathcal{R}_\mu \chi_{AQ_0\setminus Q_0}\,d\mu\right| \le \int_{Q_0} \left|\mathcal{R}_\mu(\chi_{AQ_0\setminus Q_0} - \varphi)\right| d\mu + \left|\int_{Q_0} \mathcal{R}_\mu \varphi\,d\mu\right|. \tag{3.8}$$

To bound the first integral on the right hand side note that $\chi_{AQ_0\setminus Q_0} - \varphi = \psi_1 + \psi_2$, with

$$|\psi_1| \le \chi_{AQ_0\setminus(1-\kappa)AQ_0} \quad \text{and} \quad |\psi_2| \le \chi_{(1+\kappa)Q_0\setminus Q_0}.$$

Then, we have

$$\begin{aligned}
\int_{Q_0} \left|\mathcal{R}_\mu(\chi_{AQ_0\setminus Q_0} - \varphi)\right| d\mu &\le \int_{Q_0} \left|\mathcal{R}_\mu \psi_1\right| d\mu + \int_{Q_0} \left|\mathcal{R}_\mu \psi_2\right| d\mu \\
&\le \|\mathcal{R}_\mu \psi_1\|_{L^\infty(\mu\lfloor_{Q_0})}\, \mu(Q_0) + \|\mathcal{R}_\mu \psi_2\|_{L^2(\mu\lfloor_{Q_0})}\, \mu(Q_0)^{1/2}.
\end{aligned}$$

Since $\operatorname{dist}(\operatorname{supp}(\psi_1), Q_0) \approx A\ell(Q_0)$, we have

$$\|\mathcal{R}_\mu \psi_1\|_{L^\infty(\mu\lfloor_{Q_0})} \lesssim \frac{1}{(A\ell(Q_0))^n}\, \|\psi_1\|_{L^1(\mu)} \le \frac{1}{(A\ell(Q_0))^n}\, \mu(AQ_0 \setminus (1-\kappa)AQ_0).$$

On the other hand, since $\mathcal{R}_\mu$ is bounded in $L^2(\mu\lfloor_{(1+\kappa)Q_0})$, by the assumption (d) in the Main Lemma 3.2.1, and by the thin boundary property of Q_0 (in combination with the fact that $\mu(2Q_0) \approx \mu(Q_0)$), we get

$$\|\mathcal{R}_\mu \psi_2\|_{L^2(\mu\lfloor_{Q_0})} \le C_1 \|\psi_2\|_{L^2(\mu)} \le C_1\, \mu((1+\kappa)Q_0 \setminus Q_0)^{1/2} \le C(C_0, C_1)\, \kappa^{1/2}\, \mu(Q_0)^{1/2}.$$

Therefore,

$$\int_{Q_0} \left|\mathcal{R}_\mu(\chi_{AQ_0\setminus Q_0} - \varphi)\right| d\mu \lesssim \frac{1}{A^n}\, \mu(AQ_0 \setminus (1-\kappa)AQ_0) + \kappa^{1/2}\mu(Q_0). \tag{3.9}$$

In order to estimate $\mu(AQ_0 \setminus (1-\kappa)AQ_0)$ we will use the fact that $\alpha_\mu^H(3AQ_0) \le \delta$. To this end, first consider a function $\widetilde{\varphi}$ supported on $A(1+\kappa)Q_0 \setminus (1-2\kappa)AQ_0$ which equals 1 on $AQ_0 \setminus (1-\kappa)AQ_0$, with $\|\nabla\widetilde{\varphi}\|_\infty \lesssim 1/(A\kappa\ell(Q_0))$. Then, we have

$$\begin{aligned}
\mu(AQ_0 \setminus (1-\kappa)AQ_0) &\le \int \widetilde{\varphi}\,d\mu \qquad (3.10)\\
&\le \left|\int \widetilde{\varphi}\,d(\mu - \mathcal{L}_H)\right| + \int \widetilde{\varphi}\,d\mathcal{L}_H \\
&\le \|\nabla\widetilde{\varphi}\|_\infty\, \ell(3AQ_0)^{n+1}\, \alpha_\mu^H(3AQ_0) + \mathcal{L}_H\big((1+\kappa)AQ_0 \setminus (1-2\kappa)AQ_0\big) \\
&\lesssim \left(\frac{A^n}{\kappa}\,\delta + \kappa\, A^n\right) \ell(Q_0)^n,
\end{aligned}$$

where we used the estimate for c_H in Lemma 3.3.1 for the last inequality. Hence, plugging this estimate into (3.9) we obtain

$$\int_{Q_0} \left|\mathcal{R}_\mu(\chi_{AQ_0\setminus Q_0} - \varphi)\right| d\mu \lesssim \left(\frac{1}{\kappa}\,\delta + \kappa + \kappa^{1/2}\right) \mu(Q_0) \lesssim \left(\frac{\delta}{\kappa} + \kappa^{1/2}\right) \mu(Q_0). \tag{3.11}$$

It remains to estimate the last summand in the inequality (3.8). To this end, we write

$$\begin{aligned}\left|\int_{Q_0}\mathcal{R}_\mu\varphi\,d\mu\right| &\leq \left|\int_{Q_0}\mathcal{R}_\mu\varphi\,d(\mu-\mathcal{L}_H)\right| + \left|\int_{Q_0}\mathcal{R}(\varphi\mu-\varphi\mathcal{L}_H)\,d\mathcal{L}_H\right| \\ &\quad+\left|\int_{Q_0}\mathcal{R}(\varphi\mathcal{L}_H)\,d\mathcal{L}_H\right| \\ &= \mathsf{T}_1+\mathsf{T}_2+\mathsf{T}_3.\end{aligned} \tag{3.12}$$

Since φ is even, by the antisymmetry of the Riesz kernel it easily follows that $\mathsf{T}_3=0$. To deal with T_1, consider another auxiliary function $\widehat{\varphi}$ supported on Q_0 which equals 1 on $(1-\widehat{\kappa})Q_0$, for some small constant $0<\widetilde{\kappa}<\kappa$, so that $\|\nabla\widehat{\varphi}\|_\infty \lesssim 1/(\widehat{\kappa}\ell(Q_0))$. Then, we write

$$\mathsf{T}_1 \leq \left|\int \widehat{\varphi}\,\mathcal{R}_\mu\varphi\,d(\mu-\mathcal{L}_H)\right| + \left|\int(\chi_{Q_0}-\widehat{\varphi})\,\mathcal{R}_\mu\varphi\,d(\mu-\mathcal{L}_H)\right| = \mathsf{T}_{1,a}+\mathsf{T}_{1,b}.$$

To estimate $\mathsf{T}_{1,a}$ we set

$$\mathsf{T}_{1,a} \leq \|\nabla(\widehat{\varphi}\,\mathcal{R}_\mu\varphi)\|_\infty \ell(3AQ_0)^{n+1}\,\alpha_\mu^H(3AQ_0).$$

We write

$$\|\nabla(\widehat{\varphi}\,\mathcal{R}_\mu\varphi)\|_\infty \leq \|\nabla(\mathcal{R}_\mu\varphi)\|_{\infty,Q_0} + \|\nabla\widehat{\varphi}\|_\infty\,\|\mathcal{R}_\mu\varphi)\|_{\infty,Q_0}.$$

Since $\operatorname{dist}(\operatorname{supp}\varphi,Q_0)\geq \frac{\kappa}{4}\ell(Q_0)$ and $\mu(AQ_0)\lesssim \ell(AQ_0)^n$ (by Lemma 3.3.1), we have

$$\|\mathcal{R}_\mu\varphi\|_{\infty,Q_0} \lesssim \frac{\mu(AQ_0)}{(\kappa\ell(Q_0))^n} \lesssim \frac{A^n}{\kappa^n},$$

and, analogously,

$$\|\nabla(\mathcal{R}_\mu\varphi)\|_{\infty,Q_0} \lesssim \frac{\mu(AQ_0)}{(\kappa\ell(Q_0))^{n+1}} \lesssim \frac{A^n}{\kappa^{n+1}\ell(Q_0)}.$$

Hence,

$$\|\nabla(\widehat{\varphi}\,\mathcal{R}_\mu\varphi)\|_\infty \lesssim \frac{A^n}{\kappa^{n+1}\ell(Q_0)} + \frac{A^n}{\widehat{\kappa}\kappa^n\ell(Q_0)} \lesssim \frac{A^n}{\widehat{\kappa}\,\kappa^n\ell(Q_0)},$$

and so, we have

$$\mathsf{T}_{1,a} \lesssim \frac{A^{2n+1}}{\widehat{\kappa}\,\kappa^n}\,\delta\,\mu(Q_0).$$

We consider now the term $\mathsf{T}_{1,b}$. We write

$$\mathsf{T}_{1,b} \leq \|\chi_{Q_0}-\widehat{\varphi}\|_{L^1(\mu+\mathcal{L}_H)}\,\|\mathcal{R}_\mu\varphi\|_{\infty,Q_0}.$$

Recall that $\|\mathcal{R}_\mu\varphi\|_{\infty,Q_0}\lesssim\frac{A^n}{\kappa^n}$. Also, by the construction of $\widehat{\varphi}$ and the thin boundary property of Q_0,

$$\|\chi_{Q_0}-\widehat{\varphi}\|_{L^1(\mu+\mathcal{L}_H)} \lesssim \mu(Q_0\setminus(1-\widehat{\kappa})Q_0)+\mathcal{L}_H(Q_0\setminus(1-\widehat{\kappa})Q_0) \lesssim \widehat{\kappa}\,\mu(Q_0).$$

Then, we obtain

$$\mathsf{T}_{1,b} \leq \frac{A^n}{\kappa^n}\,\widehat{\kappa}\,\mu(Q_0).$$

Thus,

$$\mathsf{T}_1 \lesssim \left(\frac{A^{2n+1}}{\kappa^n\,\widehat{\kappa}}\,\delta + \frac{A^n}{\kappa^n}\,\widehat{\kappa}\right)\mu(Q_0).$$

Choosing $\widehat{\kappa}=\delta^{1/2}$, say, we get

$$\mathsf{T}_1 \lesssim \left(\frac{A^{2n+1}}{\kappa^n}+\frac{A^n}{\kappa^n}\right)\delta^{1/2}\,\mu(Q_0) \leq \frac{A^{2n+1}}{\kappa^n}\,\delta^{1/2}\,\mu(Q_0).$$

Finally, we turn our attention to T_2. By Fubini, we have

$$\mathsf{T}_2 = \left|\int \mathcal{R}(\chi_{Q_0}\mathcal{L}_H)\,\varphi\, d(\mu - \mathcal{L}_H)\right| \leq \|\nabla\big(\mathcal{R}(\chi_{Q_0}\mathcal{L}_H)\,\varphi\big)\|_\infty\, \ell(3AQ_0)^{n+1}\,\alpha_\mu^H(3AQ_0).$$

Observe that

$$\|\nabla\big(\mathcal{R}(\chi_{Q_0}\mathcal{L}_H)\,\varphi\big)\|_\infty \leq \|\nabla\big(\mathcal{R}(\chi_{Q_0}\mathcal{L}_H)\|_{\infty,\mathrm{supp}\varphi} + \|\mathcal{R}(\chi_{Q_0}\mathcal{L}_H)\big)\|_{\infty,\mathrm{supp}\varphi}\|\nabla\varphi\|_\infty.$$

Using the fact that $\mathrm{dist}(\mathrm{supp}\varphi, Q_0) \geq \frac{\kappa}{2}\ell(Q_0)$, we derive

$$\|\mathcal{R}(\chi_{Q_0}\mathcal{L}_H)\big)\|_{\infty,\mathrm{supp}\varphi} \lesssim \frac{\mathcal{L}_H(Q_0)}{(\kappa\ell(Q_0))^n} \lesssim \frac{1}{\kappa^n}$$

and

$$\|\nabla\big(\mathcal{R}(\chi_{Q_0}\mathcal{L}_H)\|_{\infty,\mathrm{supp}\varphi} \lesssim \frac{\mathcal{L}_H(Q_0)}{(\kappa\ell(Q_0))^{n+1}} \lesssim \frac{1}{\kappa^{n+1}\ell(Q_0)},$$

so we obtain

$$\mathsf{T}_2 \lesssim \frac{A^{n+1}}{\kappa^{n+1}}\,\delta\,\mu(Q_0).$$

Gathering the estimates for T_1 and T_2, by (3.12) we infer

$$\left|\int_{Q_0} \mathcal{R}_\mu\varphi\, d\mu\right| \lesssim \frac{A^{2n+1}}{\kappa^n}\,\delta^{1/2}\,\mu(Q_0) + \frac{A^{n+1}}{\kappa^{n+1}}\,\delta\,\mu(Q_0) \lesssim \frac{A^{2n+1}}{\kappa^{n+1}}\,\delta^{1/2}\,\mu(Q_0).$$

Plugging this estimate and (3.11) into (3.8), we obtain

$$\begin{aligned}\left|\int_{Q_0} \mathcal{R}_\mu\chi_{AQ_0\setminus Q_0}\, d\mu\right| &\lesssim \left(\frac{\delta}{\kappa} + \kappa^{1/2}\right)\mu(Q_0) + \frac{A^{2n+1}}{\kappa^{n+1}}\,\delta^{1/2}\,\mu(Q_0)\\ &\lesssim \left(\frac{A^{2n+1}}{\kappa^{n+1}}\,\delta^{1/2} + \kappa^{1/2}\right)\mu(Q_0),\end{aligned}$$

so if we choose $\kappa = \delta^{1/(4n+4)}$, we get

$$\left|\int_{Q_0} \mathcal{R}_\mu\chi_{AQ_0\setminus Q_0}\, d\mu\right| \lesssim \left(A^{2n+1}\,\delta^{1/4} + \delta^{1/(8n+8)}\right)\mu(Q_0) \lesssim A^{2n+1}\delta^{1/(8n+8)}\,\mu(Q_0),$$

which yields (3.7) and finishes the proof of the lemma. □

From now on we will assume that δ is small enough, depending on A, so that the conclusion in the preceding lemma holds.

3.4 The low density cells and the stopping cells

We consider the measure $\sigma = \mu\lfloor_{Q_0}$ and the associated dyadic lattice $\mathcal{D} = \mathcal{D}_\sigma$ introduced in Section 2.3 (re-scaled appropriately, so that we can assume that Q_0 is a cell from $\mathcal{D}_\sigma$). In what follows, we allow all the constants denoted by C and all the implicit constants in the relations $\lesssim$ and $\approx$ to depend on the constants A_0 and K_0 from the construction of the lattice $\mathcal{D}_\sigma$.

Let $0 < \theta_0 < 1$ be a very small constant to be fixed later. We denote by LD the family of those cells Q from $\mathcal{D}_\sigma$ such that $\theta_\sigma(3.5B_Q) \leq \theta_0$ and have maximal side length. The main difficulty for the proof of the Main Lemma 3.2.1 consists in showing that the following holds.

Key Lemma 3.4.1. *There exists some constant $\varepsilon_0 > 0$ such that if A is big enough and θ_0, δ, ε are small enough (with δ possibly depending on A), then*

$$\mu\Bigg(\bigcup_{Q\in\mathsf{LD}} Q\Bigg) \leq (1-\varepsilon_0)\,\mu(Q_0).$$

The proof of this result will be carried out along the next sections of this paper. To do so, *we will assume from now on that*

$$\mu\Bigg(\bigcup_{Q\in\mathsf{LD}} Q\Bigg) > (1-\varepsilon_0)\,\mu(Q_0) \tag{3.13}$$

and we will get a contradiction for ε_0 small enough. To this end, first of all we need to construct another family of stopping cells which we will denote by Stop. This family is defined as follows: for each $Q\in\mathsf{LD}$, we consider the family of maximal cells contained in Q from $\mathcal{D}_\sigma^{db}$ (so they are doubling) with side length at most $t\,\ell(Q)$, where $0<t<1$ is some small parameter which will be fixed below. We denote this family by $\mathsf{Stop}(Q)$. Then, we define

$$\mathsf{Stop} = \bigcup_{Q\in\mathsf{LD}} \mathsf{Stop}(Q).$$

Note that, by Lemma C, it is immediate that, for each $Q\in\mathsf{LD}$, the cells from $\mathsf{Stop}(Q)$ cover μ-almost all Q. Therefore, the assumption (3.13) is equivalent to

$$\mu\Bigg(\bigcup_{Q\in\mathsf{Stop}} Q\Bigg) > (1-\varepsilon_0)\,\mu(Q_0)$$

We need the following auxiliary result:

Lemma 3.4.2. *If we choose $t=\theta_0^{\frac{1}{n+1}}$, then for all $Q\in\mathsf{Stop}$*

$$\theta_\mu(2B_Q)\le P_\mu(2B_Q)\lesssim \theta_0^{\frac{1}{n+1}}.$$

Proof. Let $Q\in\mathsf{Stop}$ and $R\in\mathsf{LD}$ such that $Q\subset R$. The first inequality in the lemma is trivial, so we only have to prove the second one. Let $R'\in\mathcal{D}_\sigma$ the maximal cell such that $Q\subset R'\subset R$ with $\ell(R')\le t\,\ell(R)$, so that $\ell(R')\approx t\,\ell(R)$. Then, we write

$$\begin{aligned}P_\mu(2B_Q) &\lesssim \sum_{P\in\mathcal{D}_\sigma:Q\subset P\subset R'} \theta_\mu(2B_P)\,\frac{\ell(Q)}{\ell(P)} + \sum_{P\in\mathcal{D}_\sigma:R'\subset P\subset R} \theta_\mu(2B_P)\,\frac{\ell(Q)}{\ell(P)}\\ &\quad+ \sum_{P\in\mathcal{D}_\sigma:R\subset P\subset Q_0} \theta_\mu(2B_P)\,\frac{\ell(Q)}{\ell(P)} + \sum_{k\ge1}\theta_\mu(2^kQ_0)\,\frac{\ell(Q)}{\ell(P)}\\ &= \mathsf{S}_1+\mathsf{S}_2+\mathsf{S}_3+\mathsf{S}_4.\end{aligned}$$

To deal with the sums S_1 and S_2, note that for all $P\subset R$, since $2B_P\subset 2B_R$ (assuming A_0 to be big enough), we have

$$\theta_\mu(2B_P)=\frac{\mu(2B_P)}{r(2B_P)^n}\le\frac{\mu(2B_R)}{r(2B_P)^n}=\theta_\mu(2B_R)\,\frac{r(R)^n}{r(B_P)^n}\approx\theta_\mu(2B_R)\,\frac{\ell(R)^n}{\ell(P)^n}.$$

Therefore, since $\theta_\mu(2B_R)\lesssim\theta_0$ and all the cells P appearing in S_2 satisfy $\ell(P)\ge t\,\ell(R)$, we infer that all such cells satisfy $\theta_\mu(2B_P)\lesssim\frac{1}{t^n}\,\theta_\mu(2B_R)\lesssim\frac{\theta_0}{t^n}$, and thus

$$\mathsf{S}_2\lesssim\frac{1}{t^n}\,\theta_\mu(2B_R)\lesssim\frac{\theta_0}{t^n}.$$

Also, since there are no μ-doubling cells between R' and Q, from Lemma E we obtain that

$$\theta_\mu(2B_Q)\lesssim\theta_\mu(2B_{R'})\lesssim\frac{1}{t^n}\,\theta_\mu(2B_R)\lesssim\frac{\theta_0}{t^n},$$

and therefore we also get

$$\mathsf{S}_1\lesssim\frac{\theta_0}{t^n}.$$

For the cells P in the sum S_3, we just take into account that $\theta_\mu(2B_P) \lesssim 1$, and then we get

$$\mathsf{S}_3 \lesssim \sum_{P\in\mathcal{D}_\sigma: R\subset P\subset Q_0} \frac{\ell(Q)}{\ell(P)} \lesssim \frac{\ell(Q)}{\ell(R)} \lesssim t.$$

Finally, regarding the sum S_4, note that

$$\mathsf{S}_4 = \frac{\ell(Q)}{\ell(Q_0)} \sum_{k\geq 1} \theta_\mu(2^k Q_0)\, \frac{\ell(Q_0)}{\ell(P)} = \frac{\ell(Q)}{\ell(Q_0)}\, P_\mu(Q_0) \lesssim \frac{\ell(Q)}{\ell(Q_0)} \lesssim t.$$

Hence,

$$P_\mu(2B_Q) \lesssim \frac{\theta_0}{t^n} + t \approx \theta_0^{\frac{1}{n+1}},$$

recalling that $t = \theta_0^{\frac{1}{n+1}}$. □

From now on, we will assume that we have chosen $t = \theta_0^{\frac{1}{n+1}}$, so that the conclusion of the preceding lemma holds.

The family Stop may consist of an infinite number of cells. For technical reasons, it is convenient to consider a finite subfamily of Stop which contains a very big proportion of the μ-measure of Stop. Therefore, we let Stop_0 be a *finite* subfamily of Stop such that

$$\mu\Bigg(\bigcup_{Q\in\mathsf{Stop}_0} Q\Bigg) > (1-2\varepsilon_0)\,\mu(Q_0).$$

We denote by Bad the family of the cells $P \in \mathsf{Stop}$ such that $1.1B_P \cap \partial Q_0 \neq \varnothing$.

Lemma 3.4.3. *We have*

$$\mu\Bigg(\bigcup_{Q\in\mathsf{Bad}} Q\Bigg) \lesssim \theta_0^{1/(n+1)}\,\mu(Q_0).$$

Proof. Let $I \subset \mathsf{Bad}$ an arbitrary finite family of bad cells. We apply Vitali's covering theorem of triple balls to the family $\{1.15B_Q\}_{Q\in I}$, so that we get a subfamily $J \subset I$ satisfying

- $1.15B_P \cap 1.15B_Q = \varnothing$ for different cells $P, Q \in J$.
- $\bigcup_{P\in I} 1.15B_P \subset \bigcup_{Q\in J} 3.45B_Q$.

