APLICACIONES LINEALES QUE PRESERVAN LA INVERSIBILIDAD GENERALIZADA ENTRE ÁLGEBRAS DE BANACH

LINEAR MAPS PRESERVING GENERALIZED INVERTIBILITY BETWEEN BANACH ALGEBRAS

TESIS DOCTORAL

ANTONIO CARLOS MÁRQUEZ GARCÍA

ALMERÍA, 2015

Trabajo de investigación realizado en Departamento de Matemáticas de la Universidad de Almería, bajo la dirección de los profesores D. Antonio Morales Campoy y Dª María José Burgos Navarro, para la obtención del título de Doctor en Matemáticas por la Universidad de Almería.

Visto bueno de los Directores

Fdo: Antonio Morales Campoy

Fdo: María José Burgos Navarro

El aspirante

Fdo: Antonio Carlos Márquez García

Agradecimientos

El desarrollo de este trabajo no habría sido posible sin la ayuda, directa o indirecta, de un buen número de personas.

En primer lugar quiero dar gracias a mis tutores, María Burgos y Antonio Morales, por su paciencia y entrega, máxime siendo la primera vez que llevan a cabo la dirección de una tesis. Al igual que yo me gradúo con este trabajo, ellos, en cierto sentido, también lo hacen... y con nota.

En segundo lugar, quiero agradecer a mi familia, desde la más inmediata y local hasta el sentido más amplio de la palabra. Desde Inma (y Arturo), que es el eje central de mi vida y no ha dejado de apoyarme un solo segundo en todo lo que emprendo; pasando por mis padres, hermano y abuelos (y tíos y primos), que partiendo de la nada construyeron sobre mí la persona que soy hoy; hasta mi familia "política", que hace tiempo perdió ese apelativo pues me acogió desde el primer día como uno más. Y no puedo olvidar a esa familia matemática, esos hermanos que conocí en Jaca y que han hecho de cada congreso una experiencia inolvidable.

Quiero dar gracias a Antonio Peralta, por haberme acogido tantas veces en Granada y recibirme siempre con la mejor hospitalidad y una gran sonrisa. Y a Pedro Patrício, por haberme acogido en Braga y haber movido cielo y tierra para que estuviese lo más cómodo posible.

Por último, quiero agradecer a la Universidad de Almería la oportunidad que me ha dado de dedicarme íntegramente a este trabajo gracias a la beca de doctorado que me concedió. A los buenos profesores que he tenido a lo largo de mi vida por transmitirme tanto y hacer de mí un buen matemático; a los no tan buenos, por enseñarme a no bajar la guardia. Y a los que han sido mis compañeros de departamento estos años, por haberme hecho sentir como en casa.

Contents

Introducción en español

Antecedentes en invariantes lineales

El estudio de los llamados *problemas de invariantes lineales* se ocupa de la caracterización de las aplicaciones lineales (o, más generalmente, aditivas) entre álgebras de Banach que dejan invariante cierta función, propiedad, subconjunto o relación entre sus elementos. El primer artículo sobre este problema data de 1897, y desde entonces se han destinado muchos esfuerzos al avance de esta línea de investigación. A día de hoy, el estudio de los problemas de invariantes lineales es uno de los temas de investigación más activos y fructíferos en teoría de matrices, teoría de operadores y análisis funcional.

En lo que sigue, A y B serán álgebras de Banach complejas y $T : A \to B$ una aplicación lineal entre ellas. Comenzamos presentando una revisión de las principales líneas de investigación dentro de los invariantes lineales, así como una selección de los resultados más relevantes obtenidos hasta la fecha en cada una de ellas.

(I) Invariantes de funciones

Dada una función escalar, vectorial o conjunto-valuada F, caracterizar las aplicaciones lineales T tales que $F_B(T(a)) = F_A(a)$. Por ejemplo, si tomamos $F(x) = \det(x)$ y $A = B = M_n(\mathbb{C})$, el resultado clásico de Frobenius ([59]) afirma que toda aplicación lineal $\phi : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$ que preserva el determinante, es decir, tal que $\det(\phi(A)) = \det(A)$ para toda $A \in M_n(\mathbb{C})$, toma una de las siguientes formas:

$$\phi(A) = MAN, \qquad \text{para toda } A \in M_n(\mathbb{C}) \quad \text{o}$$

$$\phi(A) = MA^tN, \qquad \text{para toda } A \in M_n(\mathbb{C}).$$

Si $F(x) = \|x\|$ y A, B son C*-álgebras, Kadison mostró en [78] que toda aplicación lineal y sobreyectiva $T : A \to B$ es una isometría si, y sólo si, T es un *-isomorfismo de Jordan multiplicado por un elemento unitario en B.

Sea F el módulo mínimo o el módulo de sobreyectividad (respectivamente, módulo mínimo reducido) y $A = B = \mathcal{B}(H)$. Mbekhta probó en [105] (respectivamente, [103]) que toda aplicación lineal, unital y sobreyectiva de $\mathcal{B}(H)$, el álgebra de los operadores lineales y acotados en un espacio de Hilbert H, en sí mismo que preserva F es un *-automorfismo (respectivamente, *-automorfismo o *-anti-automorfismo). Los autores de [18] estudiaron este problema en el contexto de las C*-álgebras unitales.

Teorema 1 *([18, Teorema 3.1]) Sean A y B C*-álgebras unitales. Si $T : A \to B$ es una aplicación lineal unital satisfaciendo $F(T(x)) = F(x)$ para todo $x \in A$, entonces T es un *-isomorfismo de Jordan isométrico.*

Teorema 2 *([18, Teorema 3.2]) Sean A un álgebra de Banach semisimple y unital y B una C*-álgebra unital. Si $T : A \to B$ es una aplicación lineal y sobreyectiva tal que $F(T(x)) = F(x)$ para todo $x \in A$, entonces A es una C*-álgebra y T es un *-homomorfismo de Jordan isométrico multiplicado por un elemento unitario.*

Otros ejemplos de funciones cuyos invariantes han sido estudiados son el rango numérico ([117]) y el ascenso y descenso de operadores ([10]).

(II) Invariantes de subconjuntos

Dado un subconjunto $S_A \subset A$, determinar el conjunto de aplicaciones lineales que dejan invariante dicho subconjunto, es decir, las aplicaciones $T : A \to B$ tales que $T(S_A) \subset S_B$. Este problema también puede ser estudiado con la condición $T(S_A) = S_B$, en cuyo caso se dirá que la aplicación preserva S en ambas direcciones. Cuando S es el conjunto de los elementos inversibles y A y B son álgebras de Banach semisimples y unitales, nos encontramos con la célebre *conjetura de Kaplansky*. Recuérdese que esta conjetura, a día de hoy, se enuncia como sigue: una aplicación lineal, biyectiva y unital $T : A \to B$ entre álgebras de Banach semisimples y unitales que preserva la inversibilidad es un isomorfismo de Jordan (nótese que, al ser T unital, preservar la inversibilidad equivale a comprimir el espectro, es decir, $\sigma(T(a)) \subset \sigma(a)$ para todo $a \in A$). Este problema sigue aún abierto en C*-álgebras. Gleason (véase [60]) y Kahane-Zelazko (véase [79]) probaron casi simultáneamente que si A y B son álgebras de Banach, con B conmutativa y semisimple, toda aplicación lineal unital que preverve la inversibilidad es un homomorfismo. En [73], Jafarian y Sourour mostraron que los isomorfismos de Jordan son las únicas aplicaciones lineales, biyectivas y unitales entre álgebras de operadores que preservan la inversibilidad en ambas direcciones. En [7], Aupetit confirmó la conjetura para álgebras de von Neumann (véanse Teorema 1.2 y Observación 2.7 en [7]). Posteriormente, Cui y Hou probaron en [48] que la conjetura tiene respuesta afirmativa si A es una C*-álgebra de rango real cero.

Teorema 3 *([48, Teorema 3.1]) Sean A una C*-álgebra unital de rango real cero y B un álgebra de Banach semisimple y unital. Supongamos que $T : A \to B$ es una aplicación lineal, sobreyectiva y unital que preserva la inversibilidad. Entonces T es un isomorfismo de Jordan.*

Existo otro ambiente favorable en el que la conjetura de Kaplansky tiene respuesta afirmativa. En [19], Brešar, Fošner y Šemrl, probaron lo siguiente:

Teorema 4 *([19, Teorema 1.1]) Sean A y B álgebras de Banach semisimples y unitales, teniendo A zócalo esencial. Sea $T : A \to B$ una aplicación lineal, biyectiva y unital que preserva la inversibilidad. Entonces T es un isomorfismo de Jordan.*

V. Mascioni y L. Molnár estudiaron las aplicaciones lineales en un factor de von Neumann A que preservan los puntos extremos de la bola unidad de A.

Teorema 5 *([101, Teorema 1]) Sea A un factor de von Neumann infinito. La aplicación lineal $T : A \to A$ preserva los puntos extremos de la bola unidad de A si, y sólo si, existe un elemento unitario $u \in A$ y un *-homomorfismo unital $S : A \to A$ tal que T es de la forma $T(a) = uS(a)$ para todo $a \in A$ o existe un elemento unitario $u' \in A$ y un *-anti-homomorfismo unital $S' : A \to A$ tal que T es de la forma $T(a) = u'S'(a)$ para todo $a \in A$.*

En [87], L. E. Labuschagne y V. Mascioni estudiaron las aplicaciones lineales entre C*-álgebras cuyas adjuntas preservan los puntos extremos de la bola dual.

Otros subconjuntos remarcables que han sido estudiados son los idempotentes ([86]), (semi-)Fredholm ([11]) y elementos con inverso generalizado ([17], [107]).

(III) Invariantes de relación

Dada una relación binaria $\sim$ en A y B, estudiar las aplicaciones lineales que preservan $\sim$, esto es, aquellas aplicaciones que satisfacen $T(a) \sim T(b)$ siempre que $a \sim b$. Ejemplos de estas relaciones son la conmutatividad ([45]), la ortogonalidad ([4, 25, 36, 38, 135]), el producto cero ([1, 39, 42]), relaciones de orden parcial ([62, 63, 89, 129]), etc. Cuando se impone la condición $T(a) \sim T(b)$ si, y sólo si, $a \sim b$ se dice que T preserva $\sim$ en ambas direcciones. De particular interés de cara a esta tesis son las relaciones de ortogonalidad y algunos órdenes parciales.

El estudio de aplicaciones lineales y continuas que preservan la ortogonalidad en C*-álgebras comenzó con el trabajo de W. Arendt [4] en el ambiente de las C*-álgebras conmutativas unitales. Los operadores lineales que preservan la ortogonalidad en C*-álgebras generales fueron considerados en primer lugar por M. Wolff en [135]. Mostró que toda aplicación lineal, autoadjunta y acotada que preserva la ortogonalidad entre C*-álgebras es múltiplo de un *-homomorfismo de Jordan.

Bajo la condición de continuidad, las aplicaciones lineales que preservan la ortogonalidad entre C*-álgebras fueron completamente descritas en [27] y [36]:

Teorema 6 *([36, Teorema 17 y Corolario 18]) Sea $T : A \to B$ una aplicación lineal y continua entre C*-álgebras. Para $h = T^{**}(1)$ y $r = r(h)$, las siguientes afirmaciones son equivalentes:*

(1) T preserva la ortogonalidad,

*(2) Existe un único *-homomorfismo de Jordan $S : A \to B_2^{**}(r)$ satisfaciendo $S^{**}(1) = r$ y $T(z) = hr^*S(z) = S(z)r^*h$ para todo $z \in A$,*

(3) T preserva producto triple cero, es decir, $\{T(x), T(y), T(z)\} = 0$ si $\{x, y, z\} = 0$.

En [38], Burgos, Garcés y Peralta probaron que toda aplicación lineal y sobreyectiva que preserva la ortogonalidad en ambas direcciones entre C*-álgebras compactas o álgebras de von Neumann es automáticamente continua. Dado que toda aplicación lineal autoadjunta entre C*-álgebras preserva la ortogonalidad si preserva el producto cero, se tiene que toda aplicación lineal autoadjunta y biseparadora entre álgebras de von Neumann es automáticamente continua.

En [25], la autora estudió las aplicaciones lineales biyectivas que preservan la ortogonalidad entre una C*-álgebra unital con zócalo esencial y una C*-álgebra cualquiera.

Teorema 7 *([25, Teorema 3.2]) Sean A y B C*-álgebras. Supongamos que A es unital y tiene zócalo esencial. Sea $T : A \to B$ una aplicación biyectiva que preserva la ortogonalidad tal que $\{T(1)\}^\perp = \{0\}$. Entonces B es unital y T es un *-isomorfismo de Jordan multiplicado por un elemento inversible.*

En 1993, Ovchinnikov estudió las aplicaciones que preservan el orden de los idempotentes en $\mathcal{B}(H)$. Obtuvo que una aplicación tal es de la forma $P \mapsto APA^{-1}$ o de la forma $P \mapsto AP^*A^{-1}$ para todo operador idempotente P, donde A es una biyección lineal o conjugado-lineal en H ([118]).

En 2001, Guterman estudió las aplicaciones lineales entre álgebras de matrices que preservan el orden star ([62]). Obtuvo el siguiente resultado:

Teorema 8 *([62]) Sea $\mathbb{K} = \mathbb{R}$ o $\mathbb{C}$. Sea $T : M_n(\mathbb{K}) \to M_n(\mathbb{K})$ una aplicación biyectiva lineal que preserva el orden star. Entonces existen $\alpha \in \mathbb{K}$ y matrices unitarias $U, V \in M_n(\mathbb{K})$ tales que $T(A) = \alpha UAV$ para toda $A \in M_n(\mathbb{K})$ o $T(A) = \alpha UA^tV$.*

Posteriormente, Šemrl obtuvo en [128] una versión mejorada del resultado de Ovchinnikov para álgebras de matrices. Observó que ciertos resultados clásicos de geometría proyectiva y extensiones de éstos en análisis funcional permiten eliminar la condición de linealidad e incluso aditividad. Legiša aprovechó las ideas de Šemrl para obtener una versión no lineal del Teorema 1.1.8.

Las aplicaciones aditivas que preservan los órdenes star, left-star y right-star entre álgebras de matrices reales y complejas fueron estudiadas por Guterman en 2007 (véase [63]). Mostró que toda aplicación aditiva $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ que preserva el orden star toma una de las siguientes formas: para toda $A \in M_n(\mathbb{C})$, $\phi(A) = \alpha UAV$, $\phi(A) = \alpha U\overline{A}V$, $\phi(A) = \alpha UA^tV$ o $\phi(A) = \alpha U\overline{A}^tV$ donde $\alpha \in \mathbb{C}$, U y V son matrices unitarias, $\bar{A}$ denota la matriz conjugada de A y A^t su traspuesta.

Artículos más recientes se han dedicado al estudio de los invariantes del orden star. En 2013, Dolinar, Guterman y Marovt estudiaron las aplicaciones aditivas, biyectivas y continuas en $K(H)$ que preservan el orden star en ambas direcciones, donde $K(H)$ deonta al ideal cerrado de los operadores compactos en un espacio de Hilbert complejo, infinito-dimensional y separable H (véase [50]). Recientemente, los autores de [51] trasladaron algunos resultados de [63] relativos a los órdenes left-star y right-star (que son versiones unilaterales del orden star) al caso infinito-dimensional, siguiendo algunas técnicas de [129]. Mostraron que toda aplicación aditiva y biyectiva $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ que preserva el orden left-star en ambas direcciones tiene la forma $\phi(A) = UAS$ para todo $A \in \mathcal{B}(H)$, donde U es un operador unitario y S es biyectivo (nótese que U y S pueden ser lineales o conjugado-lineales). Conclusiones similares son obtenidas para el orden right-star.

Las aplicaciones que preservan el orden minus fueron consideradas en primer lugar por Legiša en [89].

Teorema 9 *([89, Teorema 1]) Sea $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ una aplicación biyectiva que preserva el orden minus en ambas direcciones. Entonces existe un automorfismo φ de $\mathbb{C}$ y matrices inversibles $R, S \in M_n(\mathbb{C})$ tales que, o bien $T(A) = RA_\varphi S$ para toda $A \in M_n(\mathbb{C})$ o bien $T(A) = RA^t_\varphi S$ para toda $A \in M_n(\mathbb{C})$, donde A_φ es una matriz obtenida aplicando φ a los elementos de A, esto es, $[a_{ij}]_\varphi = [\varphi(a_{ij})]$.*

En [129], Šemrl trasladó el concepto de orden minus al caso infinito-dimensional, mediante una definición que no dependía de inversos interiores (véase Sección 2.4). En el mismo trabajo, estudió las aplicaciones (no necesariamente lineales) biyectivas que preservan el orden minus en ambas direcciones en $\mathcal{B}(H)$.

Teorema 10 *([129, Teorema 8]) Sea H un espacio de Hilbert infinito-dimensional. Supongamos que $T : \mathcal{B}(H) \to \mathcal{B}(H)$ es una aplicación biyectiva que preserva el orden minus en ambas direcciones. Entonces existen aplicaciones acotadas, ambas lineales o ambas conjugado-lineales, $R, S : H \to H$ tales que, o bien $T(A) = RAS$ para todo $A \in \mathcal{B}(H)$ o bien $T(A) = RA^*S$ para todo $A \in \mathcal{B}(H)$.*

(IV) Problemas de invariantes fuertes (problemas de tipo Hua)

Todo homomorfismo de Jordan unital *preserva fuertemente la inversibilidad*, esto es, $T(a^{-1}) = T(a)^{-1}$, para todo elemento inversible $a \in A$ (véase [131, Proposición 1.3]). En 1949 Hua probó que toda aplicación aditiva y unital $T : K \to K$ en un anillo de división K tal que $T(aba) = T(a)T(b)T(a)$ es un automorfismo o un anti-automorfismo (véase [70]). Este resultado fue reformulado por Artin en 1957 de la siguiente manera:

Teorema 11 *([5, Teorema 1.15]) Toda aplicación aditiva y unital $T : K \to K$ en un anillo de división K tal que $T(a^{-1}) = T(a)^{-1}$ es un automorfismo o un anti-automorfismo.*

El teorema de Hua fue después generalizado a álgebras de matrices ([54]) y recientemente extendido a álgebras de Banach (see [16] y [104]).

Teorema 12 *([16, Teorema 2.2]) Sean A y B álgebras de Banach unitales y $T : A \to B$ una aplicación aditiva. Entonces T preserva fuertemente la inversibilidad si, y sólo si, $T(1)T$ es un homomorfismo de Jordan unital y $T(1)$ conmuta con $T(A)$.*

De hecho, [16, 104] fueron el punto de partida de las caracterizaciones de tipo Hua en álgebras de Banach.

Teorema 13 *([104, Teorema 2.1]) Sean A y B álgebras de Banach y $T : A \to B$ un homomorfismo de Jordan. Las siguientes afirmaciones se cumplen:*

(1) T preserva fuertemente la inversibilidad generalizada,

(2) T preserva fuertemente la inversibilidad de grupo,

(3) T preserva fuertemente la inversibilidad Drazin,

(4) Si A y B son unitales y T unital, entonces T preserva fuertemente la inversibilidad.

Mediante el uso de la identidad de Hua, Boudi y Mbekhta caracterizan en [16] las aplicaciones lineales que preservan la inversibilidad generalizada, Drazin o de grupo con algunas restricciones sobre la imagen de la unidad en la aplicación.

Teorema 14 *([16, Teorema 4.2]) Sean A y B álgebras de Banach unitales y $T : A \to B$ una aplicación aditiva. Si T es unital (respectivamente, $T(1)$ es inversible, $1 \in T(A)$), entonces las siguientes afirmaciones son equivalentes:*

(1) T preserva fuertemente la inversibilidad generalizada,

(2) T preserva fuertemente la inversibilidad Drazin,

(3) T preserva fuertemente la inversibilidad de grupo,

(4) T (respectivamente, $T(1)T$) es un homomorfismo de Jordan unital y $T(1)$ conmuta con $T(A)$.

Los autores de [16] conjeturan que $T(1)T$ es un homomorfismo de Jordan sin ninguna condición sobre $T(1)$.

Conjetura 1 *([16, Conjetura 4.6]) Sean A y B álgebras de Banach unitales y $T : A \to B$ una aplicación aditiva. Entonces T preserva fuertemente la inversibilidad generalizada (respectivamente, Drazin, de grupo) si, y sólo si, $T(1)T$ es un homomorfismo de Jordan y $T(1)$ conmuta con $T(A)$.*

Querríamos mencionar que algunos autores habían considerado antes aplicaciones que preservan fuertemente ciertos tipos de inversibilidad generalizada. En [23] y [41], los autores estudian aplicaciones lineales entre álgebras de matrices sobre cuerpos con al menos cinco elementos, o anillos conexos conmutativos y unitales, que preservan fuertemente la inversibilidad Drazin y de grupo. Después, en [47], Cui estudió las aplicaciones aditivas entre álgebras de operadores definidos en un espacio de Hilbert infinito-dimensional que preservan fuertemente la inversibilidad Drazin, suponiendo que la imagen de la aplicación contiene a los idempotentes minimales.

Las aplicaciones lineales que preservan fuertemente la inversibilidad de Moore-Penrose en álgebras de matrices fueron consideradas por Zhang, Cao y Bu en [136]. En [104], Mbekhta mostró que toda aplicación lineal, unital y continua $T : A \to B$ entre C*-álgebras unitales que preserva fuertemente la inversibilidad de Moore-Penrose es un homomorfismo de Jordan y preserva proyecciones. Argumentos estándar muestran que T también preservan la ortogonalidad de las proyecciones. Teniendo en cuenta que todo elemento autoadjunto de una C*-álgebra de rango real cero se puede aproximar por combinaciones lineales finitas de idempotentes ortogonales, probó que una aplicación lineal *sobreyectiva*, *unital* y *continua* de una C*-álgebra unital de rango real cero a una C*-álgebra prima preserva fuertemente la inversibilidad de Moore-Penrose si, y sólo si, es un *-homomorfismo o un *-anti-homomorfismo. De hecho, el autor observó en [106] que la continuidad de la aplicación en [104, Teorema 3.2] es una consecuencia más que una hipótesis:

Teorema 15 *([106, Teorema 5.1]) Sean A y B C*-álgebras unitales, donde A es de rango real cero y B es prima. Sea $T : A \to B$ una aplicación lineal, unital y sobreyectiva. Las siguientes afirmaciones son equivalentes:*

(1) T preserva fuertemente la inversibilidad de Moore-Penrose,

*(2) T es un *-homomorfismo o un *-anti-homomorfismo.*

En tal caso, T es continua.

El autor conjetura que la misma conclusión es válida sin suponer condiciones adicionales para las C*-álgebras ni la unitalidad de la aplicación ([106, Conjetura 5.1]).

(V) Invariantes aproximados

Los autores de [76] y [85] consideran el problema de caracterizar aplicaciones lineales aproximadamente multiplicativas entre todos los funcionales lineales de un álgebra de Banach conmutativa en términos espectrales. En [2] (véase también [3]) Alaminos, Extremera y Villena investigan una versión aproximada de la conjetura de Kaplansky, obteniendo versiones aproximadas de los teoremas de Jafarian y Sourour ([73]) y Sourour ([131]). Consideran aplicaciones lineales que preservan aproximadamente el espectro o el radio espectral en álgebras de operadores y establecen una relación entre preservar aproximadamente el espectro (respectivamente, radio espectral) y ser "casi" un homomorfismo de Jordan (respectivamente, homomorfismo de Jordan con peso). Siguiendo el trabajo [77], miden cómo de cerca está una aplicación lineal $T : A \to B$ entre álgebras de Banach de ser multiplicativa por medio de la llamada multiplicatividad de T definida por

$$\mathrm{mult}(T) = \sup\{\|T(ab) - T(a)T(b)\| : a, b \in A, \quad \|a\| = \|b\| = 1\}.$$

Obviamente, T es un homomorfismo si, y sólo si, $\mathrm{mult}(T) = 0$. De forma similar, la anti-multiplicatividad y la Jordan multiplicatividad son, respectivamente, definidas como sigue:

$$\mathrm{amult}(T) = \sup\{\|T(ab) - T(b)T(a)\| : a, b \in A, \quad \|a\| = \|b\| = 1\},$$

$$\mathrm{jmult}(T) = \sup\{\|T(a^2) - T(a)^2\| : a, b \in A, \quad \|a\| = 1\}.$$

Finalmente, si A y B son C*-álgebras, las siguientes cantidades, respectivamente llamadas triple multiplicatividad y autoadjunción, nos permiten medir cómo de cerca está una aplicación lineal $T : A \to B$ de ser un triple homomorfismo o, respectivamente, una aplicación autoadjunta:

$$\mathrm{tmult}(T) = \sup\{\|T(\{a, b, c\}) - \{T(a), T(b), T(c)\}\| : a, b, c \in A, \quad \|a\| = \|b\| = \|c\| = 1\},$$

$$\mathrm{sa}(T) = \sup\{\|T(a^*) - T(a)^*\| : a, b \in A \quad \|a\| = 1\}.$$

Nuestra contribución

En esta sección describimos brevemente los resultados principales que se presentan en esta memoria, relacionados con los problemas de invariantes lineales presentados anteriormente.

Generalización del teorema de Hua

En [43], Chebotar, Ke, Lee y Shiao mejoraron el teorema de Hua relajando la condición $T(a^{-1}) = T(a)^{-1}$ a

$$T(a)T(a^{-1}) = T(b)T(b^{-1}),$$

para todo $a, b \in A^{-1}$. Mostraron que toda aplicación aditiva y bijectiva $T : K \to K$ en un anillo de división K, satisfaciendo la condición anterior es de la forma $T = T(1)S$, donde $S : K \to K$ es un isomorfismo o anti-isomorfismo y $T(1)$ está en el centro de K. Este resultado fue trasladado después a álgebras de matrices en [91]. La Sección 3.1 está dedicada a mover este resultado al ambiente de álgebras de Banach. Sean A y B álgebras de Banach con A unital y $T : A \to B$ una aplicación aditiva tal que $T(a)T(a^{-1}) = T(b)T(b^{-1})$ para cualesquiera $a, b \in A^{-1}$. Si $T(1)$ es inversible

Drazin (o $T(A) \cap B^{-1} \neq \emptyset$), probamos que $T(1)^D T$ es un homomorfismo de Jordan y $T(1)^D$ conmuta con $T(A)$ (Teorema 3.1.3 y Proposición 3.1.7). Además mostramos, por medio de un contraejemplo, que la condición de que $T(1)$ sea inversible Drazin no es suficiente para obtener la implicación recíproca (Ejemplo 3.1.9). Sin embargo, si $T(1)$ es inversible, el recíproco se cumple (Teorema 3.1.4).

Invariantes fuertes en álgebras de Banach y C*-álgebras

La Sección 3.2 contiene una respuesta positiva a la Conjetura 1.1.15 para los casos de inversibilidad Drazin y de grupo. En particular, probamos que una aplicación aditiva $T : A \to B$ entre álgebras de Banach, donde A es unital, preserva fuertemente inversibilidad Drazin (equivalentemente, de grupo) si, y sólo si, es un triple homomorfismo de Jordan (Teorema 3.2.4). Además, presentamos un contraejemplo que muestra que lo mismo no es cierto para el caso de inversibilidad generalizada (Ejemplo 3.2.6).

Las aplicaciones lineales entre C*-álgebras que preservan fuertemente la inversibilidad de Moore-Penrose se estudian en la Sección 4.1. Probamos que todo *-homomorfismo de Jordan entre C*-álgebras preserva fuertemente la inversibilidad de Moore-Penrose (Observación 4.1.8). Recíprocamente, si $T : A \to B$ es una aplicación lineal entre C*-álgebras, siendo A unital, que preserva fuertemente la inversibilidad de Moore-Penrose, entonces $T(1)T$ es un *-homomorfismo de Jordan y $T(1)$ conmuta con $T(A)$ en los siguientes casos:

- A está linealmente generada por sus proyecciones (Teorema 4.1.6),
- A es de rango real cero y T es continua (Corolario 4.1.9),
- A tiene zócalo esencial y T es biyectiva (Teorema 4.1.13).

Además, presentamos un ejemplo que muestra que estas conclusiones no pueden esperarse para el caso en que A es una C*-álgebra unital en general (Ejemplo 4.1.15).

En la Sección 4.3, caracterizamos las aplicaciones que preservan fuertemente la regularidad (en el sentido de los sistemas de Jordan) y, en consecuencia, determinamos las aplicaciones lineales autoadjuntas que preservan fuertemente la inversibilidad de Moore-Penrose. En particular, probamos que una aplicación lineal entre C*-álgebras $T : A \to B$, donde A es unital, preserva fuertemente la regularidad si, y sólo si, T es un triple homomorfismo (Teorema 4.3.5). Además, en este caso, T es automáticamente continua (Corolario 4.3.4).

El concepto de inverso según un elemento fue introducido recientemente por X. Mary en [99]. Recuérdese que un elemento a en un álgebra de Banach A es inversible según $d \in A$ si existen $b \in A$ y $x, y \in A \cup \{1\}$ tales que $bad = d = dab$ y $b = xd = dy$. En tal caso, b es llamado el inverso de a según d y se denota por $b = a^{\| d}$. Esta noción engloba a algunos de los conceptos de inversibilidad generalizada clásicos, como la inversibilidad Drazin, de grupo y de Moore-Penrose. Esto nos ha movido a investigar las aplicaciones que preservan la inversibilidad según un elemento. En la Sección 3.3 caracterizamos los homomorfismos triples de Jordan entre álgebras de Banach como aquellas aplicaciones aditivas que preservan fuertemente la inversibilidad según un elemento y, además, estudiamos en la Sección 4.2 las aplicaciones lineales que preservan el inverso según el adjunto. Para una aplicación aditiva $T : A \to B$ entre álgebras de Banach, donde A es unital, las siguientes afirmaciones son equivalentes (Teoremas 3.3.1 y 4.2.1):

(1) T es un homomorfismo triple de Jordan,

(2) $T(a^{\|1}) = T(a)^{\|T(1)}$ para todo $a \in A^{-1}$,

(3) $T(a^{\|a}) = T(a)^{\|T(a)}$ para todo $a \in A^{\sharp}$,

(4) $T(a^{\|d}) = T(a)^{\|T(d)}$ para todo $d \in A^{\wedge}$ y $a \in A^{\|d}$,

(5) Si A es una C*-álgebra y T es lineal, $T(a^{\|a^*}) = T(a)^{\|T(a^*)}$ para todo $a \in A^{\dagger}$.

Invariantes fuertes en JB*-triples

La noción de regularidad tiene sentido en una clase de espacios de Banach más amplia que la de las C*-álgebras: la de los llamados JB*-triples, introducidos por Kaup en [80]. En el Capítulo 5 estudiamos nuevas clases de invariantes lineales entre C*-álgebras y JB*-triples. Sean E y F JB*-triples. Demostramos que toda aplicación lineal $T : E \to F$ que preserva fuertemente la regularidad es un triple homomorfismo en los siguientes casos:

- $\partial_e(E_1) \neq \emptyset$ (Teorema 5.1.3),
- E es débilmente compacto y T es acotada (Teorema 5.2.1).

Estudiamos aplicaciones lineales $T : A \to B$ entre C*-álgebras unitales que preservan fuertemente la casi-inversibilidad de Brown-Pedersen, los pares Bergmann-cero, la casi-inversibilidad de Brown-Pedersen y los puntos extremos de la bola unidad. Además, exploramos las conexiones entre dichos invariantes.

$$\begin{array}{ccc}
\begin{array}{c} T \text{ preserva} \\ \text{pares Bergmann-cero}\end{array} & \Longrightarrow & \begin{array}{c} T \text{ preserva} \\ \text{elementos BP casi-inversibles}\end{array} \\
\Big\Downarrow & & \not\Downarrow \quad \Big\Uparrow \\
T \text{ preserva puntos extremos} & \begin{array}{c}\Longleftarrow \\ \not\Longrightarrow\end{array} & \begin{array}{c} T \text{ preserva fuertemente} \\ \text{elementos BP casi-inversibles}\end{array}
\end{array}$$

Por ejemplo, probamos que si $T = uS$, donde $S : A \to B$ es un *-homomorfismo de Jordan unital, $C^*(S(A)) = B$ y $u \in B$ es unitario, entonces T preserva puntos extremos (Proposición 5.3.3). Recíprocamente, se muestra que si T preserva puntos extremos y $T(1)$ es unitario, entonces existe un *-homomorfismo de Jordan $S : A \to B$ tal que $T = T(1)S$ (Proposición 5.3.1). Más aún, proporcionamos un contraejemplo que señala que no podemos esperar para C*-álgebras generales las mismas conclusiones que encontraron Mascioni y Mólnar para aplicaciones lineales que preservan puntos extremos entre factores de von Neumann (Observaciones 5.3.4 y 5.3.5). Finalmente, si E y F son JB*-triples con $\partial_e(E_1) \neq \emptyset$, obtenemos que toda aplicación lineal $T : E \to F$ que preserva fuertemente la casi-inversibilidad de Brown-Pedersen es un triple homomorfismo (Teorema 5.3.7).

Invariantes aproximados

Consideramos versiones aproximadas del teorema de Hua para álgebras de Banach y C*-álgebras. En la Sección 3.4, cambiamos la condición de preservar fuertemente la inversibilidad $T(a^{-1}) = T(a)^{-1}$ por

$$\sup_{||a||=1, a \in A^{-1}} ||T(a^{-1}) - T(a)^{-1}|| < \varepsilon,$$

y la condición $T(a^{\sharp}) = T(a)^{\sharp}$ por

$$\sup_{||a||=1, a \in A^{\sharp}} ||T(a^{\sharp}) - T(a)^{\sharp}|| < \varepsilon,$$

para algún $\varepsilon > 0$. Demostramos que para cualesquiera álgebras de Banach unitales A y B, si $\varepsilon \to 0$ en alguna de las desigualdades anteriores, entonces $\mathrm{jmult}(T(1)T) \to 0$, uniformemente en aplicaciones $T : A \to B$ con normas acotadas por arriba (Teoremas 3.4.4 y 3.4.6).

De manera similar, en la Sección 4.4 cambiamos la condición de preservar fuertemente la regularidad $T(a^{\wedge}) = T(a)^{\wedge}$ por

$$\sup_{||a||=1, a \in A^{\wedge}} ||T(a^{\wedge}) - T(a)^{\wedge}|| < \varepsilon,$$

y la condición $\gamma(T(a)) = \gamma(a)$ por

$$\sup_{||a||=1} ||\gamma(T(a)) - \gamma(a)|| < \varepsilon,$$

para algún $\varepsilon > 0$. Mostramos que para cualesquiera C*-álgebras unitales A y B, si $\varepsilon \to 0$ entonces $\mathrm{tmult}(T) \to 0$ en el primer caso, uniformemente en aplicaciones lineales $T : A \to B$ con normas acotadas por arriba (Teorema 4.4.3). Finalmente, también se muestra que las aplicaciones lineales entre C*-álgebras unitales que son aproximadamente unitales y preservan aproximadamente la conorma, son aproximadamente *-isomorfismos de Jordan (Teorema 4.4.8). El caso no unital y sobreyectivo conduce a conclusiones similares (Teorema 4.4.9).

Invariantes de órdenes parciales

La Sección 6.1 se centra en el estudio de las aplicaciones que preservan el orden sharp. Probamos que todo homomorfismo de Jordan entre álgebras de Banach preserva dicho orden (Lema 6.1.2). Recíprocamente, si $T : A \to B$ es una aplicación lineal entre álgebras de Banach unitales que preserva el orden sharp, probamos que:

- Si A es semisimple con zócalo esencial y T es biyectiva, entonces T es un isomorfismo de Jordan multiplicado por un elemento central e inversible (Teorema 6.1.7),
- Si A es una C*-álgebra de rango real cero y T es continua, entonces $T = T(1)S$, donde S es un homomorfismo de Jordan y $T(1)$ conmuta con $S(A)$ (Teorema 6.1.8).

Además, proporcionamos un contraejemplo para mostrar que lo resultados anteriores no son válidos para C*-álgebras en general (Ejemplo 6.1.9).

Para un álgebra de Banach A y $a, b \in A$, introducimos una nueva relación que extiende a la del orden sharp a toda el álgebra: $a \leq_s b$ si existe un idempotente $p \in A$ tal que $a = pb = bp$. Obtenemos conclusiones similares a aquéllas del Lema 6.1.2, Teorema 6.1.7 y Teorema 6.1.8.

En la Sección 6.2 introducimos la relación "$\leq$" para una C*-álgebra A: $a \leq b$ si existen p, q proyecciones en A tales que $a = pb = bq$. Esta relación es equivalente al orden star en C*-álgebras de Rickart. Para C*-álgebras generales, se tiene que $a \leq b$ implica $a \leq_* b$ y que el recíproco es cierto para elementos regulares. A fin de caracterizar los invariantes lineales de la relación "$\leq$", estudiamos bajo qué circunstancias éstos preservan la ortogonalidad, para poder describirlos utilizando el Teorema 1.1.6. Si A es una C*-álgebra unital, B es una C*-álgebra y $T : A \to B$ es una aplicación lineal que preserva la relación "$\leq$", entonces T preserva ortogonalidad si:

- A es la expansión lineal de sus proyecciones (Teorema 6.2.7),
- A es de rango real cero y T es acotada (Teorema 6.2.8).

Además, mejoramos el Teorema 1.1.7 mostrando que la condición $\{T(1)\}^{\perp} = \{0\}$ es redundante. Esto nos permite probar que toda aplicación lineal y biyectiva entre una C*-álgebra unital con zócalo esencial y una C*-álgebra que preserva "$\leq$" es un *-homomorfismo de Jordan multiplicado por un elemento central e inversible (Corolario 6.2.5).

La Sección 6.3 está dedicada al estudio de los invariantes lineales del orden minus. Adoptamos la definición de [49]: $a \leq^- b$ si existen idempotentes $p, q \in A$ tales que $\mathrm{ann}_l(a) = \mathrm{ann}_l(p)$, $\mathrm{ann}_r(a) = \mathrm{ann}_r(q)$, $pa = pb$ y $aq = bq$. Probamos algunas propiedades algebraicas de esta relación y mostramos que "$\leq^-$" define un orden parcial en el conjunto de los elementos regulares de un anillo semiprimo (Corolario 6.3.5). Caracterizamos los elementos maximales dentro de los regulares con respecto a este orden en anillos primos unitales (Proposición 6.3.13). También determinamos los elementos minimales en álgebras de Banach semisimples unitales con zócalo esencial (Proposición 6.3.15). Si A y B son álgebras de Banach semisimples unitales con zócalo esencial, probamos que toda aplicación lineal y biyectiva $T : A \to B$ tal que $T(A^\wedge) = B^\wedge$ y $a \leq^- b \Leftrightarrow T(a) \leq^- T(b)$, para todo $a, b \in A^\wedge$ es un isomorfismo de Jordan multiplicado por un elemento inversible (Teorema 6.3.20). La condición $T(A^\wedge) = B^\wedge$ puede ser eliminada cuando $B = \mathcal{B}(X)$ para un espacio de Banach complejo X (Teorema 6.3.22) o cuando B es una C*-álgebra prima (Teorema 6.3.24). También consideramos brevemente las aplicaciones lineales que preservan el orden minus en una dirección. Probamos en el Teorema 6.3.25 que, si A es una C*-álgebra de rango real cero, B es un álgebra de Banach y $T : A \to B$ es una aplicación lineal y continua que preserva el orden minus, entonces T es un homomorfismo de Jordan (respectivamente,un homomorfismo de Jordan multiplicado por un elemento inversible) si $T(1)$ es idempotente (respectivamente, $T(A) \cap B^{-1}$ y $T(1) \in B^\wedge$).

Finalmente, estudiamos en la Sección 6.4 el orden diamond en C*-álgebras y las aplicaciones lineales entre C*-álgebras unitales que lo preservan. Mostramos que éste es un orden parcial en C*-álgebras y describimos algunos elementos distinguidos como los maximales y los minimales (Proposiciones 6.4.2, 6.4.4 y 6.4.5, respectivamente). También caracterizamos las proyecciones y los múltiplos escalares de isometrías y coisometrías en términos del orden diamond (Proposiciones 6.4.3 y 6.4.7, respectivamente). Estos resultados serán aplicados después al estudio de los invariantes lineales del orden diamond en C*-álgebras. Todo *-homomorfismo de Jordan preserva el orden diamond en los elementos regulares (Proposición 6.4.10). En el Teorema 6.4.12 se prueba que toda aplicación lineal y sobreyectiva $T : A \to B$ entre C*-álgebras unitales con zócalo esencial (siendo B prima), que preserva el orden diamond en ambas direcciones, es un múltiplo apropiado de un *-homomorfismo de Jordan. Además, en el Teorema 6.4.14 se prueba que, si A es una C*-álgebra unital de rango real cero, B es una C*-álgebra y $T : A \to B$ es una aplicación lineal y continua que preserva el orden diamond, entonces T es un *-homomorfismo de Jordan (respectivamente, un *-isomorfismo de Jordan multiplicado por un elemento unitario) si $T(1)$ es una proyección (respectivamente, $T(A) \cap B^{-1}$ y $T(1)$ es una isometría parcial).

Chapter 1

Introduction

1.1 Background on linear preservers

The study of the so-called *linear preserver problems* deals with the characterization of linear (or, more generally, additive) maps between Banach algebras that leave certain properties, functions, subsets or relations invariant. The earliest paper on linear preserver problems date back to 1897, and a great deal of effort has been devoted to this research line since then. Nowadays, the study of linear preserver problems is one of the most active and fertile research topics in matrix theory, operator theory and functional analysis.

In the following, A and B will denote complex Banach algebras and $T : A \to B$ a linear map between them. We start presenting a brief survey on linear preserver problems, by reviewing the main research lines and pointing out the some of the most famous results concerning every line.

(I) Function preservers

Given a scalar, vector or set valued function F, characterize linear maps T such that $F_B(T(a)) = F_A(a)$. For $F(x) = \det(x)$ and $A = B = M_n(\mathbb{C})$, the classical result of Frobenius ([59]) states that every linear map $\phi : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$ that preservers the determinant, that is, such that $\det(\phi(A)) = \det(A)$ for all $A \in M_n(\mathbb{C})$, takes one of the following forms:

$$\phi(A) = MAN, \qquad \text{for every } A \in M_n(\mathbb{C}) \quad \text{or}$$

$$\phi(A) = MA^tN, \qquad \text{for every } A \in M_n(\mathbb{C}).$$

For $F(x) = \|x\|$ and A, B C*-algebras, Kadison proved in [78] that every surjective linear map $T : A \to B$ is an isometry if, and only if, T is a Jordan *-isomorphism multiplied by a unitary element in B.

Let F stand for the minimum or surjectivity (respectively, reduced minimum) moduli and $A = B = \mathcal{B}(H)$, the algebra of all bounded linear operators on a Hilbert space H, Mbekhta showed in [105] (respectively, [103]) that every unital surjective linear map from $\mathcal{B}(H)$ onto itself and preserving F is an *-automorphism (respectively, *-automorphism or *-anti-automorphism). The authors of [18] studied this problem in the context of unital C*-algebras.

Theorem 1.1.1 *([18, Theorem 3.1]) Let A and B be unital C*-algebras. If $T : A \to B$ is unital linear map such that $F(T(x)) = F(x)$ for all $x \in A$, then T is an isometric Jordan *-homomorphism.*

Theorem 1.1.2 *([18, Theorem 3.2]) Let A be a unital semisimple Banach algebra and let B be a unital C*-algebra. If $T : A \to B$ is surjective linear map such that $F(T(x)) = F(x)$ for all $x \in A$, then A is a C*-algebra and T is an isometric Jordan *-homomorphism multiplied by a unitary element.*

Further examples in more general structures could be numerical range ([117]) and ascent or descent of operators ([10]).

(II) Subset preservers

Given a subset $S_A \subset A$, determine the set of linear maps that leave that subset invariant, namely, those maps $T : A \to B$ such that $T(S_A) \subset S_B$. This problem can be also studied with the stronger condition $T(S_A) = S_B$, in which case it is said that the map preserves S in both directions. When S is the set of all invertible elements and A and B are unital semisimple Banach algebras, we address the acclaimed *Kaplansky's conjecture*. Recall that this conjecture is nowadays stated as follows: a unital bijective linear map $T : A \to B$ between unital semisimple Banach algebras that preserves invertibility is a Jordan isomorphism (note that, as T is unital, to preserve invertibility is equivalent to compress the spectrum, i.e., $\sigma(T(a)) \subset \sigma(a)$ for all $a \in A$). The problem is still open even for C*-algebras. Gleason (see [60]) and Kahane-Zelazko (see [79]) proved almost at the same time that if A and B are Banach algebras, with B commutative and semisimple, every unital invertibility preserving map is a homomorphism. In [73], Jafarian and Sourour showed that Jordan isomorphisms are the only bijective unital linear mappings between operator algebras that preserve invertibility in both directions. In [7], Aupetit confirmed the conjecture for von Neumann algebras (see Theorem 1.2 and Remark 2.7 in [7]). Later, Cui and Hou proved in [48] that the conjecture has positive answer when A is a real rank zero C*-algebra.

Theorem 1.1.3 *([48, Theorem 3.1]) Let A be a unital real rank zero C*-algebra and B a unital semisimple Banach algebra. Suppose that $T : A \to B$ is a surjective linear map preserving invertibility. Then T is a Jordan isomorphism.*

There is also another favourable setting in which the Kaplansky's problem has an affirmative answer. In [19], Brešar, Fošner and Šemrl, showed the following:

Theorem 1.1.4 *([19, Theorem 1.1]) Let A and B be unital semisimple Banach algebras, having A essential socle. Let $T : A \to B$ a unital bijective linear map preserving invertibility. Then T is a Jordan isomorphism.*

V. Mascioni and L. Molnár studied the linear maps on a von Neumann factor A which preserve the extreme points of the unit ball of A.

Theorem 1.1.5 *([101, Theorem 1]) Let A be an infinite factor. The linear map $T : A \to A$ preserves the extreme points of the unit ball of A if, and only if, there are a unitary element $u \in A$ and a unital *-homomorphism $S : A \to A$ such that T is of the form $T(a) = uS(a)$ for all $a \in A$ or there are a unitary element $u' \in A$ and a unital *-anti-homomorphism $S' : A \to A$ such that T is of the form $T(a) = u'S'(a)$ for all $a \in A$.*

In [87], L. E. Labuschagne and V. Mascioni studied linear maps between C*-algebras whose adjoints preserve extreme points of the dual ball.

Other important subsets that have been considered are the sets of idempotent ([86]), (semi-)Fredholm and generalized invertible elements ([11], [17], [107]).

(III) Relation preservers

Given a binary relation $\sim$ in A and B, study linear maps preserving the relation $\sim$, that is, those maps fulfilling $T(a) \sim T(b)$ whenever $a \sim b$. Examples of such relations can be commutativity ([45]), orthogonality ([4, 25, 36, 38, 135]), zero product ([1, 39, 42]), partial order relations ([62, 63, 89, 129]), etc. When we impose the condition $T(a) \sim T(b)$ if, and only if, $a \sim b$ it is said that T preserves $\sim$ in both directions. Of particular interest for this monograph are orthogonality preservers and partial order preservers.

The study of orthogonality preserving bounded linear maps between C*-algebras began with the work of W. Arendt [4] in the setting of unital abelian C*-algebras. Orthogonality preserving linear operators between general C*-algebras were first considered by M. Wolff in [135]. He proved that every orthogonality preserving bounded selfadjoint linear map between C*-algebras is a multiple of a Jordan *-homomorphism.

Under continuity assumptions, orthogonality preserving (bounded) linear maps between general C*-algebras were completely described in [27] and [36]:

Theorem 1.1.6 *([36, Theorem 17 and Corollary 18]) Let $T : A \to B$ be a bounded linear mapping between two C*-algebras. For $h = T^{**}(1)$ and $r = r(h)$ the following assertions are equivalent.*

(1) T is orthogonality preserving,

*(2) There exists a unique Jordan *-homomorphism $S : A \to B_2^{**}(r)$ satisfying that $S^{**}(1) = r$ and $T(z) = hr^*S(z) = S(z)r^*h$ for all $z \in A$,*

(3) T preserves zero triple products, that is, $\{T(x), T(y), T(z)\} = 0$ whenever $\{x, y, z\} = 0$.

In [38], Burgos, Garcés and Peralta proved that every biorthogonality preserving surjective linear map between compact C*-algebras or von Neumann algebras is automatically continuous. As every selfadjoint linear mapping between C*-algebras is orthogonality preserving whenever it preserves zero products, it follows that every symmetric biseparanting linear map between von Neumann algebras is automatically continuous.

In [25], the author studied orthogonality preserving bijective linear maps from a unital C*-algebra with essential socle to a C*-algebra.

Theorem 1.1.7 *([25, Theorem 3.2]) Let A and B be C*-algebras. Suppose that A is unital with essential socle. Let $T : A \to B$ be an orthogonality preserving bijective linear map with $\{T(1)\}^{\perp} = \{0\}$. Then B is unital and T is a Jordan *-isomorphism multiplied by an invertible element.*

In 1993, Ovchinnikov studied maps preserving the order of idempotents in $\mathcal{B}(H)$. He obtained that such a map has either the form $P \mapsto APA^{-1}$ or the form $P \mapsto AP^*A^{-1}$ for every idempotent operator P, where A is a linear or conjugate linear bijection on H ([118]).

In 2001, Guterman studied linear maps between matrix algebras that preserve the star partial order ([62]). He obtained the following result:

Theorem 1.1.8 *([62]) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $T : M_n(\mathbb{K}) \to M_n(\mathbb{K})$ be a bijective linear map preserving the star partial order. Then there exist a nonzero $\alpha \in \mathbb{K}$ and unitary matrices $U, V \in M_n(\mathbb{K})$ such that $T(A) = \alpha UAV$ for all $A \in M_n(\mathbb{K})$ or $T(A) = \alpha UA^tV$.*

Later, Šemrl obtained in [128] an improved version of Ovchinnikov's result for matrix algebras. He noted that classical results from projective geometry and extensions of these results in functional analysis allow to drop the assumption of linearity and even additivity. Legiša made use of Šemrl's idea to obtain a non-linear version of Theorem 1.1.8.

Additive maps preserving the star, left-star and right-star orders between real and complex matrix algebras have been studied by Guterman in 2007 (see [63]). He shows that every additive map $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ preserving the star partial order has one of the following forms: for every $A \in M_n(\mathbb{C})$, $\phi(A) = \alpha UAV$, $\phi(A) = \alpha U\overline{A}V$, $\phi(A) = \alpha UA^tV$ or $\phi(A) = \alpha U\overline{A}^tV$ where $\alpha \in \mathbb{C}$, U and V are unitary matrices, $\bar{A}$ denotes the conjugate matrix of A and A^t its transpose.

More recent papers have been devoted to the study of preservers of star order. In 2013, Dolinar, Guterman and Marovt, studied bijective, additive and continuous mappings on $K(H)$ that preserve the star partial order in both directions, where $K(H)$

stands for the closed ideal of all compact operators on a separable infinite dimensional complex Hilbert space H (see [50]). Recently, the authors of [51] brought some results from [63] concerning left and right star partial orders (which are one-sided versions of the star partial order) to the infinite-dimensional case, following some techniques from [129]. They showed that every bijective additive map $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ preserving the left-star partial order in both directions has the form $\phi(A) = UAS$ for all $A \in \mathcal{B}(H)$, where U is a unitary operator and S is bijective (note that both U and S can be linear or conjugate linear). The expected conclusions are obtained also for the right-partial order.

Mappings preserving the minus partial order were first considered by Legiša in [89].

Theorem 1.1.9 *([89, Theorem 1]) Let $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a bijective map preserving the minus partial order in both directions. Then there exist an automorphism φ of $\mathbb{C}$ and invertible matrices $R, S \in M_n(\mathbb{C})$ such that either $T(A) = RA_\varphi S$ for all $A \in M_n(\mathbb{C})$ or $T(A) = RA_\varphi^t S$ for all $A \in M_n(\mathbb{C})$, where A_φ is a matrix obtained from A by applying φ entrywise, that is, $[a_{ij}]_\varphi = [\varphi(a_{ij})]$.*

In [129], Šemrl brought the notion of minus partial order into the infinite-dimensional setting by dropping the regularity conditions (see Section 2.4). In the same paper, he studied (non necessarily linear) bijective mappings preserving the minus partial order in both directions in $\mathcal{B}(H)$.

Theorem 1.1.10 *([129, Theorem 8]) Let H be an infinite-dimensional Hilbert space. Assume that $T : \mathcal{B}(H) \to \mathcal{B}(H)$ is a bijective map preserving the minus order in both directions. Then there exist bounded both linear or both conjugate-linear maps $R, S : H \to H$ such that either $T(A) = RAS$ for all $A \in \mathcal{B}(H)$ or $T(A) = RA^*S$ for every $A \in \mathcal{B}(H)$.*

(IV) Strongly preserver problems (Hua type problems)

Every unital Jordan homomorphism between Banach algebras *strongly preserves invertibility*, that is, $T(a^{-1}) = T(a)^{-1}$, for every invertible element $a \in A$ (see [131, Proposition 1.3]). In 1949 Hua proved that every unital bijective additive map $T : K \to K$ on a division ring K such that $T(aba) = T(a)T(b)T(a)$ is either an automorphism or an anti-automorphism (see [70]). This result was reformulated by Artin in 1957 as follows:

Theorem 1.1.11 *([5, Theorem 1.15]) Every unital bijective additive map $T : K \to K$ on a division ring K such that $T(a^{-1}) = T(a)^{-1}$ is an automorphism or an anti-automorphism.*

Hua's theorem was generalized to matrix algebras ([54]) and recently extended to Banach algebras (see [16] and [104]).

Theorem 1.1.12 *([16, Theorem 2.2]) Let A and B be unital Banach algebras and let $T : A \to B$ be an additive map. Then T strongly preserves invertibility if, and only if, $T(1)T$ is a unital Jordan homomorphism and $T(1)$ commutes with $T(A)$.*

In fact, [16, 104] were the starting point of the study of Hua type characterizations for Banach algebras.

