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**TESI DOCTORAL**

# Uniform isochronous centers of degrees 3 and 4 and their perturbations

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*To my family.*

# Acknowledgements

Almost a century ago, my grandparents literally crossed half the world seeking for a better life in Brazil. Many years later, my parents decided to settle into the Brazilian Amazon region, more than 2,000 km away from their relatives and friends. In my journey I have always been doing my best to mirror that of my ancestors, who bravely pursued a better life no matter how far the distance or how high the hurdles.

The completion of this thesis is an important step in my journey. A journey in which I continue to strive for the best and push myself beyond my limits. Even as I stop to acknowledge the far I have come, and to take delight in this stage in life, I am inspired to keep moving toward the next challenge.

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*On life's vast ocean diversely we sail.*
*Reason's the card, but passion is the gale.*

- Pope

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# Chapter 1

# Introduction and statements of the results

Christian Huygens is credited with being one of the first scholars to investigate isochronous systems in the XVII century. He studied the cycloidal pendulum, which has isochronous oscillations, in opposition to the monotonicity of the period of the usual pendulum. It is probably the first example of a nonlinear isochronous center. For more details see [24].

Isochronicity appears in a wide variety of physical phenomena. Furthermore it is important in stability theory, since a periodic solution in the region surrounding the center type singular point is Liapunov stable if and only if the neighboring periodic solutions have the same period. For more details on these topics see [16].

We say that $p \in \mathbb{R}^2$ is a *center* if it is a singular point of a planar differential system such that there is a neighborhood $U$ of $p$ where all the orbits of $U \backslash \{p\}$ are periodic. For every $q \in U \backslash \{p\}$ let $T(q)$ denote the period of the periodic orbit through $q$. When $T(q)$ is constant for all $q \in U \backslash \{p\}$ we say that $p$ is an *isochronous center*. The fact that $p$ is isochronous does not imply that the angular velocity of the vector $\overrightarrow{pq}$ is the same for all periodic orbits in $U \backslash \{p\}$. When this happens we say that $p$ is a *uniform isochronous center* or a *rigid center*.

The study of polynomial differential systems in $\mathbb{R}^2$ with a uniform isochronous center has increased in the last decades, see for instance [1, 12, 19, 21, 30] and the bibliography therein. The relevance of investigating these systems is due, on the one hand, to their importance in the general problem of isochronicity. Indeed, any analytic system with linear part $(-y, x)^t$ has an isochronous center at the origin if and only if it is possible to be transformed, by applying an analytic change of coordinates of the form $(x, y) \to (x + P(y^2), y + Q(x, y))$ into the system

$$\dot{x} = -y + x\ H(x, y), \quad \dot{y} = x + y\ H(x, y), \tag{1.1}$$

where $H$ is an analytic function and $H(0, 0) = 0$, for more details see [51, 2]. On the other hand, system (1.1) in polar coordinates $x = r\cos\theta, y = r\sin\theta$ is

$$\dot{r} = \sum_{k\geq 1} H_k(\cos\theta, \sin\theta) r^{k+1}, \quad \dot{\theta} = 1,$$

where each $H_k$ is the homogeneous part of degree $k$ of the function $H$. These systems can be written under the form of a generalized Abel differential equation

$$\frac{dr}{d\theta} = \sum_{k\geq 1} H_k(\cos\theta, \sin\theta) r^{k+1}. \tag{1.2}$$

Equation (1.2) provides information about system (1.1), and vice versa, since the constant solution $r = 0$ of (1.2) corresponds to the origin of (1.1), and the periodic solutions of (1.2) with $r > 0$ correspond to periodic orbits of (1.1), see [5, 2].

In this work we study the planar polynomial differential systems of degree 3 and 4 with a uniform isochronous center. We provide a classification for these systems with respect to the topological equivalence of their global phase portraits in the Poincaré disc. We also investigate the bifurcation of limit cycles from the uniform isochronous centers and from the periodic orbits surrounding these centers, both for continuous and discontinuous polynomial perturbations.

## 1.1 Preliminaries on the global phase portraits

The next result characterizes when a center is a uniform isochronous center.

**Proposition 1.1.** *Assume that a planar polynomial differential system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ of degree $n$ has a center at the origin of coordinates. Then this center is uniform isochronous if and only if by doing a linear change of variables and a rescaling of time it can be written as*

$$\dot{x} = -y + x\ f(x, y), \quad \dot{y} = x + y\ f(x, y), \tag{1.3}$$

*with $f(x, y)$ a polynomial in $x$ and $y$ of degree $n - 1$, $f(0, 0) = 0$.*

Proposition 1.1 is proved in section 2.2.

The classification of the global phase portraits in the Poincaré disc for the uniform isochronous centers of the polynomial differential systems of degrees 1 and 2, and some results regarding the uniform isochronous centers of degrees 3 and 4 are summarized in what follows. Without loss of generality we assume that the uniform isochronous center is at the origin of coordinates.

### 1.1.1 Uniform isochronous centers of degree 1

A linear differential system with a uniform isochronous center after a linear change of variables and a rescaling of time becomes $\dot{x} = -y, \quad \dot{y} = x$. In this case the uniform isochronous center is global. The corresponding phase portrait is shown in Figure 1.1.

### 1.1.2 Uniform isochronous centers of degree 2

The quadratic polynomial differential system with a uniform isochronous center after a linear change of coordinates and a rescaling of time can be written into the form $\dot{x} =$

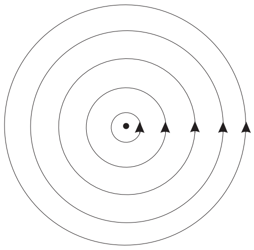


Figure 1.1: Phase portrait of the uniform isochronous center of degree 1.

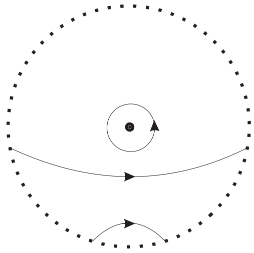


Figure 1.2: Phase portraits of the uniform isochronous center of degree 2.

$-y + x^2, \quad \dot{y} = x + xy$. The respective phase portrait for this system is shown in Figure 1.2.

This result was provided by Loud [45] in 1964. In his work Loud studied all the families of quadratic isochronous systems.

### 1.1.3 Uniform isochronous centers of degree 3

The following result, due to Collins [18] in 1997, also obtained by Devlin et al [20] in 1998, and by Gasull et al [25] in 2005, provides a characterization of the planar cubic polynomial differential systems with a uniform isochronous center.

**Theorem 1.2.** *A planar cubic polynomial differential system has a uniform isochronous center at the origin if and only if it can be written as*

$$\begin{aligned} \dot{x} &= -y + x(a_1x + a_2y + a_3x^2 + a_4xy - a_3y^2), \\ \dot{y} &= \quad x + y(a_1x + a_2y + a_3x^2 + a_4xy - a_3y^2), \end{aligned} \tag{1.4}$$

*and satisfies* $a_1^2a_3 - a_2^2a_3 + a_1a_2a_4 = 0$, $a_i \in \mathbb{R}$, $i = 1, \ldots, 4$.

The next result is due to Collins [18].

**Proposition 1.3.** *System* (1.4) *satisfying* $a_1^2a_3 - a_2^2a_3 + a_1a_2a_4 = 0$, *may be reduced to one of the following forms*

$$\dot{x} = -y(1 - x^2), \quad \dot{y} = x(1 + y^2), \tag{1.5}$$

$$\dot{x} = -y + x^2 + Ax^2y, \quad \dot{y} = x + xy + Axy^2. \tag{1.6}$$

*where* $A \in \mathbb{R}$.

Collins provided the phase portraits and the first integrals of the cubic uniform isochronous centers using (1.5) and (1.6) which present at most one parameter.

### 1.1.4 Uniform isochronous centers of degree 4

Algaba et al [4] in 1999, and Chavarriga et al [13] in 2001 independently provided the following characterization of quartic polynomial systems with a uniform isochronous center at the origin.

**Theorem 1.4.** *Consider* $f(x,y) = \sum_{i=1}^{3} f_i(x,y)$ *with* $f_i(x,y)$, $i = 1,2,3$ *homogeneous polynomials of degree* $i$, *and* $f_1^2 + f_2^2 \neq 0$, $f_3 \neq 0$ *such that* (1.3) *is a quartic polynomial differential system with a non–homogeneous nonlinear part. Then the only case of local analytic integrability in a small open neighborhood of the origin of system* (1.3) *is given, modulo a rotation, by*

$$\begin{aligned} \dot{x} &= -y + x(A_1x + B_2xy + C_1x^3 + C_3xy^2), \\ \dot{y} &= \quad x + y(A_1x + B_2xy + C_1x^3 + C_3xy^2). \end{aligned} \tag{1.7}$$

*where* $A_1, B_2, C_1, C_3 \in \mathbb{R}$.

# 1.2 Results on global phase portraits and on first integrals

We provide the global phase portraits in the Poincaré disc of all uniform isochronous centers of degree 3 and 4. We also provide the explicit expressions of the first integrals in the case of the uniform isochronous centers of degree 3.

## 1.2.1 Uniform isochronous centers of degree 3 and their first integrals

Collins [18] presented the global phase portraits and the first integrals for the uniform isochronous centers of degree 3 using systems (1.5) and (1.6). Therefore one needs to change the differential system (1.4) to these normal forms before applying Collins' results. Our results present the first integrals and the global phase portraits in the Poincaré disc for the uniform isochronous cubic centers in terms of all the parameters of system (1.4). These results have been published in [31].

In the next theorem, we present the first integrals for the uniform isochronous centers of degree 3 described by systems (1.4).

**Theorem 1.5.** *The first integrals* $H$ *of system* (1.4) *in polar coordinates* $x = r\cos\theta$, $y = r\sin\theta$ *are described in what follows.*

***Case 1:*** $\mathbf{a_1^2 - a_2^2 \neq 0}$.

***Subcase 1.1:*** $\mathbf{a_4 \neq 0}$.

***Subcase 1.1.1:*** $\mathbf{4a_4 \neq a_1^2 - a_2^2}$.

$$H = e^{-2\arctan\left[\frac{R+2a_4r(-a_2\cos\theta + a_1\sin\theta)}{RS}\right]} \left[\frac{a_4r^2}{R + r(a_2\cos\theta - a_1\sin\theta)(a_2a_4r\cos\theta - a_1a_4\sin\theta - R)}\right]^S,$$

*where* $R = a_1^2 - a_2^2$, $S = \sqrt{4a_4/R - 1}$.

*In case of a negative square root, we have a complex first integral and therefore both its real and imaginary parts are also first integrals, if not null.*

***Subcase 1.1.2:*** $\mathbf{4a_4 = a_1^2 - a_2^2}$.

$$H = \frac{re^{\frac{2}{2-a_2 r\cos\theta + a_1 r\sin\theta}}}{2 - a_2 r\cos\theta + a_1 r\sin\theta}.$$

***Subcase 1.2:*** $\mathbf{a_4 = 0}$.

$$H = \frac{r}{1 - a_2 r\cos\theta + a_1 r\sin\theta}.$$

***Case 2:*** $\mathbf{a_1^2 - a_2^2 = 0}$.

***Subcase 2.1:*** $\mathbf{a_2 = a_1}$.

***Subcase 2.1.1:*** $\mathbf{a_1 = 0}$.

$$H = \frac{r^2}{1 - a_4 r^2\cos^2\theta + a_3 r^2\sin(2\theta)}.$$

***Subcase 2.1.2:*** $\mathbf{a_1 \neq 0,\ a_4 = 0}$.

***Subcase 2.1.2.1:*** $\mathbf{a_3(a_1^2 + 4a_3) \neq 0}$.

$$\begin{aligned} H =& e^{-2\arctan\left[\frac{a_1 + 2a_3 r(\cos\theta - \sin\theta)}{a_1 R}\right]} \\ & \left[\frac{a_3 r^2(\sin(2\theta) - 1)}{(\cos\theta - \sin\theta)^2[1 + a_1 r(\sin\theta - \cos\theta) + a_3 r^2(\sin(2\theta) - 1)]}\right]^R, \end{aligned}$$

*where* $R = \sqrt{-1 - 4a_3/a_1^2}$.

***Subcase 2.1.2.2:*** $\mathbf{a_3 = 0}$.

$$H = \frac{r}{1 - a_1 r(\cos\theta - \sin\theta)}.$$

***Subcase 2.1.2.3:*** $\mathbf{a_3 = -a_1^2/4}$.

$$H = \frac{re^{\frac{2}{2 - a_1 r(\cos\theta - \sin\theta)}}}{2 - a_1 r(\cos\theta - \sin\theta)}.$$

***Subcase 2.2:*** $\mathbf{a_2 = -a_1}$.

***Subcase 2.2.1:*** $\mathbf{a_1 = 0}$. *This case becomes the subcase 2.1.1.*

***Subcase 2.2.2:*** $\mathbf{a_1 \neq 0,\ a_4 = 0}$.

***Subcase 2.2.2.1:*** $\mathbf{a_3(4a_3 - a_1^2) \neq 0}$.

$$H = \frac{e^{\frac{1}{R}\left[-2\arctan\left(\frac{a_1 + 2a_3 r(\sin\theta + \cos\theta)}{a_1 R}\right) + R\,\mathrm{arctanh}(\tan\theta)\right]} a_3 r^2(\sec(2\theta) + \tan(2\theta))}{1 + a_1 r(\sin\theta + \cos\theta) + a_3 r^2(1 + \sin(2\theta))},$$

*where* $R = \sqrt{4a_3/a_1^2 - 1}$.

***Subcase 2.2.2.2:*** $\mathbf{a_3 = 0}$.

$$H = \frac{r}{1 - a_1 r(\cos\theta - \sin\theta)}.$$

***Subcase 2.2.2.3:*** $\mathbf{a_3 = a_1^2/4}$.

$$H = \frac{r}{e^{1 + \frac{1}{2}a_1 r(\cos\theta + \sin\theta)}\left(1 + \frac{1}{2}a_1 r(\cos\theta + \sin\theta)\right)}.$$

Theorem 1.5 is Theorem 3.3 presented in chapter 3, and it is proved in section 3.3.1 of that chapter.

In the next result, we classify the global phase portraits in the Poincaré disc of the uniform isochronous centers of degree 3 described by systems (1.4) in terms of all their parameters.

**Theorem 1.6.** *The global phase portrait in the Poincaré disc of the differential system* (1.4) *is topologically equivalent to one of the four phase portraits presented in Figure 1.3.*

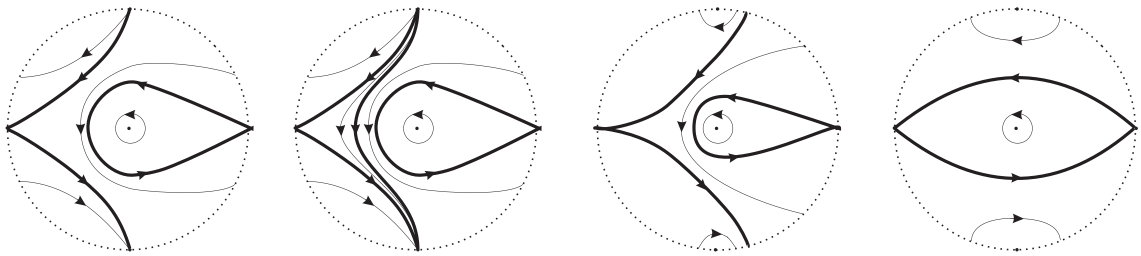



Figure 1.3: Phase portraits of cubic uniform isochronous centers.

*More precisely, the global phase portrait of* (1.4) *is topologically equivalent to either the phase portrait* $(a_1)$ *or* $(a_2)$ *of Figure 1.3 if one of the following conditions holds*

- $a_1a_2 \neq 0$, *and* $a_4(a_1^2 - a_2^2) > 0$, *and* $a_4 \leq (a_1^2 - a_2^2)/4$;
- $a_2 = -a_1 \neq 0$, *and* $0 < a_3 \leq a_1^2/4$, *and* $a_4 = 0$;
- $a_2 = a_1 \neq 0$, *and* $-a_1^2/4 \leq a_3 < 0$, *and* $a_4 = 0$;
- $a_1 = 0$, *and* $a_2 \neq 0$, *and* $-a_2^2/4 \leq a_4 < 0$;
- $a_1 \neq 0$, *and* $a_2 = 0$, *and* $0 < a_4 \leq a_1^2/4$;

*to the phase portrait (b) if one of the following conditions holds*

- $a_1a_2 \neq 0$, *and* $a_4(a_1^2 - a_2^2) > 0$, *and* $a_4 > (a_1^2 - a_2^2)/4$;
- $a_2 = -a_1 \neq 0$, *and* $a_3 > a_1^2/4$, *and* $a_4 = 0$;
- $a_2 = a_1 \neq 0$, *and* $a_3 < -a_1^2/4$, *and* $a_4 = 0$;
- $a_1 = 0$, *and* $a_2 \neq 0$ *and* $a_4 < -a_2^2/4$;
- $a_1 \neq 0$, *and* $a_2 = 0$ *and* $a_4 > a_1^2/4$;

*to the phase portrait (c) if one of the following conditions holds*

- $a_1a_2 \neq 0$, *and* $a_4(a_1^2 - a_2^2) < 0$;
- $a_2 = -a_1 \neq 0$, *and* $a_3 < 0$, *and* $a_4 = 0$;
- $a_2 = a_1 \neq 0$, *and* $a_3 > 0$, *and* $a_4 = 0$;

- $a_1 = 0$, *and* $a_2 \neq 0$, *and* $a_4 > 0$;
- $a_1 \neq 0$, *and* $a_2 = 0$, *and* $a_4 < 0$;
- $a_1 = a_2 = 0$.

The cases where $a_3 = a_4 = 0$ are omitted in Theorem 1.6 because in such cases system (1.4) is a quadratic polynomial differential system, which has already been studied.

Theorem 1.6 corresponds to Theorem 3.4, and it is proved in section 3.3.2.

### 1.2.2 Uniform isochronous centers of degree 4

We provide a topological classification of the global phase portraits in the Poincaré disc of all quartic uniform isochronous centers. We split our study into two cases, distinguishing when the nonlinear part of the planar quartic polynomial differential system is homogeneous or not.

#### Non–homogeneous nonlinear part

In this case the quartic uniform isochronous centers are of the form (1.7), according to Theorem 1.4.

**Theorem 1.7.** *Consider a quartic polynomial differential system* $X : \mathbb{R}^2 \to \mathbb{R}^2$ *and assume that* $X$ *has a uniform isochronous center at the origin such that their nonlinear part is not homogeneous. Then the global phase portrait of* $X$ *is topologically equivalent to one of the 12 phase portraits of Figure 1.4.*

*More precisely, since* $X$ *can always be written as system* (1.7)*, the global phase portrait of* $X$ *is topologically equivalent to the phase portrait*

(a) *of Figure 1.4 if either* $C_1C_3 > 0$*, or if* $C_3 = 0$, $B_2 < 0$*, or if* $C_1 = 0$, $C_3 \neq 0$ *and if either* $r_3 = r_2 = r_1$, $\forall r_1, r_2, r_3 \in \mathbb{R}^*$*, or if* $r_1 \neq 0$ *and* $r_{2,3} = a \pm bi$, $\forall r_1, b \in \mathbb{R}^*, a \in \mathbb{R}$;

(b) *of Figure 1.4 if* $C_1 = 0$, $C_3 \neq 0$ *and if either* $r_1, r_2, r_3 > 0$*, or* $r_1, r_2, r_3 < 0$*, or* $r_1r_2 > 0, r_3 = r_2$*, or* $r_2 = r_1, r_1r_3 > 0$;

(c) *of Figure 1.4 if* $C_1 = 0$, $C_3 \neq 0$ *and if either* $r_1 < 0, r_2, r_3 > 0$*, or* $r_1, r_2 < 0, r_3 > 0$*, or* $r_1 < 0, r_2 > 0, r_3 = r_2$*, or* $r_2 = r_1, r_1 < 0, r_3 > 0$;

(d) *of Figure 1.4 if* $C_3 = 0$, $C_1 \neq 0$, $B_2 > 0$, $C_1 \neq -A_1B_2$;

($e_1$) *or* ($e_2$) *of Figure 1.4 if either* $C_3 = 0$, $C_1 \neq 0$, $B_2 > 0$, $C_1 = -A_1B_2$*, or* $B_2 = C_3 = 0$;

(f) *or (g) or (h) of Figure 1.4 if* $C_1C_3 < 0$, $B_2 = 0$;

(i) *or (j) or (k) of Figure 1.4 if* $C_1C_3 < 0$, $B_2 \neq 0$;

*where in the cases with* $C_1 = 0$*, we have that* $r_1, r_2, r_3$ *are the roots of the polynomial* $-C_3 - B_2x - A_1x^2 - x^3$ *and we assume that* $r_1 \leq r_2 \leq r_3$ *when these roots are real.*

Theorem 1.7 corresponds to Theorem 4.3, and it is proved in section 4.3.1. This result has been published in [32].

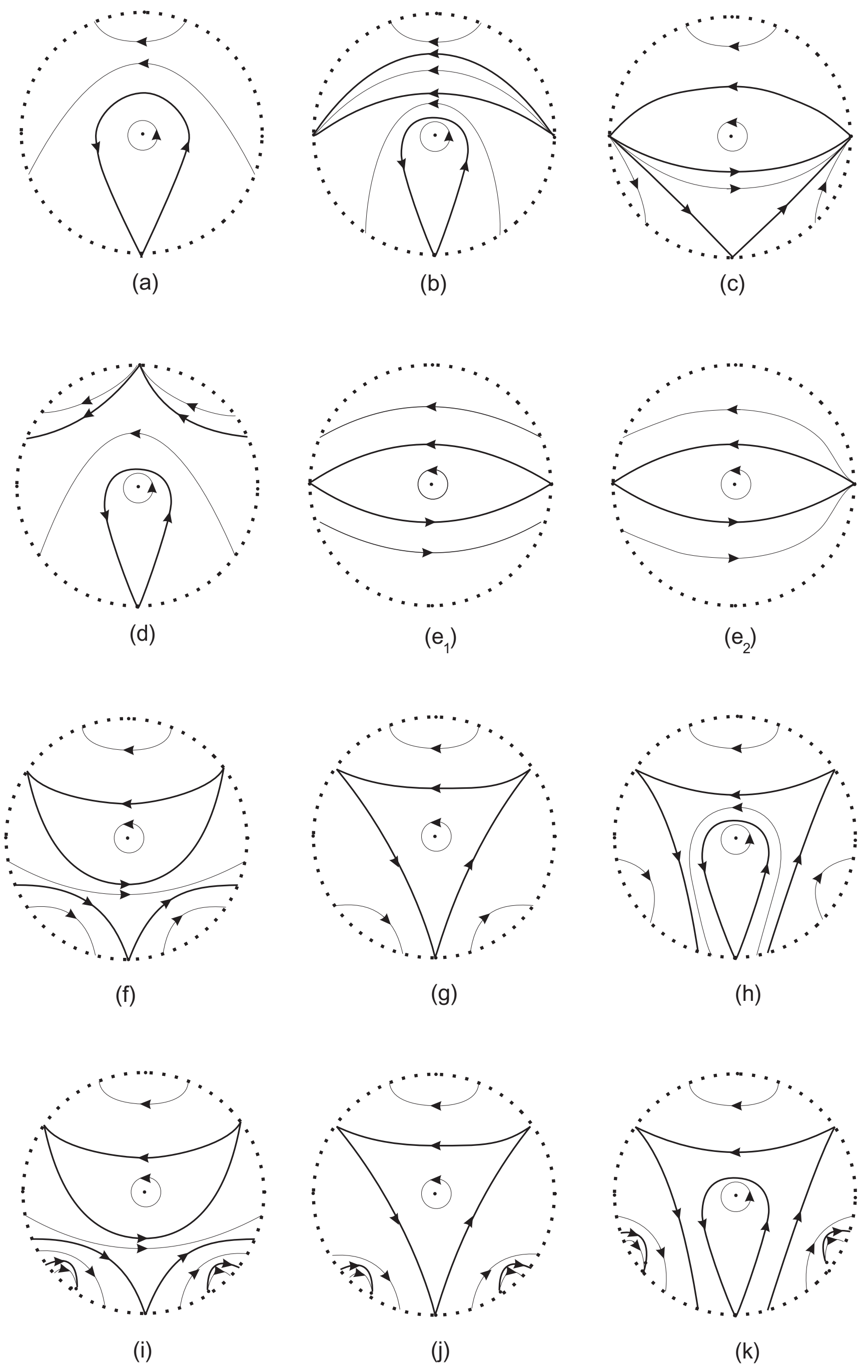



Figure 1.4: Phase portraits of the uniform isochronous centers (1.7).