Then, using the fact that

$$\mu(3.45B_Q) \leq \mu(3.5B_Q) \lesssim \theta_0^{1/(n+1)}\, r(B_Q)^n$$

for each $Q \in J$, we get

$$\mu\Bigg(\bigcup_{P\in I} P\Bigg) \leq \mu\Bigg(\bigcup_{P\in I} B_P\Bigg) \leq \sum_{Q\in J} \mu(3.45Q) \lesssim \theta_0^{1/(n+1)} \sum_{Q\in J} r(B_Q)^n.$$

Now, for each $Q \in J$ we have $1.1B_Q \cap \partial Q_0 \neq \varnothing$ and so we obtain that

$$\mathcal{H}^n(1.15B_Q \cap \partial Q_0) \gtrsim r(B_Q)^n.$$

Thus, using also the fact that the balls $1.15B_Q$, $Q \in J$, are pairwise disjoint,

$$\mu\Bigg(\bigcup_{P\in I} P\Bigg) \lesssim \theta_0^{1/(n+1)} \sum_{Q\in J} \mathcal{H}^n(1.15B_Q \cap \partial Q_0) \leq \theta_0^{1/(n+1)}\, \mathcal{H}^n(\partial Q_0) \approx \theta_0^{1/(n+1)}\,\mu(Q_0),$$

and the lemma follows. □

We will now define an auxiliary measure μ_0. First, given a small constant $0 < \kappa_0 \ll 1$ (to be fixed below) and $Q \in \mathcal{D}_\sigma$, we denote

$$I_{\kappa_0}(Q) = \{x \in Q : \operatorname{dist}(x, \operatorname{supp}(\sigma) \setminus Q) \geq \kappa_0 \ell(Q)\}, \tag{3.14}$$

so $I_{\kappa_0}(Q)$ is some kind of inner zone of Q. We set

$$\mu_0 = \mu\lfloor_{Q_0^c} + \sum_{Q \in \mathsf{Stop}_0 \setminus \mathsf{Bad}} \mu\lfloor_{I_{\kappa_0}(Q)}.$$

Observe that, by the thin boundary condition of $Q \in \mathsf{Stop}_0$ together with the fact that it is doubling, we have

$$\mu(Q \setminus I_{\kappa_0}(Q)) \lesssim \kappa_0^{1/2}\, \sigma(3.5B_Q) \lesssim \kappa_0^{1/2}\, \mu(Q).$$

Combining this estimate with the assumption (3.13) and Lemma 3.4.3, we get

$$\begin{aligned}\|\mu - \mu_0\| &= \mu(Q_0) - \mu_0(Q_0) \\ &= \mu(Q_0) - \sum_{Q \in \mathsf{Stop}_0 \setminus \mathsf{Bad}} \mu(I_{\kappa_0}(Q)) \\ &= \mu\left(Q_0 \setminus \bigcup_{Q \in \mathsf{Stop}_0} Q\right) + \sum_{Q \in \mathsf{Bad}} \mu(Q) + \sum_{Q \in \mathsf{Stop}_0 \setminus \mathsf{Bad}} \mu(Q \setminus I_{\kappa_0}(Q)) \\ &\leq 2\varepsilon_0\, \mu(Q_0) + C\theta_0^{1/(n+1)}\, \mu(Q_0) + C\kappa_0^{1/2}\, \mu(Q_0).\end{aligned} \tag{3.15}$$

Together with Lemma 3.3.2, this yields the following:

Lemma 3.4.4. *If δ is small enough (depending on A), then we have*

$$\int_{Q_0} |\mathcal{R}(\chi_{AQ_0}\mu_0)|^2\, d\mu_0 \lesssim \left(\varepsilon + \frac{1}{A^2} + \delta^{1/(8n+8)} + \varepsilon_0 + \theta_0^{1/(n+1)} + \kappa_0^{1/2}\right) \mu(Q_0).$$

Proof. We have

$$\begin{aligned}\int_{Q_0} |\mathcal{R}(\chi_{AQ_0}\mu_0)|^2\, d\mu_0 &\leq 2\int_{Q_0} |\mathcal{R}(\chi_{AQ_0}\mu)|^2\, d\mu + 2\int_{Q_0} |\mathcal{R}(\chi_{AQ_0}(\mu - \mu_0))|^2\, d\mu \\ &\lesssim \left(\varepsilon + \frac{1}{A^2} + \delta^{1/(8n+8)} + \varepsilon_0 + \theta_0^{1/(n+1)} + \kappa_0^{1/2}\right) \mu(Q_0),\end{aligned}$$

by Lemma 3.3.2, the $L^2(\mu\lfloor_{Q_0})$-boundedness of $\mathcal{R}_{\mu\lfloor_{Q_0}}$, and (3.15). □

3.5 The periodic measure $\widetilde{\mu}$

Let $\mathcal{M}$ be the lattice of cubes in $\mathbb{R}^{n+1}$ obtained by translating Q_0 in directions parallel to H, so that H coincides with the union of the n-dimensional cubes from the family $\{P \cap H\}_{P \in \mathcal{M}}$ and the cubes have disjoint interiors. For each $P \in \mathcal{M}$, denote by z_P the center of P and consider the translation $T_P : x \to x + z_P$, so that $P = T_P(Q_0)$. Note that $\{z_P : P \in \mathcal{M}\}$ coincides with the set $(\ell(Q_0)\mathbb{Z}^n) \times \{0\}$. We define

$$\widetilde{\mu} = \sum_{P \in \mathcal{M}} (T_P)_\#(\mu_0\lfloor_{Q_0}),$$

that is,

$$\widetilde{\mu}(E) = \sum_{P \in \mathcal{M}} \mu_0(Q_0 \cap T_P^{-1}(E)) = \sum_{P \in \mathcal{M}} \mu_0(Q_0 \cap (E - z_P)).$$

It is easy to check that:

(i) $\widetilde{\mu}$ is periodic with respect to $\mathcal{M}$, that is, for all $P \in \mathcal{M}$ and all $E \subset \mathbb{R}^{n+1}$, $\widetilde{\mu}(E + z_P) = \widetilde{\mu}(E)$.

(ii) $\chi_{Q_0}\widetilde{\mu} = \mu_0$.

The latter property holds because $\mu_0(\partial Q_0) = 0$.

For simplicity, from now on we will assume that A is a big enough odd natural number.

Lemma 3.5.1. *We have*

$$\alpha^H_{\widetilde{\mu}}(3AQ_0) \leq C_3\, A^{n+1} \left(\varepsilon_0 + \theta_0^{1/(n+1)} + \kappa_0^{1/2} + \delta^{1/2}\right).$$

In fact,

$$\operatorname{dist}_{3AQ_0}(\widetilde{\mu}, \mathcal{L}_H) \leq C_3 A^{n+1} \left(\varepsilon_0 + \theta_0^{1/(n+1)} + \kappa_0^{1/2} + \delta^{1/2}\right) \ell(3AQ_0)^{n+1},$$

where $\mathcal{L}_H$ is the same minimizing measure as the one for $\alpha^H_\mu(3AQ_0)$.

Proof. Let f be a Lipschitz function supported on $3AQ_0$ with Lipschitz constant at most 1, denote by $\mathcal{M}_0$ the family of cubes in $\mathcal{M}$ which are contained in $3AQ_0$, and let $\kappa > 0$ be some small parameter to be fixed below. Consider a $\mathcal{C}^1$ function φ supported on Q_0 which equals 1 on $(1-\kappa)Q_0$, with $\|\nabla\varphi\|_\infty \lesssim 1/(\kappa\ell(Q_0))$ and denote $\varphi_P(x) = \varphi(x - z_P)$. Then, we write

$$\begin{aligned}\left|\int f\, d(\widetilde{\mu} - \mathcal{L}_H)\right| &\leq \sum_{P\in\mathcal{M}_0} \left|\int_P f\, d(\widetilde{\mu} - \mathcal{L}_H)\right| \qquad (3.16)\\ &\leq \sum_{P\in\mathcal{M}_0} \left|\int \varphi_P\, f\, d(\widetilde{\mu} - \mathcal{L}_H)\right| + \sum_{P\in\mathcal{M}_0} \int \left|(\chi_P - \varphi_P)\, f\right| d(\widetilde{\mu} + \mathcal{L}_H).\end{aligned}$$

Let us estimate the first sum on the right hand side. Since $\widetilde{\mu}\lfloor_P = (T_P)_\#\mu_0\lfloor_{Q_0}$ and $\mathcal{L}_H = (T_P)_\#\mathcal{L}_H$, we have

$$\begin{aligned}\left|\int \varphi_P\, f\, d(\widetilde{\mu} - \mathcal{L}_H)\right| &= \left|\int \varphi(x)\, f(x + z_P)\, d(\mu - \mathcal{L}_H)\right|\\ &\leq \left|\int \varphi(x)\, f(x + z_P)\, d(\mu_0 - \mu)\right| + \left|\int \varphi(x)\, f(x + z_P)\, d(\mu - \mathcal{L}_H)\right|\\ &= \mathsf{I}_1 + \mathsf{I}_2.\end{aligned}$$

To estimate I_1 we use (3.15) and the fact that, by the mean value theorem, $\|\varphi\, f(\cdot + z_P)\|_\infty \lesssim \ell(3AQ_0)$. Then, we have

$$\mathsf{I}_1 \leq \left|\int \varphi(x)\, f(x + z_P)\, d(\mu_0 - \mu)\right| \lesssim \left(\varepsilon_0 + \theta_0^{1/(n+1)} + \kappa_0^{1/2}\right) \ell(3AQ_0)^{n+1}.$$

Concerning I_2, we write

$$\mathsf{I}_2 \lesssim \|\nabla(\varphi f(\cdot + z_P))\|_\infty \ell(3AQ_0)^{n+1}\, \alpha^H_\mu(3A_Q).$$

Note that

$$\|\nabla(\varphi_P f)\|_\infty \leq \|\nabla f\|_\infty + \|f\|_\infty \|\nabla\varphi_P)\|_\infty \lesssim 1 + C\, A\, \ell(Q_0)\, \frac{1}{\kappa\, \ell(Q_0)} \lesssim \frac{A}{\kappa}.$$

Thus,

$$\mathsf{I}_2 \lesssim A\, \frac{\delta}{\kappa}\, \ell(3AQ_0)^{n+1},$$

and therefore,

$$\left|\int \varphi_P\, f\, d(\widetilde{\mu} - \mathcal{L}_H)\right| \lesssim A \left(\varepsilon_0 + \theta_0^{1/(n+1)} + \kappa_0^{1/2} + \frac{\delta}{\kappa}\right) \ell(3AQ_0)^{n+1}.$$

To deal with the second sum on the right hand side of (3.16) we write

$$\begin{aligned}\int \big|(\chi_P - \varphi_P)\, f\big|\, d(\widetilde{\mu} + \mathcal{L}_H) &\leq \|\chi_P - \varphi_P\|_{L^1(\widetilde{\mu}+\mathcal{L}_H)}\, \|f\|_\infty \\ &\lesssim (\mu + \mathcal{L}_H)\big(Q_0 \setminus (1-\kappa)Q_0\big)\, \ell(3AQ_0).\end{aligned}$$

By the thin boundary condition on Q_0,

$$\mu\big(Q_0 \setminus (1-\kappa)Q_0\big) \lesssim \kappa\, \mu(Q_0) = \kappa\, \ell(Q_0)^n.$$

Clearly, the same estimate holds replacing μ by $\mathcal{L}_H$, and so we obtain

$$\int \big|(\chi_P - \varphi_P)\, f\big|\, d(\widetilde{\mu} + \mathcal{L}_H) \lesssim \kappa\, \ell(Q_0)^{n+1}.$$

Taking into account that the number of cubes $P \in \mathcal{M}_0$ is comparable to A^n, we get

$$\left|\int f\, d(\widetilde{\mu} - \mathcal{L}_H)\right| \lesssim A^{n+1} \left(\varepsilon_0 + \theta_0^{1/(n+1)} + \kappa_0^{1/2} + \frac{\delta}{\kappa} + \kappa\right) \ell(3AQ_0)^{n+1}.$$

Choosing $\kappa = \delta^{1/2}$, the lemma follows. $\square$

From now on, to simplify notation we will denote

$$\widetilde{\delta} = C_3\, A^{n+1} \left(\varepsilon_0 + \theta_0^{1/(n+1)} + \kappa_0^{1/2} + \delta^{1/2}\right), \tag{3.17}$$

so the preceding lemma ensures that $\alpha_{\mu}^{H}(3AQ_0) \leq \widetilde{\delta}$. We assume that the parameters ε_0, θ_0, κ_0, and δ are small enough so that $\widetilde{\delta} \ll 1$.

Lemma 3.5.2. *We have*

$$\int_{Q_0} |\mathcal{R}(\chi_{AQ_0}\widetilde{\mu})|^2\, d\widetilde{\mu} \leq C_4 \left(\varepsilon + \frac{1}{A^2} + \delta^{\frac{1}{8n+8}} + \varepsilon_0 + \theta_0^{\frac{1}{n+1}} + \kappa_0^{\frac{1}{2}} + A^{n+1}\, \widetilde{\delta}^{\frac{1}{2n+3}}\right) \widetilde{\mu}(Q_0). \tag{3.18}$$

Proof. Since $\widetilde{\mu}\lfloor_{Q_0} = \mu_0\lfloor_{Q_0}$, we have

$$\int_{Q_0} |\mathcal{R}(\chi_{AQ_0}\widetilde{\mu})|^2\, d\widetilde{\mu} \leq 2\int_{Q_0} |\mathcal{R}(\chi_{AQ_0}\mu_0)|^2\, d\mu_0 + 2\int_{Q_0} |\mathcal{R}(\chi_{AQ_0}(\widetilde{\mu} - \mu_0)|^2\, d\mu_0. \tag{3.19}$$

The first integral on the right hand side has been estimated in Lemma 3.4.4. Therefore, we only have to deal with the second one. The arguments that we will use will be similar to some of the ones in Lemma 3.3.2.

First, note that, using again the fact that $\widetilde{\mu}\lfloor_{Q_0} = \mu_0\lfloor_{Q_0}$ and that $\mu_0\lfloor_{Q_0^c} = \mu\lfloor_{Q_0^c}$, we have

$$\mathcal{R}\big(\chi_{AQ_0}(\widetilde{\mu} - \mu_0)\big) = \mathcal{R}\big(\chi_{AQ_0\setminus Q_0}(\widetilde{\mu} - \mu_0)\big).$$

Let $0 < \kappa < 1/10$ be some small constant to be fixed below. Let φ be a $\mathcal{C}^1$ function which equals 1 on $(1-\kappa)AQ_0 \setminus (1+\kappa)Q_0$ and vanishes out of $AQ_0 \setminus (1+\frac{\kappa}{2})Q_0$, with $\|\nabla\varphi\|_\infty \lesssim (\kappa\, \ell(Q_0))^{-1}$. Then, we split

$$\int_{Q_0} |\mathcal{R}(\chi_{AQ_0}(\widetilde{\mu} - \mu_0)|^2\, d\mu_0 \leq 2\int_{Q_0} \big|\mathcal{R}\big((\chi_{AQ_0\setminus Q_0} - \varphi)(\widetilde{\mu} - \mu_0)\big)\big|^2\, d\mu_0 + 2\int_{Q_0} \big|\mathcal{R}\big(\varphi(\widetilde{\mu} - \mu_0)\big)\big|^2\, d\mu_0. \tag{3.20}$$

Concerning the first integral on the right hand side note that $\chi_{AQ_0\setminus Q_0} - \varphi = \psi_1 + \psi_2$, with

$$|\psi_1| \leq \chi_{AQ_0\setminus(1-\kappa)AQ_0} \quad \text{and} \quad |\psi_2| \leq \chi_{(1+\kappa)Q_0\setminus Q_0}.$$

Then, we have

$$\int_{Q_0} |\mathcal{R}\big((\chi_{AQ_0} - \varphi)(\widetilde{\mu} - \mu_0)\big)|^2 \, d\mu_0 \lesssim \int_{Q_0} |\mathcal{R}\big(\psi_1(\widetilde{\mu} - \mu)\big)|^2 \, d\mu + \int_{Q_0} |\mathcal{R}\big(\psi_2(\widetilde{\mu} - \mu)\big)|^2 \, d\widetilde{\mu}$$
$$\leq \|\mathcal{R}\big(\psi_1(\widetilde{\mu} - \mu)\big)\|^2_{L^\infty(\mu\lfloor_{Q_0})} \, \mu(Q_0)$$
$$+ \|\mathcal{R}\big(\psi_2(\widetilde{\mu} - \mu)\big)\|^2_{L^4(\widetilde{\mu}\lfloor_{Q_0})} \, \mu(Q_0)^{1/2}.$$

Since $\operatorname{dist}(\operatorname{supp}(\psi_1), Q_0) \approx A\ell(Q_0)$, we get

$$\|\mathcal{R}\big(\psi_1(\widetilde{\mu} - \mu)\big)\|_{L^\infty(\mu\lfloor_{Q_0})} \lesssim \frac{1}{(A\ell(Q_0))^n} \, \|\psi_1\|_{L^1(\widetilde{\mu}+\mu)} \leq \frac{1}{(A\ell(Q_0))^n} \, (\widetilde{\mu} + \mu)(AQ_0 \setminus (1-\kappa)AQ_0).$$

Recall that in (3.10) it has been shown that

$$\mu(AQ_0 \setminus (1-\kappa)AQ_0) \lesssim \left(\frac{A^n}{\kappa} \, \delta + \kappa \, A^n \right) \ell(Q_0)^n.$$

To prove this we used the fact that $\alpha_\mu^H(3AQ_0) \leq \delta$ or, more precisely, that $\operatorname{dist}_{3AQ_0}(\mu, \mathcal{L}_H) \leq \delta$. The same inequality holds replacing μ by $\widetilde{\mu}$ and δ by $\widetilde{\delta}$, as shown in Lemma 3.5.1. Therefore, by arguments analogous to the ones in (3.10) it follows that

$$\widetilde{\mu}(AQ_0 \setminus (1-\kappa)AQ_0) \lesssim \left(\frac{A^n}{\kappa} \, \widetilde{\delta} + \kappa \, A^n \right) \ell(Q_0)^n.$$

Therefore, we obtain that

$$\|\mathcal{R}\big(\psi_1(\widetilde{\mu} - \mu)\big)\|_{L^\infty(\mu\lfloor_{Q_0})} \lesssim \frac{\delta + \widetilde{\delta}}{\kappa} + \kappa \lesssim \frac{\widetilde{\delta}}{\kappa} + \kappa,$$

taking into account that $\delta \leq \widetilde{\delta}$ for the last inequality.