Theorem 1.1.13 *([104, Theorem 2.1]) Let A and B be Banach algebras and let $T : A \to B$ be a Jordan homomorphism. The following assertions hold:*

(1) T strongly preserves generalized invertibility,

(2) T strongly preserves group invertibility,

(3) T strongly preserves Drazin invertibility,

(4) if A and B are unital and T is a unital map, then T strongly preserves invertibility.

A rather technical use of Hua's identity allows Boudi and Mbekhta to characterize in [16] those additive maps strongly preserving generalized, Drazin or group invertibility with some restrictions on the image of the identity element.

Theorem 1.1.14 *([16, Theorem 4.2]) Let A and B be unital Banach algebras and $T : A \to B$ an additive map. If T is unital (respectively $T(1)$ is invertible, $1 \in T(A)$), then the following conditions are equivalent:*

(1) T strongly preserves generalized invertibility,

(2) T strongly preserves Drazin invertibility,

(3) T strongly preserves group invertibility,

(4) T (respectively, $T(1)T$) is a unital Jordan homomorphism and $T(1)$ commutes with $T(A)$.

The authors of [16] conjecture that $T(1)T$ is a unital Jordan homomorphism without any assumption on $T(1)$.

Conjecture 1.1.15 *([16, Conjecture 4.6]) Let A and B unital Banach algebras and let $T : A \to B$ be a additive map. Then T strongly preserves generalized invertibility (respectively, Drazin invertibility, group invertibility) if, and only if, $T(1)T$ is a Jordan homomorphism and $T(1)$ commutes with $T(A)$.*

We shall mention that other authors have previously considered mappings strongly preserving generalized invertibility. In [23] and [41], the authors studied linear maps between matrix algebras over some field with at least five elements, or connected commutative unital rings, strongly preserving group invertibility or Drazin invertibility.

Later, in [47], Cui characterized additive maps between algebras of bounded linear operators on infinite dimensional Hilbert spaces strongly preserving Drazin invertibility, assuming that the image contains the set of all minimal idempotents.

Linear maps strongly preserving Moore-Penrose invertibility on matrix algebras over some fields were considered by Zhang, Cao and Bu in [136]. In [104], Mbekhta showed that every unital linear bounded map $T : A \to B$ between unital C*-algebras that strongly preserves Moore-Penrose invertibility is a Jordan homomorphism and preserves projections. Standard arguments show that T also preserves the orthogonality of projections. Having in mind that every selfadjoint element in a real rank zero C*-algebra can be approximated by finite linear combinations of orthogonal projections, he proved that every *surjective*, *unital* and *bounded* linear map from a unital real rank zero C*-algebra to a prime C*-algebra strongly preserves invertibility if, and only if, it is either a *-homomorphism or an *-anti-homomorphism. In fact, the author observes in [106] that the continuity of the map in [104, Theorem 3.2] is a consequence rather than an hypothesis:

Theorem 1.1.16 *([106, Theorem 5.1]) Let A and B be unital C*-algebras, where A is of real rank zero and B is prime. Let $T : A \to B$ be a unital surjective linear map. Then the following conditions are equivalent:*

(1) T strongly preserves Moore-Penrose invertibility,

*(2) T is either a *-homomorphism or a *-anti-homomorphism.*

In this case, T is continuous.

He conjectures that the same holds without any assumption on the C*-algebras and when T is not assumed to be unital ([106, Conjecture 5.1]).

(V) Approximate preservers

The authors of [76] and [85] consider the problem of characterizing the approximately multiplicative linear functional among all linear funtionals on a commutative Banach algebra in terms of spectra. In [2] (see also [3]) Alaminos, Extremera and Villena investigate approximate versions of Kaplansky's problem, by providing approximate formulations of the papers of Jafarian and Sourour ([73]) and Sourour ([131]). They considered linear maps that approximately preserve spectrum or spectral radius on operator algebras, and stablished the relationship between approximately preserving spectrum (respectively, spectral radius) and being "almost" a Jordan homomorphism (respectively, weighted Jordan homomorphism). Following [77], they measure how close is a linear map $T : A \to B$ between Banach algebras from being multiplicative by means of the so-called multiplicativity of T, defined by

$$\operatorname{mult}(T) = \sup\{\|T(ab) - T(a)T(b)\| : a, b \in A, \quad \|a\| = \|b\| = 1\}.$$

Obviously, T is a homomorphism if, and only if, $\mathrm{mult}(T) = 0$. Similarly, the anti-multiplicativity and the Jordan multiplicativity are, respectively, defined as follows:

$$\mathrm{amult}(T) = \sup\{\|T(ab) - T(b)T(a)\| : a, b \in A, \quad \|a\| = \|b\| = 1\},$$

$$\mathrm{jmult}(T) = \sup\{\|T(a^2) - T(a)^2\| : a, b \in A, \quad \|a\| = 1\}.$$

Moreover, if A and B are C*-algebras, it can be measured how close is a linear map $T : A \to B$ to be a triple homomorphism, respectively, a selfadjoint map, by means of the triple multiplicativity, $\mathrm{tmult}(T)$, and the selfadjointness, $\mathrm{sa}(T)$:

$$\mathrm{tmult}(T) = \sup\{\|T(\{a, b, c\}) - \{T(a), T(b), T(c)\}\| : a, b, c \in A, \quad \|a\| = \|b\| = \|c\| = 1\},$$

$$\mathrm{sa}(T) = \sup\{\|T(a^*) - T(a)^*\| : a, b \in A, \quad \|a\| = 1\}.$$

1.2 Our contribution

In this section we briefly describe the main results presented in this monograph, related with the linear preservers problems introduced above.

Generalization of Hua's theorem

In [43], Chebotar, Ke, Lee and Shiao improved Hua's theorem by relaxing the condition $T(a^{-1}) = T(a)^{-1}$ to

$$T(a)T(a^{-1}) = T(b)T(b^{-1}),$$

for all $a, b \in A^{-1}$. They showed that every bijective additive map $T : K \to K$ on a division ring K, satisfying the above condition is of the form $T = T(1)S$, where $S : K \to K$ is an automorphism or an anti-automorphism and $T(1)$ lies in the center of K. This result was later extended to matrix algebras in [91]. In Section 3.1 we consider this problem in the more general setting of unital Banach algebras. Let A and B be Banach algebras with A unital and $T : A \to B$ an additive map satifying $T(a)T(a^{-1}) = T(b)T(b^{-1})$ for all $a, b \in A^{-1}$. Whenever $T(1)$ is Drazin invertible (or $T(A) \cap B^{-1} \neq \emptyset$), we prove that $T(1)^D T$ is a Jordan homomorphism and $T(1)^D$ commutes with $T(A)$ (Theorem 3.1.3 and Proposition 3.1.7). We also show, by means of a counterexample, that the condition of $T(1)$ being Drazin invertible is not enough to get the converse of the previous result (Example 3.1.9). However, if we suppose that $T(1)$ is invertible, the converse holds (Theorem 3.1.4).

Strongly preserver problems in Banach algebras and C*-algebras

Section 3.2 contains a positive answer to Conjecture 1.1.15 for Drazin and group invertibility. In particular, we prove that an additive map $T : A \to B$ between Banach algebras, where A is supposed to be unital, strongly preserves Drazin (equivalently, group) invertibility if, and only if, it is a Jordan triple homomorphism (Theorem 3.2.4).

We also present a counterexample showing that the same does not hold for generalized invertibility (Example 3.2.6).

Linear maps between C*-algebras strongly preserving Moore-Penrose invertibility are studied in Section 4.1. We show that every Jordan *-homomorphism between C*-algebras strongly preserves Moore-Penrose invertibility (Remark 4.1.8). Conversely, if $T : A \to B$ is a linear map between C*-algebras, A being unital, that strongly preserves Moore-Penrose invertibility, then $T(1)T$ is a Jordan *-homomorphism and $T(1)$ commutes with $T(A)$ in the following cases:

- A is linearly spanned by its projections (Theorem 4.1.6),
- A is of real rank zero and T is bounded (Corollary 4.1.9),
- A has essential socle and T is bijective (Theorem 4.1.13).

Furthermore, we present an example to show that these conclusions cannot be expected when A is a general unital C*-algebra (Example 4.1.15).

In Section 4.3, we characterize the linear maps strongly preserving regularity (in the Jordan system's sense) and consequently we determine the structure of selfadjoint linear maps strongly preserving Moore-Penrose invertibility. In particular, we prove that a linear map $T : A \to B$ between C*-algebras, where A is unital, strongly preserves regularity if, and only if, T is a triple homomorphism (Theorem 4.3.5). Moreover, in this case, T is automatically continuous (Corollary 4.3.4).

The notion of inverse along an element was recently introduced by X. Mary in [99]. Recall that an element a in a Banach algebra A is invertible along $d \in A$ if there exist $b \in A$ and $x, y \in A \cup \{1\}$ such that $bad = d = dab$ and $b = xd = dy$. In this case, b is called the inverse of a along d and it is denoted by $b = a^{\| d}$. This concept gathers some of the classical notions of generalized invertibility, such as the group, Drazin and Moore-Penrose inverse. This motivates us to look into mappings strongly preserving the inverse along an element. In Section 3.3 we characterize Jordan triple homomorphisms between Banach algebras as those additive maps that strongly preserve the inverse along an element and, furthermore, we study in Section 4.2 linear maps strongly preserving the inverse along the adjoint. For an additive map $T : A \to B$ between Banach algebras, where A is unital, we find that the following conditions are equivalent (Theorems 3.3.1 and 4.2.1):

(1) T is a Jordan triple homomorphism,

(2) $T(a^{\| 1}) = T(a)^{\| T(1)}$ for all $a \in A^{-1}$,

(3) $T(a^{\| a}) = T(a)^{\| T(a)}$ for all $a \in A^{\sharp}$,

(4) $T(a^{\| d}) = T(a)^{\| T(d)}$ for all $d \in A^{\wedge}$ and $a \in A^{\| d}$,

(5) If A is a C*-algebra and T is linear, $T(a^{\| a^*}) = T(a)^{\| T(a^*)}$ for all $a \in A^{\dagger}$.

Strongly preserver problems in JB*-triples

The notion of regularity makes sense in a wider class of Banach spaces containing the C*-algebras: the so-called JB*-triples introduced by Kaup in [80]. Chapter 5 deals with the study of new classes of linear preservers between C*-algebras and JB*-triples. Let E and F be JB*-triples. We prove that every linear map $T : E \to F$ strongly preserving regularity is a triple homomorphism in the following cases:

- $\partial_e(E_1) \neq \emptyset$ (Theorem 5.1.3),
- E is weakly compact and T is bounded (Theorem 5.2.1).

We study linear maps $T : A \to B$ between unital C*-algebras strongly preserving Brown-Pedersen quasi-invertibility, Bergmann-zero pairs, Brown-Pedersen quasi-invertibility and extreme points of the unit ball. We also explore the connections between these kinds of preservers.

$$\begin{array}{ccc}
\begin{array}{c} T \text{ preserves} \\ \text{Bergmann-zero pairs} \end{array} & \Longrightarrow & \begin{array}{c} T \text{ preserves} \\ \text{BP quasi-invertible elements} \end{array} \\
\Downarrow & & \not\Downarrow \quad \Uparrow \\
T \text{ preserves extreme points} & \begin{array}{c} \Longleftarrow \\ \not\Longrightarrow \end{array} & \begin{array}{c} T \text{ strongly preserves} \\ \text{BP quasi-invertible elements} \end{array}
\end{array}$$

Namely, we prove that if $T = uS$, where $S : A \to B$ is a unital Jordan *-homomorphism satisfying $C^*(S(A)) = B$ and $u \in B$ is a unitary element, then T preserves extreme points (Proposition 5.3.3). Conversely, it is shown that if T preserves extreme points and $T(1)$ is unitary element, then there exists a Jordan *-homomorphism $S : A \to B$ such that $T = T(1)S$ (Proposition 5.3.1). We also provide a counterexample pointing out that we cannot expect for general C*-algebras the conclusions found by Mascioni and Mólnar for linear maps preserving extreme points on von Neumann factors (Remarks 5.3.4 and 5.3.5). Finally, whenever E and F are JB*-triples with $\partial_e(E_1) \neq \emptyset$, we obtain that every linear map $T : E \to F$ strongly preserving Brown-Pedersen quasi-invertibility is a triple homomorphism (Theorem 5.3.7).

Approximate preservers

We consider approximate versions of Hua's theorem for Banach algebras and C*-algebras. In Section 3.4, we translate the strongly invertibility preserving condition $T(a^{-1}) = T(a)^{-1}$ into

$$\sup_{||a||=1, a \in A^{-1}} ||T(a^{-1}) - T(a)^{-1}|| < \varepsilon,$$

and the condition $T(a^\sharp) = T(a)^\sharp$ into

$$\sup_{||a||=1, a \in A^\sharp} ||T(a^\sharp) - T(a)^\sharp|| < \varepsilon,$$

for some $\varepsilon > 0$. We prove that for every unital Banach algebras A and B, if $\varepsilon \to 0$ in one of the previous inequalities, then $\mathrm{jmult}(T(1)T) \to 0$, uniformly on any set of linear maps $T : A \to B$ with norms bounded above (Theorems 3.4.4 and 3.4.6).

Similarly, in Section 4.4 we translate the strongly regularity preserving condition $T(a^\wedge) = T(a)^\wedge$ into

$$\sup_{||a||=1, a\in A^\wedge} ||T(a^\wedge) - T(a)^\wedge|| < \varepsilon,$$

and the condition $\gamma(T(a)) = \gamma(a)$ into

$$\sup_{||a||=1} ||\gamma(T(a)) - \gamma(a)|| < \varepsilon,$$

for some $\varepsilon > 0$. It is shown that for every unital C*-algebras A and B, if $\varepsilon \to 0$ then $\mathrm{tmult}(T) \to 0$ in the first case uniformly on any set of linear maps $T : A \to B$ with norms bounded above (Theorem 4.4.3). Finally, it is also shown that linear maps between unital C*-algebras that are approximately unital and approximately preserve the conorm, are approximately Jordan *-homomorphisms (Theorem 4.4.8). The non-unital, surjective case yields a similar result (Theorem 4.4.9).

Partial order preservers

Section 6.1 is devoted to the study of linear maps preserving the sharp partial order. We show that every Jordan homomorphism between Banach algebras preserves the sharp partial order (Lemma 6.1.2). Reciprocally, if $T : A \to B$ is a linear map between unital Banach algebras that preserve the sharp partial order, we prove that:

- If A is semisimple with essential socle and T is bijective, then T is a Jordan isomorphism multiplied by a central invertible element (Theorem 6.1.7),
- If A is a real rank zero C*-algebra and T is bounded, then $T = T(1)S$, where S is a Jordan homomorphism and $T(1)$ commutes with $S(A)$ (Theorem 6.1.8).

In addition, we provide a counterexample showing that the previous results are no longer true in general C*-algebras (Example 6.1.9).

For a Banach algebra A and $a, b \in A$, we also present a new relation that extends the sharp partial order to the full algebra: $a \leq_s b$ if there exists an idempotent $p \in A$ such that $a = pb = bp$. We prove similar conclusions to those in Lemma 6.1.2, Theorem 6.1.7 and Theorem 6.1.8.

In Section 6.2 we introduce the relation "$\leq$" for a C*-algebra A: $a \leq b$ if there exist p, q projections in A such that $a = pb = bq$. This relation is equivalent to the star partial order in Rickart C*-algebras. For general C*-algebras, we have that $a \leq b$ implies $a \leq_* b$ and that the reciprocal holds for regular elements. In order to characterize linear maps preserving "$\leq$", we study under what circumstances they preserve orthogonality, so we can make use of Theorem 1.1.6. If A is a unital C*-algebra, B is a C*-algebra and

$T : A \to B$ is a linear map preserving the relation "$\leq$", then T preserves orthogonality whenever:

- A is the linear span of its projections (Theorem 6.2.7),
- A has real rank zero and T is bounded (Theorem 6.2.8).

We also improve Theorem 1.1.7 by showing that the condition $\{T(1)\}^{\perp} = \{0\}$ is redundant. This allows us to prove that every bijective linear map from a unital C*-algebra with essential socle to a C*-algebra that preserves "$\leq$" is a Jordan *-homomorphism multiplied by a central invertible element (Corollary 6.2.5).

Section 6.3 is devoted to the study of linear preserver of the minus partial order. We adopt the definition from [49]: $a \leq^- b$ if there exist idempotent elements $p, q \in A$ such that $\mathrm{ann}_l(a) = \mathrm{ann}_l(p)$, $\mathrm{ann}_r(a) = \mathrm{ann}_r(q)$, $pa = pb$ and $aq = bq$. We prove some algebraic properties of this relation and show that "$\leq^-$" defines a partial order on the set of all regular elements of a semiprime ring (Corollary 6.3.5). The space pre-order in general unital rings (Definition 6.3.6) is also considered. We characterize the maximal elements of the set of regular elements with respect to the minus order in a unital prime ring (Proposition 6.3.13). We also determine the minimal elements with respect to this relation in a unital semisimple Banach algebra with essential socle (Proposition 6.3.15). When A and B are unital semisimple Banach algebras with essential socle, we prove that every bijective linear mapping $T : A \to B$ such that $T(A^\wedge) = B^\wedge$ and $a \leq^- b \Leftrightarrow T(a) \leq^- T(b)$, for every $a, b \in A^\wedge$ is a Jordan isomorphism multiplied by an invertible element (Theorem 6.3.20). The condition $T(A^\wedge) = B^\wedge$ can be removed either when $B = \mathcal{B}(X)$ for a complex Banach space X (Theorem 6.3.22) or B is a prime C*-algebra (Theorem 6.3.24). We also consider briefly linear mappings preserving the minus partial order in just one direction. We prove in Theorem 6.3.25 that, if A is a real rank zero C*-algebra, B is a Banach algebra and $T : A \to B$ is a bounded linear map preserving the minus partial order, then T is a Jordan homomorphism (respectively, a Jordan homomorphism multiplied by an invertible element) whenever $T(1)$ is idempotent (respectively, $T(A) \cap B^{-1}$ and $T(1) \in B^\wedge$).

Finally, we study in Section 6.4 the diamond partial order in C*-algebras and linear mappings between unital C*-algebras that preserve it. We show that this is a partial order in every C*-algebra and describe some distinguished elements with respect to this relation such as the maximal and minimal elements (Propositions 6.4.2, 6.4.4 and 6.4.5, respectively). We also characterize projections and scalar multiples of isometries and co-isometries by means of the diamond partial order (Propositions 6.4.3 and 6.4.7, respectively). These results will be applied to the study of linear maps between C*-algebras preserving the diamond partial order. Every Jordan *-homomorphism preserves the diamond partial order on regular elements (Proposition 6.4.10). In Theorem 6.4.12 we prove that every surjective linear map $T : A \to B$ between unital C*-algebras with essential socle (B is assumed to be prime), that preserves the diamond partial order in both directions, is an appropriate multiple of a Jordan *-homomorphism. We also prove in Theorem 6.4.14 that, if A is a real rank zero C*-algebra, B is a C*-algebra and $T : A \to B$ is a bounded linear map preserving the diamond partial order, then T is a Jordan *-homomorphism (respectively, a Jordan *-homomorphism multiplied by a unitary element) whenever $T(1)$ is a projection (respectively, $T(A) \cap B^{-1}$ and $T(1)$ is a partial isometry).

Chapter 2

Preliminaries

In this chapter we provide the basic concepts and background needed to understand the forthcoming chapters.

2.1 Banach algebras and generalized invertibility

Let A be an (associative) algebra over $\mathbb{C}$. We say that A is a **normed algebra** if it is also a normed space and the norm $\|\cdot\|$ satisfies

$$\|ab\| \leq \|a\|\|b\| \qquad \text{for all } a, b \in A.$$

A normed algebra is called a **Banach algebra** if it is a Banach space. A Banach algebra is **unital** if it has an identity element for the multiplication. We denote by 1_A the unit element of A. When no confusion can arise, we shall write 1 instead of 1_A. Recall that an element $p \in A$ is said to be **idempotent** if $p^2 = p$. We denote the set of all idempotents of A by $A^{\bullet}$.

Let A be a unital Banach algebra and $a \in A$. Then a is said to be **left (respectively, right) invertible** if there exists $b \in A$ such that $ba = 1$ (respectively, $ab = 1$). In this case, b is called a **left (respectively, right) inverse** of a. If a is both left and right invertible, then a is said to be **invertible** and if b is both left and right inverse of a, then b is referred to as an **inverse** of a. The unique inverse of a will be denoted by a^{-1}. The set of all left invertible, right invertible and invertible elements in A will be denoted by A_l^{-1}, A_r^{-1} and A^{-1}, respectively.

Theorem 2.1.1 *([6, Theorem 3.2.1]) Let A be a unital Banach algebra. If $a \in A$ and $\|a\| < 1$, then $1 - a \in A^{-1}$ and*

$$(1-a)^{-1} = \sum_{n=1}^{\infty} a^n.$$

Theorem 2.1.2 *([84, Theorem 7.7-1] and [6, Theorem 3.2.3]) Let A be a unital Banach algebra. The set A^{-1} is an open subset of A. Moreover, the mapping $a \mapsto a^{-1}$ for A^{-1} onto A^{-1} is a homeomorphism.*

Let A be a Banach algebra. An element $a \in A$ is said to be a **left (respectively, right) zero divisor** if $ax = 0$ (respectively, $xa = 0$) for some nonzero element $x \in A$. We say that a is a **zero divisor** if a is both left and right divisor of zero. On the other hand, a is said to be a **left (respectively, right) topological zero divisor if** if there exists a sequence (z_n) in A such that $\|z_n\| = 1$ and $az_n \to 0$ (respectively, $z_n a \to 0$). We say that a is a **topological zero divisor** if a is both a left and right topological zero divisor.

An element a in a Banach algebra A is said to be **nilpotent** if $a^n = 0$ for some $n \in \mathbb{N}$. The set of all nilpotent elements of A will be denoted by $N(A)$. Obviously, every nilpotent element in A is a zero divisor. An element $a \in A$ is **quasi-nilpotent** if $\|a^n\|^{\frac{1}{n}} \to 0$ as $n \to \infty$. The set of all quasi-nilpotent elements of A will be denoted by $QN(A)$.

Let a be an element on a unital Banach algebra A. The **spectrum** of a is the set of all complex numbers λ such that $a - \lambda$ is not invertible in A. The spectrum of a will be denoted by $\sigma(a)$. The complement of the spectrum of a, $\mathbb{C} \setminus \sigma(a)$ is called the **resolvent set** of a and will be denoted by $\rho(a)$.

Theorem 2.1.3 *([6, Theorem 3.2.8(i)-(ii)]) For an element a in a unital Banach algebra A, the function $\lambda \mapsto (a - \lambda)^{-1}$, is analytic in $\mathbb{C} \setminus \sigma(a)$. Moreover, $\sigma(a)$ is a non-empty compact subset of $\mathbb{C}$.*

The **spectral radius** of an element a in a unital Banach algebra is defined by

$$r(a) = \max\{|\lambda| : \lambda \in \sigma(a)\}.$$

It is well-known that

$$r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}}$$

([6, Theorem 3.2.8(iii)]). Note also that, for every $a \in A$, the inequality $r(a) \leq \|a\|$ holds.

The next result, known as the *Hua's identity*, represents a quite useful tool in our proofs. It asserts that, for invertible elements a and b in a Banach algebra, if $a - b^{-1}$ is invertible, then so is $a^{-1} - (a - b^{-1})^{-1}$ and

$$\left(a^{-1} - (a - b^{-1})^{-1}\right)^{-1} = \left(a^{-1} + (b^{-1} - a)^{-1}\right)^{-1} = a - aba$$

A **Jordan algebra** is a (non necessarily associative) commutative algebra J whose product $\circ$ satisfies the **Jordan identity**: $(a \circ b) \circ a^2 = a \circ (b \circ a^2)$ for all $a, b \in J$. Every associative algebra can be seen as a Jordan algebra with the so-called **Jordan product** $a \circ b = \frac{1}{2}(ab + ba)$. Jordan algebras whose product comes from an associative product are called **special**. Given $a \in J$, we define the operator U_a as follows:

$$U_a(x) = 2a \circ (a \circ x) - a^2 \circ x.$$

Recall that an element in a Jordan algebra $(J, \circ)$ is **Jordan invertible** if there exists $b \in J$ such that $a \circ b = 1$ and $a^2 \circ b = a$, equivalently, the operator U_a is invertible with inverse U_b. Such b is called the **Jordan inverse** of a and, as usual, it is denoted by $b = a^{-1}$. The Hua's identity is also valid for Jordan algebras, where it takes the following form (see [72, (11)]):

$$\left(a^{-1} - (a - b^{-1})^{-1}\right)^{-1} = \left(a^{-1} + (b^{-1} - a)^{-1}\right)^{-1} = a - U_a(b). \tag{2.1}$$

Recall that, given an algebra A, a **left ideal (respectively, right ideal)** is a linear subspace I of A such that $xy \in I$ for every $x \in A$ and $y \in I$ (respectively, $yx \in I$ for every $x \in A$ and $y \in I$). A linear subspace which is left and right ideal is called **two-sided ideal**. It is clear that $\{0\}$ and A are (respectively left, right) ideals of A and that the intersecion of ideals is again an ideal of the same kind. A **maximal ideal** of A is a non-trivial ideal that is not contained within any other proper ideal of A. The intersection of all left (respectively, right) maximal ideals can be characterized in many ways:

Theorem 2.1.4 *([6, Theorem 3.1.3]) Let A be a ring with unit. Then the following sets are identical:*

(1) the intersection of all maximal left ideals of A,

(2) the intersection of all maximal right ideals of A,

(3) the set of x such that $1 - zx$ is invertible for all $z \in A$,

(4) the set of x such that $1 - xz$ is invertible for all $z \in A$.

The two-sided ideal satisfying the previous conditions is called the **Jacobson's radical** and it is denoted by $\mathrm{Rad}(A)$. If $\mathrm{Rad}(A) = \{0\}$, then A is said to be **semisimple**.

An algebra is called **prime** if for any two of its ideals I and J it follows from the equality $IJ = \{0\}$ then either $I = \{0\}$ or $J = \{0\}$. It is well-known that an algebra A is prime if $xAy = \{0\}$ implies $x = 0$ or $y = 0$, for every $x, y \in A$. Every prime algebra A is **semiprime**, that is, it has no nonzero ideal I with $I^2 = \{0\}$. Equivalently, $xAx = \{0\}$ implies $x = 0$. It is also well-known that every semisimple algebra is semiprime.

Let A be a semisimple Banach algebra. A **minimal left (respectively, right) ideal** is a nontrivial left (respectively, right) ideal that does not contain any other proper left (respectively, right) ideal of A. Every minimal left ideal of A is of the form Ae where e is a **minimal idempotent**, i.e., $e^2 = e \neq 0$ and $eAe = \mathbb{C}e$. In this case eA is a minimal right ideal of A. The sum of all minimal left ideals of A is called the **socle** of A and it coincides with the sum of all minimal right ideals of A. We denote the socle of A by $\mathrm{soc}(A)$. For example, for any Banach space X, $\mathrm{soc}(\mathcal{B}(X))$ is equal to the ideal of all finite rank operators in $\mathcal{B}(X)$. If A has no minimal one-sided ideals, then we define $\mathrm{soc}(A) = \{0\}$. We say that a nonzero element $u \in A$ has **rank one** if u belongs to some minimal left ideal of A (equivalently, $u = ue$ for some minimal idempotent e in A). By $F_1(A)$ we denote the set of all elements of rank one in A. It is easy to see ([20]) that $u \in F_1(A)$ if, and only if, $u \neq 0$ and u lies in some minimal right ideal of A, and furthermore, this is equivalent to the condition that $uAu = \mathbb{C}u \neq \{0\}$. When A is unital, another characterization is that $u \in F_1(A)$ if, and only if, $u \neq 0$ and $|\sigma(zu) \setminus \{0\}| \leq 1$ for every $z \in A$ or, equivalently, $|\sigma(uz) \setminus \{0\}| \leq 1$ for every $z \in A$ (see [20]).

Given $u \in F_1(A)$, there is $\tau(u) \in \mathbb{C}$ such that $u^2 = \tau(u)u$. Clearly $\tau(u) \in \sigma(u)$ and, moreover, either $\tau(u) = 0$ or $\tau(u)$ is the only nonzero point in $\sigma(u)$. Since $\tau(u)$ is unique, we may consider τ as a function from $F_1(A)$ to $\mathbb{C}$ and extend it by defining $\tau(0) = 0$. Using $uAu = \mathbb{C}u$, $u \in F_1(A)$ and considering $(ux)^2$ and $(xu)^2$ it follows that $\tau(ux)u = uxu = \tau(xu)u$ for any $x \in A$. Furthermore, we have

$$\tau(x_1u + x_2u) = \tau(x_1u) + \tau(x_2u)$$

for all $x_1, x_2 \in A$ (see [20]). Also, it is straightforward to check that $\tau(\lambda u) = \lambda\tau(u)$ for all $\lambda \in \mathbb{C}$ and $u \in F_1(A)$. Therefore, the restriction of τ to any minimal ideal Au is a linear functional. The reader should also note that $F_1(A)$ is a multiplicative ideal, that is, $AF_1(A)A \subset F_1(A)$ ([124, Lemma 2.7]).

Recall that an ideal in an algebra A is said to be **essential** if it has nonzero intersection with every nonzero ideal of A. For a semisimple Banach algebra A this equivalent to the condition $aI = \{0\}$, for $a \in A$, implies $a = 0$. Semisimple Banach algebras with essential socle turn out to be a natural and favourable environment where our results and techniques take place.

Generalized invertibility in Banach algebras

The concept of generalized inverse seems to have been first mentioned in the literature by Fredholm in 1903 ([57]). He introduced a particular notion of generalized inverse of an integral operator (called *pseudoinverse*). Later, Hurwitz ([71]) made use of the properties of Fredholm operators to give an algebraic construction of pseudoinverses. Generalized inverses for differential operators were studied by several authors in the next few years. On the other hand, generalized inverses of matrices where first noted by E. H. Moore in 1920 ([114]). Although in an interval of 30 years Moore's findings were almost unnoticed, Bjerhammar rediscovered Moore's inverse and noted the relationship of generalized inverses to solutions of linear systems (see, for instance, [13]). In 1955, R. Penrose sharpened and extended Bjerhammar's results on linear systems and characterized Moore's inverse via the identities that are still used today ([123]). Since 1955 thousands of papers concerning generalized inverses and their applications have been published.

Let A be a Banach algebra and $a \in A$. We say that $b \in A$ is an **inner inverse** or **G_1-inverse** of a if $aba = a$. It is clear that, in such case, ab and ba are idempotent elements.

It is said that $c \in A$ is a **generalized inverse** or G_2**-inverse** of a if $aca = a$ and $cac = c$. An element with a generalized inverse is said to be **von Neumann regular** or just **regular**. Note that every element with an inner inverse has a generalized inverse. Indeed, if $aba = a$, then $c = bab$ is a G_2-inverse of a. Notice also that, contrary to the genuine inverse, the inner inverse of a regular element is not unique in general. In fact, if $aba = a$, then every element of the form $b - x + baxab$ with $x \in A$ is an inner inverse of a. The set of all G_1-inverses (respectively, G_2-inverses) of a regular element a will be denoted by $G_1(a)$ (respectively, $G_2(a)$). We will also denote by $A^\wedge$ the set of regular elements of A. It is well-known that, for a Hilbert space H, an operator $S \in \mathcal{B}(H)$ has a generalized inverse if, and only if, S has closed range. If X is a Banach space, then $A \in \mathcal{B}(X)$ has generalized inverse if, and only if, the range and null space of A are closed and complemented subspaces in X. Moreover, any element on a finite-dimensional Banach algebra is regular.

For an operator $T \in \mathcal{B}(X)$, the reduced minimum modulus is defined by

$$\gamma(T) = \inf\{\|T(x)\| : \operatorname{dist}(x, \operatorname{Ker}(T)) = 1\},$$

with $\gamma(T) = +\infty$ if $T = 0$. It is a classical fact that $\gamma(T) > 0$ if, and only if, the range of T is closed and that $\gamma(T) = \gamma(T^*)$ (see [115]). For an element a in a Banach algebra A, let us consider the left and right multiplication operators $L_a : x \mapsto ax$ and $R_a : x \mapsto xa$, respectively. If a is regular, then so are L_a and R_a, and thus their ranges $aA = L_a(A)$ and $Aa = R_a(A)$ are both closed. The **conorm** (or **reduced minimum modulus**) of an element a in A, is defined as the reduced minimum modulus of the left multiplication operator by a, $\gamma(a) := \gamma(L_a)$. If b is a generalized inverse of a, with $a \neq 0$, then

$$\|b\|^{-1} \leq \gamma(a) \leq \|ba\| \, \|ab\| \, \|b\|^{-1}$$

(see [66, Theorem 2]).

The Drazin inverse was introduced by Drazin in 1958 ([52]) in the context of semigroups and rings. It was later moved to the sets of (bounded) operators on Banach spaces and Banach algebras. An element a in a ring A is said to be **Drazin invertible** if there exists $b \in A$ such that

$$bab = b, \qquad ab = ba \qquad \text{and} \qquad a^k ba = a^k \text{ for some } k \in \mathbb{N}. \tag{2.2}$$

In case it exists, such b is unique and it is called the **Drazin inverse** of a. The Drazin inverse is denoted of a is denoted by a^D and the set of Drazin invertible elements A is denoted by A^D. It is known that the previous conditions are equivalent to the following ones, whether A is unital:

$$ab = ba, \qquad bab = b \qquad \text{and} \qquad a(1 - ab) \in N(A). \tag{2.3}$$

(See [83, Lemma 2.1].)

The least k in (2.2) is called the **Drazin index** of a and it is denoted by $\operatorname{ind}(a)$. In spite of not being reflexive, the Drazin inverse has become a powerful tool in many different fields, such as matrix theory, partial differential equations, Markov chains, cryptography, iterative methods and more (see [12], [40], [109], [130]). Unlike the inner inverse, the Drazin inverse is unique whenever it exists. It is also well-known that $a \in A$ has a Drazin inverse if, and only if, $0 \notin \sigma(a)$ or 0 is a pole of the resolvent $\lambda \mapsto (\lambda - a)^{-1}$ and, in such case, the Drazin index coincides with the order of the pole of the resolvent. The set of all Drazin inverse elements in A is denoted by A^D and, for an element $a \in A^D$, a^D denotes its Drazin inverse.

A special case of Drazin invertible elements are the so-called group invertible elements. Given $a \in A$, a is said to be **group invertible** if there exists b satisfying (2.2) for $k = 1$. In this case, b is known as the **group inverse** of a. We will denote the group inverse of a as $a^\sharp$, while $A^\sharp$ stands for the set of group invertible elements of A. Clearly, every group invertible element is Drazin invertible, but the converse does not hold generally. In fact, Drazin showed in [52] that the group invertible elements are precisely the reflexive Drazin invertible ones.

Lemma 2.1.5 *([52, Theorems 1,2,3 and Corollary 2]) Let A be an associative ring and $a \in A^D$.*

(1) $a^D \in (\{a\}')'$, that is, a^D lies in the double commutant of a,

(2) For every $k \in \mathbb{N}$, $a^k \in A^D$ and $(a^k)^D = (a^D)^k$. Moreover, ind(a^k) is the unique positive integer q satisfying $0 \leq kq - ind(x) < k$,

(3) $a^D \in A^\sharp$ and $(a^D)^\sharp = a^2a^D$ holds,

(4) $a \in A^\sharp$ if, and only if, $a = (a^D)^D$.

Koliha introduced in [83] the concept of generalized Drazin invertibility. For $a \in A$, we say that a is **Koliha-Drazin invertible** or **generalized Drazin invertible** if there is $b \in A$ satisfying

$$bab = b, \qquad ab = ba \qquad \text{and} \qquad a(1 - ab) \in QN(A). \tag{2.4}$$

In this case, b is called the **Koliha-Drazin inverse** or **generalized Drazin inverse** of a. The Koliha-Drazin inverse it is still unique in case it exists. We will use the notation $b = a^{KD}$ and the set of all Koliha-Drazin invertible elements will be denoted by A^{KD}.

It is clear that:

$$A^{-1} \subset A^\sharp \subset A^\wedge \cap A^D \quad \text{and}$$
$$A^{-1} \subset A^\sharp \subset A^D \subset A^{KD}.$$

We would like to present now the notion of inverse along an element introduced by X. Mary in [99]. It has the advantage that it encompasses some of the generalized inverses previously known. Let S be a semigroup and let S^1 stand for the monoid generated by S. The so-called **Green's preorders** in S are the following ([61]):

$$a \leq_{\mathcal{L}} b \Leftrightarrow S^1 a \subset S^1 b \Leftrightarrow \text{ exists } x \in S^1 \text{ such that } a = xb,$$

$$a \leq_{\mathcal{R}} b \Leftrightarrow aS^1 \subset bS^1 \Leftrightarrow \text{ exists } x \in S^1 \text{ such that } a = bx,$$

$$a \leq_{\mathcal{H}} b \Leftrightarrow a \leq_{\mathcal{L}} b \text{ and } a \leq_{\mathcal{R}} b.$$

If $\leq_{\mathcal{J}}$ is one of these preorders, the relation $\mathcal{J}$ is defined by

$$a\mathcal{J}b \Leftrightarrow a \leq_{\mathcal{J}} b \text{ and } b \leq_{\mathcal{J}} a.$$

Given a and d in S, it is said that a **is invertible along** d if there exists b in S such that $bad = d = dab$ and $b \leq_{\mathcal{H}} d$. Equivalently, $bab = b$ and $b\mathcal{H}d$ ([99, Lemma 3]). If such element b exists, it is termed the **inverse of a along d**, it is unique and it is denoted by $b = a^{\|d}$. Note that the following assertions hold for every element a in a Banach algebra A:

(1) If A is unital, then a is invertible if, and only if, $a^{\|1}$ exists and, in such case, $a^{-1} = a^{\|1}$.

(2) a is Drazin invertible if, and only if, $a^{\|a^m}$ exists for some positive integer m. The least of such m is the index of a, and $a^D = a^{\|a^m}$.

(3) a is group invertible if, and only if, $a^{\|a}$ exists and, in this case, $a^{\sharp} = a^{\|a}$.

The reader is referred to [99] and [100] in order to find existence criteria and other properties concerning this kind of generalized invertibility.

Jordan homomorphisms

Let A and B be Banach algebras. An additive map $T : A \to B$ is said to be a **homomorphism** if $T(ab) = T(a)T(b)$ for all $a, b \in A$; on the other hand, T is said to be a **anti-homomorphism** if $T(ab) = T(b)T(a)$ for all $a, b \in A$. Furthermore, T is said to be a **Jordan homomorphism** if $T(a \circ b) = T(a) \circ T(b)$ for all $a, b \in A$, or, equivalently $T(a^2) = T(a)^2$ for all $a \in A$. By an **isomorphism** (respectively, **anti-isomorphism**, **Jordan isomorphism**) we mean a bijective homomorphism (respectively anti-homomorphism, Jordan homomorphism). Obviously, homomorphisms and anti-homomorphisms are Jordan homomorphisms. Besides, if $T_1 : A_1 \to B_1$ and $T_2 : A_2 \to B_2$ are respectively an homomorphism and an anti-homomorphism between non commutative algebras, then $T : A_1 \oplus A_2 \to B_1 \oplus B_2$ defined by

$$T(a_1 + a_2) = T_1(a_1) + T_2(a_2)$$

is a Jordan homomorphism that is neither an homomorphism nor an anti-homomorphism. However, Herstein proved in [68] that every surjective Jordan homomorphism from a ring onto a prime ring of characteristic different from 2 or 3 is either a homomorphism or an anti-homomorphism. The following results contain some of the main properties of Jordan homomorphisms:

Lemma 2.1.6 ([119], Lemma 6.3.2) *Let A and B be Banach algebras and $T : A \to B$ a Jordan homomorphism. Then, the following assertions hold:*

(1) $T(a^2) = T(a)^2$ for every $a \in A$,

(2) $T(a^n) = T(a)^n$ for every $n \in \mathbb{N}$, $a \in A$,

(3) $T(aba) = T(a)T(b)T(a)$ for every $a, b \in A$.

An additive map $T : A \to B$ fulfilling condition *(3)* is called a **Jordan triple homomorphism**. Equivalently, $T : A \to B$ is a triple Jordan homomorphism if it preserves the **Jordan triple product**, that is,

$$T(abc + cba) = T(a)T(b)T(c) + T(c)T(b)T(a) \tag{2.5}$$

for every $a, b, c \in A$. Recall that if A and B are unital algebras and $T : A \to B$ is a map such that $T(1_A) = 1_B$, T is said to be **unital**. The behaviour of Jordan homomorphisms with idempotent and invertible elements is also well-known:

Lemma 2.1.7 ([119], Lemma 6.3.3) *Let A and B Banach algebras and $T : A \to B$ a Jordan homomorphism. If $p \in A^\bullet$ and $a \in A$, then:*

(1) $T(p)$ is idempotent,

(2) $ap = pa$ implies $T(p)T(a) = T(a)T(p) = T(ap)$,

(3) $ap = pa = a$ implies $T(p)T(a) = T(a)T(p) = T(a)$,

(4) $ap = pa = 0$ implica $T(p)T(a) = T(a)T(p) = 0$.

Consequently, if A is unital then $T(1)$ is the identity of the subalgebra of B generated by $T(A)$. Therefore, if $T(A) \cap B^{-1} \neq 0$ then T is unital.

Proposition 2.1.8 ([131], Proposition 1.3) *Let A and B be unital Banach algebras and $T : A \to B$ a Jordan homomorphism such that $1 \in T(A)$. Then $T(a^{-1}) = T(a)^{-1}$ for all $a \in A$.*

2.2 C*-algebras and Moore-Penrose invertibility

An **involution** on a Banach algebra A is a conjugate linear isometric anti-automorphism of order two, usually denoted $x \mapsto x^*$. In other words, $(x + y)^* = x^* + y^*$, $(xy)^* = y^*x^*$, $(\lambda x)^* = \bar{\lambda}x^*$, $(x^*)^* = x$ and $\|x\| = \|x^*\|$ for all $x, y \in A$ and $\lambda \in \mathbb{C}$. A **Banach *-algebra** is a Banach algebra with an involution.

A **C*-algebra** is a Banach *-algebra A satifying the C*-axiom:

$$\|xx^*\| = \|x\|^2$$

for all $x \in A$. This axiom turns out to force a rigid structure on a C*-algebra. Namely, the norm is completely determined by the algebraic structure and, thus, it is unique, and every C*-algebra can be isometrically represented as a subalgebra of $\mathcal{B}(H)$ that is closed for the norm and the involution (**non-commutative Gelfand-Naimark theorem**). From the C*-axiom it follows that every element a in a C*-algebra A is ***-cancellable**, that is, if $a^*ax = 0$ for some $x \in A$, then $ax = 0$.

Let A be a commutative C*-algebra and $\hat{A}$ the set of homomorphisms from A to $\mathbb{C}$. The celebrated (commutative) Gelfand-Naimark theorem states that A is isometrically *-isomorphic to $C_0(\hat{A})$, that is, the space of complex-valued functions on $\hat{A}$ that vanish at infinity. The natural *-isomorphism takes the form $x \mapsto \hat{x}$ given by $\hat{x}(\varphi) = \varphi(x)$ and is called **Gelfand transform**.

Let A be a C*-algebra. An element $a \in A$ is said to be **selfadjoint** if $a^* = a$. The set of all selfadjoint elements in A is denoted by A_{sa}. It is easy to see that every element in a C*-algebra is linear combination of two selfadjoint elements. Specifically, for every $x \in A$, we have $x = h + ik$, where $h = 2^{-1}(x + x^*)$ and $k = (2i)^{-1}(x - x^*)$ are selfadjoint. An element $a \in A$ is a **partial isometry** if $aa^*a = a$. If the C*-algebra A is unital, $a \in A$ is an **isometry** if $a^*a = 1$, a **co-isometry** if $aa^* = 1$ and a **unitary element** if $aa^* = a^*a = 1$. We will denote the set of all unitary elements in A by $U(A)$. The next result is due to Russo and Dye.

Theorem 2.2.1 *([126, Theorem 1]) Let A be a unital C*-algebra. Then the convex hull of $U(A)$ is uniformly dense in the unit sphere of A.*

A direct consequence of this powerful theorem is that every unital C*-algebra A is the linear span of its unitary elements. Furthermore, the norm of every element $x \in A$ can be computated as $\|x\| = \inf \sum_i \lambda_i$ where the infimum is taken over every representation $x = \sum_{i=0}^{n} \lambda_i u_i$ with $u_i \in U(A)$ for every $i \in \{1, ..., n\}$ ([126, Lemma 2]). Moreover, if B is a normed linear space and $T : A \to B$ is a bounded linear map, then $\|T\| = \sup_{u \in U(A)} \|T(u)\|$ ([126, Corollary 1]).

Let A be a C*-algebra. An element $x \in A$ is said to be **normal** if $xx^* = x^*x$. The C*-algebra generated by a normal element x (i.e., the smallest C*-subalgebra of A containing x, $C^*(x)$) is isometrically isomorphic to $C_0(\sigma(x))$ under an isomorphism that sends x to the function $f(t) = t$. In fact, polynomials in x and x^* without constant term are uniformly dense in $C^*(x)$ and, by the Stone-Weierstrass theorem, polynomials in λ and $\bar{\lambda}$ without constant term are dense in $C_0(\sigma(x))$. Thus, if f is a complex-valued function which is continuous on $\sigma(x)$, with $f(0) = 0$ if $0 \in \sigma(x)$, then there is a corresponding element $f(x) \in C^*(x)$. This is the so-called **continuous functional calculus** for normal elements.

Let A be a C*-algebra. An element $x \in A$ is **positive** if $x = x^*$ and $\sigma(x) \subset [0, +\infty)$. We will denote this fact by $x \geq 0$ and the set of positive elements in A by A_+. It is well-known that A_+ is a cone, that is, a subset of A that is closed under addition and scalar multiplication by positive real numbers. Note also that, for every $x \in A_+$, the continuous functional calculus ensures the existence of a unique $a \in A_+$ such that $a^2 = x$. This is called the **positive square root** of x and it is denoted by $a = x^{\frac{1}{2}}$. In addition, for every $x \in A$, we have that $xx^* \in A_+$. Recall that, for every $x \in A_{\mathrm{sa}}$, there exists a unique decomposition $x = x_+ - x_-$ where $x_+, x_- \geq 0$ and $x_+ x_- = 0$. Finally, selfadjoint idempotents in C*-algebras are called **projections**. We will denote the set of projections of a C*-algebra A by $\mathrm{Proj}(A)$.

Let H be a Hilbert space. For a set $M \subset \mathcal{B}(H)$, the **commutant** of M is the set of operators that commute with every element of M. We denote the commutant of M by M'. A **von Neumann algebra** or **W*-algebra** is a *-subalgebra of $\mathcal{B}(H)$ satisfying $M = (M')'$. This notion was introduced by J. von Neumann motivated by his study of single operators, group representations, ergodic theory and quantum mechanics. Note that, by definition, a von Neumann algebra is always unital. The Double Commutant Theorem (due to von Neumann himself, [134]) states that, for a *-subalgebra M of $\mathcal{B}(H)$ containing the identity operator, its strong topology closure and the weak topology are equal and they coincide with $(M')'$. This means that a von Neumann algebra is a *-subalgebra of $\mathcal{B}(H)$ containing the identity operator that is closed in the strong (equivalently, weak) operator topology. Since the norm topology of operators is finer than the strong one, every von Neumann algebra is a C*-algebra. Indeed, a C*-algebra A is a von Neumann algebra if, and only if, A regarded as a vector space is the dual space of another vector space.

Note that a C*-algebra needs not to have any non zero projection (consider, for instance, $C_0(\mathbb{R})$, the C*-algebra of continuous complex functions of a real variable vanishing at infinity). Our techniques, however, sometimes need the presence of projections. A C*-algebra A is said to be of **real rank zero** if every selfadjoint element can be approximated by real-linear combination of orthogonal projections (see [21]). Recall that two elements $a, b \in A$ are called **orthogonal** if $a^*b = ba^* = 0$ (we denote this fact by $a \perp b$). Equivalently, every selfadjoint element of A can be approximated by selfadjoint elements with finite spectrum. Every von Neumann algebra and, in particular, $\mathcal{B}(H)$, has real rank zero. Since every element in a C*-algebra is linear combination of its real and imaginary parts, it follows that if A is of real rank zero, then the linear span of the projections in A is dense. The converse in known to be false ([122]). However, in a large number of C*-algebras every element can be expressed as a finite linear combination of projections, such as properly infinite C*-algebras, Bunce-Deddens C*-algebras, UHF C*-algebras, von Neumann algebras of type II_1, irrational rotation algebras and many others (see for instance [90], [95], [96], [97], [121], and the references therein).

For a ring R and a set $S \subset R$, the set

$$\mathrm{ann}_l(S) = \{x \in R : xs = 0 \quad \text{for all} \quad s \in S\}$$

is called the **left annihilator** of S. For an element $x \in R$, we write $\mathrm{ann}_l(x) = \mathrm{ann}_l(\{x\})$. The right annihilator of S, $\mathrm{ann}_r(S)$, is defined similarly. A C*-algebra A is called **Rickart C*-algebra** if the left annihilator of any element $a \in A$ is generated by a projection, that is, there exists $p \in \mathrm{Proj}(A)$ such that $\mathrm{ann}_l(a) = Ap$ (see [94]). It is well-known that every von Neumann algebra is a Rickart C*-algebra, and that every Rickart C*-algebra has real rank zero ([14], [127]).

A map $T : A \to B$ between C*-algebras that preserves the involution (namely, $T(x^*) = T(x)^*$ for every $x \in A$) will be called **selfadjoint**. A selfadjoint homomorphism (respectively, anti-homomorphism, Jordan homomorphism) is called ***-homomorphism** (respectively, ***-anti-homomorphism**, **Jordan *-homomorphism**). In his paper [78], Kadison describes the surjective isometries between C*-algebras:

Theorem 2.2.2 *([78, Theorem 7]) Let A and B be unital C*-algebras and $T : A \to B$ a surjective linear map. Then, there exist a unitary element $u \in B$ and a Jordan *-homomorphism $S : A \to B$ such that $T(x) = uS(x)$ for all $x \in A$.*

Note that this result remains true in the non unital case, although the unitary element u generally comes $B \oplus \mathbb{C}$ ([120]). In both cases, T satisfies

$$T(ab^*c + cb^*a) = T(a)T(b)^*T(c) + T(c)T(b)^*T(a)$$

for all $a, b, c \in A$.

Let A be a C*-algebra. A **(*-)representation** of A is a (*-)homomorphism $\rho : A \to \mathcal{B}(H)$ where H is a Hilbert space. A representation ρ is said to be **irreducible** if it has no nontrivial closed invariant subspaces. A **primitive ideal** is an ideal which is the kernel of an irreducible representation of A. If 0 is a primitive ideal, i.e., if there exists an injective irreducible representation of A, then A is called **primitive**.

Let A be a unital prime C*-algebra with nonzero socle. Then A is primitive and $\overline{\mathrm{soc}(A)}$ is its unique minimal closed ideal ([102]). In this situation, if e is a minimal projection in A, the minimal left ideal Ae can be endowed with inner product, $\langle x, y\rangle e = y^*x$, (for all $x, y \in Ae$), under which Ae becomes a Hilbert space in the algebra norm. Let $\rho : A \to \mathcal{B}(Ae)$ be the left regular representation on Ae, given by $\rho(a)(x) = ax$. The mapping ρ is an isometric (hence, injective) irreducible representation, satisfying:

(1) $\rho(\mathrm{soc}(A)) = F(Ae)$,

(2) $\rho(\overline{\mathrm{soc}(A)}) = K(Ae)$,

(3) $\sigma_A(x) = \sigma_{\mathcal{B}(Ae)}(\rho(x))$, for every $x \in A$.

(See [9, Section F.4].)

Moore-Penrose invertibility in C*-algebras

Regularity in C*-algebras were deeply studied by Harte and Mbekhta in [65] and [66]. The main results in those papers state that an element a in a C*-algebra A is regular if, and only if, aA is closed, equivalently $\gamma(a) > 0$, and that

$$\gamma(a)^2 = \gamma(a^*a) = \inf\{\lambda : \lambda \in \sigma(a^*a) \setminus \{0\}\} = \gamma(a^*)^2.$$

We say that $a \in A$ is **Moore-Penrose invertible** if there is $b \in A$ such that

$$aba = a, \qquad bab = b, \qquad ab = (ab)^* \qquad \text{and} \qquad ba = (ba)^*. \tag{2.6}$$

In case it exists, such b is called the **Moore-Penrose inverse** of a. In this case, b is unique and we denote it by $b = a^\dagger$. The set of all Moore-Penrose invertible elements of A is usually denoted by $A^\dagger$. In [65], the authors showed that every regular element in a unital C*-algebra A has a Moore-Penrose inverse, that is, $A^\wedge = A^\dagger$. Furthermore,

$$\gamma(a) = \|a^\dagger\|^{-1}$$

for every $a \in A^\wedge$. In connection with the notion of inverse along an element, it is known that $a \in A$ is Moore-Penrose invertible if, and only if, $a^{\|a^*}$ exists and, in such case $a^\dagger = a^{\|a^*}$. It should be noted that this notion of generalized inverse is just a generalization of the Moore-Penrose pseudoinverse for matrix algebras independently presented by Moore ([114]), Bjerhammar ([13]) and Penrose ([123]). The Moore-Penrose inverse is applied to computate least square solutions, minimum norm solutions and condition number of matrices (see, for instance, [12]).