### Homogeneous nonlinear part

In this case we have the following result.

**Theorem 1.8.** *Let*

$$\dot{x} = -y + xf(x,y), \quad \dot{y} = x + yf(x,y), \tag{1.8}$$

*be a polynomial differential system of degree 4, such that $f(x,y)$ is a cubic homogeneous polynomial. Then any quartic polynomial differential system which can be written into the form* (1.8) *has a uniform isochronous center at the origin and its global phase portrait is topologically equivalent to one of the 3 phase portraits of Figure 1.5.*

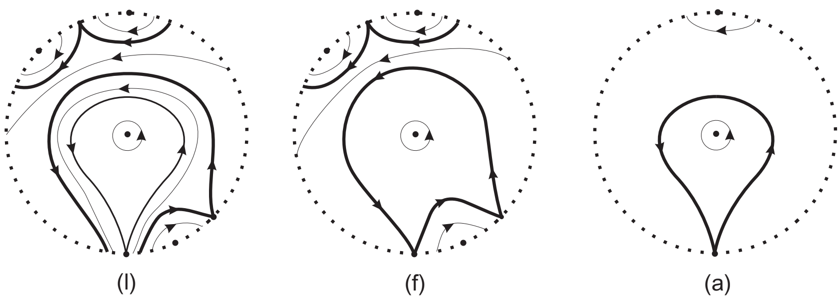



Figure 1.5: Phase portraits of (1.8) with quartic homogeneous polynomial nonlinearities.

We remark that the phase portraits (a) and (f) in Theorems 1.7 and 1.8 are topologically equivalent.

Theorem 1.8 is Theorem 4.4, and it is proved in section 4.3.2. This result has been submitted for publication, see [35].

## 1.3 Preliminaries on the bifurcation of limit cycles

Let $\mathcal{O}$ be an open subset of $\mathbb{R}^2$ and let $X : \mathcal{O} \to \mathbb{R}^2$ be a vector field. A periodic orbit $\gamma$ of $X$ is a *limit cycle* if there exists a neighborhood of $\gamma$ such that it is the only periodic orbit contained in this neighborhood. The biggest connected set of periodic solutions surrounding a center and having in its inner boundary the center itself is called the *period annulus* of the center.

A classical way to investigate limit cycles is perturbing a differential system which has a center. In this case the perturbed system can exhibit limit cycles that bifurcate, either from the center equilibrium point (having the so-called *Hopf bifurcation*), or from some of the periodic orbits surrounding the center, see for instance Pontrjagin [50], the second part of the book [17], and the hundreds of references quoted there. The problem of studying the limit cycles bifurcating from a center, or from its periodic solutions has been exhaustively studied in the last century and is closely related to the Hilbert's $16^{th}$ Problem. Nevertheless, in spite of all efforts, there is no general method to solve this problem.

Essentially there are four methods for determining the number of limit cycles which bifurcate from the periodic orbits of a period annulus of a center. The first method is based on studying the fixed points of the Poincaré return map, see for instance [10, 14]. The second method uses the Poincaré-Pontrjagin-Melnikov integrals or the Abelian integrals, which are also related with the Poincaré return map. These two integrals are equivalent in the plane, see section 6 of chapter 4 of [29], and section 5 of chapter 6 of [7]. The third method is based on the inverse integrating factor, see section 6 of [26] or [27]. The last method is based on the averaging theory, see for example [11, 53], and it is also related with the Poincaré map. From [11] one can check that in the plane the averaging method of first order is equivalent to the method of the Abelian integrals. Moreover the first two methods only give information on the number of periodic orbits of the unperturbed system that become limit cycles after the perturbation. The last two methods can also provide the shape of the bifurcated limit cycle up to some order of the perturbation parameter, see [27, 38].

The theory of averaging has a long history that starts with the classical works of Lagrange and Laplace, who provided an intuitive justification of the method. The first formalization of this theory was done in 1928 by Fatou [23]. For a more modern exposition of the averaging theory see the book of Sanders, Verhulst and Murdock [53].

Bifurcation of limit cycles in continuous planar differential systems are still largely studied. Nonetheless due to the considerable number of discontinuous phenomena in the real world, see for example [9, 54] and the references therein, a significant interest in the investigation of limit cycles of discontinuous piecewise differential systems has arisen. For instance in [44], applying the theory of regularization, the averaging theory is extended up to order 1 for studying the periodic solutions of systems of the form $x' = \varepsilon\big(F(t,x,\varepsilon) + \operatorname{sign}(h(x))G(t,x,\varepsilon)\big)$. In [41] there is a version of the averaging theorem up to order 2 for a bigger class of discontinuous piecewise differential equations $x' = \varepsilon F_1(t,x,\varepsilon)$. Finally in [42] it is stated averaging theorems for studying the periodic solutions of discontinuous piecewise differential equations of the form $x' = F_0(t,x) + \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x,\varepsilon)$.

## 1.4 Result on averaging theory

We develop the averaging theory at any order for computing the periodic solutions of discontinuous piecewise differential systems of the form

$$r' = \begin{cases} F^+(\theta,r,\varepsilon) & \text{if} \quad 0 \le \theta \le \alpha, \\ F^-(\theta,r,\varepsilon) & \text{if} \quad \alpha \le \theta \le 2\pi, \end{cases} \tag{1.9}$$

where

$$F^\pm(\theta,r,\varepsilon) = \sum_{i=1}^{k} \varepsilon^i F_i^\pm(\theta,r) + \varepsilon^{k+1} R^\pm(\theta,r,\varepsilon).$$

The set of discontinuity of system (1.9) is $\Sigma = \{\theta = 0\} \cup \{\theta = \alpha\}$ with $0 < \alpha < 2\pi$. Here $F_i^\pm : \mathbb{S}^1 \times D \to \mathbb{R}$ for $i = 1, \ldots, k$, and $R^\pm : \mathbb{S}^1 \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ are $\mathcal{C}^{k+1}$ functions, where $D$ is an open and bounded interval of $(0,\infty)$, and $\mathbb{S}^1 \equiv \mathbb{R}/(2\pi)$.

We remark that for $\alpha = 2\pi$ system (1.9) becomes continuous. So the averaging theory developed here can also be applied to continuous differential systems.

The *averaging function* $f_i : D \to \mathbb{R}$ of order $i$, for $i = 1, 2, \ldots, k$, is defined as

$$f_i(\rho) = \frac{y_i^+(\alpha,\rho) - y_i^-(\alpha - 2\pi,\rho)}{i!},$$

where $y_i^\pm : \mathbb{S}^1 \times D \to \mathbb{R}$, for $i = 1, 2, \ldots, k-1$, are defined recurrently as

$$\begin{aligned} y_i^\pm(\theta,\rho) = \quad & i! \int_0^\theta \Bigg( F_i^\pm(\phi,\rho) + \sum_{l=1}^{i} \sum_{S_l} \frac{1}{b_1!\, b_2! 2!^{b_2} \cdots b_l! l!^{b_l}} \cdot \\ & \partial^L F_{i-l}^\pm(\phi,\rho) \prod_{j=1}^{l} y_j^\pm(\phi,\rho)^{b_j} \Bigg) d\phi, \end{aligned}$$

where $S_l$ is the set of all $l$-tuples of non–negative integers $(b_1, b_2, \cdots, b_l)$ satisfying $b_1 + 2b_2 + \cdots + lb_l = l$, and $L = b_1 + b_2 + \cdots + b_l$.

The explicit expressions of $y_i^\pm$ up to order 7 are given in Appendix C.

Our result on the periodic solutions of system (1.9) is the following.

**Theorem 1.9.** *Assume that, for some $\ell \in \{1, 2, \ldots, k\}$, $f_i = 0$ for $i = 1, 2, \ldots, \ell - 1$ and $f_\ell \neq 0$. If there exists $\rho^* \in D$ such that $f_\ell(\rho^*) = 0$ and $f_\ell'(\rho^*) \neq 0$, then for $|\varepsilon| > 0$ sufficiently small there exists a $2\pi$–periodic solution $r(\theta, \varepsilon)$ of system* (1.9) *such that $r(0, \varepsilon) \to \rho^*$ when $\varepsilon \to 0$.*

Theorem 1.9 corresponds to Theorem 5.4, and it is proved in section 5.3.1. This result has been submitted for publication, see [34].

# 1.5 Results on limit cycles

We define a *small limit cycle* as a limit cycle which bifurcates from a center equilibrium point, whereas a *medium limit cycle* is one which bifurcates from a periodic orbit of the period annulus of a center. A singular point $p$ is a *weak focus* if it is a center for the linearized system at $p$ and $p$ is not a center.

We study the bifurcation of limit cycles in planar polynomial differential systems of degrees 3 and 4 with a uniform isochronous center at the origin, both for continuous and discontinuous polynomial perturbations.

## 1.5.1 Bifurcation of limit cycles from the uniform isochronous centers of degree 3

We consider the following continuous systems

$$\begin{aligned} \dot{x} &= -y + xf(x,y) + \sum_{i=1}^{6} \varepsilon^i p_i(x,y), \\ \dot{y} &= \quad x + yf(x,y) + \sum_{i=1}^{6} \varepsilon^i q_i(x,y), \end{aligned} \tag{1.10}$$

where $f(x, y)$ is as in Theorem 1.2, and the system

$$\dot{x} = -y + x^2 y + \varepsilon p_K(x,y), \quad \dot{y} = \quad x + xy^2 + \varepsilon q_K(x,y), \tag{1.11}$$

where

$$
\begin{aligned}
p_j &= \alpha_1^j x + \alpha_2^j y + \alpha_3^j x^2 + \alpha_4^j xy + \alpha_5^j y^2 + \alpha_6^j x^3 + \alpha_7^j x^2 y + \alpha_8^j xy^2 + \alpha_9^j y^3,\\
q_j &= \beta_1^j x + \beta_2^j y + \beta_3^j x^2 + \beta_4^j xy + \beta_5^j y^2 + \beta_6^j x^3 + \beta_7^j x^2 y + \beta_8^j xy^2 + \beta_9^j y^3,\\
p_K &= \alpha_0 + p_1, \qquad q_K = \beta_0 + q_1.
\end{aligned}
$$

Moreover we consider the discontinuous systems

$$
\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathcal{X}(x,y) = \begin{cases} X_1(x,y) & \text{if} \quad y > 0; \\ X_2(x,y) & \text{if} \quad y < 0. \end{cases} \tag{1.12}
$$

$$
\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathcal{Y}(x,y) = \begin{cases} Y_1(x,y) & \text{if} \quad y > 0; \\ Y_2(x,y) & \text{if} \quad y < 0. \end{cases} \tag{1.13}
$$

where

$$
\begin{aligned}
X_1(x,y) &= \begin{pmatrix} -y + xf(x,y) + \sum_{i=1}^{6} \varepsilon^i p_i(x,y) \\ x + yf(x,y) + \sum_{i=1}^{6} \varepsilon^i q_i(x,y) \end{pmatrix},\\
X_2(x,y) &= \begin{pmatrix} -y + xf(x,y) + \sum_{i=1}^{6} \varepsilon^i u_i(x,y) \\ x + yf(x,y) + \sum_{i=1}^{6} \varepsilon^i v_i(x,y) \end{pmatrix},\\
Y_1(x,y) &= \begin{pmatrix} -y + x^2 y + \varepsilon p_K(x,y) \\ x + xy^2 + \varepsilon q_K(x,y) \end{pmatrix},\\
Y_2(x,y) &= \begin{pmatrix} -y + x^2 y + \varepsilon u_K(x,y) \\ x + xy^2 + \varepsilon v_K(x,y) \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
u_j &= \gamma_1^j x + \gamma_2^j y + \gamma_3^j x^2 + \gamma_4^j xy + \gamma_5^j y^2 + \gamma_6^j x^3 + \gamma_7^j x^2 y + \gamma_8^j xy^2 + \gamma_9^j y^3,\\
v_j &= \delta_1^j x + \delta_2^j y + \delta_3^j x^2 + \delta_4^j xy + \delta_5^j y^2 + \delta_6^j x^3 + \delta_7^j x^2 y + \delta_8^j xy^2 + \delta_9^j y^3,\\
u_K &= \gamma_0 + u_1, \qquad v_K = \delta_0 + v_1.
\end{aligned}
$$

We state our results in what follows.

**Theorem 1.10.** *For $|\varepsilon| \neq 0$ sufficiently small the maximum number of small limit cycles of the differential system* (1.10) *is 3 using the averaging theory of order 6, and this number can be reached.*

Theorem 1.10 corresponds to Theorem 5.5, and it is proved in section 5.3.2.

**Theorem 1.11.** *For $|\varepsilon| \neq 0$ sufficiently small the maximum number of medium limit cycles of the differential system* (1.11) *is 3 using the first order averaging theory and this number can be reached.*

Theorem 1.11 is Theorem 5.6, and its proof is presented in section 5.3.3.

**Theorem 1.12.** *For $|\varepsilon| \neq 0$ sufficiently small the maximum number of small limit cycles of the discontinuous differential system* (1.12) *is 5 using the averaging method of order 6 and this number can be reached.*

Theorem 1.12 corresponds to Theorem 5.7, and it is proved in section 5.3.4.

**Theorem 1.13.** *For $|\varepsilon| \neq 0$ sufficiently small the maximum number of medium limit cycles of the discontinuous differential system* (1.13) *is 7 using the averaging method of first order and this number can be reached.*

Theorem 1.13 corresponds to Theorem 5.8, and its proof is presented in section 5.3.5.

These results on the bifurcation of limit cycles from the uniform isochronous centers of degree 3 have been published in [31].

### 1.5.2 Bifurcation of limit cycles from the uniform isochronous centers of degree 4

Let $H_c(n)$ denote the maximum number of limit cycles that bifurcate from the origin of system (1.3), when it is perturbed inside the class of all continuous polynomial differential systems of degree $n$, and $H_d(n)$ denotes the maximum number of limit cycles that bifurcate from the origin of system (1.3), when it is perturbed inside the class of all discontinuous piecewise polynomial differential systems of degree $n$ with two zones separated by the straight line $y = 0$.

We consider the following family of continuous differential systems

$$\begin{aligned} \dot{x} &= -y + xp(x,y) + \sum_{i=1}^{4} \varepsilon^i p_i(x,y), \\ \dot{y} &= \quad x + yp(x,y) + \sum_{i=1}^{4} \varepsilon^i q_i(x,y), \end{aligned} \tag{1.14}$$

where

$$\begin{aligned} p_j =&\alpha_0^j + \alpha_1^j x + \alpha_2^j y + \alpha_3^j x^2 + \alpha_4^j xy + \alpha_5^j y^2 + \alpha_6^j x^3 + \alpha_7^j x^2 y + \alpha_8^j xy^2 + \alpha_9^j y^3 \\ &+ \alpha_{10}^j x^4 + \alpha_{11}^j x^3 y + \alpha_{12}^j x^2 y^2 + \alpha_{13}^j xy^3 + \alpha_{14}^j y^4, \\ q_j =&\beta_0^j + \beta_1^j x + \beta_2^j y + \beta_3^j x^2 + \beta_4^j xy + \beta_5^j y^2 + \beta_6^j x^3 + \beta_7^j x^2 y + \beta_8^j xy^2 + \beta_9^j y^3 \\ &+ \beta_{10}^j x^4 + \beta_{11}^j x^3 y + \beta_{12}^j x^2 y^2 + \beta_{13}^j xy^3 + \beta_{14}^j y^4, \end{aligned}$$

and of the discontinuous differential systems

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathcal{X}(x,y) = \begin{cases} X_1(x,y) & \text{if} \quad y > 0, \\ X_2(x,y) & \text{if} \quad y < 0, \end{cases} \tag{1.15}$$

where

$$\begin{aligned} X_1(x,y) &= \begin{pmatrix} -y + xp(x,y) + \sum_{i=1}^{k} \varepsilon^i p_i(x,y) \\ x + yp(x,y) + \sum_{i=1}^{k} \varepsilon^i q_i(x,y) \end{pmatrix}, \\ X_2(x,y) &= \begin{pmatrix} -y + xp(x,y) + \sum_{i=1}^{k} \varepsilon^i u_i(x,y) \\ x + yp(x,y) + \sum_{i=1}^{k} \varepsilon^i v_i(x,y) \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} u_j =&\gamma_0^j + \gamma_1^j x + \gamma_2^j y + \gamma_3^j x^2 + \gamma_4^j xy + \gamma_5^j y^2 + \gamma_6^j x^3 + \gamma_7^j x^2 y + \gamma_8^j xy^2 + \gamma_9^j y^3 \\ &+ \gamma_{10}^j x^4 + \gamma_{11}^j x^3 y + \gamma_{12}^j x^2 y^2 + \gamma_{13}^j xy^3 + \gamma_{14}^j y^4, \\ v_j =&\delta_0^j + \delta_1^j x + \delta_2^j y + \delta_3^j x^2 + \delta_4^j xy + \delta_5^j y^2 + \delta_6^j x^3 + \delta_7^j x^2 y + \delta_8^j xy^2 + \delta_9^j y^3 \\ &+ \alpha_{10}^j x^4 + \delta_{11}^j x^3 y + \delta_{12}^j x^2 y^2 + \delta_{13}^j xy^3 + \delta_{14}^j y^4, \end{aligned}$$

with $k = 4$ or $k = 7$ depending on the order of the averaging theory that we can compute. For the continuous and the discontinuous cases we have to consider either

$$p(x, y) = t_{10}x + t_{01}y + t_{20}x^2 + t_{11}xy + t_{02}y^2 + t_{30}x^3 + t_{21}x^2y + t_{12}xy^2 + t_{03}y^3, \quad (1.16)$$

with $t_{ij} \in \mathbb{R}$, $i + j = 1, 2, 3$, $t_{30}^2 + t_{21}^2 + t_{12}^2 + t_{03}^2 \neq 0$, or

$$p(x, y) = t_{10}x + t_{11}xy + t_{30}x^3 + t_{12}xy^2, \quad (1.17)$$

with $t_{30}^2 + t_{12}^2 \neq 0$, or

$$p(x, y) = t_{30}x^3 + t_{21}x^2y + t_{12}xy^2 + t_{03}y^3. \quad (1.18)$$

We remark that the polynomials $p(x, y)$ in (1.17) and (1.18) are used to study the cases of quartic polynomial differential systems with a uniform isochronous center at the origin, either having a non–homogeneous nonlinear part (using (1.7) of Theorem 1.4), or a homogeneous nonlinear part, respectively. On the other hand, since (1.16) is a general cubic polynomial in $x$ and $y$ without constant term, it is used to study the bifurcation of limit cycles in both cases when the origin can be either a uniform isochronous center or a weak focus.

In the following we state our results.

**Theorem 1.14.** *Using averaging theory of order 4 we obtain, for* $|\varepsilon| \neq 0$ *sufficiently small,* $H_d(4) \geq 6$ *for the differential system* (1.15) *with* $p(x, y)$ *of the form* (1.16) (*i.e. system* (1.15) *has a weak focus or a uniform isochronous center at the origin*).

Theorem 1.14 is Theorem 6.2, and it is proved in section 6.3.1.

**Theorem 1.15.** *Using averaging theory of order 4 we obtain, for* $|\varepsilon| \neq 0$ *sufficiently small,* $H_d(4) \geq 5$ *for the differential system* (1.15) *with* $p(x, y)$ *either of the form* (1.17) *or* (1.18) (*i.e. system* (1.15) *has a uniform isochronous center at the origin*).

Theorem 1.15 corresponds to Theorem 6.3, and its proof is presented in section 6.3.2.

**Theorem 1.16.** *Using the averaging theory of order 7 we obtain, for* $|\varepsilon| \neq 0$ *sufficiently small,* $H_d(4) \geq 6$ *for the differential system* (1.15) *with* $p(x, y)$ *of the form* (1.17) *and* $\alpha_0^j = \beta_0^j = \gamma_0^j = \delta_0^j = 0$, $j = 1, \ldots, 7$.

Theorem 1.16 is Theorem 6.4, and is proved in section 6.3.3.

**Theorem 1.17.** *Using the averaging theory of order 4 we obtain, for* $|\varepsilon| \neq 0$ *sufficiently small,* $H_c(4) \geq 2$ *for the differential system* (1.14) *with* $p(x, y)$ *of the form* (1.16).

**Theorem 1.18.** *Using the averaging theory of order 4 we obtain, for* $|\varepsilon| \neq 0$ *sufficiently small,* $H_c(4) \geq 1$ *for the differential system* (1.14) *with* $p(x, y)$ *either of the form* (1.17) *or* (1.18).

Theorems 1.17 and 1.18 correspond to Theorems 6.5 and 6.6, respectively. They are proved in section 6.3.4.

We remark that all these results were obtained studying only the Hopf bifurcation, that is, we studied the number of small limit cycles that can bifurcate from the uniform isochronous center

We also remark that to prove Theorems 1.14 and 1.15 (respectively Theorems 1.17 and 1.18) we shall use the averaging theory of order 4 for discontinuous (respectively continuous) differential systems, together with a rescaling of the variables. In these proofs we can see, using Descartes Theorem (see Theorem 6.1 in this work), that the lower bounds which appear in the theorems are actually upper bounds for the averaging theory of order 4. From Theorems 1.14 and 1.15 (respectively Theorems 1.17 and 1.18) it follows that if applying the averaging theory of order 4 to the differential system (1.15) (respectively (1.14)) we obtain 6 (respectively 2) limit cycles, the origin of the differential system (1.15) (respectively (1.14)) is a weak focus.

These results on the bifurcation of limit cycles from the uniform isochronous centers of degree 4 have been submitted for publication, see [34].

### 1.5.3 Application of the averaging theory in a concrete planar polynomial differential system of degree 4

In this section we apply the averaging theory to study the bifurcation of limit cycles from the period annulus of the uniform isochronous center of a given planar polynomial differential system of degree 4.