Next we will estimate $\|\mathcal{R}\big(\psi_2(\widetilde{\mu} - \mu)\big)\|_{L^4(\widetilde{\mu}\lfloor_{Q_0})}$. By the triangle inequality, we have

$$\|\mathcal{R}\big(\psi_2(\widetilde{\mu} - \mu)\big)\|_{L^4(\widetilde{\mu}|_{Q_0})} \leq \|\mathcal{R}_\mu \psi_2\|_{L^4(\mu|_{Q_0})} + \|\mathcal{R}_{\widetilde{\mu}} \psi_2\|_{L^4(\widetilde{\mu}|_{Q_0})}.$$

Recall that $\mathcal{R}_\mu$ is bounded in $L^2(\mu\lfloor_{2Q_0})$, and so in $L^4(\mu\lfloor_{2Q_0})$, and that $\operatorname{supp}\psi_2 \subset (1+\kappa Q_0) \setminus Q_0 \subset 2Q_0$. Hence, using also the thin boundary property of Q_0, we obtain

$$\|\mathcal{R}_\mu \psi_2\|^4_{L^4(\mu|_{Q_0})} \lesssim \|\psi_2\|^4_{L^4(\mu|_{2Q_0})} \lesssim \mu((1+\kappa Q_0) \setminus Q_0) \lesssim \kappa \, \mu(Q_0).$$

We can apply the same argument to estimate $\|\mathcal{R}_{\widetilde{\mu}} \psi_2\|_{L^4(\widetilde{\mu}\lfloor_{Q_0})}$. This is due to the fact that $\mathcal{R}_{\widetilde{\mu}}$ is bounded in $L^2(\widetilde{\mu}\lfloor_{2Q_0})$. This is an easy consequence of the fact that, given two measures μ_1 and μ_2 with growth of order n such that, for $i = 1, 2$, $\mathcal{R}_{\mu_i}$ is bounded in $L^2(\mu_i)$, then $\mathcal{R}_{\mu_1+\mu_2}$ is bounded in $L^2(\mu_1 + \mu_2)$. For the proof, see Proposition 2.25 of [T6], for example. Then, applying this result to a finite number of translated copies of $\mu\lfloor_{Q_0}$, we infer that $\mathcal{R}_{\widetilde{\mu}}$ is bounded in $L^2(\widetilde{\mu}\lfloor_{2Q_0})$ and so in $L^4(\widetilde{\mu}\lfloor_{2Q_0})$. Therefore, we also have

$$\|\mathcal{R}_{\widetilde{\mu}} \psi_2\|^4_{L^4(\widetilde{\mu}|_{Q_0})} \lesssim \kappa \, \widetilde{\mu}(Q_0) \leq \kappa \, \mu(Q_0).$$

Gathering the estimates above, it turns out that the first integral on the right side of (3.20) satisfies the following:

$$\int_{Q_0} |\mathcal{R}(\chi_{AQ_0}(\widetilde{\mu} - \mu_0)|^2 \, d\mu_0 \lesssim \left(\frac{\widetilde{\delta}}{\kappa} + \kappa \right)^2 \mu(Q_0) + \kappa^{1/2} \, \mu(Q_0) \lesssim \left(\frac{\widetilde{\delta}}{\kappa} + \kappa^{1/4} \right)^2 \mu(Q_0). \tag{3.21}$$

It remains to estimate the second integral on the right hand side of (3.20). To this end, for any $x \in Q_0$ we set

$$\begin{aligned}|\mathcal{R}\big(\varphi(\widetilde{\mu}-\mu_0)(x)\big)| &= \left|\int K(x-y)\,\varphi(y)\,d(\widetilde{\mu}-\mu)(y)\right| \\ &\le \left|\int K(x-y)\,\varphi(y)\,d(\widetilde{\mu}-\mathcal{L}_H)(y)\right| \\ &\quad + \left|\int K(x-y)\,\varphi(y)\,d(\mu-\mathcal{L}_H)(y)\right| \\ &\le \|\nabla(K(x-\cdot)\,\varphi)\|_\infty \left[\operatorname{dist}_{3AQ_0}(\widetilde{\mu},\mathcal{L}_H)+\operatorname{dist}_{3AQ_0}(\mu,\mathcal{L}_H)\right],\end{aligned}$$

where in the first identity we used the fact that μ_0 coincides with μ on the support of φ. Taking into account the fact that $\operatorname{dist}(x,\operatorname{supp}(\varphi)) \gtrsim \kappa\ell(Q_0)$, we obtain

$$\|\nabla(K(x-\cdot)\,\varphi)\|_\infty \le \|\nabla K(x-\cdot)\|_{\infty,\operatorname{supp}(\varphi)} + \|K(x-\cdot)\|_{\infty,\operatorname{supp}(\varphi)}\,\|\nabla\varphi\|_\infty \lesssim \frac{1}{(\kappa\,\ell(Q_0))^{n+1}}.$$

By Lemma 3.5.1, $\operatorname{dist}_{3AQ_0}(\widetilde{\mu},\mathcal{L}_H) \le \widetilde{\delta}\,\ell(3AQ_0)^{n+1}$ and, by the assumption (e) in the Main Lemma, $\operatorname{dist}_{3AQ_0}(\mu,\mathcal{L}_H) \le \delta\,\ell(3AQ_0)^{n+1}$. Therefore,

$$|\mathcal{R}\big(\varphi(\widetilde{\mu}-\mu_0)(x)\big)| \lesssim \frac{1}{(\kappa\,\ell(Q_0))^{n+1}}(\delta+\widetilde{\delta})\,\ell(3AQ_0)^{n+1} \lesssim \frac{A^{n+1}}{\kappa^{n+1}}\,\widetilde{\delta},$$

so the last integral on the right hand side of (3.20) satisfies

$$\int_{Q_0} |\mathcal{R}(\chi_{AQ_0}(\widetilde{\mu}-\mu_0)|^2\,d\widetilde{\mu} \lesssim \frac{A^{n+1}}{\kappa^{n+1}}\,\widetilde{\delta}\,\mu(Q_0). \tag{3.22}$$

From (3.20), (3.21) and (3.22) we obtain that

$$\begin{aligned}\int_{Q_0} |\mathcal{R}(\chi_{AQ_0}(\widetilde{\mu}-\mu_0)|^2\,d\mu_0 &\lesssim \left(\frac{\widetilde{\delta}}{\kappa}+\kappa^{1/4}\right)^2\mu(Q_0) + \frac{A^{n+1}}{\kappa^{n+1}}\,\widetilde{\delta}\mu(Q_0) \\ &\lesssim A^{n+1}\left(\frac{\widetilde{\delta}}{\kappa^{n+1}}+\kappa^{1/2}\right)\mu(Q_0).\end{aligned}$$

Choosing $\kappa = \widetilde{\delta}^{\frac{2}{2n+3}}$, the right hand side above equals $CA^{n+1}\,\widetilde{\delta}^{\frac{1}{2n+3}}\,\mu(Q_0)$. Together with (3.19) and Lemma 3.5.2, this yields (3.18). □

To simplify notation we will write

$$\widetilde{\varepsilon} = C_4\left(\varepsilon + \frac{1}{A^2} + \delta^{\frac{1}{8n+8}} + \varepsilon_0 + \theta_0^{\frac{1}{n+1}} + \kappa_0^{\frac{1}{2}} + A^{n+1}\,\widetilde{\delta}^{\frac{1}{2n+3}}\right), \tag{3.23}$$

so that the preceding lemma guarantees that

$$\int_{Q_0} |\mathcal{R}(\chi_{AQ_0}\widetilde{\mu})|^2\,d\widetilde{\mu} \le \widetilde{\varepsilon}\,\widetilde{\mu}(Q_0).$$

We will also need the following auxiliary result below.

Lemma 3.5.3. *For all $Q \in \mathsf{Stop}_0 \setminus \mathsf{Bad}$, we have*

$$\int_{1.1B_Q\setminus Q}\int_Q \frac{1}{|x-y|^n}\,d\widetilde{\mu}(x)\,d\widetilde{\mu}(y) \lesssim \theta_0^{\frac{1}{2(n+1)^2}}\,\widetilde{\mu}(Q).$$

Proof. Since any ball $1.1B_Q$ with $Q \in \mathsf{Stop}_0 \setminus \mathsf{Bad}$ is contained in Q_0, we have that $\widetilde{\mu} = \mu_0$ in the domain of integration considered above.

Let $0 < \kappa < 1$ be some small constant to be fixed below. Then we split

$$\int_{1.1B_Q\setminus Q}\int_Q \frac{1}{|x-y|^n}\,d\widetilde{\mu}(y)\,d\widetilde{\mu}(x) = \int_{x\in 1.1B_Q\setminus Q}\int_{y\in Q:|x-y|>\kappa\ell(Q)} \frac{1}{|x-y|^n}\,d\widetilde{\mu}(y)\,d\widetilde{\mu}(x) \\ + \int_{x\in 1.1B_Q\setminus Q}\int_{y\in Q:|x-y|\leq\kappa\ell(Q)} \frac{1}{|x-y|^n}\,d\widetilde{\mu}(y)\,d\widetilde{\mu}(x). \tag{3.24}$$

First we deal with the first integral on the right hand side:

$$\begin{aligned}\int_{x\in 1.1B_Q\setminus Q}\int_{y\in Q:|x-y|>\kappa\ell(Q)} \frac{1}{|x-y|^n}\,d\widetilde{\mu}(y)\,d\widetilde{\mu}(x) &\leq \frac{1}{\kappa^n\ell(Q)^n}\,\widetilde{\mu}(1.1B_Q)\,\widetilde{\mu}(Q) \\ &\lesssim \frac{1}{\kappa^n}\,\Theta_{\widetilde{\mu}}(1.1B_Q)\,\widetilde{\mu}(Q) \lesssim \frac{\theta_0^{\frac{1}{n+1}}}{\kappa^n}\,\mu(Q),\end{aligned}$$

by Lemma 3.4.2.

Let us turn our attention to the last integral in (3.24). To estimate it, we take into account the fact that given $x \in 1.1B_Q \setminus Q$, if $y \in Q$, then $|x-y| \geq \operatorname{dist}(x,Q)$. Then, by the polynomial growth of order n of $\mu\lfloor_{Q_0}$ and standard estimates, we get

$$\int_{y\in Q:|x-y|\leq\kappa\ell(Q)} \frac{1}{|x-y|^n}\,d\widetilde{\mu}(y) \lesssim \log\left(2+\frac{\kappa\,\ell(Q)}{\operatorname{dist}(x,Q)}\right) \quad \text{for all } x\in 1.1B_Q\setminus Q.$$

For each $j \geq 0$, denote

$$U_j = \big\{x\in 1.1B_Q\setminus Q : \operatorname{dist}(x,Q)\leq 2^{-j}\,\kappa\,\ell(Q)\big\}.$$

By the thin boundary property of Q and the fact that Q is doubling,

$$\mu(U_j) \lesssim (2^{-j}\,\kappa)^{1/2}\,\mu(3.5B_Q) \lesssim (2^{-j}\,\kappa)^{1/2}\,\mu(Q).$$

Then, we obtain

$$\begin{aligned}\int_{x\in 1.1B_Q\setminus Q}\int_{y\in Q:|x-y|\leq\kappa\ell(Q)} \frac{1}{|x-y|^n}\,d\widetilde{\mu}(y)\,d\widetilde{\mu}(x) &\leq \sum_{j\geq 0}\int_{U_j\setminus U_{j+1}} \log\left(2+\frac{\kappa\,\ell(Q)}{\operatorname{dist}(x,Q)}\right)d\mu(x) \\ &\lesssim \sum_{j\geq 0}\log\left(2+\frac{\kappa\,\ell(Q)}{2^{-j-1}\kappa\ell(Q)}\right)\mu(U_j) \\ &\lesssim \sum_{j\geq 0}(j+1)(2^{-j}\,\kappa)^{1/2}\,\mu(Q) \\ &\lesssim \kappa^{1/2}\,\mu(Q).\end{aligned}$$

Therefore, we have

$$\int_{1.1B_Q\setminus Q}\int_Q \frac{1}{|x-y|^n}\,d\widetilde{\mu}(x)\,d\widetilde{\mu}(y) \lesssim \big(\theta_0^{1/(n+1)}\,\kappa^{-n} + \kappa^{1/2}\big)\,\mu(Q).$$

Choosing $\kappa = \theta_0^{\frac{1}{(n+1)^2}}$, the lemma follows. □

It is easy to check that

$$\widetilde{\mu}(B(x,r)) \lesssim r^n \quad \text{for all } x\in\mathbb{R}^{n+1} \text{ and all } r>0. \tag{3.25}$$

This follows easily from the analogous estimate for $\mu\lfloor_{Q_0}$ and the periodicity of $\widetilde{\mu}$, and is left for the reader. On the other hand, in general, we cannot guarantee that the estimates for the coefficients $P_\mu(2B_Q)$ in Lemma 3.4.2 also hold with μ replaced by $\widetilde{\mu}$. However, we have following substitute:

Lemma 3.5.4. *The function*

$$p_{\widetilde{\mu}}(x) = \sum_{Q \in \mathsf{Stop}_0 \setminus \mathsf{Bad}:\, x \in Q} \chi_Q\, P_{\widetilde{\mu}}(2B_Q)$$

satisfies

$$\int_{Q_0} p_{\widetilde{\mu}}^2\, d\widetilde{\mu} \lesssim \theta_0^{\frac{1}{2(n+1)}}\, \widetilde{\mu}(Q_0).$$

Proof. Let $0 < \kappa < 1$ be some small constant to be fixed below. We split

$$\int_{Q_0} p_{\widetilde{\mu}}^2\, d\widetilde{\mu} = \int_{x \in Q_0 : \mathrm{dist}(x, \partial Q_0) \leq \kappa\, \ell(Q_0)} p_{\widetilde{\mu}}(x)^2\, d\widetilde{\mu}(x) + \int_{x \in Q_0 : \mathrm{dist}(x, \partial Q_0) > \kappa\, \ell(Q_0)} p_{\widetilde{\mu}}(x)^2\, d\widetilde{\mu}(x). \tag{3.26}$$

For the first integral on the right hand side we just take into account that $p_{\widetilde{\mu}}(x) \lesssim 1$ by (3.25), and thus

$$\begin{aligned}\int_{x \in Q_0 : \mathrm{dist}(x, \partial Q_0) \leq \kappa\, \ell(Q_0)} p_{\widetilde{\mu}}(x)^2\, d\widetilde{\mu}(x) &\lesssim \mu\big(\{x \in Q_0 : \mathrm{dist}(x, \partial Q_0) \leq \kappa\, \ell(Q_0)\}\big) \\ &\lesssim \kappa\, \mu(Q_0) \approx \kappa\, \widetilde{\mu}(Q_0).\end{aligned}$$

Let us deal with the the last integral on the right hand side of (3.26). Consider $x \in Q \in \mathsf{Stop}_0$ such that $\mathrm{dist}(x, \partial Q_0) > \kappa\, \ell(Q_0)$. We assume that $\kappa \gg t = \theta_0^{\frac{1}{(n+1)}}$. Since $\ell(Q) \leq t\, \ell(Q_0)$,

$$\mathrm{dist}(x, \partial Q_0) \approx \mathrm{dist}(2B_Q, \partial Q_0) \gtrsim \kappa\, \ell(Q_0).$$

Then, we can write

$$p_{\widetilde{\mu}}(x) \lesssim P_\mu(2B_Q) + \sum_{j \geq 1 : 2^j B_Q \cap \partial Q_0 \neq \varnothing} 2^{-j}\, \theta_{\widetilde{\mu}}(2^j B_Q) \lesssim \theta_0^{\frac{1}{(n+1)}} + \sum_{j \geq 1 : 2^j B_Q \cap \partial Q_0 \neq \varnothing} 2^{-j},$$

by Lemma 3.4.2. For the last sum we have

$$\sum_{j \geq 1 : 2^j B_Q \cap \partial Q_0 \neq \varnothing} 2^{-j} \approx \frac{\ell(Q)}{\mathrm{dist}(x, \partial Q_0)} \lesssim \frac{t\, \ell(Q_0)}{\kappa\, \ell(Q_0)} = \frac{\theta_0^{\frac{1}{(n+1)}}}{\kappa},$$

and so we obtain

$$p_{\widetilde{\mu}}(x) \lesssim \theta_0^{\frac{1}{(n+1)}} + \frac{\theta_0^{\frac{1}{(n+1)}}}{\kappa} \approx \frac{\theta_0^{\frac{1}{(n+1)}}}{\kappa}.$$

Therefore,

$$\int_{x \in Q_0 : \mathrm{dist}(x, \partial Q_0) \geq \kappa\, \ell(Q_0)} p_{\widetilde{\mu}}(x)^2\, d\widetilde{\mu}(x) \lesssim \frac{\theta_0^{\frac{1}{(n+1)}}}{\kappa}\, \widetilde{\mu}(Q_0).$$

Gathering the estimates above, we obtain

$$\int_{Q_0} p_{\widetilde{\mu}}^2\, d\widetilde{\mu} \lesssim \left(\kappa + \frac{\theta_0^{\frac{1}{(n+1)}}}{\kappa}\right) \widetilde{\mu}(Q_0).$$

Choosing $\kappa = \theta_0^{\frac{1}{2(n+1)}}$, the lemma follows. $\square$

3.6 The approximating measure η

We consider the measure

$$\eta_0 = \sum_{Q \in \mathsf{Stop}_0 \setminus \mathsf{Bad}} \mu_0(Q)\, \frac{\mathcal{H}^{n+1}\lfloor_{\frac{1}{4}B(Q)}}{\mathcal{H}^{n+1}\big(\frac{1}{4}B(Q)\big)}.$$

In a sense, η_0 can be considered as an approximation of $\mu_0\lfloor_{Q_0}$ which is absolutely continuous with respect to $\mathcal{H}^{n+1}$. Furthermore, since the family Stop_0 is finite, the density of η with respect to $\mathcal{H}^{n+1}$ is bounded.

Recall that, by Remark 1, the balls $\frac{1}{2}B(Q)$, $Q \in \mathcal{D}_\sigma$, are pairwise disjoint, so the balls $\frac{1}{4}B(Q)$ in the sum above satisfy

$$\operatorname{dist}(\tfrac{1}{4}B(Q), \tfrac{1}{4}B(Q')) \geq \frac{1}{4}\left[r(B(Q)) + r(B(Q'))\right] \quad \text{ if } Q \neq Q'.$$

Now we define the following periodic version of η_0: let $\mathcal{M}$ be the lattice of cubes from $\mathbb{R}^{n+1}$ introduced in Section 3.5. Recall that for $P \in \mathcal{M}$, z_P stands for the center of P, and T_P is the translation defined by $T_P(x) = x + z_P$. We define

$$\eta = \sum_{P \in \mathcal{M}} (T_P)_{\#}\eta_0,$$

So η can be considered as a kind of approximation of $\widetilde{\mu}$.

The following result should be compared to Lemma 3.5.2:

Lemma 3.6.1. *We have*

$$\int_{Q_0} |\mathcal{R}(\chi_{AQ_0}\eta)|^2 \, d\eta \lesssim \varepsilon' \, \eta(Q_0),$$

where $\varepsilon' = \widetilde{\varepsilon} + A^n \, \kappa_0^{-2n-2} \, \theta_0^{\frac{1}{2(n+1)^2}}$.

Proof. To simplify notation, we denote $\mathcal{S} = \mathsf{Stop}_0 \setminus \mathsf{Bad}$. We consider the function

$$f = \sum_{Q \in \mathcal{S}} m_{\widetilde{\mu},Q}(\mathcal{R}(\chi_{AQ_0}\widetilde{\mu}))\, \chi_Q.$$

It is clear that

$$\|f\|^2_{L^2(\widetilde{\mu})} \leq \|\mathcal{R}(\chi_{AQ_0}\widetilde{\mu})\|^2_{L^2(\widetilde{\mu}\lfloor_{Q_0})} \leq \widetilde{\varepsilon}\,\widetilde{\mu}(Q_0) = \widetilde{\varepsilon}\,\eta(Q_0). \tag{3.27}$$

For all $x \in \frac{1}{4}B(Q)$, $Q \in \mathcal{S}$, we write

$$\begin{aligned} \left|\mathcal{R}(\chi_{AQ_0}\eta)(x)\right| &\leq \left|\mathcal{R}(\chi_{\frac{1}{4}B(Q)}\eta)(x)\right| + \left|\mathcal{R}(\chi_{AQ_0\setminus\frac{1}{4}B(Q)}\eta)(x) - \mathcal{R}(\chi_{AQ_0\setminus Q}\widetilde{\mu})(x)\right| \\ &\quad + \left|\mathcal{R}(\chi_{AQ_0\setminus Q}\widetilde{\mu})(x) - m_{\widetilde{\mu},Q}(\mathcal{R}(\chi_{AQ_0}\widetilde{\mu}))\right| + \left|m_{\widetilde{\mu},Q}(\mathcal{R}(\chi_{AQ_0}\widetilde{\mu}))\right| \\ &=: \mathsf{T}_1 + \mathsf{T}_2 + \mathsf{T}_3 + \left|m_{\widetilde{\mu},Q}(\mathcal{R}(\chi_{AQ_0}\widetilde{\mu}))\right|. \end{aligned} \tag{3.28}$$

Using the fact that

$$\eta\lfloor_{\frac{1}{4}B(Q)} = \widetilde{\mu}(Q)\, \frac{\mathcal{H}^{n+1}\lfloor_{\frac{1}{4}B(Q)}}{\mathcal{H}^{n+1}\big(\frac{1}{4}B(Q)\big)},$$

it follows easily that

$$\mathsf{T}_1 = \left|\mathcal{R}(\chi_{\frac{1}{4}B(Q)}\eta)(x)\right| \lesssim \frac{\widetilde{\mu}(Q)}{r(B(Q))^n} \lesssim \theta_0^{1/(n+1)}.$$

Now we will deal with the term T_3 in (3.28). To this end, for $x \in \frac{1}{4}B(Q)$ we set

$$\begin{aligned} &\left|\mathcal{R}(\chi_{AQ_0\setminus Q}\widetilde{\mu})(x) - m_{\widetilde{\mu},Q}(\mathcal{R}(\chi_{AQ_0}\widetilde{\mu}))\right| \\ &\quad \leq \left|\mathcal{R}(\chi_{1.1B_Q\setminus Q}\widetilde{\mu})(x)\right| + \left|\mathcal{R}(\chi_{AQ_0\setminus 1.1B_Q}\widetilde{\mu})(x) - m_{\widetilde{\mu},Q}(\mathcal{R}(\chi_{AQ_0\setminus 1.1B_Q}\widetilde{\mu}))\right| \\ &\qquad + \left|m_{\widetilde{\mu},Q}(\mathcal{R}(\chi_{1.1B_Q\setminus Q}\widetilde{\mu}))\right|, \end{aligned} \tag{3.29}$$

taking into account he fact that $m_{\widetilde{\mu},Q}(\mathcal{R}(\chi_Q\widetilde{\mu})) = 0$, by the antisymmetry of the Riesz kernel.

It is immediate to check that

$$\left|\mathcal{R}(\chi_{1.1B_Q\setminus Q}\widetilde{\mu})(x)\right| \lesssim \int_{1.1B_Q\setminus Q} \frac{1}{|x-y|^n}\,d\widetilde{\mu}(y) \lesssim \frac{\widetilde{\mu}(1.1B_Q)}{r(B(Q))^n} \lesssim \theta_0^{1/(n+1)},$$

recalling the fact that $x \in \frac{1}{4}B(Q)$ and $\theta_\mu(1.1B_Q) \lesssim \theta_0^{1/(n+1)}$ for the last estimate.