2.3 JB*-triples, regularity and Brown-Pedersen quasi-invertibility

There exists a wider class of complex Banach spaces containing all the C*-algebras in which the notion of regularity makes sense and extends the concept given for C*-algebras. We refer to the class of JB*-triples introduced by Kaup in [80]. A **JB*-triple** is a complex Banach space E together with a continuous triple product

$$\{.,.,.\} : E \times E \times E \to E,$$

which is conjugate linear in the middle variable and symmetric and bilinear in the outer variables satisfying that,

(1) $L(a,b)L(x,y) = L(x,y)L(a,b) + L(L(a,b)x,y) - L(x, L(b,a)y)$, where $L(a,b)$ is the operator on E given by $L(a,b)x = \{a,b,x\}$;

(2) $L(a,a)$ is an hermitian operator with non-negative spectrum;

(3) $\|L(a,a)\| = \|a\|^2$.

For each x in a JB*-triple E, $Q(x)$ will stand for the conjugate linear operator on E defined by $Q(x)(y) = \{x, y, x\}$. For $a, b \in E$, $B(a, b)$ denotes the **Bergmann operator** on A associated with (a, b), which is defined by

$$B(a, b) = I_E - 2L(a, b) + Q(a)Q(b).$$

A **triple homomorphism** is a linear map preserving the triple product, namely, $T(\{x, y, z\}) = \{T(x)T(y)T(z)\}$ for all $x, y, z \in E$.

Every C*-algebra A is a JB*-triple endowed with the triple product

$$\{x, y, z\} = \frac{xy^*z + zy^*x}{2}$$

for all $x, y, z \in A$. For a C*-algebra A, we denote by E_A the underlying JB*-triple.

A complex Jordan algebra J with a conjugate linear algebra involution $*$ is called a **JB*-algebra** if it is also a Banach space in which the norm satisfies $\|x \circ y\| \leq \|x\|\|y\|$ and $\|U_x(x^*)\| = \|x\|^3$ for all $x, y \in J$. Every JB*-algebra is also a JB*-triple under the triple product:

$$\{x, y, z\} = (x \circ y^*) \circ z + x \circ (y^* \circ z) - (x \circ z) \circ y^*.$$

Also, $\mathcal{B}(H, K)$, the Banach space of bounded linear operators between two complex Banach spaces H, K, is a JB*-triple via the triple product

$$\{R, S, T\} = \frac{RS^*T + TS^*R}{2}$$

for all $R, S, T \in \mathcal{B}(H, K)$.

An element e in a JB*-triple E is said to be a **tripotent** if $\{e, e, e\} = e$ (note that tripotents in C*-algebras are just partial isometries). Each tripotent e in E gives raise to the so-called **Peirce decomposition** of E associated to e, that is,

$$E = E_2(e) \oplus E_1(e) \oplus E_0(e),$$

where for $i = 0, 1, 2$, $E_i(e)$ is the $\frac{i}{2}$ eigenspace of $L(e, e)$. The Peirce decomposition satisfies certain rules known as **Peirce arithmetic**:

$$\{E_i(e), E_j(e), E_k(e)\} \subseteq E_{i-j+k}(e),$$

if $i - j + k \in \{0, 1, 2\}$ and is zero otherwise. In addition,

$$\{E_2(e), E_0(e), E\} = \{E_0(e), E_2(e), E\} = 0.$$

The Peirce space $E_2(e)$ is a JB*-algebra with product $x \bullet y := \{x, e, y\}$ and involution $x^\# := \{e, x, e\}$. Moreover the triple product induced on $E_2(e)$ by this Jordan *-algebra structure coincides with its original triple product, that is

$$\{x, y, z\} = (x \bullet y^\#) \bullet z + (z \bullet y^\#) \bullet x - (x \bullet z) \bullet y^\#. \tag{2.7}$$

A tripotent e in E is called **complete** if the equality $E_0(E) = 0$ holds. When $E_2(e) = \mathbb{C}e \neq \{0\}$, we say that e is **minimal**.

For an element x in a JB*-triple E, it is known that E_x, the JB*-subtriple generated by x is JB*-triple isomorphic (and hence isometric) to $C_0(\Omega)$ for some locally compact Hausdorff space Ω contained in $(0, \|x\|]$, such that $\Omega \cup \{0\}$ is compact, where $C_0(\Omega)$ denotes the Banach space of all complex-valued continuous functions vanishing at 0. It is also known that if Ψ denotes the triple isomorphism from E_x onto $C_0(\Omega)$, then $\Psi(x)(t) = t$ $(t \in \Omega)$ (cf. [80, Corollary 1.15] and [58]). The set $\overline{\Omega} = \mathrm{Sp}(x)$ is called the **triple spectrum** of x. We should note that $C_0(\mathrm{Sp}(x)) = C(\mathrm{Sp}(x))$, whenever $0 \notin \mathrm{Sp}(x)$.

For every element x in a JB*-triple E, we write $x^{[1]} := x$, $x^{[3]} := \{x, x, x\}$ and $x^{[2n+1]} := \left\{x, x, x^{[2n-1]}\right\}$ $(n \in \mathbb{N})$. For each $x \in J$, there exists a unique y in E_x such that $x = y^{[3]}$. Such y is called the **cubic root** of x and it is denoted by $y = x^{[\frac{1}{3}]}$. We can inductively define $x^{[\frac{1}{3^n}]} = \left(x^{[\frac{1}{3^{n-1}}]}\right)^{\frac{1}{3}}$, $n \in \mathbb{N}$. The sequence $\left(x^{[\frac{1}{3^n}]}\right)_n$ converges in the weak* topology of E^{**} (the bidual space of E) to a tripotent denoted by $r(x)$ and called **range tripotent** of x.

Regular elements in Jordan triple systems and JB*-triples were originally studied by A. Férnandez López, E. García Rus, E. Sánchez Campos, M. Siles Molina ([55]), O. Loos ([93]) and Kaup ([81]). An element a in a JB*-triple E is called **von Neumann regular** if there exists (a unique) $b \in E$ such that $Q(a)(b) = a$, $Q(b)(a) = b$ and $Q(a)Q(b) = Q(b)Q(a)$, or equivalently $Q(a)(b) = a$ and $Q(a)(b^{[3]}) = b$. The element b is called the **generalized inverse** of a. We observe that every tripotent e in a JB*-triple E is von Neumann regular and its generalized inverse coincides with it. We denote by $E^{\wedge}$ the set of regular elements in the JB*-triple E and, for an element $a \in E^{\wedge}$, let $a^{\wedge}$ denote its generalized inverse. For a C*-algebra A and $a \in A$, a has Moore-Penrose inverse b in A if, and only if, a has generalized inverse b^* in E_A. That is, $A^{\dagger} = E_A^{\wedge} = A^{\dagger}$ and $a^{\wedge} = (a^{\dagger})^*$. We refer to [37, 55, 82, 93] for basics results on von Neumann regularity in JB*-triples.

The set, A_q^{-1}, of quasi-invertible elements in a unital C*-algebra A was introduced by L. Brown and G.K. Pedersen as the set $A^{-1}\mathfrak{A}_1 A^{-1}$, where A^{-1} and $\mathfrak{A}_1$ denote the set of invertible elements in A and the set of extreme points of the closed unit ball of A, respectively (see [22]). It is known that $a \in A_q^{-1}$ if, and only if, there exists $b \in A$ such that $B(a, b) = 0$ (cf. [22, Theorem 1.1] and [75, Theorem 11]).

The notion of quasi-invertible element was extended by F.B. Jamjoom, A.A. Siddiqui, and H.M. Tahlawi to the wider setting of JB*-triples. An element x in a JB*-triple E is called **Brown-Pedersen quasi-invertible** if there exists $y \in E$ such that $B(x, y) = 0$ (cf. [75]). The element y is called a **Brown-Pedersen quasi-inverse** of x. It is known that $B(x, y) = 0$ implies $B(y, x) = 0$. Moreover, the Brown-Pedersen quasi-inverse of an element is not unique. Indeed, if $B(x, y) = 0$ then it can be checked that $B(x, Q(y)(x)) = 0$, so for any Brown-Pedersen quasi-inverse y of x, $Q(y)(x)$ also is a Brown-Pedersen quasi-inverse of x. It is established in [75, Theorems 6 and 11] that an element x in E is Brown-Pedersen quasi-invertible if, and only if, it is (von Neumann) regular and its range tripotent is an extreme point of the closed unit ball of E, equivalently, there exists a complete tripotent $v \in E$ such that x is positive and invertible in $E_2(v)$. In particular, the set, E_q^{-1}, of all Brown-Pedersen quasi-invertible elements in E contains all extreme points of the closed unit ball of E.

2.4 Partial orders

Since the 80's, many authors have focused on the study of some partial orders defined in abstract structures, such as semigroups, rings of matrices and, more specifically, algebras (see [53], [64], [110], [113]). These partial orders are usually generalizations of the classical one for idempotents and projections. We recall the basic definitions of the orders that will be considered in Chapter 6.

Let $M_n(\mathbb{C})$ be the algebra of all $n \times n$ complex matrices. The **star partial order** on $M_n(\mathbb{C})$ was introduced by Drazin in [53], as follows:

$$A \leq_* B \qquad \text{if, and only if,} \qquad A^*A = A^*B \text{ and } AA^* = BA^*,$$

where as usual A^* denotes the conjugate transpose of A. It was proved that $A \leq_* B$ if, and only if, $A^\dagger A = A^\dagger B$ and $AA^\dagger = BA^\dagger$.

Hartwig ([64]) introduced the **rank substractivity order** on $M_n(\mathbb{C})$:

$$A \leq^- B \qquad \text{if, and only if,} \qquad \operatorname{rank}(B - A) = \operatorname{rank}(B) - \operatorname{rank}(A).$$

He proved that

$$A \leq^- B \qquad \text{if, and only if,} \qquad A^- A = A^- B \text{ and } AA^- = BA^-,$$

where A^- denotes a G_1-inverse of A. This partial order is usually named the **minus partial order**.

Later, Mitra introduced in [112] the **space pre-order** on $M_n(\mathbb{C})$:

$$M \leq_s N \quad \text{if, and only if,} \quad \mathcal{C}(M) \subseteq \mathcal{C}(N) \quad \text{and} \quad \mathcal{C}(N^*) \subseteq \mathcal{C}(M^*),$$

where $\mathcal{C}(M)$ denotes the column space of the matrix M.

In [125], Rakić and Djordjević extend the definition of space pre-order to the class of bounded linear operators on Banach spaces and generalize some well-known properties of this partial order to the new setting.

Let H be an infinite-dimensional complex Hilbert space. Having into account that an operator in $\mathcal{B}(H)$ is regular if, and only if, it has closed range, Šemrl ([129]) extended the minus partial order from $M_n(\mathbb{C})$ to $\mathcal{B}(H)$, finding an appropriate equivalent definition of the minus partial order on $M_n(\mathbb{C})$ which does not involve G_1-inverses: for

$A, B \in \mathcal{B}(H)$, $A \preceq B$ if, and only if, there exist idempotent operators $P, Q \in \mathcal{B}(H)$ such that

$$R(P) = \overline{R(A)}, \quad N(A) = N(Q), \quad PA = PB, \quad AQ = BQ.$$

Šemrl proved that the relation $\preceq$ is a partial order in $\mathcal{B}(H)$ extending the minus partial order of matrices. In 2013, Djordjević, Rakić and Marovt ([49]) generalized Šemrl's definition to the environment of Rickart rings and generalized some well-known results. Recall that a ring is a **Rickart ring** if the left and right annihilator of any element are generated by idempotent elements. For a Rickart ring A, the minus partial order is defined as follows: $a \leq^- b$ if there exist $p, q \in A^\bullet$ such that

$$\mathrm{ann}_l(a) = \mathrm{ann}_l(p), \quad \mathrm{ann}_r(a) = \mathrm{ann}_r(q), \quad pa = pb, \quad aq = bq.$$

Note that this definition is still valid even if the ring is not Rickart, but the possible lack of idempotents makes this relation non reflexive (in fact, a ring is Rickart if, and only if, the minus relation is reflexive).

On the other hand, Mitra used in [111] the group inverse of a matrix to define the **sharp order** on group invertible matrices:

$$A \leq_\sharp B \qquad \text{if, and only if,} \qquad A^\sharp A = A^\sharp B \text{ and } AA^\sharp = BA^\sharp.$$

In this work, the author compared the star and the sharp order and provided many equivalent formulations to these and other partial orders.

Finally, in [88], Lebtahi, Patrício and Thome introduce the **diamond partial order** on a *-regular ring, extending a partial order defined in the matrix setting by Baksalary and Hauke in [8]. Although it can be considered in a more general setting, we will focus on the framework of C*-algebras. For a C*-algebra A and $a, b \in A$, we say that $a \leq_\diamond b$ if, and only if, $aA \subset bA$, $Aa \subset Ab$ and $aa^*a = ab^*a$. Let us recall some basic properties and relations concerning the previous notions of order for regular elements. These will be used in Chapter 6.

Lemma 2.4.1 *([88]) Let A be a Banach algebra (C*-algebra, if necessary) and $a, b \in A^\wedge$. The following assertions hold:*

(1) $a \leq^- b$ if, and only if, there exists $b^- \in G_1(b)$ such that $a = ab^-a = bb^-a = ab^-b$,

(2) $a \leq^- b$ if, and only if, $b - a \leq^- b$,

(3) $a \leq^- b$ if, and only if, $a^\dagger \leq_\diamond b^\dagger$,

(4) $a \leq_* b$ if, and only if, $b - a \leq_* b$,

(5) $a \leq_* b$ implies $a \leq_\diamond b$,

(6) $a \leq_\sharp$ implies $a \leq^- b$.

Chapter 3

Additive preservers of generalized inverses in Banach algebras

This chapter is concerned with Hua type theorems for generalized inverses in the setting of Banach algebras. We study additive and linear maps strongly preserving Drazin and group invertibility. We characterize Jordan triple homomorphisms between general Banach algebras as those additive maps strongly preserving the inverse along an element. Finally, we provide approximate versions of Hua's theorem for invertibility and group invertibility. All the results appearing in this chapter can be found in [29], [30] and [31].

3.1 A generalization of Hua's theorem

In this section we present a generalized version of Hua's theorem for Banach algebras, where the condition $T(a^{-1}) = T(a)^{-1}$ for all invertible element $a \in A$ is replaced by

$$T(a)T(a^{-1}) = T(b)T(b^{-1}), \tag{3.1}$$

for all invertible elements $a, b \in A$. As we have mentioned, every unital Jordan homomorphism between Banach algebras strongly preserves invertibility, that is, $T(a^{-1}) = T(a)^{-1}$ for every invertible element $a \in A$. Hua's theorem shows that every unital additive map between division rings that strongly preserves invertibility is an isomorphism or an anti-isomorphism. In [43], Chebotar, Ke, Lee and Shiao improved Hua's theorem by relaxing the strongly invertibility condition to (3.1).

Theorem 3.1.1 *([43, Theorem 2.1]) Let K be a division ring with center Z and let $T : K \to K$ be a bijective additive map such that*

$$T(a)T(a^{-1}) = T(b)T(b^{-1})$$

for all non zero $a, b \in K$. Then $T = T(1)S$, where $S : K \to K$ is either an automorphism or an anti-automorphism.

Later, Lin and Wong extended this result to the ring $M_n(K)$ of $n \times n$ matrices over a division ring K.

Theorem 3.1.2 *([91, Theorem 1.3])Let K be a division ring with center Z, $n \geq 2$ and $T : M_n(K) \to M_n(K)$ a bijective additive map satisfying (3.1), such that $T(1)^2 \neq 0$. Then $T(1)$ is a central (invertible) element in $M_n(K)$ and $T = T(1)S$ where $S : K \to K$ is an automorphism or an anti-automorphism.*

Taking into account that Hua's theorem has been successfully adapted from matrix algebras ([54]) to Banach algebras ([104]), our aim to move Theorem 3.1.2 to the environment of Banach algebras. We present now the main results in this section. They describe additive mappings between unital Banach algebras satisfying Condition (3.1). The first one consider the case when the range contains some invertible element (in particular, when the map is surjective) and the second one deals with the case when the image of the unit is invertible.

Theorem 3.1.3 *Let A and B be unital Banach algebras and $T : A \to B$ be an additive map such that $T(a)T(a^{-1}) = T(b)T(b^{-1})$ for all $a, b \in A^{-1}$ and $T(A) \cap B^{-1} \neq \emptyset$. Then $T(1)$ is Drazin invertible, $T(1)^D T$ is a Jordan homomorphism and $T(1)^D$ commutes with $T(A)$.*

Note that the requirement of $T(1)$ being Drazin invertible is trivially satisfied in the finite-dimensional setting (Theorem 3.1.2). Whether $T(1)$ is, in fact, invertible, we can obtain the converse of the above theorem.

Theorem 3.1.4 *Let A and B be unital Banach algebras and $T : A \to B$ be an additive map such that $T(1)$ is invertible. Then the following conditions are equivalent:*

(1) $T(a)T(a^{-1}) = T(b)T(b^{-1})$ for all $a, b \in A^{-1}$,

(2) $T = T(1)S$, where S is a unital Jordan homomorphism and $T(1)$ commutes with $S(A)$.

In order to prove them we will need some technical results. We make use of the Hua's identity. Recall that, if $a, b \in A$ are invertible elements such that $a - b^{-1}$ is invertible, then so is $a^{-1} - (a - b^{-1})^{-1}$ and

$$\left(a^{-1} - (a - b^{-1})^{-1}\right)^{-1} = \left(a^{-1} + (b^{-1} - a)^{-1}\right)^{-1} = a - aba. \tag{3.2}$$

In what follows, A and B are supposed to Banach algebras, A is assumed to be unital and $T : A \to B$ is an additive map fulfilling Condition (3.1). Actually, it is equivalent to

$$T(a)T(a^{-1}) = T(1)^2 \text{ for all } a \in A^{-1}. \tag{3.3}$$

Lemma 3.1.5 *For every $x \in A$ the following identities hold.*

(1) $T(1)^2T(x) = T(x)T(1)^2$,

(2) $T(1)^3T(x) = T(x)T(1)^3$.

Proof Since for every element $x \in A$ and $\lambda \in \mathbb{Q}$, with $|\lambda| > ||x||$, $x + \lambda$ is invertible, it is clear that it is enough to prove *(1)* and *(2)* for every invertible element x in A. Pick $a \in A^{-1}$. By hypothesis $T(a)T(a^{-1}) = T(1)^2$. Multiplying by $T(a)$ on the right, we obtain $T(a)T(a^{-1})T(a) = T(1)^2T(a)$. As $T(a^{-1})T(a) = T(1)^2$ it follows that $T(a)T(1)^2 = T(1)^2T(a)$ (for every invertible element $a \in A^{-1}$). By the Hua's identity, if b and $1 + b$ are invertible, then

$$1 = (1 + b)^{-1} + (1 + b^{-1})^{-1}.$$

Applying T to the last equation we get

$$T(1) = T((1 + b)^{-1}) + T((1 + b^{-1})^{-1}).$$

Multiplying by $T(1 + b^{-1})$ on the left and having in mind (3.3) it follows that

$$T(1 + b^{-1})T(1) = T(1 + b^{-1})T((1 + b)^{-1}) + T(1)^2.$$

Then

$$T(b^{-1})T(1) = T(1 + b^{-1})T((1 + b)^{-1}).$$

We multiply the previous equation now by $T(1 + b)$ on the right to obtain

$$T(b^{-1})T(1)T(1 + b) = T(1 + b^{-1})T(1)^2.$$

Hence

$$T(b^{-1})T(1)T(b) = T(1)^3.$$

Finally, multiplying by $T(b)$ on the left, we have

$$T(1)^3T(b) = T(b)T(1)^3,$$

for all $b \in A$ such that b and $1 + b$ are invertible.

Given any $a \in A^{-1}$, and $\alpha \in \mathbb{Q}$ with $0 < |\alpha| < \|a\|^{-1}$, it is clear that αa and $1 + \alpha a$ are invertible. Therefore

$$T(1)^3T(\alpha a) = T(\alpha a)T(1)^3,$$

and so,

$$T(1)^3T(a) = T(a)T(1)^3$$

for all $a \in A^{-1}$. ■

Lemma 3.1.6 *For every $x \in A$,*

(1) $T(1)^4T(x^3) = T(1)^2T(x)^3$,

(2) $T(1)^4T(x^2) = T(1)^3T(x)^2$.

Proof Given $a \in A^{-1}$ and $\lambda \in \mathbb{Q}$ with $0 < |\lambda| < ||a^{-1}||^{-2}$, it is clear that $b = \lambda^{-1}a$, and $a - b^{-1} = a - \lambda a^{-1}$ are invertible elements in A. Therefore, by Hua's identity (3.2)

$$\left(a^{-1} - (a - \lambda a^{-1})^{-1}\right)^{-1} = a - a(\lambda^{-1}a)a = a - \lambda^{-1}a^3. \tag{3.4}$$

Let $a \in A^{-1}$. Applying T to Identity (3.4) we have

$$T(a) - \lambda^{-1}T(a^3) = T((a^{-1} - \left(a - \lambda a^{-1})^{-1}\right)^{-1})$$

Now, multiplying this equation by $T(a^{-1} - (a - \lambda a^{-1})^{-1})$ on the left and having in mind (3.3), we get

$$T(a^{-1} - (a - \lambda a^{-1})^{-1}) \left(T(a) - \lambda^{-1}T(a^3)\right) = T(1)^2.$$

Since $T(a^{-1})T(a) = T(1)^2$ it follows that

$$\lambda^{-1}T(a^{-1})T(a^3) + T\left((a - \lambda a^{-1})^{-1}\right)\left(T(a) - \lambda^{-1}T(a^3)\right) = 0$$

Multiplying now by $T(a - \lambda a^{-1})$ on the left, and using (3.3), we have

$$\lambda^{-1}T(a - \lambda a^{-1})T(a^{-1})T(a^3) + T(1)^2\left(T(a) - \lambda^{-1}T(a^3)\right) = 0.$$

Direct calculations on the previous equation, together with (3.3) yield

$$T(a^{-1})^2T(a^3) = T(1)^2T(a).$$

Since from Lemma 3.1.5 (1), $T(1)^2$ commutes with $T(A)$, we can multiply the last equation by $T(a)^2$ on the left to get

$$T(1)^4T(a^3) = T(1)^2T(a)^3,$$

for all $a \in A^{-1}$.

Finally, let $x \in A$ and $\lambda \in \mathbb{Q}$ such that $x + \lambda$ is invertible. From the above equality

$$T(1)^4T\left((x + \lambda)^3\right) = T(1)^2T(x + \lambda)^3.$$

Expanding the previous identity and having in mind Lemma 3.1.5 we have

$$\begin{aligned} T(1)^4&\left(T(x^3) + 3\lambda T(x^2) + 3\lambda^2T(x) + \lambda^3T(1)\right) = \\ &T(1)^2T(x)^3 + \lambda T(1)^2\left(T(x)^2T(1) + T(x)T(1)T(x) + T(1)T(x)^2\right) + \\ &\qquad \lambda^2T(1)^2\left(2T(1)^2T(x) + T(1)T(x)T(1)\right) + \lambda^3T(1)^5. \end{aligned}$$

A simple identification of the coefficients of λ^0 and λ in the above equation leads, respectively, to the following identities

$$T(1)^4T(x^3) = T(1)^2T(x)^3,$$

$$3T(1)^4T(x^2) = T(1)^2T(x)^2T(1) + T(1)^2T(x)T(1)T(x) + T(1)^3T(x)^2.$$

Since $T(1)^2$ and $T(1)^3$ commute with $T(x)$, it is clear that

$$T(1)^4T(x^2) = T(1)^3T(x)^2,$$

which completes the proof. ■

While for an additive map $T : A \to B$ strongly preserving invertibility, the codomain B must to be unital and $T(1)$ must be invertible (with $T(1)^{-1} = T(1)$), this is not the case for an additive map $T : A \to B$ fulfilling Condition (3.1). The following proposition describes additive mappings from a unital Banach algebra A on a Banach algebra B satisfying (3.1) and such that $T(1)$ is just Drazin invertible. It will be used in the proof of the main result in Section 3.3 (Theorem 3.3.1).

Proposition 3.1.7 *Let A and B be Banach algebras, where A is assumed to be unital, and $T : A \to B$ be an additive map such that*

$$T(a)T(a^{-1}) = T(b)T(b^{-1}) \qquad \textit{for all } a, b \in A^{-1}.$$

Suppose that $T(1)$ is Drazin invertible. Then $T(1)^D T$ is a Jordan homomorphism and $T(1)^D$ commutes with $T(A)$.

Proof Let $x \in A$. From Lemma 3.1.5 *(1)*, we know that $T(1)^2T(x) = T(x)T(1)^2$. Multiplying the previous identity by $(T(1)^D)^3$ on the left we get

$$T(1)^D T(x) = (T(1)^D)^3 T(x) T(1)^2.$$

In view of Lemma 3.1.5 *(2)* and Lemma 2.1.5, $(T(1)^3)^D = (T(1)^D)^3$ commutes with $T(x)$. Consequently

$$T(1)^D T(x) = T(x)(T(1)^D)^3 T(1)^2 = T(x)T(1)^D,$$

and thus $T(1)^D$ commutes with $T(A)$. Now, from Lemma 3.1.6 *(2)* we know $T(1)^4T(x^2) = T(1)^3T(x)^2$. Multiplying by $(T(1)^D)^5$ on the left, this yields

$$T(1)^D T(x^2) = (T(1)^D)^2 T(x)^2 = (T(1)^D T(x))^2,$$

which shows that $T(1)^D T$ is a Jordan homomorphism, as desired. ■

Proof of Theorem 3.1.3 Let A and B be unital Banach algebras, and $T : A \to B$ be an additive map such that $T(a)T(a^{-1}) = T(1)^2$, for every invertible element $a \in A$. Let $u \in A$ with $T(u) \in B^{-1}$. By Lemma 3.1.6 *(2)*, $T(1)^4T(u^3) = T(1)^2T(u)^3$. Multiplying by $T(u)^{-3}$ on the right and having in mind that $T(1)^2$ commutes with $T(A)$ we obtain

$$T(1)^2 T(u^3) T(u)^{-3} T(1)^2 = T(1)^2.$$

Thus, $T(1)^2$ is regular with generalized inverse

$$T(u^3)T(u)^{-3}T(1)^2T(u^3)T(u)^{-3},$$

commuting with $T(1)^2$, so it is its group inverse. Therefore, $T(1)$ is Drazin invertible (with Drazin inverse $T(1)^D = \left(T(1)^2\right)^\sharp T(1)$) and the conditions in Proposition 3.1.7 are fulfilled. ■

Proof of Theorem 3.1.4 Let $T : A \to B$ be an additive map satisfying (3.1), or equivalently, (3.3). Suppose that $T(1)$ is invertible. From Proposition 3.1.7 we know that $S = T(1)^{-1}T$ is a unital Jordan homomorphism and $T(1)^{-1}$ commutes with $T(A)$.

Reciprocally, suppose that $T = T(1)S$, where S is a unital Jordan homomorphism and $T(1)$ commutes with $S(A)$. Take $a \in A^{-1}$. Since S is a unital Jordan homomorphism, it strongly preserves invertibility, and thus

$$T(a)T(a^{-1}) = T(1)S(a)T(1)S(a^{-1}) = T(1)^2S(a)S(a^{-1}) = T(1)^2. \blacksquare$$

As a direct consequence of Theorem 3.1.4, we can recover one of the main results in [16].

Corollary 3.1.8 *([16, Theorem 2.2]) Let A and B be unital Banach algebras and let $T : A \to B$ be an additive map strongly preserving invertibility. Then $T(1)$ commutes with $T(A)$ and $T(1)T$ is a unital Jordan homomorphism.*

Example 3.1.9 *The condition $T(1) \in B^{-1}$ cannot be replaced by $T(1) \in B^D$ nor by $T(1) \in B^\sharp$ to obtain the converse of Theorem 3.1.4. The following example shows why.*

Let $\mathcal{T}_2(\mathbb{C})$ be the unital Banach algebra of upper tringular complex matrices of order 2 and $T : \mathcal{T}_2(\mathbb{C}) \to \mathcal{T}_2(\mathbb{C})$ defined by

$$T\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} b & d-b \\ 0 & d \end{pmatrix}$$

We have:

$$T(1) = T\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = T(1)^2$$

$$T(1)T\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} 0 & d \\ 0 & d \end{pmatrix} = T\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}T(1)$$

From this it is clear that $T(1) = T(1)^D$ commutes with $T(\mathcal{T}_2(\mathbb{C}))$ and $T(1)T$ is homomorphism. However, for $ad \neq 0$:

$$T\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^{-1}\right) = \frac{1}{ad}T\begin{pmatrix} d & -b \\ 0 & a \end{pmatrix} = \frac{1}{ad}\begin{pmatrix} -b & a+b \\ 0 & a \end{pmatrix}$$

Thus,

$$T\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^{-1}\right)T\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} \frac{-b^2}{ad} & \frac{b^2}{ad}+1 \\ 0 & 1 \end{pmatrix}$$

which clearly depends on the elements $a, b, c \in \mathbb{C}$ chosen.

3.2 Additive preservers of Drazin and group inverses

In this section we provide a positive answer to Conjecture 1.1.15 for Drazin and group invertibility. In particular, we prove that an additive map $T : A \to B$ between Banach algebras, where A is supposed to be unital, strongly preserves Drazin (equivalently, group) invertibility if, and only if, it is a Jordan triple homomorphism. We also present a counterexample showing that the same does not hold for generalized invertibility.

Let A be a unital Banach algebra. Given $a \in A^{-1}$ and $\lambda \in \mathbb{Q}$ with $0 < |\lambda| < ||a^{-1}||^{-2}$, it is clear that $b = \lambda^{-1}a$, and $a - b^{-1} = a - \lambda a^{-1}$ are invertible elements in A. Therefore, by Hua's identity (see (3.2))

$$\left(a^{-1} - (a - \lambda a^{-1})^{-1}\right)^{-1} = a - a(\lambda^{-1}a)a = a - \lambda^{-1}a^3. \tag{3.5}$$

Lemma 3.2.1 *Let A and B be Banach algebras, and let $T : A \to B$ be an additive map strongly preserving group invertibility. Then $T(u^3) = T(u)^3$ for every $u \in A^\sharp$.*

Proof Let $u \in A^\sharp$, $u \neq 0$, and $p = uu^\sharp = u^\sharp u$ its associated idempotent. Then pAp is a unital Banach algebra (with unit element p) and $u \in pAp$ is invertible with inverse $u^\sharp \in pAp$. Thus, for every $\lambda \in \mathbb{Q}$ with $0 < |\lambda| < ||u^\sharp||^{-2}$, it is clear that $u - \lambda u^\sharp$ is an invertible element of the local algebra pAp. Moreover if $x \in pAp$ is invertible in pAp with inverse $y \in pAp$, then x has group inverse y in A. Therefore, the inverse of $u - \lambda u^\sharp$ regarded as an element of pAp is its group inverse in A. In the same way, the inverse of $u^\sharp - (u - \lambda u^\sharp)^\sharp$ in pAp is its group inverse in A. According to Equation (3.5), we get

$$u - \lambda^{-1}u^3 = (u^\sharp - (u - \lambda u^\sharp)^\sharp)^\sharp.$$

We may assume that $T(u) \neq 0$ (otherwise the result is trivial). As T strongly preserves group invertibility, we know that $T(u)$ has group inverse $T(u)^\sharp$. Given $\lambda \in \mathbb{Q}$ such that $0 < |\lambda| < \min\{||u^\sharp||^{-2}, ||T(u)^\sharp||^{-2}\}$, the above arguments applied to $T(u)$ yield

$$T(u) - \lambda^{-1}T(u)^3 = (T(u)^\sharp - (T(u) - \lambda T(u)^\sharp)^\sharp)^\sharp.$$

Since T is additive (hence $\mathbb{Q}$-linear) and strongly preserves group inverses, it follows that

$$\begin{aligned} T(u) - \lambda^{-1}T(u)^3 &= (T(u^\sharp) - T(u - \lambda u^\sharp)^\sharp)^\sharp \\ &= T(u^\sharp - (u - \lambda u^\sharp)^\sharp)^\sharp = T((u^\sharp - (u - \lambda u^\sharp)^\sharp)^\sharp) \\ &= T(u - \lambda^{-1}u^3) = T(u) - \lambda^{-1}T(u^3). \end{aligned}$$

Hence $T(u^3) = T(u)^3$, as desired. ■

It is clear that the zero map strongly preserves group invertibility; we prove that no other additive map annihilating the identity element strongly preserves group invertibility.

Proposition 3.2.2 *Let A and B be Banach algebras, and let $T : A \to B$ be an additive map strongly preserving group invertibility. Then either $T(1) \neq 0$ or $T = 0$.*

Proof If b and $1+b$ are invertible elements in A, as consequence of Hua's identity (3.2)

$$1 = (1+b)^{-1} + (1+b^{-1})^{-1}.$$

Since T strongly preserves group invertibility,

$$T(1) = T((1+b)^{-1}) + T((1+b^{-1})^{-1}) = (T(1)+T(b))^\sharp + (T(1)+T(b)^\sharp)^\sharp.$$

If we assume that $T(1) = 0$, then we get

$$T(b)^\sharp + T(b) = 0,$$

for every $b \in A^{-1}$ with $1 + b \in A^{-1}$. Thus, given $a \in A^{-1}$, and $\alpha \in \mathbb{Q} \setminus \{0\}$ such that $|\alpha| < ||a||^{-1}$, it is clear that αa and $1 + \alpha a$ are invertible, and therefore $T(a)^\sharp = -\alpha^2 T(a)$. By the uniqueness of the group inverse, we have $T(a) = 0$. Thus T is the zero map. ■

Proposition 3.2.3 *Let A and B be Banach algebras, A is assumed to be unital. Let $T : A \to B$ be an additive map strongly preserving group invertibility. For every $x \in A$,*

(1) $3T(x) = T(1)^2T(x) + T(x)T(1)^2 + T(1)T(x)T(1)$,

(2) $3T(x^2) = T(x)^2T(1) + T(1)T(x)^2 + T(x)T(1)T(x)$.

Proof Let $u \in A^{-1}$ and $\alpha \in \mathbb{Q}$ be such that $0 < |\alpha| < ||u^{-1}||^{-1}$. Then $u + \alpha \in A^{-1}$, and by Lemma 3.2.1, we know that $T(u^3) = T(u)^3$, $T(1) = T(1)^3$ and $T((u+\alpha)^3) = T(u+\alpha)^3$. As T is additive, these three equalities can be combined in order to obtain

$$\begin{aligned} 3T(u^2) + 3\alpha T(u) \quad &= \quad T(u)^2T(1) + \alpha T(1)^2T(u) + T(u)T(1)T(u) \\ &\quad\quad +\alpha T(u)T(1)^2 + T(1)T(u)^2 + \alpha T(1)T(u)T(1). \end{aligned}$$

From this it is clear that

$$3T(u) = T(1)^2T(u) + T(u)T(1)^2 + T(1)T(u)T(1), \tag{3.6}$$

and

$$3T(u^2) = T(u)^2T(1) + T(1)T(u)^2 + T(u)T(1)T(u), \tag{3.7}$$

for every invertible element u in A.

Given $x \in A$, let $\mu \in \mathbb{Q}$ be such that $u = x + \mu$ is invertible. Since $T(1)^3 = T(1)$ and T is additive, statement *(1)* follows directly from Equation (3.6). Moreover, by taking $u = x + \mu$ in Equation (3.7) and rearranging terms we get

$$\begin{aligned} 3T(x^2) + 3\mu^2T(1) + 6\mu T(x) \quad &= \quad T(x)^2T(1) + T(1)T(x)^2 + 3\mu^2T(1)^3 \\ &\quad + \quad 2\mu(T(1)^2T(x) + T(x)T(1)^2 + T(1)T(x)T(1)) \\ &\quad + \quad T(x)T(1)T(x). \end{aligned}$$

Having in mind the assertion *(1)* just proved and that $T(1) = T(1)^3$, statement *(2)* can be deduced immediately from the above expression. ■

We present now the main result in this section.

Theorem 3.2.4 *Let A and B be Banach algebras and let $T : A \to B$ be an additive map. Suppose that A is unital. The following conditions are equivalent:*

(1) T strongly preserves Drazin invertibility,

(2) T strongly preserves group invertibility,

(3) $T(1) = T(1)^3$ and $T = T(1)S$ for a Jordan homomorphism $S : A \to B$ such that $T(1)$ commutes with the range of S,

(4) T is a Jordan triple homomorphism.

Proof *(1) ⇒ (2).* Assume that T strongly preserves Drazin invertibility. Recall that, if $x \in A$ is Drazin invertible, then x^D is Drazin invertible and $(x^D)^D = x^2x^D$. In particular, x has a group inverse if, and only if, $x = (x^D)^D$ (see Lemma 2.1.5). Thus, for every group invertible element $u \in A$,

$$T(u) = T((u^D)^D) = T(u^D)^D = (T(u)^D)^D,$$

which shows that $T(u)$ has group inverse

$$T(u)^\sharp = T(u)^D = T(u^D) = T(u^\sharp).$$

That is, T strongly preserves group invertibility.

(2) ⇒ (3) Suppose that T strongly preserves group invertibility. By the preceding proposition we know that

$$3T(x) = T(1)^2T(x) + T(x)T(1)^2 + T(1)T(x)T(1), \tag{3.8}$$

and

$$3T(x^2) = T(x)^2T(1) + T(1)T(x)^2 + T(x)T(1)T(x), \tag{3.9}$$

for all $x \in A$.

Multiplying (3.8) on the left and right by $T(1)$ and having in mind that $T(1)^3 = T(1)$ we deduce

$$T(1)T(x)T(1) = T(1)^2T(x)T(1)^2,$$

for all $x \in A$. If we multiply again this equation by $T(1)$ we get

$$T(1)^2T(x)T(1) = T(1)T(x)T(1)^2, \tag{3.10}$$

for all $x \in A$. Also from Equation (3.8) (by multiplying by $T(1)$ on the left and right, respectively) and Equation (3.10) it follows

$$2T(1)T(x) = T(1)T(x)T(1)^2 + T(1)^2T(x)T(1) = 2T(1)T(x)T(1)^2,$$

and

$$2T(x)T(1) = T(1)^2T(x)T(1) + T(1)T(x)T(1)^2 = 2T(1)T(x)T(1)^2.$$

That is,

$$T(x)T(1) = T(1)T(x), \tag{3.11}$$

for every x in A. Taking into account this last equality in (3.8) and (3.9) we deduce

$$T(x) = T(1)^2 T(x) = T(x)T(1)^2, \tag{3.12}$$

and

$$T(x^2) = T(x)^2 T(1), \tag{3.13}$$

for all $x \in A$. Thus, $S(x) = T(1)T(x)$ defines a Jordan homomorphism, and $T(x) = S(x)T(1) = T(1)S(x)$, for all $x \in A$.

(3) $\Rightarrow$ *(1)* Take $T(1) = T(1)^3$ and $T = T(1)S$ for a Jordan homomorphism $S : A \to B$ such that $T(1)$ commutes with the range of S. As S is a Jordan homomorphism, it strongly preserves Drazin invertibility (see Theorem 1.1.13). Let $a \in A^D$ and $b = a^D$, that is $ab = ba$, $bab = b$ and $a^k = a^k ba$, where $k = \mathrm{ind}(a)$. Since S is a Jordan homomorphism, $T(1) = T(1)^3$ and $T(1)$ commutes with the image of S, it is clear that $T(a)T(b) = T(b)T(a)$ and

$$\begin{aligned} T(b) &= T(bab) = T(1)S(bab) = T(1)S(b)S(a)S(b) = T(1)^3 S(b)S(a)S(b) \\ &= T(1)S(b)T(1)S(a)T(1)S(b) = T(b)T(a)T(b). \end{aligned}$$

Similar arguments yield

$$\begin{aligned} T(a)^k &= T(1)^k S(a)^k = T(1)^k S(a)^k S(b)S(a) = T(1)^k S(a)^k T(1)S(b)T(1)S(a) \\ &= T(a)^k T(b)T(a). \end{aligned}$$

This proves that T strongly preserves Drazin invertibility.

To conclude the proof, we show that *(3)* and *(4)* are equivalent. Notice that if $T = T(1)S$ for a Jordan homomorphism $S : A \to B$ such that $T(1)$ commutes with the range of S and $T(1) = T(1)^3$, it is clear that for every $a, b \in A$,

$$\begin{aligned} T(a)T(b)T(a) &= T(1)S(a)T(1)S(b)T(1)S(a) = T(1)^3 S(a)S(b)S(a) \\ &= T(1)S(aba) = T(aba), \end{aligned}$$

which shows that T is a Jordan triple homomorphism. Reciprocally, if T is a Jordan triple homomorphism, for every $x \in A$, as

$$x = \frac{1}{2}(x11 + 11x) = 1x1$$

it follows that

$$T(x) = \frac{1}{2}(T(x)T(1)^2 + T(1)^2 T(x)) = T(1)T(x)T(1).$$

Multiplying on the right, respectively left, by $T(1)$ and having in mind that $T(1)^3 = T(1)$ we get

$$T(x)T(1) = T(1)^2 T(x)T(1) = T(1)T(x)T(1)^2 = T(1)T(x).$$

This proves that $T(1)$ commutes with the range of T. Moreover, since $x^2 = x1x$ we also have $T(x^2) = T(x)^2T(1)$. Therefore $S = T(1)T$ is a Jordan homomorphism such that $T = T(1)S$. ■

Remark 3.2.5 *Let A and B be unital Banach algebras and $T : A \to B$ be an additive map strongly preserving group invertibility. From Equation (3.12) it is clear that if $T(A) \cap B^{-1} \neq \emptyset$, then $T(1)^2 = 1$. Hence T is a unital Jordan homomorphism multiplied by an invertible element commuting with the range of T. (Compare with [16, Lemma 3.6, Lemma 3.7] which also holds when generalized invertibility is replaced by group invertibility.)*

Let us discuss about strongly preservers of Koliha-Drazin invertibility. Recall that an element $a \in A$ is said to be Koliha-Drazin invertible if there exists $b \in A$ such that $ab = ba$, $bab = b$ and $a - a^2b$ is quasi-nilpotent. Moreover, by [83, Theorem 5.4], if a has Koliha-Drazin inverse, then $(a^{KD})^{KD} = a^2a^{KD}$. Hence a has group inverse if, and only if, $a = (a^{KD})^{KD}$. Therefore if $T : A \to B$ is an additive map between unital complex Banach algebras strongly preserving Koliha-Drazin invertibility, the same arguments employed in *(1) ⇒ (2)* of Theorem 3.2.4, show that T strongly preserves group invertibility, and hence there exists a Jordan homomorphism $S : A \to B$ with $T(x) = T(1)S(x) = S(x)T(1)$, for all $x \in A$.

Now, let $T(x) = T(1)S(x) = S(x)T(1)$, for all $x \in A$, with S a unital Jordan homomorphism and $T(1)^3 = T(1)$. Given a Koliha-Drazin invertible element $a \in A$ with $a^{KD} = b$, arguing as in *(3) ⇒ (1)* of Theorem 3.2.4 it follows that $T(a)T(b) = T(b)T(a)$, $T(b)T(a)T(b) = T(b)$. Let us show that $T(a) - T(a)^2T(b)$ is quasi-nilpotent. Notice that since S a unital Jordan homomorphism, it preserves quasi-nilpotent elements. Moreover, since $S(a^2b) = S(aba) = S(a)^2S(b)$ it is clear that

$$T(a) - T(a)^2T(b) = T(1)(S(a) - S(a)^2S(b)) = T(1)S(a - a^2b)$$

is quasi-nilpotent.

We conclude this section providing a negative answer concerning additive strongly preservers of generalized invertibility in Conjecture 1.1.15.

Example 3.2.6 *Let $T : A \to B$, where $A = C([0,1])$ and $B = M_4(\mathbb{C})$ defined by*

$$T(f) = \begin{pmatrix} f(0) & f(1) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & f(0) & 0 \\ 0 & 0 & f(1) & 0 \end{pmatrix}$$

It can be checked that T strongly preserves generalized invertibility, since every matrix of the form

$$\begin{pmatrix} a & b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & b & 0 \end{pmatrix}$$

with $a \neq 0$ *admits as a generalized inverse every matrix of the form*

$$\begin{pmatrix} a^{-1} & c & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & d & 0 \end{pmatrix}$$

for every $b, c, d \in \mathbb{C}$. *It is straightforward to see that* $T(1)$ *does not commute with* $T(A)$ *and that neither* $T(1)T$ *nor* $TT(1)$ *are Jordan homomorphisms.*

3.3 Additive preservers of inverses along an element

One of the main goals in this section is to obtain Hua type theorems concerning invertibility along an element. In particular we characterize Jordan triple homomorphisms between Banach algebras as those additive mappings strongly preserving the inverse along an element. As we have pointed out in Section 2.1, this notion was recently introduced by X. Mary in [99] and it gathers the classical notions of group, Drazin and Moore-Penrose invertibility.

Recall that an element a in a Banach algebra A is invertible along $d \in A$, if there exist $b = a^{\|d} \in A$, $x, y \in A \cup \{1\}$ such that

$$d = dab = bad, \qquad b = xd = dy.$$

It is known that $a^{\|d} = d(ad)^\sharp = (da)^\sharp d$ ([99, Theorem 7]). Notice that if the inverse of an element a along d exists, then $d = dab = d(ax)d$ (also, $d = bad = d(ya)d$) which shows that d is regular. Moreover, $d \in A^\wedge$ if, and only if, $d = a^{\|d}$, for some $a \in A$. For a regular element d, let us denote by $A^{\|d}$ the set of elements of A that are invertible along d.

In the next theorem we characterize Jordan triple homomorphism between Banach algebras by means of invertibility along an element preserving.

Theorem 3.3.1 *Let* A *and* B *be Banach algebras with* A *unital and* $T : A \to B$ *an additive map. The following conditions are equivalent:*

(1) $T(a^{\|d}) = T(a)^{\|T(d)}$ *for every* $d \in A^\wedge$ *and* $a \in A^{\|d}$.

(2) $T(a^{\|1}) = T(a)^{\|T(1)}$ *for all* $a \in A^{-1}$.

(3) $T(a^{\|a}) = T(a)^{\|T(a)}$ *for all* $a \in A^\sharp$.

(4) T *is a Jordan triple homomorphism.*

Proof Notice that *(1)* clearly implies *(2)* and *(3)*, and that *(3)* $\Leftrightarrow$ *(4)* is merely a reformulation of *(2)* $\Leftrightarrow$ *(3)* in Theorem 3.2.4.

Let us prove that *(2)* implies *(4)*. By hypothesis, given $a \in A^{-1}$, there exist $u, v \in B$ such that

$$T(a)T(a^{-1})T(1) = T(1) = T(1)T(a^{-1})T(a), \tag{3.14}$$

and

$$T(a) = T(1)u = vT(1). \tag{3.15}$$

Choosing $a = 1$ in (3.14) we get $T(1)^3 = T(1)$. Multiplying (3.15) by $T(1)^2$ on the left we obtain

$$T(1)^2T(a) = T(1)^3u = T(1)u = T(a).$$

Analogously it can be checked that $T(a) = T(a)T(1)^2$, for every $a \in A^{-1}$. It is clear now that

$$T(x) = T(1)^2T(x) = T(x)T(1)^2, \quad \text{for all } x \in A.$$

Given $a \in A^{-1}$,

$$T(a^{-1})T(a) = T(1)^2T(a^{-1})T(a) = T(1)\left(T(1)T(a^{-1})T(a)\right) = T(1)^2$$

and, as $T(1)^\sharp = T(1)$, Proposition 3.1.7 allows us to conclude that $S = T(1)T$ is a Jordan homomorphism and $T(1)$ commutes with $T(A)$. As $T(x) = T(1)^2T(x) = T(1)S(x)$ for all $x \in A$, it is clear that *(3)* holds.

Finally, suppose that T is a Jordan triple homomorphism, equivalently, $S = T(1)T$ is a Jordan homomorphism, $T(1)$ commutes with $S(A)$, $T = T(1)S$ and $T(1) = T(1)^3$ (see *(3)* $\Leftrightarrow$ *(4)* Theorem in 3.2.4). Let $a, b, d \in A$ be such that $b = a^{\|d}$. Hence $bad = d = dab$, $b = xd = dy$, for some $x, y \in A$. Let us show that $T(b) = T(a)^{\|T(d)}$. Since T is a Jordan triple homomorphism and $bab = b$, it is clear that $T(b)T(a)T(b) = T(b)$. Also,

$$2T(d) = T(bad + dab) = T(b)T(a)T(d) + T(d)T(a)T(b).$$

Multiplying this equation by $T(b)T(a)$ on the left (having in mind that $T(b)T(a)$ is an idempotent), we obtain

$$T(b)T(a)T(d) = T(b)T(a)T(d)T(a)T(b) = T(badab) = T(d).$$

Analogously it can be checked that $T(d)T(a)T(b) = T(d)$. Moreover, since $b = bab = d(yax)d$, it follows $T(b) = T(d)T(yax)T(d)$, which shows that $T(b) \leq_{\mathcal{H}} T(d)$, as desired. ■

It is worth to mention what happens when a map $T : A \to B$ between unital Banach algebras satisfies the condition

$$T(1^{\|a}) = T(1)^{\|T(a)} \text{ for all } a \in A^\sharp. \tag{3.16}$$

By [100, Corollary 3.4] $a \in A$ is group invertible if, and only if, $1^{\|a}$ exists. In this case, it can be checked that $1^{\|a} = aa^\sharp$. In particular, if $p \in A^\bullet$, by assumption, $T(p) = T(1^{\|p}) = T(1)^{\|T(p)}$, that is, $T(p)T(1)T(p) = T(p)$.

This shows that $S = T(1)T$ preserves idempotents. It is straightforward to prove that an idempotent preserving linear map must send a set of mutually orthogonal idempotents to a set of mutually orthogonal idempotents, where two idempotents p and q are said to be **orthogonal** if $pq = qp = 0$. The following observation has become a standard tool in the study of Jordan homomorphisms (see [7]).

Lemma 3.3.2 *Let A be a real rank zero C^*-algebra, B a Banach algebra and let $T : A \to B$ be a bounded linear mapping sending projections to idempotents. Then T is a Jordan homomorphism. Moreover, if B is a C^*-algebra and T sends projections to projections, it is in fact a Jordan *-homomorphism.*

The next theorem follows easily from the previous lemma.

Theorem 3.3.3 *If $T : A \to B$ is a continuous linear mapping from a C^*-algebra with real rank zero into a Banach algebra such that $T(1^{\|a}) = T(1)^{\|T(a)}$ for all $a \in A^\sharp$, then $T(1)T$ is a Jordan homomorphism.*

Suppose now that $T : A \to B$ is a surjective linear map between unital Banach algebras satisfying (3.16). Choose an invertible element $a \in A$. It is easy to see that $1^{\|a} = 1$. Hence, $T(1) = T(1^{\|a}) = T(1)^{\|T(a)}$ and, by definition of inverse along $T(a)$ we have

$$T(1)^2T(a) = T(a) = T(a)T(1)^2, \tag{3.17}$$

$$T(1) = T(a)y = zT(a) \text{ for some } y, z \in B. \tag{3.18}$$

From Equation (3.17), it follows that $T(1)^2T(x) = T(x)T(1)^2 = T(x)$, for all $x \in A$. Since T is surjective $T(1)^2 = 1$ and from (3.18) we conclude that $T(a)$ is invertible. Hence $S = T(1)T$ is a unital surjective linear map preserving invertibility. In particular, S is bounded providing that B is semisimple ([6, Theorem 5.5.2]).

Theorem 3.3.4 *Let A and B be unital semisimple Banach algebras, and $T : A \to B$ be a bijective linear mapping such that $T(1^{\|a}) = T(1)^{\|T(a)}$ for all $a \in A^\sharp$. If A has essential socle, then $T(1)T$ is a Jordan isomorphism.*

Let A and B be Banach algebras. As usual A is assumed to be unital. In Theorem 3.3.1 we have shown in particular that a Jordan triple homomorphism $T : A \to B$ strongly preserves the inverse along an element, that is, $T(a)^{\|T(d)} = T(a^{\|d})$ whenever $a \in A^{\|d}$. As we have already mentioned, $a \in A^\sharp$ if, and only if, $1^{\|a}$ exists, and in such case $1^{\|a} = aa^\sharp$. Therefore,

$$T(1)^{\|T(a)} = T(1^{\|a}) = T(aa^\sharp).$$

As $T(1)T$ is a Jordan homomorfism, $T(1)$ commutes with $T(A)$ and $T(x) = T(1)^2T(x)$ for all $x \in A$, it follows that

$$T(1)T(1)^{\|T(a)} = T(1)T(aa^\sharp) = T(1)T(a)T(1)T(a^\sharp) = T(a)T(a^\sharp). \tag{3.19}$$

Clearly the role of a and $a^\sharp$ can be swapped. If now $a \in A^D$, it is known that $a^D \in A^\sharp$ with $(a^D)^\sharp = a^2 a^D$ (see Lemma 2.1.5). Hence 1 is invertible along a^D and

$$1^{\|a^D} = a^D(a^2 a^D) = aa^D.$$

Since T a Jordan triple homomorphism, we also obtain

$$T(1)^{\|T(a^D)} = T(1^{\|a^D}) = T(aa^D),$$

and

$$T(1)T(1)^{\|T(a^D)} = T(1)T(aa^D) = T(a)T(a^D). \tag{3.20}$$

Besides, if $m = \mathrm{ind}(a)$, then a^m is group invertible with $(a^m)^\sharp = \left(a^D\right)^m$. Thus, from Equation (3.19) we deduce

$$\begin{aligned} T(1)T(1)^{\|T(a^m)} &= T(1)T(a^m\,(a^m)^\sharp) = T(1)T(a^m\left(a^D\right)^m) \\ &= T(1)T(aa^D) = T(a)T(a^D). \end{aligned} \tag{3.21}$$

The following result shows how Equations (3.19), (3.20) and (3.21) characterize Jordan triple homomorphism.

Theorem 3.3.5 *Let A and B be Banach algebras with A unital and $T : A \to B$ be an additive map. The following condition are equivalent:*

(1) $T(a)T(a^D) = T(1)T(1)^{\|T(a^m)} = T(1)^{\|T(a^D)}T(1)$ for all $a \in A^D$ where $m = ind(a)$,

(2) $T(a)T(a^\sharp) = T(1)T(1)^{\|T(a)} = T(1)^{\|T(a^\sharp)}T(1)$ for all $a \in A^\sharp$,

(3) T is a Jordan triple homomorphism.