Peng and Feng studied in [48] the following quartic polynomial differential system with a uniform isochronous center at the origin

$$\dot{x} = -y + xy(x^2 + y^2), \quad \dot{y} = x + y^2(x^2 + y^2). \tag{1.19}$$

They show that under any quartic homogeneous polynomial perturbations, at most 2 limit cycles bifurcate from the period annulus of system (1.19) using averaging theory of first order, and this upper bound can be reached. In addition these authors prove that for the family of perturbed quartic polynomial differential systems

$$\begin{aligned} \dot{x} =& - y + xy(x^2 + y^2) + \varepsilon(a_{10}x + a_{01}y + a_{11}xy + a_{21}x^2y + a_{03}y^3 \\ & + a_{40}x^4 + a_{31}x^3y + a_{22}x^2y^2 + a_{13}xy^3 + a_{04}y^4), \\ \dot{y} =& x + y^2(x^2 + y^2) + \varepsilon(b_{10}x + b_{01}y + b_{20}x^2 + b_{02}y^2 + b_{30}x^3 \\ & + b_{12}xy^2 + b_{40}x^4 + b_{31}x^3y + b_{22}x^2y^2 + b_{13}xy^3 + b_{04}y^4), \end{aligned} \tag{1.20}$$

there are at most 3 limit cycles bifurcating from the period annulus of (1.19) using averaging theory of first order, and this upper bound is sharp.

We remark that the perturbed system (1.20) studied by Peng and Feng do not consider all the quartic polynomial differential systems because they omit the coefficients $a_{00}$, $a_{20}$, $a_{02}$, $a_{30}$, $a_{12}$, $b_{00}$, $b_{11}$, $b_{21}$, $b_{03}$.

We consider the polynomial differential systems

$$\begin{aligned} \dot{x} =& -y + xy(x^2 + y^2) + \varepsilon \sum_{i=0}^{4} p_i(x, y), \\ \dot{y} =& \quad x + y^2(x^2 + y^2) + \varepsilon \sum_{i=0}^{4} q_i(x, y), \end{aligned} \tag{1.21}$$

where $p_i = \sum_{j+k=i} a_{jk} x^j y^k$ and $q_i = \sum_{j+k=i} b_{jk} x^j y^k$ are real homogeneous polynomials of degree $i$.

The following result completes the preliminary study presented in [48] using averaging theory of first order.

**Theorem 1.19.** *For $|\varepsilon| \neq 0$ sufficiently small there are quartic polynomial differential systems* (1.21) *having at least 8 limit cycles bifurcating from the periodic orbits of the uniform isochronous center* (7.1).

Theorem 1.19 is Theorem 7.1. The proof is presented in section 7.3. This result has been submitted for publication, see [33].

Note that in Theorem 1.19 we study medium limit cycles, i.e. limit cycles bifurcating from the periodic orbits surrounding the uniform isochronous center of the differential system (1.19), whereas in the previous subsection we have studied the small limit cycles of all quartic uniform isochronous centers, i.e. the limit cycles bifurcating from the center equilibrium point.

# Chapter 2

# Preliminaries

In this chapter we present some preliminary concepts, definitions and results that we shall use throughout this work.

## 2.1 Definitions and concepts

Let $\mathcal{O}$ be an open subset of $\mathbb{R}^2$. *A vector field of class* $\mathcal{C}^r$ on $\mathcal{O}$ is a $\mathcal{C}^r$ map $X : \mathcal{O} \to \mathbb{R}^2$, where $X(x)$ represents a vector attached at the point $x \in \mathcal{O}$. We can associate a differential equation to the vector field $X$ as the following

$$\dot{x} = X(x), \tag{2.1}$$

where $x \in \mathcal{O}$ and the dot denotes the derivative with respect to the variable $t$. The variables $x$ and $t$ are called the *dependent variable* and the *independent variable* of (2.1), respectively.

Let $p \in \mathcal{O}$ and $J$ an open interval containing the origin. Then $\varphi_p : J \to \mathcal{O}$ denotes the solution of (2.1) (i.e. $\dot{\varphi}_p(t) = X(\varphi_p(t))$ such that $\varphi_p(0) = p$. The solution $\varphi_p$ is called *maximal* if for every solution $\xi_p : K \to \mathcal{O}$ such that $J \subset K$ and $\varphi_p = \xi_p\big|_J$ then $J = K$ and, consequently $\varphi_p = \xi_p$. The *orbit* $\gamma_p$ of a vector field $X$ through the point $p$ is the image of the maximal solution $\varphi_p : J \to \mathcal{O}$ endowed with an orientation if the solution is regular. The *phase portrait* of the vector field $X : \mathcal{O} \to \mathbb{R}^2$ is the description of $\mathcal{O}$ as union of all orbits of $X$.

A point $p \in \mathcal{O}$ such that $X(p) = 0$ (respectively $\neq 0$) is called a *singular point* (respectively *regular point*) of $X$. If a singular point has a neighborhood that does not contain any other singular point, than such singular point is called an *isolated singular point*.

The *linear part* of $X$ at the point $p$ is the Jacobian matrix of $X$ calculated at that point. A singular point $p$ is *non-degenerate* if zero is not an eigenvalue of the linear part of the vector field at $p$. If both eigenvalues of the linear part of the vector field at that point have nonzero real part, the singular point $p$ is called *hyperbolic*. The singular point $p$ is called *semi-hyperbolic* if exactly one eigenvalue of the linear part of the vector field at $p$ is equal to zero. Hyperbolic and semi-hyperbolic singularities are also known as *elementary singular points*. If the linear part of the vector field at $p$ is not identically zero but both eigenvalues are zero, then $p$ is a *nilpotent singular point*. The singular point $p$ is called *linearly zero* if the linear part of the vector field at this point is identically zero.

Polynomial differential systems can be extended to infinity, compactifying the plane by adding a circle at the infinity, and analytically extending the flow to this boundary circle. This is done by the so called *Poincaré compactification*, which allows to study the behavior of the orbits near infinity. The singular points that are on the circle at infinity are the *infinite singular points* of the initial polynomial differential system, and the other singular points are called *finite singular points*. For further details about the Poincaré compactification, see Appendix A.

The concepts of *node*, *cusp*, *saddle* and *node* are the usual ones found in the literature, for more details see for instance pp. 7 and 110 of [22].

The singular point $p$ is called a *center* if there exists an open neighborhood consisting, besides the singularity, only of the periodic orbits.

Let $p \in \mathbb{R}^2$ be a singular point of an analytic differential system in $\mathbb{R}^2$, and assume that $p$ is a center. Without loss of generality we can assume that $p$ is the origin of coordinates (if necessary we do a translation of coordinates sending $p$ at the origin). Then, after a linear change of variables and a rescaling of the time variable (if necessary), the system can be written in one of the following three forms

$$\dot{x} = -y + F_1(x, y), \qquad \dot{y} = x + F_2(x, y); \tag{2.2}$$

$$\dot{x} = y + F_1(x, y), \qquad \dot{y} = F_2(x, y); \tag{2.3}$$

$$\dot{x} = F_1(x, y), \qquad \dot{y} = F_2(x, y); \tag{2.4}$$

where $F_1(x, y)$ and $F_2(x, y)$ are real analytic functions without constant and linear terms, defined in a neighborhood of the origin.

A center of an analytic differential system in $\mathbb{R}^2$ is called *linear type*, *nilpotent* or *degenerate* if after an affine change of variables and a rescaling of the time it can be written as system (2.2), (2.3) or (2.4), respectively.

The *period annulus* of a center is the biggest connected set of periodic solutions surrounding a center and having in its inner boundary the center itself. Compactifying $\mathbb{R}^2$ to the Poincaré disc (see Appendix A), the boundary of the period annulus of a center has two connected components: the center itself and a graphic, except perhaps in the case of a *global center*, where the two connected components are the center itself and the boundary of the Poincaré disc, which can be a periodic orbit or a graphic.

We say that $p$ is a *weak focus* if it is a center for the linearized system at $p$ and $p$ is not a center.

An orbit $\gamma(t)$ is a *periodic orbit* if there exists a constant $k > 0$ such that $\gamma(t+k) = \gamma(t)$, for all $t \in \mathbb{R}$. A periodic orbit $\gamma$ is called a *limit cycle* if there exists a neighborhood of $\gamma$ such that $\gamma$ is the only periodic orbit contained in this neighborhood. In this work a *small limit cycle* is one which bifurcates from either a focus or a center, and a *medium limit cycle* is one which bifurcates from a periodic orbit of the period annulus of a center.

Let $p$ be a center of a polynomial differential system in $\mathbb{R}^2$. Without loss of generality we can assume that $p$ is the origin of coordinates. We say that $p$ is an *isochronous center* if it is a center having a neighborhood such that all the periodic orbits in this neighborhood have the same period. We say that $p$ is a *uniform isochronous center*, also known in the literature as a *rigid center* if the system, in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, takes the form $\dot{r} = G(r,\theta)$, $\dot{\theta} = k$, $k \in \mathbb{R} \setminus \{0\}$. For more details, see Conti [19].

Let

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y), \tag{2.5}$$

be a real polynomial differential system. The vector field associated to the differential system (2.5) is defined by

$$X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}.$$

The polynomial differential system (2.5) is *integrable* on an open subset $\mathcal{O} \in \mathbb{R}^2$ if there exists a nonconstant function $H : \mathcal{O} \to \mathbb{R}$, called a *first integral* of the system on $\mathcal{O}$, which is constant on all solution curves $(x(t), y(t))$ of (2.5) contained in $\mathcal{O}$. Clearly $H$ is a first integral of (2.5) on the open subset $\mathcal{O}$ if and only if $XH = P\dfrac{\partial H}{\partial x} + Q\dfrac{\partial H}{\partial y} \equiv 0$ on $\mathcal{O}$.

Let $\mathcal{O}$ be an open subset $\mathbb{R}^2$ and let $R : \mathcal{O} \to \mathbb{R}$ be an analytic function which is not identically zero on $\mathcal{O}$. The function $R$ is an *integrating factor* of the differential system (2.5) on $\mathcal{O}$ if one of the following three equivalent conditions holds on $\mathcal{O}$.

$$\frac{\partial(RP)}{\partial x} = -\frac{\partial(RQ)}{\partial y}, \qquad \operatorname{div}(RP, RQ) = 0, \qquad XR = -R\operatorname{div}(P,Q).$$

The divergence of the vector field $X$ is defined, as usual, by

$$\operatorname{div}(X) = \operatorname{div}(P,Q) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

The first integral $H$ associated to the integrating factor $R$ is given by

$$H(x,y) = \int R(x,y)P(x,y)dy \; + h(x),$$

where $h$ is chosen such that $\partial H/\partial x = -RQ$, and we suppose that the domain of integration $\mathcal{O}$ is well adapted to the specific expression.

In Appendices A and B we discuss about the Poincaré compactification and the topological equivalence, respectively.

## 2.2 Some results on the uniform isochronous centers

In this section we present some results about uniform isochronous centers for polynomial differential systems in $\mathbb{R}^2$.

The next result characterizes when a center is a uniform isochronous one.

**Proposition 2.1.** *Assume that a planar polynomial differential system $\dot{x} = P(x,y)$, $\dot{y} = Q(x,y)$ of degree $n$ has a center at the origin of coordinates. Then this center is uniform isochronous if and only if by doing a linear change of variables and a rescaling of time it can be written as*

$$\dot{x} = -y + x\, f(x,y), \quad \dot{y} = x + y\, f(x,y), \tag{2.6}$$

*with $f(x,y)$ a polynomial in $x$ and $y$ of degree $n-1$, $f(0,0) = 0$.*

In what follows we provide a proof for this proposition.

**Proof:** Using blow up techniques we know that neither nilpotent nor degenerate centers can have uniform isochronous centers. This is due to the fact that, after the blow up, nilpotent and degenerate centers become a graphic, and the periodic orbits of those centers tending to the graphic have period tending to the infinity, and thus, the period cannot be constant. Hence, only systems having linear-type centers can present a uniform isochronous center

$$\dot{x} = -y + p(x, y), \quad \dot{y} = x + q(x, y),$$

where $p(x, y)$ and $q(x, y)$ are polynomials starting with at least terms of second order.

By doing a polar change of coordinates in this system, we have

$$\begin{aligned} \dot{r} &= \left. \frac{x\, p(x, y) + y\, q(x, y)}{\sqrt{x^2 + y^2}} \right|_{(x,y)=(r\cos\theta, r\sin\theta)}, \\ \dot{\theta} &= 1 + \left. \frac{x\, q(x, y) - y\, p(x, y)}{x^2 + y^2} \right|_{(x,y)=(r\cos\theta, r\sin\theta)}. \end{aligned}$$

But by hypothesis, such system has a uniform isochronous center at the origin, that is, $\dot{\theta} = 1$. Hence, $x\, q(x, y) - y\, p(x, y) = 0$, and thus

$$p(x, y) = x\, f(x, y), \quad q(x, y) = y\, f(x, y),$$

where $f(x, y)$ is a polynomial.

Reciprocally, if a polynomial differential system is of the form (2.6), then by doing a polar change of coordinates we obtain

$$\dot{r} = r\, f(r\cos\theta, r\sin\theta), \quad \dot{\theta} = 1.$$

Hence, it has a uniform isochronous center at the origin. ∎

The assumption of the existence of a center in proposition 2.1 cannot be removed. There are polynomial differential systems in $\mathbb{R}^2$ under the form (2.6) which do not have a uniform isochronous center at the origin. In what follows we present an example provided by Conti [19]. Consider the system

$$\dot{x} = -y - x(x^2 + y^2), \quad \dot{y} = x - y(x^2 + y^2).$$

In polar coordinates this system is written as $\dot{r} = -r^3$, $\dot{\theta} = 1$ and therefore the origin of the coordinates is a "uniform isochronous" stable focus.

**Proposition 2.2.** *If a planar polynomial differential system has a uniform isochronous center, then this center is the unique finite singular point of the differential system.*

**Proof:** Without loss of generality we can assume that the uniform isochronous center is the origin of coordinates.

The result follows by Proposition 2.1, since $(-y + xf(x, y))^2 + (x + yf(x, y))^2 = (x^2 + y^2)(1 + f^2(x, y)) > 0$, for $x^2 + y^2 > 0$, and therefore the origin of coordinates is the only finite singular point of the differential system. ∎

In short, no planar polynomial differential system can have more than one uniform isochronous center. On the other hand, there are differential systems with more than one isochronous center, as illustrated by the following example (see [45])

$$\dot{x} = -y + xy, \quad \dot{y} = x - \frac{x^2}{2} + 2y^2,$$

which has isochronous centers at the origin and at $(2, 0)$.

**Proposition 2.3.** *The global phase portrait in the Poincaré disc of any planar polynomial differential system of degree $n \geq 2$ with a uniform isochronous center has the infinity filled of singular points.*

**Proof:** Consider a planar polynomial differential system $\dot{x} = \sum_{i=0}^{n} P_i \quad \dot{y} = \sum_{i=0}^{n} Q_i$ where each $P_i, Q_i$ is a homogeneous polynomial of degree $i$, for $i = 0, \ldots, n$. It is well known that the infinite singular points in the Poincaré disc of this system are given by the end points of the real linear factors of the homogeneous polynomial $xQ_n - yP_n$. For system (2.6) we have that $xQ_n - yP_n = xyf(x, y) - yxf(x, y) \equiv 0$. Therefore every point at the boundary of the Poincaré disc is an infinite singular point for system (2.6). ∎

The following result can be found in Theorem 3.1 of [19].

**Proposition 2.4.** *Consider the polynomial differential system* (2.6) *of degree $n$. If $n > 1$ and the origin is a uniform isochronous center then it cannot be a global center.*

**Proof:** Under the hypotheses of the proposition we can write $f(x, y) = \sum_{n=1}^{n-1} f_h(x, y)$, with each $f_h(x, y)$ a homogeneous polynomial of degree $h$. The trajectories of system (2.6) in polar coordinates correspond to the solutions $\theta \mapsto r(\theta)$ of

$$\frac{dr}{d\theta} = \sum_{h=1}^{n-1} f_h(\cos\theta, \sin\theta) r^{h+1}. \tag{2.7}$$

Each periodic solution of (2.6) has associated a positive $2\pi-$periodic solution of (2.7) and vice-versa. Then the origin of (2.6) is a global center if and only if all the positive solutions of (2.7) are $2\pi-$periodic. In order to prove that this is not possible, unless in the case of $f_h \equiv 0$, for $h = 1, \ldots, n-1$, we replace $r$ by $\sigma = r^{-1}$, $r > 0$, sending the solutions $r$ of (2.7) into the solutions $\sigma$ of

$$\frac{d\sigma}{d\theta} = -\sum_{h=1}^{n-1} f_h(\cos\theta, \sin\theta) \sigma^{1-h}. \tag{2.8}$$

So the periodic orbits of (2.6) correspond to the positive $2\pi-$periodic solutions of (2.8). Now if the origin is a uniform isochronous center of (2.6) the family $\mathcal{F}$ of the periodic solutions of system (2.8) is non-empty. We shall show that there exist solutions that actually do not belong to $\mathcal{F}$.

If the uniform isochronous center is global, let $\sigma_k$ for $k \in \mathbb{N}$ be the periodic solutions of system (2.8) such that $\sigma_k(0) = 1/k$. So $\sigma_{k+1}(\theta) < \sigma_k(\theta)$, $0 \leq \theta \leq 2\pi$ because by the

*Existence and Uniqueness Theorem* of solutions of a differential system (see, for example, Theorem 1.2.4 of [53]), the orbits of $\sigma_k$ and $\sigma_{k+1}$ cannot intersect and by hypothesis $\sigma_{k+1}(0) < \sigma_k(0)$. Then from (2.8) we have

$$\begin{aligned} \sigma_k^{n-1}(\theta) = \frac{1}{k^{n-1}} - (n-1)\int_0^\theta f_1(\cos\varphi, \sin\varphi)\sigma_k^{n-2}(\varphi)d\varphi - \dots \\ - (n-1)\int_0^\theta f_{n-2}(\cos\varphi, \sin\varphi)\sigma_k(\varphi)d\varphi \\ - (n-1)\int_0^\theta f_{n-1}(\cos\varphi, \sin\varphi)d\varphi. \end{aligned} \tag{2.9}$$

On the other hand, since $\{\sigma_k(\theta)\}_{k\in\mathbb{N}}$ is a decreasing sequence converging to the function zero, it follows by *Dini's Theorem* (see for instance [52] for further details) that $\sigma_k(\theta) \to 0$ uniformly. Therefore from (2.9) we have

$$\int_0^\theta f_{n-1}(\cos\varphi, \sin\varphi)d\varphi = 0, \quad \text{for} \quad 0 \le \theta \le 2\pi,$$

that is,

$$f_{n-1}(\cos\varphi, \sin\varphi) = 0, \quad \text{for} \quad 0 \le \theta \le 2\pi$$

which is a contradiction. ■

Recall that a polynomial differential system is *Hamiltonian* if there exist a map $H$ defined in $\mathbb{R}^2$, such that the differential system can be written as

$$\dot{x} = \partial H/\partial y, \quad \dot{y} = -\partial H/\partial x.$$

**Proposition 2.5.** *If a planar polynomial differential system of degree $n \ge 2$ has a uniform isochronous center than this system is not Hamiltonian.*

**Proof:** A polynomial differential system $\dot{x} = -y + P(x,y)$, $\dot{y} = x + Q(x,y)$ is Hamiltonian if and only if

$$\frac{\partial P(x,y)}{\partial x} + \frac{\partial Q(x,y)}{\partial y} = 0. \tag{2.10}$$

In the case of the differential system (2.6), we write $f(x,y) = \displaystyle\sum_{i=1}^{n-1} f_i(x,y)$ where each $f_i(x,y) = \displaystyle\sum_{j+k=i} a_{j,k}x^j y^k$ and then we have

$$\frac{\partial(xf(x,y))}{\partial x} + \frac{\partial(yf(x,y))}{\partial y} =$$
$$\frac{\partial\left(\displaystyle\sum_{i=1}^{n-1}\sum_{j+k=i} a_{j,k}x^{j+1}y^k\right)}{\partial x} + \frac{\partial\left(\displaystyle\sum_{i=1}^{n-1}\sum_{j+k=i} a_{j,k}x^j y^{k+1}\right)}{\partial y} =$$

$$\sum_{i=1}^{n-1}\sum_{j+k=i}(j+1)a_{j,k}x^jy^k+\sum_{i=1}^{n-1}\sum_{j+k=i}(k+1)a_{j,k}x^jy^k=$$
$$\sum_{i=1}^{n-1}\sum_{j+k=i}(j+k+2)a_{j,k}x^jy^k.$$

The only way to vanish this expression in order to satisfy condition (2.10) is that $a_{j,k}\equiv 0$ for all $j,k=0,1,\ldots,n-1$ such that $1\le j+k\le n-1$. But this implies that $f(x,y)\equiv 0$, which is a contradiction. Therefore, system (2.6) is not Hamiltonian. ∎

A planar polynomial differential system $\dot{x}=P(x,y),\quad \dot{y}=Q(x,y)$ is *complex* if after performing the change of coordinates $z=x+iy$ the resulting differential system $\dot{z}=u(x,y)+iv(x,y)$ satisfies the *Cauchy-Riemann equations*

$$\frac{\partial u(x,y)}{\partial x}=\frac{\partial v(x,y)}{\partial y},\qquad \frac{\partial u(x,y)}{\partial y}=-\frac{\partial v(x,y)}{\partial x}. \tag{2.11}$$

**Proposition 2.6.** *If a planar polynomial differential system of degree $n\ge 2$ has a uniform isochronous center than this system is not complex.*

**Proof:** According to Theorem 2.1 a planar polynomial differential system with a uniform isochronous center can always be written as (2.6). We show that if the polynomial differential system (2.6) has degree $n\ge 2$, then it does not satisfy (2.11) unless $f(x,y)$ is a constant polynomial, and therefore system (2.6) is not a complex system.

Let $\dot{z}=u(x,y)+iv(x,y)$ be the resulting differential system from (2.6) by the change of variables $z=x+iy$. Since $u(x,y)=-y+xf(x,y)$, $v(x,y)=x+yf(x,y)$ with $f(x,y)$ a polynomial of degree $n-1$, for $n\ge 2$ we have

$$\frac{\partial u(x,y)}{\partial x}=f(x,y)+x\frac{\partial f(x,y)}{\partial x},\quad \frac{\partial v(x,y)}{\partial y}=f(x,y)+y\frac{\partial f(x,y)}{\partial y},$$
$$\frac{\partial u(x,y)}{\partial y}=-1+x\frac{\partial f(x,y)}{\partial y},\qquad \frac{\partial v(x,y)}{\partial x}=1+y\frac{\partial f(x,y)}{\partial x}.$$

In order to fulfill (2.11) the following equations must hold

$$x\frac{\partial f(x,y)}{\partial x}=y\frac{\partial f(x,y)}{\partial y},\qquad x\frac{\partial f(x,y)}{\partial y}=y\frac{\partial f(x,y)}{\partial x},$$

and consequently we must have $(x^2+y^2)\dfrac{\partial f(x,y)}{\partial x}=0$ and $(x^2+y^2)\dfrac{\partial f(x,y)}{\partial y}=0$. That is, $\dfrac{\partial f(x,y)}{\partial x}=\dfrac{\partial f(x,y)}{\partial y}=0$ for $(x,y)\neq(0,0)$. Thus to satisfy the Cauchy-Riemann equations (2.11), $f(x,y)$ needs to be constant, which is a contradiction, because by hypothesis it is a polynomial of degree at least 1. ∎

In the case of homogeneous uniform isochronous centers, Conti provided the following result in Theorem 2.1 of [19]. For the sake of completeness we provide a proof in what follows.