Now we turn our attention to the second term in the right hand side of (3.29). For $x' \in Q \in \mathcal{S}$,

$$\begin{aligned}\left|\mathcal{R}(\chi_{AQ_0\setminus 1.1B_Q}\widetilde{\mu})(x) - \mathcal{R}(\chi_{AQ_0\setminus 1.1B_Q}\widetilde{\mu})(x')\right| &\leq \int_{AQ_0\setminus 1.1B_Q} \left|K(x-y) - K(x'-y)\right| d\widetilde{\mu}(y)\\ &\lesssim P_{\widetilde{\mu}}(2B_Q).\end{aligned}$$

since the distance both from x and x' to $(1.1B_Q)^c$ is larger than $c\,r(B_Q)$. Averaging on $x' \in Q$ with respect to $\widetilde{\mu}$ we get

$$\left|\mathcal{R}(\chi_{AQ_0\setminus 1.1B_Q}\widetilde{\mu})(x) - m_{\widetilde{\mu},Q}(\mathcal{R}(\chi_{AQ_0\setminus 1.1B_Q}\widetilde{\mu}))\right| \lesssim P_{\widetilde{\mu}}(2B_Q).$$

To estimate the last term in (3.29) we just apply Lemma 3.5.3:

$$\left|m_{\widetilde{\mu},Q}(\mathcal{R}(\chi_{1.1B_Q\setminus Q}\widetilde{\mu}))\right| \leq \frac{1}{\widetilde{\mu}(Q)} \int_{1.1B_Q\setminus Q}\int_Q \frac{1}{|x-y|^n}\,d\widetilde{\mu}(x)\,d\widetilde{\mu}(y) \lesssim \theta_0^{\frac{1}{2(n+1)^2}}.$$

Then, we obtain

$$\mathsf{T}_3 = \left|\mathcal{R}(\chi_{AQ_0\setminus Q}\widetilde{\mu})(x) - m_{\widetilde{\mu},Q}(\mathcal{R}(\chi_{AQ_0}\widetilde{\mu}))\right| \lesssim \theta_0^{\frac{1}{n+1}} + P_{\widetilde{\mu}}(2B_Q) + \theta_0^{\frac{1}{2(n+1)^2}} \lesssim \theta_0^{\frac{1}{2(n+1)^2}} + P_{\widetilde{\mu}}(2B_Q). \quad (3.30)$$

To deal with the term T_2 in (3.28) we need to introduce some additional notation. We set

$$J = \bigcup_{P\in\mathcal{M}} \left\{T_P(R') : R' \in \mathsf{Stop}_0 \setminus \mathsf{Bad}\right\}.$$

For $R \in J$ such that $R = T_P(R')$, $R' \in \mathsf{Stop}_0 \setminus \mathsf{Bad}$, we set $B(R) = T_P(B(R'))$ and $B_R = T_P(B_{R'})$. Also, we denote by J_A the family of cells $R \in J$ which are contained in AQ_0. This way, we have

$$\chi_{AQ_0}\widetilde{\mu} = \sum_{R\in J_A} \widetilde{\mu}\lfloor_R, \qquad \text{and} \qquad \chi_{AQ_0}\eta = \sum_{R\in J_A} \widetilde{\mu}(R)\,\frac{\mathcal{H}^{n+1}\lfloor_{\frac{1}{4}B(R)}}{\mathcal{H}^{n+1}\big(\frac{1}{4}B(R)\big)}.$$

Note that the cells $R \in J$ are pairwise disjoint. Furthermore, by the definition of the family Bad, if $R \in J$ is contained in some cube $T_P(Q_0)$, then the ball $1.1B_R$ is also contained in $T_P(Q_0)$. This guarantees that for all $R \in J$,

$$\widetilde{\mu}(1.1B_R) \lesssim C_0\,\widetilde{\mu}(R).$$

Now for $x \in \frac{1}{4}B(Q)$ we write

$$\begin{aligned}\mathsf{T}_2 &= \left|\mathcal{R}(\chi_{AQ_0\setminus\frac{1}{4}B(Q)}\eta)(x) - \mathcal{R}(\chi_{AQ_0\setminus Q}\widetilde{\mu}))(x)\right| \qquad (3.31)\\ &\leq \sum_{R\in J_A:R\neq Q} \left|\int K(x-y)\,d(\eta\lfloor_{\frac{1}{4}B(R)} - \widetilde{\mu}\lfloor_R)\right|\\ &\leq \sum_{P\in J_A:P\neq Q} \int |K(x-y) - K(x-z_R)|\,d(\eta\lfloor_{\frac{1}{4}B(R)} + \widetilde{\mu}\lfloor_R),\end{aligned}$$

using the fact that $\eta(\frac{1}{4}B(R)) = \widetilde{\mu}(R)$ for the last inequality.

We claim that, for $x \in \frac{1}{4}B(Q)$ and $y \in \frac{1}{4}B(R) \cup \operatorname{supp}(\widetilde{\mu}\lfloor_R)$,

$$|K(x-y) - K(x-z_R)| \lesssim \frac{\ell(R)}{\kappa_0^{n+1}\,D(Q,R)^{n+1}}, \qquad (3.32)$$

where

$$D(Q,R) = \ell(Q) + \ell(R) + \operatorname{dist}(Q,R).$$

To show (3.32) note first that

$$x \in \tfrac{1}{4}B(Q),\;\; z_R \in \tfrac{1}{4}B(R) \quad \Rightarrow \quad |x - z_R| \gtrsim D(Q,R), \tag{3.33}$$

since $\frac{1}{2}B(Q) \cap \frac{1}{2}B(R) = \varnothing$. Analogously, because of the same reason,

$$x \in \tfrac{1}{4}B(Q),\;\; y \in \tfrac{1}{4}B(R) \quad \Rightarrow \quad |x - y| \gtrsim D(Q,R). \tag{3.34}$$

Also,

$$x \in \frac{1}{4}B(Q),\;\; y \in \operatorname{supp}(\widetilde{\mu}\lfloor_R) \quad \Rightarrow \quad |x - y| \gtrsim \kappa_0 D(Q,R), \tag{3.35}$$

To prove this, note that

$$y \in \operatorname{supp}(\widetilde{\mu}\lfloor_R) = I_{\kappa_0}(R) \subset R, \tag{3.36}$$

which implies that $y \notin B(Q)$ and thus $|x-y| \geq \frac{1}{2}r(B(Q)) \approx \ell(Q)$. In the case $r(B(Q)) \geq 2\kappa_0\ell(R)$, this implies that

$$|x - y| \gtrsim \ell(Q) + \kappa_0\ell(R).$$

Otherwise, from the first inclusion in (3.36), since $z_Q \in \operatorname{supp}(\widetilde{\mu})$ and $y \in R$, by the definition of $I_{\kappa_0}(R)$,

$$|z_Q - y| \geq \kappa_0\ell(R),$$

and then, as $|z_Q - x| \leq \frac{1}{4}r(B(Q)) \leq \frac{1}{2}\kappa_0\ell(R)$, we infer that

$$|x - y| \geq |z_Q - y| - |z_Q - x| \geq \frac{\kappa_0}{2}\ell(R).$$

Therefore, in any case we have $|x-y| \gtrsim \kappa_0(\ell(Q)+\ell(R))$ and it is easy to obtain (3.35) from this estimate. We leave the details for the reader.

From (3.33), (3.34), and (3.35), and the fact that K is a standard Calderón-Zygmund kernel, we get (3.32). Plugging this estimate into (3.31), we obtain

$$\mathsf{T}_2 \lesssim \frac{1}{\kappa_0^{n+1}} \sum_{R \in J_A} \frac{\ell(R)\,\widetilde{\mu}(R)}{D(Q,R)^{n+1}}.$$

Therefore, from (3.28) and the estimates for the terms T_1, T_2 and T_3, we infer that for all $x \in \frac{1}{4}B(Q)$ with $Q \in \mathcal{S}$

$$\big|\mathcal{R}(\chi_{AQ_0}\eta)(x)\big| \lesssim \big|m_{\widetilde{\mu},Q}(\mathcal{R}(\chi_{AQ_0}\widetilde{\mu}))\big| + \theta_0^{\overline{\frac{1}{2(n+1)^2}}} + P_{\widetilde{\mu}}(2B_Q) + \frac{1}{\kappa_0^{n+1}} \sum_{R \in J_A} \frac{\ell(R)\,\widetilde{\mu}(R)}{D(Q,R)^{n+1}}. \tag{3.37}$$

Denote

$$\widetilde{p}_{\widetilde{\mu}}(x) = \sum_{Q \in J} \chi_{\frac{1}{4}B(Q)}\, P_{\widetilde{\mu}}(2B_Q) \quad \text{and} \quad \widetilde{g}(x) = \sum_{Q \in \mathcal{S}} \sum_{R \in J_A} \frac{\ell(R)}{D(Q,R)^{n+1}}\, \widetilde{\mu}(R)\, \chi_{\frac{1}{4}B(Q)}(x).$$

Squaring and integrating (3.37) with respect to η on Q_0, we get

$$\begin{aligned} \big\|\mathcal{R}(\chi_{AQ_0}\eta)\big\|_{L^2(\eta)}^2 \lesssim &\sum_{Q \in \mathcal{S}} \big|m_{\widetilde{\mu},Q}(\mathcal{R}(\chi_{AQ_0}\widetilde{\mu}))\big|^2\, \eta(\tfrac{1}{4}B(Q)) \\ &+ \theta_0^{\overline{\frac{1}{(n+1)^2}}}\, \eta(Q_0) + \|\widetilde{p}_{\widetilde{\mu}}\|_{L^2(\eta)}^2 + \frac{1}{\kappa_0^{2n+2}} \|\widetilde{g}\|_{L^2(\eta)}^2, \end{aligned} \tag{3.38}$$

Note that, since $\eta(\frac{1}{4}B(Q)) = \mu(Q)$, the first sum on the right hand side of (3.38) equals $\|f\|^2_{L^2(\widetilde{\mu})}$, which does not exceed $\widetilde{\varepsilon}\,\eta(Q_0)$, by (3.27). By an analogous argument we obtain that $\|\widetilde{p}_{\widetilde{\mu}}\|^2_{L^2(\eta)} = \|p_{\widetilde{\mu}}\|^2_{L^2(\widetilde{\mu}\lfloor_{Q_0})}$ and $\|\widetilde{g}\|^2_{L^2(\eta)} = \|g\|^2_{L^2(\widetilde{\mu}\lfloor_{Q_0})}$, where

$$p_{\widetilde{\mu}}(x) = \sum_{Q\in J} \chi_Q\, P_{\widetilde{\mu}}(2B_Q) \quad \text{ and } \quad g(x) = \sum_{Q\in\mathcal{S}}\sum_{R\in J_A} \frac{\ell(R)}{D(Q,R)^{n+1}}\,\widetilde{\mu}(R)\,\chi_Q(x).$$

We will estimate $\|g\|_{L^2(\widetilde{\mu}\lfloor_{Q_0})}$ by duality: for any non-negative function $h \in L^2(\widetilde{\mu}\lfloor_{Q_0})$, we set

$$\int g\,h\,d\widetilde{\mu} = \sum_{Q\in\mathcal{S}}\sum_{R\in J_A} \frac{\ell(R)}{D(Q,R)^{n+1}}\,\widetilde{\mu}(R)\int_Q h\,d\widetilde{\mu} = \sum_{R\in J_A}\widetilde{\mu}(R)\sum_{Q\in\mathcal{S}}\frac{\ell(R)}{D(Q,R)^{n+1}}\int_Q h\,d\widetilde{\mu}. \tag{3.39}$$

For each $z \in R \in J_A$ we have

$$\begin{aligned}
\sum_{Q\in\mathcal{S}} \frac{\ell(R)}{D(Q,R)^{n+1}}\int_Q h\,d\widetilde{\mu} &\lesssim \int \frac{\ell(R)\,h(y)}{\big(\ell(R)+|z-y|\big)^{n+1}}\,d\widetilde{\mu}(y)\\
&= \int_{|z-y|\le\ell(R)} \frac{\ell(R)\,h(y)}{\big(\ell(R)+|z-y|\big)^{n+1}}\,d\widetilde{\mu}(y)\\
&\quad + \sum_{j\ge1}\int_{2^{j-1}\ell(R)<|z-y|\le 2^{j-1}\ell(R)} \frac{\ell(R)\,h(y)}{\big(\ell(R)+|z-y|\big)^{n+1}}\,d\widetilde{\mu}(y)\\
&\lesssim \sum_{j\ge0} ⨍_{B(z,2^j\ell(R))} h\,d\widetilde{\mu}\;\frac{2^{-j}\,\widetilde{\mu}(B(z,2^j\ell(R)))}{\big(2^j\ell(R)\big)^n}\\
&\lesssim M_{\widetilde{\mu}}h(z)\,P_{\widetilde{\mu}}\big(B(z,\ell(R))\big),
\end{aligned}$$

where $M_{\widetilde{\mu}}$ stands for the centered maximal Hardy-Littlewood operator with respect to $\widetilde{\mu}$. Then, by (3.39),

$$\begin{aligned}
\int g\,h\,d\widetilde{\mu} &\lesssim \sum_{R\in J_A}\inf_{z\in R}\big[M_{\widetilde{\mu}}h(z)\,P_{\widetilde{\mu}}\big(B(z,\ell(R))\big)\big]\widetilde{\mu}(R) \le \int_{AQ_0} M_{\widetilde{\mu}}h\;p_{\widetilde{\mu}}\,d\widetilde{\mu}\\
&\lesssim \|M_{\widetilde{\mu}}h\|_{L^2(\widetilde{\mu})}\,\|p_{\widetilde{\mu}}\|_{L^2(\widetilde{\mu})} \lesssim \|h\|_{L^2(\widetilde{\mu})}\,\|p_{\widetilde{\mu}}\|_{L^2(\widetilde{\mu}\lfloor_{AQ_0})}.
\end{aligned}$$

Then, by Lemma 3.5.4 and recalling that $\widetilde{\mu}$ is $\mathcal{M}$-periodic,

$$\|g\|^2_{L^2(\widetilde{\mu})} \lesssim \|p_{\widetilde{\mu}}\|_{L^2(\widetilde{\mu}|_{AQ_0})} = A^n\,\|p_{\widetilde{\mu}}\|_{L^2(\widetilde{\mu}|_{Q_0})} \lesssim A^n\,\theta_0^{\frac{1}{2(n+1)}}\,\widetilde{\mu}(Q_0).$$

Plugging this estimate into (3.38) and using the fact that that $\|g\|_{L^2(\eta)} = \|g\|_{L^2(\widetilde{\mu})}$, we obtain

$$\big\|\mathcal{R}(\chi_{AQ_0}\eta)\big\|^2_{L^2(\eta)} \lesssim \left(\widetilde{\varepsilon} + \theta_0^{\frac{1}{(n+1)^2}} + \frac{A^n}{\kappa_0^{2n+2}}\,\theta_0^{\frac{1}{2(n+1)}}\right)\eta(Q_0) \lesssim \left(\widetilde{\varepsilon} + \frac{A^n}{\kappa_0^{2n+2}}\,\theta_0^{\frac{1}{(n+1)^2}}\right)\eta(Q_0),$$

as wished. □

Note that the Riesz kernel is locally integrable with respect to η (recall that the number of cells from Stop_0 is finite). Then, for any bounded function f with compact support the integral $\int K(x-y)\,f(y)\,d\eta(y)$ is absolutely convergent for all $x\in\mathbb{R}^{n+1}$.

Now we wish to extend the definition of $\mathcal{R}_\eta f(x)$ for $\mathcal{M}$-periodic functions $f\in L^\infty(\eta)$ in a pointwise way (not only in a BMO sense, say). We consider a non-negative radial $\mathcal{C}^1$ function ϕ supported on $B(0,2)$ which equals 1 on $B(0,1)$, and we set $\phi_r(x) = \phi\left(\frac{x}{r}\right)$ for $r>0$. We denote $\widetilde{K}_r(x-y) = K(x-y)\phi_r(x-y)$ and we define

$$\widetilde{\mathcal{R}}_{\eta,r}f(x) = \widetilde{\mathcal{R}}_r(f\eta)(x) = \int \widetilde{K}_r(x-y)\,f(y)\,d\eta(y),$$

and

$$\text{p.v.}\mathcal{R}_\eta f(x) = \text{p.v.}\mathcal{R}(f\eta)(x) = \lim_{r\to\infty} \widetilde{\mathcal{R}}_{\eta,r} f(x), \tag{3.40}$$

whenever the limit exists. Let us remark that one may also define the principal value in a more typical way by

$$\lim_{r\to\infty} \int_{|x-y|<r} K(x-y)\, f(y)\, d\eta(y). \tag{3.41}$$

However, the definition (3.40) has some technical advantages and simplifies the exposition. Nevertheless, one can show that both definitions (3.40) and (3.41) coincide, at least for the $\mathcal{M}$-periodic functions $f \in L^\infty(\eta)$ (we will not prove this fact because it will be not needed below).

Lemma 3.6.2. *Let $f \in L^\infty(\eta)$ be $\mathcal{M}$-periodic, that is, $f(x+z_P) = f(x)$ for all $x \in \mathbb{R}^{n+1}$ and all $P \in \mathcal{M}$. Then:*

- *p.v.$\mathcal{R}_\eta f(x)$ exists for all $x \in \mathbb{R}^{n+1}$ and $\widetilde{\mathcal{R}}_{\eta,r} f \to \mathcal{R}_\eta f$ as $r \to \infty$ uniformly in compact subsets of $\mathbb{R}^{n+1}$. The convergence is also uniform on supp(η). Furthermore, given any compact set $F \subset \mathbb{R}^{n+1}$, there is $r_0 = r_0(F) > 0$ such that for $s > r \geq r_0$,*

$$\big\|\widetilde{\mathcal{R}}_s(f\eta) - \widetilde{\mathcal{R}}_r(f\eta)\big\|_{\infty,F} \lesssim \frac{c_F}{r}\, \|f\|_\infty, \tag{3.42}$$

  where c_F is some constant depending on F.

- *The function p.v.$\mathcal{R}_\eta f$ is $\mathcal{M}$-periodic and continuous in $\mathbb{R}^{n+1}$, and harmonic in $\mathbb{R}^{n+1} \setminus$ supp$(f\eta)$.*

The arguments needed to prove the lemma are standard. However, for the reader's convenience we will show the details.

Proof. By the $\mathcal{M}$-periodicity of the measure $\nu := f\,\eta$, it is immediate that the functions $\widetilde{\mathcal{R}}_r(f\eta)(x)$, $r > 0$, are $\mathcal{M}$-periodic. On the other hand, using the fact that η is absolutely continuous with respect to Lebesgue measure on a compact set with a uniformly bounded density, it is straightforward to check that each $\widetilde{\mathcal{R}}_r(f\eta)$ is also continuous and bounded in $\mathbb{R}^{n+1}$. Then, except for harmonicity, all the statements in the lemma follow if we show that the family of functions $\{\widetilde{\mathcal{R}}_r(f\eta)\}_{r>0}$ satisfies (3.42) for any compact subset $F \subset \mathbb{R}^{n+1}$. Indeed, this clearly implies the uniform convergence on compact subsets, and since supp(η) is periodic, also the uniform convergence on supp(η).

Let $s > r \geq r_0$, and denote $\widetilde{K}_{r,s}(x-y) = \widetilde{K}_s(x-y) - \widetilde{K}_r(x-y)$. Notice that $\widetilde{K}_{r,s}$ is a standard Calderón-Zygmund kernel (with constants independent of r and s). We write

$$\nu = \sum_{P\in\mathcal{M}} (T_P)_\#(\chi_{Q_0}\nu),$$

so that

$$\widetilde{\mathcal{R}}_s(f\eta)(x) - \widetilde{\mathcal{R}}_r(f\eta)(x) = \int \widetilde{K}_{r,s}(x-y)\, d\Big(\sum_{P\in\mathcal{M}} (T_P)_\#(\chi_{Q_0}\nu)\Big)(y).$$

Since the support of $\widetilde{K}_{r,s}(x-y)$ is compact, the last sum only has a finite number of non-zero terms, and so we can change the order of summation and integration:

$$\begin{aligned} \widetilde{\mathcal{R}}_s(f\eta)(x) - \widetilde{\mathcal{R}}_r(f\eta)(x) &= \sum_{P\in\mathcal{M}} \int \widetilde{K}_{r,s}(x-y)\, d\big[(T_P)_\#(\chi_{Q_0}\nu)\big](y) \\ &= \sum_{P\in\mathcal{M}} \int_{Q_0} \widetilde{K}_{r,s}(x-y-z_P)\, d\nu(y). \end{aligned} \tag{3.43}$$

Note now that by the antisymmetry of the kernel $\widetilde{K}_{r,s}$, from the last equation we derive

$$\widetilde{\mathcal{R}}_s(f\eta)(x) - \widetilde{\mathcal{R}}_r(f\eta)(x) = -\sum_{P\in\mathcal{M}} \int_{Q_0} \widetilde{K}_{r,s}(z_P - (x-y))\,d\nu(y).$$

Also, by the definition of $\mathcal{M}$, it is clear that $P\in\mathcal{M}$ if and only if $-P\in\mathcal{M}$. Then, replacing z_P by $-z_P$ does not change the last sum in (3.43). Hence, we have

$$\widetilde{\mathcal{R}}_s(f\eta)(x) - \widetilde{\mathcal{R}}_r(f\eta)(x) = \sum_{P\in\mathcal{M}} \int_{Q_0} \widetilde{K}_{r,s}(z_P + (x-y))\,d\nu(y).$$

Averaging the last two equations we get

$$\widetilde{\mathcal{R}}_s(f\eta)(x) - \widetilde{\mathcal{R}}_r(f\eta)(x) = \frac{1}{2}\sum_{P\in\mathcal{M}} \int_{Q_0} \big[\widetilde{K}_{r,s}(z_P + (x-y)) - \widetilde{K}_{r,s}(z_P - (x-y))\big]\,d\nu(y). \tag{3.44}$$

Note that if x belongs to a compact set $F\subset\mathbb{R}^{n+1}$ and $y\in Q_0$, then both $(x-y)$ and $-(x-y)$ lie in a compact set $\widetilde{F}$. Observe also that $\widetilde{K}_{r,s}$ vanishes in $B(0,r)$. Then, if we assume $r_0 \geq 2\,\mathrm{diam}(\widetilde{F})$, say, then both $\widetilde{K}_{r,s}(z_P + (x-y))$ and $\widetilde{K}_{r,s}(z_P - (x-y))$ vanish unless $|z_P|\geq r$. For such x, y we have $|x-y|\leq \mathrm{diam}(\widetilde{F}) \leq \frac{1}{2}\,r \leq |z_P|$, and so

$$|z_P + (x-y)| \approx |z_P + (x-y)| \approx |z_P| \geq r.$$

Then, we obtain

$$\big|\widetilde{K}_{r,s}(z_P + (x-y)) - \widetilde{K}_{r,s}(z_P - (x-y))\big| \lesssim \frac{|x-y|}{|z_P|^{n+1}} \lesssim \frac{\mathrm{diam}(\widetilde{F})}{|z_P|^{n+1}}.$$

Plugging this estimate into (3.44) we get

$$\big|\widetilde{\mathcal{R}}_s(f\eta)(x) - \widetilde{\mathcal{R}}_r(f\eta)(x)\big| \lesssim \sum_{P\in\mathcal{M}:|z_P|\geq r} \frac{\mathrm{diam}(\widetilde{F})}{|z_P|^{n+1}}\,|\nu|(Q_0) \leq \sum_{P\in\mathcal{M}:|z_P|\geq r} \frac{\mathrm{diam}(\widetilde{F})}{|z_P|^{n+1}}\,\ell(P)^n\,\|f\|_\infty.$$

It is easy to check that

$$\sum_{P\in\mathcal{M}:|z_P|\geq r} \frac{\ell(P)^n}{|z_P|^{n+1}} \lesssim \frac{1}{r},$$

so we infer

$$\big\|\widetilde{\mathcal{R}}_s(f\eta) - \widetilde{\mathcal{R}}_r(f\eta)\big\|_{\infty,F} \lesssim \frac{\mathrm{diam}(\widetilde{F})}{r}\,\|f\|_\infty \to 0 \quad \text{ as } r\to\infty,$$

as wished.