Proof *(1) ⇒ (2)* is trivial since, for $a \in A^\sharp$, we have $a^D = a^\sharp$ and $m = 1$.

For *(2) ⇒ (3)*, take $a \in A^\sharp$. Since $T(a)T(a^\sharp) = T(1)T(1)^{\|T(a)}$, multiplying the previous identity by $T(a)$ on the left we get

$$T(a)^2T(a^\sharp) = T(a)T(1)T(1)^{\|T(a)} = T(a).$$

Replacing a with $a^\sharp$, we obtain $T(a^\sharp)^2T(a) = T(a^\sharp)$. Now, from

$$T(a)T(a^\sharp) = T(1)^{\|T(a^\sharp)}T(1),$$

multiplying by $T(a^\sharp)$ on the right, we get

$$T(a)T(a^\sharp)^2 = T(1)^{\|T(a^\sharp)}T(1)T(a^\sharp) = T(a^\sharp).$$

As above, this also gives us $T(a^\sharp)T(a)^2 = T(a)$. This shows that

$$T(a^\sharp) = T(a)^\sharp \quad \text{for every} \quad a \in A^\sharp.$$

By Theorem 3.2.4, *(3)* holds. (We have used here the fact that if x, y satisfy that $x^2y = x$, $y^2x = y$, $yx^2 = x$ and $xy^2 = y$, then $x^\sharp = y$.)

Finally, *(3) ⇒ (1)* follows from Equations (3.20) and (3.21), having in mind that $T(1)$ commutes with $T(A)$. ∎

Remark 3.3.6 *Observe that for a Jordan triple homomorphism $T: A \to B$, assertion (1) in Theorem 3.3.5 holds for every $n \geq ind(a)$. Indeed, from [99, Corollary 12] we know that for $a \in A^D$ with $m = ind(a)$, $a^m \mathcal{H} a^n$ for every $n \geq m$, and thus Corollary 2.4 in [100] entails that $a^{\|a^m} = a^{\|a^n}$ for all $n \geq m$. Since T is a Jordan triple homomorphism*

$$T(a)^{\|T(a^m)} = T(a^{\|a^m}) = T(a^{\|a^n}) = T(a)^{\|T(a^n)}.$$

3.4 Approximate preservers in Banach algebras

In this section we provide approximate versions of Hua's theorem and Theorem 3.2.4. We show that, if an additive map almost strongly preserves invertibility, Drazin invertibility or group invertibility then it is almost a certain multiple of a Jordan homomorphism.

Let A and B be Banach algebras and $T : A \to B$ be a bounded linear map. Following [2] and [77], the **multiplicativity**, **anti-multiplicativity** and **Jordan-multiplicativity** of T can be measured by considering the values

$$\text{mult}(T) = \sup\{||T(ab) - T(a)T(b)||\colon a, b \in A,\ ||a|| = ||b|| = 1\},$$

$$\text{amult}(T) = \sup\{||T(ab) - T(b)T(a)||\colon a, b \in A,\ ||a|| = ||b|| = 1\},$$

$$\text{jmult}(T) = \sup\{||T(a^2) - T(a)^2||\colon a \in A,\ ||a|| = 1\},$$

respectively. Obviously, T is a homomorphism (anti-homomorphism, Jordan homomorphism) if, and only if, $\text{mult}(T) = 0$ (respectively, $\text{amult}(T) = 0$, $\text{jmult}(T) = 0$).

Ultraproducts of Banach algebras and C*-algebras

Let us introduce the main tool in this section. Given a set $X \neq \emptyset$, recall that a **filter** X is a family $\mathcal{F} \neq \emptyset$ of subsets of X, fulfilling the folowing properties:

(1) $\emptyset \notin \mathcal{F}$,

(2) If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$,

(3) If $A \in \mathcal{F}$ and $A \subset B$ then $B \in \mathcal{F}$.

An **ultrafilter** is a filter that is maximal respect to the inclusion, that is, $\mathcal{F}$ is an ultrafilter if $\mathcal{F} \subset \mathcal{G}$ for another filter $\mathcal{G}$ implies $\mathcal{F} = \mathcal{G}$. This is equivalent to $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$ for every $A \subset X$. It is easy to see that, for any non-empty set X and $x \in X$, the family $\mathcal{U}_x = \{A \subset X : x \in A\}$ is an ultrafilter. This kind of ultrafilter is said to be **fixed** or **trivial**. Otherwise, the ultrafilter is said to be **free**. Note that, according to Zorn's lemma, every filter is contained in an ultrafilter.

Given a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and a sequence of Banach spaces $\{X_n\}_{n\in\mathbb{N}}$, the so-called **ultraproduct** of the sequence is defined as follows:

$$(X_n)^{\mathcal{U}} := \frac{\ell^\infty(\mathbb{N}, X_n)}{\mathcal{N}_{\mathcal{U}}},$$

where $\ell^\infty(\mathbb{N}, X_n)$ is the Banach space of all bounded sequences $\{x_n\}_{n\in\mathbb{N}}$ equipped with the ℓ_∞ norm and

$$\mathcal{N}_{\mathcal{U}} := \{\{x_n\}_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, X_n) : \lim_{\mathcal{U}} \|x_n\| = 0\}.$$

If the sequence $\{X_n\}_{n\in\mathbb{N}} = \{X\}$ is constant, $X^{\mathcal{U}} := \frac{\ell^\infty(\mathbb{N},X)}{\mathcal{N}_{\mathcal{U}}}$, is called the **ultrapower** of X with respect to the ultrafilter $\mathcal{U}$. We will denote by $\mathbf{x} = [x_n]$ the equivalence class of the sequence $\{x_n\}_{n\in\mathbb{N}}$. The ultrapower of a Banach space is also a Banach space provided with the norm

$$\|\mathbf{x}\| := \lim_{\mathcal{U}} \|x_n\|.$$

This norm is well defined by the properties of the ultrafilters: every bounded sequence has a limit through the ultrafilter ($\lim_{\mathcal{U}} y_n = y$ if for every $\varepsilon > 0$ there exists $U \in \mathcal{U}$ such that $|y_n - y| < \varepsilon$ for all $n \in \mathcal{U}$). Of course, the ultrapower $A^{\mathcal{U}}$ of a Banach algebra (respectively, C*-algebra) is also a Banach algebra (respectively, C*-algebra), with respect to the pointwise operations.

Finally, for every Banach spaces X and Y, the canonical linear isometry $\mathcal{B}(X,Y)^{\mathcal{U}} \to \mathcal{B}(X^{\mathcal{U}}, Y^{\mathcal{U}})$ given by

$$\mathbf{T}(\mathbf{x}) = [T_n(x_n)],$$

for every $\mathbf{T} = [T_n] \in \mathcal{B}(X,Y)^{\mathcal{U}}$ and $\mathbf{x} = [x_n] \in X^{\mathcal{U}}$, allows us to consider $\mathcal{B}(X,Y)^{\mathcal{U}}$ as a closed subspace of $\mathcal{B}(X^{\mathcal{U}}, Y^{\mathcal{U}})$. For $X = Y$, the canonical map gives an isometric unital homomorphism from $\mathcal{B}(X)^{\mathcal{U}}$ to $\mathcal{B}(X^{\mathcal{U}})$. An important fact is to recall, as the authors showed in [2], Jordan multiplicativity is stable under the limit throught an ultrafilter, that is, if $\mathbf{T} = [T_n]$, then $jmult(\mathbf{T}) = \lim_{\mathcal{U}} jmult(T_n)$. The reader can see [67] in order to find basic results on ultraproducts.

Approximate preservers

Let A be a unital Banach algebra and $\mathcal{U}$ a free ultrafilter in $\mathbb{N}$. The following proposition is devoted to the description of invertible elements in $A^{\mathcal{U}}$ through certain coset representatives. The result is probably well-known, but the lack of an adequate reference moves us to include it here.

Proposition 3.4.1 *Let $\boldsymbol{a} \in A^{\mathcal{U}}$. The following assertions are equivalent.*

(1) $\boldsymbol{a}$ is invertible.

(2) $\boldsymbol{a}$ has a coset representative $[u_n]$ such that $u_n \in A^{-1}$ for all $n \in \mathbb{N}$ and $\{u_n^{-1}\}_{n\in\mathbb{N}}$ is bounded.

Proof For *(2)* $\Rightarrow$ *(1)*, just note that $[u_n^{-1}] \in A^{\mathcal{U}}$ is an inverse for $[u_n]$. Reciprocally, assume that $\boldsymbol{a} = [a_n]$ is invertible. Then there exists $\mathbf{b} = [b_n] \in A^{\mathcal{U}}$ such that $\boldsymbol{ab} = \boldsymbol{ba} = \mathbf{1}$. That is

$$\lim_{\mathcal{U}} \|a_n b_n - 1\| = 0,$$

and

$$\lim_{\mathcal{U}} \|b_n a_n - 1\| = 0.$$

Fix $0 < \delta < 1$. The above identities imply that

$$R := \{n \in \mathbb{N} : \|a_n b_n - 1\| < \delta\} \in \mathcal{U},$$

and

$$L := \{n \in \mathbb{N} : \|b_n a_n - 1\| < \delta\} \in \mathcal{U}.$$

In particular, a_n is right invertible for every $n \in R$ and a_n is left invertible for every $n \in L$. Thus, a_n is invertible for every $n \in I := R \cap L \in \mathcal{U}$. Moreover,

$$\|a_n^{-1}\| = \|b_n(a_n b_n)^{-1}\| \leq \|b_n\| \, \|(a_n b_n)^{-1}\| \leq \frac{\|b_n\|}{1 - \|1 - a_n b_n\|} \leq \frac{\|b_n\|}{1-\delta},$$

which shows that $\{a_n^{-1} \colon n \in I\}$ is bounded. Therefore, we can assume without loss of generality that $\{a_n\}_{n\in\mathbb{N}}$ consists of invertible elements and $\{a_n^{-1}\}_{n\in\mathbb{N}}$ is bounded. (Otherwise, we chose

$$a'_n = \begin{cases} a_n & \text{if } n \in I \\ 1 & \text{if } n \notin I. \end{cases}$$

Clearly $[a_n] = [a'_n]$.) ■

Remark 3.4.2 *It is clear that for $\boldsymbol{a} = [a_n] \in A^{\mathcal{U}}$ with $\|\boldsymbol{a}\| = 1$, we can choose a coset representative $\boldsymbol{a} = [b_n]$ such that $\|b_n\| = 1$ for all $n \in \mathbb{N}$:*

$$b_n = \begin{cases} \frac{a_n}{\|a_n\|} & \text{if } a_n \neq 0 \\ 1 & \text{if } a_n = 0. \end{cases}$$

Hence, for every invertible element $\boldsymbol{a}$ in $A^{\mathcal{U}}$, we can find a coset representative $\boldsymbol{a} = [a_n]$ fulfilling the conditions in Proposition 3.4.1 and satisfying $\|a_n\| = \|\boldsymbol{a}\|$ for all $n \in \mathbb{N}$. We will name this one a ***normalized representative*** *for $\boldsymbol{a}$.*

Let A and B be unital Banach algebras. Recall that, Boudi and Mbekhta proved in [16, Theorem 2.2] that an additive map $T : A \to B$ strongly preserves invertibility if, and only if, $T(1)T$ is a unital Jordan homomorphism and $T(1)$ commutes with the range of T. Hence, for a bounded linear map $T : A \to B$ between unital Banach algebras, we consider the **unit-commutativity** of T, defined as

$$\mathrm{ucomm}(T) = \sup_{\|a\|=1} \|T(a)T(1) - T(1)T(a)\|,$$

in order to measure how close is our “approximately preserving invertibility” map from fulfilling that property.

Obviuosly, every bounded linear map satisfies $\mathrm{ucomm}(T) \leq 2\|T\|^2$. The next lemma shows the good behaviour of this concept with the ultraproduct of operators.

Some arguments in this section are inspired in [2].

Lemma 3.4.3 *Let $\{T_n\}_{n\in\mathbb{N}}$ be a bounded sequence of linear maps between Banach algebras A and B, where A is supposed to be unital. Consider $\mathbf{T} = [T_n] : A^{\mathcal{U}} \to B^{\mathcal{U}}$. Then*

$$\lim_{\mathcal{U}} \mathrm{ucomm}(T_n) = \mathrm{ucomm}(\mathbf{T}).$$

Proof Given $\boldsymbol{a} \in A^{\mathcal{U}}$ with $\|\boldsymbol{a}\| = 1$, we can choose $\boldsymbol{a} = [a_n]$ with $\|a_n\| = 1$ for every $n \in \mathbb{N}$. Therefore,

$$\|\mathbf{T}(\boldsymbol{a})\mathbf{T}(\mathbf{1}) - \mathbf{T}(\mathbf{1})\mathbf{T}(\boldsymbol{a})\| = \lim_{\mathcal{U}} \|T_n(a_n)T_n(1) - T_n(1)T_n(a_n)\| \leq \lim_{\mathcal{U}} \mathrm{ucomm}(T_n)$$

and hence,

$$\mathrm{ucomm}(\mathbf{T}) \leq \lim_{\mathcal{U}} \mathrm{ucomm}(T_n).$$

Reciprocally, for each $n \in \mathbb{N}$, there exists $a_n \in A$ with $\|a_n\| = 1$ such that

$$\mathrm{ucomm}(T_n) - \frac{1}{n} < \|T_n(a_n)T_n(1) - T_n(1)T_n(a_n)\|.$$

Taking limits through $\mathcal{U}$ we obtain

$$\lim_{\mathcal{U}} \mathrm{ucomm}(T_n) \leq \lim_{\mathcal{U}} \|T_n(a_n)T_n(1) - T_n(1)T_n(a_n)\| =$$

$$= \|\mathbf{T}(\boldsymbol{a})\mathbf{T}(\mathbf{1}) - \mathbf{T}(\mathbf{1})\mathbf{T}(\boldsymbol{a})\| \leq \mathrm{ucomm}(\mathbf{T}). \blacksquare$$

The next result provides an approximate version of the Hua's theorem for Banach algebras [16, Theorem 2.2] above mentioned.

Theorem 3.4.4 *Let A and B be unital Banach algebras and $K, \varepsilon > 0$. Then there exists $\delta > 0$ such that for every linear map $T : A \to B$ with $\|T\| < K$, the condition*

$$\sup_{\|a\|=1, a\in A^{-1}} \left\|T(a^{-1}) - T(a)^{-1}\right\| < \delta$$

implies

$$\mathrm{jmult}(T(1)T) < \varepsilon \quad \textit{and} \quad \mathrm{ucomm}(T) < \varepsilon.$$

Proof Suppose that the assertion of the theorem is false. Then we can find $K_0, \varepsilon_0 > 0$ and a sequence $\{T_n\}_{n\in\mathbb{N}}$ of linear maps from A to B such that, for every $n \in \mathbb{N}$,

- $\|T_n\| < K_0$,
- $\sup_{\|a\|=1} \left\|T_n(a^{-1}) - T_n(a)^{-1}\right\| < \frac{1}{n}$ and
- $\mathrm{jmult}(T_n(1)T_n) \geq \varepsilon_0$ or $\mathrm{ucomm}(T_n) \geq \varepsilon_0$.

Consider $\mathbf{T} = [T_n] : A^{\mathcal{U}} \to B^{\mathcal{U}}$. We claim that $\mathbf{T}$ strongly preserves invertibility. Indeed, let $\boldsymbol{a} \in A^{\mathcal{U}}$ be an invertible element. We can suppose, without loss of generality, that $\|\boldsymbol{a}\| = 1$. Let $[a_n]$ be its normalized representative, with $\|a_n^{-1}\| < \alpha$, for some $\alpha > 0$ (see Proposition 3.4.1 and Remark 3.4.2). As

$$\left\|T_n(a_n^{-1})\right\| \leq \|T_n\| \left\|a_n^{-1}\right\| < K_0\, \alpha,$$

and

$$\left\|T_n(a_n^{-1}) - T_n(a_n)^{-1}\right\| < 1,$$

we get

$$\left\|T_n(a_n)^{-1}\right\| < K_0\, \alpha + 1,$$

for all $n \in \mathbb{N}$. Hence, $\mathbf{T}(\boldsymbol{a})$ is invertible and $[T_n(a_n)^{-1}]$ is its inverse. This yields

$$\|\mathbf{T}(\boldsymbol{a}^{-1}) - \mathbf{T}(\boldsymbol{a})^{-1}\| = \lim_{\mathcal{U}} \|T_n(a_n^{-1}) - T_n(a_n)^{-1}\| \leq \lim_{\mathcal{U}} \frac{1}{n} = 0.$$

Thus, $\mathbf{T}(\boldsymbol{a}^{-1}) = \mathbf{T}(\boldsymbol{a})^{-1}$ for every invertible element $\boldsymbol{a} \in A^{\mathcal{U}}$. By [16, Theorem 2.2], $\mathbf{T}(\mathbf{1})\mathbf{T}(\boldsymbol{a}^2) = (\mathbf{T}(\mathbf{1})\mathbf{T}(\boldsymbol{a}))^2$ and $\mathbf{T}(\mathbf{1})\mathbf{T}(\boldsymbol{a}) = \mathbf{T}(\boldsymbol{a})\mathbf{T}(\mathbf{1})$, for every $\boldsymbol{a} \in A^{\mathcal{U}}$. We apply [2, Lemma 3.4] and Lemma 3.4.3 to obtain, respectively,

$$0 = \operatorname{jmult}(\mathbf{T}(\mathbf{1})\mathbf{T}) = \lim_{\mathcal{U}} \operatorname{jmult}(T_n(1)T_n)\,,$$

and

$$0 = \operatorname{ucomm}(\mathbf{T}) = \lim_{\mathcal{U}} \operatorname{ucomm}(T_n)\,.$$

Consequently,

$$I = \{n \in \mathbb{N} : \operatorname{jmult}(T_n(1)T_n) < \varepsilon_0\} \in \mathcal{U},$$

$$J = \{n \in \mathbb{N} : \operatorname{ucomm}(T_n) < \varepsilon_0\} \in \mathcal{U}.$$

Finally, $I \cap J \in \mathcal{U}$ gives us the desired contradiction. ■

Our goal now is to achieve a group invertibility version for the previous theorem. Recall that given an additive map $T : A \to B$ from a unital Banach algebra A into a Banach algebra B, by Theorem 3.2.4, if T strongly preserves group invertibility, then $T(1)T$ is a Jordan homomorphism and $T(1)$ commutes with the range of T. In order to take advantage of Proposition 3.4.1, our first step is to improve Theorem 3.2.4 by showing that all the information required is located in A^{-1}.

Theorem 3.4.5 *Let A and B be Banach algebras, where A is supposed to be unital, and $T : A \to B$ be an additive map such that $T(a^{-1}) = T(a)^\sharp$ for all $a \in A^{-1}$. Then $T(1)T$ is a Jordan homomorphism and $T(1)$ commutes with $T(A)$.*

Proof A look to the arguments employed in Lemma 3.2.1, allows us to show that T preserves the cubes of the invertible elements. Indeed, given $u \in A^{-1}$ and $\lambda \in \mathbb{Q}$ with

$0 < |\lambda| < ||u^{-1}||^{-2}$, as $\lambda^{-1}u$ and $u - \lambda u^{-1}$ are invertible elements, we know the following weak Hua's identity

$$\left(u^{-1} - (u - \lambda u^{-1})^{-1}\right)^{-1} = u - u(\lambda^{-1}u)u = u - \lambda^{-1}u^3.$$

Let us assume that $T(u) \neq 0$. Since $T(u) \in B^\sharp$, it follows that $T(u)$ is invertible in the unital Banach algebra pBp for $p = T(u)T(u)^\sharp$, with inverse $T(u)^\sharp$. The weak Hua's identity above applied to $T(u)$ and $0 < |\lambda| < ||T(u)^\sharp||^{-2}$ give

$$T(u) - \lambda^{-1}T(u)^3 = (T(u)^\sharp - (T(u) - \lambda T(u)^\sharp)^\sharp)^\sharp.$$

Hence, for every $\lambda \in \mathbb{Q}$ such that $0 < |\lambda| < \min\{||u^{-1}||^{-2}, ||T(u)^\sharp||^{-2}\}$ we get

$$\begin{aligned} T(u) - \lambda^{-1}T(u)^3 &= (T(u^{-1}) - T(u - \lambda u^{-1})^\sharp)^\sharp \\ &= T(u^{-1} - (u - \lambda u^{-1})^{-1})^\sharp = T((u^{-1} - (u - \lambda u^{-1})^{-1})^{-1}) \\ &= T(u - \lambda^{-1}u^3) = T(u) - \lambda^{-1}T(u^3). \end{aligned}$$

Therefore, $T(u^3) = T(u)^3$, as desired. From this last identity, reasoning as in Proposition 3.2.3 we deduce that the following equalities hold for every $x \in A$,

$$3T(x) = T(1)^2T(x) + T(x)T(1)^2 + T(1)T(x)T(1),$$

$$3T(x^2) = T(x)^2T(1) + T(1)T(x)^2 + T(x)T(1)T(x).$$

Finally, it only remains to repeat the arguments in *(2)* $\Rightarrow$ *(3)* in Theorem 3.2.4 to conclude the proof. ■

Now, we can state the following result.

Theorem 3.4.6 *Let A and B be Banach algebras where A is unital and $K, \varepsilon > 0$. Then there exists $\delta > 0$ such that for every linear map $T : A \to B$ with $\|T\| < K$, the condition*

$$\sup_{\|a\|=1, a \in A^\sharp} \left\| T(a^\sharp) - T(a)^\sharp \right\| < \delta$$

implies

$$\operatorname{jmult}(T(1)T) < \varepsilon \quad \textit{and} \quad \operatorname{ucomm}(T) < \varepsilon.$$

Proof First, notice that if $\boldsymbol{b} \in A^{\mathcal{U}}$ has a coset representative $\boldsymbol{b} = [b_n]$ where b_n is group invertible for every $n \in \mathbb{N}$ and $\{b_n^\sharp\}_{n \in \mathbb{N}}$ is bounded, then $\boldsymbol{b}$ is group invertible and $\boldsymbol{b}^\sharp = [b_n^\sharp]$. Hence, the same arguments used in Theorem 3.4.4 produce an operator $\mathbf{T} = [T_n] : A^{\mathcal{U}} \to B^{\mathcal{U}}$ satisfying $\mathbf{T}(\boldsymbol{a}^{-1}) = \mathbf{T}(\boldsymbol{a})^\sharp$ for every invertible element $\boldsymbol{a} \in A^{\mathcal{U}}$. Now, Theorem 3.4.5 proves that $\mathbf{T}(\mathbf{1})\mathbf{T}$ is a Jordan homomorphism and $\mathbf{T}(\mathbf{1})$ commutes with $\mathbf{T}(A^{\mathcal{U}})$. Again, the final argument in Theorem 3.4.4 completes the proof. ■

In Theorem 3.1.4 we proved that if an additive map $T : A \to B$ between unital Banach algebras satisfies

$$T(a)T(a^{-1}) = T(1)^2, \quad \text{for every}\, a \in A^{-1},$$

and $T(1)$ is invertible, then $T(1)^{-1}T$ is a Jordan homomorphism and $T(1)$ commutes with $T(A)$. It is clear now that for a sequence of linear operators $T_n : A \to B$ satisfying that $\|T_n\|, \|T_n(1)^{-1}\| < K$ for all $n \in \mathbb{N}$, and

$$\|T_n(a)T_n(a^{-1}) - T_n(1)^2\| < \frac{1}{n} \quad \text{for all } n \in \mathbb{N},$$

its ultrapoduct $\mathbf{T} : A^{\mathcal{U}} \to B^{\mathcal{U}}$ fulfills $\mathbf{T}(\boldsymbol{a})\mathbf{T}(\boldsymbol{a}^{-1}) = \mathbf{T}(\mathbf{1})^2$ for every invertible $\boldsymbol{a} \in A^{\mathcal{U}}$. Therefore, $\mathbf{T}(\mathbf{1})^{-1}\mathbf{T}$ is a Jordan homomorphism and $\mathbf{T}(\mathbf{1})$ commutes with $\mathbf{T}(A^{\mathcal{U}})$. This leads us to the following approximate formulation of Theorem 3.1.4.

Theorem 3.4.7 *Let A and B be unital Banach algebras and $K, \varepsilon > 0$. Then there exists $\delta > 0$ such that for every linear map $T : A \to B$ with $\|T\|, \|T(1)^{-1}\| < K$, the condition*

$$\sup_{a \in A^{-1}} \left\|T(a)T(a^{-1}) - T(1)^2\right\| < \delta$$

implies

$$\mathrm{jmult}(T(1)^{-1}T) < \varepsilon \quad \textit{and} \quad \mathrm{ucomm}(T) < \varepsilon.$$

Chapter 4

Linear preservers of generalized inverses in C*-algebras

In this chapter we first study the linear maps $T : A \to B$ between C*-algebras strongly preserving Moore-Penrose invertibility when A has a rich structure of projections, that is, A is either the linear span of its projections, or A has real rank zero (and T is bounded) or A has essential socle (and T is bijective). We connect with the problem of linear maps strongly preserving the inverse along the adjoint.

We also characterize the linear maps strongly preserving generalized invertibility (in the Jordan system's sense) and consequently we determine the structure of selfadjoint linear maps strongly preserving Moore-Penrose invertibility. We conclude the chapter by providing approximate versions of these results and also by considering linear maps approximately preserving the conorm. The results in this chapter can be found in [28], [29], [30] and [31].

4.1 Linear preservers of Moore-Penrose inverses in C*-algebras

Recall that, in [104], Mbekhta proved that every unital linear bounded map $T : A \to B$ between unital C*-algebras that strongly preserves Moore-Penrose invertibility is a Jordan homomorphism and preserves the orthogonality of projections. We begin this section by showing that every linear map strongly preserving Moore-Penrose invertibility preserves orthogonality of regular elements.

Proposition 4.1.1 *Let A and B be C*-algebras and let $T : A \to B$ be a linear map strongly preserving Moore-Penrose invertibility. Then $a \perp b$ implies $T(a) \perp T(b)$ for all $a, b \in A^\dagger$.*

Proof Let $a, b \in A^\dagger$ with $a \perp b$. For every $\alpha \in \mathbb{Q} \setminus \{0\}$ it is easy to see that $(a + \alpha b)^\dagger = a^\dagger + \alpha^{-1} b^\dagger$. By assumption,

$$(T(a) + \alpha T(b))(T(a)^\dagger + \alpha^{-1} T(b)^\dagger)(T(a) + \alpha T(b)) = T(a) + \alpha T(b),$$

which yields

$$\begin{aligned}\alpha^{-1}T(a)T(b)^\dagger T(a) &+ (T(a)T(b)^\dagger T(b) + T(b)T(b)^\dagger T(a))\\ &+ \alpha(T(b)T(a)^\dagger T(a) + T(a)T(a)^\dagger T(b))\\ &+ \alpha^2 T(b)T(a)^\dagger T(b) = 0,\end{aligned}$$

for every $\alpha \in \mathbb{Q}\setminus\{0\}$. Hence

$$T(a)T(b)^\dagger T(b) + T(b)T(b)^\dagger T(a) = 0.$$

Multiplying the last equation on the right and on the left, respectively, by $T(b)^\dagger$ it follows that

$$T(a)T(b)^\dagger = -T(b)T(b)^\dagger T(a)T(b)^\dagger, \tag{4.1}$$

and

$$T(b)^\dagger T(a) = -T(b)^\dagger T(a)T(b)^\dagger T(b). \tag{4.2}$$

As

$$(T(a)^\dagger + \alpha^{-1}T(b)^\dagger)(T(a) + \alpha T(b))(T(a)^\dagger + \alpha^{-1}T(b)^\dagger) = T(a)^\dagger + \alpha^{-1}T(b)^\dagger$$

for every $\alpha \in \mathbb{Q}\setminus\{0\}$, we get analogously

$$T(b)^\dagger T(a)T(b)^\dagger = 0. \tag{4.3}$$

From Equations (4.1), (4.2) and (4.3) we deduce that $T(a)T(b)^\dagger = 0$ and $T(b)^\dagger T(a) = 0$. Equivalently, $T(a)T(b)^* = 0$ and $T(b)^*T(a) = 0$, that is, $T(a) \perp T(b)$. ■

It is clear that the zero map strongly preserves Moore-Penrose invertibility. In the next proposition we show that this is the only map strongly preserving Moore-Penrose invertibility that annihilates the identity element.

Proposition 4.1.2 *Let A and B be C*-algebras, A being unital, and let $T : A \to B$ be a linear map strongly preserving Moore-Penrose invertibility. Then either $T(1) \neq 0$ or $T = 0$.*

Proof If b and $1 + b$ are invertible elements in A, as we have previously noted

$$1 = (1 + b)^{-1} + (1 + b^{-1})^{-1}.$$

Since T strongly preserves Moore-Penrose invertibility,

$$T(1) = T((1 + b)^{-1}) + T((1 + b^{-1})^{-1}) = (T(1) + T(b))^\dagger + (T(1) + T(b)^\dagger)^\dagger.$$

If we assume that $T(1) = 0$, then we get

$$T(b)^\dagger + T(b) = 0,$$

for every invertible element $b \in A$ with $1 + b$ invertible. Thus, let a be an invertible element in A, and $\alpha \in \mathbb{Q}\setminus\{0\}$ be such that $|\alpha| < ||a||^{-1}$. It is clear that αa and $1 + \alpha a$ are invertible, and therefore $T(a)^\dagger = -\alpha^2 T(a)$. By the uniqueness of the Moore-Penrose inverse, it follows that $T(a) = 0$. Thus T is the zero map. ■

Notice that if $T : A \to B$ is a non zero linear map strongly preserving Moore-Penrose invertibility, and $T(1)$ commutes with $T(A)$, then $B' = T(1)^2 B T(1)^2$ is a C*-algebra with identity $T(1)^2$ ($T(1)^2 \neq 0$ in view of the preceding proposition), the map $S = T(1)^2 T$ from A to B' strongly preserves Moore-Penrose invertibility, and $S(1)$ is invertible. A closer look at the arguments employed in Theorem 3.5, Lemma 3.7 and Proposition 3.10 in [16], where the authors only required the Hua's identity and the inner relation of the generalized inverse on invertible elements, reveals that the same reasoning works with Moore-Penrose invertibility. Obviously B' has an identity element even if B is not unital.

Proposition 4.1.3 *Let A and B be C*-algebras and let T be a linear map such that $T(1)$ commutes with the range of T. If T strongly preserves Moore-Penrose invertibility, then $T(1)T$ is a Jordan homomorphism.*

The following result is a technical lemma which, together with Proposition 4.1.3, will be the key tool for the next ones. It describes the behaviour of a linear map strongly preserving Moore-Penrose invertibility with respect to the projections.

Lemma 4.1.4 *Let A and B be C*-algebras, where A is unital. Let $T : A \to B$ be a linear map strongly preserving Moore-Penrose invertibility. For every projection $p \in A$:*

(1) $T(p)T(1)^* = T(1)T(p)^*$ *and* $T(1)^*T(p) = T(p)^*T(1)$,

(2) $T(p) = T(p)T(1)^2 = T(1)^2T(p)$,

(3) $T(p)T(1) = T(1)T(p) = (T(p)T(1))^*$.

Proof For the sake of simplicity, write $h = T(1)$. Let p be a non zero projection in A. As $p \perp (1-p)$, and by Proposition 4.1.1, T preserves orthogonality of regular elements, then $T(p) \perp (h - T(p))$, that is, $T(p)h^* = T(p)T(p)^*$ and $h^*T(p) = T(p)^*T(p)$. In particular, $T(p)h^* = hT(p)^*$ and $h^*T(p) = T(p)^*h$. Since $h = h^\dagger$, it is clear that $h^3 = h$, and $h^2 = (h^2)^*$. Hence

$$\begin{aligned} T(p)^*T(p)h^2 &= h^*T(p)h^2 = T(p)^*hh^2 = T(p)^*h \\ &= h^*T(p) = T(p)^*T(p). \end{aligned}$$

Again, $T(p) = T(p)^\dagger$ gives $T(p)^3 = T(p)$ and thus $T(p)^*T(p)(h^2 - T(p)^2) = 0$. By the cancellation law, $T(p)h^2 = T(p)^3 = T(p)$. In the same way, $T(p) = h^2T(p)$, $T(p)^* = h^2T(p)^*$ and $T(p)^*h^2 = T(p)^*$. Also,

$$\begin{aligned} T(p)h &= h^2T(p)h = h^*h^*T(p)h = h^*T(p)^*h^2 = h^*T(p)^* \\ &= (T(p)h)^*. \end{aligned}$$

Analogously, $(hT(p))^* = hT(p)$.

It only remains to prove that $hT(p) = T(p)h$. Since $T(p) = h^2T(p) = T(p)h^2$, it suffices to show that $T(p) = hT(p)h$. Having in mind the uniqueness of the Moore-Penrose inverse, and that $T(p)^\dagger = T(p^\dagger) = T(p)$ we proceed by checking that $hT(p)h$ is the Moore-Penrose inverse of $T(p)$. As $T(p)h = h^*T(p)^*$, $hT(p) = T(p)^*h^*$, $T(p)^*T(p) = h^*T(p)$, $h^2 = (h^2)^*$ and $h^3 = h$, we get

$$\begin{aligned} T(p)(hT(p)h)T(p) &= T(p)hh^*T(p)^*T(p) = T(p)hh^*h^*T(p) \\ &= T(p)hT(p) = T(p)T(p)^*h^* = T(p)(h^*)^2 = T(p). \end{aligned}$$

From this it is clear that, $(hT(p)h)T(p)(hT(p)h) = hT(p)h$, and since

$$T(p)(hT(p)h) = (T(p)h)T(p)h = h^*T(p)^*T(p)h = h^*h^*T(p)h = T(p)h,$$

and similarly $(hT(p)h)T(p) = hT(p)$, are selfadjoint, this shows that $hT(p)h = T(p)^\dagger$, as desired. ■

Remark 4.1.5 *Note that, as every additive map $T : A \to B$ between Banach algebras is $\mathbb{Q}$-linear, the preceding results also hold if we change the linearity with additivity.*

C*-algebras linearly spanned by their projections and real rank zero C*-algebras

The following theorem describes linear maps strongly preserving Moore-Penrose invertibility from C*-algebras linearly spanned by their projections. In particular, by [121, Corollary 2.3], it applies to the algebra of all bounded linear operators on a complex infinite dimensional Hilbert space (compare with Theorem 3.3 $(i) \Rightarrow (ii)$ in [104], where T is assumed to be unital and bijective).

Theorem 4.1.6 *Let $T : A \to B$ be a linear map strongly preserving Moore-Penrose invertibility between C*-algebras, where A is unital. Assume that every element of A is a finite linear combination of projections. Then $T(1)T$ is a Jordan *-homomorphism and $T(1)$ commutes with the range of T.*

Proof From A being linearly spanned by its projections, by Lemma 4.1.4 it is clear that $T(x)T(1) = (T(x^*)T(1))^* = T(1)T(x)$, for every $x \in A$. The conclusions can be obtained directly by applying Proposition 4.1.3. ■

When A is of real rank zero and $T : A \to B$ is a *bounded* linear map strongly preserving Moore-Penrose invertibility, we can similarly obtain:

Theorem 4.1.7 *Let A and B be C*-algebras, and $T : A \to B$ be a bounded linear map strongly preserving Moore-Penrose invertibility. Suppose that A is unital of real rank zero. Then:*

(1) $T(1)$ commutes with the range of T,

(2) $T(1)T$ is a Jordan $*$-homomorphism.

Remark 4.1.8 *Let A and B be C*-algebras and let $T : A \to B$ be a Jordan *-homomorphism. Then T strongly preserves Moore-Penrose invertibility. Indeed if $a \in A^\dagger$, and $b = a^\dagger$, as T preserves triple products, it is clear that $T(a) = T(a)T(b)T(a)$ and $T(b) = T(b)T(a)T(b)$. Thus it remains to show that $T(b)T(a)$ and $T(a)T(b)$ are selfadjoint. As $a = b^*a^*a = aa^*b^*$, in particular $2a = b^*a^*a + aa^*b^*$, and since T is a Jordan *-homomorphism, it is clear that*

$$2T(a) = T(b)^*T(a)^*T(a) + T(a)T(a)^*T(b)^*.$$

Multiplying on the left by $T(a)^*$, we get that

$$T(a)^*T(a) = T(a)^*T(a)T(a)^*T(b)^*,$$

or equivalently $T(a)^*T(a)(T(b)T(a) - T(a)^*T(b)^*) = 0$, which implies that $T(a) = T(a)T(a)^*T(b)^*$, and hence $T(b)T(a) = T(b)T(a)T(a)^*T(b)^*$ is selfadjoint.

*Moreover if $T : A \to B$ is a Jordan *-homomorphism between C*-algebras and u is a regular element in B such that $u = u^\dagger$, and u commutes with the range if T, it is clear that uT also strongly preserves Moore-Penrose invertibility.*

Consequently, we are able to characterize bounded linear maps (not necessarily surjective nor unital) strongly preserving Moore-Penrose invertibility on a real rank zero unital C*-algebra.

Corollary 4.1.9 *Let A and B be C*-algebras, and let $T : A \to B$ be a bounded linear map. Suppose that A is unital of real rank zero. The following are equivalent:*

(1) T strongly preserves Moore-Penrose invertibility,

(2) $T(1)^\dagger = T(1)$, $T = ST(1) = T(1)S$ for a Jordan $*$-homomorphism S.

Proof The implication *(1) ⇒ (2)* is a direct consequence of Theorem 4.1.7, and the converse follows from Remark 4.1.8. ■

C*-algebras of large socle

Recall that every element in the socle of a C*-algebra is a linear combination of minimal projections. It is also well-known that $A^\dagger + \mathrm{soc}(A) \subset A^\dagger$ ([92, Theorem 6.3]). This fact, together with Proposition 4.1.1, allows us to employ the techniques on orthogonality preserving maps on C*-algebras with large socle in order to determine the structure of strongly Moore-Penrose invertibility linear preservers. The following lemma is inspired in [25] (see also [26]).

Lemma 4.1.10 *Let A and B be C*-algebras, where A is unital and has nonzero socle, and let $T : A \to B$ a linear map strongly preserving Moore-Penrose invertibility. Then, for every $a \in A$ and $x \in soc(A)$, the following identities hold:*

(1) $2T(a\circ x)T(1)^* = T(a)T(x^*)^* + T(x)T(a^*)^*$ *and*

$2T(1)^*T(a\circ x) = T(x^*)^*T(a) + T(a^*)^*T(x)$,

(2) $T(x)T(1)^*T(a) = T(x)T(a^*)^*T(1)$ *and*

$T(a)T(1)^*T(x) = T(1)T(a^*)^*T(x)$,

(3) $T(x)T(1)T(a) = T(x)T(a^*)^*T(1)^*$ *and*

$T(a)T(1)T(x) = T(1)^*T(a^*)^*T(x)$,

(4) $\{T(x)\,T(a)\,T(x)\} = T(\{x\,a\,x\})T(1)^*T(1)$,

Proof As above denote $T(1)$ by h. In view of Lemma 4.1.4, since every element of the socle is a linear combination of minimal projections, it follows directly that,

$$T(x)h^* = hT(x^*)^*, \quad h^*T(x) = T(x^*)^*h,$$

$$T(x)h = hT(x) = h^*T(x^*)^* = T(x^*)^*h^*$$

and $T(x) = T(x)h^2$ for every $x \in \mathrm{soc}(A)$.

Let p, q be minimal projections in A. Since qp and $(1-q)(1-p) = 1-p-q+qp$ are mutually orthogonal regular elements, by Proposition 4.1.1, $T(qp) \perp T(1-q-p+qp)$. Therefore

$$T(qp)h^* - T(qp)T(q)^* - T(qp)T(p)^* + T(qp)T(qp)^* = 0.$$

As $q(1-p) \perp (1-q)p$, we also have $T(q-qp) \perp T(p-qp)$, that is

$$T(q)T(p)^* - T(q)T(qp)^* - T(qp)T(p)^* + T(qp)T(qp)^* = 0.$$

Taking into account these equations and $\mathrm{soc}(A)$ being linearly spanned by the minimal projections, we can prove

$$T(yx+xy)h^* = T(y)T(x^*)^* + T(x)T(y^*)^*, \tag{4.4}$$

for all $x, y \in \mathrm{soc}(A)$ (compare with the proof of Theorem 14 in [26]). Besides, given a minimal projection p in A and an invertible element b in A, p and $(1-p)b(1-p) = b - bp - pb + pbp$ are mutually orthogonal regular elements. Thus

$$T(p)^*T(b) = T(p)^*T(bp+pb-pbp)$$

and

$$T(b)T(p)^* = T(bp+pb-pbp)T(p)^*.$$

Equation (4.4) yields

$$\begin{aligned} T(bp+pb)h^* &= T((bp+pb)p + p(bp+pb) - 2pbp)h^* \\ &= T(bp+pb)T(p)^* + T(p)T(b^*p+pb^*)^* \\ &\quad -T(pbp)T(p)^* - T(p)T(pb^*p)^* \\ &= T(bp+pb-pbp)T(p)^* + T(p)T(b^*p+pb^*-pb^*p)^* \\ &= T(b)T(p)^* + T(p)T(b^*)^*. \end{aligned}$$

As $T(p)h^* = hT(p)^*$, given $a \in A$ and $\alpha \in \mathbb{C}$ such that $a - \alpha$ is invertible, the last equation gives $T(ap+pa)h^* = T(a)T(p)^* + T(p)T(a^*)^*$, and by the linearity of T

$$T(ax+xa)h^* = T(a)T(x^*)^* + T(x)T(a^*)^* \qquad (a \in A, x \in \mathrm{soc}(A)). \tag{4.5}$$

The other equality of (1) can be proved analogously.

Again for a minimal projection p in A, and an invertible element $b \in A$, from $p \perp (1-p)b(1-p)$, we obtain

$$\begin{aligned} T(p)h^*T(b) &= T(p)T(p)^*T(b) = T(p)T(p)^*T(bp+pb-pbp) \\ &= T(p)h^*T(bp+pb-pbp) = T(p)T((bp+pb-pbp)^*)^*h \\ &= T(p)T(b^*)^*h, \end{aligned}$$

and

$$\begin{aligned} T(p)hT(b) &= h^*T(p)^*T(b) = h^*T(p)^*T(bp+pb-pbp) \\ &= T(p)hT(bp+pb-pbp) = T(p)T((bp+pb-pbp)^*)^*h^* \\ &= T(p)T(b^*)^*h^*. \end{aligned}$$

This proves that

$$T(x)h^*T(a) = T(x)T(a^*)^*h, \tag{4.6}$$

and

$$T(x)hT(a) = T(x)T(a^*)^*h^*, \tag{4.7}$$

for all $x \in \mathrm{soc}(A)$ and $a \in A$. The other relations of *(2)* and *(3)* can be deduced in an obvious way.

In order to prove equality *(4)*, let $x \in \mathrm{soc}(A)$ and $a \in A$. By the definition of the triple product in a C*-algebra and the statements just proved we get

$$\begin{aligned} T(\{x\,a\,x\})h^*h &= 2T((x\circ a^*)\circ x)h^*h - T(x^2\circ a^*)h^*h \\ &= (T(x\circ a^*)T(x^*)^* + T(x)T(x^*\circ a)^*)\,h \\ &- \frac{1}{2}\left(T(x^2)T(a)^* + T(a^*)T((x^2)^*)^*\right)h \\ &= T(x\circ a^*)h^*T(x) + T(x)h^*T(x\circ a^*) \\ &- \frac{1}{2}\left(T(x^2)h^*T(a^*) + T(a^*)T((x^2)^*)^*h\right) \\ &= \frac{1}{2}((T(x)T(a)^* + T(a^*)T(x^*)^*)T(x)) \\ &+ \frac{1}{2}(T(x)(T(x^*)^*T(a^*) + T(a)^*T(x))) \\ &- \frac{1}{2}((T(x)T(x^*)^*T(a^*) + T(a^*)T(x^*)^*T(x)) \\ &= \{T(x)\,T(a)\,T(x)\}. \blacksquare \end{aligned}$$

Remark 4.1.11 *From the preceding lemma, it is clear that*

$$T(1)T(x) = (T(1)T(x^*))^*$$

and

$$T(1)T(x^2) = T(1)T(x^2)(T(1)^*)^2 = T(1)T(x)T(x^*)^*T(1)^* = (T(1)T(x))^2,$$

*for every element x in the socle of A. This shows that the mapping $x \mapsto T(1)T(x)$ is a Jordan *-homomorphism from soc(A) to B.*

Recall that an ideal I of a C*-algebra A is essential if $aI = \{0\}$ implies $a = 0$.

Theorem 4.1.12 *Let A and B be C*-algebras, where A is unital and has nonzero socle, and let $T : A \to B$ a linear map strongly preserving Moore-Penrose invertibility. Assume that T does not annihilate rank-one elements.*

(1) If $T(a)T(1) - T(1)T(a) \in T(A)$ for every $a \in A$, then $T^{-1}(T(a)T(1) - T(1)T(a))soc(A) = \{0\}$, for every $a \in A$.

(2) If $T(a)T(1) - (T(a)T(1))^* \in T(A)$ for every selfadjoint element $a \in A$, then $T^{-1}(T(a)T(1) - T(1)^*T(a^*)^*)soc(A) = \{0\}$, for every $a \in A$.

In particular, if soc(A) is essential, then $T(1)T$ is a Jordan $*$-homomorphism, and $T(1)$ commutes with the range of T.

Proof Again write $h = T(1)$. Let $x \in \mathrm{soc}(A)$ and $a \in A$. From *(4)* of Lemma 4.1.10, by multiplying on the right by hh^*

$$\begin{aligned} T(\{x\,a\,x\}) &= T(x)T(a)^*T(x)hh^* = T(x)T(a)^*hT(x)h^* \\ &= T(x)T(a)^*h^2T(x^*)^* = T(x)T(a)^*T(x^*)^*. \end{aligned}$$

Moreover, since $T(\{x\,a\,x\})h^*h = hh^*T(\{x\,a\,x\})$, we also get (by multiplying on the left by h^*h)

$$T(\{x\,a\,x\}) = h^*T(x)hT(a)^*T(x) = T(x^*)^*h^2T(a)^*T(x) = T(x^*)^*T(a)^*T(x).$$

Therefore,

$$\begin{aligned} \{T(x)\,(T(a)h)\,T(x)\} &= T(x)h^*T(a)^*T(x) = hT(x^*)^*T(a)^*T(x) \\ &= hT(\{x\,a\,x\}) = T(\{x\,a\,x\})h = T(x)T(a)^*T(x^*)^*h \\ &= T(x)T(a)^*h^*T(x) = \{T(x)\,(hT(a))\,T(x)\}. \end{aligned}$$

If $T(a)h - hT(a) \in T(A)$, there exists $b \in A$ such that $T(b) = T(a)h - hT(a)$. The last identities show that

$$0 = \{T(x)\,T(b)\,T(x)\} = T(\{x\,b\,x\})h^*h,$$

and hence $T(\{x\,b\,x\}) = 0$. In particular $T(\{u\,b\,u\}) = 0$ for every $u \in \mathcal{F}_1(A)$. As T does not annihilate rank-one elements, and for every $u \in \mathcal{F}_1(A)$, $ubu = 0$ or ubu has rank-one, it follows that $ubu = 0$ for all $u \in \mathcal{F}_1(A)$. This implies that $bu = 0$ for every

$u \in \mathcal{F}_1(A)$ (see for instance the proof of Theorem 1.1 in [19]). Hence $b\,\mathrm{soc}(A) = \{0\}$, that is,

$$T^{-1}(T(a)h - hT(a))\mathrm{soc}(A) = \{0\}. \tag{4.8}$$

From Lemma 4.1.10 *(3)*, it follows that

$$\begin{aligned}\{T(x)\,(T(a)h)\,T(x)\} &= T(x)h^*T(a)^*T(x) = T(x)T(a^*)hT(x)\\ &= \{T(x)\,(h^*T(a^*)^*)\,T(x)\}.\end{aligned}$$

Whenever $T(z)h - (T(z)h)^* \in T(A)$, for every selfadjoint element $z \in A$, it is clear that $T(a)h - h^*T(a^*)^*$ lies in $T(A)$, for every $a \in A$. Then, as above, we can prove

$$T^{-1}(T(a)h - h^*T(a^*)^*)\mathrm{soc}(A) = \{0\}. \tag{4.9}$$

If $\mathrm{soc}(A)$ is essential, by Equation (4.8), h commutes with $T(A)$, and by Proposition 4.1.3, $S = hT$ is a Jordan homomorphism. Besides, Equation (4.9) gives $T(a)h = h^*T(a^*)^* = (T(a^*)h)^*$ for all $a \in A$, which shows that S is selfadjoint. ■

Notice that if $T : A \to B$ is a bijective linear map strongly preserving Moore-Penrose invertibility, and $\mathrm{soc}(A)$ is essential, since

$$\{T(x)\,(T(a)T(1_A)^2)\,T(x)\} = \{T(x)\,T(a)\,T(x)\} = \{T(x)\,(T(1_A)^2T(a))\,T(x)\},$$

we can obtain that $T(a)T(1_A)^2 = T(a) = T(1_A)^2T(a)$, for every $a \in A$, and hence B is unital with identity element $1_B = T(1_A)^2$. The following result can be derived now as a consequence.

Theorem 4.1.13 *Let A and B be C*-algebras. Suppose that A is unital with essential socle. Let $T : A \to B$ be a bijective linear map. The following are equivalent:*

(1) T strongly preserves Moore-Penrose invertibility,

(2) $T(1_A)^2 = 1_B$, $T = T(1_A)S = ST(1_A)$ for a Jordan $*$-isomorphism S.

We consider now the case of linear mappings from prime C*-algebras with non zero socle. Recall that every prime C*-algebra A with non zero socle is primitive (see [102]) and hence its socle is a simple algebra which is contained in every non zero (Jordan) ideal of A (see [72, IV §9] and [69, Theorem 1.1]). As we have noted in Remark 4.1.11, if $T : A \to B$ is a linear map strongly preserving Moore-Penrose invertibility, then $T(1)T|_{\mathrm{soc}(A)} : \mathrm{soc}(A) \to B$ is a Jordan $*$-homomorphism and hence $\mathrm{Ker}(T) \cap \mathrm{soc}(A)$ is a Jordan ideal of A. Therefore, if A is prime, either $\mathrm{Ker}(T) \cap \mathrm{soc}(A) = \{0\}$ or $T(\mathrm{soc}(A)) = \{0\}$.

Having in mind these considerations and the proof of Theorem 4.1.12, we get the following result.

Corollary 4.1.14 *Let A and B be C*-algebras, and let $T : A \to B$ be a surjective linear map strongly preserving Moore-Penrose invertibility. Suppose that A is prime, unital, with non zero socle. If $T(soc(A)) \neq \{0\}$, then $T(1)T$ is a Jordan $*$-homomorphism, and $T(1)$ commutes with the range of T.*

Finally, the following example shows that we cannot expect to obtain the selfadjointness of linear strongly preservers of Moore-Penrose invertibility in general unital C*-algebras.

Example 4.1.15 *Let $T : A \to B$ where $A = C([0,1])$ y $B = M_2(\mathbb{C})$ defined by*

$$T(f) = \begin{pmatrix} f(0) & f(1) - f(0) \\ 0 & f(1) \end{pmatrix}$$

It is straightforward that T is a unital homomorphism, so it strongly preserves invertibility. Note that $C([0,1])$ is a commutative C-algebra with no nontrivial projections, that is, $A^\dagger = A^{-1} \cup \{0\}$. Hence, T strongly preserves Moore-Penrose invertibility. However, it is easy to see that T is not selfadjoint.*

4.2 Linear preservers of the inverse along the adjoint

We recover the study of linear maps preserving the inverse along an element initiated in Section 3.3). Recall that an element a in a C*-algebra A is Moore-Penrose invertible if, and only if, a is invertible along a^*. In this case, $a^\dagger = a^{||a^*}$. In the setting of unital C*-algebras we can state the following dealing invertibility along the adjoint.

Theorem 4.2.1 *Let A be a unital C*-algebra, B a Banach algebra and $T : A \to B$ be a linear map. Then T is a Jordan triple homomorphism if, and only if, $T(a^{||a^*}) = T(a)^{||T(a^*)}$ for every $a \in A^\dagger$.*

Proof Sufficiency follows directly from *(1)* $\Leftrightarrow$ *(4)* in Theorem 3.3.1. For the necessity, we proceed to show that T preserves the cubes of selfadjoint group invertible elements. Notice that if $a = a^* \in A^\dagger$, by assumption $T(a^{||a}) = T(a)^{||T(a)}$. That is, $T(a^\sharp) = T(a)^\sharp$ for every selfadjoint and group invertible $a \in A$. Let $u \in A^\sharp$ with $u \neq 0$. Following the arguments in Lemma 3.2.1 we get, for $\lambda \in \mathbb{R}$ such that $0 < |\lambda| < \min\{||u^\sharp||^{-2}, ||T(u)^\sharp||^{-2}\}$,

$$u - \lambda^{-1}u^3 = (u^\sharp - (u - \lambda u^\sharp)^\sharp)^\sharp$$

and

$$T(u) - \lambda^{-1}T(u)^3 = (T(u)^\sharp - (T(u) - \lambda T(u)^\sharp)^\sharp)^\sharp.$$

Since T is linear and strongly preserves group inverses for selfadjoint elements, it follows that

$$\begin{aligned} T(u) - \lambda^{-1}T(u)^3 &= (T(u^\sharp) - T(u - \lambda u^\sharp)^\sharp)^\sharp \\ &= T(u^\sharp - (u - \lambda u^\sharp)^\sharp)^\sharp = T((u^\sharp - (u - \lambda u^\sharp)^\sharp)^\sharp) \\ &= T(u - \lambda^{-1}u^3) = T(u) - \lambda^{-1}T(u^3). \end{aligned}$$

Hence $T(u^3) = T(u)^3$, as claimed.