**Theorem 2.7.** *Let* $f(x,y) = \sum_{i+j=n-1} p_{i,j}x^i y^j$ *be a homogeneous polynomial of degree* $n-1$. *Then system* (2.6) *has a uniform isochronous center at the origin if either* $n$ *is even, or if* $n$ *is odd and*
$$\sum_{\nu=0}^{n-1}\left[p_{n-1-\nu,\nu}\int_0^{2\pi}\cos^{n-1-\nu}\theta\sin^{\nu}\theta\ d\theta\right] = 0.$$

**Proof:** Let $n \geq 2$ and $f(x,y) = \sum_{i+j=n-1} p_{i,j}x^i y^j$ be a homogeneous polynomial of degree $n-1$, $f(0,0) = 0$ in system (2.6). Then system (2.6) has either a center or a focus at the origin and in polar coordinates it is written as

$$\dot{r} = r^n f(\cos\theta, \sin\theta), \quad \dot{\theta} = 1,$$

where $f(\cos\theta,\sin\theta) = \sum_{\nu=0}^{n-1} p_{n-1-\nu,\nu}\cos^{n-1-\nu}\theta\sin^{\nu}\theta$. Then

$$\frac{dr}{d\theta} = r^n f(\cos\theta,\sin\theta).$$

and therefore

$$H = H(r,\theta) = \frac{r^{n-1}}{1+(n-1)r^{n-1}\displaystyle\int_0^{\theta} f(\cos\varphi,\sin\varphi)\ d\varphi} \tag{2.12}$$

is a first integral for the differential system if $H(r,0) = H(r,2\pi)$. Thus in order to determine the conditions for a center at the origin we have to consider two cases.

If $n$ is even, then $\int_0^{2\pi} f(\cos\varphi,\sin\varphi)\ d\varphi = \sum_{\nu=0}^{n-1} p_{n-1-\nu,\nu}\int_0^{2\pi}\cos^{n-1-\nu}\varphi\sin^{\nu}\varphi d\varphi = 0$, because $f(\cos(\varphi+\pi),\sin(\varphi+\pi)) = -f(\cos\varphi,\sin\varphi)$. Therefore $H(r,0) = H(r,2\pi)$.

If $n$ is odd, then it is required that $\sum_{\nu=0}^{n-1}\left[p_{n-1-\nu,\nu}\int_0^{2\pi}\cos^{n-1-\nu}\theta\sin^{\nu}\theta\ d\theta\right] = 0$, in order to satisfy $H(r,0) = H(r,2\pi)$. ■

In what follows we present results on the uniform isochronous centers in planar polynomial differential systems of degrees 1 and 2. Without loss of generality, we assume that the uniform isochronous center is at the origin of coordinates (if necessary, we do a translation of coordinates sending the singular point to the origin).

### 2.2.1 Uniform isochronous centers of degree 1

A linear differential system with a uniform isochronous center after a linear change of variables and a rescaling of time becomes

$$\dot{x} = -y, \quad \dot{y} = x,$$

and its phase portrait is presented in Figure 2.1.

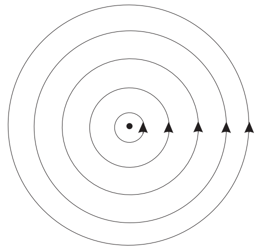


Figure 2.1: Phase portrait of the uniform isochronous center of degree 1.

### 2.2.2 Uniform isochronous centers of degree 2

The quadratic polynomial differential systems with a uniform isochronous center after a linear change of coordinates and a rescaling of time can be written into the form

$$\dot{x} = -y + x^2, \quad \dot{y} = x + xy.$$

This is part of a more general result provided by Loud [45] in 1964, which covers not only the uniform isochronous systems, but the family of all quadratic isochronous systems. The following theorem summarizes these results.

**Theorem 2.8.** *Consider a system of the form*

$$\dot{x} = -y + P_2(x, y) \quad \dot{y} = x + Q_2(x, y), \tag{2.13}$$

*where $P_2$ and $Q_2$ are homogeneous quadratic polynomials and at least one of them is non-vanishing. Assume that system* (2.13) *has a center at the origin.*

*Then the origin is an isochronous center of the system* (2.13) *if and only if this system can be brought to one of the following systems $S_1$, $S_2$, $S_3$, $S_4$ through a linear change of coordinates and rescaling of time. The first integral for each system is provided in table 2.1.*

| *Name* | *System* | *First integral* |
|---|---|---|
| $S_1$ | $\dot{x} = -y + \frac{x^2}{2} - \frac{y^2}{2}$<br>$\dot{y} = x(1+y)$ | $\frac{x^2+y^2}{1+y}$ |
| $S_2$ | $\dot{x} = -y + x^2$<br>$\dot{y} = x(1+y)$ | $\frac{x^2+y^2}{(1+y)^2}$ |
| $S_3$ | $\dot{x} = -y + \frac{x^2}{4}$<br>$\dot{y} = x(1+y)$ | $\frac{(x^2+4y+8)^2}{1+y}$ |
| $S_4$ | $\dot{x} = -y + 2x^2 - \frac{y^2}{2}$<br>$\dot{y} = x(1+y)$ | $\frac{4x^2-2(y+1)^2+1}{(1+y)^4}$ |

Table 2.1: First integrals of the uniform isochronous centers of degree 2.

*The respective phase portraits for systems* $S_1$, $S_2$, $S_3$ *and* $S_4$ *are shown in Figure 2.2.*

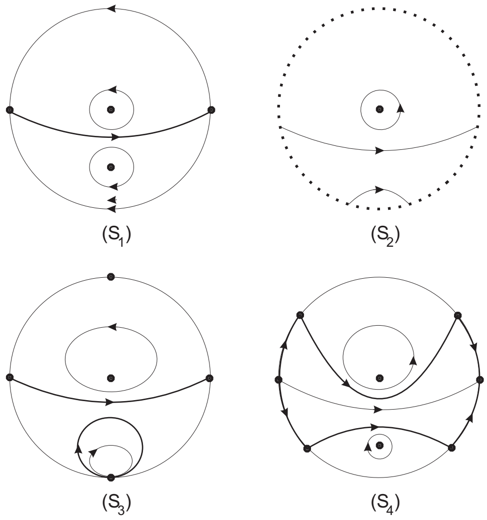



Figure 2.2: Phase portraits of isochronous quadratic systems.

Clearly, $S_2$ is the only one having a uniform isochronous center according to Theorem 2.1.

# Chapter 3

# Global phase portraits and first integrals of the uniform isochronous centers of degree 3

We provide a topological classification of the global phase portraits in the Poincaré disc of all uniform isochronous centers of degree 3. We also provide the explicit expressions of the first integrals for these systems.

## 3.1 Background

The uniform isochronous centers of degree 3 has been studied since at least the last decade of the past century. The following result, due to Collins [18] in 1997, also obtained by Devlin et al [20] in 1998, and by Gasull et al [25] in 2005, characterizes these differential systems.

**Theorem 3.1.** *A planar cubic polynomial differential system has a uniform isochronous center at the origin if and only if it can be written as*

$$\begin{aligned} \dot{x} &= -y + x(a_1x + a_2y + a_3x^2 + a_4xy - a_3y^2), \\ \dot{y} &= \phantom{-y+{}} x + y(a_1x + a_2y + a_3x^2 + a_4xy - a_3y^2), \end{aligned} \tag{3.1}$$

*and satisfies* $a_1^2a_3 - a_2^2a_3 + a_1a_2a_4 = 0$*,* $a_i \in \mathbb{R}$*,* $i = 1, \dots, 4$*.*

The following result is due to Collins [18].

**Proposition 3.2.** *System* (3.1) *satisfying* $a_1^2a_3 - a_2^2a_3 + a_1a_2a_4 = 0$*, may be reduced to either one of the following forms*

$$\dot{x} = -y(1 - x^2), \quad \dot{y} = x(1 + y^2), \tag{3.2}$$

$$\dot{x} = -y + x^2 + Ax^2y, \quad \dot{y} = x + xy + Axy^2, \tag{3.3}$$

*where* $A \in \mathbb{R}$*.*

Using (3.2) and (3.3), which present at most one parameter, Collins was able to provide the phase portraits and the first integrals of the cubic uniform isochronous centers. For the sake of completeness we present these first integrals in table 3.1.

| System | Condition | First integral |
|---|---|---|
| (3.2) | - | $\dfrac{x^2+y^2}{1+y^2}$ |
| (3.3) | $0 \neq A < 1/4$ | $(x^2+y^2)\left[y+\dfrac{1-K}{2A}\right]^{-\frac{1}{K}-1}\left[y+\dfrac{1+K}{2A}\right]^{\frac{1}{K}-1}$ |
| | $A > 1/4$ | $\dfrac{(x^2+y^2)^2}{Ay^2+y+1}e^{\left[\frac{2}{L}\arctan\frac{L}{2Ay+1}\right]}$ |
| | $A = 1/4$ | $\dfrac{(x^2+y^2)e^{\frac{4}{2+y}}}{(2+y)^2}$ |

where $K=\sqrt{1-4A}$ and $L=\sqrt{4A-1}$.

Table 3.1: First integrals of the uniform isochronous centers of degree 3.

## 3.2 Main results

Collins [18] presented the phase portraits and first integrals for systems (3.2) and (3.3) and therefore one needs to change the differential systems (3.1) to such normal forms before applying Collins' results. Our theorems present the first integrals and the global phase portraits in the Poincaré disc for the uniform isochronous cubic centers in terms of all the parameters of system (3.1) for the uniform isochronous centers. These results are published in [31].

### 3.2.1 First integrals

In the next theorem, we present the first integrals for the uniform isochronous centers of degree 3, in terms of all their parameters.

**Theorem 3.3.** *The first integrals $H$ of system* (3.1) *in polar coordinates $x = r\cos\theta,\ y = r\sin\theta$ are described in what follows.*

***Case 1:*** $\mathbf{a_1^2 - a_2^2 \neq 0}$.

***Subcase 1.1***: $\mathbf{a_4 \neq 0}$.

***Subcase 1.1.1***: $\mathbf{4a_4 \neq a_1^2 - a_2^2}$.

$$H = e^{-2\arctan\left[\frac{R+2a_4r(-a_2\cos\theta+a_1\sin\theta)}{RS}\right]}$$
$$\left[\frac{a_4r^2}{R+r(a_2\cos\theta-a_1\sin\theta)(a_2a_4r\cos\theta-a_1a_4\sin\theta-R)}\right]^S,$$

*where* $R = a_1^2 - a_2^2,\ S = \sqrt{4a_4/R - 1}$.

*In case of a negative square root, we have a complex first integral and therefore both its real and imaginary parts are also first integrals, if not null.*

***Subcase 1.1.2:*** $\mathbf{4a_4 = a_1^2 - a_2^2}$.

$$H = \frac{re^{\frac{2}{2-a_2 r\cos\theta + a_1 r\sin\theta}}}{2 - a_2 r\cos\theta + a_1 r\sin\theta}.$$

***Subcase 1.2:*** $\mathbf{a_4 = 0}$.

$$H = \frac{r}{1 - a_2 r\cos\theta + a_1 r\sin\theta}.$$

***Case 2:*** $\mathbf{a_1^2 - a_2^2 = 0}$.

***Subcase 2.1:*** $\mathbf{a_2 = a_1}$.

***Subcase 2.1.1:*** $\mathbf{a_1 = 0}$.

$$H = \frac{r^2}{1 - a_4 r^2\cos^2\theta + a_3 r^2\sin(2\theta)}.$$

***Subcase 2.1.2:*** $\mathbf{a_1 \neq 0,\ a_4 = 0}$.

***Subcase 2.1.2.1:*** $\mathbf{a_3(a_1^2 + 4a_3) \neq 0}$.

$$\begin{aligned} H =& e^{-2\arctan\left[\frac{a_1+2a_3 r(\cos\theta-\sin\theta)}{a_1 R}\right]} \\ &\left[\frac{a_3 r^2(\sin(2\theta)-1)}{(\cos\theta-\sin\theta)^2[1 + a_1 r(\sin\theta-\cos\theta) + a_3 r^2(\sin(2\theta)-1)]}\right]^R, \end{aligned}$$

*where* $R = \sqrt{-1 - 4a_3/a_1^2}$.

***Subcase 2.1.2.2:*** $\mathbf{a_3 = 0}$.

$$H = \frac{r}{1 - a_1 r(\cos\theta - \sin\theta)}.$$

***Subcase 2.1.2.3:*** $\mathbf{a_3 = -a_1^2/4}$.

$$H = \frac{re^{\frac{2}{2-a_1 r(\cos\theta-\sin\theta)}}}{2 - a_1 r(\cos\theta - \sin\theta)}.$$

***Subcase 2.2:*** $\mathbf{a_2 = -a_1}$.

***Subcase 2.2.1:*** $\mathbf{a_1 = 0}$. *This case becomes the subcase 2.1.1.*

***Subcase 2.2.2:*** $\mathbf{a_1 \neq 0,\ a_4 = 0}$.

***Subcase 2.2.2.1:*** $\mathbf{a_3(4a_3 - a_1^2) \neq 0}$.

$$H = \frac{e^{\frac{1}{R}\left[-2\arctan\left(\frac{a_1+2a_3 r(\sin\theta+\cos\theta)}{a_1 R}\right) + R\,\mathrm{arctanh}(\tan\theta)\right]} a_3 r^2(\sec(2\theta) + \tan(2\theta))}{1 + a_1 r(\sin\theta + \cos\theta) + a_3 r^2(1 + \sin(2\theta))},$$

*where* $R = \sqrt{4a_3/a_1^2 - 1}$.
***Subcase 2.2.2.2:*** $\mathbf{a_3 = 0}$.

$$H = \frac{r}{1 - a_1 r(\cos\theta - \sin\theta)}.$$

***Subcase 2.2.2.3:*** $\mathbf{a_3 = a_1^2/4}$.

$$H = \frac{r}{e^{1+\frac{1}{2}a_1 r(\cos\theta+\sin\theta)}\left(1 + \frac{1}{2}a_1 r(\cos\theta + \sin\theta)\right)}.$$

Theorem 3.3 is proved in section 3.3.1.

### 3.2.2 Global phase portraits

We classify the global phase portraits in the Poincaré disc of the uniform isochronous centers of degree 3, in terms of all their parameters.

**Theorem 3.4.** *The global phase portrait in the Poincaré disc of the differential system* (3.1) *is topologically equivalent to one of the four phase portraits presented in Figure 3.1.*

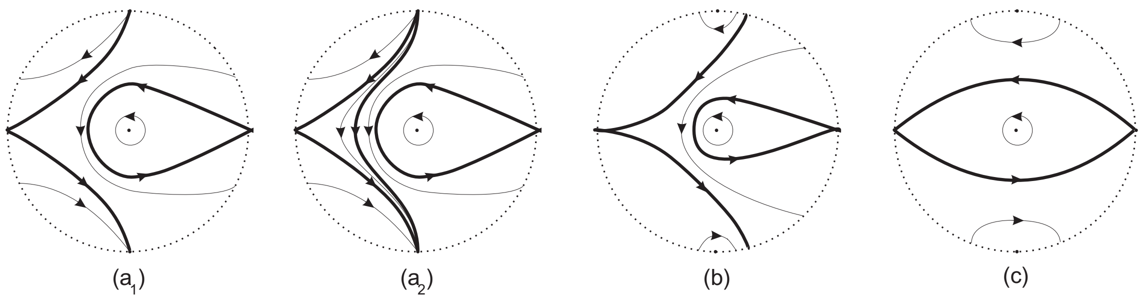



Figure 3.1: Phase portraits of cubic uniform isochronous centers.

*More precisely, the global phase portrait of* (3.1) *is topologically equivalent to either the phase portrait* ($a_1$) *or* ($a_2$) *of Figure 3.1 if one of the following conditions holds*

- $a_1 a_2 \neq 0$, *and* $a_4(a_1^2 - a_2^2) > 0$, *and* $a_4 \leq (a_1^2 - a_2^2)/4$;
- $a_2 = -a_1 \neq 0$, *and* $0 < a_3 \leq a_1^2/4$, *and* $a_4 = 0$;
- $a_2 = a_1 \neq 0$, *and* $-a_1^2/4 \leq a_3 < 0$, *and* $a_4 = 0$;
- $a_1 = 0$, *and* $a_2 \neq 0$, *and* $-a_2^2/4 \leq a_4 < 0$;
- $a_1 \neq 0$, *and* $a_2 = 0$, *and* $0 < a_4 \leq a_1^2/4$;

*to the phase portrait (b) if one of the following conditions holds*

- $a_1 a_2 \neq 0$, *and* $a_4(a_1^2 - a_2^2) > 0$, *and* $a_4 > (a_1^2 - a_2^2)/4$;
- $a_2 = -a_1 \neq 0$, *and* $a_3 > a_1^2/4$, *and* $a_4 = 0$;

- $a_2 = a_1 \neq 0$, *and* $a_3 < -a_1^2/4$, *and* $a_4 = 0$;
- $a_1 = 0$, *and* $a_2 \neq 0$ *and* $a_4 < -a_2^2/4$;
- $a_1 \neq 0$, *and* $a_2 = 0$ *and* $a_4 > a_1^2/4$;

*to the phase portrait (c) if one of the following conditions holds*

- $a_1a_2 \neq 0$, *and* $a_4(a_1^2 - a_2^2) < 0$;
- $a_2 = -a_1 \neq 0$, *and* $a_3 < 0$, *and* $a_4 = 0$;
- $a_2 = a_1 \neq 0$, *and* $a_3 > 0$, *and* $a_4 = 0$;
- $a_1 = 0$, *and* $a_2 \neq 0$, *and* $a_4 > 0$;
- $a_1 \neq 0$, *and* $a_2 = 0$, *and* $a_4 < 0$;
- $a_1 = a_2 = 0$.

The cases where $a_3 = a_4 = 0$ are omitted in Theorem 3.4 because in such cases system (3.1) is a quadratic polynomial differential system, which has already been exhaustively studied, see for instance system $S_2$ at p.38 of [12].

Theorem 3.4 is proved in section 3.3.2.

Our results have been checked with the software *P4*, see for more details on this software the chapters 9 and 10 of [22].

## 3.3 Proofs of the results

### 3.3.1 Proof of Theorem 3.3

We analyze each distinct case in order to compute the first integrals, considering the condition

$$a_1^2a_3 - a_2^2a_3 + a_1a_2a_4 = 0 \tag{3.4}$$

for the equation

$$\dot{x} = -y + xf(x,y), \quad \dot{y} = x + yf(x,y), \tag{3.5}$$

where $f(x,y) = a_1x + a_2y + a_3x^2 + a_4xy - a_3y^2$, presented in Theorem 3.1.

**Case 1:** $\mathbf{a_1^2 - a_2^2 \neq 0}$. The condition (3.4) can be expressed as

$$a_3 = -\frac{a_1a_2a_4}{a_1^2 - a_2^2},$$

and in polar coordinates the system can be written as

$$\frac{dr}{d\theta} = r^2(a_1\cos\theta + a_2\sin\theta) + \frac{a_4r^3(-a_2\cos\theta + a_1\sin\theta)(a_1\cos\theta + a_2\sin\theta)}{a_1^2 - a_2^2}. \tag{3.6}$$

**Subcase 1.1**: $\mathbf{a_4 \neq 0}$.

**Subcase 1.1.1**: $\mathbf{4a_4 \neq a_1^2 - a_2^2}$. It is easy to verify that

$$H = e^{-2\arctan\left[\frac{a_1^2-a_2^2+2a_4 r(-a_2\cos\theta+a_1\sin\theta)}{(a_1^2-a_2^2)R}\right]}$$
$$\left[\frac{a_4 r^2}{a_1^2-a_2^2+r(a_2\cos\theta-a_1\sin\theta)(a_2^2-a_1^2+a_2a_4r\cos\theta-a_1a_4\sin\theta)}\right]^R,$$

is a first integral of system (3.6), where $R = \sqrt{4a_4/(a_1^2-a_2^2)-1}$. This first integral is defined at $r = 0$. Therefore, the origin is a center.

We note that, in case of a negative square root, we have a complex first integral and therefore both its real and imaginary parts are also first integrals, if not null.

**Subcase 1.1.2**: $\mathbf{4a_4 = a_1^2 - a_2^2}$. In polar coordinates system (3.5) is written as

$$\frac{dr}{d\theta} = Ar^3 + Br^2.$$

where $A = 1/4(a_1a_2\sin^2\theta + (a_1^2-a_2^2)\sin\theta\cos\theta - a_1a_2\cos^2\theta)$, $B = a_1\cos\theta + a_2\sin\theta$. This is an *Abel differential equation* satisfying

$$\frac{dA(\theta)}{d\theta}B(\theta) - A(\theta)\frac{dB(\theta)}{d\theta} = aB(\theta)^3,$$

with $a = 1/4$. Applying the results presented in [37], the equation is integrable with the first integral

$$H = \frac{re^{\frac{2}{2+r(a_1\sin\theta-a_2\cos\theta)}}}{2+r(a_1\sin\theta-a_2\cos\theta)}.$$

Since it is defined at $r = 0$, we have a center.

**Subcase 1.2**: $\mathbf{a_4 = 0}$. System (3.6) is reduced to

$$\frac{dr}{d\theta} = r^2(a_1\cos\theta + a_2\sin\theta),$$

and

$$H = \frac{r}{1-a_2r\cos\theta+a_1r\sin\theta}$$

is a first integral for this system, and thus, the origin is a center.

**Case 2:** $\mathbf{a_1^2 - a_2^2 = 0}$

**Subcase 2.1**: $\mathbf{a_2 = a_1}$. The expression (3.4) is reduced to $a_1^2a_4 = 0$. Therefore we have the following possibilities.

**Subcase 2.1.1**: $\mathbf{a_1 = 0}$. Applying the condition $a_1 = a_2 = 0$ in system (3.5), we obtain in polar coordinates

$$\frac{dr}{d\theta} = r^3(a_3\cos^2\theta + a_4\sin\theta\cos\theta - a_3\sin^2\theta).$$

The following expression is a first integral of this system

$$H = \frac{r^2}{1-a_4r^2\cos^2\theta+a_3r^2\sin(2\theta)}.$$

**Subcase 2.1.2**: $\mathbf{a_4 = 0}$. Under this condition, system (3.5) has in polar coordinates the following expression

$$\frac{dr}{d\theta} = r^2 a_1(\cos\theta + \sin\theta) + r^3[a_3(\cos^2\theta - \sin^2\theta)].$$

**Subcase 2.1.2.1**: $\mathbf{a_3(a_1^2 + 4a_3) \neq 0}$. The following expression is a first integral of the system

$$\begin{aligned} H =& e^{-2\arctan\left[\frac{a_1+2a_3 r(\cos\theta-\sin\theta)}{a_1 R}\right]} \\ & \left[\frac{a_3 r^2(\sin(2\theta)-1)}{(\cos\theta-\sin\theta)^2[1+a_1 r(\sin\theta-\cos\theta)+a_3 r^2(\sin(2\theta)-1)]}\right]^R, \end{aligned}$$

where $R = \sqrt{-1 - 4a_3/a_1^2}$.