It remains to prove that $\mathrm{p.v.}\mathcal{R}_\nu f$ is harmonic in $\mathbb{R}^{n+1}\setminus \mathrm{supp}(f\eta)$. Consider again a compact set $F\subset\mathbb{R}^{n+1}$ and $x\in F$. Then, we have

$$\mathcal{R}(f\phi_r\eta)(x) - \widetilde{\mathcal{R}}_r(f\eta)(x) = \int K(x-y)\,\big(\phi_r(y) - \phi_r(x-y)\big)\,f(y)\,d\eta(y).$$

We write

$$|\phi_r(y) - \phi_r(x-y)| \lesssim \|\nabla\phi_r\|_\infty\,|x| \lesssim \frac{|x|}{r}.$$

For $r\geq 4\,\mathrm{diam}\,F$, it is easy to check that $\phi_r(y) - \phi_r(x-y) = 0$ unless $|x-y|\approx|y|\approx r$. Thus

$$\begin{aligned}\big|\mathcal{R}(f\phi_r\eta)(x) - \widetilde{\mathcal{R}}_r(f\eta)(x)\big| &\lesssim \int_{\substack{|y|\leq Cr\\ C^{-1}r\leq|x-y|\leq Cr}} \frac{1}{|x-y|^n}\,\frac{|x|}{r}\,|f(y)|\,d\eta(y)\\ &\lesssim \frac{|x|}{r^{n+1}}\,\|f\|_\infty\,\eta(B(0,Cr)) \lesssim \frac{\mathrm{diam}\,F + \mathrm{dist}(0,F)}{r}\,\|f\|_\infty,\end{aligned}$$

that is,

$$\big\|\mathcal{R}(f\phi_r\eta)-\widetilde{\mathcal{R}}_r(f\eta)\big\|_{\infty,F}\lesssim \frac{\operatorname{diam}F+\operatorname{dist}(0,F)}{r}\,\|f\|_\infty\to 0 \quad \text{as } r\to\infty.$$

Since $\widetilde{\mathcal{R}}_r(f\eta)$ converges uniformly to p.v.$\mathcal{R}_\eta f$ in F as $r\to\infty$, it follows that $\mathcal{R}(f\phi_r\eta)$ also converges uniformly to p.v.$\mathcal{R}_\eta f$ in F.

Note now that, for all $r>0$, $\mathcal{R}(f\phi_r\eta)$ is harmonic out of $\operatorname{supp}(f\eta)$, because $f\phi_r\eta$ has compact support, and so by their local uniform convergence to p.v.$\mathcal{R}_\eta f$, we obtain that p.v.$\mathcal{R}_\eta f$ is harmonic out of $\operatorname{supp}(f\eta)$ too. $\square$

From now on, to simplify notation we will denote p.v.$\mathcal{R}_\eta f$ just by $\mathcal{R}_\eta f$.

Lemma 3.6.3. *Let $L^\infty_{\mathcal{M}}(\eta)$ denote the Banach space of the $\mathcal{M}$-periodic functions which belong to $L^\infty(\eta)$ equipped with the norm $\|\cdot\|_{L^\infty(\eta)}$. The map $\mathcal{R}_\eta: L^\infty_{\mathcal{M}}(\eta)\to L^\infty_{\mathcal{M}}(\eta)$ is bounded. Furthermore, for all $f\in L^\infty_{\mathcal{M}}(\eta)$ and all sufficiently big $r>0$ we have*

$$\|\mathcal{R}(f\eta)-\widetilde{\mathcal{R}}_r(f\eta)\|_{L^\infty(\eta)}\lesssim \frac{\|f\|_{L^\infty(\eta)}}{r}. \tag{3.45}$$

We remark that the bound on the norm of $\mathcal{R}_\eta$ from $L^\infty_{\mathcal{M}}(\eta)$ to $L^\infty_{\mathcal{M}}(\eta)$ depends strongly on the construction of η. It is finite due to the fact that the number of cells from Stop_0 is finite, but it may explode as this number grows. The precise value of the norm will not play any role in the estimates below, we just need to know that it is finite.

Proof. Since f is $\mathcal{M}$-periodic, from (3.42) we infer that for $s>r\geq r_0=r_0(Q_0)$,

$$\big\|\widetilde{\mathcal{R}}_s(f\eta)-\widetilde{\mathcal{R}}_r(f\eta)\big\|_{\infty,F}\lesssim \frac{c_F}{r}\,\|f\|_\infty,$$

Letting $s\to\infty$, $\widetilde{\mathcal{R}}_s(f\eta)$ converges uniformly to $\mathcal{R}\nu$ and so we get (3.45).

To prove the boundedness of $\mathcal{R}_\eta: L^\infty_{\mathcal{M}}(\eta)\to L^\infty_{\mathcal{M}}(\eta)$, note first that $\widetilde{K}_{r_0}$ is compactly supported and η is absolutely continuous with respect to Lebesgue measure on a compact set with a uniformly bounded density. Then, we deduce that $\widetilde{\mathcal{R}}_{\eta,r_0}: L^\infty_{\mathcal{M}}(\eta)\to L^\infty_{\mathcal{M}}(\eta)$ is bounded, which together with (3.45) applied to $\widetilde{\mathcal{R}}_{r_0}$ implies that $\mathcal{R}_\eta: L^\infty_{\mathcal{M}}(\eta)\to L^\infty_{\mathcal{M}}(\eta)$ is bounded. $\square$

From now on, given $x\in\mathbb{R}^{n+1}$, we denote

$$x_H=(x_1,\cdots,x_n),$$

so that $x=(x_H,x_{n+1})$. Also, we write

$$\mathcal{R}^H=(\mathcal{R}_1,\ldots,\mathcal{R}_n),$$

where $\mathcal{R}_j$ stands for the j-th component of $\mathcal{R}$, so that $\mathcal{R}=(\mathcal{R}^H,\mathcal{R}_{n+1})$.

For simplicity, in the arguments below we will assume that the function ϕ defined slightly above (3.40) is of the form $\phi(x)=\widetilde{\phi}(x^2)$, for some $\mathcal{C}^1$ function $\widetilde{\phi}$ which equals 1 on $B(0,1)$ and vanishes out of $B(0,2^{1/2})$.

Lemma 3.6.4. *Let $f\in L^1_{loc}(\eta)$ be $\mathcal{M}$-periodic. Then,*

(a) Let $\widetilde{A}\geq 3$ be some odd natural number. For all $x\in 2Q_0$,

$$\big|\mathcal{R}(\chi_{(\widetilde{A}Q_0)^c}f\eta)(x)\big|\lesssim \frac{1}{\widetilde{A}\,\ell(Q_0)^n}\int_{Q_0}|f|\,d\eta.$$

(b) For all $x \in \mathbb{R}^{n+1}$ such that $\operatorname{dist}(x,H) \geq \ell(Q_0)$,

$$\big|\mathcal{R}(f\eta)(x)\big| \lesssim \frac{1}{\ell(Q_0)^n} \int_{Q_0} |f|\, d\eta \tag{3.46}$$

and

$$\big|\mathcal{R}^H(f\eta)(x)\big| \lesssim \frac{1}{\operatorname{dist}(x,H)\,\ell(Q_0)^{n-1}} \int_{Q_0} |f|\, d\eta \tag{3.47}$$

Proof. We denote $\nu = f\eta$. The arguments to prove the estimate in (a) are quite similar to the ones used in the proof of Lemma 3.6.2. Since we are assuming that $\widetilde{A}$ is some odd number, there is a subset $\mathcal{M}_{\widetilde{A}} \subset \mathcal{M}$ such that

$$\chi_{(\widetilde{A}Q_0)^c}\nu = \sum_{P \in \mathcal{M}_{\widetilde{A}}} (T_P)_\#(\chi_{Q_0}\nu).$$

Furthermore, the cubes from $P \in \mathcal{M}_{\widetilde{A}}$ satisfy $|z_P| \gtrsim \widetilde{A}\ell(Q_0)$. Then, for all $x \in Q_0$ and all $r > 0$ we have

$$\widetilde{\mathcal{R}}_r(\chi_{(\widetilde{A}Q_0)^c}\nu)(x) = \int \widetilde{K}_r(x-y)\, d\Big(\sum_{P \in \mathcal{M}_{\widetilde{A}}} (T_P)_\#(\chi_{Q_0}\nu)\Big)(y).$$

Since the support of $\widetilde{K}_r(x-y)$ is compact, the last sum only has a finite number of non-zero terms, and so we can change the order of summation and integration, and thus

$$\begin{aligned}\widetilde{\mathcal{R}}_r(\chi_{(\widetilde{A}Q_0)^c}\nu)(x) &= \sum_{P \in \mathcal{M}_{\widetilde{A}}} \int \widetilde{K}_r(x-y)\, d\big[(T_P)_\#(\chi_{Q_0}\nu)\big](y) \\ &= \sum_{P \in \mathcal{M}_{\widetilde{A}}} \int_{Q_0} \widetilde{K}_r(x-y-z_P)\, d\nu(y).\end{aligned} \tag{3.48}$$

By the antisymmetry of the kernel $\widetilde{K}_r$, from the last equation we get

$$\widetilde{\mathcal{R}}_r(\chi_{(\widetilde{A}Q_0)^c}\nu)(x) = -\sum_{P \in \mathcal{M}_{\widetilde{A}}} \int_{Q_0} \widetilde{K}_r(z_P - (x-y))\, d\nu(y).$$

Also, by the definition of $\mathcal{M}_{\widetilde{A}}$, it follows that $P \in \mathcal{M}_{\widetilde{A}}$ if and only if $-P \in \mathcal{M}_{\widetilde{A}}$, so replacing z_P by $-z_P$ does not change the last sum in (3.48), and then we have

$$\widetilde{\mathcal{R}}_r(\chi_{(\widetilde{A}Q_0)^c}\nu)(x) = \sum_{P \in \mathcal{M}_{\widetilde{A}}} \int_{Q_0} \widetilde{K}_r(z_P + (x-y))\, d\nu(y).$$

Averaging the last two equations we get

$$\widetilde{\mathcal{R}}_r(\chi_{(\widetilde{A}Q_0)^c}\nu)(x) = \frac{1}{2} \sum_{P \in \mathcal{M}_{\widetilde{A}}} \int_{Q_0} \big[\widetilde{K}_r(z_P + (x-y)) - \widetilde{K}_r(z_P - (x-y))\big]\, d\nu(y). \tag{3.49}$$

Note now that $x, y \in Q_0$ and, recalling that $|z_P| \gtrsim \widetilde{A}\ell(Q_0)$ for $P \in \mathcal{M}_{\widetilde{A}}$, we have

$$|z_P + (x-y)| \approx |z_P - (x-y)| \approx |z_P|.$$

Thus,

$$\big|\widetilde{K}_r(z_P + (x-y)) - \widetilde{K}_r(z_P - (x-y))\big| \lesssim \frac{|x-y|}{|z_P|^{n+1}} \lesssim \frac{\ell(Q_0)}{|z_P|^{n+1}}.$$

Then, from this estimate and (3.49) we obtain that

$$\big|\widetilde{\mathcal{R}}_r(\chi_{(\widetilde{A}Q_0)^c}\nu)(x)\big| \lesssim \sum_{P \in \mathcal{M} : |z_P| \geq C^{-1}\widetilde{A}\ell(Q_0)} \frac{\ell(Q_0)}{|z_P|^{n+1}}\, |\nu|(Q_0) \lesssim \frac{|\nu|(Q_0)}{\widetilde{A}\,\ell(Q_0)^n}.$$

as wished.

To prove the first estimate in (b), let $x \in \mathbb{R}^{n+1}$ be such that $\operatorname{dist}(x, H) \geq \ell(Q_0)$. Since $\mathcal{R}\nu$ is $\mathcal{M}$-periodic, we may assume that $x_H \in Q_0 \cap H$. As in (3.49), for any $r > 0$ we have

$$\widetilde{\mathcal{R}}_r\nu(x) = \frac{1}{2} \sum_{P \in \mathcal{M}} \int_{Q_0} \left[\widetilde{K}_r(z_P + (x-y)) - \widetilde{K}_r(z_P - (x-y))\right] d\nu(y). \tag{3.50}$$

We claim that for x as above and $y \in Q_0$,

$$\left|\widetilde{K}_r(z_P + (x-y)) - \widetilde{K}_r(z_P - (x-y))\right| \lesssim \frac{\operatorname{dist}(x,H)}{\big(\operatorname{dist}(x,H) + |z_P|\big)^{n+1}}. \tag{3.51}$$

Indeed, if $|z_P| \geq 2|x-y|$, then $\operatorname{dist}(x,H) + |x - z_P| \approx |x-y| + |z_P| \approx |z_P|$, and thus

$$\left|\widetilde{K}_r(z_P + (x-y)) - \widetilde{K}_r(z_P - (x-y))\right| \lesssim \frac{|x-y|}{|z_P|^{n+1}} \approx \frac{\operatorname{dist}(x,H)}{|z_P|^{n+1}}.$$

Since $|z_P| \geq 2|x-y|$, we have $|z_P| \approx |z_P| + |x-y| \approx |z_P| + \operatorname{dist}(x,H)$, and then (3.52) holds in this case.

On the other hand, if $|z_P| < 2|x-y|$, then

$$\left|\widetilde{K}_r(z_P + (x-y)) - \widetilde{K}_r(z_P - (x-y))\right| \lesssim \frac{1}{\left|(z_P - y) + x\right|^n} + \frac{1}{\left|(z_P + y) - x\right|^n}.$$

It is immediate to check that $\operatorname{dist}(x, y - z_P) \approx \operatorname{dist}(x, z_P + y) \gtrsim \operatorname{dist}(x,H) \approx |x-y|$, and so we obtain

$$\left|\widetilde{K}_r(z_P + (x-y)) - \widetilde{K}_r(z_P - (x-y))\right| \lesssim \frac{1}{|x-y|^n}.$$

Furthermore, from the condition $|z_P| < 2|x-y|$ we infer that

$$|x-y| \approx |x-y| + |z_P| \approx \operatorname{dist}(x,H) + |z_P|,$$

and thus

$$\left|\widetilde{K}_r(z_P + (x-y)) - \widetilde{K}_r(z_P - (x-y))\right| \lesssim \frac{|x-y|}{\big(|x-y| + |z_P|\big)^{n+1}} \approx \frac{\operatorname{dist}(x,H)}{\big(\operatorname{dist}(x,H) + |z_P|\big)^{n+1}},$$

which completes the proof of (3.52).

From (3.50) and (3.52) we obtain

$$\begin{aligned}\left|\widetilde{\mathcal{R}}_r\nu(x)\right| &\lesssim \sum_{P \in \mathcal{M}} \int_{Q_0} \frac{\operatorname{dist}(x,H)}{\big(\operatorname{dist}(x,H) + |z_P|\big)^{n+1}}\, d|\nu|(y) \\ &= |\nu|(Q_0)\, \frac{\operatorname{dist}(x,H)}{\ell(P)^n} \sum_{P \in \mathcal{M}} \frac{\ell(P)^n}{\big(\operatorname{dist}(x,H) + |z_P|\big)^{n+1}}.\end{aligned}$$

It is easy to check that

$$\sum_{P \in \mathcal{M}} \frac{\ell(P)^n}{\big(\operatorname{dist}(x,H) + |z_P|\big)^{n+1}} \lesssim \frac{1}{\operatorname{dist}(x,H)},$$

and so (3.46) follows.

We turn now our attention to the last estimate from (b). Again let $x \in \mathbb{R}^{n+1}$ be such that $\operatorname{dist}(x,H) \geq \ell(Q_0)$, so that the identity (3.50) is still valid. We claim that for $y \in Q_0$ and r big enough,

$$\left|\widetilde{K}^H_r(z_P + (x-y)) - \widetilde{K}^H_r(z_P - (x-y))\right| \lesssim \frac{\ell(Q_0)}{\big(\operatorname{dist}(x,H) + |z_P|\big)^{n+1}}, \tag{3.52}$$

where $\widetilde{K}_r^H$ is the kernel of $\widetilde{\mathcal{R}}_r^H$. To prove this, we write

$$\widetilde{K}_r^H(z) = z_H\,\psi_r(|z|^2), \qquad \text{with } \ \psi_r(t) = \frac{\widetilde{\phi}_r(t)}{t^{\frac{n+1}{2}}}.$$

Then, we have

$$\begin{aligned}
&\big|\widetilde{K}_r^H(z_P + (x-y)) - \widetilde{K}_r^H(z_P - (x-y))\big| \\
&\quad = \big|((z_{P,H} + (x_H - y_H))\,\psi_r\big(|z_P + (x-y)|^2\big) - (z_{P,H} - (x_H - y_H))\,\psi_r\big(|z_P - (x-y)|^2\big)\big| \\
&\quad \le 2\,\big|x_H - y_H\big|\,\psi_r\big(|z_P + (x-y)|^2\big) \\
&\qquad + \big|z_{P,H} - (x_H - y_H)\big|\;\big|\psi_r\big(|z_P - (x-y)|^2\big) - \psi_r\big(|z_P + (x-y)|^2\big)\big| \\
&\quad =: \mathsf{T}_1 + \mathsf{T}_2.
\end{aligned}$$

To deal with T_1 we write

$$\mathsf{T}_1 \le \frac{2\,\big|x_H - y_H\big|}{|z_P + (x-y)|^{n+1}}.$$

Note then that $\big|x_H - y_H\big| \le \ell(Q_0)$, while $|x-y| \approx \operatorname{dist}(x,H)$. Furthermore, it is easy to check that

$$|z_P + (x-y)| \approx |z_P - (x-y)| \approx |z_P| + \operatorname{dist}(x,H), \tag{3.53}$$

which implies that

$$\mathsf{T}_1 \lesssim \frac{\ell(Q_0)}{\big(\operatorname{dist}(x,H) + |z_P|\big)^{n+1}}.$$

Now we will estimate T_2. To this end we intend to apply the Mean Value Theorem. It is easy to check that for all $t>0$,

$$|\psi_r'(t)| \lesssim \frac{1}{t^{\frac{n+3}{2}}},$$

and then, by (3.53),

$$\big|\psi_r\big(|z_P - (x-y)|^2\big) - \psi_r\big(|z_P + (x-y)|^2\big)\big| \lesssim \frac{\big||z_P - (x-y)|^2 - |z_P + (x-y)|^2\big|}{\big(\operatorname{dist}(x,H) + |z_P|\big)^{n+3}}.$$

Now we have

$$\begin{aligned}
\big||z_P - (x-y)|^2 - |z_P + (x-y)|^2\big| &= \big|\big[(z_{P,H} - (x_H - y_H))^2 + (x_{n+1} - y_{n+1})^2\big] \\
&\qquad - \big[(z_{P,H} + (x_H - y_H))^2 + (x_{n+1} - y_{n+1})^2\big]\big| \\
&= 2\,\big|z_{P,H}\,(x_H - y_H)\big| \le 2\,|z_P|\,\ell(Q_0).
\end{aligned}$$

Therefore, we infer that

$$\mathsf{T}_2 \lesssim \frac{\big|z_{P,H} - (x_H - y_H)\big|\,|z_P|\,\ell(Q_0)}{\big(\operatorname{dist}(x,H) + |z_P|\big)^{n+3}} \lesssim \frac{\ell(Q_0)}{\big(\operatorname{dist}(x,H) + |z_P|\big)^{n+1}}.$$

Together with the estimate above for T_1 ,this yields (3.52), as wished.