Pick now a selfadjoint element $x \in A$. For every $\alpha \in \mathbb{R}$ with $|\alpha| > \|x\|$, $x + \alpha$ is invertible and selfadjoint. By the previous assertion,

$$T((x+\alpha)^3) = T(x+\alpha)^3.$$

Repeating the arguments in Proposition 3.2.3 we obtain

$$3T(x) = T(1)^2T(x) + T(x)T(1)^2 + T(1)T(x)T(1), \tag{4.10}$$

and

$$3T(x^2) = T(x)^2T(1) + T(1)T(x)^2 + T(x)T(1)T(x), \tag{4.11}$$

for every selfadjoint element x in A. A linearization of Equation (4.10), yields to

$$3T(x) = T(1)^2T(x) + T(x)T(1)^2 + T(1)T(x)T(1) \tag{4.12}$$

for all $x \in A$. Hence, as in *(2)* $\Rightarrow$ *(3)* of Theorem 3.2.4, we can get

$$T(x)T(1) = T(1)T(x) \tag{4.13}$$

for every x in A. Therefore,

$$T(x) = T(1)^2T(x) \quad \text{for every } x \in A, \tag{4.14}$$

and

$$T(x^2) = T(1)T(x)^2 \quad \text{for every selfadjoint element } x \in A. \tag{4.15}$$

Now, for $x \in A$, write $x = h + ik$, where h, k are selfadjoint elements in A. From Equation (4.15) it is clear that

$$T((h+k)^2) = T(1)T(h+k)^2, \quad T(h^2) = T(1)T(h)^2, \quad T(k^2) = T(1)T(k)^2$$

which together leads to $T(hk + kh) = T(1)(T(h)T(k) + T(k)T(h))$. Now:

$$\begin{array}{rcl} T(x^2) & = & T((h+ik)^2) = T(h^2) - T(k^2) + iT(hk+kh) \\ & = & T(1)T(h)^2 - T(1)T(k)^2 + iT(1)(T(h)T(k) + T(k)T(h)) \\ & = & T(1)T(x)^2. \end{array}$$

Hence, $S = T(1)T$ is Jordan homomorphism and, by Equation (4.14), $T = T(1)S$ as we wanted. ■

Recall from Section 3.3 that if $T : A \to B$ is a linear map between unital Banach algebras such that $T(1^{\|a}) = T(1)^{\|T(a)}$ for all $a \in A^\sharp$, then $T(1)T$ preserves idempotents. Having in mind that a continuous linear mapping from a C*-algebra with real rank zero into a Banach algebra sending orthogonal projections to mutually orthogonal idempotents is a Jordan homomorphism (see Lemma 3.3.2), we have the following result.

Theorem 4.2.2 *If $T : A \to B$ is a continuous linear mapping from a C*-algebra with real rank zero into a Banach algebra such that $T(1^{\|a}) = T(1)^{\|T(a)}$ for all $a \in A^\sharp$, then $T(1)T$ is a Jordan homomorphism.*

Remark 4.2.3 *When A is a unital C*-algebra, the conditions in Theorem 3.3.5 are also equivalent to*

$$T(a)T(a^\dagger) = T(1)T(1)^{\|T(a^*)} = T(1)^{\|T(a^\dagger)}T(1) \text{ for all } a \in A^\dagger.$$

Indeed, for a linear map $T : A \to B$ satisfying this condition, it is clear that

$$T(a)T(a^\sharp) = T(1)T(1)^{\|T(a)} = T(1)^{\|T(a^\sharp)}T(1) \text{ for all } a^* = a \in A^\sharp.$$

As in the proof of (2) $\Rightarrow$ (3) in Theorem 3.3.5, we conclude that $T(a^\sharp) = T(a)^\sharp$ for every selfadjoint and group invertible element $a \in A$. Theorem 4.2.1, ensures that T is a Jordan triple homomorphism.

4.3 Linear maps strongly preserving (triple) regularity in C*-algebras

Recall from Section 2.3 that for each von Neumann regular element a in a JB*-triple E, there exists a tripotent $e \in E$ satisfying that a is a selfadjoint invertible element in the JB*-algebra $E_2(e)$. Moreover $L(a, a^\wedge) = L(a^\wedge, a) = L(e, e)$. Moreover, if a and b are invertible elements in the Jordan algebra J such that $a - b^{-1}$ is also invertible, then $a^{-1} + (b^{-1} - a)^{-1}$ is invertible and the Hua's identity

$$\left(a^{-1} + (b^{-1} - a)^{-1}\right)^{-1} = a - U_a(b)$$

holds.

Lemma 4.3.1 *Let A and B be C*-algebras and T be a linear map strongly preserving regularity. Then $T(u^{[3]}) = T(u)^{[3]}$, for every $u \in A^\wedge$.*

Proof Let $u \in A^\wedge \setminus \{0\}$. Then there exists a unique partial isometry e, such that u is selfadjoint and invertible in the Jordan algebra $(E_A)_2(e) = ee^*Ae^*e$, with inverse $u^\wedge$. Hence for every $\lambda \in \mathbb{Q}$ with $0 < |\lambda| < ||u^\wedge||^{-2}$, the element $u - \lambda u^\wedge$ is invertible in ee^*Ae^*e. Reciprocally, the inverses of $u - \lambda u^\wedge$ and $u^\wedge - (u - \lambda u^\wedge)^\wedge$ in ee^*Ae^*e are their generalized inverses in E_A (recall that the triple product induced on $(E_A)_2(e) = ee^*Ae^*e$ by the Jordan *-algebra structure coincides with its original triple product, (2.7), and $Q(u) = U_u \circ \#$, for every $u \in A$). By the Hua's identity we obtain

$$u - \lambda^{-1}u^{[3]} = \left(u^\wedge - (u - \lambda u^\wedge)^\wedge\right)^\wedge.$$

Let $u \in A^\wedge$. We may assume that $T(u) \neq 0$. Since T strongly preserves regularity, $T(u)^\wedge = T(u^\wedge)$ and thus, for $\lambda \in \mathbb{Q}$ with $0 < |\lambda| < \min\{||u^\wedge||^{-2}, ||T(u)^\wedge||^{-2}\}$, we get

$$T(u) - \lambda^{-1}T(u)^{[3]} = \left(T(u)^\wedge - (T(u) - \lambda T(u)^\wedge)^\wedge\right)^\wedge.$$

Arguing as in Lemma 3.2.1 we deduce that

$$T(u) - \lambda^{-1}T(u)^{[3]} = (T(u^\wedge) - T(u - \lambda u^\wedge)^\wedge)^\wedge = T(u) - \lambda^{-1}T(u^{[3]}).$$

Hence $T(u^{[3]}) = T(u)^{[3]}$. ■

Remark 4.3.2 *As in Proposition 3.2.2 it is clear that the zero map between C*-algebras strongly preserves regularity and that this is the only map strongly preserving regularity annihilating the identity element.*

Proposition 4.3.3 *Let A and B be C*-algebras, A unital, and $T : A \to B$ be a linear map strongly preserving regularity. For every $a \in A$,*

(1) $T(1)^*T(a) = T(a^*)^*T(1)$, *and* $T(a)T(1)^* = T(1)T(a^*)^*$,

(2) $T(a) = T(1)T(1)^*T(a) = T(a)T(1)^*T(1) = T(1)T(a^*)^*T(1)$,

(3) $T(1)^*T(a^2) = T(a^*)^*T(a)$.

Proof Let $u \in A^{-1}$ and $\alpha \in \mathbb{Q}$ be such that $0 < |\alpha| < ||u^{-1}||^{-1}$. Then $u + \alpha \in A^{-1}$, and by the above lemma, we know that $T(u^{[3]}) = T(u)^{[3]}$, $T(1) = T(1)^{[3]}$ and $T((u + \alpha)^{[3]}) = T(u + \alpha)^{[3]}$. Hence

$$(u + \alpha)^{[3]} = u^{[3]} + \alpha^3 + \alpha(uu^* + u^*u) + 2\alpha^2 u + \alpha u^2 + \alpha^2 u^*,$$

and

$$\begin{aligned}(T(u) + \alpha T(1))^{[3]} &= T(u)^{[3]} + \alpha^3 T(1)\\ &+ 2\alpha\{T(u), T(u), T(1)\} + \alpha\{T(u), T(1), T(u)\}\\ &+ 2\alpha^2\{T(1), T(1), T(u)\} + \alpha^2\{T(1), T(u), T(1)\}.\end{aligned}$$

By merging these two equations we deduce

$$\begin{aligned}&T(uu^* + u^*u) + 2\alpha T(u) + T(u^2) + \alpha T(u^*) =\\ &\qquad 2\{T(u), T(u), T(1)\} + \{T(u), T(1), T(u)\}\\ &\qquad\qquad + 2\alpha\{T(1), T(1), T(u)\} + \alpha\{T(1), T(u), T(1)\},\end{aligned}$$

for all $\alpha \in \mathbb{Q}$ with $0 < |\alpha| < ||u^{-1}||^{-1}$. This shows that

$$2T(u) + T(u^*) = 2\{T(1), T(1), T(u)\} + \{T(1), T(u), T(1)\}, \tag{4.16}$$

and

$$T(uu^* + u^*u) + T(u^2) = 2\{T(u), T(u), T(1)\} + \{T(u), T(1), T(u)\}. \tag{4.17}$$

for every invertible element $u \in A$.

Given $x \in A$, let $\mu \in \mathbb{Q}$ such that $u = x + \mu$ is invertible. From Equation (4.16), as T is linear and $T(1)$ is a tripotent, it is clear that

$$2T(x) + T(x^*) = 2\{T(1), T(1), T(x)\} + \{T(1), T(x), T(1)\}.$$

Let us assume that $x = x^*$. Then

$$3T(x) = 2\{T(1), T(1), T(x)\} + \{T(1), T(x), T(1)\}. \tag{4.18}$$

Moreover, since $\{T(1), \{T(1), T(1), T(x)\}, T(1)\} = \{T(1), T(x), T(1)\}$, we deduce

$$\{T(1), T(x), T(1)\} = \{T(1), \{T(1), T(x), T(1)\}, T(1)\},$$

or equivalently

$$T(1)T(x)^*T(1) = T(1)T(1)^*T(x)T(1)^*T(1).$$

We now multiply this equation by $T(1)^*$ on the left and right, respectively, to obtain

$$T(1)^*T(1)T(x)^*T(1) = T(1)^*T(x)T(1)^*T(1),$$

and

$$T(1)T(x)^*T(1)T(1)^* = T(1)T(1)^*T(x)T(1)^*.$$

Also, by multiplying (4.18) on the left and right by $T(1)^*$, and having in mind these last two equations we have, respectively

$$T(1)^*T(x) = T(1)^*T(x)T(1)^*T(1) = T(1)^*T(1)T(x)^*T(1),$$

and

$$T(x)T(1)^* = T(1)T(1)^*T(x)T(1)^* = T(1)T(x)^*T(1)T(1)^*.$$

In particular

$$T(1)^*T(x) = (T(1)^*T(x))^* = T(x)^*T(1),$$

and

$$T(x)T(1)^* = (T(x)T(1)^*)^* = T(1)T(x)^*.$$

By linearizing these expressions we obtain

$$T(1)^*T(a) = T(a^*)^*T(1) \qquad T(a)T(1)^* = T(1)T(a^*)^*. \tag{4.19}$$

Equations (4.18) and (4.19) imply

$$T(x) = T(1)T(1)^*T(x) = T(x)T(1)^*T(1) = T(1)T(x^*)^*T(1), \tag{4.20}$$

for all $x \in A$. In particular, if u is a selfadjoint invertible element in A, Equations (4.17) and (4.19) lead to

$$T(u^2) = T(u)T(u)^*T(1) = T(1)T(u)^*T(u) = T(u)T(1)^*T(u). \tag{4.21}$$

Thus, given a selfadjoint element $h \in A$, and $\mu \in \mathbb{Q}$ such that $u = h + \mu$ is invertible, since $T(1)^{[3]} = T(1)$ and T is linear, Equation (4.21) applied to $u = h + \mu$, together with Equation (4.19) yield

$$T(h^2) = T(h)T(h)^*T(1) = T(1)T(h)^*T(h) = T(h)T(1)^*T(h). \tag{4.22}$$

Thus, for every h, k selfadjoint elements in A, Equation (4.22) ensures that

$$T(hk + kh) = T(h)T(1)^*T(k) + T(k)T(1)^*T(h).$$

Having in mind (4.20)

$$T(1)^*T(hk + kh) = T(h)^*T(k) + T(k)^*T(h). \tag{4.23}$$

As T is linear, from Equations (4.22) and (4.23) we deduce that

$$T(1)^*T(a^2) = T(a^*)^*T(a),$$

for every $a \in A$, and the proof is completed. ■

Corollary 4.3.4 *Let A and B be C^*-algebras and $T : A \to B$ be a linear map strongly preserving regularity. Then T is continuous.*

Proof From the third assertion of the preceding proposition it is clear that the linear mapping $S : A \to B$, $S(x) := T(x)T(1)^*$, is positive, and hence continuous. Indeed, given a positive element $a \in A$, there exists a selfadjoint element $x \in A$ such that $a = x^2$. Then

$$S(a) = T(x^2)T(1)^* = T(x)T(x)^* \geq 0.$$

Finally, if $x = x^*$ then

$$\|T(x)\|^2 = \|T(x)T(x)^*\| = \|S(x^2)\| \leq \|S\|\|x\|^2,$$

which implies that T is bounded on selfadjoint elements, and thus continuous. ■

We can now state the main result of this section.

Theorem 4.3.5 *Let A and B be C^*-algebras and $T : A \to B$ be a linear map. The following are equivalent:*

(1) T strongly preserves regularity,

(2) $T(1)$ is a partial isometry and $T = T(1)S$ for a Jordan $*$-homomorphism $S : A \to B$, with $ST(1)^*T(1) = T(1)^*T(1)S$,

(3) T is a triple homomorphism.

Proof *(1) ⇒ (2)* Let us assume that T strongly preserves regularity. Then $T(1)^{[3]} = T(1)$ and by Proposition 4.3.3 the linear mapping $S : A \to B$ defined by $S(x) = T(1)^*T(x)$ is a Jordan *-homomorphism, $T(x) = T(1)S(x)$ and $S(x)T(1)^*T(1) = S(x) = T(1)^*T(1)S(x)$.

(2) ⇒ (3) Pick $x \in A$. Then

$$\begin{aligned}
T(x^{[3]}) &= T(1)S(x^{[3]}) = T(1)^{[3]}S(x)^{[3]} \\
&= T(1)T(1)^*T(1)S(x)S(x)^*S(x) = T(1)S(x)T(1)^*T(1)S(x^*)S(x) \\
&= T(1)S(x)S(x^*)T(1)^*T(1)S(x) = T(1)S(x)S(x)^*T(1)^*T(1)S(x) \\
&= T(x)T(x)^*T(x) = T(x)^{[3]},
\end{aligned}$$

which shows that T is a triple homomorphism.

(3) ⇒ (1) It is enough to recall that b is the generalized inverse if a if, and only if, $Q(a)(b) = a$ and $Q(a)(b^{[3]}) = b$. ■

4.4 Approximate preservers in C*-algebras

The aim of this section is twofold. On the one hand, we prove that linear maps approximately preserving (triple) generalized invertibility in C*-algebras are close to be triple homomorphisms. On the other hand, we study linear maps approximately preserving the conorm.

Linear maps approximately preserving (triple) generalized inverses

We start this section with a refinement of Lemma 4.3.1.

Theorem 4.4.1 *Let A and B be C*-algebras, being A unital, and $T : A \to B$ a bounded linear map satisfying $T(x^{-1}) = T(x)^\wedge$ for every selfadjoint invertible element $x \in A$. Then T is a triple homomorphism.*

Proof Arguing as in Lemma 3.2.1, pick a selfadjoint invertible element $u \in A$. We may assume that $T(u) \neq 0$. Then, given $\lambda \in \mathbb{C}$ with

$$0 < |\lambda| < \min\{||u^{-1}||^{-2}, ||T(u)^\wedge||^{-2}\},$$

identity (3.4) and the fact that $T(u^{-1}) = T(u)^\wedge$ imply

$$T(u) - \lambda^{-1}T(u)^{[3]} = \left(T(u)^\wedge - (T(u) - \lambda T(u)^\wedge)^\wedge\right)^\wedge = T(u) - \lambda^{-1}T(u^3),$$

which shows that $T(u^3) = T(u)^{[3]}$. Once we have proved that T preserves the cubes of selfadjoint invertible elements, given a selfadjoint element $a \in A$, and $\alpha \in \mathbb{R}$ with $|\alpha| > \|a\|$, as the element $a + \alpha$ is selfadjoint and invertible, we get $T((a+\alpha)^3) = T(a+\alpha)^{[3]}$. Expanding this last equation we obtain:

$$T(a^3) + \alpha T(a^2) + \alpha^2 T(a) + \alpha^3 T(1) =$$

$$= T(a)^{[3]} + \alpha^3 T(1) + 2\alpha\{T(a), T(a), T(1)\} + \alpha\{T(a), T(1), T(a)\} +$$
$$+ 2\alpha^2\{T(1), T(1), T(a)\} + \alpha^2\{T(1), T(a), T(1)\},$$

for every $a \in A$, and $|\alpha| > \|a\|$. From this we deduce that $T(a^3) = T(a)^{[3]}$. That is, T preserves the triple cubes of selfadjoint elements. By [36, Theorem 20], T is a triple homomorphism. ■

As we proceed in Section 3.4 with Jordan homomorphisms, we measure how close is a linear map $T : A \to B$ between C*-algebras to being a triple homomorphism or selfadjoint by the **triple multiplicativity** and the **selfadjointness** of T, respectively:

$$\mathrm{tmult}(T) := \sup\{\|T(\{a,\, b,\, c\}) - \{T(a),\, T(b),\, T(c)\}\| : \|a\| = \|b\| = \|c\| = 1\},$$

$$\mathrm{sa}(T) := \sup_{\|a\|=1} \|T(a^*)^* - T(a)\|.$$

Remark 4.4.2 *Let A and B be C*-algebras. Clearly, every Jordan *-homomorphism $T : A \to B$ is a triple homomorphism. We ask whether* $\mathrm{jmult}(T)$ *and* $\mathrm{sa}(T)$ *being small imply* $\mathrm{tmult}(T)$ *being small.*

Let $T : A \to B$ be a bounded linear map. Define

$$T_J(a, b) := T(a \circ b) - T(a) \circ T(b),$$

$$T_T(a,\, b,\, c) := T(\{a,\, b,\, c\}) - \{T(a),\, T(b),\, T(c)\},$$

$$T_*(a) := T(a^*)^* - T(a),$$

for every $a, b, c \in A$. Then

$$\begin{aligned}
T(\{a,\, b,\, c\}) &= T((a \circ b^*) \circ c) + T(a \circ (b^* \circ c)) - T((a \circ c) \circ b^*) \\
&= T((a \circ b^*) \circ T(c) + T_J(a \circ b^*, c) \\
&+ T(a) \circ T(b^* \circ c) + T_J(a, b^* \circ c) \\
&- T(a \circ c) \circ T(b^*) - T_J(a \circ c, b^*) \\
&= (T(a) \circ T(b^*)) \circ T(c) + T_J(a, b^*) \circ T(c) + T_J(a \circ b^*, c) \\
&+ T(a) \circ (T(b^*) \circ T(c)) + T(a) \circ T_J(b^*, c) + T_J(a, b^* \circ c) \\
&- (T(a) \circ T(c)) \circ T(b^*) - T_J(a, c) \circ T(b^*) - T_J(a \circ c, b^*) \\
&= \{T(a),\, T(b^*)^*,\, T(c)\} + T_J(a, b^*) \circ T(c) + T_J(a \circ b^*, c) \\
&+ T(a) \circ T_J(b^*, c) + T_J(a, b^* \circ c) - T_J(a, c) \circ T(b^*) - T_J(a \circ c, b^*)
\end{aligned}$$

Therefore,

$$\begin{aligned}
T_T(a,\, b,\, c) &= \{T(a),\, T_*(b),\, T(c)\} + T_J(a, b^*) \circ T(c) + T_J(a \circ b^*, c) \\
&+ T(a) \circ T_J(b^*, c) + T_J(a, b^* \circ c) - T_J(a, c) \circ T(b^*) - T_J(a \circ c, b^*).
\end{aligned}$$

This implies that

$$\mathrm{tmult}(T) \le \|T\|^2 \mathrm{sa}(T) + 3(\|T\| + 1)\mathrm{jmult}(T).$$

As in Lemma 3.4.3, it can be shown that, for an operator $\mathbf{T} = [T_n]$, the following hold:

$$\lim_{\mathcal{U}} \operatorname{tmult}(T_n) = \operatorname{tmult}(\mathbf{T})$$

and

$$\lim_{\mathcal{U}} \operatorname{sa}(T_n) = \operatorname{sa}(\mathbf{T}).$$

We will omit the proof of the next result in order to avoid repetition. The argument is analogous to the one used in the proofs of Theorem 3.4.4 and 3.4.6: assuming the contrary we can construct a map $\mathbf{T} = [T_n] : A^{\mathcal{U}} \to B^{\mathcal{U}}$ between the ultrapowers fulfilling $\mathbf{T}(\mathbf{x}^{-1}) = \mathbf{T}(\mathbf{x})^{\wedge}$ for every selfadjoint invertible $\mathbf{x} \in A^{\mathcal{U}}$. By Theorem 4.4.1, $\mathbf{T}$ is a triple homomorphism.

Theorem 4.4.3 *Let A and B be C*-algebras where A is unital and $K, \varepsilon > 0$. Then there exists $\delta > 0$ such that for every linear map $T : A \to B$ with $\|T\| < K$, the condition*

$$\sup_{\|a\|=1, a=a^*} \left\| T(a^{\wedge}) - T(a)^{\wedge} \right\| < \delta$$

implies

$$\operatorname{tmult}(T) < \varepsilon.$$

Remark 4.4.4 *Notice that in Theorem 4.4.1 it can be also obtained that $T(1)^*T$ is a Jordan *-homomorphism. Hence, the hypothesis of Theorem 4.4.3 also yield to* $\operatorname{jmult}(T(1)^*T) < \varepsilon$ *and* $\operatorname{sa}(T(1)^*T) < \varepsilon$.

Corollary 4.4.5 *Let A and B be C*-algebras where A is unital and $K, \varepsilon > 0$. Then there exists $\delta > 0$ such that for every linear map $T : A \to B$ with $\|T\| < K$, the conditions*

$$\sup_{\|a\|=1, a\in A^{\dagger}} \left\| T(a^{\dagger}) - T(a)^{\dagger} \right\| < \delta \quad \textit{and} \quad \operatorname{sa}(T) < \delta$$

imply

$$\operatorname{tmult}(T) < \varepsilon.$$

Proof Let us briefly sketch the proof: assuming the contrary, there exist $K_0, \varepsilon_0 > 0$ and a sequence $\{T_n\}_{n\in\mathbb{N}}$ of linear maps from A to B such that, for every $n \in \mathbb{N}$,

$$\|T_n\| < K_0, \sup_{\|a\|=1} \left\| T_n(a^{\dagger}) - T_n(a)^{\dagger} \right\| < \frac{1}{n}, \quad \operatorname{sa}(T_n) < \frac{1}{n},$$

and

$$\operatorname{tmult}(T_n) \geq \varepsilon_0.$$

Then $\mathbf{T} = [T_n] : A^{\mathcal{U}} \to B^{\mathcal{U}}$ is a selfadjoint map such that $\mathbf{T}(\boldsymbol{a}^{-1}) = \mathbf{T}(\boldsymbol{a})^{\dagger}$, for every invertible element $\boldsymbol{a} \in A^{\mathcal{U}}$. In particular, $\boldsymbol{T}(\boldsymbol{a}^{-1}) = \mathbf{T}(\boldsymbol{a})^{\wedge}$, for every invertible selfadjoint element $\boldsymbol{a} \in A^{\mathcal{U}}$. From Theorem 4.4.1, $\boldsymbol{T}$ is a triple homomorphism. Contradiction. ■

Maps approximately preserving the conorm

Given an element a of a Banach algebra A, the **minimum modulus** and the **surjectivity modulus** of a are defined respectively by

$$\mathrm{m}(a) := \inf\{\|ax\| \colon x \in A,\ \|x\| = 1\},$$

and

$$\mathrm{q}(a) := \inf\{\|xa\| \colon x \in A,\ \|x\| = 1\}.$$

Obviously, $\mathrm{m}(a) = 0$ (respectively $\mathrm{q}(a) = 0$) if, and only if, a is a left (respectively right) topological divisor of zero. It is well-known that for any invertible element $a \in A$,

$$\mathrm{m}(a) = \mathrm{q}(a) = \|a^{-1}\|^{-1}.$$

Moreover,

$$\mathrm{m}(a)\mathrm{m}(b) \leq \mathrm{m}(ab) \leq \|a\|\mathrm{m}(b), \text{ and } |\mathrm{m}(a) - \mathrm{m}(b)| \leq \|a - b\|,$$

for every $a, b \in A$. If A is a C*-algebra, then $\mathrm{m}(a) = \mathrm{q}(a^*)$. In particular, $\mathrm{m}(a) > 0$ (respectively $\mathrm{q}(a) > 0$) if and only if a is left (respectively, right) invertible.

Recall also from Section 2.2 that the conorm of an element a in a Banach algebra A, is defined as

$$\gamma(a) := \begin{cases} \inf\{\|ax\| \colon \mathrm{dist}(x, \ker(L_a)) \geq 1\} & \text{if } a \neq 0 \\ \infty & \text{if } a = 0. \end{cases}$$

For a regular element a in a C*-algebra A,

$$\gamma(a) = \|a^\dagger\|^{-1}.$$

Recall that, in [18], the authors address the question of characterizing unital or surjective linear maps between C*-algebras preserving some spectral quantities (see Theorems 1.1.1 and 1.1.2). They showed that if $T : A \to B$ is a linear map preserving any of certain spectral quantities then T is an isometric Jordan *-homomorphism whenever T is unital, and T is an isometric Jordan *-homomorphism multiplied by a unitary element, whenever T is surjective. In the next results we show that the same holds if we just impose the preserving condition for invertible elements. Notice that we focus our attention in the conorm but identical results can be established for the minimum and surjective modulus.

Theorem 4.4.6 *Let A and B be unital C*-algebras and $T : A \to B$ a unital linear map satisfying $\gamma(T(x)) = \gamma(x)$ for all $x \in A^{-1}$. Then T is a Jordan *-homomorphism.*

Proof First, let us prove that T is injective. Take $a_0 \in A$ such that $T(a_0) = 0$ and let $\alpha \in \mathbb{C}$ be sufficiently small so that $1 + \alpha a_0$ is invertible. Then

$$1 = \gamma(T(1)) = \gamma(T(1 + \alpha a_0)) = \gamma(1 + \alpha a_0)$$

In particular, we get

$$1 \leq \gamma(1 + ita_0) \quad \text{and} \quad 1 \leq \gamma(1 - ta_0)$$

as $t \to 0$. Hence, by [18, Lemma 4.1], both ia_0 and a_0 are selfadjoint and, consequently, $a_0 = 0$.

We claim that T is positive. Indeed, given a selfadjoint element $a \in A$, we know that

$$1 + o(t) \leq \gamma(1 + ita) \quad (\text{as} \quad t \to 0)$$

([18, Lemma 4.1]). Since $1 + ita$ is invertible for $t \in \mathbb{R}$ small enough, it follows

$$1 + o(t) \leq \gamma(1 + ita) = \gamma(1 + itT(a)) \quad (\text{as} \quad t \to 0).$$

This implies that $T(a)$ is selfadjoint (see [18, Lemma 4.1]).

Moreover, given $x \in A$ and $\lambda \notin \sigma(x)$, there exists a neighborhood U_λ of λ such that $x - \mu$ is invertible for every $\mu \in U_\lambda$ and, hence, there exists $\epsilon > 0$ such that $\gamma(x-\mu) > \epsilon$, for every $\mu \in U_\lambda$. Therefore,

$$\gamma(T(x) - \mu) = \gamma(x - \mu) > \epsilon$$

for every $\mu \in U_\lambda$. Consequently,

$$\lim_{\mu \to \lambda} \gamma(T(x) - \mu) \geq \epsilon > 0.$$

This means that $\lambda \notin \sigma_{\mathrm{K}}(T(x))$, where $\sigma_{\mathrm{K}}(x)$ denotes the Kato spectrum of x

$$\sigma_{\mathrm{K}}(a) := \{\lambda \in \mathbb{C} \colon \lim_{\mu \to \lambda} \gamma(a - \mu) = 0\}.$$

Since $\partial\sigma(a) \subseteq \sigma_K(a) \subseteq \sigma(a)$ for every $a \in A$ (see [115, Sections 12,13]), we have just proved that

$$\partial\sigma(T(x)) \subseteq \sigma_K(T(x)) \subseteq \sigma(x)$$

for every $x \in A$. Being T selfadjoint, this implies that T is positive and hence, $\|T\| = 1$.

Arguing as in [18, Theorem 5.1], given a selfadjoint element $a \in A$ and t sufficiently small so that $u = e^{ita}$ is a unitary element with spectrum strictly contained in the unit circle $\mathbb{T}$, since

$$\partial\sigma(T(u)) \subsetneqq \mathbb{T}, \ \|T(u)\| \leq 1, \text{ and } \|T(u)^{-1}\| = \gamma(T(u))^{-1} = \gamma(u)^{-1} = 1,$$

the element $T(u)$ is unitary. From

$$\begin{aligned} 1 &= T(u)T(u)^* = T(u)T(u^*) \\ &= \left(1 + itT(a) - \frac{1}{2}t^2T(a^2) + \cdots\right)\left(1 - itT(a) - \frac{1}{2}t^2T(a^2) + \cdots\right), \end{aligned}$$

we deduce that $T(a^2) = T(a)^2$ as desired. ■

Theorem 4.4.7 *Let A and B be unital C^*-algebras and $T : A \to B$ a surjective linear map satisfying $\gamma(T(x)) = \gamma(x)$ for all $x \in A^{-1}$. Then T is a Jordan *-homomorphism multiplied by a unitary element in B.*

Proof First, let us prove that $b = T(1)$ is invertible. Since $\gamma(T(1)) = \gamma(1) = 1$, b is regular. Let $y = 1 - bb^\dagger$, and $x \in A$ such that $y = T(x)$. Notice that $b^*y = y^*b = 0$.

For $\alpha \in \mathbb{C}$ sufficiently small such that $1 + \alpha x \in A^{-1}$

$$\gamma(1 + \alpha x)^2 = \gamma(b + \alpha y)^2 = \gamma(bb^* + |\alpha|^2 yy^*).$$

Hence,

$$\lim_{|\alpha| \to 0} \gamma(bb^* + |\alpha|^2 yy^*) = \lim_{|\alpha| \to 0} \gamma(1 + \alpha x)^2 = 1 = \gamma(bb^*).$$

Reasoning in a similar way to [18, Theorem 6.2] we get

$$\gamma(1 + \alpha x)^2 = \gamma(bb^* + |\alpha| yy^*) \geq 1 - |\alpha|^2 \|y\|^2,$$

and therefore

$$\gamma(1 + itx) \geq (1 - t^2\|y\|^2)^{\frac{1}{2}} \quad \text{and} \quad \gamma(1 - tx) \geq (1 - t^2\|y\|^2)^{\frac{1}{2}},$$

for enough small $t \in \mathbb{R}$. From these inequalities we get, respectively, that x and ix are selfadjoint. This shows that $x = 0$ and thus $y = 0$. Consequently $1 = bb^\dagger$, that is, b is right invertible. Similarly it can be proved that b is left invertible.

Note that, as in the previous theorem, T is injective. Therefore $S := b^{-1}T$ is a unital and bijective linear map satisfying

$$\mathrm{m}(S(x)) = \mathrm{m}(b^{-1}T(x)) \leq \|b^{-1}\|\mathrm{m}(T(x)) = \mathrm{m}(T(x)) \leq \gamma(T(x)) = \gamma(x),$$

for all $x \in A^{-1}$.

Let y be a selfadjoint element in B and $t \in \mathbb{R}$ small such that $1 + itS^{-1}(y)$ is invertible. Taking $x = 1 + itS^{-1}(y)$ in the previous identity, we have

$$\mathrm{m}(1 + ity) \leq \gamma(1 + itS^{-1}(y)).$$

It follows that S^{-1} is selfadjoint and so is S.

We claim that S is positive. Note that for every $x \in A^\dagger$ and $u \in A^{-1}$, it is clear that $ux \in A^\dagger$, with

$$(ux)(x^\dagger u^{-1})(ux) = ux \quad \text{and} \quad (x^\dagger u^{-1})(ux)(x^\dagger u^{-1}) = x^\dagger u^{-1}.$$

This implies, by [66, Theorem 2], the following:

$$\frac{1}{\|x^\dagger u^{-1}\|} \leq \gamma(ux) \leq \frac{\|x^\dagger u^{-1} ux\| \|uxx^\dagger u^{-1}\|}{\|x^\dagger u^{-1}\|} \leq \frac{\|u\|\|u^{-1}\|}{\|x^\dagger u^{-1}\|}.$$

Hence, for every $x \in A^{-1}$ we have

$$\gamma(S(x)) = \gamma(b^{-1}T(x)) \geq \frac{1}{\|T(x)^\dagger b\|} \geq \frac{1}{\|b\|\|T(x)^\dagger\|} = \|b\|^{-1}\gamma(T(x))$$

and

$$\gamma(S(x)) = \gamma(b^{-1}T(x)) \le \frac{\|b^{-1}\|\|T(1)\|}{\|T(x)^{\dagger}b\|} \le \|b\|\|b^{-1}\|^2\gamma(T(x)).$$

So, we have shown so far:

$$\|b\|^{-1}\gamma(x) \le \gamma(S(x)) \le \|b\|\|b^{-1}\|^2\gamma(x) \text{ for all } x \in A^{-1}. \tag{4.24}$$

The first inequality can be used to show

$$\partial\sigma(S(x)) \subset \sigma_K(S(x)) \subseteq \sigma(x)$$

in a similar way as in the previous theorem. As a consequence, S is positive. In order to conclude that S is an isometric Jordan *-isomorphism, it suffices to prove that S^{-1} is also positive (see for instance [78, Corollary 5]). So, let $h = S(a)$ be a positive element. As S^{-1} is selfadjoint, a is selfadjoint. We can therefore write $a = x - y$, where x, y are positive elements and $xy = yx = 0$. For every $\mu \in \mathbb{C}$, we have

$$\partial\sigma(S(x) + \mu S(y)) \subset \sigma_K(S(x) + \mu S(y)) \subset \sigma(x + \mu y) \subset \mathbb{R} \cup \mu\mathbb{R}.$$

(Recall that if $wz = zw = 0$, then $\sigma(w + z) \setminus \{0\} = (\sigma(w) \setminus \{0\}) \cup (\sigma(z) \setminus \{0\})$.) The previous spectral inclusion gives

$$\sigma(S(x) + \mu S(y)) \subset \mathbb{R} \cup \mu\mathbb{R}.$$

By Lemmas B and C in [44] we get $S(y) = 0$ and so $y = 0$. Consequently, a is positive as desired. We conclude the proof by showing that b is unitary. Indeed, since S is a Jordan $*$-isomorphism and $T(x) = bS(x)$, it is clear that, for every $x \in A^{-1}$, $T(x)$ is invertible with inverse $T(x)^{-1} = S(x^{-1})b^{-1}$. Moreover $\gamma(T(x)) = \gamma(x)$, that is,

$$\|T(x)^{-1}\| = \|x^{-1}\|,$$

for every $x \in A^{-1}$. Since S is an isometry,

$$\|S(x^{-1})\| = \|x^{-1}\| = \|S(x^{-1})b^{-1}\|,$$

or equivalently,

$$\|y\| = \|yb^{-1}\|,$$

for every $y \in B^{-1}$. This yields that b is unitary. ■

Note that, for every invertible element $\boldsymbol{a} \in A^{\mathcal{U}}$, where $\boldsymbol{a} = [a_n]$ is a normalized representative,

$$\gamma(\boldsymbol{a}) = \|\boldsymbol{a}^{-1}\|^{-1} = \lim_{\mathcal{U}} \|a_n^{-1}\|^{-1} = \lim_{\mathcal{U}} \gamma(a_n).$$

We are now in position to prove the main results in this section.

Theorem 4.4.8 *Let A and B be unital C^*-algebras and $K, \varepsilon > 0$. Then there exists $\delta > 0$ such that for every linear map $T : A \to B$ with $\|T\| < K$, the conditions*

$$\sup_{\|a\|=1} |\gamma(T(a)) - \gamma(a)| < \delta \quad \textit{and} \quad \|T(1) - 1\| < \delta$$

imply

$$\mathrm{jmult}(T) < \varepsilon \quad \textit{and} \quad \mathrm{sa}(T) < \varepsilon$$

Proof As we did above, suppose the assertion false, that is, we can find $K_0, \varepsilon_0 > 0$ and a sequence $\{T_n\}_{n\in\mathbb{N}}$ of linear maps from A to B satisfying

- $\|T_n\| \leq K_0$,
- $\sup_{\|a\|=1} |\gamma(T_n(a)) - \gamma(a)| < \frac{1}{n}$, $\|T_n(1) - 1\| < \frac{1}{n}$
- $\mathrm{jmult}(T_n) \geq \varepsilon_0$ or $\mathrm{sa}(T_n) \geq \varepsilon_0$,

for every $n \in \mathbb{N}$. Consider $\mathbf{T} = [T_n] : A^{\mathcal{U}} \to B^{\mathcal{U}}$. We claim that $\mathbf{T}$ is unital and preserves the conorm of invertible elements. On the one hand,

$$\|\mathbf{T}(\mathbf{1}) - \mathbf{1}\| = \lim_{\mathcal{U}} \|T_n(1) - 1\| \leq \lim_{\mathcal{U}} \frac{1}{n} = 0,$$

so $\mathbf{T}(\mathbf{1}) = \mathbf{1}$. On the other hand, for an invertible element $\boldsymbol{a} \in A^{\mathcal{U}}$ with norm 1, let $[a_n]$ be its normalized representative: $\|a_n\| = 1$ and $\|a_n^{-1}\| < \alpha$ for some positive α. We know that

$$\gamma(a_n) - \frac{1}{n} < \gamma(T_n(a_n)) \quad \text{for every } n \in \mathbb{N}.$$

Hence $T_n(a_n) \in A^\dagger$, with $\|T_n(a_n)^\dagger\| < 2\alpha$, for $n > 2\alpha$. That is, $\{T_n(a_n)\}$ is Moore-Penrose invertible almost everywhere, with their respective Moore-Penrose inverses uniformly bounded in norm. This ensures that $\mathbf{T}(\boldsymbol{a})$ is Moore-Penrose invertible and $\gamma(\mathbf{T}(\boldsymbol{a})) = \lim_{\mathcal{U}} \gamma(T_n(a_n))$. Finally:

$$|\gamma(\mathbf{T}(\boldsymbol{a})) - \gamma(\boldsymbol{a})| = \lim_{\mathcal{U}} |\gamma(T_n(a_n)) - \gamma(a_n)| \leq \lim_{\mathcal{U}} \frac{1}{n} = 0.$$

By Theorem 4.4.6, $\mathbf{T}$ is a Jordan *-homomorphism, which gives the contradiction. ■

Let X and Y be complex Banach spaces, and let $\mathcal{B}(X,Y)$ be the algebra of all bounded linear operators from X to Y. Recall that the surjectivity modulus of T is given by $\mathrm{q}(T) := \sup\{\varepsilon \geq 0\colon \varepsilon B_Y \subseteq T(B_X)\}$, where as usual B_X denotes the closed unit ball of X. Note that $\mathrm{q}(T) > 0$ if, and only if, T is surjective, and $\mathrm{q}(T) = \inf\{\|ST\|\colon S \in \mathcal{B}(X),\ \|S\| = 1\}$ (see [115, Theorem II.9.11]).

Theorem 4.4.9 *Let A and B be unital C^*-algebras and $K, \varepsilon > 0$. Then there exists $\delta > 0$ such that for every linear map $T : A \to B$ with $\|T\| < K$ and $q(T) > K^{-1}$, the condition*

$$\sup_{\|a\|=1} |\gamma(T(a)) - \gamma(a)| < \delta$$

implies

$$\mathrm{jmult}(T(1)^*T) < \varepsilon, \quad \mathrm{sa}(T(1)^*T) < \varepsilon.$$

Proof If we assume the contrary, hence, as above we find $K_0, \varepsilon_0 > 0$ and a sequence $\{T_n\}_{n\in\mathbb{N}}$ of linear maps from A to B satisfying

- $\|T_n\| \leq K_0$,
- $\sup_{\|a\|=1} |\gamma(T_n(a)) - \gamma(a)| < \frac{1}{n}, \quad \mathrm{q}(T_n) \geq K_0^{-1}$
- $\mathrm{jmult}(T_n(1)^* T_n) \geq \varepsilon_0$ or $\mathrm{sa}(T_n(1)^* T_n) \geq \varepsilon_0$,

for every $n \in \mathbb{N}$.

As in the previous theorem, the map $\mathbf{T} = [T_n]$ preserves the conorm of all invertible elements. Moreover,

$$\mathrm{q}(\mathbf{T}) = \lim_{\mathcal{U}} \mathrm{q}(T_n) \geq K_0^{-1} > 0$$

and thus $\mathbf{T}$ is surjective. By Theorem 4.4.7, $\mathbf{T}(\mathbf{1})$ is unitary and $\mathbf{T}(\mathbf{1})^*\mathbf{T}$ is a unital Jordan *-isomorphism. ■

Chapter 5

Linear maps strongly preserving Brown-Pedersen quasi-invertibility in JB*-triples

In this chapter we study linear maps between JB*-triples strongly preserving regularity and Brown-Pedersen quasi-invertibility. We also explore the connections between linear maps strongly preserving Brown-Pedersen quasi-invertibility and other classes of linear preservers between C*-algebras like Bergmann-zero pair preservers, extreme point preservers and Brown-Pedersen quasi-invertibility preservers. The content within this chapter can be found in [35].

5.1 Linear maps strongly preserving regularity in JB*-triples

A linear map $T : E \to F$ between JB*-triples **strongly preserves regularity** if $T(x^\wedge) = T(x)^\wedge$ for every $x \in E^\wedge$. Every triple homomorphism $T : E \to F$ between JB*-triples strongly preserves regularity. In Theorem 4.3.5 we have characterized the triple homomorphisms between C*-algebras as the linear maps strongly preserving regularity. As consequence, we have proved that a selfadjoint linear map from a unital C*-algebra A into a C*-algebra B is a triple homomorphism if, and only if, it strongly preserves Moore-Penrose invertibility.

Recall that a nonzero element a in a JB*-triple E is von Neumann regular if, and only if, the range tripotent $r(a)$ of a lies in E and a is positive in the JB*-algebra $E_2(e)$. Moreover, when a is von Neumann regular,

$$L(a, a^\wedge) = L(a^\wedge, a) = L(r(a), r(a)).$$

Two elements a and b in a JB*-triple E are said to be orthogonal, and denoted by $a \perp b$, if $L(a, b) = 0$ (several equivalent definitions can be found in [36, Lemma 1]). The next result is inspired in Lemma 4.3.1.

Lemma 5.1.1 *Let E and F be JB*-triples, and let $T : E \to F$ be a linear map such that $T(x^\wedge) = T(x)^\wedge$ for every $x \in E^\wedge$. Then*

$$T(x^{[3]}) = T(x)^{[3]},$$

for every $x \in E^\wedge$.

Proof Let $x \in E^\wedge \setminus \{0\}$. Let $e = r(x)$ the range tripotent of x. As we have just mentioned, x is positive and invertible in the JB*-algebra $E_2(e)$, with inverse $x^\wedge$, and $0 \notin Sp(x)$. As usual we identify E_x with $C(\mathrm{Sp}(x))$ in such a way that x corresponds to the function $t \mapsto t$. Hence for every $\lambda \in \mathbb{C}$ with $0 < |\lambda| < ||x^\wedge||^{-2}$, the element $\lambda x^\wedge - x$ is invertible in E_x, and hence invertible in $E_2(e)$, with inverse $(\lambda x^\wedge - x)^\wedge$. In this case, $x^\wedge + (\lambda x^\wedge - x)^\wedge$ is invertible in E_x (and in $E_2(e)$).

Further, the inverses of $x - \lambda x^\wedge$ and $x^\wedge - (x - \lambda x^\wedge)^\wedge$ in E_x (or in $E_2(e)$) are their generalized inverses in E (let us recall that the triple product induced on $E_2(e)$ by the Jordan *-algebra structure coincides with its original triple product, and $Q(x) = U_x \circ \sharp$, for every $x \in E$). By Hua's identity (cf. (2.1)), applied to $a = x$ and $b = \lambda^{-1}x$, we obtain

$$x - \lambda^{-1}x^{[3]} = \left(x^\wedge - (x - \lambda x^\wedge)^\wedge\right)^\wedge.$$

Let $x \in E^\wedge$. We may assume that $T(x) \neq 0$. Since T strongly preserves regularity, $T(x)^\wedge = T(x^\wedge)$. Thus, for $\lambda \in \mathbb{C}$ with $0 < |\lambda| < \min\{||x^\wedge||^{-2}, ||T(x)^\wedge||^{-2}\}$, we have

$$T(x) - \lambda^{-1}T(x)^{[3]} = \left(T(x)^\wedge - (T(x) - \lambda T(x)^\wedge)^\wedge\right)^\wedge.$$

Since T is linear and strongly preserves regularity, it follows that

$$T(x) - \lambda^{-1}T(x)^{[3]} = (T(x)^\wedge - (T(x) - \lambda T(x)^\wedge)^\wedge)^\wedge = (T(x^\wedge) - T(x - \lambda x^\wedge)^\wedge)^\wedge$$

$$= T(\left(x^\wedge - (x - \lambda x^\wedge)^\wedge\right)^\wedge) = T(x) - \lambda^{-1}T(x^{[3]}),$$

and thus $T(x^{[3]}) = T(x)^{[3]}$. ■

Remark 5.1.2 *Let $T : E \to F$ be a linear map between JB*-triples. Assume that T strongly preserves regularity. Then T preserves the orthogonality relation on regular elements. Indeed, given $a, b \in E^\wedge$, such that $a \perp b$, it can be easily seen that*

$$(a + \alpha b)^\wedge = a^\wedge + \alpha^{-1}b^\wedge,$$

for every $\alpha \in \mathbb{R} \setminus \{0\}$. By assumption $T(a^\wedge + \alpha^{-1}b^\wedge) = T(a + \alpha b)^\wedge$. In particular

$$\{T(a) + \alpha T(b), T(a)^\wedge + \alpha^{-1}T(b)^\wedge, T(a) + \alpha T(b)\} = T(a) + \alpha T(b).$$

It follows from the above identity that

$$2\alpha\{T(a), T(a)^\wedge, T(b)\} + 2\{T(a), T(b)^\wedge, T(b)\}$$
$$-\alpha^{-1}\{T(a), T(b)^\wedge, T(a)\} + \alpha^2\{T(b), T(a)^\wedge, T(b)\} = 0,$$

for every $\alpha \in \mathbb{R} \setminus \{0\}$. *Therefore*

$$\{T(a), T(a)^\wedge, T(b)\} = 0, \qquad \{T(a), T(b)^\wedge, T(b)\} = 0.$$

Since $L(T(a), T(a)^\wedge) = L(r(T(a)))$, *and* $L(T(b), T(b)^\wedge) = L(r(T(b)))$, *it follows that* $T(a) \perp T(b)$.

Notice that a JB*-triple might contain no non-trivial tripotents (consider, for example, the C*-algebra $C_0(0,1]$ of all complex-valued continuous functions on $[0,1]$ vanishing at 0). However, since the complete tripotents of a JB*-triple E coincide with the extreme points of its closed unit ball (see [46, Theorem 3.2.3]), every JBW*-triple contains a large set of complete tripotents.

Let us recall that, by Lemma 2.1 in [74], an element a in a JB*-triple E is Brown-Pedersen quasi-invertible if, and only if, a is regular and $\{a\}^\perp = \{0\}$, where $\{a\}^\perp = \{b \in E \colon a \perp b = 0\}$. For a JB*-triple E, we will denote by $\partial_e(E_1)$ the set of extreme points of the unit ball of E.

Theorem 5.1.3 *Let E and F be JB*-triples with $\partial_e(E_1) \neq \emptyset$. Let $T : E \to F$ be a linear map strongly preserving regularity. Then T is a triple homomorphism.*

Proof Pick a complete tripotent $e \in E$. For every $x \in E$, let $\lambda \in \mathbb{C}$, with $|\lambda| > ||P_2(e)(x)||$. It is clear that $P_2(e)(x - \lambda e) = P_2(e)(x) - \lambda e$ is invertible in the unital JB*-algebra $E_2(e)$. It follows from [74, Lemma 2.2] that $x - \lambda e$ is Brown-Pedersen quasi-invertible. We know, by Proposition 5.1.1, that

$$T\left((x - \lambda e)^{[3]}\right) = T(x - \lambda e)^{[3]}.$$

Since the above identity holds for every $\lambda \in \mathbb{C}$, with $|\lambda| > ||P_2(e)(x)||$, we deduce that

$$T(x^{[3]}) = T(x)^{[3]},$$

for every $x \in E$. The polarization formula

$$8\{x, y, z\} = \sum_{k=0}^{3} \sum_{j=1}^{2} i^k (-1)^j \left(x + i^k y + (-1)^j z\right)^{[3]}, \tag{5.1}$$

and the linearity of T assure that T is a triple homomorphism. ■

Particularizing the previous result to the setting of C*-algebras we obtain the following result.

Corollary 5.1.4 *Let $T : A \to B$ be a linear map strongly preserving regularity between C*-algebras. Suppose that $\partial_e(A_1) \neq \emptyset$. Then T is a triple homomorphism.* ■

5.2 Maps strongly preserving regularity on weakly compact JB*-triples

The notions of compact and weakly compact elements in JB*-triples is due to L. Bunce and Ch.-H. Chu [24]. Recall that an element a in a JB*-triple E is said to be **compact** or **weakly compact** if the mapping $Q(a)$ is compact or weakly compact, respectively. These notions extend, in a natural way, the corresponding definitions in the settings of C*-algebras. A JB*-triple E is weakly compact (respectively, compact) if every element in E is weakly compact (respectively, compact).

In a JB*-triple, the set of weakly compact elements is, in general, strictly bigger than the set of compact elements (cf. [24, Theorem 3.6]). A nonzero tripotent e in E is called **minimal** whenever $E_2(e) = \mathbb{C}e$. The **socle**, $\mathrm{soc}(E)$, of a JB*-triple E is the linear span of all minimal tripotents in E. Following [24], the symbol $K_0(E)$ denotes the norm-closure of $\mathrm{soc}(E)$. By [24, Lemma 3.3 and Proposition 4.7], the triple ideal $K_0(E)$ coincides with the set of all weakly compact elements in E. Hence a JB*-triple E is weakly compact whenever $E = K_0(E)$. Every finite sum of mutually orthogonal minimal tripotents in a JB*-triple E lies in the socle of E. It is also known that an element a in a JB*-triple E is weakly compact if, and only if, $L(a,a)$ is a weakly compact operator (see [24]). Therefore, for each tripotent e in the socle of E, $P_1(e) = 2L(e,e) - P_2(e) = 2L(e,e) - Q(e)^2$ is a weakly compact operator on E (cf. [56, §2]).

It is well-known that every element in the socle of a JB*-triple is regular. Moreover, for every JB*-triple E,

$$E^\wedge + \mathrm{soc}(E) \subseteq E^\wedge.$$

Indeed, given $a \in E^\dagger$ and $x \in \mathrm{soc}(E)$,

$$(a+x) - Q(a+x)(a^\wedge) = x - 2\{a, a^\wedge, x\} - \{x, a^\wedge, x\} \in \mathrm{soc}(E) \subseteq E^\dagger.$$

By Mc Coy's Lemma (see [108]), $a + x \in E^\wedge$.

Let E, F be JB*-triples. Let us assume that E has nonzero socle, and let $T : E \to F$ be a linear map strongly preserving regularity. The polarization formula (5.1) and Proposition 5.1.1 show that $T(\{x,y,z\}) = \{T(x), T(y), T(z)\}$, whenever one of the elements x, y, or z is regular and the others lie in the socle.

Theorem 5.2.1 *Let E, F be JB*-triples, with E weakly compact. Let $T : E \to F$ be a bounded linear map strongly preserving regularity. Then T is a triple homomorphism.*

Proof We know, from Proposition 5.1.1, that T preserves cubes of regular elements. Since every element in the socle of a JB*-triple is regular, is follows that $T(x^{[3]}) = T(x)^{[3]}$, for every $x \in \mathrm{soc}(E)$. Since $E = K_0(E) = \overline{\mathrm{soc}(E)}$, the continuity of T, together with the norm continuity of the triple product prove that T is a triple homomorphism. ■

In the next example we show that the continuity assumption cannot be dropped from the hypothesis in the previous theorem (even in the setting of C*-algebras).

Remark 5.2.2 *Let c_0 denote the C*-algebra of all scalar null sequences. It is clear that c_0 is a weakly compact JB*-triple, with $soc(c_0) = c_{00}$, i.e. the subspace of eventually zero sequences in c_0. Let $\{e_n\}$ denote the standard coordinate (Schauder) basis of c_0. We extend this basis, via Zorn's lemma, to an algebraic (Hamel) basis of c_0, say $B = \{e_n\} \cup \{z_n\}$.*

We define $T : c_0 \to c_0$ as the linear (unbounded) mapping given by

$$T(e_n) = e_n, \qquad T(z_n) = nz_n.$$

Clearly T is not a triple homomorphism but it strongly preserves regularity. Let us notice that $c_0^\wedge = c_{00}$ and $T(c_{00}) = c_{00}$.