**Subcase 2.1.2.2**: $\mathbf{a_3 = 0}$. In this case system (3.5) becomes in polar coordinates

$$\frac{dr}{d\theta} = r^2 a_1(\cos\theta + \sin\theta).$$

A first integral for this system is

$$H = \frac{r}{1 - a_1 r(\cos\theta - \sin\theta)}.$$

**Subcase 2.1.2.3**: $\mathbf{a_3 = -a_1^2/4}$. In polar coordinates system (3.5) is written as

$$\frac{dr}{d\theta} = -\frac{1}{4}a_1 r^2[a_1\cos(2\theta)r - 4(\cos\theta + \sin\theta)].$$

This is an *Abel differential equation* satisfying

$$\frac{dA(\theta)}{d\theta}B(\theta) - A(\theta)\frac{dB(\theta)}{d\theta} = aB(\theta)^3, \ a \in \mathbb{R},$$

where $A(\theta) = -\frac{1}{4}a_1^2\cos(2\theta)$, $B(\theta) = a_1(\cos\theta + \sin\theta)$ and $a = \frac{1}{4}$. Using the results presented in [37], the equation is integrable with the first integral

$$H = \frac{2re^{\frac{2}{2-a_1 r(\cos\theta-\sin\theta)}}}{2 - a_1 r(\cos\theta - \sin\theta)}.$$

Since it is defined at $r = 0$, we have a center.

**Subcase 2.2**: $\mathbf{a_2 = -a_1}$.

**Subcase 2.2.1**: $\mathbf{a_1 = 0}$. This case becomes the subcase 2.1.1.

**Subcase 2.2.2**: $\mathbf{a_4 = 0}$. System (3.5) becomes in polar coordinates

$$\frac{dr}{d\theta} = r^2 a_1(\cos\theta - \sin\theta) + r^3[a_3(\cos^2\theta - \sin^2\theta)].$$

**Subcase 2.2.2.1**: $\mathbf{a_3(-a_1^2+4a_3)\neq 0}$. The following expression is a first integral of the system

$$H=\frac{e^{\frac{1}{R}\left[-2\arctan\left(\frac{a_1+2a_3r(\sin\theta+\cos\theta)}{a_1R}\right)+R\,\mathrm{arctanh}(\tan\theta)\right]}a_3r^2(\sec(2\theta)+\tan(2\theta))}{1+a_1r(\sin\theta+\cos\theta)+a_3r^2(1+\sin(2\theta))},$$

Where $R=\sqrt{4a_3/a_1^2-1}$.

**Subcase 2.2.2.2**: $\mathbf{a_3=0}$. System (3.5) becomes in in polar coordinates

$$\frac{dr}{d\theta}=r^2a_1(\cos\theta-\sin\theta).$$

A first integral of this system is the following

$$H=\frac{r}{1-a_1r(\cos\theta-\sin\theta)}.$$

**Subcase 2.2.2.3**: $\mathbf{a_3=a_1^2/4}$. In polar coordinates system (3.5) can be written as

$$\frac{dr}{d\theta}=a_1r^2(\cos\theta-\sin\theta)+\frac{1}{4}[a_1^2r^3\cos(2\theta)].$$

This is an *Abel differential equation* satisfying

$$\frac{dA(\theta)}{d\theta}B(\theta)-A(\theta)\frac{dB(\theta)}{d\theta}=aB(\theta)^3,\ a\in\mathbb{R},$$

where $A(\theta)=\dfrac{1}{4}a_1^2\cos(2\theta)$, $B(\theta)=a_1(\cos\theta-\sin\theta)$ and $a=\dfrac{1}{4}$. Using the results presented in [37], we conclude that a first integral for the system is

$$H=\frac{r}{e^{1+\frac{1}{2}a_1r(\cos\theta+\sin\theta)}\left(1+\frac{1}{2}a_1r(\cos\theta+\sin\theta)\right)}.$$

### 3.3.2 Proof of Theorem 3.4

We provide all the possible phase portraits for the planar cubic differential systems with a uniform isochronous center at the origin in the Poincaré disc, by studying the finite and infinite singular points of such systems. Consider the condition

$$a_1^2a_3-a_2^2a_3+a_1a_2a_4=0 \tag{3.7}$$

for the equation

$$\dot{x}=-y+xf(x,y),\quad \dot{y}=x+yf(x,y), \tag{3.8}$$

where $f(x,y)=a_1x+a_2y+a_3x^2+a_4xy-a_3y^2$, presented in Theorem 3.1.

**Finite singular points**

By proposition 2.2 (see section 2.2) the differential system (3.8) has no finite singular points except the origin.

Proposition 2.4 (see section 2.2) together with the fact that the origin is the unique finite singular point in system (3.8) imply that the boundary of the period annulus of the uniform isochronous center at the origin is a graphic formed by infinite singular points and their separatrices.

**Infinite singular points**

For studying the infinite singular points in the Poincaré disc, we use the concepts and formulae provided in the Appendix A.

We perform the analysis of the vector field at infinity. In the chart $U_1$ the differential system (3.8) becomes

$$\dot{u} = (1+u^2)v^2, \quad \dot{v} = (-a_3 - a_4 u + a_3 u^2 - a_1 v - a_2 uv + uv^2)v. \tag{3.9}$$

We remark that $(u, 0)$ for all $u \in \mathbb{R}$ is an infinite singular point of the differential system (3.8) in $U_1$, and this result was expected due to proposition 2.3. In order to obtain the phase portraits, we perform a change of coordinates of the form $dt = vds$, and system (3.9) becomes

$$u' = (1+u^2)v, \quad v' = -a_3 - a_4 u + a_3 u^2 - a_1 v - a_2 uv + uv^2, \tag{3.10}$$

where the prime denotes derivative with respect to s.

In the chart $U_2$ system (3.8) becomes

$$\dot{u} = -(1+u^2)v^2, \quad \dot{v} = (a_3 - a_4 u - a_3 u^2 - a_2 v - a_1 uv - uv^2)v.$$

We only need to study the point $(0, 0)$ of $U_2$. By performing a change of coordinates of the form $dt = vds$ we obtain the system

$$u' = -(1+u^2)v, \quad v' = a_3 - a_4 u - a_3 u^2 - a_2 v - a_1 uv - uv^2. \tag{3.11}$$

In order to study the singular points at infinity of systems (3.10) and (3.11), we have to consider several cases. We shall apply Theorems 2.15, 2.19 and 3.15 of [22] to obtain the local phase portraits at each singular point.

**Case I**: $\mathbf{a_1^2 - a_1^2 \neq 0}$. The condition (3.7) is written as $a_3 = -a_1 a_2 a_4/(a_1^2 - a_2^2)$. If $a_4 = 0$, then $a_3 = 0$, and hence system (3.8) degenerates to a quadratic differential system, which has already been studied, as previously mentioned in this article. Therefore, we are going to omit the cases in which $a_4 = 0$.

**Subcase I.1**: $\mathbf{a_1 a_2 \neq 0}$. The expression (3.10) for our system in $U_1$ becomes

$$\begin{aligned} u' &= (1+u^2)v, \\ v' &= \frac{a_1 a_2 a_4}{a_1^2 - a_2^2} - a_4 u - \frac{a_1 a_2 a_4}{a_1^2 - a_2^2} u^2 - a_1 v - a_2 uv + uv^2. \end{aligned} \tag{3.12}$$

The singular points at the infinity are $p_1 = (-a_1/a_2, 0)$ and $p_2 = (a_2/a_1, 0)$. The linear parts of system (3.12) at $p_1$ and $p_2$ are, respectively

$$\begin{pmatrix} 0 & \left(\dfrac{a_1}{a_2}\right)^2 \\ \dfrac{(a_1^2 + a_2^2)a_4}{a_1^2 - a_2^2} & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \left(\dfrac{a_2}{a_1}\right)^2 \\ -\dfrac{(a_1^2 + a_2^2)a_4}{a_1^2 - a_2^2} & -\dfrac{a_1^2 + a_2^2}{a_1} \end{pmatrix}.$$

These singularities are studied later on. For $U_2$ the expression (3.11) becomes

$$\begin{aligned} u' &= -(1+u^2)v, \\ v' &= -\frac{a_1a_2a_4}{a_1^2-a_2^2} - a_4u + \frac{a_1a_2a_4}{a_1^2-a_2^2}u^2 - a_2v - a_1uv - uv^2. \end{aligned}$$

Since we are assuming $a_1a_2 \neq 0$, the origin of $U_2$ is not a singular point.

**Subcase I.1.1**: $\mathbf{a_4(a_1^2 - a_2^2) > 0}$.

**Subcase I.1.1.1**: $\mathbf{a_4 \leq \frac{a_1^2 - a_2^2}{4}}$.

**Subcase I.1.1.1.1**: $\mathbf{a_1 > 0}$. $p_1$ is a saddle and $p_2$ is a stable node.

**Subcase I.1.1.1.2**: $\mathbf{a_1 < 0}$. $p_1$ is a saddle and $p_2$ is an unstable node.

**Subcase I.1.1.2**: $\mathbf{a_4 > \frac{a_1^2 - a_2^2}{4}}$. $p_1$ is a saddle and $p_2$ is a focus.

**Subcase I.1.2**: $\mathbf{a_4(a_1^2 - a_2^2) < 0}$. $p_1$ is a focus/center and $p_2$ is saddle.

**Subcase I.2**: $\mathbf{a_1 = 0}$. In chart $U_1$, we have

$$u' = (1+u^2)v, \quad v' = -a_4u - a_2uv + uv^2, \tag{3.13}$$

and therefore the only infinite singular point is the origin, which we will designate by $O_{U_1}$. Similarly, in chart $U_2$ we have the origin $O_{U_2}$ as the unique infinite singular point, since the expression of the vector field becomes

$$u' = -(1+u^2)v, \quad v' = -a_4u - a_2v - uv^2. \tag{3.14}$$

The linear parts of systems (3.13) and (3.14) at the origin are respectively

$$\begin{pmatrix} 0 & 1 \\ -a_4 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ -a_4 & -a_2 \end{pmatrix}.$$

Hence we have the following cases.

**Subcase I.2.1**: $\mathbf{a_4 > 0}$. $O_{U_1}$ is a focus/center and $O_{U_2}$ is a saddle.

**Subcase I.2.2**: $\mathbf{-\frac{a_2^2}{4} \leq a_4 < 0}$.

**Subcase I.2.2.1**: $\mathbf{a_2 > 0}$. $O_{U_1}$ is a saddle and $O_{U_2}$ is a stable node.

**Subcase I.2.2.2**: $\mathbf{a_2 < 0}$. $O_{U_1}$ is a saddle and $O_{U_2}$ is an unstable node.

**Subcase I.2.3**: $\mathbf{a_4 < -\frac{a_2^2}{4}}$. $O_{U_1}$ is a saddle and $O_{U_2}$ is a focus.

**Subcase I.3**: $\mathbf{a_2 = 0}$. In chart $U_1$, we have

$$u' = (1+u^2)v, \quad v' = -a_4u - a_1v + uv^2, \tag{3.15}$$

and therefore the only infinite singular point is the origin, which we will designate by $O_{U_1}$. Similarly, in chart $U_2$ we have the origin $O_{U_2}$ as the unique infinite singular point, since the expression of the vector field becomes

$$u' = -(1+u^2)v, \quad v' = -a_4u - a_1uv - uv^2. \tag{3.16}$$

The linear parts of systems (3.15) and (3.16) at the origin are respectively

$$\begin{pmatrix} 0 & 1 \\ -a_4 & -a_1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ -a_4 & 0 \end{pmatrix}.$$

Hence we have the following cases.

**Subcase I.3.1**: $\mathbf{a_4 < 0}$. $O_{U_1}$ is a saddle and $O_{U_2}$ is a focus/center.

**Subcase I.3.2**: $\mathbf{0 < a_4 \leq \frac{a_1^2}{4}}$.

**Subcase I.3.2.1**: $\mathbf{a_1 > 0}$. $O_{U_1}$ is a stable node and $O_{U_2}$ is a saddle.

**Subcase I.3.2.2**: $\mathbf{a_1 < 0}$. $O_{U_1}$ is an unstable node and $O_{U_2}$ is a saddle.

**Subcase I.3.3**: $\mathbf{a_4 > \frac{a_1^2}{4}}$. $O_{U_1}$ is a focus and $O_{U_2}$ is a saddle.

**Case II**: $\mathbf{a_1^2 - a_2^2 = 0}$. The condition (3.7) is simplified to $a_1a_2a_4 = 0$ and therefore the following cases might occur.

**Subcase II.1**: $\mathbf{a_1 = a_2 = 0}$ **and** $\mathbf{a_4 \neq 0}$.
**Subcase II.1.1**: $\mathbf{a_3 \neq 0}$. $p_1$ is a focus/center and $p_2$ is a saddle. In fact the expression (3.10) for our system in $U_1$ becomes

$$u' = (1 + u^2)v, \quad v' = -a_3 - a_4u + a_3u^2 + uv^2. \tag{3.17}$$

The singular points at the infinity are $p_{1,2} = ((a_4 \mp \sqrt{4a_3^2 + a_4^2})/2a_3, 0)$. The linear parts of system (3.17) at $p_1$ and $p_2$ are, respectively

$$\begin{pmatrix} 0 & 2 + \dfrac{a_4(a_4 - \sqrt{4a_3^2 + a_4^2})}{2a_3^2} \\ -\sqrt{4a_3^2 + a_4^2} & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 2 + \dfrac{a_4(a_4 + \sqrt{4a_3^2 + a_4^2})}{2a_3^2} \\ \sqrt{4a_3^2 + a_4^2} & 0 \end{pmatrix}.$$

It is easy to see that $p_1$ is a focus/center and $p_2$ is a saddle.

For $U_2$ the expression (3.11) becomes

$$u' = -(1 + u^2)v, \quad v' = a_3 - a_4u - a_3u^2 - uv^2.$$

The singular points at the infinity are $p_{3,4} = ((-a_4 \mp \sqrt{4a_3^2 + a_4^2})/2a_3,\ 0)$. Since $-a_4 \mp \sqrt{4a_3^2 + a_4^2} \neq 0$ for all $a_3, a_4 \in \mathbb{R}\backslash\{0\}$, the origin of $U_2$ is not a singular point and hence, the only infinite singular points are $p_1$ and $p_2$.

**Subcase II.1.2**: $\mathbf{a_3 = 0}$. The expression (3.10) for our system in $U_1$ becomes

$$u' = (1 + u^2)v, \quad v' = -a_4u + uv^2, \tag{3.18}$$

and therefore the origin $O_{U_1}$ is the unique infinite singular point in $U_1$. Similarly, in the chart $U_2$ the origin $O_{U_2}$ is an infinite singular point because system (3.11) becomes

$$u' = -(1+u^2)v, \quad v' = -a_4 u - uv^2. \tag{3.19}$$

The linear parts of systems (3.18) and (3.19) at the origin are respectively

$$\begin{pmatrix} 0 & 1 \\ -a_4 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ -a_4 & 0 \end{pmatrix}.$$

Hence we have the following cases.

**Subcase II.1.2.1**: $\mathbf{a_4 < 0}$. $O_{U_1}$ is a saddle and $O_{U_2}$ is a focus/center.

**Subcase II.1.2.2**: $\mathbf{a_4 > 0}$. $O_{U_1}$ is a focus/center and $O_{U_2}$ is a saddle.

**Subcase II.2**: $\mathbf{a_2 = -a_1 \neq 0}$ **and** $\mathbf{a_4 = 0}$. We are only interested in the cases that $a_3 \neq 0$, because as previously mentioned, when $a_3 = a_4 = 0$ system (3.8) becomes a quadratic differential system, which has already been exhaustively studied.

The expression (3.10) for our system in $U_1$ becomes

$$u' = (1+u^2)v, \quad v' = -a_3 - a_1 v + a_3 u^2 + a_1 uv + uv^2.$$

The singular points at the infinity are $p_{1,2} = (\mp 1, 0)$. The linear parts this system at $p_1$ and $p_2$ are, respectively

$$\begin{pmatrix} 0 & 2 \\ -2a_3 & -2a_1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 \\ 2a_3 & 0 \end{pmatrix}.$$

For $U_2$ the expression (3.11) becomes

$$u' = -(1+u^2)v, \quad v' = a_3 + a_1 v - a_3 u^2 - a_1 uv - uv^2.$$

The singular points at infinity are $p_{3,4} = (\mp 1, 0)$. The origin of $U_2$ is not a singular point and hence, the only infinite singular points are $p_1$ and $p_2$. These singularities are studied in what follows.

**Subcase II.2.1**: $\mathbf{a_3 < 0}$. $p_1$ is a saddle and $p_2$ is a focus/center.

**Subcase II.2.2**: $\mathbf{0 < a_3 \leq a_1^2/4}$.

**Subcase II.2.2.1**: $\mathbf{a_1 > 0}$. $p_1$ is a stable node and $p_2$ is a saddle.

**Subcase II.2.2.2**: $\mathbf{a_1 < 0}$. $p_1$ is an unstable node and $p_2$ is a saddle.

**Subcase II.2.3**: $\mathbf{a_3 > a_1^2/4}$. $p_1$ is a focus and $p_2$ is a saddle.

**Subcase II.3**: $\mathbf{a_2 = a_1 \neq 0}$ **and** $\mathbf{a_4 = 0}$. Again we are only interested in the cases that $a_3 \neq 0$.

The expression (3.10) for our system in $U_1$ becomes

$$u' = (1+u^2)v, \quad v' = -a_3 - a_1 v + a_3 u^2 - a_1 uv + uv^2.$$

The singular points at infinity are $p_{1,2} = (\mp 1, 0)$. The linear parts of the system at $p_1$ and $p_2$ are, respectively

$$\begin{pmatrix} 0 & 2 \\ -2a_3 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 \\ 2a_3 & -2a_1 \end{pmatrix}.$$

These singularities are studied later on.

For $U_2$ the expression (3.11) becomes

$$u' = -(1+u^2)v, \quad v' = a_3 + a_1 v - a_3 u^2 - a_1 uv - uv^2. \tag{3.20}$$

The singular points at infinity for (3.20) are $p_{3,4} = (\mp 1, 0)$. The origin of $U_2$ is not a singular point.

**Subcase II.3.1**: $\mathbf{a_3 > 0}$. $p_1$ is a focus/center and $p_2$ is a saddle.

**Subcase II.3.2**: $\mathbf{-a_1^2/4 \leq a_3 < 0}$.

**Subcase II.3.2.1**: $\mathbf{a_1 > 0}$. $p_1$ is a saddle and $p_2$ is a stable node.

**Subcase II.3.2.2**: $\mathbf{a_1 < 0}$. $p_1$ is a saddle and $p_2$ is an unstable node.

**Subcase II.3.3**: $\mathbf{a_3 < -a_1^2/4}$. $p_1$ is a saddle and $p_2$ is a focus.

**Subcase II.4**: $\mathbf{a_1 = a_2 = a_4 = 0}$. Again we are only interested in the cases that $a_3 \neq 0$. In this case system (3.8) has the particular form

$$\dot{x} = -y + a_3 x^3 - a_3 xy^2, \quad \dot{y} = x + a_3 x^3 - a_3 xy^2.$$

The expression (3.10) for our system in $U_1$ becomes

$$u' = (1+u^2)v, \quad v' = -a_3 + a_3 u^2 + uv^2. \tag{3.21}$$

The singular points at the infinity are $p_{1,2} = (\mp 1, 0)$. The linear parts of system (3.21) at $p_1$ and $p_2$ are, respectively

$$\begin{pmatrix} 0 & 2 \\ -2a_3 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 \\ 2a_3 & 0 \end{pmatrix}.$$

These singularities are studied in the next subcases.

For $U_2$ the expression (3.11) becomes

$$u' = -(1+u^2)v, \quad v' = a_3 - a_3 u^2 - uv^2.$$

The singular points at infinity are $p_{3,4} = (\mp 1, 0)$. The origin of $U_2$ is not a singular point.

**Subcase II.4.1**: $\mathbf{a_3 > 0}$. $p_1$ is a focus/center and $p_2$ is a saddle.

**Subcase II.4.2**: $\mathbf{a_3 < 0}$. $p_1$ is a saddle and $p_2$ is a focus/center.

Finally, the global phase portraits in the Poincaré disc for the planar cubic polynomial differential systems with a uniform isochronous center at the origin are obtained using the study of the finite and infinite singular points in the local phase portraits and the first integrals calculated in Theorem 3.3. Hence Theorem 3.4 is proved.

# Chapter 4

# Global phase portraits of the uniform isochronous centers of degree 4

We provide a topological classification of the global phase portraits in the Poincaré disc of all planar quartic polynomial differential systems with a uniform isochronous center at the origin.

## 4.1 Background

Algaba et al [4] in 1999, and Chavarriga et al [13] in 2001, independently provided the following characterization of quartic polynomial systems with a uniform isochronous center at the origin.

**Theorem 4.1.** *Consider $f(x,y)=\sum_{i=1}^{3} f_i(x,y)$ with $f_i(x,y)$, for $i=1,2,3$ homogeneous polynomials of degree $i$, $f_1^2+f_2^2\neq 0$ and $f_3\neq 0$. Then* (2.6) *is a quartic polynomial differential system having a non–homogeneous nonlinear part. Then the only case of local analytic integrability in a small open neighborhood of the origin of system* (2.6) *is given, modulo a rotation, by the time-reversible system.*

$$
\begin{aligned}
\dot{x} &= -y + x(A_1x + B_2xy + C_1x^3 + C_3xy^2),\\
\dot{y} &= \quad x + y(A_1x + B_2xy + C_1x^3 + C_3xy^2).
\end{aligned}
\tag{4.1}
$$

*where $A_1, B_2, C_1, C_3 \in \mathbb{R}$.*

By the following classical result due to Poincaré [49] and Liapunov [46] Theorem 4.1 characterizes the quartic uniform isochronous centers, except the ones for which the polynomial $f(x,y)$ is a homogeneous polynomial of degree 3.

**Theorem 4.2.** *An analytic differential system $\dot{x}=-y+F_1(x,y)$, $\dot{y}=x+F_2(x,y)$, with $F_1(x,y)$ and $F_2(x,y)$ real analytic functions without constant and linear terms defined in a neighborhood of the origin, has a center at the origin if and only if there exists a local analytic first integral of the form $H=x^2+y^2+G(x,y)$ defined in a neighborhood of the origin, where $G$ starts with terms of order higher than two.*

The proof of Theorem 4.1 by Chavarriga et al is made by calculating the Liapunov constants to find necessary conditions and by applying time-reversibility for sufficiency. For a complete proof of Theorem 4.1, see [13] or [4].

In the case $C_1 = 0$, Algaba et al [4], and Chavarriga et al [12] provided the first integral for system (4.1).

$$H(x,y) = (x^2+y^2)e^{-2\int \frac{A_1+B_2y+C_3y^2}{1+A_1y+B_2y^2+C_3y^3}dy}.$$

In the first paper, polynomial commutators were used to prove the result, and in the latter it considers that system (4.1) presents the invariant algebraic curve $h_2(x,y) \doteq 1 + A_1y + B_2y^2 + C_3y^3 = 0$. Algaba et al [4] also provided the phase portraits for (4.1) for the case $C_1 = 0$. In such case system (4.1) has a polynomial commutator, allowing to get the bifurcation diagram of the system.

## 4.2 Main results

The classification of the global phase portraits in the Poincaré disc of the quartic uniform isochronous centers consists of 13 topologically different phase portraits. Our investigation consider two cases, distinguishing when the nonlinear part of the differential system is homogeneous or not.