From (3.50) and (3.52) we obtain

$$\big|\widetilde{\mathcal{R}}_r\nu(x)\big| \lesssim |\nu|(Q_0) \sum_{P\in\mathcal{M}} \frac{\ell(Q_0)}{\big(\operatorname{dist}(x,H) + |z_P|\big)^{n+1}}.$$

It is easy to check that

$$\sum_{P\in\mathcal{M}} \frac{\ell(Q_0)^{n+1}}{\big(\operatorname{dist}(x,H) + |z_P|\big)^{n+1}} \lesssim \frac{\ell(Q_0)}{\operatorname{dist}(x,H)},$$

and then (3.47) follows. $\square$

Lemma 3.6.5. *We have*

$$\int_{Q_0} |\mathcal{R}\eta|^2 \, d\eta \lesssim \left(\varepsilon' + \frac{1}{A^2}\right) \eta(Q_0).$$

Proof. By Lemma 3.6.1 it is enough to show that

$$\int_{Q_0} |\mathcal{R}(\chi_{(AQ_0)^c}\eta)|^2 \, d\eta \lesssim \frac{1}{A^2} \, \eta(Q_0).$$

This estimate is an immediate consequence of Lemma 3.6.4 (a). □

Remark 2. *By taking A big enough and δ, ε small enough in the assumptions of the Main Lemma 3.2.1, and then choosing the parameters $\varepsilon_0, \kappa_0, \theta_0$ appropriately, it follows that*

$$\int_{Q_0} |\mathcal{R}\eta|^2 \, d\eta \ll \eta(Q_0). \tag{3.54}$$

Indeed, the preceding lemma asserts that

$$\int_{Q_0} |\mathcal{R}\eta|^2 \, d\eta \lesssim \left(\varepsilon' + \frac{1}{A^2}\right) \eta(Q_0),$$

with ε' given in Lemma 3.6.1 by

$$\varepsilon' = \widetilde{\varepsilon} + A^n \, \kappa_0^{-2n-2} \, \theta_0^{\frac{1}{2(n+1)^2}},$$

where $\widetilde{\varepsilon}$ is defined in (3.23) by

$$\widetilde{\varepsilon} = C_4 \left(\varepsilon + \frac{1}{A^2} + \delta^{\frac{1}{8n+8}} + \varepsilon_0 + \theta_0^{\frac{1}{n+1}} + \kappa_0^{\frac{1}{2}} + A^{n+1} \, \widetilde{\delta}^{\frac{1}{2n+3}} \right),$$

and $\widetilde{\delta}$ in (3.17) by

$$\widetilde{\delta} = C_3 \, A^{n+1} \left(\varepsilon_0 + \theta_0^{1/(n+1)} + \kappa_0^{1/2} + \delta^{1/2} \right).$$

Hence, if we take first A big enough and then $\varepsilon_0, \kappa_0, \delta, \theta_0$ small enough (depending on A), so that moreover $\theta_0 \ll \kappa_0$ (to ensure that $A^n \kappa_0^{-2n-2} \, \theta_0^{\frac{1}{2(n+1)^2}} \ll 1$), then (3.54) follows.

3.7 Proof of the Key Lemma by contradiction

3.7.1 A variational argument and an almost everywhere inequality

Lemma 3.7.1. *Suppose that, for some $0 < \lambda \leq 1$, the inequality*

$$\int_{Q_0} |\mathcal{R}\eta|^2 d\eta \leq \lambda \, \eta(Q_0)$$

holds. Then, there is a function $b \in L^\infty(\eta)$ such that

(i) $0 \leq b \leq 2$,

(ii) b *is periodic with respect to* $\mathcal{M}$,

(iii) $\displaystyle\int_{Q_0} b \, d\eta = \eta(Q_0)$,

and such that the measure $\nu = b\eta$ satisfies

$$\int_{Q_0} |\mathcal{R}\nu|^2 d\nu \leq \lambda \, \nu(Q_0) \tag{3.55}$$

and

$$|\mathcal{R}\nu(x)|^2 + 2\mathcal{R}^*((\mathcal{R}\nu)\nu)(x) \leq 6\lambda \quad \textit{for } \nu\textit{-a.e. } x \in \mathbb{R}^{n+1}. \tag{3.56}$$

Proof. In order to find such a function b, we consider the following class of admissible functions

$$\mathcal{A} = \left\{ a \in L^\infty(\eta) : \; a \geq 0, \;\; a \text{ is } \mathcal{M}\text{-periodic, and } \textstyle\int_{Q_0} a\, d\eta = \eta(Q_0) \right\} \tag{3.57}$$

and we define a functional J on $\mathcal{A}$ by

$$J(a) = \lambda \|a\|_{L^\infty(\eta)}\, \eta(Q_0) + \int_{Q_0} |\mathcal{R}(a\eta)|^2 a\, d\eta. \tag{3.58}$$

Observe that $1 \in \mathcal{A}$ and

$$J(1) = \lambda\, \eta(Q_0) + \int_{Q_0} |\mathcal{R}\eta|^2 d\eta \leq 2\lambda\, \eta(Q_0),$$

Thus

$$\inf_{a\in\mathcal{A}} J(a) \leq 2\lambda\, \eta(Q_0).$$

Since $J(a) \geq \lambda \|a\|_{L^\infty(\eta)}\, \eta(Q_0)$, it is clear that

$$\inf_{a\in\mathcal{A}} J(a) = \inf_{a\in\mathcal{A}: \|a\|_{L^\infty(\eta)}\leq 2} J(a).$$

Hence, by standard arguments one can prove that J attains a global minimum on $\mathcal{A}$, i.e., there is a function $b \in \mathcal{A}$ such that $J(b) \leq J(a)$ for all $a \in \mathcal{A}$. Indeed, by the Banach-Alaoglu theorem there exists a sequence $\{a_k\}_k \subset L^\infty(\eta)$, with $\|a_k\|_{L^\infty(\eta)} \leq 2$ which converges weakly-* in $L^\infty(\eta)$ to some function $b \in L^\infty(\eta)$. It is clear that b satisfies (i), (ii) and (iii). Also, since $y \mapsto \frac{x-y}{|x-y|^{n+1}}$ belongs to $L^1(\eta)$ (recall that η has bounded density with respect to Lebesgue measure), it follows that $\mathcal{R}(a_k\eta) \to \mathcal{R}(b\eta)$ pointwise. Taking into account that

$$|\mathcal{R}(a_k\eta)(x)| \leq 2 \int \frac{1}{|x-y|^n}\, d\eta(y) \leq C(\eta),$$

by the dominated convergence theorem, $J(a_k) \to J(b)$.

The estimate (3.55) for $\nu = b\,\eta$ follows from the fact that $J(b) \leq J(1)$, because the property (iii) implies that $\|b\|_{L^\infty(\eta)} \geq 1$.

In order to prove that (3.56) holds, we perform a blow-up argument taking advantage of the fact that b is a minimizer for J. Let B be any ball contained in Q_0 and centered on $\operatorname{supp}(\nu) \cap Q_0$. Denote by

$$P_{\mathcal{M}}(B) = \bigcup_{R\in\mathcal{M}} (B + z_R) \tag{3.59}$$

the "periodic extension" of B with respect to $\mathcal{M}$. Now, for every $0 \leq t < 1$, define

$$b_t = (1 - t\chi_{P_{\mathcal{M}}(B)})b + t\,\frac{\nu(B)}{\nu(Q_0)}\, b. \tag{3.60}$$

It is clear that $b_t \in \mathcal{A}$ for all $0 \leq t < 1$ and $b_0 = b$. Therefore,

$$\begin{aligned} J(b) \leq J(b_t) &= \lambda \|b_t\|_\infty \eta(Q_0) + \int_{Q_0} |\mathcal{R}(b_t\eta)|^2 b_t\, d\eta \\ &\leq \lambda \left(1 + t\frac{\nu(B)}{\nu(Q_0)}\right) \|b\|_\infty \eta(Q_0) + \int_{Q_0} |\mathcal{R}(b_t\eta)|^2 b_t\, d\eta := h(t). \end{aligned} \tag{3.61}$$

Since $h(0) = J(b)$, we have that $h(0) \leq h(t)$ for $0 \leq t < 1$ and, thus $h'_+(0) \geq 0$ (assuming that $h'_+(0)$ exists). Notice that

$$\frac{db_t}{dt}\bigg|_{t=0} = -\chi_{P_{\mathcal{M}}(B)} b + \frac{\nu(B)}{\nu(Q_0)}\, b,$$

Therefore,

$$
\begin{aligned}
0 \le h'_+(0) &= \lambda \frac{\nu(B)}{\nu(Q_0)} \|b\|_\infty \eta(Q_0) + \frac{d}{dt}\Big|_{t=0} \int_{Q_0} |\mathcal{R}(b_t\eta)|^2 b_t d\eta \\
&= \lambda \frac{\nu(B)}{\nu(Q_0)} \|b\|_\infty \eta(Q_0) + 2\int_{Q_0} \mathcal{R}\left(\frac{db_t}{dt}\Big|_{t=0}\eta\right) \cdot \mathcal{R}\nu \, b \, d\eta + \int_{Q_0} |\mathcal{R}\nu|^2 \, \frac{db_t}{dt}\Big|_{t=0} d\eta \\
&= \lambda \frac{\nu(B)}{\nu(Q_0)} \|b\|_\infty \eta(Q_0) + 2\int_{Q_0} \mathcal{R}\left(\left(-\chi_{P_{\mathcal{M}}(B)} b + \frac{\nu(B)}{\nu(Q_0)} b\right)\eta\right) \cdot \mathcal{R}\nu \, b \, d\eta \\
&\quad + \int_{Q_0} |\mathcal{R}\nu|^2 \left(-\chi_{P_{\mathcal{M}}(B)} b + \frac{\nu(B)}{\nu(Q_0)} b\right) d\eta \\
&= \lambda \frac{\nu(B)}{\nu(Q_0)} \|b\|_\infty \eta(Q_0) - 2\int_{Q_0} \mathcal{R}(\chi_{P_{\mathcal{M}}(B)}\nu) \cdot \mathcal{R}\nu \, d\nu + 2\frac{\nu(B)}{\nu(Q_0)} \int_{Q_0} |\mathcal{R}\nu|^2 \, d\nu \\
&\quad - \int_B |\mathcal{R}\nu|^2 \, d\nu + \frac{\nu(B)}{\nu(Q_0)} \int_{Q_0} |\mathcal{R}\nu|^2 \, d\nu,
\end{aligned}
$$

where we used the fact that $P_{\mathcal{M}}(B) \cap Q_0 = B$ in the last identity. The fact that the derivatives above commute with the integral sign and with the operator $\mathcal{R}$ is guaranteed by the fact that b_t is an affine function of t and then one can expand the integrand $|\mathcal{R}(b_t\eta)|^2 b_t$ and obtain a polynomial on t. Using also the fact that $\lambda \le 1$ and that $J(b) \le 2\lambda\,\nu(Q_0)$, we get

$$
\begin{aligned}
\int_B |\mathcal{R}\nu|^2 \, d\nu + 2\int_{Q_0} \mathcal{R}(\chi_{P_{\mathcal{M}}(B)}\nu) \cdot \mathcal{R}\nu \, d\nu &\le \frac{\nu(B)}{\nu(Q_0)} \left[\lambda \|b\|_\infty \eta(Q_0) + 3\int_{Q_0} |\mathcal{R}\nu|^2 \, d\nu\right] \\
&\le 3\, J(b)\, \nu(B) \le 6\lambda\, \nu(B).
\end{aligned} \tag{3.62}
$$

We claim now that

$$
\int_{Q_0} \mathcal{R}(\chi_{P_{\mathcal{M}}(B)}\nu) \cdot \mathcal{R}\nu \, d\nu = \int_B \mathcal{R}^*((\mathcal{R}\nu)\nu) \, d\nu. \tag{3.63}
$$

Assuming this to be true for the moment, from (3.62) and (3.63), dividing by $\nu(B)$, we obtain

$$
\frac{1}{\nu(B)} \int_B |\mathcal{R}\nu|^2 d\nu + \frac{2}{\nu(B)} \int_B \mathcal{R}^*((\mathcal{R}\nu)\nu) \, d\nu \le 6\lambda,
$$

and so, letting $\nu(B) \to 0$ and applying Lebesgue's Differentiation Theorem, we obtain

$$
|\mathcal{R}\nu(x)|^2 + 2\mathcal{R}^*((\mathcal{R}\nu)\nu)(x) \le 6\,\lambda \quad \text{ for } \nu\text{-a.e. } x \in \mathbb{R}^{n+1},
$$

as desired.

It remains to prove the claim (3.63). By the uniform convergence of $\widetilde{\mathcal{R}}_r(\chi_{P_{\mathcal{M}}(B)}\nu)$ and $\widetilde{\mathcal{R}}_r\nu$ to $\mathcal{R}(\chi_{P_{\mathcal{M}}(B)}\nu)$ and $\mathcal{R}\nu$, respectively, we have

$$
\int_{Q_0} \mathcal{R}(\chi_{P_{\mathcal{M}}(B)}\nu) \cdot \mathcal{R}\nu \, d\nu = \lim_{r\to\infty} \int_{Q_0} \widetilde{\mathcal{R}}_r(\chi_{P_{\mathcal{M}}(B)}\nu) \cdot \widetilde{\mathcal{R}}_r\nu \, d\nu. \tag{3.64}
$$

Since $\widetilde{K}_r(x - \cdot)$ has compact support, for all $x \in Q_0$,

$$
\mathcal{R}_r(\chi_{P_{\mathcal{M}}(B)}\nu)(x) = \int_{P_{\mathcal{M}}(B)} \widetilde{K}_r(x-y) \, d\nu(y) = \sum_{P\in\mathcal{M}} \int \widetilde{K}_r(x-y) \, d((T_P)_\#(\chi_B\nu)(y).
$$

For the last identity we have used the fact that the sum above runs only over a finite number of $P \in \mathcal{M}$ because there is only a finite number of non-zero terms (in fact, we may assume these $P \in \mathcal{M}$ to be independent of $x \in Q_0$). Thus we have

$$
\mathcal{R}_r(\chi_{P_{\mathcal{M}}(B)}\nu)(x) = \sum_{P\in\mathcal{M}} \int_B \widetilde{K}_r(x-y-z_P) \, d\nu(y) = \sum_{P\in\mathcal{M}} \widetilde{\mathcal{R}}_r(\chi_B\nu)(x-z_P),
$$

and so

$$\begin{aligned}\int_{Q_0} \widetilde{\mathcal{R}}_r(\chi_{P_{\mathcal{M}}(B)}\nu)(x)\cdot\widetilde{\mathcal{R}}_r\nu\,d\nu(x) &= \sum_{P\in\mathcal{M}}\int_{Q_0}\widetilde{\mathcal{R}}_r(\chi_B\nu)(x-z_P)\cdot\widetilde{\mathcal{R}}_r\nu(x)\,d\nu(x)\\ &= \sum_{P\in\mathcal{M}}\int_{Q_0-z_P}\widetilde{\mathcal{R}}_r(\chi_B\nu)(x)\cdot\widetilde{\mathcal{R}}_r\nu(x+z_P)\,d((T_P)^{-1}_{\#}\nu)(x)\end{aligned}$$

Since $\widetilde{\mathcal{R}}_r\nu$ is $\mathcal{M}$-periodic, $\widetilde{\mathcal{R}}_r\nu(x+z_P)=\widetilde{\mathcal{R}}_r\nu(x)$ and $(T_P)^{-1}_{\#}\nu=\nu$, and then applying also Fubini's theorem we get

$$\begin{aligned}\int_{Q_0} \widetilde{\mathcal{R}}_r(\chi_{P_{\mathcal{M}}(B)}\nu)(x)\cdot\widetilde{\mathcal{R}}_r\nu(x)\,d\nu(x) &= \sum_{P\in\mathcal{M}}\int_{Q_0-z_P}\widetilde{\mathcal{R}}_r(\chi_B\nu)(x)\cdot\widetilde{\mathcal{R}}_r\nu(x)\,d\nu(x)\\ &= \int \widetilde{\mathcal{R}}_r(\chi_B\nu)(x)\cdot\widetilde{\mathcal{R}}_r\nu(x)\,d\nu(x)\\ &= \int_B \widetilde{\mathcal{R}}_r^*((\widetilde{\mathcal{R}}_r\nu)\nu)(y)\,d\nu(y).\end{aligned}\tag{3.65}$$

Since $\widetilde{\mathcal{R}}_r\nu$ converges uniformly to $\mathcal{R}\nu$ as $r\to\infty$ and $\widetilde{\mathcal{R}}_r^*$ tends to $\mathcal{R}^*$ in operator norm in $L^\infty_{\mathcal{M}}(\eta)\to L^\infty_{\mathcal{M}}(\eta)$, we deduce that

$$\lim_{r\to\infty}\int_B \widetilde{\mathcal{R}}_r^*((\widetilde{\mathcal{R}}_r\nu)\nu)(y)\,d\nu(y)=\int_B \mathcal{R}^*((\mathcal{R}\nu)\nu)\,d\nu.$$

Together with (3.64) and (3.65) this yields (3.63). □

3.7.2 A maximum principle

Lemma 3.7.2. *Assume that, for some $0<\lambda\le 1$, the inequality*

$$\int_{Q_0}|\mathcal{R}\eta|^2\,d\eta\le\lambda\eta(Q_0)$$

is satisfied, and let b and ν be as in Lemma 3.7.1. Let $K_S>0$ be a big constant ($K_S\gg 10$) and let S be the horizontal strip

$$S=\big\{x\in\mathbb{R}^{n+1}\colon |x_{n+1}|\le K_S\ell(Q_0)\big\}.$$

Also, set

$$f(x)=c_S\,x_{n+1}e_{n+1}=c_S(0,\dots,0,x_{n+1}),\qquad \textit{with}\;\; c_S=\int\frac{1}{\big(|y_H|^2+(K_S\ell(Q_0))^2\big)^{\frac{n+1}{2}}}\,d\nu(y).$$

Then, we have

$$|\mathcal{R}\nu(x)-f(x)|^2+4\mathcal{R}^*((\mathcal{R}\nu)\nu)(x)\lesssim\lambda^{1/2}+\frac{1}{K_S^2}\qquad\textit{for all } x\in S.\tag{3.66}$$

Furthermore,

$$c_S\lesssim\frac{1}{K_S\ell(Q_0)}.\tag{3.67}$$

Proof. The inequality (3.67) is very easy. Indeed, we just have to use the fact that $\nu(B(x,r))\lesssim r^n$ for all $x\in\mathbb{R}^{n+1}$ and $r>0$, and use standard estimates which we leave for the reader.

To prove (3.66), we denote

$$F(x)=|\mathcal{R}\nu(x)-f(x)|^2+4\mathcal{R}^*((\mathcal{R}\nu)\nu)(x).$$

It is clear that F is subharmonic in $\mathbb{R}^{n+1}\setminus\operatorname{supp}(\nu)$ and continuous in the whole space $\mathbb{R}^{n+1}$, by Lemma 3.6.2. THen, if we show that the estimate in (3.66) holds for all $x\in\operatorname{supp}(\nu)\cup\partial S$, then this will be also satisfied in the whole S. Indeed, since F is $\mathcal{M}$-periodic and continuous in S, it is clear that the maximum of F in S is attained, and since F is subharmonic in $\mathring{S}\setminus\operatorname{supp}(\nu)$, it should be attained in $\operatorname{supp}(\nu)\cup\partial S$.

First we check that the inequality in (3.66) holds for all $x \in \operatorname{supp}(\nu)$. To this end, recall that by Lemma 3.7.1, we have

$$|\mathcal{R}\nu(x)|^2 + 2\mathcal{R}^*((\mathcal{R}\nu)\nu)(x) \leq 6\lambda \quad \nu\text{-almost everywhere in } \operatorname{supp}(\nu),$$

and this inequality extends to the whole $\operatorname{supp}(\nu)$ by continuity. Therefore we have, for all $x \in \operatorname{supp}(\nu)$,

$$\begin{aligned} F(x) &= |\mathcal{R}\nu(x) - f(x)|^2 + 4\mathcal{R}^*((\mathcal{R}\nu)\nu)(x) \leq 2|\mathcal{R}\nu(x)|^2 + 2|f(x)|^2 + 4\mathcal{R}^*((\mathcal{R}\nu)\nu)(x) \\ &\leq 12\lambda + 2|f(x)|^2 \leq 12\lambda + \big(c_S \ell(Q_0)\big)^2 \lesssim \lambda + \frac{1}{K_S^2}, \end{aligned}$$

where we took into account that $|x_{n+1}| \leq \frac{1}{2}\ell(Q_0)$ for $x \in \operatorname{supp}\nu$ and we used (3.67).