5.3 Linear maps strongly preserving Brown-Pedersen quasi-invertibility

Let X be a Banach space. In many favorable cases, the set $\partial_e(X_1)$, of all extreme points of the closed unit ball, X_1, of X, reveals many of the geometric properties of the whole Banach space X. There are spaces X with $\partial_e(X_1) = \emptyset$, however, the Krein-Milman theorem guarantees that $\partial_e(X_1)$ is non-empty when X is a dual space.

Let A be a C*-algebra. It is known that $\partial_e(A_1) \neq \emptyset$ if and only if A is unital (see [133, Theorem I.10.2(i)]). When A is commutative, the unitary elements in A are precisely the extreme points of the closed unit ball of A. The same statement remains true when A is a finite von Neumann algebra (cf. [101, Lemma 2]). For a general unital C*-algebra A, every unitary element in A is an extreme point of the closed unit ball of A, however, the reciprocal statement is, in general, false (for example a non-surjective isometry in $B(H)$ is not a unitary element in this C*-algebra). A theorem due to R.V. Kadison establishes that the extreme points of the closed unit ball of a C*-algebra A are precisely the maximal partial isometries of A, i.e., partial isometries $e \in A$ satisfying $(1 - ee^*)A(1 - e^*e) = 0$ (cf. [133, Theorem I.10.2]).

Let A and B be unital C*-algebras. One of the consequences derived from the Russo-Dye theorem assures that a linear mapping $T : A \to B$ mapping unitary elements in A to unitary elements in B admits a factorization of the form $T(a) = uS(a)$ $(a \in A)$, where u is a unitary in B and S is a unital Jordan *-homomorphism (cf. [126, Corollary 2]).

Consequently, the problem of studying the linear maps $T : A \to B$ such that $T(\partial_e(A_1)) \subseteq \partial_e(B_1)$ is a more general challenge, which is directly motivated by the just mentioned consequence of the Russo-Dye theorem. We only know partial answers to this problem. Concretely, V. Mascioni and L. Molnár studied the linear maps on a von Neumann factor M which preserve the extreme points of the unit ball of M. They prove that if M is infinite, then every linear mapping T on M preserving extreme points admits a factorization of the form $T(a) = uS(a)$ $(a \in M)$, where u is a (fixed) unitary in M and S either is a unital *-homomorphism or a unital *-anti-homomorphism (see Theorem 1.1.5). Theorem 2 in [101] states that, for a finite von Neumann algebra M, a linear map $T : M \to M$ preserves extreme points of the unit ball of M if and only if there exist a unitary operator $u \in M$ and a unital Jordan *-homomorphism $S : M \to M$ such that $T(a) = uS(a)$ $(a \in A)$. In [87], L.E. Labuschagne and V. Mascioni study linear maps between C*-algebras whose adjoints preserve extreme points of the dual ball.

The above results of Mascioni and L. Molnár are the most conclusive answers we know about linear maps between unital C*-algebras preserving extreme points. In this note we shall revisit the problem in full generality. We present several counter-examples to illustrate that the conclusions proved by Mascioni and Molnár for von Neumann factors need not be true for linear mappings preserving extreme points between unital C*-algebras (compare Remarks 5.3.4 and 5.3.5). It seems natural to ask whether a different class of linear preservers satisfies the same conclusions found by Mascioni and Molnár.

In [56], the authors proved that Bergmann operators can be used to characterize the relation of being orthogonal in JB*-triples. More concretely, it is proved in [56, Proposition 7] that, for any element x in a JB*-triple E with $||x|| < \sqrt{2}$, the orthogonal annihilator of x in E coincides with the set of all fixed points of the Bergmann operator $B(x, x)$. It is also obtained, in the just quoted paper, that a norm one element e in a JB*-triple E is a tripotent if, and only if, $B(e, e)(E) = \{e\}^{\perp}$ (cf. [56, Proposition 9]).

Having in mind all the characterizations of tripotents and Brown-Pedersen quasi-invertible elements commented above, and recalling that extreme points of the closed unit ball of a JB*-triple E are precisely the complete tripotents in E, it can be deduced that the equivalence

$$e \in \partial_e(E_1) \Leftrightarrow B(e, e) = 0, \tag{5.2}$$

holds for every $e \in E_1$.

Let $T : E \to F$ be a linear map between JB*-triples. We introduce some definitions concerning linear preservers. We say that T **preserves Brown-Pedersen quasi-invertibility** if $T(E_q^{-1}) \subseteq F_q^{-1}$, that is, T maps Brown-Pedersen quasi-invertible elements in E to Brown-Pedersen quasi-invertible elements in F. T is said to **preserve Bergmann-zero pairs** if

$$B(x, y) = 0 \Rightarrow B(T(x), T(y)) = 0.$$

Furthermore, we say that T **strongly preserves Brown-Pedersen quasi-invertibility** if T preserves Brown-Pedersen quasi-invertibility and $T(x^\wedge) = T(x)^\wedge$ for every $x \in E_q^{-1}$. Finally, T is said to **preserve extreme points** if $T(\partial_e(E_1)) \subseteq \partial_e(F_1)$. It is worth to notice that all definitions above make sense for linear operators between C*-algebras. In this section we employ Jordan techniques to study these kind of mappings and so, we set the above definitions in the most general setting.

Suppose $T : E \to F$ is a linear mapping strongly preserving Brown-Pedersen quasi-invertibility between two JB*-triples. Suppose $u \in \partial_e(E_1)$. Then u is Brown-Pedersen quasi-invertible with $u^\wedge = u$. It follows from our assumptions that $T(u)$ is Brown-Pedersen quasi-invertible and

$$T(u)^\wedge = T(u^\wedge) = T(u).$$

In such a case,

$$\{T(u), T(u), T(u)\} = Q(T(u))(T(u)) = T(u)$$

is a tripotent and Brown-Pedersen quasi-invertible, which implies that $T(u) \in \partial_e(E_1)$ (cf. [74, Lemma 2.1]). We have therefore shown that every linear mapping between JB*-triples strongly preserving Brown-Pedersen quasi-invertibility also preserves extreme points points.

The characterization of the extreme points of the closed unit ball of a JB*-triple given in (5.2) implies that every linear mapping between JB*-triples preserving Bergmann-zero pairs also preserves extreme points.

Clearly, a linear mapping $T : E \to F$ preserving Bergmann-zero pairs maps Brown-Pedersen quasi-invertible elements in E to Brown-Pedersen quasi-invertible elements in F.

Therefore, for every linear mapping T between JB*-triples the following implications hold:

$$\begin{array}{ccc}
\begin{array}{c}T \text{ preserves}\\ \text{Bergmann-zero pairs}\end{array} & \Longrightarrow & \begin{array}{c}T \text{ preserves}\\ \text{BP quasi-invertible elements}\end{array}\\
\Downarrow & & {\scriptstyle\text{(Remark 5.3.5)}}\ \not\Downarrow \quad \Uparrow\\
T \text{ preserves extreme points} & \begin{array}{c}\Longleftarrow\\ \stackrel{\text{(Remark 5.3.5)}}{\not\Longrightarrow}\end{array} & \begin{array}{c}T \text{ strongly preserves}\\ \text{BP quasi-invertible elements}\end{array}
\end{array}$$

The other implications are, for the moment, unknown. V. Mascioni and L. Molnár characterized the linear maps on a von Neumann factor M preserving the extreme points of the unit ball of M in [101]. According to our terminology, they prove that, for a von Neumann factor M, a linear map $T : M \to M$ such that $B(T(a), T(a)) = 0$ whenever $B(a, a) = 0$, is a unital Jordan *-homomorphism multiplied by a unitary element (see [101, Theorem 1, Theorem 2]).

Suppose $T : E \to E$ is a linear mapping between JB*-triples which preserves Bergmann-zero pairs. Given a Brown-Pedersen quasi-invertible element x, with generalized inverse $x^\wedge$, we have

$$B(x, x^\wedge) = B(x^\wedge, x) = B(r(x), r(x)) = 0,$$

and hence $B(T(x),T(x^\wedge))=0$. This shows that

$$Q(T(x))(T(x^\wedge))=T(x), \quad \text{and} \quad Q(T(x^\wedge))(T(x))=T(x^\wedge).$$

However $T(x^\wedge)$ may not coincide, in general, with $T(x)^\wedge$. So, we cannot conclude that every linear Bergmann-zero pairs preserving is a strongly Brown-Pedersen quasi-invertibility preserver (cf. Remark 5.3.5).

We mainly focus our study on maps between C*-algebras. Let A be a unital C*-algebra A. It is easy to see that, for an element a in A

$$B(a,a)(x)=(1-aa^*)x(1-a^*a), \quad \text{for all } x\in A.$$

Moreover it is also a well-known fact that the extreme points of the closed unit ball of A are precisely those elements v in A for which

$$(1-vv^*)A(1-v^*v)=\{0\}$$

(see [133, Theorem I.10.2]).

Let $T:A\to B$ be a linear map between unital C*-algebras which preserves extreme points. Since for every unitary element $u\in A$, $B(u,u)=0$ it follows that $B(T(u),T(u))=0$, which, in particular, shows that $T(u)$ is a partial isometry. Hence, T is automatically bounded and $\|T\|=1$ (cf. [126, §3]). Therefore, for every selfadjoint element $a\in A$, we have

$$\{T(e^{ita}),T(e^{ita}),T(e^{ita})\}=T(e^{ita}) \qquad (t\in\mathbb{R}).$$

Differentiating both sides of the above identity with respect to t, we deduce that

$$2\{iT(ae^{ita}),T(e^{ita}),T(e^{ita})\}+\{T(e^{ita}),iT(ae^{ita}),T(e^{ita})\}=iT(ae^{ita}),$$

and hence

$$2\{T(ae^{ita}),T(e^{ita}),T(e^{ita})\}-\{T(e^{ita}),T(ae^{ita}),T(e^{ita})\}=T(ae^{ita}), \tag{5.3}$$

for every $t\in\mathbb{R}$. For $t=0$, we get

$$2\{T(a),T(1),T(1)\}-\{T(1),T(a),T(1)\}=T(a),$$

equivalently

$$T(a)=T(a)T(1)^*T(1)+T(1)T(1)^*T(a)-T(1)T(a)^*T(1), \tag{5.4}$$

for every $a=a^*$ in A.

Differentiating (5.3) with respect to t, we obtain

$$T(a^2e^{ita})=2\{T(a^2e^{ita}),T(e^{ita}),T(e^{ita})\}-4\{T(ae^{ita}),T(ae^{ita}),T(e^{ita})\}$$
$$+2\{T(ae^{ita}),T(e^{ita}),T(ae^{ita})\}+\{T(e^{ita}),T(a^2e^{ita}),T(e^{ita})\},$$

for every $t \in \mathbb{R}$. In the case $t = 0$ we get

$$T(a^2) = 2\{T(a^2), T(1), T(1)\} - 4\{T(a), T(a), T(1)\}$$
$$+2\{T(a), T(1), T(a)\} + \{T(1), T(a^2), T(1)\},$$

or equivalently,

$$T(a^2) = T(a^2)T(1)^*T(1) + T(1)T(1)^*T(a^2) - 2T(a)T(a)^*T(1) \tag{5.5}$$
$$-2T(1)T(a)^*T(a) + 2T(a)T(1)^*T(a) + T(1)T(a^2)^*T(1),$$

for every $a = a^*$ in A.

Multiplying identity (5.4) by $T(1)^*$ from both sides, and taking into account that $T(1)$ is a (maximal) partial isometry, we deduce that

$$T(1)^*T(a)T(1)^* = T(1)^*T(1)T(a)^*T(1)T(1)^*, \tag{5.6}$$

for every selfadjoint element $a \in A$.

Proposition 5.3.1 *Let A and B be unital C*-algebras. Let $T : A \to B$ be a linear map preserving extreme points. Suppose that $T(1)$ is a unitary in B. Then there exists a unital Jordan *-homomorphism $S : A \to B$ satisfying $T(a) = T(1)S(a)$, for every $a \in A$.*

Proof By hypothesis $v = T(1)$ is a unitary in B. We deduce from (5.4) that

$$T(a) = vT(a)^*v,$$

for every selfadjoint element $a \in A$, and hence, by linearity,

$$T(a) = vT(a^*)^*v, \text{ or equivalently, } v^*T(a) = T(a^*)^*v, \tag{5.7}$$

for every $a \in A$. Therefore, the mapping $S : A \to B$, defined by $S(x) := v^*T(x)$, is selfadjoint, and $S(1) = v^*T(1) = v^*v = 1$.

Now, since $v^*v = 1 = vv^*$, we deduce from (5.5) and (5.7) that

$$T(a^2) = vT(a)^*T(a),$$

for every $a = a^*$ in A. Multiplying on the left by v^* we obtain:

$$S(a^2) = v^*vT(a)^*T(a) = T(a)^*T(a) = S(a)^*S(a) = S(a)^2,$$

for every $a = a^*$ in A, and hence S is a Jordan *-homomorphism. It is also clear that $T(a) = vv^*T(a) = vS(a)$, for every a in A. ■

We recall that, according to [133, Theorem 10.2], for a C*-algebra, A, the intersection $\partial_e(A_1) \cap A_{sa}$ is precisely the set of all selfadjoint unitary elements of A.

Corollary 5.3.2 *Let A and B be unital C*-algebras. Let $T : A \to B$ be a selfadjoint linear map. If T preserves extreme points then $T(1)$ is a selfadjoint unitary element in B and there exists a unital Jordan *-homomorphism $S : A \to B$ satisfying $T(a) = T(1)S(a)$, for every $a \in A$.*

Proof Suppose that T preserves extreme points. Since T is selfadjoint, the element $T(1)$ must be a selfadjoint extreme point of the closed unit ball of B, and hence a selfadjoint unitary element. Proposition 5.3.1 assures that $S(a) := T(1)T(a)$ $(a \in A)$ is a unital Jordan *-homomorphism and $T(a) = T(1)S(a)$, for every $a \in A$. ■

The next result gives sufficient conditions for the reciprocal statement of Proposition 5.3.1 and Corollary 5.3.2.

Proposition 5.3.3 *Let $T : A \to B$ be a linear map between unital C*-algebras. Suppose that T writes in the form $T = vS$, where v is a unitary element in B and $S : A \to B$ is a unital Jordan *-homomorphism such that B equals the C*-algebra generated by $S(A)$. Then T preserves extreme points.*

Proof Suppose that $T = vS$, where v is a unitary element in B and that $S : A \to B$ is a unital Jordan *-homomorphism. Since $S^{**} : A^{**} \to B^{**}$ is a unital Jordan *-homomorphism between von Neumann algebras (cf. [132, Lemma 3.1]), Theorem 3.3 in [132] implies the existence of two orthogonal central projections E and F in B^{**} such that $S_1 = S^{**} : A^{**} \to B^{**}E$ is a *-homomorphism, $S_2 = S^{**} : A^{**} \to B^{**}F$ is a *-anti-homomorphism, $E + F = 1$ and $S^{**} = S_1 + S_2$. The equality $1 = S(1) = S_1(1) + S_2(1)$ implies that $S_1(1) = E$ and $S_2(1) = F$.

Take $e \in \partial_e(A_1)$. We claim that $S(e) \in \partial_e(B_1)$. Indeed, the equalities

$$\begin{aligned}
&(1 - S(e)S(e)^*)S(A)(1 - S(e)^*S(e)) \\
&= (1 - S_1(e)S_1(e)^* - S_2(e)S_2(e)^*)S(A)(1 - S_1(e)^*S_1(e) - S_2(e)^*S_2(e)) \\
&= (E - S_1(e)S_1(e)^*)S_1(A)(E - S_1(e)^*S_1(e)) \\
&\quad +(F - S_2(e)S_2(e)^*)S_2(A)(F - S_2(e)^*S_2(e)) \\
&= S_1((1 - ee^*)A(1 - e^*e)) + S_2((1 - e^*e)A(1 - ee^*)) = \{0\},
\end{aligned}$$

together with the fact that B equals the C*-algebra generated by $S(A)$ show that $S(e) \in \partial_e(B_1)$.

Finally, given $e \in \partial_e(A_1)$ we know that $S(e) \in \partial_e(B_1)$, and hence

$$\begin{aligned}
(1 - T(e)T(e)^*)B(1 - T(e)^*T(e)) &= (1 - vS(e)S(e)^*v)B(1 - S(e^*)v^*vS(e)) \\
&= v(1 - S(e)S(e)^*)v^*B(1 - S(e^*)S(e)) \\
&\subseteq v(1 - S(e)S(e)^*)B(1 - S(e^*)S(e)) = \{0\},
\end{aligned}$$

because $S(e) \in \partial_e(B_1)$. We have therefore shown that $T(e) \in \partial_e(B_1)$. ■

Henceforth, $T : A \to B$ will denote a linear map between unital C*-algebras which preserves extreme points, and we assume that B is prime. Let $v = T(1) \in \partial_e(B_1)$. The assumption B being prime implies that $vv^* = 1$ or $v^*v = 1$. We shall assume that $v^*v = 1$. When $vv^* = 1$ we can apply Proposition 5.3.1, otherwise (i.e. in the case $vv^* \neq 1$), we cannot deduce the same conclusions. Indeed, from (5.4) and (5.6) we deduce that $vv^*T(a) = vT(a)^*v$, for every $a = a^*$ in A, and consequently $v^*T(a) = T(a)^*v$, for every $a \in A_{sa}$. Therefore, the operator $S = v^*T$ is unital and selfadjoint. Moreover, it follows from (5.5) that

$$vv^*T(a^2) = 2T(a)T(a)^*v + 2vT(a)^*T(a) - 2T(a)v^*T(a) - vT(a^2)^*v,$$

for every $a = a^*$ in A. Multiplying on the left by v^*, and having in mind that S is selfadjoint, we obtain

$$S(a^2) = 2S(a)S(a^*) + 2T(a)^*T(a) - 2S(a)S(a) - S((a^2)^*) = 2T(a)^*T(a) - S(a^2),$$

which proves that $S(a^2) = T(a)^*T(a) \geq S(a)^2$, for every $a = a^*$ in A.

When M is an infinite von Neumann factor, a linear map $T : M \to M$ preserves extreme points if, and only if, there exist a unitary u in M and a linear map $\Phi : M \to M$ which is either a unital *-homomorphism or a unital *-anti-homomorphism such that $T(a) = u\Phi(a)$ $(a \in A)$ [101, Theorem 1]. When M is a finite von Neumann algebra, a linear map T on M preserves extreme points if, and only if, there exist a unitary u in M and a Jordan *-homomorphism $\Phi : M \to M$ satisfying $T(a) = u\Phi(a)$ $(a \in A)$ [101, Theorem 2]. Motivated by these results it is natural to ask whether a similar conclusion remains true for operators preserving extreme points between unital C*-algebras. The next simple examples show that the answer is, in general, negative.

Remark 5.3.4 *Let H be an infinite dimensional complex Hilbert space. Suppose v is a maximal partial isometry in $\mathcal{B}(H)$ which is not a unitary. The operator $T : \mathbb{C} \to \mathcal{B}(H)$, $\lambda \mapsto \lambda v$, preserves extreme points, but we cannot write T in the form $T = u\Phi$, where u is a unitary in $\mathcal{B}(H)$ and Φ is a unital Jordan *-homomorphism.*

Remark 5.3.5 *Under the assumptions of Remark 5.3.4, let v and w be extreme points in $\mathcal{B}(H)_1$ such that $v^*v = 1 = w^*w$ and $vv^* \perp ww^*$. Let $A = \mathbb{C} \oplus^\infty \mathbb{C}$. We consider the following operator*

$$T : A \to \mathcal{B}(H)$$

$$T(\lambda, \mu) = \frac{\lambda}{2}(v + w) + \frac{\mu}{2}(v - w).$$

Clearly $T(1,1) = v$. Furthermore, every extreme point of the closed unit ball of A writes in the form (λ_0, μ_0) with $|\lambda_0| = |\mu_0| = 1$. Therefore

$$T(\lambda_0, \mu_0) = \frac{\lambda_0}{2}(v + w) + \frac{\mu_0}{2}(v - w) = \frac{\lambda_0 + \mu_0}{2}v + \frac{\lambda_0 - \mu_0}{2}w$$

satisfies

$$T(\lambda_0,\mu_0)^*T(\lambda_0,\mu_0) = \left(\frac{\lambda_0+\mu_0}{2}v + \frac{\lambda_0-\mu_0}{2}w\right)^* \left(\frac{\lambda_0+\mu_0}{2}v + \frac{\lambda_0-\mu_0}{2}w\right)$$

$$= \frac{|\lambda_0+\mu_0|^2}{4}v^*v + \frac{|\lambda_0-\mu_0|^2}{4}w^*w = \left(\frac{|\lambda_0+\mu_0|^2}{4} + \frac{|\lambda_0-\mu_0|^2}{4}\right)1$$

$$= \frac{2(|\lambda_0|^2+|\mu_0|^2)}{4}1 = 1,$$

which proves that $T(\lambda_0,\mu_0)\in\partial_e(\mathcal{B}(H)_1)$*, and hence* T *preserves extreme points.*

The mapping T *satisfies a stronger property. Elements* a *and* b *in the* C^**-algebra* A *satisfy* $B(a,b)=0$ *if, and only if,* $ab^*=1$*. We observe that* $A_q^{-1}=A^{-1}$*, and hence an element* $(\lambda,\mu)\in A_q^{-1}$ *if, and only if,* $\lambda\mu\neq 0$*. Let us pick* $a=(\lambda_0,\mu_0)\in A_q^{-1}$ *(with* $\lambda_0\mu_0\neq 0$*). Clearly,* $a^\wedge = \left(\overline{\lambda_0^{-1}},\overline{\mu_0^{-1}}\right)$*. It is easy to check that*

$$T(a^\wedge)^*T(a) = \left(\frac{\overline{\lambda_0}^{-1}+\overline{\mu_0}^{-1}}{2}v + \frac{\overline{\lambda_0}^{-1}-\overline{\mu_0}^{-1}}{2}w\right)^* \left(\frac{\lambda_0+\mu_0}{2}v + \frac{\lambda_0-\mu_0}{2}w\right)$$

$$= \left(\frac{\lambda_0^{-1}+\mu_0^{-1}}{2}v^* + \frac{\lambda_0^{-1}-\mu_0^{-1}}{2}w^*\right)\left(\frac{\lambda_0+\mu_0}{2}v + \frac{\lambda_0-\mu_0}{2}w\right)$$

$$= \frac{1}{4}\frac{(\lambda_0+\mu_0)^2-(\lambda_0-\mu_0)^2}{\lambda_0\mu_0}1 = 1,$$

and hence $B(T(a),T(a^\wedge))=0$*, which shows that* T *preserves Bergmann-zero pairs.*

It is easy to check that $T(1,-1)=w$*, and hence* $v^*T(1,-1)=v^*w=0$*, and* $vv^*T(1,-1)=0$*. For* $S=v^*T$ *we have* $S(1,-1)^2=0$ *but*

$$S((1,-1)^2) = S(1,1) = v,$$

that is, S *is not a Jordan homomorphism. We can further check that* T *is not a triple homomorphism, for example,* $(1,0)$ *is a tripotent in* A *but* $\|T(1,0)\| = \frac{1}{\sqrt{2}}$*, and hence* $T(1,0)$ *is not a tripotent in* $\mathcal{B}(H)$*.*

Finally, for $a=(\lambda_0,\mu_0)\in A_q^{-1}$ *(with* $\lambda_0\mu_0\neq 0$*),*

$$T(a^\wedge) = \frac{\overline{\lambda_0}^{-1}}{2}(v+w) + \frac{\overline{\mu_0}^{-1}}{2}(v-w)$$

need not coincide with

$$T(a)^\wedge = \left(\frac{\lambda_0}{2}(v+w) + \frac{\mu_0}{2}(v-w)\right)^\wedge.$$

Indeed, $T(2,1)=\frac{3}{2}v+\frac{1}{2}w=\frac{\sqrt{10}}{2}r$*, where* $r=\frac{3}{\sqrt{10}}v+\frac{1}{\sqrt{10}}w$ *is the range tripotent of* $T(2,1)$*, and thus* $T(2,1)^\wedge = \frac{2}{\sqrt{10}}r = \frac{3}{5}v+\frac{1}{5}w$*. Clearly,*

$$T((2,1)^\wedge) = T(1/2,1) = \frac{3}{4}v - \frac{1}{4}w.$$

The counter-examples provided by Remark 5.3.5 point out that the conclusions found by Mascioni and Molnár for linear maps preserving extreme points on infinite von Neumann factor (cf. [101]) are not expectable for general C*-algebras. We shall show that a more tractable description is possible for linear maps strongly preserving Brown-Pedersen quasi-invertibility. The proofs are based on the JB*-triple structure underlying every C*-algebra.

The following variant of Proposition 5.1.1 follows with similar arguments, its proof is outlined here.

Proposition 5.3.6 *Let E and F be JB*-triples, and let $T : E \to F$ be a nonzero linear map strongly preserving Brown-Pedersen quasi-invertible elements, that is, $T(x^\wedge) = T(x)^\wedge$ for every $x \in E_q^{-1}$. Then*

$$T(x^{[3]}) = T(x)^{[3]},$$

for every $x \in E_q^{-1}$.

Proof Let x be an element in E_q^{-1}, and let $e = r(x) \in \partial_e(E_1)$ denote its range tripotent. For each $0 < \lambda < \|x^\wedge\|^{-2}$ the element $\lambda x^\wedge - x$ is Brown-Pedersen quasi-invertible in E. Indeed, if we regard $\lambda x^\wedge - x$ as an element in $E_x \equiv C(\mathrm{Sp}(x))$, the JB*-subtriple of E generated by x, then $x - \lambda x^\wedge$ is invertible and positive in E_x, and its range tripotent is $r(x - \lambda x^\wedge) = e \in \partial_e(E_1)$. By Hua's identity (cf. (2.1)), we have

$$x - \lambda^{-1} x^{[3]} = \left(x^\wedge - (x - \lambda x^\wedge)^\wedge\right)^\wedge .$$

Given $0 < \lambda < \min\{||x^\wedge||^{-2}, ||T(x)^\wedge||^{-2}\}$, since T strongly preserves Brown-Pedersen quasi-invertible elements, and $x, \lambda x^\wedge - x$, $T(x)$ and $T(\lambda x^\wedge - x)$ are Brown-Pedersen quasi-invertible, we deduce that

$$T(x) - \lambda^{-1} T(x)^{[3]} = (T(x)^\wedge - (T(x) - \lambda T(x)^\wedge)^\wedge)^\wedge$$

$$= (T(x^\wedge) - (T(x) - \lambda T(x^\wedge))^\wedge)^\wedge = T(\left(x^\wedge - (x - \lambda x^\wedge)^\wedge\right)^\wedge) = T(x) - \lambda^{-1} T(x^{[3]}),$$

for every $0 < \lambda$ as above, which proves the desired statement. ■

The full meaning of Theorem 5.1.3 (and the role played by [74, Lemma 2.2] in its proof) is more explicit in the following result, whose proof follows the lines we gave in the just mentioned theorem but replacing Proposition 5.1.1 with Proposition 5.3.6.

Theorem 5.3.7 *Let E and F be JB*-triples with $\partial_e(E_1) \neq \emptyset$. Suppose $T : E \to F$ is a linear map strongly preserving Brown-Pedersen quasi-invertible elements. Then T is a triple homomorphism.* ■

We can state now our conclusions on linear maps strongly preserving Brown-Pedersen quasi-invertibility.

Theorem 5.3.8 *Let A and B be unital C*-algebras. Let $T : A \to B$ be a linear map strongly preserving Brown-Pedersen quasi-invertible elements. Then there exists a Jordan *-homomorphism $S : A \to B$ satisfying $T(x) = T(1)S(x)$, for every $x \in A$. We further know that*

$$T(A) \subseteq T(1)T(1)^*BT(1)^*T(1), \ \ S(A) \subseteq T(1)^*T(1)BT(1)^*T(1),$$

*and $S : A \to T(1)^*T(1)BT(1)^*T(1)$ is a unital Jordan *-homomorphism.*

Proof Since T preserves extreme points, $v = T(1) \in \partial_e(B_1)$ is a partial isometry with

$$(1 - vv^*)T(x)(1 - v^*v) = 0 \tag{5.8}$$

for every $x \in A$. It follows from (5.6) that $vT(a)^*v = vv^*T(a)v^*v$, for every $a = a^* \in A$.

Now, Theorem 5.3.7 assures that T is a triple homomorphism. Thus, we have

$$T(x) = T\{x, 1, 1\} = \{T(x), v, v\} = \frac{1}{2}(T(x)v^*v + vv^*T(x)), \tag{5.9}$$

and

$$T(x^*) = T\{1, x, 1\} = \{v, T(x), v\} = vT(x)^*v, \tag{5.10}$$

for every $x \in A$. Identities (5.8) and (5.9) give:

$$T(x) = vv^*T(x)v^*v = vv^*T(x) = T(x)v^*v, \tag{5.11}$$

for every $x \in A$. Multiplying on the left by v^* we get

$$v^*T(x) = v^*T(x)v^*v = (\text{by } (5.10)) = T(x^*)^*v,$$

for every $x \in A$, which proves that $S = v^*T : A \to B$ is a selfadjoint operator. Furthermore, since T is a triple homomorphism, we have

$$S(x^2) = v^*T\{x, 1, x\} = v^*\{T(x), v, T(x)\} = v^*T(x)v^*T(x) = S(x)^2,$$

for all $x \in A$, which guarantees that S is a Jordan *-homomorphism. The identity in (5.11) gives $T(x) = vv^*T(x) = vS(x)$, for every $x \in A$. The rest is clear. ■

Remark 5.3.9 *Under the hypothesis of Theorem 5.3.8 we can similarly prove that the mapping $S_1 : A \to B$, $S_1(x) = T(x)T(1)^*$ is a Jordan *-homomorphism and $T(x) = S_1(x)v$, for every x in A.*

If v is an extreme point of the closed unit ball of a prime unital C*-algebra B, then $1 = vv^*$ or $v^*v = 1$. Therefore, the next result is a straight consequence of the previous Theorem 5.3.8.

Corollary 5.3.10 *Let A and B be unital C*-algebras with B prime. Let $T : A \to B$ be a linear map strongly preserving Brown-Pedersen quasi-invertible elements. Then one of the following statements holds:*

(1) $T(1)^*T(1) = 1$, $T(1)T(1)^*T(a) = T(a)$, *for every* $a \in A$, *and there exists a unital Jordan *-homomorphism* $S : A \to B$ *satisfying* $T(a) = T(1)S(a)$, *for every* $a \in A$;

(2) $T(1)T(1)^* = 1$, $T(a)T(1)^*T(1) = T(a)$, *for every* $a \in A$, *and there exists a unital Jordan *-homomorphism* $S : A \to B$ *satisfying* $T(a) = S(a)T(1)$, *for every* $a \in A$.

Chapter 6

Linear maps preserving partial orders in Banach algebras and C*-algebras

This final chapter is devoted to the study of linear preservers of some kinds of order relations. In particular, we adress the issue of characterizing linear maps preserving the sharp, star, minus and diamond partial orders in semisimple Banach algebras and C*-algebras. In addition, we find some new and interesting properties of some of these orders. The contents in this chapter appear in [32, 33, 34].

6.1 Linear preservers of the sharp partial order

Let us recall the definition of sharp partial order (see Section 2.4). For a ring R, we denote by $R^\sharp$ the set of its group invertible elements. Recall that the group inverse $a^\sharp$ of an element $a \in R^\sharp$ is the unique element satisfying

$$aa^\sharp a = a, \quad a^\sharp aa^\sharp = a^\sharp \quad \text{and} \quad aa^\sharp = a^\sharp a.$$

For $a \in R^\sharp$ and $b \in A$, we say that $a \leq_\sharp b$ if $a^\sharp a = a^\sharp b = ba^\sharp$. The following lemma collects some useful algebraic properties of the sharp relation.

Lemma 6.1.1 *Let R be a unital (associative) ring. The following assertions hold:*

(1) $p \in R^\sharp$ is an idempotent if, and only if, $p \leq_\sharp 1$.

(2) The maximal elements with respect to the partial order $\leq_\sharp$ in $R^\sharp$ are precisely the invertible elements.

(3) Let $a \in R^\sharp$, $b \in R$ and u a group invertible element commuting with a and b. If $a \leq_\sharp b$ then $ua \leq_\sharp ub$.

Proof The first assertion is clear.

Let $a \in R^\sharp$. It is straightforward to prove that $a \leq_\sharp (a - 1 + aa^\sharp)$. If a is a maximal element with respect to $\leq_\sharp$, then $a = (a - 1 + aa^\sharp)$, which means that $1 = aa^\sharp = a^\sharp a$. Reciprocally, if $a \in R$ is invertible and $a \leq_\sharp b$, we have $1 = a^{-1}a = a^{-1}b$, which clearly implies that $a = b$.

Finally, pick $a, b \in R$ and $u \in R^\sharp$ with $a \leq_\sharp b$, $ua = au$ and $ub = bu$. Since u also commutes with $a^\sharp$ and $b^\sharp$, and $(ua)^\sharp = u^\sharp a^\sharp$, it follows that

$$(ua)^\sharp ua = u^\sharp ua^\sharp a = u^\sharp ua^\sharp b = (ua)^\sharp ub.$$

Similarly we show that $(ua)(ua)^\sharp = (ub)(ua)^\sharp$. ■

Recall that a linear (additive) map $T : A \to B$ between Banach algebras preserves the sharp order if $T(a) \leq_\sharp T(b)$ whenever $a \leq_\sharp b$. We begin this section by noticing that every Jordan homomorphism preserves the sharp relation.

Lemma 6.1.2 *Let A and B be Banach algebras and $T : A \to B$ be a Jordan homomorphism. Then T preserves the relation $\leq_\sharp$.*

Proof For $a \in A^\sharp$ and $b \in A$ such that $a \leq_\sharp b$, let us prove that $T(a) \leq_\sharp T(b)$. Recall that, by Theorem 1.1.13, if T is a Jordan homomorphism, then T strongly preserves group invertibility. As part of the proof of this result, Mbekhta showed that a Jordan homomorphism preserves the commutativity of G_1-inverses, that is, if $xyx = x$ and $xy = yx$, then $T(x)T(y) = T(y)T(x)$. Having this facts in mind, since $T(aa^\sharp + a^\sharp a) = T(a)T(a^\sharp) + T(a^\sharp)T(a)$, we obtain $T(aa^\sharp) = T(a)T(a)^\sharp$. Moreover, as $a^\sharp = a^\sharp ba^\sharp$ and $a^\sharp b = ba^\sharp$, the same arguments show that $T(a^\sharp b) = T(a)^\sharp T(b)$. Consequently

$$T(a)^\sharp T(a) = T(a^\sharp a) = T(a^\sharp b) = T(a)^\sharp T(b).$$

The identity $T(a)T(a)^\sharp = T(b)T(a)^\sharp$ can be obtained in the same way and, thus, $T(a) \leq_\sharp T(b)$. ■

From Lemmas 6.1.1 and 6.1.2 it is clear that every Jordan homomorphism multiplied by an invertible element commuting with its range, also preserves the sharp relation. We address the question whether the reciprocal result holds. First we will study linear preservers of the sharp relation in the environment of semisimple Banach algebras with non zero socle.

Remark 6.1.3 *Notice that, for every $a \in A^\sharp$ and $b \in A$, $ab = ba = 0$ is equivalent to $a \leq_\sharp (a + b)$. Hence, given Banach algebras A, B and a linear map $T : A \to B$ preserving the sharp relation, for every $a \in A^\sharp$ and $b \in A$, $ab = ba = 0$ implies that $T(a)T(b) = T(b)T(a) = 0$.*

The initial step for the description of zero product preserving linear maps in [39] consists in describing the behaviour of the mapping on Jordan products of minimal idempotents $p, q \in A$. In this sense, our aim is to achieve the identities from [39, Lemma 2.5, Lemma 2.6], through rank-one group invertible elements, that is, rank-one elements with nonzero trace (see Section 2.1).

Lemma 6.1.4 *Let A and B be Banach algebras. Assume that A is unital. Let $T : A \to B$ be a linear map preserving the sharp relation. For every idempotent element $p \in A$, the following holds:*

(1) $T(p)^2 = T(p)T(1) = T(1)T(p)$,

(2) $T(p) = T(1)T(1)^\sharp T(p) = T(p)T(1)^\sharp T(1)$.

Proof Since $1 \leq_\sharp 1$, $T(1) \leq_\sharp T(1)$, which in particular implies that $T(1)$ has group inverse.

The first identity follows from Remark 6.1.3. Indeed, as

$$p(1-p) = (1-p)p = 0,$$

we have

$$T(p)\big(T(1) - T(p)\big) = \big(T(1) - T(p)\big)T(p) = 0,$$

which proves *(1)*.

Now, by using that

$$T(p)^\sharp T(p) = (T(p)^\sharp T(p))^\sharp = (T(p)^\sharp T(1))^\sharp = T(1)^\sharp T(p)$$

we get

$$T(p) = T(p)^2 T(1) = T(p)T(1)^\sharp T(1) = T(1)T(1)^\sharp T(p). \blacksquare$$

Proposition 6.1.5 *Let A be a unital semisimple Banach algebra with nonzero socle, B be a Banach algebra and $T : A \to B$ be a linear map preserving the sharp relation. Then*

$$T(p \circ q)T(1) = T(p) \circ T(q),$$

for every minimal idempotents $p, q \in A$.

Proof In order to simplify the notation, we write $h = T(1)$. Take minimal idempotents $p, q \in A$. Then pq is a rank-one element. We must consider different cases:
Case 1: $(pq)^2 \neq 0$, that is, $\tau(pq) = \tau(qp) \neq 0$.

If we assume that $p = pq = qp$, then $p(1-q) = (1-q)p = 0$ and, as we have noticed in Remark 6.1.3, $T(p)\,(h - T(q)) = (h - T(q))\,T(p) = 0$. This leads to $T(p)h = T(p)T(q)$ and $hT(p) = T(q)T(p)$, which in particular gives $T(p \circ q)h = T(p) \circ T(q)$.

Suppose now that either $p \neq pq$ or $p \neq qp$. If $\tau(p(1-q)) \neq 0$ (for $\tau(q(1-p)) \neq 0$ the proof is similar), then pq, $p(1-q)$ and $(1-q)p$ are rank-one group invertible elements. Since $pq\,(1-q)(1-p) = (1-q)(1-p)\,pq = 0$, $p(1-q)\,q(1-p) = q(1-p)\,p(1-q) = 0$ and $(1-q)p\,(1-p)q = (1-p)q\,(1-q)p = 0$, we obtain, respectively:

$$T(pq)h = T(pq)T(p) + T(pq)T(q) - T(pq)T(qp), \tag{6.1}$$

$$T(p)T(q) = T(p)T(qp) + T(pq)T(q) - T(pq)T(qp), \tag{6.2}$$

$$T(q)T(p) = T(q)T(qp) + T(pq)T(p) - T(pq)T(qp). \tag{6.3}$$

From (6.1) and (6.2) it follows that

$$T(pq)h + T(p)T(qp) = T(p)T(q) + T(pq)T(p). \tag{6.4}$$

Analogously (6.1) and (6.3) gives

$$T(pq)h + T(q)T(qp) = T(q)T(p) + T(pq)T(q). \tag{6.5}$$

Note that in (6.1), (6.2) and (6.3), the roles of p and q can be exchanged. Thus, we can process in this way in (6.5) to obtain

$$T(qp)h + T(p)T(pq) = T(p)T(q) + T(qp)T(p). \tag{6.6}$$

From (6.4) and (6.6) we get

$$T(pq + qp)h + T(p)T(pq + qp) = 2T(p)T(q) + T(pq + qp)T(p). \tag{6.7}$$

Using the other side identities of the zero product and proceeding similarly, it follows that

$$hT(pq + qp) + T(pq + qp)T(p) = 2T(q)T(p) + T(p)T(pq + qp). \tag{6.8}$$

From (6.7) and (6.8) we get $T(p \circ q)h = T(p) \circ T(q)$.

Now, suppose that $\tau(p(1-q)) = \tau(q(1-p)) = 0$ (being $\tau(pq) \neq 0$, and $p \neq pq$ or $p \neq qp$). From $pq\,(1-q)(1-p) = (1-q)(1-p)\,pq = 0$ and $qp\,(1-p)(1-q) = (1-p)(1-q)\,qp = 0$ we get

$$T(pq)h = T(pq)T(p) + T(pq)T(q) - T(pq)T(qp),$$

$$T(qp)h = T(qp)T(q) + T(qp)T(p) - T(qp)T(pq).$$

As $\tau(p(1-q)) = \tau(q(1-p)) = 0$, we have $p(1-q)p = q(1-p)q = 0$, that is, $pqp = p$ and $qpq = p$. When $p = pq$ (the case $p = qp$ is similar), it follows that $qp = q$, and $p \circ q = \frac{1}{2}(p+q)$ is an idempotent. Having into account Lemma 6.1.4,

$$\begin{aligned} T(p+q)h &= 2T(\frac{1}{2}(p+q))h = 2T(\frac{1}{2}(p+q))^2 \\ &= \frac{1}{2}(T(p)^2 + T(q)^2 + T(p)T(q) + T(q)T(p)). \end{aligned}$$

This yields

$$2T(p)h + 2T(q)h = T(p)^2 + T(q)^2 + T(p)T(q) + T(q)T(p)$$

and, consequently, $T(p \circ q)h = \frac{1}{2}T(p+q)h = T(p) \circ T(q)$. Finally, suppose that $pqp = p$, $qpq = q$, $pq \neq q$ and $qp \neq p$. Then

$$(p + pq)^2 = 2(p + pq),$$

$$(p+qp)^2 = 2(p+qp),$$
$$(q+pq)^2 = 2(q+pq),$$
$$(q+qp)^2 = 2(q+qp).$$

Therefore

$$(p+pq)^\sharp = \frac{1}{4}(p+pq),$$
$$(p+qp)^\sharp = \frac{1}{4}(p+qp),$$
$$(q+pq)^\sharp = \frac{1}{4}(q+pq),$$
$$(q+qp)^\sharp = \frac{1}{4}(q+qp).$$

Arguing as above, the following identities are easily obtained:

$$T(p+pq)h = T(p)T(pq) + T(pq)T(p), \tag{6.9}$$
$$T(p+qp)h = T(p)T(qp) + T(qp)T(p), \tag{6.10}$$
$$T(q+pq)h = T(q)T(pq) + T(pq)T(q), \tag{6.11}$$
$$T(q+qp)h = T(q)T(qp) + T(qp)T(q). \tag{6.12}$$

Notice that, for an idempotent p and $x \in A$, such that $pxp = 0$, then

$$p\,(x - px - xp) = (x - px - xp)\,p = 0,$$

and thus

$$T(p)\,(T(x) - T(px) - T(xp)) = (T(x) - T(px) - T(xp))\,T(p) = 0.$$

Applying this fact to $x = 1 - q$ we have:

$$\begin{aligned} T(p)h - T(p)T(q) &= T(p)\,(T(p(1-q)) + T((1-q)p)) \\ &= 2T(p)^2 - T(p)T(pq) - T(p)T(qp). \end{aligned}$$

That is,

$$T(p)h = T(p)T(pq) + T(p)T(qp) - T(p)T(q). \tag{6.13}$$

Similarly,

$$hT(p) = T(pq)T(p) + T(qp)T(p) - T(q)T(p). \tag{6.14}$$

From (6.13) and (6.14), we obtain

$$2T(p)h = T(p)T(pq) + T(pq)T(p) + T(p)T(qp) + \tag{6.15}$$
$$+ \; T(qp)T(p) - (T(p)T(q) + T(q)T(p)). \tag{6.16}$$

The identities (6.15), (6.9) and (6.10), produce $T(p \circ q)h = T(p) \circ T(q)$.

Case 2: $\tau(pq) = \tau(qp) = 0$. In this case $pqp = qpq = 0$ and, since every rank-one element is single it must be $pq = 0$ or $qp = 0$.

Suppose that $pq = 0$ and $qp \neq 0$. As

$$p\big(q(1-p)\big) = \big(q(1-p)\big)p \quad \text{and} \quad q\big((1-q)p\big) = \big((1-q)p\big)q = 0,$$

we obtain, respectively

$$T(p)T(q) = T(p)T(qp), \qquad T(q)T(p) = T(qp)T(p),$$

$$T(q)T(p) = T(q)T(qp), \qquad T(p)T(q) = T(qp)T(q).$$

As $pq = 0$, $p+q-qp$ is an idempotent element. Hence by Lemma 6.1.4, $T(p+q-qp)^2 = T(p+q-qp)h$, that is

$$T(p)^2 + T(q)^2 + T(p)T(q) + T(q)T(p) + T(qp)^2 - T(p)T(qp) - T(q)T(qp)$$

$$-T(qp)T(p) - T(qp)T(q) = T(p)h + T(q)h - T(qp)h.$$

Having in mind the previous identities we deduce that

$$T(qp)h = T(p)T(q) + T(q)T(p) - T(qp)^2.$$

It only remains to prove that $T(qp)^2 = 0$. To this end, we will prove that, for every rank-one element $u \in A$ with $\tau(u) = 0$, we have $T(u)^2 = 0$. As we know, given $x \in A$ and $\lambda \in \mathbb{C}$ such that $uxu = u$ and $x - \lambda 1$ is invertible, $e_1 = ux$ and $e_2 = u(x-\lambda)$ are minimal idempotents such that $\lambda u = e_1 - e_2$, $e_1e_2 = e_2$ and $e_2e_1 = e_1$. Therefore

$$\begin{aligned} \lambda^2 T(u)^2 &= T(e_1 - e_2)^2 = T(e_1)^2 + T(e_2)^2 - T(e_1)T(e_2) - T(e_2)T(e_1) \\ &= T(e_1 + e_2)h - (T(e_1)T(e_2) + T(e_2)T(e_1)). \end{aligned}$$

As $e_1e_2 = e_2$ and $e_2e_1 = e_1$, then $(e_1 + e_2)^\sharp = \frac{1}{4}(e_1 + e_2)$ and

$$T(e_1 + e_2)h = T(e_1e_2 + e_2e_1)h = T(e_1)T(e_2) + T(e_2)T(e_1).$$

Hence, $T(u)^2 = 0$ as wanted. ■

Since every element of the socle is a linear combination of minimal idempotents, once we have obtained Lemma 6.1.4 and Proposition 6.1.5, it can be checked that the rest of calculations shown in Lemma 2.5, Lemma 2.6 and Theorem 2.7 in [39] still work for our setting.

Proposition 6.1.6 *Let A and B be Banach algebras. Assume that A is unital, with nonzero socle. Let $T : A \to B$ be a linear map preserving the sharp relation. Let $h = T(1)$. For $a \in A$ and $x, y \in soc(A)$, the following identities hold.*

(1) $T(x)h = hT(x)$.

(2) $T(a \circ x)h = T(a) \circ T(x)$.

(3) $T(x)hT(a) = T(x)T(a)h$, *and* $T(a)hT(x) = hT(a)T(x)$.

(4) $\{T(x), T(a), T(y)\} = T(\{x, a, y\})h^2$.

(5) $\{T(x), T(a)^2, T(y)\} = T(\{x, a^2, y\})h^3$.

Theorem 6.1.7 *Let A and B be unital Banach algebras, A having essential socle. Let $T : A \to B$ be a bijective linear map. Then the following conditions are equivalent:*

(1) T preserves the sharp relation,

(2) T is a Jordan isomorphism multiplied by a central invertible element.

Proof Assume that T preserves the sharp relation. Taking into account the preceding proposition, we argue as in [39, Theorem 2.7].

Let us first make an easy observation: we know from Lemma 6.1.4 that $T(p) = hh^\sharp T(p) = T(p)h^\sharp h$, for every minimal idempotent $p \in A$, where $h = T(1)$. By linearizing, the same holds for every elements in the socle, that is

$$T(x) = T(x)h^\sharp h \tag{6.17}$$

for all $x \in \mathrm{soc}(A)$. Let $a \in A$, and $x \in \mathrm{soc}(A)$. By the surjectivity of T, there exists $b \in A$ such that $T(b) = T(a)h - hT(a)$. From Proposition 6.1.6 (iii), it follows that $T(xbx)h^2 = 0$, or equivalently (multiplying by $h^\sharp$), $T(xbx)h = 0$. From Equation (6.17) we deduce that $T(xbx) = 0$. Since T is injective, it follows that $xbx = 0$, for all $x \in \mathrm{soc}(A)$, and thus, $b = 0$. This proves that

$$T(a)h = hT(a), \qquad \text{for every } a \in A.$$

Similarly, since $T(x)h^\sharp hT(a) = T(x)T(a)$, and

$$T(x)hT(a^2)T(x) = hT(x)T(a^2)T(x) = h^3T(xa^2x) = T(x)T(a)^2T(x),$$

for every $x \in \mathrm{soc}(A)$, we can prove that

$$T(a) = T(a)h^\sharp h \quad \text{ and } \quad T(a^2)h = T(a)^2,$$

for every $a \in A$ (compare if necessary with the proof of [39, Theorem 2.7]). By the surjectivity of T, it is clear that h is invertible, and that $h^{-1}T$ is a Jordan isomorphism.

The reciprocal statement follows from Lemmas 6.1.1 and 6.1.2. ■

Now we consider linear preservers of the sharp order in real rank zero C*-algebras. Recall that every bounded linear map $T : A \to B$ from a real rank zero C*-algebra A into a Banach algebra B that preserves idempotents is a Jordan homomorphism (see Lemma 3.3.2).

Theorem 6.1.8 *Let A and B be unital Banach algebras. Assume that A is a real rank zero C*-algebra. Let $T : A \to B$ be a bounded linear map. The following conditions are equivalent:*

(1) T preserves the relation $\leq_\sharp$,

(2) $T = T(1)S$ where S is a Jordan homomorphism, $T(1)$ is group invertible and it commutes with $S(A)$.

Proof Let $h = T(1)$. Suppose that T preserves the sharp relation. Since $1 \leq_\sharp 1$, we have $h \leq_\sharp h$ and, thus, h is group invertible. From Lemma 6.1.4 we know that

$$T(p)^2 = T(p)h = hT(p)$$

and

$$T(p) = hh^\sharp T(p) = T(p)h^\sharp h,$$

for every idempotent $p \in A$. As every selfadjoint element in A can be approximated by real linear combinations of (orthogonal) idempotents, and T is bounded, we get $hT(x) = T(x)h$, and $T(x) = hh^\sharp T(x)$ for every selfadjoint element $x \in A$. Moreover, since for every $x \in A$ there exists $x_1, x_2 \in A$ selfadjoint elements such that $x = x_1 + ix_2$, it is clear that $hT(x) = T(x)h$ and $T(x) = hh^\sharp T(x)$ for every $x \in A$.

Now, from $hT(p) = T(p)^2$, multiplying by $\left(h^\sharp\right)^2$ and taking into account the commutativity of h (which implies the commutativity of $h^\sharp$), we deduce that $h^\sharp T(p) = \left(h^\sharp T(p)\right)^2$. Let $S : A \to B$ be the map defined as $S(x) := h^\sharp T(x)$, for all $x \in A$. The previous identity gives $S(p) = S(p)^2$, for every idempotent $p \in A$. As we have mentioned, this guarantees that S is a Jordan homomorphism. Finally, note that $T = hh^\sharp T = hS$.

The converse can be checked straightforwardly combining Lemmas 6.1.2 and 6.1.1. ■

One may wonder if the previous result is true for general C*-algebras.

Example 6.1.9 *Let $A = C([0,1])$, $B = M_2(\mathbb{C})$ and $T : A \to B$ the map given by*

$$T(f) = \begin{pmatrix} f(0) & f(1) \\ 0 & 0 \end{pmatrix}.$$

Notice that $A^\sharp = A^{-1} \cup \{0\}$. Hence $0 \leq_\sharp g$ for every $g \in A$ and for $f \in A^{-1}$, $f \leq_\sharp g$ if, and only if, $f = g$. Trivially, T fulfills $T(0) \leq_\sharp T(f)$ for every $f \in A$. To see that $T(f) \leq_\sharp T(f)$ for every $f \in A^\sharp$ it is enough to show that T sends invertible functions to group invertible matrices. This last assert follows from the fact that every matrix

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$

with $a \neq 0$ has group inverse

$$\begin{pmatrix} a^{-1} & a^{-2}b \\ 0 & 0 \end{pmatrix}$$

for every $b \in \mathbb{C}$. Finally, $T = T(1)^\sharp T$ and it can be easily seen that T is not a Jordan homomorphism. However, it can be checked that $TT(1)^\sharp$ is a Jordan homomorphism.

Remark 6.1.10 *Let A and B be unital semisimple Banach algebras and $T : A \to B$ be a surjective linear map. Notice that, in view of Lemma 6.1.1* (3)*, if T preserves the sharp relation in both directions, that is,*

$$a \leq_\sharp b \quad \text{if, and only if,} \quad T(a) \leq_\sharp T(b),$$

then T preserves invertibility in both directions. Obviously, T is injective because $T(a) = 0$ implies that $a \leq_\sharp 0$ and hence $a = 0$. Therefore, the mapping $S = T(1)^{-1}T$ is a unital bijective linear map preserving invertibility in both directions. Thus, if A has essential socle (see Theorem 1.1.4) or A has real rank zero (see Theorem 1.1.3), then S is a Jordan isomorphism.

Let R be a (unital associative) ring. The sharp relation $a \leq_\sharp b$ makes sense only when a is group invertible. Note that, from $a^\sharp a = a^\sharp b = ba^\sharp$, we get $a = aa^\sharp b = ba^\sharp a$, that is, $a = bp = pb$ where $p = aa^\sharp = a^\sharp a \in R^\bullet$ (recall that $R^\bullet$ stands for the set of idempotent elements in R). So it makes sense to extend the sharp relation to the whole ring, in the following way: for $a, b \in A$, we say that

$$(R1) \qquad a \leq_s b \quad \text{if, and only if, there is } p \in A^\bullet \text{ such that } a = pb = bp.$$

This last definition provides a natural extension of $\leq_\sharp$ in the following sense: if $a \leq_s b$ and a is group invertible, then $a \leq_\sharp b$. Indeed, if $a = bp = pb$ for some $p \in A^\bullet$, then $a = ap = pa$. As a is a group invertible, we get $a^\sharp = pa^\sharp = a^\sharp p$. Thus, $a^\sharp a = a^\sharp pb = a^\sharp b$ and, similarly, $aa^\sharp = ba^\sharp$. Observe also that $a \in A$ is group invertible if, and only if, $a \leq_s u$ for some invertible element u. Indeed, if a is group invertible, then $a - 1 + aa^\sharp$ is invertible and $a \leq_s (a - 1 + aa^\sharp)$. Reciprocally, if if $a \leq_s u$ for some invertible element u, then $a = pu = up$, for certain $p \in A^\bullet$. Therefore, $a^\sharp = u^{-1}p$. Notice also that for every $a, b \in A$, and $u \in A$ commuting with a and b, if $a \leq_s b$ then $ua \leq_s ub$. Indeed, let p be an idempotent in A such that $a = bp = pb$. It follows that $ua = u(bp) = (ub)p$ and $ua = au = (pb)u = p(bu) = p(ub)$.