### 4.2.1 Non–homogeneous nonlinear part

In this case our result is

**Theorem 4.3.** *Consider a quartic polynomial differential system $X : \mathbb{R}^2 \to \mathbb{R}^2$ and assume that $X$ has a uniform isochronous center at the origin such that their nonlinear part is not homogeneous. Then the global phase portrait of $X$ is topologically equivalent to one of the 12 phase portraits of Figure 4.1.*

*More precisely, since $X$ can always be written as system* (4.1), *the global phase portrait of $X$ is topologically equivalent to the phase portrait*

*(a) of Figure 4.1 if either $C_1C_3 > 0$, or if $C_3 = 0$, $B_2 < 0$, or if $C_1 = 0$, $C_3 \neq 0$ and if either $r_3 = r_2 = r_1$, $\forall r_1, r_2, r_3 \in \mathbb{R}^*$, or if $r_1 \neq 0$ and $r_{2,3} = a \pm bi$, $\forall r_1, b \in \mathbb{R}^*, a \in \mathbb{R}$;*

*(b) of Figure 1.4 if $C_1 = 0$, $C_3 \neq 0$ and if either $r_1, r_2, r_3 > 0$, or $r_1, r_2, r_3 < 0$, or $r_1r_2 > 0, r_3 = r_2$, or $r_2 = r_1, r_1r_3 > 0$;*

*(c) of Figure 1.4 if $C_1 = 0$, $C_3 \neq 0$ and if either $r_1 < 0, r_2, r_3 > 0$, or $r_1, r_2 < 0, r_3 > 0$, or $r_1 < 0, r_2 > 0, r_3 = r_2$, or $r_2 = r_1, r_1 < 0, r_3 > 0$;*

*(d) of Figure 4.1 if $C_3 = 0$, $C_1 \neq 0$, $B_2 > 0$, $C_1 \neq -A_1B_2$;*

*($e_1$) or ($e_2$) of Figure 4.1 if either $C_3 = 0$, $C_1 \neq 0$, $B_2 > 0$, $C_1 = -A_1B_2$, or $B_2 = C_3 = 0$;*

*(f) or (g) or (h) of Figure 4.1 if $C_1C_3 < 0$, $B_2 = 0$;*

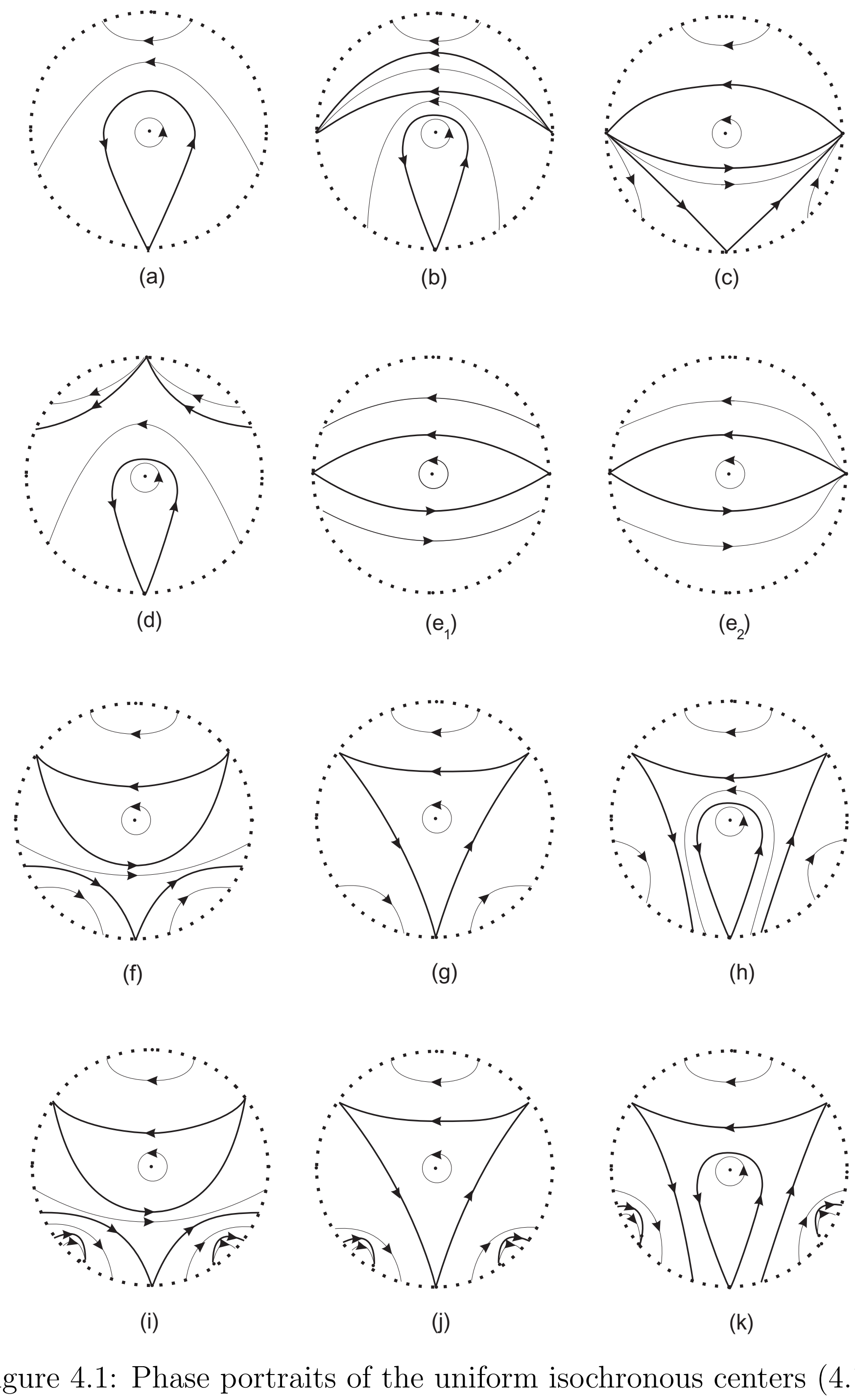



Figure 4.1: Phase portraits of the uniform isochronous centers (4.1).

*(i) or (j) or (k) of Figure 4.1 if* $C_1C_3 < 0$, $B_2 \neq 0$;

*where in the cases with* $C_1 = 0$, *we have that* $r_1, r_2, r_3$ *are the roots of the polynomial* $-C_3 - B_2x - A_1x^2 - x^3$ *and we assume that* $r_1 \leq r_2 \leq r_3$ *when these roots are real.*

Theorem 4.3 is proved in section 4.3.1 and it has been published in [32].

### 4.2.2 Homogeneous nonlinear part

Our result is the following.

**Theorem 4.4.** *Let*

$$\dot{x} = -y + xf(x,y), \quad \dot{y} = x + yf(x,y), \tag{4.2}$$

*be a polynomial differential system of degree 4, such that* $f(x,y)$ *is a cubic homogeneous polynomial. Then any quartic polynomial differential system which can be written into the form* (4.2) *has a uniform isochronous center at the origin and its global phase portrait is topologically equivalent to one of the 3 phase portraits of Figure 4.2.*

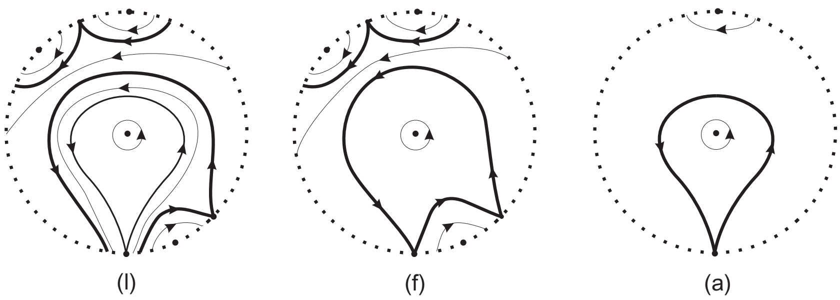



Figure 4.2: Phase portraits of (4.2) with quartic homogeneous polynomial nonlinearities.

Note that the phase portraits (a) and (f) in Theorems 4.3 and 4.4 are topologically equivalent.

Theorem 4.4 is proved in section 4.3.2. This result has been submitted for publication, see [35].

Our results have been checked with the software *P4*, see the chapters 9 and 10 of [22] for more details on this software.

## 4.3 Proofs of the results

### 4.3.1 Proof of Theorem 4.3

For providing all the possible global phase portraits in the Poincaré disc for the planar quartic polynomial differential systems with a uniform isochronous center at the origin such that their nonlinear part is not homogeneous, we shall start studying all the finite and infinite singular points of such systems. We remark that in this proof we never consider the quartic polynomial differential systems (4.2) with a uniform isochronous center such that $f(x,y)$ is a homogeneous polynomial.

Every planar quartic polynomial differential system with a uniform isochronous center at the origin such that their nonlinear part is not homogeneous can always be written as

$$\begin{aligned} \dot{x} &= -y + x(A_1x + B_2xy + C_1x^3 + C_3xy^2), \\ \dot{y} &= \ \ x + y(A_1x + B_2xy + C_1x^3 + C_3xy^2). \end{aligned} \tag{4.3}$$

where $A_1, B_2, C_1, C_3 \in \mathbb{R}$, see Theorem 4.1. These systems are invariant under the transformation $(x, y, t) \mapsto (-x, y, -t)$, so all their phase portraits are symmetric with respect to the $y-$axis.

**Finite singular points**

By proposition 2.2 (see section 2.2) the differential system (4.3) has no finite singular points except the origin.

The fact that the origin is the unique finite singular point in system (4.2) together with proposition 2.4 (see section 2.2) imply that the boundary of the period annulus of the uniform isochronous center at the origin is a graphic formed by infinite singular points and their separatrices.

**Infinite singular points**

In the chart $U_1$ the differential system (4.3) becomes

$$\begin{aligned} \dot{u} &= (1 + u^2)v^3, \\ \dot{v} &= (-C_1 - C_3u^2 - B_2uv - A_1v^2 + uv^3)v, \end{aligned} \tag{4.4}$$

and therefore the points $(u, 0)$ for all $u \in \mathbb{R}$ are infinite singular points of the differential system (4.3) in $U_1$. Due to proposition 2.3 this result was already expected. In order to obtain the local phase portraits at these points, after the rescaling of time $ds = vdt$ system (4.4) becomes

$$\begin{aligned} u' &= (1 + u^2)v^2, \\ v' &= -C_1 - C_3u^2 + v(-B_2u - A_1v + uv^2), \end{aligned} \tag{4.5}$$

where the prime denotes derivative with respect to s.

In chart $U_2$, system (4.3) becomes

$$\begin{aligned} \dot{u} &= -(1 + u^2)v^3, \\ \dot{v} &= (-C_3u - C_1u^3 - B_2uv - A_1uv^2 - uv^3)v. \end{aligned} \tag{4.6}$$

We only need to study the point $(0, 0)$ of $U_2$. Doing the rescaling of time $ds = vdt$, we obtain the system

$$\begin{aligned} u' &= -(1 + u^2)v^2, \\ v' &= -C_3u - C_1u^3 - B_2uv - A_1uv^2 - uv^3. \end{aligned} \tag{4.7}$$

We shall apply the well known results for the hyperbolic, semi–hyperbolic and nilpotent singular points for the characterization of the local phase portraits at each singular point of systems (4.5) and (4.7), for further information see for instance Theorems 2.15, 2.19 and 3.15 of [22].

**Case I**: $\mathbf{C_1 = 0}$. We remark that if $C_1 = C_3 = 0$ system (4.3) degenerates to a cubic polynomial differential system, which their first integrals and phase portraits are given in Theorems 3.3 and 3.4, respectively in this work. Therefore in Case I we shall assume $C_3 \neq 0$.

We first analyze the chart $U_2$. We denote by $O_{U_2}$ the origin of the chart $U_2$. The corresponding linear part of system (4.7) at $O_{U_2}$ is

$$\begin{pmatrix} 0 & 0 \\ -C_3 & 0 \end{pmatrix}.$$

Therefore $O_{U_2}$ is a nilpotent singularity and applying Theorem 3.5 of [22] we conclude that it is a cusp, whose behavior depends on the sign of the coefficient $C_3$. Hence, the local phase portrait at the origin for system (4.7) might be one of the two shown in Figure 4.3.

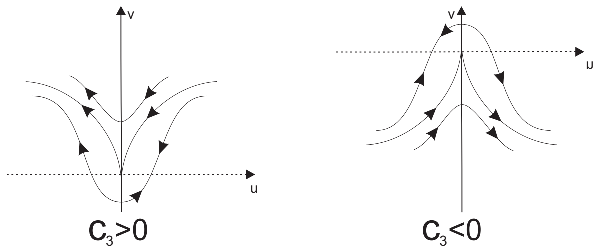



Figure 4.3: Local phase portrait at the origin of system (4.6). The horizontal axis is filled of singular points.

We now perform the study for the chart $U_1$. Clearly the only singular point at infinity in the chart $U_1$ is the origin, which we denote by $O_{U_1}$.

The corresponding linear part of system (4.5) at $O_{U_1}$ is identically zero. So it is necessary to apply a directional blow up $(u, v) \mapsto (u, w)$ where $v = uw$, and we obtain the system

$$\begin{aligned} u' &= (1 + u^2)u^2w^2, \\ w' &= u(-C_3 - B_2w - A_1w^2 - w^3). \end{aligned} \tag{4.8}$$

Performing a change of the independent variable of the form $dT = u\,ds$ in system (4.8), we get the system

$$\begin{aligned} u' &= (1 + u^2)uw^2, \\ w' &= -C_3 - B_2w - A_1w^2 - w^3, \end{aligned} \tag{4.9}$$

where the prime now denotes derivative with respect to T. The singular points of system (4.9) are of the form $(0, r_i)$, $i = 1, 2, 3$, where $r_1, r_2, r_3$ are the roots of the polynomial $-C_3 - B_2w - A_1w^2 - w^3$, that is, $A_1 = -(r_1+r_2+r_3), B_2 = r_1r_2+r_1r_3+r_2r_3, C_3 = -r_1r_2r_3$. We observe that, since we are assuming $C_3 \neq 0$, we have $r_1r_2r_3 \neq 0$. Hence, we have the following cases. Of course, of these roots we only need to take into account the real ones.

**Subcase I.1: Three simple real roots**. Without loss of generality we assume that $r_1 < r_2 < r_3$. The singular points at the infinity are $p_1 = (0, r_1)$, $p_2 = (0, r_2)$, and $p_3 = (0, r_3)$. The corresponding linear part of system (4.9) at each of these points is respectively

$$\begin{pmatrix} r_1^2 & 0 \\ 0 & -(r_1 - r_2)(r_1 - r_3) \end{pmatrix}, \quad \begin{pmatrix} r_2^2 & 0 \\ 0 & (r_1 - r_2)(r_2 - r_3) \end{pmatrix},$$

$$\begin{pmatrix} r_3^2 & 0 \\ 0 & -(r_1 - r_3)(r_2 - r_3) \end{pmatrix}.$$

Applying Theorem 2.15 of [22] and the hypotheses $r_1 < r_2 < r_3$, $r_1 r_2 r_3 \neq 0$ in the above expressions we conclude that $p_1$ and $p_3$ are saddles, and $p_2$ is an unstable node. The resulting singularity obtained from the blow down of $p_1, p_2$ and $p_3$ depends on the position of these singular points with respect to the origin of the $u-$axis. Hence we have the following subcases.

**Subcase I.1.1**: $\mathbf{0 < r_1 < r_2 < r_3}$. The local phase portraits at the singularities $p_1, i = 1, 2, 3$ for system (4.9) and system (4.8) are shown in Figures 4.4 and 4.5, respectively.

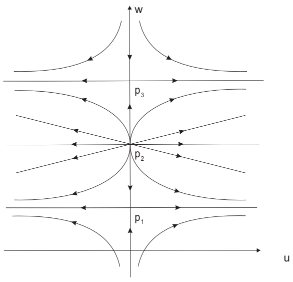



Figure 4.4: Phase portrait of system (4.9) for $0 < r_1 < r_2 < r_3$.

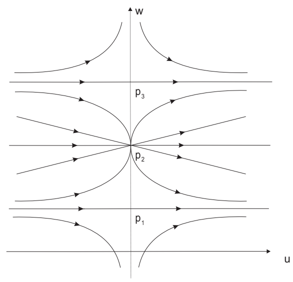



Figure 4.5: Phase portrait of system (4.8) for $0 < r_1 < r_2 < r_3$. The vertical axis is filled of singular points.

Going back through the blow up we get the local phase portrait at the origin of system (4.5), see Figure 4.6. Finally, taking into account the rescaling of time $ds = vdt$, we obtain that the phase portrait at the origin of system (4.4) is topologically equivalent to the one of Figure 4.7.

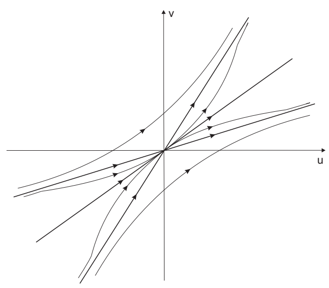



Figure 4.6: Phase portrait of system (4.5) for $0 < r_1 < r_2 < r_3$.

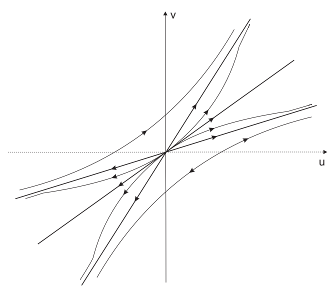



Figure 4.7: Phase portrait of system (4.4) for $0 < r_1 < r_2 < r_3$. The horizontal axis is filled of singular points.

For the chart $U_2$, since $r_1, r_2, r_3 > 0$ then $C_3 = -r_1 r_2 r_3 < 0$, and we obtain a local phase portrait as the one in Figure 4.3($C_3 < 0$).

In short, the global phase portrait in this case is obtained taking into account all the local phase portraits of the finite and infinite singular points, the *Existence and Uniqueness Theorem* of solutions (see, for example, Theorem 1.2.4 of [53]), the fact that all the phase portraits of planar quartic polynomial differential systems with a uniform isochronous center at the origin are symmetric with respect to the $y-$axis, and that the graphic at the boundary of the period annulus of the uniform isochronous center at the origin is formed by separatrices of infinite singular points. We conclude that the global phase portrait for Subcase I.1.1 is topologically equivalent to the one of Figure 4.1(b) of Theorem 4.3.

**Subcase I.1.2**: $\mathbf{r_1 < 0 < r_2 < r_3}$. The resulting local phase portrait at the origin of system (4.4) is given in Figure 4.8. This local phase portrait is obtained proceeding in a similar way to Case I.1.1.

For the chart $U_2$, since $r_1 < 0$ and $r_2, r_3 > 0$ then $C_3 = -r_1 r_2 r_3 > 0$ and we have a local phase portrait topologically equivalent to the one of Figure 4.3($C_3 > 0$). Therefore the global phase portrait for Subcase I.1.2 is shown in Figure 4.1(c) of Theorem 4.3.

**Subcase I.1.3**: $\mathbf{r_1 < r_2 < 0 < r_3}$. The phase portrait at the origin of system (4.4) is given in Figure 4.9. This local phase portrait is obtained proceeding in a similar way to Case I.1.1.

For the chart $U_2$, since $C_3 = -r_1 r_2 r_3 < 0$, we have a local phase portrait topologically equivalent to the one of Figure 4.3($C_3 < 0$). Then the global phase portrait for Subcase I.1.3 is shown in Figure 4.1(c) of Theorem 4.3.

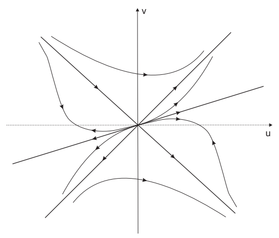



Figure 4.8: Phase portrait of system (4.4) for $r_1 < 0 < r_2 < r_3$.

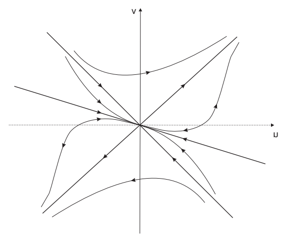



Figure 4.9: Phase portrait of system (4.4) for $r_1 < r_2 < 0 < r_3$.

**Subcase I.1.4**: $\mathbf{r_1 < r_2 < r_3 < 0}$. The resulting phase portrait at the origin of system (4.4) is given in Figure 4.10, obtained as in case I.1.1.

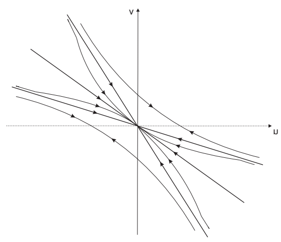



Figure 4.10: Phase portrait of system (4.4) for $r_1 < r_2 < r_3 < 0$.

For the chart $U_2$, since $C_3 = -r_1 r_2 r_3 > 0$, we have a local phase portrait topologically equivalent to the one of Figure 4.3($C_3 > 0$). So the global phase portrait for Subcase I.1.3 is shown in Figure 4.1(b) of Theorem 4.3.

**Subcase I.2: One simple real root and one double real root**. Without loss of generality we consider two distinct cases depending on the relative position of the simple and the double real roots: $r_1 < r_2 = r_3$ and $r_1 = r_2 < r_3$. We start with the first case.

The singular points at infinity are $p_1 = (0, r_1)$ and $p_2 = (0, r_2)$. The corresponding linear part of system (4.9) at each of these points is respectively

$$\begin{pmatrix} r_1^2 & 0 \\ 0 & -(r_1 - r_2)^2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} r_2^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Now we assume $\mathbf{r_1 < r_2 = r_3}$, and $r_1, r_2 \neq 0$ in the above expressions and applying Theorems 2.15 and 2.19 of [22], we conclude that $p_1$ is a saddle and $p_2$ is a saddle-node. The resulting singularity obtained from the blow down of $p_1$ and $p_2$ depends on the position of such singular points with respect to the horizontal axis. Hence we have the following cases.

**Subcase I.2.1**: $\mathbf{0 < r_1 < r_2}$. In this case the local phase portrait at the origin of system (4.4) is given in Figure 4.11, obtained as in case I.1.1.

For the chart $U_2$, since $C_3 = -r_1 r_2^2 < 0$, we have a local phase portrait similar to the one in Figure 4.3($C_3 < 0$). Consequently the global phase portrait for Subcase I.2.1 is shown in Figure 4.1(b) of Theorem 4.3.

**Subcase I.2.2**: $\mathbf{r_1 < 0 < r_2}$. The local phase portrait at the origin of system (4.4) is given in Figure 4.12, obtained as in case I.1.1.

For the chart $U_2$, since $C_3 = -r_1 r_2^2 > 0$, we have a local phase portrait topologically equivalent to the one of Figure 4.3($C_3 > 0$), and the global phase portrait for Subcase I.2.2 is shown in Figure 4.1(c) of Theorem 4.3.

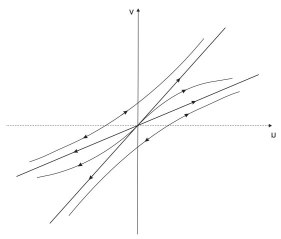



Figure 4.11: Phase portrait of system (4.4) for $0 < r_1 < r_2 = r_3$.

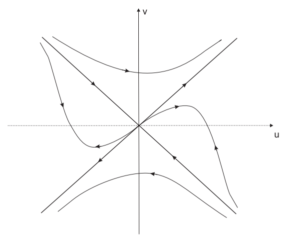



Figure 4.12: Phase portrait of system (4.4) for $r_1 < 0 < r_2 = r_3$.