Our next objective consists in getting an upper bound for F in ∂S. By applying Lemma 3.6.4 to the function $\mathcal{R}\nu$ (which is $\mathcal{M}$-periodic), with $\mathcal{R}^*$ instead of $\mathcal{R}$ (since $\mathcal{R}$ is antisymmetric we are allowed to do this) we obtain

$$\big|\mathcal{R}^*((\mathcal{R}\nu)\nu)(x)\big| \lesssim \frac{1}{\ell(Q_0)^n}\int_{Q_0} |\mathcal{R}\nu|\,d\nu \lesssim \frac{1}{\ell(Q_0)^n}\left(\int_{Q_0}|\mathcal{R}\nu|^2\,d\nu\right)^{1/2}\nu(Q_0)^{1/2} \lesssim \lambda^{1/2}.$$

It suffices to show now that $|\mathcal{R}\nu(x) - f(x)| \lesssim \frac{1}{K_S}$ for all $x \in \partial S$. We write $\mathcal{R}\nu(x) = (\mathcal{R}^H\nu(x), \mathcal{R}_{n+1}\nu(x))$. From (3.47) we infer that

$$\big|\mathcal{R}^H\nu(x)\big| \lesssim \frac{1}{K_S\,\ell(Q_0)^n}\,\nu(Q_0) \lesssim \frac{1}{K_S}.$$

Hence, it only remains to prove that

$$\big|\mathcal{R}_{n+1}\nu(x)\,e_{n+1} - f(x)\big| \lesssim \frac{1}{K_S} \qquad \text{for all } x \in \partial S. \tag{3.68}$$

To prove this estimate we can assume without loss of generality that $x_{n+1} = K_S\ell(Q_0)$ and that $x_H \in Q_0 \cap H$, by the $\mathcal{M}$-periodicity of $\mathcal{R}_{n+1}\nu$. Since $f(x) = c_S\,K_S\,\ell(Q_0)\,e_{n+1}$ for this point x, (3.68) is equivalent to

$$\big|\mathcal{R}_{n+1}\nu(x) - c_S\,K_S\,\ell(Q_0)\big| \lesssim \frac{1}{K_S}. \tag{3.69}$$

Note first that

$$\mathcal{R}_{n+1}\nu(x) = \lim_{r\to 0}\int \phi_r(x-y)\frac{x_{n+1}-y_{n+1}}{|x-y|^{n+1}}\,d\nu(y) = \int \frac{x_{n+1}-y_{n+1}}{|x-y|^{n+1}}\,d\nu(y),$$

by an easy application of the dominated convergence theorem (using the fact that $|x_{n+1} - y_{n+1}| \leq \operatorname{dist}(x,H) + \ell(Q_0)$). Consider the point $x_0 = (0, K_S\ell(Q_0))$. Since for all $y \in \operatorname{supp}(\nu)$,

$$|x - x_0| + |y - y_H| \leq \ell(Q_0) \leq \frac{1}{2}|x-y|,$$

and since the $(n+1)$ component of $K(\cdot)$, which we denote by $K_{n+1}(\cdot)$, is a standard Calderón-Zygmund kernel,

$$\big|K_{n+1}(x-y) - K_{n+1}(x_0 - y_H)\big| \lesssim \frac{|x-x_0| + |y-y_H|}{|x-y|^{n+1}} \lesssim \frac{\ell(Q_0)}{|x-y|^{n+1}}.$$

Therefore,

$$\big|\mathcal{R}_{n+1}\nu(x) - c_S\,K_S\,\ell(Q_0)\big| = \left|\int \big(K_{n+1}(x-y) - K_{n+1}(x_0-y_H)\big)\,d\nu(y)\right| \lesssim \int \frac{\ell(Q_0)}{|x-y|^{n+1}}\,d\nu(y).$$

Since $\operatorname{dist}(x, \operatorname{supp}(\nu)) \gtrsim K_S\,\ell(Q_0)$ and ν is a measure with polynomial growth of order n, by standard estimates it follows that

$$\int \frac{\ell(Q_0)}{|x-y|^{n+1}}\,d\nu(y) \lesssim \frac{1}{K_S},$$

which proves (3.69) and finishes the proof of the lemma. $\square$

The next result is an immediate consequence of Lemma 3.7.2.

Lemma 3.7.3. *Assume that, for some* $0 < \lambda \leq 1$, *the inequality*

$$\int_{Q_0} |\mathcal{R}\eta|^2 \, d\eta \leq \lambda \eta(Q_0)$$

is satisfied, and let b *and* ν *be as in Lemma 3.7.1. Then, we have*

$$|\mathcal{R}\nu(x)|^2 + 4\mathcal{R}^*((\mathcal{R}\nu)\nu)(x) \lesssim \lambda^{1/2} \quad \text{for all } x \in \mathbb{R}^{n+1}. \tag{3.70}$$

Proof. This follows by letting $K_S \to \infty$ in the inequality (3.66), taking into account that $c_S \to 0$, by (3.67). □

3.7.3 The contradiction

Lemma 3.7.4. *Suppose that, for some* $0 < \lambda \leq 1$, *the inequality*

$$\int_{Q_0} |\mathcal{R}\eta|^2 d\eta \leq \lambda \, \eta(Q_0) \tag{3.71}$$

is satisfied, and let b *and* ν *be as in Lemma 3.7.1. Then, the exists some constant* $c_3 > 0$ *depending only*[1] *on* n, C_0, C_1 *such that*

$$\lambda \geq c_3.$$

Proof. By Lemma 3.7.2, we have

$$|\mathcal{R}\nu(x)|^2 + 4\mathcal{R}^*((\mathcal{R}\nu)\nu)(x) \lesssim \lambda^{1/2} \tag{3.72}$$

for all $x \in \mathbb{R}^{n+1}$. Now pick a smooth function φ with $\chi_{Q_0} \leq \varphi \leq \chi_{2Q_0}$ and $\|\nabla\varphi\|_\infty \lesssim \frac{1}{\ell(Q_0)}$. Set $\psi = C_5 \nabla\varphi$, so that $\mathcal{R}^*(\psi\mathcal{H}^{n+1}) = \varphi$. Then, we have

$$\begin{aligned}
\eta(Q_0) = \nu(Q_0) &\leq \int \varphi \, d\nu = \int \mathcal{R}^*(\psi\mathcal{H}^{n+1}) \, d\nu \\
&= \int \mathcal{R}\nu \, \psi d\mathcal{H}^{n+1} \leq \left(\int |\mathcal{R}\nu|^2 |\psi| \, d\mathcal{H}^{n+1} \right)^{1/2} \left(\int |\psi| \, d\mathcal{H}^{n+1} \right)^{1/2}.
\end{aligned}$$

First of all, observe that

$$\|\psi\|_\infty \lesssim \frac{1}{\ell(Q_0)} \quad \text{and} \quad \int |\psi| \, d\mathcal{H}^{n+1} \lesssim \ell(Q_0)^n$$

and so

$$\eta(Q_0) \lesssim \left(\int |\mathcal{R}\nu|^2 |\psi| d\mathcal{H}^{n+1} \right)^{1/2} \ell(Q_0)^{n/2}. \tag{3.73}$$

Furthermore, by (3.72) we have

$$\begin{aligned}
\int |\mathcal{R}\nu|^2 |\psi| \, d\mathcal{H}^{n+1} &\leq C \, \lambda^{1/2} \int |\psi| d\mathcal{H}^{n+1} + 4 \left| \int \mathcal{R}^*((\mathcal{R}\nu)\nu) |\psi| \, d\mathcal{H}^{n+1} \right| \qquad (3.74) \\
&\lesssim \lambda^{1/2} \ell(Q_0)^n + \left| \int \mathcal{R}^*(\chi_{(3Q_0)^c}(\mathcal{R}\nu)\nu) |\psi| \, d\mathcal{H}^{n+1} \right| + \left| \int \mathcal{R}^*\big(\chi_{3Q_0}(\mathcal{R}\nu)\nu\big) |\psi| \, d\mathcal{H}^{n+1} \right|.
\end{aligned}$$

[1] In fact, keeping track of the dependencies, one can check that c_3 depends only on n and C_0, and not on C_1. However, this is not necessary for the proof of the Key Lemma.

To estimate the first integral on the right hand side we apply Lemma 3.6.4 (a) with $\widetilde{A} = 3$ and $f = \mathcal{R}\nu\, b$ (where b is such that $b\eta = \nu$), and then we deduce that for all $x \in 2Q_0$,

$$\begin{aligned}\big|\mathcal{R}^*\big(\chi_{(3Q_0)^c}(\mathcal{R}\nu)\nu\big)(x) &\lesssim \frac{1}{\ell(Q_0)^n}\int_{Q_0} |\mathcal{R}\nu\, b|\, d\eta \\ &= \frac{1}{\ell(Q_0)^n}\int_{Q_0} |\mathcal{R}\nu|\, d\nu \lesssim \left(\fint_{Q_0} |\mathcal{R}\nu|^2\, d\nu\right)^{1/2} \lesssim \lambda^{1/2}.\end{aligned}$$

Thus, recalling that ψ is supported in $2Q_0$,

$$\left|\int \mathcal{R}^*(\chi_{(3Q_0)^c}(\mathcal{R}\nu)\nu)|\psi|\, d\mathcal{H}^{n+1}\right| \lesssim \lambda^{1/2}\, \|\psi\|_1 \lesssim \lambda^{1/2}\, \nu(Q_0).$$

Concerning the last integral on the right hand side of (3.74), we have

$$\begin{aligned}\left|\int \mathcal{R}^*\big(\chi_{3Q_0}(\mathcal{R}\nu)\nu\big)|\psi|\, d\mathcal{H}^{n+1}\right| &= \left|\int_{3Q_0} \mathcal{R}\nu \cdot \mathcal{R}(|\psi|\, d\mathcal{H}^{n+1})\, d\nu\right| \\ &\leq \left(\int_{3Q_0} |\mathcal{R}\nu|^2\, d\nu\right)^{1/2}\left(\int_{3Q_0} |\mathcal{R}(|\psi|\, d\mathcal{H}^{n+1})|\, d\nu\right)^{1/2}.\end{aligned}$$

The first integral on the right hand side does not exceed $c\lambda\, \nu(Q_0)$ (by (3.55) and the periodicity of $\mathcal{R}\nu$). For the second one, using the fact that $|\psi| \lesssim \frac{1}{\ell(Q_0)}\chi_{2Q_0}$, it follows easily that $\|\mathcal{R}(|\psi|\, \mathcal{H}^{n+1})\|_\infty \lesssim 1$. Therefore, we get

$$\left|\int \mathcal{R}^*\big(\chi_{3Q_0}(\mathcal{R}\nu)\nu\big)|\psi|\, d\mathcal{H}^{n+1}\right| \lesssim \lambda^{1/2}\nu(Q_0).$$

Then, from (3.74) and the last estimates we deduce that

$$\int |\mathcal{R}\nu|^2|\psi| d\mathcal{H}^{n+1} \lesssim \lambda^{1/2}\nu(Q_0).$$

Thus, by (3.73),

$$\nu(Q_0) \lesssim \lambda^{1/4}\nu(Q_0),$$

that is, $\lambda \gtrsim 1$. □

Now, in order to prove the Key Lemma 3.4.1 we only have to recall that, by Remark 2, $\int_{Q_0} |\mathcal{R}\eta|^2\, d\eta \ll \eta(Q_0)$ if A is big enough and $\delta, \varepsilon, \kappa_0, \theta_0$ are small enough and chosen appropriately, under the assumption that ε_0 is small enough too. This contradicts Lemma 3.7.4. Hence, (3.13) cannot hold and thus we are done.

3.8 Construction of the AD-regular measure ζ and the uniformly rectifiable set $\varGamma$ in the Main Lemma

Denote

$$F = Q_0 \cap \operatorname{supp}(\mu) \setminus \bigcup_{Q \in \mathsf{LD}} Q. \tag{3.75}$$

It is easy to check that $0 < \theta^n_*(x,\mu) \leq \theta^{n,*}(x,\mu) < \infty$ for μ-a.e. $x \in F$. Since $\mathcal{R}_\mu$ is bounded on $L^2(\mu\lfloor_F)$, it follows that $\mu|_F$ is n-rectifiable, by the Nazarov-Tolsa-Volberg theorem [NToV2]. However, to get a big piece of a set contained in a uniformly n-rectifiable set $\varGamma$ as the one required in the Main Lemma and in Theorem 3.1.1 we have to argue more carefully. To this end, first we will construct an auxiliary AD-regular measure ζ such that $\zeta(F) \gtrsim \mu(F)$, and then we will apply the Nazarov-Tolsa-Volberg theorem [NToV1] for AD-regular measures.

Next we are going to construct the aforementioned auxiliary measure ζ. The arguments for this construction can be considered as quantitative version of the ones from [NToV2], which rely on a covering theorem of Pajot (see [P]).

Recall the notation $\sigma = \mu\lfloor_{Q_0}$. Consider the maximal dyadic operator

$$\mathcal{M}_{\mathcal{D}_\sigma} f(x) = \sup_{Q\in\mathcal{D}_\sigma : x\in Q} \frac{1}{\sigma(Q)} \int_Q |f|\, d\sigma,$$

where $\mathcal{D}_\sigma$ is the David-Mattila lattice associated σ. Let F be as in (3.75) and set

$$\widetilde{F} = \Big\{x \in F : \mathcal{M}_{\mathcal{D}_\sigma}(\chi_{F^c})(x) \le 1 - \frac{\varepsilon_0}{2}\Big\}.$$

We wish to show that

$$\sigma(\widetilde{F}) \ge \frac{1}{2}\,\sigma(F). \tag{3.76}$$

To this end, note that

$$F\setminus\widetilde{F} = \Big\{x \in F : \mathcal{M}_{\mathcal{D}_\sigma}(\chi_{F^c})(x) > 1 - \frac{\varepsilon_0}{2}\Big\}$$

and consider a collection of maximal (and thus disjoint) cells $\{Q_i\}_{i\in J} \subset \mathcal{D}_\sigma$ such that $\sigma(Q_i \setminus F) > (1 - \frac{\varepsilon_0}{2})\sigma(Q_i)$. Observe that

$$F\setminus\widetilde{F} = \bigcup_{i\in J} Q_i \cap F.$$

Clearly, the cells Q_i satisfy $\sigma(Q_i \cap F) \le \frac{\varepsilon_0}{2}\sigma(Q_i)$ and so we have

$$\sigma(F\setminus\widetilde{F}) \le \sum_{i\in J}\sigma(Q_i\cap F) \le \sum_{i\in J}\frac{\varepsilon_0}{2}\,\sigma(Q_i) \le \frac{\varepsilon_0}{2}\,\sigma(Q_0) \le \frac{1}{2}\,\sigma(F),$$

which proves (3.76).

For each $i \in J$ we consider the family $\mathcal{A}_i$ of maximal doubling cells from $\mathcal{D}_\sigma^{db}$ which cover Q_i, and we define

$$\mathcal{A} = \bigcup_{i\in J}\mathcal{A}_i.$$

Finally, we denote by $\mathcal{A}_0$ the subfamily of the cells $P \in \mathcal{A}$ such that $\sigma(P\cap F) > 0$. Now, for each $Q \in \mathcal{A}_0$ we consider an n-dimensional sphere $S(Q)$ concentric with $B(Q)$ and with radius $\frac{1}{4}r(B(Q))$. We define

$$\zeta = \sigma\lfloor_{\widetilde{F}} + \sum_{Q\in\mathcal{A}_0}\mathcal{H}^n\lfloor_{S(Q)}.$$

Remark 3. *If $P \in \mathcal{A}_0$ and $P \subset Q_i$ for some $i \in J$, then*

$$\ell(P) \approx_{\theta_0,C_0} \ell(Q_i).$$

Indeed, since P is a maximal doubling cell contained in Q_i, by Lemma E and the fact that $3.5B_P \subset 100B(P)$,

$$\theta_\sigma(3.5B_P) \lesssim \theta_\sigma(100B(P)) \lesssim A_0^{-9n(J(P)-J(Q_i))}\,\theta_\sigma(100B(Q_i)) \lesssim_{C_0} A_0^{-9n(J(P)-J(Q_i))}.$$

Since $\sigma(P\cap F) > 0$, it turns out that P is not contained in any cell from LD, *and so $\Theta_\sigma(3.5B_P) > \theta_0$. Then, we have*

$$\theta_0 \lesssim_{C_0} A_0^{-9n(J(P)-J(Q_i))},$$

which implies that $|J(P) - J(Q_i)| \lesssim_{\theta_0,C_0} 1$.

A very similar argument shows that if $P \in \mathcal{D}_\sigma$ satisfies $P \cap F \neq \varnothing$ (and so it is not contained in any cell from LD*), then there exists some $Q \in \mathcal{D}_\sigma^{db}$ which contains Q and such that*

$$\ell(P) \approx_{\theta_0, C_0} \ell(Q).$$

The details are left for the reader.

From the two statements above, if follows that for any cell $P \in \mathcal{D}_\sigma$ which is not strictly contained in any cell from $\mathcal{A}_0$ there exists some cell $\widehat{P} \in \mathcal{D}_\sigma^{db}$ which is not contained in any cell Q_i, $i \in J$, so that $P \subset \widehat{P}$ and $\ell(P) \approx_{\theta_0, C_0} \ell(Q)$.

Lemma 3.8.1. *The measure ζ is AD regular, with the AD-regularity constant depending on C_0, θ_0, and ε_0.*

Proof. First we will show the upper AD-regularity of ζ, that is, we will prove that $\zeta(B(x,r)) \leq C(C_0, \theta_0)\, r^n$ for all x, r. By the upper AD-regularity of σ, it is enough to show that the measure

$$\nu = \sum_{Q \in \mathcal{A}_0} \mathcal{H}^n \lfloor_{S(Q)}$$

is also upper AD-regular, so we have to prove that

$$\nu(B(x,r)) \leq C(C_0, \theta_0)\, r^n \quad \text{ for all } x \in \textstyle\bigcup_{Q \in \mathcal{A}_0} S(Q) \text{ and all } r > 0. \tag{3.77}$$

Take $x \in S(Q)$, for some $Q \in \mathcal{A}_0$. Clearly, the estimate above holds if the only sphere $S(P)$, $P \in \mathcal{A}_0$, that intersects $B(x,r)$ is just $S(Q)$ itself, so assume that $B(x,r)$ intersects a sphere $S(P)$, $P \in \mathcal{A}_0$, with $P \neq Q$. Recall that $\frac{1}{2}B(Q) \cap \frac{1}{2}B(P) = \varnothing$, by Remark 1, and thus for some constant C_6, $P \subset B(x, C_6 r)$. Hence,

$$\nu(B(x,r)) \leq \sum_{P \in \mathcal{A}_0 : P \subset B(x, C_6 r)} \nu(\tfrac{1}{4} S(P)) \lesssim \sum_{P \in \mathcal{A}_0 : P \subset B(x, C_6 r)} \ell(P)^n.$$

Note now that by the definition of $\mathcal{A}_0$, $\sigma(F \cap P) > 0$, which implies that $P \notin \mathsf{LD}$ and that P is not contained in any other cell from LD, and thus taking also into account that $P \in \mathcal{D}^{db}$,

$$\sigma(P) \gtrsim \sigma(3.5 B_P) \gtrsim \theta_0\, \ell(P)^n. \tag{3.78}$$

Together with the upper AD-regularity of σ, this yields

$$\nu(B(x,r)) \lesssim \frac{1}{\theta_0} \sum_{P \in \mathcal{A}_0 : P \subset B(x, C_6 r)} \sigma(P) \lesssim \frac{1}{\theta_0}\, \sigma(B(x, C_6 r)) \lesssim_{C_0, \theta_0} r^n,$$

which concludes the proof of (3.77).

It remains now to show the lower AD-regularity of ζ. First we will prove that

$$\zeta(2 B_Q) \gtrsim_{\theta_0, \varepsilon_0, C_0} \ell(Q)^n \quad \text{ if } Q \in \mathcal{D}_\sigma^{db} \text{ is not contained in any cell } Q_i,\ i \in J. \tag{3.79}$$

Indeed, note that by the definition of the cells Q_i, $i \in J$,

$$\sigma(Q \setminus F) \leq \Big(1 - \frac{\varepsilon_0}{2}\Big)\, \sigma(Q),$$

or equivalently,

$$\sigma(Q \cap F) \geq \frac{\varepsilon_0}{2}\, \sigma(Q).$$

Since Q is not contained in any cell from LD (by the definitions of F and $\mathcal{A}_0$) and is doubling,

$$\sigma(Q \cap F) \gtrsim_{\varepsilon_0} \sigma(3.5 B_Q) \gtrsim_{\theta_0, \varepsilon_0} \ell(Q)^n. \tag{3.80}$$

On the other hand, by the construction of ζ,

$$\sigma(Q\cap F) = \sigma(Q\cap \widetilde{F}) + \sum_{P\in\mathcal{A}_0: P\subset Q} \sigma(P\cap F) \lesssim_{C_0} \zeta(Q\cap\widetilde{F}) + \sum_{P\in\mathcal{A}_0:P\subset Q} \mathcal{H}^n(S(P)).$$

We may assume that all the cells $P\subset Q$ satisfy $S(P)\subset 2B_Q$, just by choosing the constant A_0 in the construction of the lattice $\mathcal{D}_\sigma$ big enough. Then we get

$$\sigma(Q\cap F) \lesssim_{C_0} \zeta(Q\cap\widetilde{F}) + \sum_{P\in\mathcal{A}_0:S(P)\subset 2B_Q} \zeta(S(P)) \lesssim_{C_0} \zeta(2B_Q).$$

Together with (3.80), this gives (3.79).