In the next lemma, we prove that every Jordan homomorphism preserves the relation $(R1)$; whence so does every Jordan homomorphism multiplied by an element commuting with its range.

Lemma 6.1.11 *Let A and B be Banach algebras and $T : A \to B$ be a Jordan homomorphism. If $a \leq_s b$, then $T(a) \leq_s T(b)$.*

Proof For every $p \in A^\bullet$, $T(p) = T(p)^2$ holds. Let $a, b \in A$ and suppose that $a \leq_s b$, that is, there exists $p \in A^\bullet$ such that $a = bp = pb$. As $a = ap = pa$, we get

$$2T(a) = T(pa + ap) = T(p)T(a) + T(a)T(p).$$

Multiplying this last equation by $T(p)$ on the left and on the right, respectively, and combining their results, it yields $T(a)T(p) = T(p)T(a)$, which gives $T(a) = T(p)T(a)$. From $a = bp = pb$ we can also write

$$2T(a) = T(bp + pb) = T(b)T(p) + T(p)T(b).$$

We multiply this expression by $T(p)$ on the right, to produce

$$2T(a) = 2T(a)T(p) = T(b)T(p) + T(p)T(b)T(p).$$

Since T preserves triple products, it follows

$$2T(a) = T(b)T(p) + T(pbp) = T(b)T(p) + T(a),$$

which finally gives $T(a) = T(b)T(p)$. The identity $T(a) = T(p)T(b)$ can be obtained similarly. This proves that $T(a) \leq_s T(b)$ as desired. ■

It is a natural question to ask if multiples of Jordan homomorphisms arise from linear maps preserving the relation $(R1)$. We focus on the two settings that we are already dealing with, that is, unital semisimple Banach algebras with large socle and real rank zero C*-algebras.

Lemma 6.1.12 *Let A and B be Banach algebras and $T : A \to B$ be a linear map preserving the relation $(R1)$. Then, for every $a \in A^\sharp$, $b \in B$, the condition $ab = ba = 0$ implies $T(a)T(b) = T(b)T(a) = 0$.*

Proof Take $a \in A^\sharp$, $b \in B$. Then it is clear that

$$ab = ba = 0 \quad \text{if, and only if,} \quad a^\sharp b = ba^\sharp = 0 \quad \text{if, and only if,} \quad a \leq_s a + b.$$

Therefore, if $ab = ba = 0$, then $T(a) \leq_s T(a) + T(b)$, that is,

$$T(a) = pT(a) = T(a)p = p(T(a) + T(b)) = (T(a) + T(b))p$$

for some $p \in B^\bullet$. In particular, $pT(b) = T(b)p = 0$, which gives

$$T(a)T(b) = T(a)pT(b) = 0 \quad \text{and}$$

$$T(b)T(a) = T(b)pT(a) = 0,$$

as desired. ■

Remark 6.1.13 *Let A and B be Banach algebras and $T : A \to B$ be a linear map preserving the relation $(R1)$. We assume that A is unital. Let $p \in A^\bullet$. The previous lemma implies, in particular, that*

$$T(p)^2 = T(p)T(1) = T(1)T(p).$$

Moreover, since $T(p) \leq_s T(1)$, there exists $q \in B^\bullet$ such that

$$T(p) = T(1)q = qT(1).$$

If we assume moreover that $T(1)$ is group invertible, then it is clear that

$$T(p) = T(1)T(1)^\sharp T(p) = T(p)T(1)^\sharp T(1).$$

From Lemma 6.1.12 it follows that the conclusions in Propositions 6.1.5 and 6.1.6 still hold. Having in mind these facts and the previous remark, the proof of the next theorem runs in the same way as that of Theorem 6.1.7.

Theorem 6.1.14 *Let A and B be unital Banach algebras, A having essential socle. Let $T : A \to B$ be a bijective linear map. Assume that $T(1)$ has group inverse. Then, the following conditions are equivalent:*

(1) T preserves the relation $(R1)$,

(2) T is a Jordan isomorphism multiplied by a central invertible element.

To conclude this section, we consider a continuous linear mapping defined on a unital real rank zero C*-algebra that preserves the relation $(R1)$.

Theorem 6.1.15 *Let A be a unital real rank zero C*-algebra and B be a Banach algebra. Let $T : A \to B$ be a continuous linear map. Assume that $T(1)$ is group invertible. Then, the following conditions are equivalent:*

(1) T preserves the relation $(R1)$,

(2) $T = T(1)S$ where S is a Jordan homomorphism, and $T(1)$ commutes with $S(A)$.

Proof From Remark 6.1.13 we know that

$$T(p)^2 = T(p)h = hT(p),$$

and

$$T(p) = hh^\sharp T(p) = T(p)h^\sharp h,$$

for every idempotent p in A. For every real linear combination of mutually orthogonal idempotents $x = \sum_{k=1}^n \lambda_k p_k$, we have

$$hT(x^2) = hT\left(\sum_{k=1}^n \lambda_k^2 p_k\right) = \sum_{k=1}^n \lambda_k^2 hT(p_k) = \sum_{k=1}^n \lambda_k^2 T(p_k)^2 = T(x)^2.$$

Since A has real rank zero and T is continuous, it is clear that

$$T(a)h = hT(a), \qquad T(a) = hh^\sharp T(a),$$

and

$$hT(a^2) = T(a)^2,$$

for every selfadjoint element a in A. Since for every element $a \in A$, there exists selfadjoint elements $x, y \in A$ such that $a = x + iy$ and

$$a^2 = (x+iy)^2 = x^2 - y^2 + i\big((x+y)^2 - x^2 - y^2\big),$$

we have

$$T(a)h = hT(a), \qquad T(a) = hh^{\sharp}T(a),$$

and

$$hT(a^2) = hT((x+iy)^2) = hT\big(x^2 - y^2 + i\big((x+y)^2 - x^2 - y^2\big)\big) = T(a)^2.$$

From these identities, it is clear that $S(x) = h^{\sharp}T(x)$ is a Jordan homomorphism, and $T(x) = hS(x)$ for every $x \in A$ (compare with Theorem 6.1.8). ■

6.2 Linear preservers of the star partial order

Recall that two elements a, b in a C*-algebra A are called orthogonal (denoted by $a \perp b$) if $ab^* = b^*a = 0$. A linear mapping $T : A \to B$ between C*-algebras is said to be **orthogonality preserving** if $T(a) \perp T(b)$ whenever $a \perp b$. The map T is **biorthogonality preserving** if $T(a) \perp T(b)$ if, and only if, $a \perp b$. A linear mapping $T : A \to B$ between C*-algebras **preserves the star order** if $a \leq_* b$ implies that $T(a) \leq_* T(b)$.

Notice that given a, b in a C*-algebra A, $a \leq_* b$ if, and only if, $a \perp (a+b)$, so the problem of preserving star partial order is in fact equivalent to that of preserving orthogonality.

In the setting of complex matrix algebras the star partial order can be stated as follows:

$$A \leq_* B \quad \text{if, and only if,} \quad A = PB = BQ,$$

for some selfadjoint idempotent matrices P, Q. This characterization is still true for the more general context of Rickart C*-algebras ([98]). Recall that if A is a Rickart C*-algebra, for every element $a \in A$, there exists a unique projection p such that

$$\operatorname{ann}_l(a) = \{x \in A \colon xa = 0\} = A(1-p).$$

We denote it by $p = \operatorname{lp}(a)$. Similarly, we denote by $q = \operatorname{rp}(a)$ the unique projection such that

$$\operatorname{ann}_r(a) = \{x \in A \colon ax = 0\} = (1-q)A.$$

If a and b are elements in a Rickart C*-algebra A, such that $a \leq_* b$, or equivalently $a \perp (a+b)$, then $(a+b)^* \in \operatorname{ann}_l(a)$ and $(a+b)^* \in \operatorname{ann}_r(a)$, which show that $(a+b)^*\operatorname{lp}(a) = 0$ and $\operatorname{rp}(a)(a+b)^* = 0$. Having in mind that $a = \operatorname{lp}(a)a = a\operatorname{rp}(a)$, we conclude that $a = \operatorname{lp}(a)b = b\operatorname{rp}(a)$. (Compare with [98, Theorem 1].)

Let A be a C*-algebra and $a, b \in A$. Motivated by the previous characterization, we will study the following relation:

$$(R2) \qquad a \leq b \quad \text{if, and only if,} \quad a = pb = bq \text{ for some projections } p, q \in A.$$

Even for non Rickart C*-algebras, the notion just presented is deeply related to the star partial order. As a matter of fact, if $a = pb = bq$ for some projections $p, q \in A$, then

$$a^*a = b^*ppb = b^*pb = a^*b \quad \text{and}$$

$$aa^* = bqqb^* = bqb^* = ba^*.$$

As a consequence, $a \leq_* b$. Reciprocally, if $a \in A$ is regular and $a^*a = a^*b$, then it can be checked that $a^\dagger a = a^\dagger b$. Hencefore $a = aa^\dagger a = aa^\dagger b$, where $p = aa^\dagger$ is a projection. Similarly, from $aa^* = ba^*$ we get $a = ba^\dagger a$, where $q = a^\dagger a$ is a projection. We have proved the following:

Lemma 6.2.1 *Let A be a C*-algebra. Then $a \leq b$ implies $a \leq_* b$. If a is regular, $a \leq_* b$ implies $a \leq b$.*

The previous lemma shows that, for a regular element a in a C*-algebra A, $a \perp b$ if, and only if, $a \leq (a + b)$.

Because every element in the socle C*-algebra A is regular, we can employ the techniques on orthogonality preserving maps on C*-algebras with large socle (see [25]) in order to determine the structure of linear maps preserving the relation $(R2)$, due to the crucial role played by the regular elements within our proofs.

Lemma 6.2.2 *Let A and B be C*-algebras. Assume that A is unital with nonzero socle. Let $T : A \to B$ be a linear map preserving the relation $(R2)$. Let $h = T(1)$. For every $a \in A$ and $x, y \in soc(A)$, the following identities hold:*

(1) $T(x)h^* = hT(x^*)^*$ *and* $h^*T(x) = T(x^*)^*h$,

(2) $T(ax + xa)h^* = T(a)T(x^*)^* + T(x)T(a^*)^*$ *and*
$h^*T(ax + xa) = T(x^*)^*T(a) + T(a^*)^*T(x)$,

(3) $T(x)h^*T(a) = T(x)T(a^*)^*h$ *and*
$T(a)h^*T(x) = hT(a^*)^*T(x)$,

(4) $\{T(x)\, T(a)\, T(y)\} = T(\{x\, a\, y\})h^*h$,

(5) $\{T(x)\, \{T(a)\, h\, T(a)\}\, T(y)\} = \{h\, \{h\, T(\{x\, a^2\, y\})\, h\}\, h\}$.

Proof Let a be a regular element in A and $a \perp b$. As we have pointed out, $a \leq (a+b)$. By hypothesis, $T(a) \leq T(a) + T(b)$, and hence $T(a) \perp T(b)$. That is, T sends mutually orthogonal elements into mutually orthogonal elements, when one of them is regular. A quickly inspection of the proof of [25, Lemma 2.1], allows us to see that one of the elements appearing in all the orthogonality relations is always regular. Hence, the identities obtained there hold when orthogonality is replaced by the relation $(R2)$. Thus *(1)*, *(2)* and *(3)* are clear. We deduce *(4)* and *(5)* from them arguing as in [25, Proposition 2.2]. ■

Proposition 6.2.3 *Let A and B be C^*-algebras. Assume that A is unital with nonzero socle. Let $T : A \to B$ be a linear map preserving orthogonality or the relation $(R2)$. Then, for $x \in soc(A)$, the condition $T(x) \perp T(1)$ implies $T(x) = 0$.*

Proof For the sake of simplicity, let us denote $h = T(1)$. Pick $x \in \operatorname{soc}(A)$ satisfying $T(x) \perp T(1)$. By [25, Lemma 2.1], if T preserves orthogonality, or by the previous lemma if T preserves the relation $(R2)$, we get

$$0 = T(x)h^* = hT(x^*)^*,$$

$$0 = h^*T(x) = T(x^*)^*h,$$

and hence $T(x^*) \perp h$. Moreover

$$T(xx^* + x^*x)h^*h = T(x)T(x)^*h + T(x^*)T(x^*)^*h = 0,$$

or equivalently $T(xx^* + x^*x)h^* = 0$. This leads us to

$$T(x)T(x)^* + T(x^*)T(x^*)^* = 0,$$

which clearly implies that $T(x) = T(x^*) = 0$. ∎

Proposition 6.2.4 *Let A and B be C^*-algebras, where A is unital and has essential socle. Let $T : A \to B$ be an injective linear map preserving orthogonality or preserving the relation $(R2)$. Then $T(A) \cap \{T(1)\}^\perp = \{0\}$.*

Proof Let $a \in A$ be such that $T(a) \perp h$. We claim that, for every $x \in \operatorname{soc}(A)$, we have:

(1) $T(a \circ x) \perp h$,

(2) $T(a) \perp T(x)$.

Indeed, given $x \in \operatorname{soc}(A)$, taking into account [25, Lemma 2.1] or Lemma 6.2.2, we obtain

$$\begin{aligned} T(ax + xa)h^*h &= T(a)T(x^*)^*h + T(x)T(a^*)^*h \\ &= T(a)h^*T(x) + T(x)h^*T(a) = 0. \end{aligned}$$

Similarly, $hh^*T(ax + xa) = 0$, which proves *(1)*.

In order to show that *(2)* holds, let p be a minimal projection in A. From *(1)*,

$$0 = T(ap + pa)h^*h = T(a)T(p)^*h + T(p)T(a^*)^*h = T(a)T(p)^*h.$$

In the same way, we prove $hT(p)^*T(a) = 0$. Now, as

$$T(p)T(p)^* = T(p)h^* = hT(p)^*$$

and $T(p)^*T(p) = T(p)^*h$, it follows that

$$T(a)T(p)^*T(p) = T(p)T(p)^*T(a) = 0,$$

and by cancellation, $T(a) \perp T(p)$. Since $\mathrm{soc}(A)$ is linearly spanned by its minimal projections, we get *(2)*.

Finally, take $a \in A$ such that $T(a) \perp h$. Again by [25, Lemma 2.1] (respectively, Lemma6.2.2)

$$hh^*T(\{x,a,y\}) = T(\{x,a,y\})h^*h = T(x)T(a)^*T(y) + T(y)T(a)^*T(x) = 0$$

for every $x, y \in \mathrm{soc}(A)$. That is, $T(\{x,a,y\}) \perp h$ for every $x, y \in \mathrm{soc}(A)$. By the previous proposition, $T(\{x,a,y\}) = 0$, and since T is injective, $\{x,a,y\} = 0$ for every $x, y \in \mathrm{soc}(A)$. The essentiality of the socle of A gives $a = 0$ and finishes the proof. ■

The next result improves the main conclusion of Theorem 1.1.7. Notice that we have shown in Proposition 6.2.4 that the orthogonal of $\{T(1)\}$ does not contains elements of the image of T, the rest of the proof of Theorem 3.2 in [25] runs in the same way.

Corollary 6.2.5 *Let A and B be C*-algebras. Suppose that A is unital and has essential socle. Let $T : A \to B$ be a bijective linear map preserving orthogonality or the relation $(R2)$. Then B is unital and T is a Jordan *-homomorphism multiplied by an invertible element.*

In order to describe linear preservers of the relation $(R2)$, we consider under what circumstances we can obtain a bounded linear map preserving orthogonality, so that [36, Theorem 17 and Corollary 18] can be used to conclude its description.

Lemma 6.2.6 *Let A and B be C*-algebras, where A is unital, and $T : A \to B$ be a bounded linear map. Suppose that*

(1) $T(x)h^* = hT(x^*)^*$,

(2) $T(xy + yx)h^* = T(x)T(y^*)^* + T(y)T(x^*)^*$,

for every $x, y \in A$, where $h = T(1)$. Then T is bounded and preserves orthogonality.

Proof Indeed, it is clear that T is bounded: from the second identity, if follows that

$$T(x^2)h^* = T(x)T(x^*)^* \qquad (x \in A).$$

Therefore, the linear mapping $S : A \to B$, given by $S(x) = T(x)h^*$, is positive and hence continuous. So T is also bounded.

Let us write $k = h^*h$. For every x, y, z in A,

$$\begin{aligned}
2T\big((x \circ y^*) \circ z\big)k &= \big(T(x \circ y^*)T(z^*)^* + T(z)T(x^* \circ y)^*\big)h \\
&= T(x \circ y^*)h^*T(z) + T(z)T(x^* \circ y)^*h \\
&= \frac{1}{2}\big(T(x)T(y)^*T(z) + T(y^*)T(x^*)^*T(z) \\
&+ T(z)T(x^*)^*T(y^*) + T(z)T(y)^*T(x)\big) \\
&= \{T(x), T(y), T(z)\} + \{T(y^*), T(x^*), T(z)\}.
\end{aligned}$$

Similarly

$$2T\left((z\circ y^*)\circ x\right)k=\{T(x),T(y),T(z)\}+\{T(y^*),T(z^*),T(x)\},$$

$$2T\left((x\circ z)\circ y^*\right)k=\{T(x),T(z^*),T(y^*)\}+\{T(z),T(x^*),T(y^*)\}.$$

From these equalities we get

$$\begin{aligned}T(\{x,y,z\})k &= T\big((x\circ y^*)\circ z+(z\circ y^*)\circ x-(x\circ z)\circ y^*\big)\\ &= \{T(x),T(y),T(z)\},\end{aligned}$$

for $x,y,z\in A$.

It is clear now that T preserves zero triple products or, equivalently, orthogonality (Theorem 1.1.6). ■

In the next theorem, we show that if A is linearly spanned by its projections, and $T:A\to B$ preserves the relation $(R2)$, then T satisfies the conditions in Lemma 6.2.6 and hence T preserves orthogonality. In particular, from [38, Theorem 14], T is automatically bounded.

Theorem 6.2.7 *Let A be a unital C^*-algebra linearly spanned by its projections, B a C^*-algebra and $T:A\to B$ be a linear map preserving the relation $(R2)$. Then T preserves orthogonality.*

Proof For any projections $p,q\in A$, it is easy to show that

$$qp\le qp+(1-q)(1-p)\quad\text{and}\quad q(1-p)\le q(1-p)+(1-q)p.$$

By hypothesis,

$$T(qp)\le T(qp)+T((1-q)(1-p))\quad\text{and}\quad T(q(1-p))\le T(q(1-p))+T((1-q)p).$$

In particular

$$T(qp)\perp T((1-q)(1-p))\quad\text{and}\quad T(q(1-p))\perp T((1-q)p).$$

With these identities in mind, we can argue as in [38, Theorem 14] to obtain

(1) $T(x)h^*=hT(x^*)^*$,

(2) $T(xy+yx)h^*=T(x)T(y^*)^*+T(y)T(x^*)^*$,

for every $x,y\in A$. The conclusion follows by applying Lemma 6.2.6. ■

It is not difficult to realize that if A is not linearly spanned by its projections but it has enough projections, in the sense that A has real rank zero, and the map T is assumed to be continuous, then the previous line of arguments provides the following result.

Theorem 6.2.8 *Let A be a unital real rank zero C*-algebra, B be a C*-algebra and $T: A \to B$ be a bounded linear map preserving the relation $(R2)$. Then T preserves orthogonality.*

Remark 6.2.9 *Let A be a von Neumann algebra, B be a C*-algebra and $T: A \to B$ be a bijective linear map preserving the relation $(R2)$. As every von Neumann algebra is a Rickart C*-algebra, the relations $(R2)$ and $\leq_*$ are equivalent in A. Hence, T preserves orthogonality. From [116, Corollary 4.6] T is automatically bounded. Hence Theorem 1.1.6 implies that T is an appropriate multiple of a Jordan *-homomorphism.*

6.3 Linear preservers of the minus partial order

The minus partial order

Let H be an infinite dimensional complex Hilbert space. Recall that Šemrl ([129]) extended the minus partial order from $M_n(\mathbb{C})$ to $\mathcal{B}(H)$, finding an appropriate equivalent definition of the minus partial order on $M_n(\mathbb{C})$ which does not involve G_1-inverses. More recently, Djordjević, Rakić and Marovt ([49]) generalized Šemrl's definition to the environment of Rickart rings and generalized some well-known results. Recall that a ring A is a Rickart ring if the left and right annihilators of any element are generated by idempotent elements.

We will adopt the definition from [49]:

Definition 6.3.1 *We say that $a \leq^- b$ if there exist $p, q \in A^\bullet$ such that $ann_l(a) = ann_l(p)$, $ann_r(a) = ann_r(q)$, $pa = pb$ and $aq = bq$.*

In the next proposition we collect some properties of the relation $\leq^-$ in a unital ring that we will need in the sequel.

Proposition 6.3.2 *Let A be a unital ring. The following assertions hold:*

(1) If $a \in A^\wedge$, then $a \leq^- b$ if, and only if, there exists $a^- \in G_1(a)$ such that $a^-a = a^-b$ and $aa^- = ba^-$.

(2) If $b \in A^\wedge$ and $a \in A$ satisfy that $a \leq^- b$, then $a \in A^\wedge$ and $G_1(b) \subset G_1(a)$.

(3) If $a, b \in A^\wedge$, then $a \leq^- b$ if, and only if, there exists $b^- \in G_1(b)$ such that $a = ab^-b = bb^-a = ab^-a$.

(4) For every invertible element $u \in A$,

$$a \leq^- b \Leftrightarrow ua \leq^- ub \quad \textit{and}$$

$$a \leq^- b \Leftrightarrow au \leq^- bu,$$

for every $a, b \in A$.

(5) If $p \in A^\bullet$ and $a \leq^- p$ then $a \in A^\bullet$ and $a = ap = pa$.

Proof *(1)* Let $a, b \in A$. If $a \leq^- b$ and p, q are the idempotents appearing in Definition 6.3.1, since $(1-p)p = 0$, then $(1-p)a = 0$ and, consequently, $a = pa$. Similarly, $a = aq$. Note also that, if $a \in A$ is a regular element, then $a \leq^- b$ if, and only if, there exists $a^- \in G_1(a)$ such that $a^- a = a^- b$ and $aa^- = ba^-$. Indeed, let $b \in A$ such that $a \leq^- b$, and let $p, q \in A^\bullet$ as in the definition. Take $x \in G_1(a)$. Since $a = pa$ and $a = aq$ it is clear that $a^- := qxp$ is an inner inverse of a. Moreover $aa^- = aqxp = bqxp = ba^-$. Analogously, it can be checked that $a^- a = a^- b$. Notice that we can actually choose $a^+ \in G_2(a)$ satisfying $a^+ a = a^+ b$ and $aa^+ = ba^+$ by putting $a^+ = a^- aa^-$.

Reciprocally, suppose that $a \in A^\wedge$ and $b \in A$ satisfy that $aa^- = ba^-$ and $a^- a = a^- b$. Then $p = aa^-$ and $q = a^- a$ are idempotents, $\mathrm{ann}_l(a) = \mathrm{ann}_l(p)$, $\mathrm{ann}_r(a) = \mathrm{ann}_r(q)$,

$$a = pa = aa^- a = aa^- b = pb \quad \text{and}$$

$$a = aq = aa^- a = ba^- a = bq.$$

This shows that $a \leq^- b$.

(2) Let $b \in A^\wedge$ and $a \in A$ such that $a \leq^- b$. There exist $p, q \in A^\bullet$ verifying $a = pa = pb$ and $a = aq = bq$. For an arbitrary $b^- \in G_1(b)$, multiplying the first identity by $b^- b$ on the right we obtain

$$ab^- b = pbb^- b = pb = a.$$

Multiplying now by q on the right it yields $ab^- bq = aq$, that is, $ab^- a = a$ and, consequently, $b^- \in G_1(a)$.

(3) By looking at the proof of [88, Lemma 2 (a)], it can be seen that the same statement holds only assuming that the elements a and b are regular. In other words, the hypothesis of R being regular can be relaxed to a, b regular. Notice also that if $a \leq^- b$, then $a = ab^- b = bb^- a = ab^- a$, for every $b^- \in G_1(b)$.

(4) If $a \leq^- b$, there exist $p, q \in A^\bullet$ such that $\mathrm{ann}_l(a) = \mathrm{ann}_l(p)$, $\mathrm{ann}_r(a) = \mathrm{ann}_r(q)$, $pa = pb$ and $aq = bq$. Let $p_u = upu^{-1}$. Then $p_u \in A^\bullet$, $\mathrm{ann}_l(ua) = \mathrm{ann}_l(p_u)$, $p_u ua = p_u ub$, and $uaq = ubq$. This shows that $a \leq^- b \Leftrightarrow ua \leq^- ub$. Similarly, it can be proved that for any invertible element $u \in A$, $a \leq^- b \Leftrightarrow au \leq^- bu$, for every $a, b \in A$.

(5) We know from *(2)* that $a \in A^\wedge$ and $G_1(p) \subset G_1(a)$. From *(3)*, $a = ap = pa = apa$. In particular $a^2 = apa = a$. ■

There are many characterizations of the minus partial order. In [110] it is proved that for complex matrices M and N of the same order, $M \leq^- N$ if, and only if, $G_1(N) \subseteq G_1(M)$. This result was latter extended to the setting of regular rings in [15]. In the next proposition we show that the relation "$\leq^-$" is equivalent to the inclusion of the set of G_1-inverses for regular elements on a unital semiprime ring. For a regular element $a \in A$, we define $D_1(a) = \{x - y \colon x, y \in G_1(a)\}$.

Lemma 6.3.3 *Let A be a unital ring and $a \in A^\wedge$. Then*

$$D_1(a) = \{x \in A \colon axa = 0\}.$$

Proof Pick $x \in D_1(a)$. Then $x = a^- - a^=$ for some a^-, $a^= \in G_1(a)$. Hence

$$axa = a(a^- - a^=)a = a - a = 0.$$

For the reciprocal inclusion, suppose that $axa = 0$ and take $a^- \in G_1(a)$. As $a(a^- - x)a = aa^-a - axa = a$, it is clear that $a^-, a^- - x \in G_1(a)$ and, consequently, $x = a^- - (a^- - x) \in D_1(a)$, as desired. ■

Proposition 6.3.4 *Let A be a unital semiprime ring and $a, b \in A^\wedge$. The following assertions are equivalent:*

(1) $a \leq^- b$,

(2) $G_1(b) \subset G_1(a)$,

(3) $G_1(b) \cap G_1(a) \neq \emptyset$ *and* $D_1(b) \subset D_1(a)$.

Proof Let $a, b \in A^\wedge$. We know from Proposition 6.3.2 that $G_1(b) \subset G_1(a)$ whenever $a \leq^- b$, and hence *(1)* $\Rightarrow$ *(2)*.

It is clear that *(2)*$\Rightarrow$*(3)*.

Finally suppose that *(3)* holds and let $b^- \in G_1(b) \cap G_1(a)$. Since a and b are regular, in order to prove that $a \leq^- b$, it is enough to show that $a = ab^-b = bb^-a$. Taking into account that $b(1 - b^-b)Ab = \{0\}$ for every $b^- \in G_1(b)$ and that $D_1(b) \subset D_1(a)$, we conclude by Lemma 6.3.3 that $a(1 - b^-b)Aa = \{0\}$. Therefore,

$$a(1 - b^-b)Aa(1 - b^-b) = \{0\},$$

which, being A a semiprime algebra, gives $a = ab^-b$. Similar arguments can be applied to get $a = bb^-a$. ■

In [49, Theorem 3.3] the authors showed that the relation "$\leq^-$" is a partial order on a Rickart ring. Also, from [125, Theorem 3.3], "$\leq^-$" is a partial order on the class of relatively regular operators on Banach spaces. As a consequence of the above proposition we generalize this result to the setting of unital semiprime rings.

Corollary 6.3.5 *Let A be a unital semiprime ring. The relation "$\leq^-$" is a partial order on $A^\wedge$.*

Proof Reflexivity and transitivity of the relation "$\leq^-$" follow directly from Proposition 6.3.4. In order to prove the anti-symmetry, take $a, b \in A^\wedge$ with $a \leq^- b$ and $b \leq^- a$. There exists $a^- \in G_1(a)$ and $b^- \in G_1(b)$ such that

$$aa^- = ba^-,\ a^-a = a^-b,$$

$$bb^- = ab^-,\ b^-b = b^-a.$$

Since $G_1(a) = G_1(b)$, it follows that $b^- \in G_1(a)$. That is

$$a = ab^-a = ab^-b = bb^-b = b. \blacksquare$$

Definition 6.3.6 *Let A be a ring. We say that $a \leq_{sp} b$ if $aA \subset bA$ and $Aa \subset Ab$.*

This definition is analogous to the definition of the space pre-order on complex matrices introduced by Mitra in [112]. Recall that $M \leq_{sp} N$ if $\mathcal{C}(M) \subseteq \mathcal{C}(N)$ and $\mathcal{C}(N^*) \subseteq \mathcal{C}(M^*)$, where $\mathcal{C}(M)$ denotes the column space of the matrix M and M^* denotes the conjugate transpose of M. Notice that the condition $\mathcal{C}(N^*) \subseteq \mathcal{C}(M^*)$ can be replaced by $\mathcal{N}(N) \subseteq \mathcal{N}(M)$, where $\mathcal{N}(N)$ is the null space of the matrix N.

In [125], Rakić and Djordjević extend the definition of space pre-order to the class of bounded linear operators on Banach spaces, and generalize some well-known properties of this partial order to the new setting.

Observe that, whenever A is unital, $a \leq_{sp} b$ if, and only if, there exist $x, y \in A$ such that $a = bx = yb$. It is easy to see that this relation is a partial order in every unital ring and that $a \leq_{sp} b$ whenever $a \leq^- b$.

The following results are partially motivated by Theorem 2.5 and Theorem 3.7 in [125].

Proposition 6.3.7 *Let A be a unital semiprime ring, $a \in A$ and $b \in A^\wedge$. The following conditions are equivalent:*

(1) $a \leq_{sp} b$,

(2) $ann_l(b) \subset ann_l(a)$ *and* $ann_r(b) \subset ann_r(a)$,

(3) $a = bb^-a = ab^-b$ *for every* $b^- \in G_1(b)$,

(4) $aD_1(b)a = \{0\}$.

Proof It is clear that *(1)* $\Rightarrow$ *(2)*. In order to prove *(2)* $\Rightarrow$ *(3)*, observe that $b(1-b^-b) = 0$ for all $b^- \in G_1(b)$ and hence, by assumption, $a(1 - b^-b) = 0$ for every $b^- \in G_1(b)$. That is, $a = ab^-b$, for all $b^- \in G_1(b)$. Similarly, it can be proved that $a = bb^-a$ for every $b^- \in G_1(b)$.

Now suppose that *(3)* holds and pick $x \in D_1(b)$. Then, $x = b^- - b^=$ for some $b^-, b^= \in G_1(b)$. By hypothesis we have $ab^-a = ab^-bb^=a = ab^=a$ and, consequently, $axa = ab^-a - ab^=a = 0$. This proves that *(3)* holds.

Finally, assume that $aD_1(b)a = \{0\}$. By Lemma 6.3.3, $a(1 - b^-b)xa = 0$ for all $x \in A$. Hence,

$$a(1 - b^-b)xa(1 - b^-b) = 0 \qquad (x \in A)$$

and, being A semiprime, it yields $a = ab^-b$. Similarly, we obtain $a = bb^-a$ and therefore $a \leq_{sp} b$. $\blacksquare$

Corollary 6.3.8 *Let A be a unital semiprime ring and $a, b \in A^\wedge$. Then $a \leq_{sp} b$ if, and only if, $D_1(b) \subset D_1(a)$.*

As a direct consequence of Proposition 6.3.2 *(3)*, and *(1)* $\Leftrightarrow$ *(3)* in Proposition 6.3.7, we obtain the following characterization of the minus partial order.

Corollary 6.3.9 *Let A be a unital semiprime ring and $a, b \in A^\wedge$. The following are equivalent:*

(1) $a \leq^- b$

(2) $a \leq_{sp} b$ *and* $G_1(a) \bigcap G_1(b) \neq \emptyset$.

For a ring A and $a, b \in A^\wedge$, we define

$$G_1^b(a) := \{a^- \in G_1(a) \colon aa^- = ba^-, a^-a = a^-b\}.$$

The following results provide algebraic adaptations for Theorems 3.8, 3.9 and 3.10 in [125].

Proposition 6.3.10 *Let A be a ring and $a, b \in A^\wedge$ satisfying $a \leq^- b$. Then*

$$G_1^b(a) = \{b^- - b^-(b-a)b^- : b^- \in G_1(b)\}.$$

Proof Since $a \leq^- b$, it follows from [88, Lemma 2] that

$$a = ab^-b = bb^-a = ab^-a$$

for every $b^- \in G_1(b)$. Accordingly, an easy computation shows that, for every $b^- \in G_1(b)$, $(b-a)b^-(b-a) = b-a$. In particular, $b-a \in A^\wedge$ and by Proposition 6.3.4, $b-a \leq^- b$.

Let $a^- \in G_1^b(a)$ and $(b-a)^+ \in G_2(b-a)$ such that

$$(b-a)(b-a)^+ = b(b-a)^+ \quad \text{and} \quad (b-a)^+(b-a) = (b-a)^+b.$$

Then

$$(b-a)a^- = 0 = a^-(b-a) \quad \text{and} \quad (b-a)^+a = 0 = a(b-a)^+.$$

Let $b^- = a^- + (b-a)^+$. From above it follows that $b^- \in G_1(b)$. Moreover,

$$\begin{aligned} b^- \quad - \quad & b^-(b-a)b^- = \\ = \quad & a^- + (b-a)^+ - \left(a^- + (b-a)^+\right)(b-a)\left(a^- + (b-a)^+\right) = a^-. \end{aligned}$$

Conversely, for every $b^- \in G_1(b)$

$$\begin{aligned} a\left(b^- - b^-(b-a)b^-\right) &= ab^- - ab^-bb^- + ab^-ab = ab^- \\ &= ab^- = b\left(b^- - b^-(b-a)b^-\right). \end{aligned}$$

Similarly,

$$\left(b^{-}-b^{-}\left(b-a\right)b^{-}\right)a=b^{-}a=\left(b^{-}-b^{-}\left(b-a\right)b^{-}\right)b.$$

Furthermore,

$$a\left(b^{-}-b^{-}\left(b-a\right)b^{-}\right)a=ab^{-}a-\left(ab^{-}b\right)b^{-}a+\left(ab^{-}a\right)b^{-}a=a.$$

Therefore, $b^{-}-b^{-}(b-a)b^{-}\in G_1^b(a)$, as desired. ■

Proposition 6.3.11 *Let A be a ring and $a,b\in A^{\wedge}$ such that $a\leq^{-}b$. The following assertions hold:*

(1) For every $a^{-}\in G_1^b(a)$, there exists $b^{-}\in G_1(b)$ satisfying $b^{-}a=a^{-}a$ and $ab^{-}=aa^{-}$,

(2) For every $b^{-}\in G_1(b)$, there exists $a^{-}\in G_1^b(a)$ satisfying $b^{-}a=a^{-}a$ and $ab^{-}=aa^{-}$.

Proof In order to prove *(1)*, pick $a^{-}\in G_1^b(a)$. By Proposition 6.3.10 there is $b^{-}\in G_1(b)$ such that

$$a^{-}=b^{-}-b^{-}(b-a)b^{-}.$$

Hence,

$$a^{-}a=(b^{-}-b^{-}(b-a)b^{-})a=b^{-}a-b^{-}bb^{-}a+b^{-}ab^{-}a=b^{-}a.$$

Similarly, $aa^{-}=ab^{-}$.

Now we prove *(2)*. Let $b^{-}\in G_1(b)$. Again by Proposition 6.3.10, we know that $a^{-}=b^{-}-b^{-}(b-a)b^{-}\in G_1^b(a)$. As $a\leq^{-}b$, it follows

$$aa^{-}=a(b^{-}-b^{-}(b-a)b^{-})=ab^{-}-ab^{-}bb^{-}+ab^{-}ab^{-}=ab^{-}.$$

The identity $b^{-}a=a^{-}a$ can be obtained in the same way. ■

Proposition 6.3.12 *Let A be a unital complex algebra, $a,b\in A^{\wedge}$ such that $a\leq^{-}b$ and $c_1,c_2\in\mathbb{C}$ with $c_2\neq 0$ and $c_1+c_2\neq 0$. Then $c_1a+c_2b\in A^{-1}$ if, and only if, $b\in A^{-1}$. Moreover, in such case*

$$(c_1a+c_2b)^{-1}=c_2^{-1}b^{-1}+((c_1+c_2)^{-1}-c_2^{-1})b^{-1}ab^{-1}.$$

Proof Suppose that $b\in A^{-1}$. As $a\leq^{-}b$, by the previous proposition, we have $G_1^b(a)=\{b^{-1}ab^{-1}\}$. In particular, this implies that $ab^{-1}ab^{-1}=ab^{-1}$ and $b^{-1}ab^{-1}a=b^{-1}a$. Now, by a direct computation

$$\begin{aligned}(c_1a+c_2b)\left(c_2^{-1}b^{-1}+\left((c_1+c_2)^{-1}-c_2^{-1}\right)b^{-1}ab^{-1}\right)&=\\ =c_1c_2^{-1}ab^{-1}+c_1\left((c_1+c_2)^{-1}-c_2^{-1}\right)ab^{-1}ab^{-1}+1+&\\ c_2\left((c_1+c_2)^{-1}-c_2^{-1}\right)ab^{-1}&=\\ =1+\left(c_1c_2^{-1}+c_1\left(c_1+c_2\right)^{-1}-c_1c_2^{-1}+c_2\left(c_1+c_2\right)^{-1}-1\right)ab^{-1}&=1.\end{aligned}$$

Similarly,

$$\left(c_2^{-1}b^{-1} + ((c_1+c_2)^{-1} - c_2^{-1})b^{-1}ab^{-1}\right)(c_1a + c_2b) = 1.$$

Conversely, if $c_1a + c_2b \in A^{-1}$, as $a = ab^-b = bb^-a$ for every $b^- \in G_1(b)$, we get

$$c_1a + c_2b = (c_1ab^- + c_2)b = b(c_1b^-a + c_2).$$

Hence, b is (left and right) invertible. ■

Recall that, as we have proved in Corollary 6.3.5, the relation "$\leq^-$" is a partial order on the set of regular elements of every unital semiprime ring.

Our next goal is describing the maximal and minimal elements of the minus partial order.

Recall that, for a unital prime ring A, A_l^{-1} and A_r^{-1} denote the sets of all left and right invertible elements of A, respectively.

Proposition 6.3.13 *Let A be a unital prime ring. The following conditions are equivalent:*

(1) $a \in A^\wedge$ and a is maximal with respect to the relation "$\leq^-$",

(2) $a \in A_l^{-1} \cup A_r^{-1}$.

Proof First, given $a \in A^\wedge$ and $a^- \in G_1(a)$, it is easy to see that

$$a \leq^- a + (1 - aa^-)x(1 - a^-a),$$

for every $x \in A$. If we suppose that a is maximal, we get

$$(1 - aa^-)x(1 - a^-a) = 0,$$

for every $x \in A$. As A is a prime algebra, it yields $1 = aa^-$ or $1 = a^-a$.

Reciprocally, we may assume without loss of generality that a is left invertible in A. If $a \leq^- b$, there exists $q \in A^\bullet$ such that $a = aq = bq$. Since $a \in A_l^{-1}$ it is clear that $q = 1$ and hence, $a = b$. This shows that a is maximal with respect to the relation "$\leq^-$". ■

Remark 6.3.14 *Note that the condition of primality cannot be dropped in order to characterize maximal regular elements as left or right invertible elements. For instance, take $A = \mathcal{B}(X) \oplus \mathcal{B}(X)$ for an infinite dimensional Banach space X. This algebra is not prime but it is, in fact, semiprime. Take operators $L, R \in \mathcal{B}(X)$ which are, respectively, left invertible and right invertible but none of them are invertible. The element $L \oplus R \in A$ is clearly maximal with respect to "$\leq^-$" but it is neither left nor right invertible.*

Proposition 6.3.15 *Let A be a unital semisimple Banach algebra with essential socle. Then, for every nonzero $a \in A$, there exists $u \in \mathcal{F}_1(A)$ such that $u \leq^- a$. Furthermore, $u \in \mathcal{F}_1(A)$ if, and only if, for every $v \leq^- u$ we have $u = v$ or $v = 0$. In other words, the elements in $\mathcal{F}_1(A)$ are precisely the nonzero minimal elements with respect to "$\leq^-$".*

Proof Fix $a \in A \setminus \{0\}$. Since A is semisimple and has essential socle, there exists $w \in \mathcal{F}_1(A)$ such that $aw \neq 0$. Given $(aw)^- \in G_1(aw)$, set $v = w(aw)^-$. It is clear that av is a minimal idempotent and, in particular, $u = ava \in \mathcal{F}_1(A)$. We claim that $u \leq^- a$. Indeed, let $p = av$ and $q = va$. These are idempotent elements in A, such that

$$pu = avava = pa \quad \text{and} \quad uq = avava = aq.$$

Moreover, since $u = pa = aq$ it can be easily checked that $\mathrm{ann}_l(p) = \mathrm{ann}_l(u)$ and $\mathrm{ann}_r(q) = \mathrm{ann}_r(u)$.

Now, let $u \in \mathcal{F}_1(A)$ and $0 \neq v \leq^- u$. By Proposition 6.3.2, $v \in A^\wedge$ and there exists $v^+ \in G_2(v)$ such that $v^+u = v^+v$ and $uv^+ = vv^+$. Hence $v = vv^+u = uv^+v$ and, multiplying by v^+u on the right, we get

$$v = (uv^+v)v^+u = uv^+u = \tau(uv^+)u = u. \blacksquare$$

Let A be a unital semisimple Banach algebra with nonzero socle. For every $u \in \mathcal{F}_1(A)$ we define

$$L_u := \{ux : x \in A\} \qquad \text{and} \qquad R_u := \{xu : x \in A\}.$$

Remark 6.3.16 *These definitions are the algebraic analogue to the ones given in [129, Theorem 8]: for a rank one operator $S = x \otimes y^* \in \mathcal{B}(H)$, we have $L_x = L_S$ and $R_y = R_S$. Indeed, if $R \in L_S$, then $R = ST$ for some $T \in \mathcal{B}(H)$. Consequently,*

$$R = S(T(\cdot)) =< T(\cdot), y > x =< \cdot\,, T^*(y) > x = x \otimes (T^*(y))^*$$

and hence, $R \in L_x$.

Now let $w^* \in \mathcal{B}(H)^*$ and $R = x \otimes w^* \in L_x$. Take an arbitrary operator U such that $U(y) = w$ and set $T = U^*$. Then it can be proved that $R = ST$, that is, $R \in L_S$.

The equality $R_y = R_S$ is proven similarly.

Notice that, for every $u \in \mathcal{F}_1(A)$, $L_u = uA$ and $R_u = Au$ are the right minimal ideal and the left minimal ideal generated by u, respectively. It is also clear that, for every $u \in \mathcal{F}_1(A)$, L_u and R_u are subspaces of $\mathrm{soc}(A)$ consisting of elements of rank at most one. Moreover, if $0 \neq v \in L_u$ then $L_u = L_v$. Indeed, if $v = ux$ for some $x \in A$, then

$$v(ux)^-u = ux(ux)^-u = \tau(ux(ux)^-)u = u,$$

which gives $u \in L_v$. (That is, $w = va$ for some $a \in A$ if, and only if, $w = ub$ for some $b \in A$). Similarly, if $0 \neq v \in R_u$, then $R_u = R_v$.

Lemma 6.3.17 *The maximal linear subspaces of soc(A) consisting of elements with rank at most one are precisely L_u or R_u where $u \in \mathcal{F}_1(A)$.*

Proof As we have just mentioned, for every $u \in \mathcal{F}_1(A)$, L_u and R_u are linear subspaces of soc(A).

Let u, v be nonzero elements such that $u, v, u+v \in \mathcal{F}_1(A)$. For every $x \in A$, we have $(u+v)x(u+v) = \tau((u+v)x)(u+v)$, and by the additivity properties of the trace (see Section 2.1) we know that $\tau((u+v)x) = \tau(ux) + \tau(vx)$. Hence

$$(u+v)x(u+v) = (\tau(ux) + \tau(vx))\,(u+v),$$

which implies

$$uxv + vxu = \tau(ux)v + \tau(vx)u.$$

Equivalently,

$$(ux - \tau(ux))v = (\tau(vx) - vx)u,$$

for every $x \in A$.

Assume that $z_0 = (ux_0 - \tau(ux_0))v = (\tau(vx_0) - vx_0)u$ is nonzero for some $x_0 \in A$. Then we can write

$$v = vz_0^- z_0 = v((ux_0 - \tau(ux_0))v)^-(\tau(vx_0) - vx_0)u,$$

for $z_0^- \in G_1(z_0)$. This yields $v \in R_u$ and, consequently $R_u = R_v$.

Otherwise, we have $uxv = \tau(ux)v$ for all $x \in A$. Therefore, we have

$$uu^- v = \tau(uu^-)v = v$$

for any $u^- \in G_1(u)$. Thus, $v \in L_u$, which finally gives $L_u = L_v$. This shows that, for every linear subspace M of soc(A) with $M \subset \mathcal{F}_1(A) \cup \{0\}$ and $0 \neq u \in M$, we have $M \subset L_u \cup R_u$ and hence, $M \subset L_u$ or $M \subset R_u$. ■

Proposition 6.3.18 *Let A be a unital semisimple Banach algebra with essential socle and $a \in A$. The following conditions are equivalent:*

(1) $a \in A^{-1}$,

(2) $a \in A^\wedge$ and for every $u \in \mathcal{F}_1(A)$, there exist $x \in L_u \setminus \{0\}$ and $y \in R_u \setminus \{0\}$ such that $x, y \leq^- a$.

Proof Let $a \in A^{-1}$, $u \in \mathcal{F}_1(A)$ and $u^- \in G_1(u)$. It is clear that $x = uu^- a$ belongs to $L_u \setminus \{0\}$. Let us show that $x \leq^- a$. Set $p = uu^-$ and $q = a^{-1}uu^- a$. Then $p, q \in A^\bullet$,

$$px = uu^- uu^- a = uu^- a = pa \quad \text{and}$$

$$xq = uu^- aa^{-1} uu^- a = uu^- a = aq.$$

Besides, it can be easily checked that

$$\mathrm{ann}_l(x) = \mathrm{ann}_l(p) \quad \text{and} \quad \mathrm{ann}_r(x) = \mathrm{ann}_r(q).$$

Thus, $x \leq^- a$. The existence of $y \in R_u \setminus \{0\}$ satisfying $y \leq^- a$ is guaranteed in the same way. This shows that *(1)* $\Rightarrow$ *(2)*.

Conversely, let $a \in A$ satisfying *(2)*. Given $u \in \mathcal{F}_1(A)$, there exists $x \in A$ such that $ux \leq^- a$. As ux is regular, there exists $(ux)^- \in G_1(ux)$ such that $(ux)^-(ux) = (ux)^-a$ and $(ux)(ux)^- = a(ux)^-$. Multiplying the last identity by u on the right, it yields $(ux)(ux)^-u = a(ux)^-u$. As $u \in \mathcal{F}_1(A)$, we have $(ux)(ux)^-u = \tau((ux)(ux)^-)u = u$ and, hence, $u = a(ux)^-u$. Similarly, given $y \in A$ such that $yu \leq^- a$, we get $u = u(yu)^-a$.

Suppose that $za = 0$. In such case $za(ux)^-u = zu = 0$, for every $u \in \mathcal{F}_1(A)$. Since A is semisimple and has essential socle, it gives $z = 0$. We have proved that $\mathrm{ann}_l(a) = \{0\}$. Similarly, it can be checked that $\mathrm{ann}_r(a) = \{0\}$. Therefore, a is a regular element which is not a zero divisor, that is, a is invertible. ■

Linear maps preseving the minus order

Recall that every Jordan homomorphism $T : A \to B$ is a Jordan triple homomorphism, that is

$$T(aba) = T(a)T(b)T(a) \quad \text{for all} \quad a, b \in A.$$

In particular, it is clear that every Jordan triple homomorphism $T : A \to B$ strongly preserves regularity, that is, if $a \in A^\wedge$ and $a^- \in G_1(a)$, then $T(a) \in B^\wedge$ and $T(a^-) \in G_1(T(a))$. (Obviously, if $a^+ \in G_2(a)$ then $T(a^+) \in G_2(T(a))$.) The next proposition shows that every Jordan triple homomorphism preserves the minus partial order on regular elements.

Proposition 6.3.19 *Let A, B be Banach algebras and $T : A \to B$ a Jordan triple homomorphism. Then, $a \leq^- b$ implies $T(a) \leq^- T(b)$, for every $a, b \in A^\wedge$.*

Proof Let $a, b \in A^\wedge$. By Proposition 6.3.2 *(3)*, $a \leq^- b$ if, and only if, there exists $b^- \in G_1(b)$ such that $a = ab^-a = ab^-b = bb^-a$. We may assume that $b^- \in G_2(b)$. Since T is a Jordan triple homomorphism, and $a = ab^-a$ and $2a = ab^-b + bb^-a$, we have

$$T(a) = T(a)T(b)^-T(a) \quad \text{and} \quad 2T(a) = T(a)T(b)^-T(b) + T(b)T(b)^-T(a).$$

Multiplying the last identity by $T(b)^-T(a)$ on the right, and havind in mind that, as we have previously point out, $T(b)^- \in G_2(T(b))$, we get

$$\begin{aligned} 2T(a) &= T(a)T(b)^-T(b)T(b)^-T(a) + T(b)T(b)^-T(a)T(b)^-T(a) \\ &= T(a) + T(b)T(b)^-T(a). \end{aligned}$$

Consequently, $T(a) = T(b)T(b)^-T(a)$. Similarly, it can be obtained that $T(a) = T(a)T(b)^-T(b)$, which completes the proof. ■

We present now our main result in this section. It is inspied in [129]. Notice that in [129, Theorem 8] the map is not assumed to be linear. By adding linearity, we are able to extend this result to the more general environment of unital semisimple Banach algebras with large socle. In its proof we will extensively use some of the results appearing in the first part of this section.

Theorem 6.3.20 *Let A and B be unital semisimple Banach algebras with essential socle. Let $T : A \to B$ be a bijective linear map. The following conditions are equivalent:*

(1) $T(A^\wedge) = B^\wedge$, and $a \leq^- b \Leftrightarrow T(a) \leq^- T(b)$, for every $a, b \in A^\wedge$.

(2) T is a Jordan isomorphism multiplied by an invertible element.

Proof It is clear from Proposition 6.3.2 *(4)* and Proposition 6.3.19 that *(2)* $\Rightarrow$ *(1)*.

Suppose now that $T(A^\wedge) = B^\wedge$ (that is, a is regular if, and only if, $T(a)$ is regular) and

$$a \leq^- b \Leftrightarrow T(a) \leq^- T(b), \qquad \text{for every } a, b \in A^\wedge.$$

We will show that $T(\mathcal{F}_1(A)) = \mathcal{F}_1(B)$. Let $u \in \mathcal{F}_1(A)$. Hence $T(u)$ is a nonzero regular element, and by Proposition 6.3.15 there exists $T(v) \in \mathcal{F}_1(B)$ such that $T(v) \leq^- T(u)$. From Proposition 6.3.2 *(2)*, $T(v) \in B^\wedge$ and $G_1(T(u)) \subseteq G_1(T(v))$. By hypothesis, $v \leq^- u$. Since $u \in \mathcal{F}_1(A)$ and $v \neq 0$, again by Proposition 6.3.15, $v = u$. That is $T(v) = T(u)$, which shows that $T(v) \in \mathcal{F}_1(B)$. Taking into account that T^{-1} satisfies the same conditions, we get $T(\mathcal{F}_1(A)) = \mathcal{F}_1(B)$.

Now, since the sets L_u and R_u are the maximal linear subspaces of $\mathrm{soc}(A)$ consisting of elements with rank at most one (see Lemma 6.3.17), we conclude that $T(L_u), T(R_u) \in \{L_{T(u)}, R_{T(u)}\}$ for every $u \in \mathcal{F}_1(A)$.

Let $a \in A^{-1}$. We claim that $T(a) \in B^{-1}$. By hypothesis, $T(a) \in B^\wedge$. Given $T(u) \in \mathcal{F}_1(B)$, since $a \in A^{-1}$, we know by Proposition 6.3.18 that there exist $x_0 \in L_u \setminus \{0\}$ and $y_0 \in R_u \setminus \{0\}$ such that $x_0, y_0 \leq^- a$. If $T^{-1}(L_{T(u)}) = L_u$, take $x = x_0$. Otherwise, take $x = y_0$. Then $T(x) \in L_{T(u)}$ and $T(x) \leq^- T(a)$. Similarly, we find $T(y) \in R_{T(u)}$ with $T(y) \leq^- T(a)$. This shows that $T(a) \in B^{-1}$.