**Subcase I.2.3**: $\mathbf{r_1 < r_2 < 0}$. The resulting phase portrait at the origin of system (4.4) is given in Figure 4.13. This local phase portrait is obtained proceeding in a similar way to the case I.1.1.

For the chart $U_2$, since $C_3 = -r_1 r_2^2 > 0$, we have a local phase portrait topologically equivalent to the one of Figure 4.3($C_3 > 0$).

The resulting global phase portrait for Subcase I.2.3 is shown in Figure 4.1(b) of Theorem 4.3.

Now we analyze the case $\mathbf{r_1 = r_2 < r_3}$. The singular points at the infinity are $p_1 = (0, r_1)$, $p_2 = (0, r_3)$. The corresponding linear part of system (4.9) at each of these points is respectively

$$\begin{pmatrix} r_1^2 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} r_3^2 & 0 \\ 0 & -(r_1 - r_3)^2 \end{pmatrix},$$

Considering that we assume $r_1 < r_3$, and $r_1, r_3 \neq 0$ in the above expressions and applying Theorems 2.15 and 2.19 of [22], we conclude that $p_1$ is a saddle-node, and $p_2$ is a saddle. The resulting singularity obtained from the blow down of $p_1$, and $p_2$ depends on the position of such singular points to the horizontal axis. Hence, we have the following cases.

**Subcase I.2.4**: $\mathbf{0 < r_1 < r_3}$. In this case the local phase portrait at the origin of system (4.4) is given in Figure 4.14, obtained as in case I.1.1.

We remark that the dynamics in the above phase portrait is almost topologically equivalent to the one of Case I.2.1, except between the two separatrices.

For the chart $U_2$, since $C_3 = -r_1^2 r_3 < 0$, we have a local phase portrait topologically equivalent to the one of Figure 4.3($C_3 < 0$). Then the global phase portrait for Subcase I.2.4 is shown in Figure 4.1(b) of Theorem 4.3.

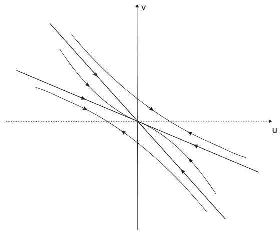


Figure 4.13: Phase portrait of system (4.4) for $r_1 < r_2 = r_3 < 0$.

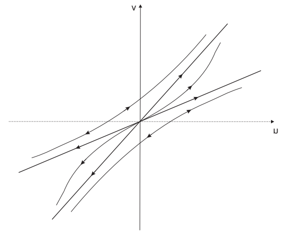


Figure 4.14: Phase portrait of system (4.4) for $0 < r_1 = r_2 < r_3$.

**Subcase I.2.5**: $\mathbf{r_1 < 0 < r_3}$. In this case the local phase portrait at the origin of system (4.4) is given in Figure 4.15.

For the chart $U_2$, since $C_3 = -r_1^2 r_3 < 0$, we have a local phase portrait topologically equivalent to the one of Figure 4.3($C_3 < 0$). Hence the global phase portrait for Subcase I.2.5 is shown in Figure 4.1(c) of Theorem 4.3.

**Subcase I.2.6**: $\mathbf{r_1 < r_3 < 0}$. In this case the local phase portrait at the origin of system (4.4) is given in Figure 4.16.

We remark that the dynamics in the above phase portrait is almost topologically equivalent to the one of Case I.2.1, except between the two separatrices.

For the chart $U_2$, since $C_3 = -r_1^2 r_3 > 0$, we have a local phase portrait topologically equivalent to the one of Figure 4.3($C_3 > 0$). Therefore the global phase portrait for Subcase I.2.6 is shown in Figure 4.1(b) of Theorem 4.3

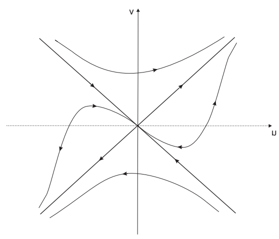


Figure 4.15: Phase portrait of system (4.4) for $r_1 = r_2 < 0 < r_3$.

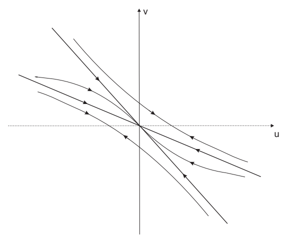


Figure 4.16: Phase portrait of system (4.4) for $r_1 = r_2 < r_3 < 0$.

**Subcase I.3: One triple real root**. In this case we have $r_1 = r_2 = r_3$. Hence the only singular point at infinity is $p_1 = (0, r_1)$. The corresponding linear part of system (4.9) at

$p_1$ is

$$\begin{pmatrix} r_1^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $r_1 \neq 0$ from Theorem 2.19 of [22], it follows that $p_1$ is a saddle. Then the resulting singularity obtained from the blow down of $p_1$ depends on the position of such singular point with respect to the horizontal axis. So we distinguish the following cases.

**Subcase I.3.1**: $\mathbf{r_1 > 0}$. In this case the local phase portrait at the origin of system (4.4) is given in Figure 4.17.

For the chart $U_2$, since $C_3 = -r_1^3 < 0$, we have a local phase portrait topologically equivalent to the one of Figure 4.3($C_3 < 0$). So the global phase portrait for Subcase I.3.1 is shown in Figure 4.1(a) of Theorem 4.3.

**Subcase I.3.2**: $\mathbf{r_1 < 0}$. In this case the local phase portrait at the origin of system (4.4) is given in Figure 4.18.

For the chart $U_2$, since $C_3 = -r_1^3 > 0$, we have a local phase portrait topologically equivalent to the one of Figure 4.3($C_3 > 0$). Then the global phase portrait for Subcase I.3.2 is shown in Figure 4.1(a) of Theorem 4.3.

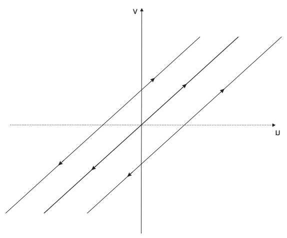



Figure 4.17: Phase portrait of system (4.4) for $0 < r_1 = r_2 = r_3$.

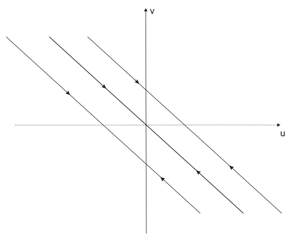



Figure 4.18: Phase portrait of system (4.4) for $r_3 = r_2 = r_1 < 0$.

**Subcase I.4: One simple real root and two complex conjugate roots**. We denote the real root as $r_1$ and the two complex conjugate roots as $r_{2,3} = a \pm ib$. Note that, if at least one of the roots $r_i, i = 1, 2, 3$ is zero, we have $C_3 = -r_1(a^2 + b^2) = 0$ and as already commented, the case $C_1 = C_3 = 0$ leads to a cubic polynomial differential system, which has already been studied. Since we are only interested in analyzing the quartic systems, we assume $C_3 \neq 0$, which leads to $r_1, b \neq 0$.

The unique real singular point at infinity is $p_1 = (0, r_1)$. The linear part of system (4.9) at $p_1$ is

$$\begin{pmatrix} r_1^2 & 0 \\ 0 & -(b^2 + (a - r_1)^2) \end{pmatrix}.$$

Since $r_1, b \neq 0$ by Theorem 2.15 of [22], we get that $p_1$ is a saddle. Then the singularity obtained from the blow down of $p_1$ depends on the position of such singular point with respect to the horizontal axis. So we have the following cases.

**Subcase I.4.1**: $\mathbf{r_1 > 0}$. In this case the local phase portrait at the origin of system (4.4) is given in Figure 4.19.

For the chart $U_2$, since $C_3 = -r_1(a^2 + b^2) < 0$, we have a local phase portrait topologically equivalent to the one of Figure 4.3($C_3 < 0$). Therefore the resulting global phase portrait for Subcase I.4.1 is shown in Figure 4.1(a) of Theorem 4.3.

**Subcase I.4.2**: $\mathbf{r_1 < 0}$. In this case the local phase portrait at the origin of system (4.4) is given in Figure 4.20.

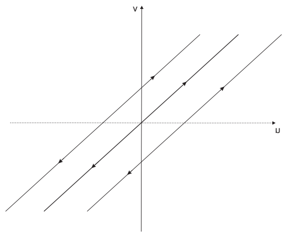


Figure 4.19: Phase portrait of system (4.4) for $r_1 > 0, r_{2,3} = a \pm ib$.

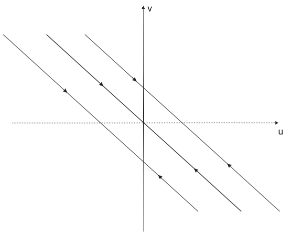


Figure 4.20: Phase portrait of system (4.4) for $r_1 < 0, r_{2,3} = a \pm ib$.

For the chart $U_2$, since $C_3 = -r_1(a^2 + b^2) > 0$, we have a local phase portrait topologically equivalent to the one of Figure 4.3($C_3 > 0$), and the global phase portrait for Subcase I.4.2 is shown in Figure 4.1(a) of Theorem 4.3.

**Case II**: $\mathbf{C_3 = 0}$. We assume $C_1 \neq 0$ since otherwise system (4.3) becomes a cubic differential system.

We first study the chart $U_1$. Then system (4.5) in this chart becomes

$$\begin{aligned} u' &= (1 + u^2)v^2, \\ v' &= -C_1 - B_2uv - A_1v^2 + uv^3. \end{aligned} \tag{4.10}$$

Analyzing system (4.10) we obtain that there is no singular point at infinity in the chart $U_1$.

For the chart $U_2$ we only need to study the origin $O_{U_2}$. The system (4.7) in that chart writes

$$\begin{aligned} u' &= -(1 + u^2)v^2, \\ v' &= -C_1u^3 - B_2uv - A_1uv^2 - uv^3, \end{aligned} \tag{4.11}$$

The linear part of system (4.11) at $O_{U_2}$ is identically zero. Thus it is necessary to apply a directional blow up $v = uw$ to it, resulting the following system

$$\begin{aligned} u' &= -(1 + u^2)u^2w^2, \\ v' &= u(-C_1u - B_2w - A_1uw^2 + w^3). \end{aligned} \tag{4.12}$$

Doing the rescaling of time $dT = uds$ in system (4.12) and we get

$$\begin{aligned} u' &= -(1 + u^2)uw^2, \\ v' &= -C_1u - B_2w - A_1uw^2 + w^3, \end{aligned} \tag{4.13}$$

where the prime now denotes derivative with respect to $T$. The singular points of system (4.13) are $p_1 = (0, 0), p_2 = (0, -\sqrt{B_2})$ and $p_3 = (0, \sqrt{B_2})$. Hence we consider the following cases.

**Subcase II.1:** $\mathbf{B_2 > 0}$. We have $p_1, p_2, p_3$ as three distinct real singular points. The corresponding linear part of system (4.13) at $p_1$ is

$$\begin{pmatrix} 0 & 0 \\ -C_1 & -B_2 \end{pmatrix}.$$

Applying Theorem 2.19 of [22] we conclude that $p_1$ is an unstable node.

The linear parts of system (4.13) at $p_2$ and $p_3$ are identical, namely

$$\begin{pmatrix} -B_2 & 0 \\ -A_1B_2 - C_1 & 2B_2 \end{pmatrix}.$$

Applying Theorem 2.15 of [22] it follows that $p_2$ and $p_3$ are saddles. Going back with the blow down we get the local phase portrait at the origin of system (4.6) topologically equivalent to the one of Figure 4.21.

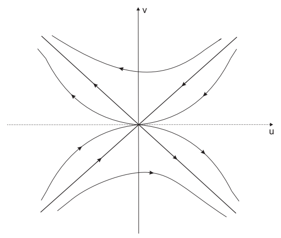



Figure 4.21: Phase portrait of system (4.6) for $B_2 > 0$.

All global phase portraits of planar quartic polynomial differential systems with a uniform isochronous center at the origin are symmetric with respect to the $y-$axis. Moreover, the graphic at the boundary of the period annulus of the uniform isochronous center at the origin is formed by separatrices of infinite singular points. Considering these results and the above calculations, we shall have two distinct global phase portraits for Subcase II.1.

**Subcase II.1.1**: $\mathbf{C_1 = -A_1B_2}$. Under this hypothesis we have the following result.

**Lemma 4.5** (Invariant straight lines)**.** *If $B_2 > 0$, $C_1 = -A_1B_2$ and $C_3 = 0$ in the quartic polynomial differential system* (4.3) *of Theorem 4.1, then the system has the two invariant straight lines $x = \pm\sqrt{1/B_2}$.*

**Proof:** If $B_2 > 0$, $C_1 = -A_1B_2$ and $C_3 = 0$ in system (4.3), then it writes $\dot{x} = (B_2x^2 - 1)(-A_1x^2 + y)$, $\dot{y} = x(1 + A_1y - A_1B_2x^2y + B_2y^2)$. Hence $x = \pm\sqrt{1/B_2}$ are invariant. ■

Using lemma 4.5 we obtain the global phase portrait for Subcase II.1.1 shown in Figure 4.1(e) of Theorem 4.3.

**Subcase II.1.2**: $\mathbf{C_1 \neq -A_1B_2}$. The global phase portrait is shown in Figure 4.1(d) of Theorem 4.3.

**Subcase II.2:** $\mathbf{B_2 < 0}$. The only real singular point is the origin, $p_1 = (0, 0)$. The linear part of system (4.13) at $p_1$ is

$$\begin{pmatrix} 0 & 0 \\ -C_1 & -B_2 \end{pmatrix}.$$

Applying Theorem 2.19 of [22] we conclude that $p_1$ is a saddle.

The local phase portrait at the origin of system (4.6) depends on the sign of the coefficient $C_1$ as shown in Figure 4.22, obtained as in case I.1.1.

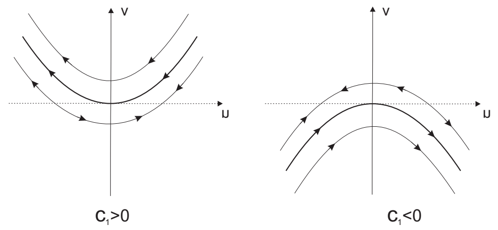



Figure 4.22: Phase portrait of system (4.6) for $B_2 < 0$.

Although the system might present two distinct local phase portraits at the origin, the corresponding global phase portraits are topologically equivalent, and it is shown in Figure 4.1(a) of Theorem 4.3.

**Subcase II.3: $\mathbf{B_2} = \mathbf{0}$**. The only singular point is the origin, $p_1 = (0, 0)$. The linear part of system (4.13) at $p_1$ is

$$\begin{pmatrix} 0 & 0 \\ -C_1 & 0 \end{pmatrix}.$$

Therefore $p_1$ is a nilpotent singular point. Applying Theorem 3.5 of [22], we conclude that $p_1$ is a saddle, similar to the one illustrated in Figure 4.23.

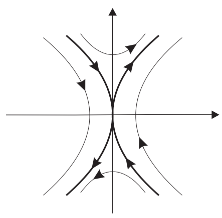


Figure 4.23: saddle of a nilpotent singularity.

The local phase portrait at the origin of system (4.6) in this case is given in Figure 4.24. Then the global phase portrait is shown in Figure 4.1(e) of Theorem 4.3.

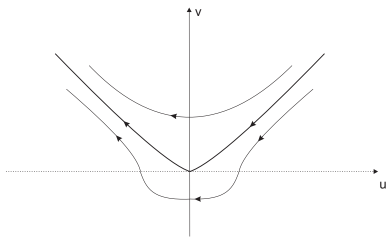



Figure 4.24: Phase portrait of system (4.6) for $C_3 = B_2 = 0$.

**Case III**: $\mathbf{C_1C_3} > \mathbf{0}$. There are only two possible singular points in the chart $U_1$, $(-\sqrt{-C_1/C_3}, 0)$ and $(\sqrt{-C_1/C_3}, 0)$. Since $C_1C_3 > 0$ system (4.5) in $U_1$ has no real singular points.

In the chart $U_2$ the origin, which we denote by $O_{U_2}$ is the only real singular point of system (4.7). Its linear part is

$$\begin{pmatrix} 0 & 0 \\ -C_3 & 0 \end{pmatrix}.$$

Therefore $O_{U_2}$ is a nilpotent singularity and by Theorem 3.5 of [22] it is a cusp, whose behavior depends on the sign of the coefficient $C_3$. Hence the local phase portrait at the origin for system (4.6) might be one of the two shown in Figure 4.3. Then the global phase portrait is shown in Figure 4.1(a) of Theorem 4.3.

**Case IV**: $\mathbf{B_2 = 0, C_1C_3 < 0}$. The expression (4.5) for the system in the local chart $U_1$ is

$$\begin{aligned} u' &= (1+u^2)v^2, \\ v' &= -C_1 - C_3u^2 - A_1v^2 + uv^3, \end{aligned} \tag{4.14}$$

So there are two singular points at infinity in $U_1$, $p_{1,2} = (\pm\sqrt{-C_1/C_3}, 0)$. Similarly in the chart $U_2$ the origin $O_{U_2}$ is a singularity, because the system in that chart is

$$\begin{aligned} u' &= -(1+u^2)v^2, \\ v' &= -C_3u - C_1u^3 - A_1uv^2 - uv^3, \end{aligned} \tag{4.15}$$

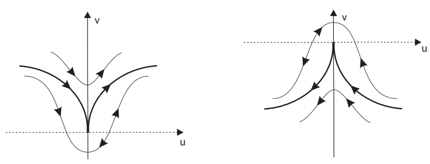
The linear parts of system (4.14) at $p_1$ and $p_2$, and of system (4.15) at $O_{U_2}$ are

$$\begin{pmatrix} 0 & 0 \\ -2C_3\sqrt{-C_1/C_3} & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 2C_3\sqrt{-C_1/C_3} & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ -C_3 & 0 \end{pmatrix},$$

respectively.

The point $O_{U_2}$ in the chart $U_2$ is a nilpotent singularity and by Theorem 3.5 of [22] it is a cusp, whose local phase portrait depends on the sign of the coefficient $C_3$. Hence the local phase portrait at the origin for system (4.6) might be one of the two shown in Figure 4.3.

Since both $p_1$ and $p_2$ are also nilpotent singularities we apply the same theorem to determine that $p_1$ and $p_2$ are cusps, whose behavior also depend on the sign of $C_3$, but are slightly distinct from that one of $O_{U_2}$. For $C_3 < 0$ the local phase portraits at $p_1$ and $p_2$ are topologically equivalent to Figure 4.25(I) and (II) respectively, whereas for $C_3 > 0$, the local phase portraits at $p_1$ and $p_2$ are topologically equivalent to Figure 4.25(II) and (I), respectively.




(I) (II)

Figure 4.25: Phase portrait of system (4.6) for $B_2 = 0$, $c_1C_3 < 0$.

Applying similar arguments as in the previous cases, we shall have three possible configurations for the global phase portraits, they are shown in Figures 4.1(f), (g) and (h) of Theorem 4.3.

**Case V**: $\mathbf{B_2 \neq 0, C_1C_3 < 0}$. The expression (4.5) for the system in the local chart $U_1$ is

$$\begin{aligned} u' &= (1+u^2)v^2, \\ v' &= -C_1 - C_3u^2 - B_2uv^2 - A_1v^2 + uv^3, \end{aligned} \tag{4.16}$$

So there are two singular points at infinity, $p_1 = (\sqrt{-C_1/C_3}, 0)$ and $p_2 = (-\sqrt{-C_1/C_3}, 0)$. Similarly in the chart $U_2$ the origin $O_{U_2}$ is a singularity, because this system writes

$$\begin{aligned} u' &= -(1+u^2)v^2, \\ v' &= -C_3 u - C_1 u^3 - B_2 uv - A_1 uv^2 - uv^3, \end{aligned} \tag{4.17}$$

The linear parts of system (4.16) at $p_1$ and $p_2$, and of system (4.17) at $O_{U_2}$ are respectively

$$\begin{pmatrix} 0 & 0 \\ -2C_3\sqrt{-C_1/C_3} & -B_2\sqrt{-C_1/C_3} \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 \\ 2C_3\sqrt{-C_1/C_3} & B_2\sqrt{-C_1/C_3} \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 \\ -C_3 & 0 \end{pmatrix}.$$

The singular points $p_1$ and $p_2$ are semi-hyperbolic singularities. By Theorem 2.19 of [22], $p_1$ and $p_2$ are saddle-nodes. We remark that in **Case IV,** which only differs from the present one by the vanishing of the coefficient $B_2$, the two singular points $p_1$ and $p_2$ in the chart $U_1$ are cusps. Both cusps and saddle-nodes are singular points of index 0. The cusps in the previous case bifurcate to saddle-nodes by changing $B_2$ from zero to non-zero values.

The local phase portraits at $p_1$ and $p_2$ depend on the values of the coefficients $A_1$, $B_2$, $C_1$ and $C_3$. The possible local phase portraits are shown from figures 4.26 to 4.30. These local phase portraits are obtained as in case I.1.1. The local phase portraits to the cases $A_1, B_2, C_3 > 0$, $C_1 < 0$ and $B_2, C_3 > 0$, $A_1, C_1 < 0$ are topologically equivalent to the ones in Figures 4.30 and 4.27, respectively.

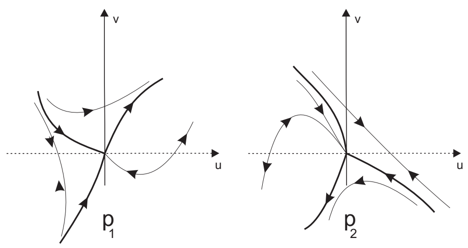



Figure 4.26: Phase portrait of system (4.4) for $A_1, C_1 > 0$, $B_2, C_3 < 0$.

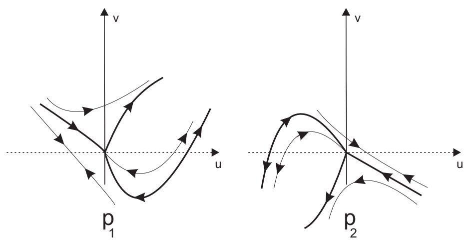



Figure 4.27: Phase portrait of system (4.4) for $C_1 > 0$, $A_1, B_2, C_3 < 0$.

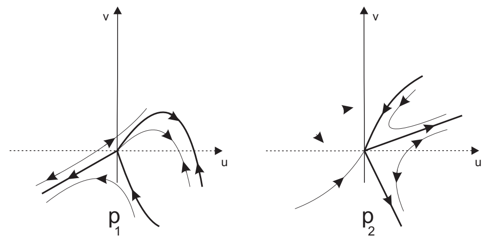



Figure 4.28: Phase portrait of system (4.4) for $A_1, C_3 > 0$, $B_2, C_1 < 0$.

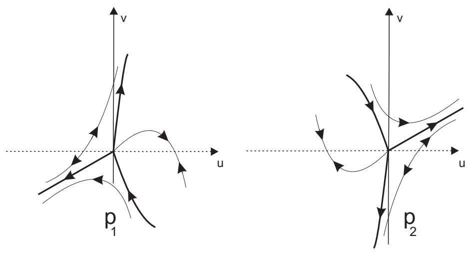



Figure 4.29: Phase portrait of system (4.4) for $C_3 > 0$, $A_1, B_2, C_1 < 0$.

The point $O_{U_2}$ in the chart $U_2$ is a nilpotent singular point. Applying Theorem 3.5 of [22] we see that it is a cusp, whose behavior depends on the sign of the coefficient $C_3$. Hence the local phase portrait at the origin for system (4.6) might be one of the two shown in Figure 4.3.