To prove the lower AD-regularity of ζ, note that by Remark 3 there exists some constant $C'(C_0,\theta_0)$ such that if $x\in S(Q)$, $Q\in\mathcal{A}_0$, and $C'(C_0,\theta_0)\,\ell(Q) < r \leq \operatorname{diam}(Q_0)$, then there exists $P\in\mathcal{D}_\sigma^{db}$ not contained in any cell Q_i, $i\in J$, such that $2B_P\subset B(x,r)$, with $\ell(P)\approx_{\theta_0,C_0} r$. The same holds for $0<r\leq \operatorname{diam}(Q_0)$ if $x\in\widetilde{F}$. From (3.79) we infer that

$$\zeta(B(x,r)) \geq \zeta(2B_P) \gtrsim_{\theta_0,\varepsilon_0,C_0} \ell(P)^n \approx_{\theta_0,\varepsilon_0,C_0} r^n.$$

In the case that $r\leq C'(C_0,\theta_0)\,\ell(Q)$ for $x\in S(Q)$, $Q\in\mathcal{A}_0$, the lower AD-regularity of $\mathcal{H}^n\lfloor_{S(Q)}$ gives the required lower estimate for $\zeta(B(x,r))$. □

Lemma 3.8.2. *The Riesz transform $\mathcal{R}_\zeta$ is bounded in $L^2(\zeta)$, with a bound on the norm depending on C_0, C_1, θ_0, and ε_0.*

To prove this result we will follow very closely the arguments in the last part of the proof of the Main Lemma 2.1 of [NToV2]. For completeness, we will show all the details.

For technical reasons, it will be convenient to work with an ε-regularized version $\widehat{\mathcal{R}}_{\nu,\varepsilon}$ of the Riesz transform $\mathcal{R}_\nu$. For a measure ν with polynomial growth of order n, we set

$$\widehat{\mathcal{R}}_{\nu,\varepsilon} f(x) = \int \frac{x-y}{\max(|x-y|,\varepsilon)^{n+1}}\, f(y)\, d\nu(y).$$

It is easy to check that

$$|\widehat{\mathcal{R}}_{\nu,\varepsilon} f(x) - \mathcal{R}_{\nu,\varepsilon} f(x)| \leq c\, M_n f(x) \qquad \text{for all } x\in\mathbb{R}^{n+1},$$

where c is independent of ε and M_n is the following maximal operator with respect to ν:

$$M_n f(x) = \sup_{r>0} \frac{1}{r^n}\int_{B(x,r)} |f|\, d\nu.$$

Since M_n is bounded in $L^2(\nu)$ (because ν has growth of order n), it turns out that $\mathcal{R}_\nu$ is bounded in $L^2(\nu)$ if and only if the operators $\widehat{\mathcal{R}}_{\nu,\varepsilon}$ are bounded in $L^2(\nu)$ uniformly on $\varepsilon>0$. The advantage of $\widehat{\mathcal{R}}_{\nu,\varepsilon}$ over $\mathcal{R}_{\nu,\varepsilon}$ is that the kernel

$$\widehat{K}_\varepsilon(x) = \frac{x}{\max(|x|,\varepsilon)^{n+1}}$$

is continuous and satisfies the smoothness condition

$$|\nabla\widehat{K}_\varepsilon(x)| \leq \frac{c}{|x|^{n+1}}, \quad |x|\neq\varepsilon$$

(with c independent of ε), which implies that $\widehat{K}_\varepsilon(x-y)$ is a standard Calderón-Zygmund kernel (with constants independent of ε), unlike the kernel of $\mathcal{R}_{\nu,\varepsilon}$.

Proof of Lemma 3.8.2. To shorten notation, in the arguments below we will allow all the implicit constants in the relations $\lesssim$ and $\approx$ to depend on $C_0, C_1, \theta_0, \varepsilon_0$.

Denote

$$\nu = \sum_{Q\in\mathcal{A}_0} \mathcal{H}^n\lfloor_{S(Q)},$$

so that $\zeta = \sigma\lfloor_{\widetilde{F}}+\nu$. Since $\mathcal{R}_\sigma$ is bounded in $L^2(\sigma)$, it is enough to show that $\mathcal{R}_\nu$ is bounded in $L^2(\nu)$. Indeed, the boundedness of both operators implies the boundedness of $\mathcal{R}_{\sigma+\nu}$ in $L^2(\sigma+\nu)$ (see Proposition 2.25 of [T6], for example).

As in (3.14), given $\kappa>0$, for each $Q\in\mathcal{A}_0$, we consider the set

$$I_\kappa(Q) = \{x\in Q : \operatorname{dist}(x, \operatorname{supp}\sigma\setminus Q) \geq \kappa\ell(Q)\}.$$

By the thin boundary condition of Q, the fact that Q is doubling, and that $\sigma(Q)\gtrsim\theta_0\,\ell(Q)^n$ (as shown in (3.78)), we obtain that there exists some $\kappa>0$ small enough such that

$$\sigma(I_\kappa(Q)) \geq \frac{1}{2}\,\sigma(Q) \gtrsim \theta_0\,\ell(Q)^n. \tag{3.81}$$

We consider the measure

$$\widetilde{\sigma} = \sum_{Q\in\mathcal{A}_0} c_Q\,\sigma\lfloor_{I_\kappa(Q)},$$

with $c_Q = \mathcal{H}^n(S(Q))/\sigma(I_\kappa(Q))$. By (3.81), it follows that the constants c_Q, $Q\in\mathcal{A}_0$, have a uniform bound depending on θ_0, and thus $\mathcal{R}_{\widetilde{\sigma}}$ is bounded in $L^2(\widetilde{\sigma})$ (with a norm possibly depending on θ_0). Furthermore, $\nu(S(Q)) = \widetilde{\sigma}(Q)$ for each $Q\in\mathcal{A}_0$.

It is clear that, in a sense, $\widetilde{\sigma}$ can be considered as an approximation of ν (and conversely). To prove the boundedness of $\mathcal{R}_\nu$ in $L^2(\nu)$, we will prove that $\widehat{\mathcal{R}}_{\nu,\varepsilon}$ is bounded in $L^2(\nu)$ uniformly on $\varepsilon>0$ by comparing it to $\widehat{\mathcal{R}}_{\widetilde{\sigma},\varepsilon}$. First we need to introduce some local and non local operators: given $z\in\bigcup_{Q\in\mathcal{A}_0} S(Q)$, we denote by $S(z)$ the sphere $S(Q), Q\in\mathcal{A}_0$, that contains z. Then we write, for $z\in\bigcup_{Q\in\mathcal{A}_0} S(Q)$,

$$\mathcal{R}^{loc}_{\nu,\varepsilon} f(z) = \widehat{\mathcal{R}}_{\nu,\varepsilon}(f\chi_{S(z)})(z), \qquad \mathcal{R}^{nl}_{\nu,\varepsilon} f(z) = \widehat{\mathcal{R}}_{\nu,\varepsilon}(f\chi_{\mathbb{R}^{n+1}\setminus S(z)})(z).$$

We define analogously $\mathcal{R}^{loc}_{\sigma,\varepsilon} f$ and $\mathcal{R}^{nl}_{\sigma,\varepsilon} f$: given $z\in\bigcup_{Q\in\mathcal{A}_0} Q$, we denote by $Q(z)$ the cell $Q\in\mathcal{A}_0$ that contains z. Then for $z\in\bigcup_{Q\in\mathcal{A}_0} Q$, we set

$$\mathcal{R}^{loc}_{\widetilde{\sigma},\varepsilon} f(z) = \widehat{\mathcal{R}}_{\widetilde{\sigma},\varepsilon}(f\chi_{Q(z)})(z), \qquad \mathcal{R}^{nl}_{\widetilde{\sigma},\varepsilon} f(z) = \widehat{\mathcal{R}}_{\widetilde{\sigma},\varepsilon}(f\chi_{\mathbb{R}^{n+1}\setminus Q(z)})(z).$$

It is straightforward to check that $\mathcal{R}^{loc}_{\nu,\varepsilon}$ is bounded in $L^2(\nu)$, and that $\mathcal{R}^{loc}_{\sigma,\varepsilon}$ is bounded in $L^2(\sigma)$, both uniformly on ε (in other words, $\mathcal{R}^{loc}_\nu$ is bounded in $L^2(\nu)$ and $\mathcal{R}^{loc}_\sigma$ is bounded in $L^2(\sigma)$). Indeed,

$$\|\mathcal{R}^{loc}_{\nu,\varepsilon} f\|^2_{L^2(\nu)} = \sum_{Q\in\mathcal{A}_0} \|\chi_{S(Q)}\widehat{\mathcal{R}}_{\nu,\varepsilon}(f\chi_{S(Q)})\|^2_{L^2(\nu)} \lesssim \sum_{Q\in\mathcal{A}_0} \|f\chi_{S(Q)}\|^2_{L^2(\nu)} = \|f\|^2_{L^2(\nu)},$$

by the boundedness of the n-Riesz transforms on $S(Q)$. Using the boundedness of $\mathcal{R}_\sigma$ in $L^2(\sigma)$, it follows analogously that $\mathcal{R}^{loc}_{\widetilde{\sigma},\varepsilon}$ is bounded in $L^2(\widetilde{\sigma})$.

Boundedness of $\mathcal{R}^{nl}_\nu$ in $L^2(\nu)$. We must show that $\mathcal{R}^{nl}_\nu$ is bounded in $L^2(\nu)$. To this end, we will compare $\mathcal{R}^{nl}_\nu$ to $\mathcal{R}^{nl}_{\widetilde{\sigma}}$. Observe first that, since $\mathcal{R}^{nl}_{\widetilde{\sigma},\varepsilon} = \widehat{\mathcal{R}}_{\widetilde{\sigma},\varepsilon} - \mathcal{R}^{loc}_{\widetilde{\sigma},\varepsilon}$, and both $\widehat{\mathcal{R}}_{\widetilde{\sigma},\varepsilon}$ and $\mathcal{R}^{loc}_{\widetilde{\sigma},\varepsilon}$ are bounded in $L^2(\widetilde{\sigma})$, it follows that $\mathcal{R}^{nl}_{\widetilde{\sigma},\varepsilon}$ is bounded in $L^2(\widetilde{\sigma})$ (all uniformly on $\varepsilon>0$).

Note also that for two different cells $P,Q\in\mathcal{A}_0$, we have

$$\operatorname{dist}(S(P),S(Q)) \approx \operatorname{dist}(I_\kappa(P),I_\kappa(Q)) \approx \operatorname{dist}(S(P),I_\kappa(Q)) \approx D(P,Q), \tag{3.82}$$

where $D(P,Q) = \ell(P) + \ell(Q) + \operatorname{dist}(P,Q)$ and the implicit constants may depend on κ. The arguments to prove this are exactly the same as the ones for (3.33), (3.34) and (3.35), and so we omit them. In particular, (3.82) implies that $\big(S(P) \cup I_\kappa(P)\big) \cap \big(S(Q) \cup I_\kappa(Q)\big) = \varnothing$, and thus for every $z \in \mathbb{R}^{n+1}$ there is at most one cell $Q \in \mathcal{A}_0$ such that $z \in S(Q) \cup I_\kappa(Q)$, which we denote by $Q(z)$. Hence we can extend $\mathcal{R}^{nl}_{\nu,\varepsilon}$ and $\mathcal{R}^{nl}_{\widetilde{\sigma},\varepsilon}$ to $L^2(\widetilde{\sigma}+\nu)$ by setting

$$\mathcal{R}^{nl}_{\nu,\varepsilon} f(z) = \widehat{\mathcal{R}}_{\nu,\varepsilon}(f\chi_{\mathbb{R}^{n+1}\setminus S(Q(z))})(z), \qquad \mathcal{R}^{nl}_{\widetilde{\sigma},\varepsilon} f(z) = \widehat{\mathcal{R}}_{\widetilde{\sigma},\varepsilon}(f\chi_{\mathbb{R}^{n+1}\setminus Q(z)})(z).$$

We will prove below that, for all $f \in L^2(\widetilde{\sigma})$ and $g \in L^2(\nu)$ satisfying

$$\int_{I_\kappa(P)} f\, d\widetilde{\sigma} = \int_{S(P)} g\, d\nu \qquad \text{for all } P \in \mathcal{A}_0, \tag{3.83}$$

we have

$$I(f,g) := \int |\mathcal{R}^{nl}_{\widetilde{\sigma},\varepsilon} f - \mathcal{R}^{nl}_{\nu,\varepsilon} g|^2\, d(\widetilde{\sigma}+\nu) \lesssim \|f\|^2_{L^2(\widetilde{\sigma})} + \|g\|^2_{L^2(\nu)}, \tag{3.84}$$

uniformly on ε. Let us see how the boundedness of $\mathcal{R}^{nl}_\nu$ in $L^2(\nu)$ follows from this estimate. As a preliminary step, we show that $\mathcal{R}^{nl}_\nu : L^2(\nu) \to L^2(\widetilde{\sigma})$ is bounded. To this end, given $g \in L^2(\nu)$, we consider a function $f \in L^2(\widetilde{\sigma})$ satisfying (3.83) that is constant on each ball B_j. It is straightforward to check that

$$\|f\|_{L^2(\widetilde{\sigma})} \leq \|g\|_{L^2(\nu)}.$$

Then from the $L^2(\widetilde{\sigma})$ boundedness of $\mathcal{R}^{nl}_{\widetilde{\sigma}}$ and (3.84), we obtain

$$\|\mathcal{R}^{nl}_{\nu,\varepsilon} g\|_{L^2(\widetilde{\sigma})} \leq \|\mathcal{R}^{nl}_{\widetilde{\sigma},\varepsilon} f\|_{L^2(\widetilde{\sigma})} + I(f,g)^{1/2} \lesssim \|f\|_{L^2(\widetilde{\sigma})} + \|g\|_{L^2(\nu)} \lesssim \|g\|_{L^2(\nu)},$$

which proves that $\mathcal{R}^{nl}_\nu : L^2(\nu) \to L^2(\widetilde{\sigma})$ is bounded.

It is straightforward to check that the adjoint of $(\mathcal{R}^{nl}_{\nu,\varepsilon})_j : L^2(\nu) \to L^2(\widetilde{\sigma})$ (where $(\mathcal{R}^{nl}_{\nu,\varepsilon})_j$ stands for the j-th component of $(\mathcal{R}^{nl}_{\nu,\varepsilon})_j)$ equals $-(\mathcal{R}^{nl}_{\widetilde{\sigma},\varepsilon})_j : L^2(\widetilde{\sigma}) \to L^2(\nu)$. So, by duality, we obtain that $\mathcal{R}^{nl}_{\widetilde{\sigma}} : L^2(\widetilde{\sigma}) \to L^2(\nu)$ is also bounded.

To prove now the $L^2(\nu)$-boundedness of $\mathcal{R}^{nl}_\nu$, we consider an arbitrary function $g \in L^2(\nu)$, and we construct $f \in L^2(\widetilde{\sigma})$ satisfying (3.83) which is constant in each ball P. Again, we have $\|f\|_{L^2(\widetilde{\sigma})} \leq \|g\|_{L^2(\nu)}$. Using the boundedness of $\mathcal{R}^{nl}_{\widetilde{\sigma}} : L^2(\widetilde{\sigma}) \to L^2(\nu)$ together with (3.84), we obtain

$$\|\mathcal{R}^{nl}_{\nu,\varepsilon} g\|_{L^2(\nu)} \leq \|\mathcal{R}^{nl}_{\widetilde{\sigma},\varepsilon} f\|_{L^2(\nu)} + I(f,g)^{1/2} \lesssim \|f\|_{L^2(\widetilde{\sigma})} + \|g\|_{L^2(\nu)} \lesssim \|g\|_{L^2(\nu)},$$

as wished.

It remains to prove that (3.84) holds for $f \in L^2(\widetilde{\sigma})$ and $g \in L^2(\nu)$ satisfying (3.83). For $z \in \bigcup_{P\in\mathcal{A}_0} P$, we have

$$|\mathcal{R}^{nl}_{\widetilde{\sigma},\varepsilon} f(z) - \mathcal{R}^{nl}_{\nu,\varepsilon} g(z)| \leq \sum_{P\in\mathcal{A}_0 : P \neq Q(z)} \left| \int \widehat{K}_\varepsilon(z-y)(f(y)\, d\widetilde{\sigma}|_{I_\kappa(P)}(y) - g(y)\, d\nu|_{S(P)}(y)) \right|,$$

where $\widehat{K}_\varepsilon(z)$ is the kernel of the ε-regularized n-Riesz transform. By standard estimates, using (3.83) and (3.82), and the smoothness of $\widehat{K}_\varepsilon$, it follows that

$$\begin{aligned}
&\left| \int \widehat{K}_\varepsilon(z-y)(f(y)\, d\widetilde{\sigma}|_{I_\kappa(P)}(y) - g(y)\, d\nu|_{S(P)}(y)) \right| \\
&\qquad = \left| \int_P (\widehat{K}_\varepsilon(z-y) - K_\varepsilon(z-x))(f(y)\, d\widetilde{\sigma}|_{I_\kappa(P)}(y) - g(y)\, d\nu|_{S(P)}(y)) \right| \\
&\qquad \lesssim \int \frac{|x-y|}{|z-y|^{n+1}} (|f(y)|\, d\widetilde{\sigma}|_{I_\kappa(P)}(y) + |g(y)|\, d\nu|_{S(P)}(y)) \\
&\qquad \approx \frac{\ell(P)}{D(Q(z),P)^{n+1}} \int (|f(y)|\, d\widetilde{\sigma}|_{I_\kappa(P)}(y) + |g(y)|\, d\nu|_{S(P)}(y)).
\end{aligned}$$

Recall that $Q(z)$ stands for the cell $Q, Q \in \mathcal{A}_0$, such that $z \in S(Q) \cup I_\kappa(Q)$.

We consider the operators

$$T_{\widetilde{\sigma}}(f)(z) = \sum_{P\in\mathcal{A}_0: P\neq Q(z)} \frac{\ell(P)}{D(Q(z),P)^{n+1}} \int f\, d\widetilde{\sigma}|_{I_\kappa(P)}\,,$$

and T_ν, which is defined in the same way with $\widetilde{\sigma}_{I_\kappa(P)}$ replaced by $\nu|_{S(P)}$. Observe that

$$\begin{aligned} I(f,g) &\leq c\,\|T_{\widetilde{\sigma}}(|f|) + T_\nu(|g|)\|^2_{L^2(\widetilde{\sigma}+\nu)} \\ &\leq 2c\,\|T_{\widetilde{\sigma}}(|f|)\|^2_{L^2(\widetilde{\sigma}+\nu)} + 2c\,\|T_\nu(|g|)\|^2_{L^2(\widetilde{\sigma}+\nu)} \\ &= 4c\,\|T_{\widetilde{\sigma}}(|f|)\|^2_{L^2(\widetilde{\sigma})} + 4c\,\|T_\nu(|g|)\|^2_{L^2(\nu)}, \end{aligned}$$

where, for the last equality, we took into account that both $T_{\widetilde{\sigma}}(|f|)$ and $T_\nu(|g|)$ are constant on $I_\kappa(P) \cup S(P)$ and that $\widetilde{\sigma}(I_\kappa(P)) = \nu(S(P))$ for all $P \in \mathcal{A}_0$.

To complete the proof of (3.84) it is enough to show that $T_{\widetilde{\sigma}}$ is bounded in $L^2(\widetilde{\sigma})$ and T_ν in $L^2(\nu)$. We only deal with $T_{\widetilde{\sigma}}$, since the arguments for T_ν are analogous. We argue by duality again, so we consider non-negative functions $f, h \in L^2(\widetilde{\sigma})$ and we write

$$\begin{aligned} \int T_{\widetilde{\sigma}}(f)\, h\, d\widetilde{\sigma} &= \int \left(\sum_{P\in\mathcal{A}_0: P\neq Q(z)} \frac{\ell(P)}{D(P,Q(z))^{n+1}} \int_P f\, d\widetilde{\sigma} \right) h(z)\, d\widetilde{\sigma}(z) \\ &\lesssim \sum_{P\in\mathcal{A}_0} \ell(P) \int_P f\, d\widetilde{\sigma} \int_{\mathbb{R}^{n+1}\setminus P} \frac{1}{(\operatorname{dist}(z,P)+\ell(P))^{n+1}}\ h(z)\, d\widetilde{\sigma}(z). \end{aligned}$$

From the growth of order n of $\widetilde{\sigma}$, it follows easily that

$$\int_{\mathbb{R}^{n+1}\setminus P} \frac{1}{(\operatorname{dist}(z,P)+\ell(P))^{n+1}}\ h(z)\, d\widetilde{\sigma}(z) \lesssim \frac{1}{\ell(P)}\, M_{\widetilde{\sigma}} h(y) \quad \text{ for all } y \in P,$$

where $M_{\widetilde{\sigma}}$ stands for the (centered) maximal Hardy-Littlewood operator (with respect to $\widetilde{\sigma}$). Then we deduce that

$$\int T_{\widetilde{\sigma}}(f)\, h\, d\widetilde{\sigma} \lesssim \sum_{P\in\mathcal{A}_0} \int_P f(y)\, M_{\widetilde{\sigma}} h(y)\, d\widetilde{\sigma}(y) \lesssim \|f\|_{L^2(\widetilde{\sigma})} \|h\|_{L^2(\widetilde{\sigma})},$$

by the $L^2(\widetilde{\sigma})$ boundedness of $M_{\widetilde{\sigma}}$. Thus, $T_{\widetilde{\sigma}}$ is bounded in $L^2(\widetilde{\sigma})$. □

Proof of the Main Lemma 3.2.1. By Lemmas 3.8.1, 3.8.2, and the Nazarov-Tolsa-Volberg theorem of [NToV1], ζ is a uniformly rectifiable measure, so it only remains to note that the set $\Gamma := \widetilde{F}$ satisfies the required properties from the Main Lemma: it is contained in $\operatorname{supp}(\zeta)$, which is uniformly rectifiable and, by (3.76), $\mu(\Gamma) = \sigma(\widetilde{F}) \geq \frac{\varepsilon_0}{2}\, \mu(Q_0)$. □

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