Let $S : A \to B$ be the linear mapping given by $S(x) = T(1)^{-1}T(x)$, for all $x \in A$. It is clear that S is unital, bijective and preserves invertibility. By Theorem 1.1.4 S is a Jordan isomorphism, which concludes the proof. ■

Remark 6.3.21 *Let X be a complex Banach space. $\mathcal{B}(X)$ is a unital semisimple Banach algebra with essential socle. Notice that soc$(\mathcal{B}(X)) = F(X)$ is the ideal of finite rank operators on X.*

Let $T \in \mathcal{B}(X)$. By looking at the proof of (2) $\Rightarrow$ (1) *in Proposition 6.3.18 it can be seen that, if T satisfies that, for every rank one operator $U \in \mathcal{F}_1(X)$, there exist $L \in L_U \setminus \{0\}$ and $R \in R_U \setminus \{0\}$ with $L, R \leq^- T$, then T is invertible.*

Let A be a unital semisimple Banach algebra with essential socle, and X be a complex Banach space. Let $T : A \to \mathcal{B}(X)$ be a surjective linear map such that

$$a \leq^- b \quad \text{if, and only if,} \quad T(a) \leq^- T(b).$$

Notice that T is injective: if $T(x) = 0$, then $T(x) \leq^- T(0)$, which by assumption, gives that $x \leq^- 0$, and finally $x = 0$.

A direct application of Proposition 6.3.15 shows that $T(\mathcal{F}_1(A)) = \mathcal{F}_1(B)$, and by Lemma 6.3.17 $T(L_u), T(R_u) \in \{L_{T(u)}, R_{T(u)}\}$ for every $u \in \mathcal{F}_1(A)$. From this facts and Remark 6.3.21, it is clear now that T preserves invertibility. As in the previous theorem, it follows that the linear mapping given by $S(x) = T(1)^{-1}T(x)$, is a Jordan isomorphism. By the Herstein's theorem ([68]) S is either an isomorphism or an anti-isomorphism.

On the other hand, it is straightforward to check that every isomorphism and every anti-isomorphism preserves the minus partial relation in both directions. In view of Proposition 6.3.2 *(4)*, this is also the case for every isomorphism or anti-isomorphism multiplied by an invertible element. This proves the next result.

Theorem 6.3.22 *Let A be a unital semisimple Banach algebra with essential socle, and X be a complex Banach space. Let $T : A \to \mathcal{B}(X)$ be a surjective linear map. The following are equivalent:*

(1) $a \leq^- b$ if, and only if, $T(a) \leq^- T(b)$, for every $a, b \in A$.

(2) T is either an isomorphism multiplied by an invertible element or an anti-isomorphism multiplied by an invertible element.

Let A be a unital prime C*-algebra with nonzero socle. We know that A is primitive and has essential socle. Let $e \in A$ be a minimal projection and $\rho : A \to \mathcal{B}(H)$ the left regular *-respresentation (see Section 2.2). Let $a \in A$ (non necessarily regular) such that, for every $u \in \mathcal{F}_1(A)$, there exist $x \in L_u \setminus \{0\}$ and $y \in R_u \setminus \{0\}$ such that $x, y \leq^- a$. As we have proved in Proposition 6.3.18, given $u \in \mathcal{F}_1(A)$, we can find $w, z \in A$ such that $u = awu = uza$, which in particular shows that a is not a zero divisor. We claim that $\rho(a)$ is invertible, which, in this setting, shows that a is invertible. Indeed, since $\mathrm{ann}_r(a) = \{0\}$ it is clear that $\rho(a)$ is injective. Moreover, given $ze \in Ae$, by hypothesis, there exist $w \in A$ such that $ze = awze = \rho(a)(wze)$. This shows that $\rho(a)$ is surjective, and hence $\rho(a)$ is invertible. We have just proved the following:

Proposition 6.3.23 *Let A be a unital prime C*-algebra with nonzero socle and $a \in A$. The following conditions are equivalent:*

(1) $a \in A^{-1}$,

(2) For every $u \in \mathcal{F}_1(A)$, there exist nonzero $x \in L_u$ and $y \in R_u$ such that $x, y \leq^- a$.

The proof of the next theorem follows the lines of Theorems 6.3.20 and 6.3.22, by using Propositions 6.3.23, 6.3.15 and Theorem 1.1.4.

Theorem 6.3.24 *Let A be a unital semisimple Banach algebra with essential socle, B a unital prime C*-algebra with nonzero socle and $T : A \to B$ a surjective linear map. The following are equivalent:*

(1) $a \leq^- b$ if, and only if, $T(a) \leq^- T(b)$, for every $a, b \in A$,

(2) T is either an isomorphism multiplied by an invertible element or an anti-isomorphism multiplied by an invertible element.

We would like to shed some light on the study of mappings preserving the relation "$\leq^-$" just in one direction. In order to do that, we will focus our aim in real rank zero C*-algebras.

Theorem 6.3.25 *Let A be a real rank zero C*-algebra and B be unital Banach algebra. Let $T : A \to B$ be a bounded linear map satisfying that*

$$a \leq^- b \quad \textit{implies} \quad T(a) \leq^- T(b), \quad \textit{for all } a, b \in A.$$

The following assertions hold:

(1) If $T(1) \in B^\bullet$ then T is a Jordan homomorphism,

(2) If $T(A) \cap B^{-1}$ and $T(1) \in B^\wedge$ then T is a Jordan homomorphism multiplied by an invertible element.

Proof *(1)* Assume that $T(1) \in B^\bullet$. For every $p \in A^\bullet$, as $p \leq^- 1$ it follows that $T(p) \leq^- T(1)$. Having in mind Proposition 6.3.2 *(5)*, we conclude that $T(p) \in B^\bullet$ for all $p \in A^\bullet$.

This shows that T preserves idempotents. It only remains to apply Lemma 3.3.2.

(2) Suppose that $T(A) \cap B^{-1}$ and $T(1) \in B^\wedge$. As above, $T(e) \leq^- T(1)$ for every $e \in A^\bullet$. In particular, $T(e) \in T(1)B \cap BT(1)$ for every $e \in A^\bullet$. Since $T(1)$ is regular, it is well-known that $T(1)B$ and $BT(1)$ are closed. Taking into account that every selfadjoint element in A can be approximated by linear combinations of mutually orthogonal projections, and that T is linear and bounded, we conclude that $T(x) \in T(1)B \cap BT(1)$ for every $x \in A$. Therefore, as $T(A) \cap B^{-1}$, we deduce that $T(1)$ is invertible. Finally, let $S : A \to B$ be the linear mapping defined as $S(x) = T(1)^{-1}T(x)$, for all $x \in A$. We conclude the proof by proving that S preserves idempotents: given $e \in A^\bullet$, since $T(e) \leq^- T(1)$, there exists $p \in B^\bullet$ such that $T(e) = T(1)p$, that is $S(e) = T(1)^{-1}T(e) = p \in B^\bullet$. ■

We conclude this section with two remarks. The first one shows that it is not always possible to characterize Jordan homomorphims in terms of minus partial order preserving conditions. The second one deals with linear maps preserving the space preorder (see Definition 6.3.6).

Remark 6.3.26 *Let A be a Rickart ring. By [49, Theorem 3.3], the relation "$\leq^-$" defines a partial order in A. Every linear mapping $T : \mathbb{C} \to A$ preserves the minus partial order. Notice that $a \leq^- b$ in $\mathbb{C}$ if, and only if, $a = 0$ or $a = b$, and that, by reflexivity, $T(a) \leq^- T(a)$ for every $a \in \mathbb{C}$.*

Observe that the same conclusions hold when $\mathbb{C}$ is replaced by any Banach algebra A in which the only idempotents are the trivial ones, namely, the identity and zero. For instance $A = C([0,1])$.

Remark 6.3.27 *Let A and B be unital semisimple Banach algebras. Let $T : A \to B$ be a linear mapping such that*

$$a \leq_{sp} b \quad \textit{implies} \quad T(a) \leq_{sp} T(b), \quad \textit{for all} \quad a, b \in A.$$

Notice that $b \in A^{-1}$ if, and only if, $a \leq_{sp} b$ for every $a \in A$. Hence, if T is surjective then T preserves invertibility. This shows that T is a Jordan homomorphism multiplied by an invertible element in the following settings:

(1) If A has essential socle (see Theorem 1.1.4),

(2) If A has real rank zero (see Theorem 1.1.3).

6.4 Linear preservers of the diamond partial order

The last section of this chapter is devoted to the study of the diamond partial order in C*-algebras and the description of linear maps preserving this partial order. The techniques used in this section are analogous to the ones in Section 6.3, with the exception that the diamond partial order is related to some specific concepts from the theory of C*-algebras, such as projections, isometries and unitaries.

The diamond partial order

Let A be a unital C*-algebra and $a, b \in A$. Recall that $a \leq_\diamond b$ if, and only if, $aA \subset bA$, $Aa \subset Ab$ and $aa^*a = ab^*a$. The following proposition collects some algebraic properties of the relation "$\leq_\diamond$" that will we need in the sequel. It is implicitly proved in [88].

Proposition 6.4.1 *Let A be a unital C*-algebra.*

(1) If $a \in A^\wedge$ and $b \in A$, $a \leq_\diamond b$ if, and only if, $a \leq_{sp} b$ and $a^\dagger b a^\dagger = a^\dagger$.

(2) If $a \in A^\wedge$ and $b \in A$, then $a \leq_\diamond b$ whenever $a \leq_* b$.

(3) Given $a, b \in A^\wedge$, $a \leq_\diamond b$ if, and only if, $a^\dagger \leq^- b^\dagger$.

Proof *(1)* See [88, Theorem 1].

(2) See [88, Proposition 2 (a)].

(3) See [88, Theorem 2]. ■

It follows from Proposition 6.4.1 *(3)* and the fact that "$\leq^-$" is a partial order on the set of all regular elements (see Corollary 6.3.5) that the relation "$\leq_\diamond$ " is a partial order on $A^\wedge$. Besides, we can state the following:

Proposition 6.4.2 *Let A be a unital C*-algebra. The relation "$\leq_\diamond$" is a partial order on A.*

Proof Reflexivity of the relation "$\leq_\diamond$ " is clear.

Let $a, b \in A$ such that $a \leq_\diamond b$ and $b \leq_\diamond a$. In particular, $aa^*a = ab^*a$, $bb^*b = ba^*b$, and there exist $x, y \in A$ such that $a = xb = by$. Since $bb^*b = bb^*x^*b$, it follows by cancellation that, $b^*b = b^*x^*b = a^*b$. That is, $b^*b = y^*b^*b$, which shows that $b^* = y^*b^* = a^*$, equivalently $a = b$. This proves that the relation "$\leq_\diamond$ " is anti-symmetric.

Finally, in order to prove the transitivity of "$\leq_\diamond$ ", take $a, b, c \in A$ such that $a \leq_\diamond b$ and $b \leq_\diamond c$. Clearly, $a \leq_{sp} c$. Let $x, y \in A$ be such that $a = xb = by$. If follows that

$$aa^*a = ab^*a = xbb^*by = xbc^*by = ac^*a,$$

and hence $a \leq_\diamond c$, as desired. ■

In the next proposition we characterize projections in terms of the diamond partial order.

Proposition 6.4.3 *Let A be a unital C*-algebra. The following conditions are equivalent:*

(1) $p \in \mathit{Proj}(A)$,

(2) $p \leq_\diamond 1$ *and* $1 - p \leq_\diamond 1$,

(3) *there is* $q \in \mathit{Proj}(A)$, *such that* $p \leq_\diamond q$ *and* $q - p \leq_\diamond q$.

Proof It is clear that *(1)* $\Rightarrow$ *(2)* $\Rightarrow$ *(3)*.

Assume that *(3)* holds. Let $q \in \mathrm{Proj}(A)$ such that $p \leq_\diamond q$ and $q - p \leq_\diamond q$. There exist $x, y \in A$ such that $p = qx = yq$, which shows that

$$p = qp = pq. \tag{6.18}$$

Hence

$$pp^*p = pqp = p^2. \tag{6.19}$$

Moreover, by transitivity, since $q - p \leq_\diamond q$, we have $q - p \leq_\diamond 1$. In particular,

$$(q-p)(q-p)^*(q-p) = (q-p)^2. \tag{6.20}$$

From Equations (6.18), (6.19) and (6.20), we deduce

$$p^2 + p^* = pp^*p + p^* = pp^* + p^*p.$$

Multiplying this identity by p on the left and on the right, and havind in mind Equation (6.19), we get

$$p^4 + p^2 = p^3 + p^3.$$

Equivalently,

$$p^2(1-p)^2 = 0.$$

From the last identity, and Equations (6.19) and (6.20) it is clear that

$$0 = p^2(1-p)^2q^2 = p^2(q-p)^2 = pp^*p(q-p)(q-p)^*(q-p).$$

By cancellation, we get $p(q-p) = 0$. That is $pp^*p = p^2 = pq = p$, which shows that $p \in \mathrm{Proj}(A)$, as claimed. ■

Next our aim is to characterize the maximal and minimal elements on a unital C*-algebra with respect to the diamond partial order.

Proposition 6.4.4 *Let A be a unital prime C*-algebra. The following conditions are equivalent:*

(1) $a \in A^\wedge$ and a is maximal with respect to the diamond partial order,

(2) $a \in A_l^{-1} \cup A_r^{-1}$.

Proof Let $a \in A^\wedge$. It is straightforward to see that

$$a \leq_\diamond a + (1 - aa^\dagger)x(1 - a^\dagger a),$$

for every $x \in A$. If we suppose that a is maximal with respect to "$\leq_\diamond$", this gives $(1 - aa^\dagger)x(1 - a^\dagger a) = 0$ for every $x \in A$. Since A is prime, it yields to $1 = aa^\dagger$ or $1 = a^\dagger a$.

Reciprocally, assume that $a \in A_l^{-1}$. Let $b \in A$ with $a \leq_\diamond b$. Then, $aa^*a = ab^*a$ and there exist $x, y \in A$ satisfying $a = bx = yb$. Being a left invertible, from the first identity we get $a^*a = b^*a = a^*b$. Multiplying by x on the right, we obtain $a^*ax = a^*bx = a^*a$ which, by *-cancellation, shows $ax = a$. Since $a \in A_l^{-1}$, this finally gives $x = 1$ and, hence, $a = b$. Similar considerations can be made if we suppose $a \in A_r^{-1}$. ■

Proposition 6.4.5 *Let A be a unital C*-algebra with essential socle. Then $\mathcal{F}_1(A) = \mathrm{Minimals}_{\leq_\diamond}(A \setminus \{0\})$.*

Proof Let us first show that for every $a \in A \setminus \{0\}$, there exists $u \in \mathcal{F}_1(A)$ such that $u \leq_\diamond a$.

Since A is semisimple and has essential socle, given $a \in A \setminus \{0\}$, there exists $w \in \mathcal{F}_1(A)$ such that $aw \neq 0$. Let $v = w(aw)^\dagger \in \mathcal{F}_1(A)$. Then av is a minimal projection in A. Set $u = ava$. Clearly, $uA \subset aA$ and $Au \subset Aa$. Moreover,

$$uu^*u = (ava)(ava)^*(ava) = avaa^*avava = (ava)a^*(ava) = ua^*u,$$

that is, $u \leq_\diamond a$.

To finish the proof, we show that, given $u, v \in \mathcal{F}_1(A)$, with $u \leq_\diamond v$ then $u = v$. Indeed, since $u \leq_{sp} v$ (and $u, v \in \mathcal{F}_1(A)$), it is clear that $uA = vA$ and $Au = Av$. In particular, $v = uz = wu$ for some $w, z \in A$. Accordingly, from $uu^*u = uv^*u$ we get $uu^*u = uu^*w^*u$. By *-cancellation, we get $u^*u = u^*w^*u = z^*u^*u$, and hence $u^* = z^*u^* = v^*$. That is, $u = v$. ∎

As we did in the previous section (see Theorem 6.3.24), we make use of the representation theory of C*-algebras to get the following result.

Proposition 6.4.6 *Let A be a unital prime C*-algebra with non zero socle and $a \in A$. The following conditions are equivalent:*

(1) $a \in A^{-1}$,

(2) For every $u \in \mathcal{F}_1(A)$, there exist non zero $x \in uA$ and $y \in Au$ such that $x, y \leq_\diamond a$.

Proof Notice that for every $a \in A^{-1}$ (even though A is non necessarily prime), and every $p \in \mathrm{Proj}(A)$, $pa \leq_\diamond a$ and $ap \leq_\diamond a$. In particular, for every $a \in A^{-1}$ and every $u \in \mathcal{F}_1(A)$, $uu^\dagger a \leq_\diamond a$ and $au^\dagger u \leq_\diamond a$. This proves that *(1)* $\Rightarrow$ *(2)*.

Reciprocally, assume that condition *(2)* is fulfilled. For any $u \in \mathcal{F}_1(A)$, there exist $x, y \in A$ such that $uxA \subset aA$ and $Ayu \subset Aa$. Consequently, $ux = az$ and $yu = wa$ for some $z, w \in A$. Therefore,

$$u = \tau(ux(ux)^\dagger)u = ux(ux)^\dagger u = a\left(z(ux)^\dagger u\right) \quad \text{and}$$

$$u = \tau((yu)^\dagger yu)u = u(yu)^\dagger yu = \left(u(yu)^\dagger w\right)a.$$

In particular, $\mathrm{ann}_l(a) \subseteq \mathrm{ann}_l(u)$ and $\mathrm{ann}_r(a) \subseteq \mathrm{ann}_r(u)$, for every $u \in \mathcal{F}_1(A)$. Since A has essential socle, we conclude that $\mathrm{ann}_l(a) = \{0\}$ and $\mathrm{ann}_r(a) = \{0\}$. That is, a is not a zero divisor. Fix e a minimal projection in A and let ρ denote the left regular representation on $\mathcal{B}(Ae)$. From $\mathrm{ann}_r(a) = \{0\}$ it is clear that $\rho(a)$ is injective. Moreover, given $ze \in Ae$, by hypothesis, there exists $w \in A$ such that $ze = awze = \rho(a)(wze)$. This shows that $\rho(a)$ is surjective, and hence $\rho(a)$ is invertible. That is, $a \in A^{-1}$. ∎

It is straightforward to show that for every unitary element u in a C*-algebra A, $a \leq_\diamond b$ if, and only if, $ua \leq_\diamond ub$, for every $a, b \in A$.

Proposition 6.4.7 *Let A be a unital C*-algebra, and $u \in A$.*

(1) If $u^*u = \lambda 1$, for some $\lambda \in \mathbb{R}^+$, then

$$a \leq_\diamond b \Rightarrow ua \leq_\diamond ub, \quad \textit{for every } a, b \in A.$$

(2) If $uu^* = \lambda 1$*, for some* $\lambda \in \mathbb{R}^+$*, then*

$$a \leq_\diamond b \Rightarrow au \leq_\diamond bu, \quad \textit{for every } a, b \in A.$$

Proof We only prove the first assertion (the second can be shown in a similar way). Suppose that $u^*u = \lambda 1$, and let $a, b \in A$ with $a \leq_\diamond b$. As $aA \subseteq bA$ obviously $uaA \subseteq ubA$, and since u is left invertible and $Aa \subseteq Ab$, we get $Aua \subseteq Aub$. Moreover,

$$(ua)(ua)^*(ua) = uaa^*u^*ua = \lambda uaa^*a = \lambda uab^*a = (ua)(ub)^*(ua),$$

which shows that $ua \leq_\diamond ub$. ■

We conclude this section by characterizing the scalar multiples of isometries and co-isometries in a unital prime C*-algebra with non zero socle.

Proposition 6.4.8 *Let* A *be a unital prime C*-algebra with non zero socle and* $u \in A^\wedge$.

(1) The condition

$$a \leq_\diamond b \Leftrightarrow au \leq_\diamond bu, \quad \textit{for every } a, b \in A,$$

implies that $uu^* = \lambda 1$*, with* $\lambda \in \mathbb{R}^+$.

(2) The condition

$$a \leq_\diamond b \Leftrightarrow ua \leq_\diamond ub, \quad \textit{for every } a, b \in A,$$

implies that $u^*u = \lambda 1$*, with* $\lambda \in \mathbb{R}^+$.

Proof As in the previous proposition we only need to prove the first assertion. Assume that

$$a \leq_\diamond b \Leftrightarrow au \leq_\diamond bu, \quad \text{for every } a, b \in A. \tag{6.21}$$

It is clear that $\mathrm{ann}_l(u) = \{0\}$. Since $u \in A^\wedge$, we conclude that u is right invertible, that is, $uu^\dagger = 1$.

Notice that,

$$u^\dagger p \leq_\diamond u^\dagger, \quad \text{for every } p \in \mathrm{Proj}(A).$$

Indeed, let $p \in \mathrm{Proj}(A)$. Then $p \leq_\diamond 1$. It is clear that $u^\dagger pA \subseteq u^\dagger A$ and since $u^\dagger$ is left invertible $Au^\dagger p \subseteq Au^\dagger$. Finally

$$(u^\dagger p)(u^\dagger p)^*(u^\dagger p) = u^\dagger p(u^\dagger)^* u^\dagger p,$$

gives $u^\dagger p \leq_\diamond u^\dagger$. In the same way,

$$u^\dagger - u^\dagger p = u^\dagger(1-p) \leq_\diamond u^\dagger, \quad \text{for every } p \in \mathrm{Proj}(A).$$

Let us apply the condition (6.21) with $a = u^\dagger p$ and $b = u^\dagger$. Therefore,

$$u^\dagger pu \leq_\diamond u^\dagger u. \tag{6.22}$$

Applying now the condition (6.21) with $a = u^\dagger - u^\dagger p$ and $b = u^\dagger$, we obtain

$$u^\dagger u - u^\dagger p u \leq_\diamond u^\dagger u. \tag{6.23}$$

Having in mind Proposition 6.4.3 and Equations (6.22) and (6.23), we conclude that $u^\dagger p u \in \mathrm{Proj}(A)$, for every $p \in \mathrm{Proj}(A)$. That is,

$$u^\dagger p u = u^* p (u^\dagger)^*,$$

for every $p \in \mathrm{Proj}(A)$. Multiplying this last identity by u on the left, and by u^* on the right, we deduce that

$$puu^* = uu^* p, \quad \text{ for every } \ p \in \mathrm{Proj}(A).$$

In particular, uu^* commutes with every minimal projection, and hence

$$xuu^* = uu^* x, \quad \text{ for every } \ x \in \mathrm{soc}(A).$$

Being $\mathrm{soc}(A)$ essential, uu^* lies in the center of A, $\mathrm{Z}(A)$. As A is prime, $\mathrm{Z}(A) = \mathbb{C}1$, that is, $uu^* = \lambda 1$, for some $\lambda \in \mathbb{R}^+$. ■

Remark 6.4.9 *Notice that the same conclusions hold when A is a unital C*-algebra with trivial center and either A is linearly spanned by its projections, or A has real rank zero.*

Linear maps presrving the diamond partial order

In the next proposition we show that every Jordan *-homomorphism preserves the diamond partial order in the set of all regular elements. It can be proved by using Remark 4.1.8 and Proposition 6.3.19.

Proposition 6.4.10 *Let A and B be C*-algebras. If $T : A \to B$ is a Jordan *-homomorphism, then*

$$a \leq_\diamond b \quad \textit{implies} \quad T(a) \leq_\diamond T(b), \quad \textit{for all } \ a, b \in A^\dagger.$$

We wonder now whether Jordan *-homomorphisms arise from linear maps preserving the diamond partial order. Recall that two elements a, b in a C*-algebra A are orthogonal ($a \perp b$) if, and only if, $a \leq_* (a + b)$. From Proposition 6.4.1 *(2)* it follows that for a regular element $a \in A$, if $a \perp b$, then $a \leq_\diamond (a + b)$. The following example shows that the reciprocal does not hold. Hence we cannot expect to apply the same orthogonality arguments used in Section 6.2 in order to describe linear maps between C*-algebras preserving the diamond partial order.

Example 6.4.11 *(A. Peralta, private communication) Let $A = M_2(\mathbb{C})$ and*

$$a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad u = \begin{pmatrix} 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{pmatrix}.$$

It is clear that a is a projection and u is a partial isometry in A. It can be checked that

$$au^*a = 0, \quad aA \subseteq (a+u)A \quad \textit{and} \quad Aa \subset A(a+u)$$

where

$$aA = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} : x, y \in \mathbb{C} \right\},$$

$$(a+u)A = \left\{ \begin{pmatrix} x + z/\sqrt{2} & y + t/\sqrt{2} \\ z/\sqrt{2} & t/\sqrt{2} \end{pmatrix} : x, y, z, t \in \mathbb{C} \right\},$$

$$Aa = \left\{ \begin{pmatrix} x & 0 \\ z & 0 \end{pmatrix} : x, z \in \mathbb{C} \right\} \quad \textit{and}$$

$$A(a+u) = \left\{ \begin{pmatrix} x & (x+y)/\sqrt{2} \\ z & (z+t)/\sqrt{2} \end{pmatrix} : x, y, z, t \in \mathbb{C} \right\}.$$

This shows that $a \leq_\diamond (a+u)$. However a and u are not orthogonal since $u^*a \neq 0$.

Our first main result in this topic partially uses similar arguments to those of Theorems 6.3.20 and 6.3.24.

Theorem 6.4.12 *Let A and B be unital C^*-algebras with essential socle. Assume that B is prime. Let $T : A \to B$ be a surjective linear map and $h = T(1)$. The following conditions are equivalent:*

(1) $a \leq_\diamond b \Leftrightarrow T(a) \leq_\diamond T(b)$, for every $a, b \in A$,

*(2) $hh^* = h^*h = \lambda 1$, with $\lambda \in \mathbb{R}^+$, and $T = hS$, where $S : A \to B$ is either a *-isomorphism or a *-anti-isomorphism.*

Proof We only need to prove that *(1)* $\Rightarrow$ *(2)*, since the converse is straightforward. Suppose then that

$$a \leq_\diamond b \Leftrightarrow T(a) \leq_\diamond T(b), \qquad \text{for every } a, b \in A.$$

Notice that T is injective: if $T(x) = 0$, then $T(x) \leq_\diamond T(0)$, which by assumption, gives that $x \leq_\diamond 0$, and finally $x = 0$.

We claim that $T(\mathcal{F}_1(A)) = \mathcal{F}_1(B)$. Indeed, pick $u \in \mathcal{F}_1(A)$. From Proposition 6.4.5 there exists $T(v) \in \mathcal{F}_1(B)$ such that $T(v) \leq_\diamond T(u)$. By hypothesis we have $v \leq_\diamond u$. As $u \in \mathcal{F}_1(A)$, and $v \neq 0$, Proposition 6.4.5 implies that $v = u$. That is, $T(v) = T(u)$, which shows that $T(u) \in \mathcal{F}_1(B)$. The same arguments applied to T^{-1} gives $T(\mathcal{F}_1(A)) = \mathcal{F}_1(B)$. From Lemma 6.3.17, the maximal linear subspaces of $\mathrm{soc}(A)$ consisting of elements of rank at most one are either of the form uA or Au, for some $u \in \mathcal{F}_1(A)$. Therefore, $T(uA), T(Au) \in \{T(u)B, BT(u)\}$, for every $u \in \mathcal{F}_1(A)$.

Next we prove that T preserves invertibility. For this purpose, take $a \in A^{-1}$. Given $T(u) \in \mathcal{F}_1(B)$, by Proposition 6.4.6, there exist non zero elements $x_0 \in uA$ and

$y_0 \in Au$, such that $x_0, y_0 \leq_\diamond a$. If $T^{-1}(T(u)B) = uA$, take $x = x_0$. Otherwise, take $x = y_0$. Then $T(x) \in T(u)B$ and $T(x) \leq_\diamond T(a)$. Similarly, we find $T(y) \in BT(u)$ with $T(y) \leq_\diamond T(a)$. By Proposition 6.4.6, $T(a) \in B^{-1}$. In particular $h = T(1) \in B^{-1}$.

Let us define the linear mapping $S : A \to B$ as $S(x) = h^{-1}T(x)$ for every $x \in A$. It is clear that S is unital, bijective and preserves invertibility. By Theorem 1.1.4, S is a Jordan isomorphism. Since B is prime, we known that S is either an isomorphism or an anti-isomorphism. We may assume, without loss of generality, that S is an isomorphism. Then

$$T(xy) = T(x)h^{-1}T(y), \quad \text{for all}\, x, y \in A.$$

Let u be a unitary element in A. It is clear that $au \leq_\diamond bu$ if, and only if, $a \leq_\diamond b$. By hypothesis,

$$T(a) \leq_\diamond T(b) \Leftrightarrow T(au) \leq_\diamond T(bu) \Leftrightarrow T(a)h^{-1}T(u) \leq_\diamond T(b)h^{-1}T(u).$$

Taking into account Proposition 6.4.8, we conclude that $S(u)S(u)^* = \lambda 1$, with $\lambda \in \mathbb{R}^+$. As $S(u) \in B^{-1}$, it follows that $S(u)S(u)^* = S(u)^*S(u) = \lambda 1$. In particular, $S(u)$ is normal, for every unitary element $u \in A$. Consequently, as S is a unital Jordan homomorphism, it follows that

$$||S(u)|| = r(S(u)) = r(u) = 1, \quad \text{for every unitary element } u \in A.$$

This shows that S is selfadjoint (see [126, Corollary 2]).

Finally, $T(x) = hS(x)$, for every $x \in A$, where S is either a *-isomorphism or a *-anti-isomorphism. Since T and S both preserve the diamond partial order, we have

$$T(a) \leq_\diamond T(b) \Leftrightarrow a \leq_\diamond b \Leftrightarrow S(a) \leq_\diamond S(b) \Leftrightarrow h^{-1}T(a) \leq_\diamond h^{-1}T(b).$$

By Proposition 6.4.8, h^{-1} is a scalar multiple of an isometry and, hence, h is a scalar multiple of a unitary element. ■

The next corollary can be obtained directly from Theorem 6.4.12 and the well-known structure of surjective linear isometries of $\mathcal{B}(H)$.

Corollary 6.4.13 *Let H be a complex Hilbert space. If $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ is a surjective linear map that preserves the diamond partial order in both directions, then there are unitary operators U, V on H, and $\lambda \in \mathbb{R}^+$, such that Φ is either of the form*

$$\Phi(A) = \lambda UAV \qquad \textit{for all } A \in \mathcal{B}(H),$$

or of the form

$$\Phi(A) = \lambda UA^{tr}V \qquad \textit{for all } A \in \mathcal{B}(H).$$

In the last theorem we consider linear maps preserving the diamond partial order on a real rank zero C*-algebra under few additional conditions involving the image of the identity.

Theorem 6.4.14 *Let A and B be unital C^*-algebras. Assume that A has real rank zero. Let $T : A \to B$ be a bounded linear map satisfying that*

$$a \leq_\diamond b \quad \text{implies} \quad T(a) \leq_\diamond T(b), \quad \text{for all } a, b \in A^\wedge.$$

The following assertions hold.

*(1) If $T(1) \in Proj(B)$ then T is a Jordan *-homomorphism.*

*(2) If $T(A) \cap B^{-1}$ and $T(1)$ is a partial isometry then T is a Jordan *-homomorphism multiplied by a unitary element.*

Proof Notice that $p \leq_\diamond 1$ and $1 - p \leq_\diamond 1$, for every $p \in \mathrm{Proj}(A)$. Therefore,

$$T(p) \leq_\diamond T(1) \quad \text{and } T(1) - T(p) \leq_\diamond T(1), \quad \text{for every } \ p \in \mathrm{Proj}(A). \tag{6.24}$$

In order to prove *(1)* assume that $T(1) \in \mathrm{Proj}(B)$. Proposition 6.4.3 and (6.24) allow us to conclude that $T(p) \in \mathrm{Proj}(B)$, for every $p \in \mathrm{Proj}(A)$. Therefore T is a Jordan *-homomorphism.

Now assume that $T(A) \cap B^{-1}$ and that $T(1)$ is a partial isometry. Since $T(p) \leq_\diamond T(1)$, in particular, $T(p) \in T(1)B \cap BT(1)$ for every $p \in \mathrm{Proj}(A)$. Moreover, $T(1)B$ and $BT(1)$ are closed in view of the regularity of $T(1)$. As T is linear and bounded, and every selfadjoint element in A can be approximated by linear combinations of mutually orthogonal projections, we conclude that $T(A) \subseteq T(1)B \cap BT(1)$. This fact together with $T(A) \cap B^{-1}$, imply that $T(1) \in B^{-1}$, and therefore, $T(1)$ is unitary.

Let $S : A \to B$ be the linear mapping given by $S(x) = T(1)^*T(x)$, for all $x \in A$. Hence $T(x) = T(1)S(x)$, for all $x \in A$. Taking into account that T preserves the diamond partial order and $T(1)$ is unitary, it is clear that S is a unital, bounded, linear mapping preserving the diamond partial order. As consequence, S preserves projections and hence it is a Jordan *-homomorphism (Lemma 3.3.2). ■

Bibliography

[1] J. Alaminos, M. Brešar, J. Extremera, A. R. Villena, *Maps preserving zero products*, Studia Math. **193** (2009), 131-159.

[2] J. Alaminos, J. Extremera, A. R. Villena, *Approximately spectrum-preserving maps*, J. Funct. Anal. **261** (2011), 233-266.

[3] J. Alaminos, J. Extremera, A. R. Villena, *Spectral preservers and approximate spectral preservers on operator algebras*, Private Communication.

[4] W. Arendt, *Spectral properties of Lamperti operators*, Indiana Univ. Math. J. **32** (1983), 199-215.

[5] E. Artin, *Geometric Algebra*, Interscience Publishers, New York, 1957.

[6] B. Aupetit, *A Primer on Spectral Theory*, Springer-Verlag (1991).

[7] B. Aupetit, *Spectrum-preserving linear mappings between Banach algebras or Jordan-Banach algebras*, J. London Math. Soc. **62** (2000), 917-924.

[8] J. K. Baksalary, J. Hauke, *A further algebraic version of Cochran's theorem and matrix partial orderings*, Lin. Alg. Appl. **127** (1990), 157-169.

[9] B. A. Barnes, G. J. Murphy, M. R. F. Smyth, T. T. West, *Riesz and Fredholm theory in Banach algebras*, Pitman, London (1982).

[10] M. Bendaoud, M. Sarih, *Additive maps preserving the ascent and descent of operators*, Lin. Alg. Appl. **431** No. 10 (2009), 1740-1744.

[11] M. Bendaoud, A. Bourhim, M. Burgos, M. Sarih, *Linear maps preserving Fredholm and Atkinson elements in C*-algebras*, Lin. Mult. Alg. **57** No. 8 (2009), 823-838.

[12] A. Ben-Israel, T. T. E. Greville, *Generalized inverses: Theory and Applications*, Springer-Verlag, New York (2003).

[13] A. Bjerhammar, *Application of calculus of matrices to method of least squares with special reference to geodetic calculations*, Trans. Roy. Inst. Tech. Stockholm **49** (1951), 36 pp.

[14] B. Blackadar, *Projections in C*-algebras*, C*-algebras:1943-1993. A Fifty Year Celebration (1994), 131-150.

[15] B. Blackwood, S. K. Jain, K. M. Prasad, A. K. Srivastava, *Shorted operators relative to a partial order in a regular ring*, Comm. Algebra **37** No. 11 (2009), 4141-4152.

[16] N. Boudi, M. Mbekhta, *Additive maps preserving strongly generalized inverses*, J. Operator Theory **64** (2010), 117-130.

[17] A. Bourhim, M. Burgos, *Linear maps preserving regularity in C*-algebras*, Illinois J. Math. **53** No. 3 (2009), 899-914.

[18] A. Bourhim, M. Burgos, V. S. Shulman, *Linear maps preserving the minimum and reduced minimum moduli*, J. Funct. Anal. **258** (2010), 50-66.

[19] M. Brešar, A. Fošner, P. Šemrl, *A note on invertibility preservers on Banach algebras*, Proc. Amer. Math. Soc. **131** (2003), 3833-3837.

[20] M. Brešar, P. Šemrl, *Finite rank elements in Banach algebras*, Studia Math. **128** (1998), 287-298.

[21] L. G. Brown, G. K. Pedersen, *C*-algebras of real rank zero*, J. Funct. Anal. **99** (1991), 131-149.

[22] L. G. Brown, G. K. Pedersen, *On the geometry of the unit ball of a C*-algebra*, J. Reine Angew. Math. **469** (1995), 113-147.

[23] C. Bu, *Linear maps preserving Drazin inverses of matrices over fields*, Lin. Alg. Appl. **396** (2005), 159-173.

[24] L. J. Bunce, C.-H. Chu, *Compact operations, multipliers and Radon-Nikodym property in JB*-triples*, Pacific J. Math. **153** (1992), 249-265.

[25] M. Burgos, *Orthogonality preserving linear maps on C*-algebras with nonzero socles*, J. Math. Anal. Appl. **401** (2013), 479-487.

[26] M. Burgos, J. Garcés, A. M. Peralta, *Automatic continuity of biorthogonality preservers between compact C*-algebras and von Neumann algebras*, J. Math. Anal. Appl. **376** (2011), 221-230.

[27] M. Burgos, F.J. Fernández-Polo, J.J. Garcés, A.M. Peralta, *Orthogonality preservers revisited*, Asian-Eur. J. Math. **2** (2009), 387-405.

[28] María Burgos, A. C. Márquez-García, A. Morales-Campoy, *Linear maps preserving strongly Moore-Penrose invertibility*, Oper. Mat. **6**, No. 4 (2012), 819-831.

[29] María Burgos, A. C. Márquez-García, A. Morales-Campoy, *Strongly preserver problems in Banach algebras and C*-algebras*, Lin. Alg. Appl. **437** No. 5 (2012) 1183-1193.

[30] M. Burgos, A. C. Márquez-García, A. Morales-Campoy, *Approximate Preservers on Banach Algebras and C*-Algebras*, Abs. Appl. Anal. Volume 2013 (2013), Article ID 757646.

[31] M. Burgos, A. C. Márquez-García, A. Morales-Campoy, *Determining Jordan (triple) homomorphisms by invertibility preserving conditions*, Lin. Mult. Alg., **63** No. 7 (2015), 1294-1308.

[32] M. Burgos, A. C. Márquez-García, A. Morales-Campoy, *Minus partial order and linear preservers*, submitted.

[33] M. Burgos, A. C. Márquez-García, A. Morales-Campoy, *Maps preserving the diamond partial order*, submitted.

[34] M. Burgos, A. C. Márquez-García, P. Patrício, *On mappings preserving the sharp and star orders*, Lin. Alg. Appl. **483** (2015), 268-292.

[35] M. Burgos, A. C. Márquez-García, A. Morales-Campoy, A. Peralta, *Linear maps between C*-algebras preserving extreme points and strongly linear preservers*, submitted.

[36] María Burgos, Francisco J. Fernández-Polo, Jorge J. Garcés, Juan Martínez Moreno, Antonio M. Peralta, *Orthogonality preservers in C*-algebras, JB*-algebras and JB*-triples*, J. Math. Anal. Appl. **348** No. 1 (2008), 220-233.

[37] M. Burgos, A. Kaidi, A. Morales, A. M. Peralta, M. Ramírez, *von Neumann regularity and quadratic conorms in JB*-triples and C*-algebras*, Acta Mathematica Sinica **24** (2008), 185-200.

[38] M. Burgos, J. Garcés, A. Peralta, *Automatic continuity of biorthogonality preservers between compact C*-algebras and von Neumann algebras*, J. Math. Anal. Appl. **376** (2011), 221-230.

[39] M. Burgos, J. Sánchez-Ortega, *On maps preserving zero products*, Lin. Mult. Alg. **61** No. 3 (2013), 323-335.

[40] S.L. Campbell, *Recent Applications of Generalized Inverses*, Academic Press, New York (1982).

[41] C. Cao, X. Zhang, *Linear preservers between matrix modules over connected commutative rings*, Lin. Alg. Appl. **397** (2005), 355-366.

[42] M. A. Chebotar, W.-F. Ke, P. H. Lee, *Maps characterized by action on zero products*, Pacific J. Math. **216** (2004), 217-228,

[43] M. A. Chebotar, W.-F. Ke, P.-H. Lee, L.-S. Shiao, *On maps preserving products*, Canad. Math. Bull. **48** (2005), 355-396.

[44] M. D. Choi, D. Hadwin, E. Nordgren, H. Radjavi, P. Rosenthal, *On Positive Linear Maps Preserving Invertibility*, J. Funct. Anal. **59** (1984), 462-469.

[45] M. D. Choi, A. A. Jafarian, H. Radjavi, *Linear maps preserving commutativity*, Lin. Alg. Appl. **87** (1987), 227-241.

[46] Ch.-H. Chu, *Jordan Structures in Geometry and Analysis*, Cambridge Tracts in Mathematics 190, Cambridge Universiy Press, Cambridge, 2012.

[47] J. Cui, *Additive Drazin inverse preservers*, Lin. Alg. Appl. **426** (2007), 448-453.

[48] J. L. Cui, J. C. Hou, *Linear maps between Banach algebras compressing certain spectral functions*, Rocky Mountain J. Math. **34** No. 5 (2004), 565-585.

[49] D. S. Djordjević, D. S. Rakić, J. Marovt, *Minus partial order in Rickart rings*, IMFM Preprint series **51** (2013) 1191.

[50] G. Dolinar, A. Guterman, J. Marovt, *Automorphisms of $\mathcal{K}(H)$ with respect to the star partial order*, Oper. Mat. **7** No. 1 (2013), 225-239.

[51] G. Dolinar, A. Guterman, J. Marovt, *Monotone transformations on $\mathcal{B}(H)$ with respect to the left-star and the right-star partial order*, Math. Ineq. Appl. **17** No. 2 (2014), 573-589.

[52] M.P. Drazin, *Pseudo-inverse in associative rings and semigroups*, Amer. Math. Monthly **65** (1958), 506-514.

[53] M. P. Drazin, *Natural structures on semigroups with involution*, Bull. Amer. Nath. Soc. **84** (1978), 139-141.

[54] H. Essannouni, A. Kaidi, *Le théoreme de Hua pour les algebres artiniennes simples*, Lin. Alg. Appl. **297** (1999), 9-22.

[55] A. Férnandez López, E. García Rus, E. Sánchez Campos, M. Siles Molina, *Strong regularity and generalized inverses in Jordan systems*, Comm. Alg. **20** No. 7 (1992), 1917-1936.

[56] F. J. Fernandez-Polo, J. Martínez Moreno, A. M. Peralta, *Contractive Perturbations in JB*-triples*, J. London Math. Soc. **85** No. 2 (2012), 349-364. doi:10.1112/jlms/jdr048

[57] I. Fredholm, *Sur une classe d'équations fonctionelles*, Acta Math. **27**, 365-390.

[58] Y. Friedman, B. Russo, *Structure of the predual of a JBW*-triple*, J. Reine u. Angew. Math. **356** (1985), 67-89.

[59] G. Frobenius, *Uber die Darstellung der endlichen Gruppen durch Linear Substitutionen*, Sitzungsber Deutsch. Akad. Wiss., Berlin (1897) 994-1015.

[60] A. Gleason, *A characterization of maximal ideals*, J. Analyse Math. **19** (1967), 171-172.

[61] J. A. Green, *On the structure of semigroups*, Ann. Math. **54** (1951), 163-172.

[62] A. Guterman, *Linear preservers for Drazin star partial order*, Comm. Alg. **29** (2001), 3905-3917.

[63] A. Guterman, *Monotone additive transformations on matrices*, Mat. Zametki **81** (2007), 681-692.

[64] R. E. Hartwig, *How to partially order regular elements*, Math. Japon. **25** (1980), 1-13.

[65] R. Harte, M. Mbekhta, *On generalized inverses in C*-algebras*, Studia Math. **103** (1992), 71-77.

[66] R. Harte, *M. Mbekhta, Generalized inverses in C*-algebras II*, Studia Math. **106** (1993), 129-138.

[67] S. Heinrich, *Ultraproducts in Banach space theory*, J. Reine Angew. Math. **313** (1980), 72-104.

[68] I. N. Herstein, *Jordan homomorphisms*, Trans. Amer. Math. Soc. **81** (1956), 331-341.

[69] I. N. Herstein, *Topics in ring theory* ,Univ. of Chicago Press, Chicago, 1969.

[70] L. K. Hua, *On the automorphisms of a sfield*, Proc. Nat. Acad. Sci. U.S.A., **35** (1949), 386-389.

[71] W. A. Hurwitz, *On the pseudo-resolvent to the kernel of an integral equation*, Trans. Amer. Math. Soc. **13** (1912), 405-418.

[72] N. Jacobson, *Structure and representation of Jordan algebras*, Amer. Math. Coll. Publ. **39**, Providence, Rhode Island, 1968.

[73] A. A. Jafarian, A. R. Sourour, *Spectrum-preserving linear maps*, J. Funct. Anal. **66** (1986), 255-261.

[74] F. Jamjoom, A. Peralta, A. Siddiqui, H. Tahlawi, *Approximation and convex decomposition by extremals and the λ-function in JBW*-triples*, Quart. J. Math. **66** (2015), 583-603.

[75] F. B. Jamjoom, A. A. Siddiqui, H. M. Tahlawi, *On the geometry of the unit ball of a JB*-triple*, Abs. Appl. Anal. **2013**, Article ID 891249, 8 pages, 2013.

[76] B. E. Johnson, *Approximately multiplicative functionals*, J. Lond. Math. Soc. **34** (1986), 489-510.

[77] B. E. Johnson, *Approximately multiplicative maps between Banach algebras*, J. London Math. Soc., **s2-37** No. 2 (1988), 294-316.

[78] R. V. Kadison, *Isometries of operator algebras*, Ann. Math. **56** No. 3 (1951), 325-338.

[79] J.P. Kahane, W. Żelazko, *A characterization of maximal ideals in commutative Banach algebras*, Studia Math. **29** (1968), 339-343.

[80] W. Kaup, *A Riemann Mapping Theorem for bounded symmentric domains in complex Banach spaces*, Math. Z. **183** (1983), 503-529.

[81] W. Kaup, *On spectral and singular values in JB*-triples*, Proc. Roy. Irish Acad. Sect. A **96** (1996), 95-103.

[82] W. Kaup, *On Grassmannians associated with JB*-triples*, Math. Z. **236** (2001), 567-584.

[83] J. J. Koliha, *A generalized Drazin inverse*, Glasgow Math. J. **38** (1996), 367-381.

[84] E. Kreyszig, *Introductory Functional Analysis with Applications*, John Wiley & Sons Inc., 1978.

[85] S. H. Kulkarni, D. Sukumar, *Almost multiplicative functions on commutative Banach algebras*, Studia Math. **152** (2002), 187-199.

[86] B. Kuzma, *Additive idempotence preservers*, Lin. Alg. Appl. **355** (2002), 103-117.

[87] L.E. Labuschagne, V. Mascioni, *Linear maps between C* -algebras whose adjoint preserve extreme points of the dual ball*, Adv. Math. **138** (1) (1998), 15-45.

[88] L. Lebtahi, P. Patricio, N. Thome, *The diamond partial order in rings*, Lin. Mult. Alg. **62** No. 3 (2014), 386-395.

[89] P. Legiša, *Automorphisms of M_n, partially ordered by rank subtractivity ordering*, Lin. Alg. Appl. **389** (2004), 147-158.

[90] Y.-F. Lin, M. Mathieu, *Jordan isomorphism of purely infinite C*-algebras*, Quart. J. Math. **58** (2007), 249-253.

[91] Y.-F. Lin, T.-L. Wong, *A note on 2-local maps*, Proc. Edinb. Math. Soc. **49** (2006), 701-708.

[92] L. Lindeboom, H. Raubenheimer, *On regularities and Fredholm theory*, Czechoslovak Math. J. **52** (2002), 565-574.

[93] O. Loos, *Jordan Pairs*, Lecture Notes in Math. **460**, Springer-Verlag, Berlin, 1975.

[94] S. Maeda, *On the lattice of projections of a Baer *-ring*, J . Sci. Hiroshima Univ. Ser. A **22** (1958), 75-88.

[95] L. W. Marcoux, *On the linear span of the projections in certain simple C*-algebras*, Indiana Univ. Math. J. **51** (2002), 753-771.

[96] L. W. Marcoux, *Sums of small number of commutators*, J. Operator Theory **56** (2006), 111-142.

[97] L. W. Marcoux, G.J. Murphy, *Unitarily-invariant linear spaces in C*-algebras*, Proc. Amer. Math. Soc. **126** (2002), 3597-3605.

[98] J. Marovt, D. S. Rakić, D. S. Djordjević, *Star, Left-star, and right-star partial orders in Rickart *-rings*, Lin. Mult. Alg. **63** No. 2 (2015), 343-365.

[99] X. Mary, *On generalized inverses and Green's relations*, Lin. Alg. Appl. **343** (2011), 1836-1844.

[100] X. Mary, P. Patricio, *Generalized inverses modulo $\mathcal{H}$ in semigroups and rings*, Lin. Mult. Alg. **61** No. 8 (2013), 1130-1135.

[101] V. Mascioni, L. Molnár, *Linear maps on factors which preserve the extreme points of the unit ball*, Canad. Math. Bull. **41** (1998), 434-441.

[102] M. Mathieu, *Elementary operators on prime C*-algebras II*, Glasgow Math. J. **30** (1988), 275-284.

[103] M. Mbekhta, *Linear maps preserving the generalized spectrum*, Extracta Math. **22** (2007) 45-54.

[104] M. Mbekhta, *A Hua type theorems and linear preserver strongly problems*, Proc. Roy. Irish Acad. **109A** No. 2 (2009), 109-121.

[105] M. Mbekhta, *Linear maps preserving the minimum and surjectivity moduli of operators*, Oper. Mat. **4** No. 4 (2010), 511-518.

[106] M. Mbekhta, *A survey on linear (additive) preserver problems*, Advanced Courses of Mathematical Analysis IV (2011), 174-195.

[107] M. Mbekhta, P. Šemrl, *Linear maps preserving semi-Fredholm operators and generalized invertibility*, Lin. Mult. Alg. **57** (2009), 55-65.

[108] K. Meyberg, *Von Neumann regularity in Jordan triple systems*, Arch Math. (Basel) **23** (1972), 589-593.

[109] C.D. Meyer Jr., J.M. Shoaf, *Updatind finite Markov chains by using techniques of group inverse*, J. Statist. Comput. Simulation **11** (1980), 163-181.

[110] S. K. Mitra, *The minus partial order and the shorted matrix*, Lin. Alg. Appl. **83** (1986), 1-27.

[111] S. K. Mitra, *On Group Inverses and the Sharp Order*, Lin. Alg. Appl. **92** (1987), 17-37.

[112] S. K. Mitra, *Matrix partial orders through generalized inverses: unified theory*, Lin. Alg. Appl. **148** (1991), 237-263.

[113] S. K. Mitra, P. Bhimasankaram, S. B. Malik, *Matrix partiar orders, shorted operators and applications*, Hackensack, NJ World Scientific Publishing Company (2010).

[114] E. H. Moore, *On the reciprocal of a general algebraic matrix*, Bull. Amer. Math. Soc. **26** (1920), 394-395, (Abstract).

[115] V. Müller, *Spectral theory of linear operators and spectral systems in Banach algebras*, Operator Theory: Advances and Applications **139**, Birkhäuser Verlag, Basel, 2003.

[116] T. Oikhberg, A. M. Peralta, M. Ramírez, *Automatic continuity of M-norms on C*-algebras*, J. Math. Anal. Appl. **381** (2011), 799-811.

[117] M. Omladič, *On operators preserving the numerical range*, Lin. Alg. Appl. **135** (1990),

[118] P. G. Ovchinnikov, *Automorphisms of the poset of skew projections*, J. Funct. Anal. **115** No. 1 (1993), 184-189.

[119] T. Palmer, *Banach Algebras and the General Theory of *-Algebras Vol. 1*, Cambridge University Press, 1994.

[120] A. L. T. Paterson, A. M. Sinclair, *Characterization of isometries between C*-algebras*, J. London Math. Soc. **5** (1972), 755-761.

[121] C. Pearcy, D. Topping, *Sums of small numbers of idempotents*, Michigan Math. J. **14** (1967), 453-465.

[122] G. Pederser, N. Petersen, *Ideals in a C*-algebra*, Math. Scand. **27** (1970), 193-204.

[123] R. Penrose, *A generalized inverse for matrices*, Proceedings of the Cambridge Philosophical Society **51** (1955), 406-413.

[124] J. Puhl, *The trace of finite and nuclear elements in Banach algebras*, Czechoslovak Math. J. **28** No. 4 (1978), 656-676.

[125] D. S. Rakić, D. S. Djordjević, *Space pre-order and minus partial order for operators on Banach spaces*, Aequat. Math. **85** No. 3 (2013), 429-448.

[126] B. Russo, H.A. Dye,*A note on unitary operators in C*-algebras*, Duke Math. J. **33** (1966), 413-416.

[127] S. Sakai, *C*- and W*-algebras*, Springer-Verlag, Berlin-New York, 1971.

[128] P. Šemrl, *Hua's fundamental theorem of the geometry of matrices*, J. Alg. **272** (2004), 801-837.

[129] P. Šemrl, *Automorphisms of $\mathcal{B}(H)$ with respect to minus partial order*, J. Math. Anal. Appl. **369** No. 1 (2010), 205-213.

[130] B. Simeon, C. Fhrer, P. Rentrop, *The Drazin inverse in multibody system dynamics*, Numer. Math. **64** (1993), 521-539.

[131] A. R. Sourour, *Invertibility preserving linear maps on $\mathcal{L}(X)$*, Trans. Amer. Math. Soc. **348** (1996), 13-30.

[132] E. Størmer, On the Jordan Structure of C*-Algebras, *Trans. Amer. Math. Soc.*, **120** No. 3 (1965), 438-447.

[133] M. Takesaki, *Theory of operator algebras I*, Springer Verlag, New York, 1979.

[134] J. von Neumann, *Zur Algebra der Funktionaloperatoren und Theorie der normalen Operatoren*, Math. Ann. **102** (1930), 370-427.

[135] M. Wolff, *Disjointness preserving operators in C*-algebras*, Arch. Math. **62** (1994), 248-253.

[136] X. Zhang, C. Cao, C. Bu, *Additive maps preserving M-P inverses of matrices over fields*, Lin. Mult. Alg. **46** (1999), 199-211.