Using similar arguments as in the previous cases, we shall prove that there exist three possible configurations for the global phase portraits. They are shown in Figures 4.1(i), (j) and (k) of Theorem 4.3. We remark that all three configurations are possible, setting $B_2 = \varepsilon > 0$ in the examples presented in **Case IV**.

### 4.3.2 Proof of Theorem 4.4

The fact that any quartic polynomial differential system of the form (4.2) has a uniform isochronous center at the origin is a direct consequence of Theorem 2.7 (see section 2.2).

In order to provide all possible phase portraits in the Poincaré disc for the uniform isochronous system (4.2) with homogeneous nonlinearities of degree 4, we shall study the finite and infinite singular points of such systems.

**Finite singular points**

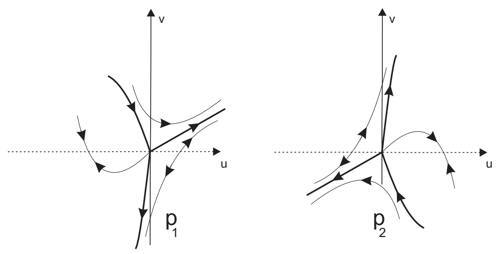



Figure 4.30: Phase portrait of system (4.4) for $A_1 \in \mathbb{R}$, $B_2, C_1 > 0$, $C_3 < 0$.

By proposition 2.2 (see section 2.2) the differential system (4.2) has no finite singular points except the origin.

Propositions 2.2 and 2.4 imply that the boundary of the period annulus of the uniform isochronous center at the origin is a graphic formed by infinite singular points and their separatrices.

**Infinite singular points**

In the chart $U_1$ the differential system (4.2) when $f(x,y)$ is a homogeneous polynomial of degree 3 becomes

$$\begin{aligned} \dot{u} &= v^3(1+u^2), \\ \dot{v} &= v(uv^4 - f(1,u)). \end{aligned} \tag{4.18}$$

Therefore all the points $(u,0)$, for all $u \in \mathbb{R}$ are infinite singular points in $U_1$. In order to obtain the local phase portraits near the infinity, we rescale system (4.18) doing $ds = vdt$ and we obtain

$$\begin{aligned} u' &= v^2(1+u^2), \\ v' &= uv^4 - f(1,u), \end{aligned} \tag{4.19}$$

where the prime denotes derivative with respect to $s$.

Now the infinite singular points of system (4.19) are $(u^*,0)$ with $u^*$ a zero of $f(1,u)$. So at the chart $U_1$ we have at most 3 infinite singular points.

Since at most one more additional infinite singular point can appear, which is the origin of the chart $U_2$, without loss of generality we can assume that all the infinite singular points of system (4.2) after the rescaling $ds = vdt$ are in the local chart $U_1$, otherwise doing a rotation in the coordinates $(x,y)$ this would be the case. So in what follows we do not need to study whether the origin of the chart $U_2$ is an infinite singular point.

To investigate the infinite singular points of system (4.2), we need to split our study into several cases, according to the cubic homogeneous polynomial $f(x,y)$. Taking into account that we can assume that all the infinite singular points after the rescaling $ds = vdt$ are in the local chart $U_1$, the polynomial $f(x,y)$ must have one of the following expressions, with $a \neq 0$

$f_1 = a(y - r_1x)(y - r_2x)(y - r_3x)$, $r_1 < r_2 < r_3$,
$f_2 = a(y - r_1x)^2(y - r_2x)$, $r_1 < r_2$,
$f_3 = a(y - r_1x)^3$,
$f_4 = (\alpha x^2 + \beta xy + \gamma y^2)(y - r_1x)$, $\beta^2 - 4\alpha\gamma < 0$,
$f_5 = (\alpha x^2 + \beta xy + \gamma y^2)y$, $\beta^2 - 4\alpha\gamma < 0$.

In the polynomial $f_k$, $k = 1,2,3$ we can assume that $a = 1$ in system (4.2) by doing the rescaling $(x,y) \mapsto (x/\sqrt[3]{a}, y/\sqrt[3]{a})$, and otherwise we can assume $\gamma = 1$ applying the rescaling $(x,y) \mapsto (x/\sqrt[3]{\gamma}, y/\sqrt[3]{\gamma})$.

In short we must study the phase portraits of the uniform isochronous system (4.2) with $f(x,y)$ in one of the following cases

**Case I:** $f(x,y) = (y - r_1x)(y - r_2x)(y - r_3x)$, $r_1 < r_2 < r_3$;
**Case II:** $f(x,y) = (y - r_1x)^2(y - r_2x)$, $r_1 < r_2$;

**Case III:** $f(x,y) = (y - r_1 x)^3$;

**Case IV:** $f(x,y) = (\alpha x^2 + \beta xy + y^2)(y - r_1 x)$, with $\beta^2 - 4\alpha < 0$;

**Case V:** $f(x,y) = (\alpha x^2 + \beta xy + y^2)y$, with $\beta^2 - 4\alpha < 0$.

Except for the Cases **I** and **IV**, in which system (4.2) depends of 3 parameters, in all other cases it depends at most of 2 parameters.

For the characterization of each local phase portrait, we shall apply the well known results for the hyperbolic and nilpotent singular points, see for instance Theorems 2.15 and 3.5 of [22]. In what follows we study each case in detail.

**Case I.** In the chart $U_1$ the differential system (4.2) becomes

$$\begin{aligned} \dot{u} &= (1+u^2)v^3, \\ \dot{v} &= v[r_1r_2r_3 - (r_1r_2 + r_1r_3 + r_2r_3)u + (r_1+r_2+r_3)u^2 - u^3 + uv^3], \end{aligned} \tag{4.20}$$

Performing the rescaling of time $ds = vdt$ system (4.20) writes as

$$\begin{aligned} u' &= (1+u^2)v^2, \\ v' &= r_1r_2r_3 - (r_1r_2 + r_1r_3 + r_2r_3)u + (r_1+r_2+r_3)u^2 - u^3 + uv^3. \end{aligned} \tag{4.21}$$

The singular points at infinity are $p_1 = (r_1, 0)$, $p_2 = (r_2, 0)$ and $p_3 = (r_3, 0)$. The linear parts of system (4.21) at each of these points are respectively

$$\begin{pmatrix} 0 & 0 \\ (r_2 - r_1)(r_1 - r_3) & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ (r_1 - r_2)(r_2 - r_3) & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 \\ (r_1 - r_3)(r_3 - r_2) & 0 \end{pmatrix}.$$

Since $r_1 < r_2 < r_3$ the terms $r_i - r_j$ for $i \neq j$, $i, j = 1, 2, 3$ never vanish. Consequently the corresponding linear parts of system (4.21) at $p_1$, $p_2$ and $p_3$ are never identically zero and thus they are nilpotent singular points. For each singular point, we perform appropriate translations and rescalings of time to have system (4.21) under the normal form necessary to apply Theorem 3.5 of [22]. Taking into account the hypothesis $r_1 < r_2 < r_3$, we conclude that each one of these 3 singular points is a cusp. Therefore modulus a translation to the origin and undoing the rescaling of time $ds = vdt$, the local phase portrait for each singular point of system (4.20) might be one of the two shown in Figure 4.31.

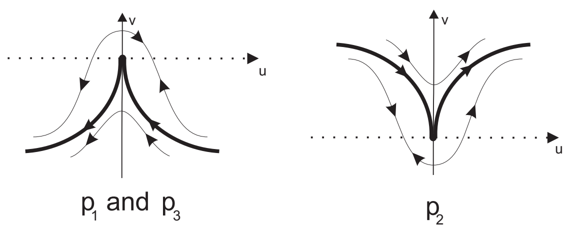



Figure 4.31: Local phase portraits at $p_1$, $p_2$ and $p_3$ of system (4.20). The horizontal axis is filled of singular points.

Each global phase portrait for this case is obtained taking into account: all the local phase portraits of the finite and infinite singular points; the *Existence and Uniqueness Theorem* of solutions; the fact that the boundary of the Poincaré disc consists entirely of singular points; and that the graphic at the boundary of the period annulus of the uniform isochronous center at the origin is formed by separatrices of infinite singular points. Hence the global phase portrait for Case I is topologically equivalent to the ones of Figure 4.2(a) or (b) of Theorem 4.4. We remark that the two configurations are possible by setting convenient values to the real parameters $r_1$, $r_2$ and $r_3$.

**Case II.** In the chart $U_1$ system (4.2) is written as

$$\begin{aligned} \dot{u} &= (1+u^2)v^3, \\ \dot{v} &= v[r_1^2 r_2 - (r_1^2 + 2r_1 r_2)u + (2r_1 + r_2)u^2 - u^3 + uv^3], \end{aligned} \tag{4.22}$$

and after the rescaling of time $ds = vdt$ system (4.22) becomes

$$\begin{aligned} u' &= (1+u^2)v^2, \\ v' &= r_1^2 r_2 - (r_1^2 + 2r_1 r_2)u + (2r_1 + r_2)u^2 - u^3 + uv^3. \end{aligned} \tag{4.23}$$

The singular points at infinity are $p_1 = (r_1, 0)$ and $p_2 = (r_2, 0)$. We first analyze $p_1$. The corresponding linear part of system (4.23) at this singular point is identically zero. Thus it is necessary to apply a directional blow up $(u, v) \mapsto (u, w)$ where $v = uw$, obtaining the following system, modulo a translation of $p_1$ to the origin

$$\begin{aligned} \dot{u} &= u^2(1 + r_1^2 + 2r_1 u + u^2)w^2, \\ \dot{w} &= u[r_2 - r_1 - u - (1 + r_1^2)w^3 - r_1 u w^3]. \end{aligned} \tag{4.24}$$

Performing a change of the independent variable of the form $dT = u\,ds$ in (4.24), we get the system

$$\begin{aligned} u' &= u(1 + r_1^2 + 2r_1 u + u^2)w^2, \\ w' &= r_2 - r_1 - u - (1 + r_1^2)w^3 - r_1 u w^3, \end{aligned} \tag{4.25}$$

where the prime now denotes derivative with respect to T. On the axis $u = 0$ there is a unique singularity $q_1 = (0, \sqrt[3]{(r_2 - r_1)/(1 + r_1^2)})$. The corresponding linear part of system (4.25) at $q_1$ is

$$\begin{pmatrix} -3(1+r_1^2)^{1/3}(r_2 - r_1)^{2/3} & 0 \\ 0 & (1+r_1^2)^{1/3}(r_2 - r_1)^{2/3} \end{pmatrix}.$$

Applying Theorem 2.15 of [22] and the hypothesis $r_1 < r_2$ we conclude that $q_1$ is a saddle. The local phase portrait at $q_1$ for system (4.25) and system (4.24) are shown in Figures 4.32 and 4.33, respectively.

Going back through the blow up we get the local phase portrait at the origin of system (4.23), see Figure 4.34. Finally, taking into account the rescaling of time $ds = vdt$, we obtain that the phase portrait at the origin of system (4.22) is topologically equivalent to the one of Figure 4.35.

Now we perform the study for $p_2$. The corresponding linear part of system (4.23) at this singular point is

$$\begin{pmatrix} 0 & 0 \\ -(r_1 - r_2)^2 & 0 \end{pmatrix}.$$

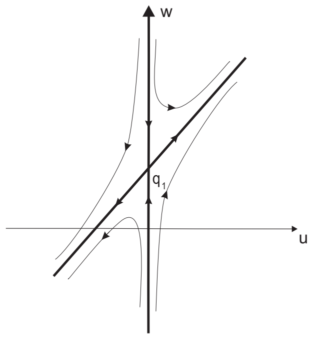



Figure 4.32: Phase portrait of system (4.25).

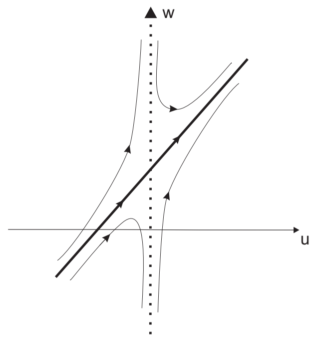



Figure 4.33: Phase portrait of system (4.24). The vertical axis is filled of singular points.

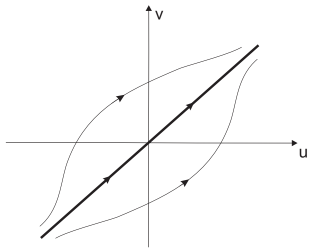



Figure 4.34: Phase portrait of system (4.23).

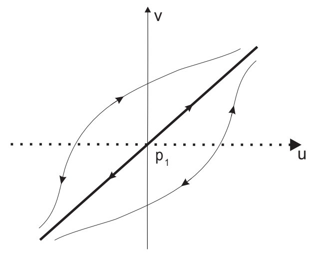



Figure 4.35: Phase portrait of system (4.20). The horizontal axis is filled of singular points.

Since $r_1 < r_2$ by hypothesis, $(r_1 - r_2)$ never vanishes. Therefore $p_2$ is a nilpotent singular point. By performing convenient translation and rescaling of time to have system (4.23) under the normal form necessary to apply Theorem 3.5 of [22], and taking into account the hypothesis $r_1 < r_2$, we conclude that this singular point is a cusp. Therefore modulus a translation to the origin and undoing the rescaling of time $ds = vdt$, the local phase portrait of system (4.22) for $p_2$ is topologically equivalent to the picture on the left side of Figure 4.31. The global phase portrait for Case II is topologically equivalent to the one of Figure 4.2(c) of Theorem 4.4.

**Case III.** In the chart $U_1$ system (4.2) becomes

$$\begin{aligned} \dot{u} &= (1+u^2)v^3, \\ \dot{v} &= v(r_1^3 - 3r_1^2 u + 3r_1 u^2 - u^3 + uv^3), \end{aligned} \tag{4.26}$$

and after the rescaling of time $ds = vdt$ system (4.26) is written as

$$\begin{aligned} u' &= (1+u^2)v^2, \\ v' &= r_1^3 - 3r_1^2 u + 3r_1 u^2 - u^3 + uv^3. \end{aligned} \tag{4.27}$$

The only singular point at infinity is $p_1 = (r_1, 0)$. The corresponding linear part of system (4.27) at this singular point is identically zero. Thus it is necessary to apply a directional blow up $(u, v) \mapsto (u, w)$ where $v = uw$, obtaining the following system, after performing a translation of $p_1$ to the origin

$$\begin{aligned} \dot{u} &= u^2(1+r_1^2+2r_1u+u^2)w^2, \\ \dot{w} &= -u[u+(1+r_1^2)w^3+r_1uw^3]. \end{aligned} \tag{4.28}$$

Performing a change of the independent variable of the form $dT = u\,ds$ in (4.28), we get the system

$$\begin{aligned} u' &= u(1+r_1^2+2r_1u+u^2)w^2, \\ w' &= -(u+(1+r_1^2)w^3+r_1uw^3). \end{aligned} \tag{4.29}$$

On the axis $u=0$ there is a unique singularity $q_1=(0,0)$. The corresponding linear part of system (4.29) at $q_1$ is

$$\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

Applying Theorem 3.5 of [22] we conclude that $q_1$ is a saddle. The local phase portrait at $q_1$ for system (4.29) and system (4.28) are shown in Figures 4.36 and 4.37, respectively

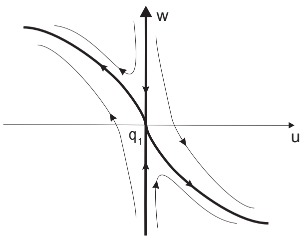



Figure 4.36: Phase portrait of system (4.29).

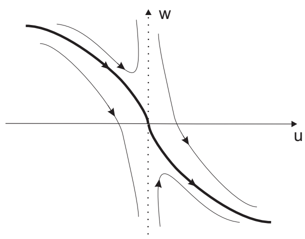



Figure 4.37: Phase portrait of system (4.28). The vertical axis is filled of singular points.

Going back through the blow up we get the local phase portrait at the origin of system (4.27), see Figure 4.38. Finally, we obtain that the phase portrait at the origin of system (4.26) is topologically equivalent to the one of Figure 4.39. Thus the global phase portrait

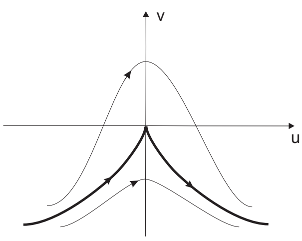



Figure 4.38: Phase portrait of system (4.27).

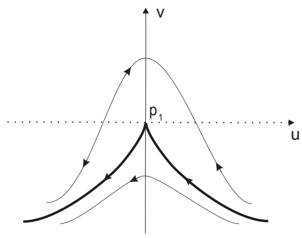



Figure 4.39: Phase portrait of system (4.26). The horizontal axis is filled of singular points.

for this case is topologically equivalent to the one of Figure 4.2(c) of Theorem 4.4.

**Case IV.** In the chart $U_1$ system (4.2) is

$$\begin{aligned} \dot{u} &= (1+u^2)v^3, \\ \dot{v} &= v[r_1\alpha+(r_1\beta-\alpha)u+(r_1-\beta)u^2-u^3+uv^3]. \end{aligned} \tag{4.30}$$

We perform the rescaling of time $ds = vdt$ to obtain

$$\begin{aligned} u' &= (1+u^2)v^2, \\ v' &= r_1\alpha + (r_1\beta - \alpha)u + (r_1 - \beta)u^2 - u^3 + uv^3. \end{aligned} \tag{4.31}$$

The unique singular point at infinity is $p_1 = (r_1, 0)$ and the corresponding linear part of system (4.31) at $p_1$ is

$$\begin{pmatrix} 0 & 0 \\ -\alpha - r_1(r_1+\beta) & 0 \end{pmatrix}.$$

Due to the hypothesis $\beta^2 - 4\alpha < 0$, the expression $-\alpha - r_1(r_1+\beta)$ never vanishes. In fact, if $\alpha = -r_1(r_1+\beta)$ then by the hypothesis we would have $(\beta + 2r_1)^2 < 0$ which is obviously a contradiction. Thus $p_1$ is a nilpotent singular point. Applying Theorem 3.5 of [22] we conclude that the resulting local phase portrait at the origin of system (4.30) is topologically equivalent to the one on the left of Figure 4.31. This local phase portrait is obtained using a similar method applied in the previous cases. The global phase portrait for Case IV is topologically equivalent to the one of Figure 4.2(c) of Theorem 4.4.

**Case V.** In the chart $U_1$ system (4.2) is written as

$$\begin{aligned} \dot{u} &= (1+u^2)v^3, \\ \dot{v} &= uv(-\alpha - \beta u - u^2 + v^3). \end{aligned} \tag{4.32}$$

We perform the rescaling of time $ds = vdt$ to obtain

$$\begin{aligned} u' &= (1+u^2)v^2, \\ v' &= u(-\alpha - \beta u - u^2 + v^3). \end{aligned} \tag{4.33}$$

The origin is the unique singular point at the infinity and the linear part of system (4.33) at $(0,0)$ is

$$\begin{pmatrix} 0 & 0 \\ -\alpha & 0 \end{pmatrix}.$$

Since $\alpha > 0$, due to the hypothesis $\beta^2 - 4\alpha < 0$, the linear part of system (4.33) at $(0,0)$ is never identically zero and therefore the origin is a nilpotent singular point. Applying Theorem 3.5 of [22] and a similar procedure as those applied in the previous cases, we conclude that the resulting local phase portrait at the origin of system (4.32) is topologically equivalent to the one on the left of Figure 4.31. The global phase portrait for this case is topologically equivalent to the one of Figure 4.2(c) of Theorem 4.4.

**Remark 4.6.** *From the proof of Theorem 4.4 it follows that the global phase portrait of any quartic polynomial differential system which can be written into the form* (4.2) *is topologically equivalent to the phase portrait (l) or (m) of Figure 4.2 if we are in Case I, and to the phase portrait (a) of Figure 4.2 otherwise.*

# Chapter 5

# Limit cycles bifurcating from continuous and discontinuous perturbations of uniform isochronous centers of degree 3

In this chapter we develop the averaging theory at any order for computing the periodic solutions of discontinuous piecewise differential system of the form

$$r' = \begin{cases} F^+(\theta, r, \varepsilon) & \text{if} \quad 0 \le \theta \le \alpha, \\ F^-(\theta, r, \varepsilon) & \text{if} \quad \alpha \le \theta \le 2\pi, \end{cases}$$

where $F^{\pm}(\theta, r, \varepsilon) = \sum_{i=1}^{k} \varepsilon^i F_i^{\pm}(\theta, r) + \varepsilon^{k+1} R^{\pm}(\theta, r, \varepsilon)$ with $\theta \in \mathbb{S}^1$ and $r \in D$, where D is an open interval of $\mathbb{R}^+$, and $\epsilon$ is a small real parameter.

Applying this theory we study the bifurcation of limit cycles in planar cubic polynomial differential systems with a uniform isochronous center at the origin when they are perturbed, either inside the class of all continuous cubic polynomial differential systems, or inside the class of all discontinuous piecewise cubic polynomial differential systems with two zones separated by the straight line $y = 0$. Later on in chapter 6 we apply this theory to analyze the number of limit cycles which bifurcate from the uniform isochronous centers of planar quartic polynomial differential systems.

## 5.1 Background

One of the main open problems in the qualitative theory of polynomial differential systems in $\mathbb{R}^2$ is the determination of their limit cycles. Bifurcations of limit cycles have been exhaustively studied in the last century and is closely related to the Hilbert's $16^{th}$ Problem. However, in spite of all efforts, up to now there is no general method to solve this problem.

Bifurcation of limit cycles in continuous planar differential systems are still largely studied. Nonetheless due to the considerable number of discontinuous phenomena in the real world, see for example [9, 54] and the references therein, a significant interest in the investigation of limit cycles of discontinuous piecewise differential systems has arisen. For instance in [44], applying the theory of regularization, the averaging theory is extended up to order 1 for studying the periodic solutions of systems of the form $x' = \varepsilon\big(F(t,x,\varepsilon) + \mathrm{sign}(h(x))G(t,x,\varepsilon)\big)$. In [41] there is a version of the averaging theorem up to order 2 for a bigger class of discontinuous piecewise differential equations $x' = \varepsilon F_1(t,x,\varepsilon)$. Finally in [42] it is stated averaging theorems for studying the periodic solutions of discontinuous piecewise differential equations of the form $x' = F_0(t,x) + \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x,\varepsilon)$.

### 5.1.1 Result on averaging theory

We develop the averaging theory at any order for computing the periodic solutions of discontinuous piecewise differential system of the form

$$r' = \begin{cases} F^+(\theta,r,\varepsilon) & \text{if} \quad 0 \le \theta \le \alpha, \\ F^-(\theta,r,\varepsilon) & \text{if} \quad \alpha \le \theta \le 2\pi, \end{cases} \tag{5.1}$$

where

$$F^\pm(\theta,r,\varepsilon) = \sum_{i=1}^{k} \varepsilon^i F_i^\pm(\theta,r) + \varepsilon^{k+1} R^\pm(\theta,r,\varepsilon). \tag{5.2}$$

The set of discontinuity of system (5.1) is $\Sigma = \{\theta = 0\} \cup \{\theta = \alpha\}$ with $0 < \alpha < 2\pi$. Here $F_i^\pm : \mathbb{S}^1 \times D \to \mathbb{R}$ for $i = 1,\ldots,k$, and $R^\pm : \mathbb{S}^1 \times D \times (-\varepsilon_0,\varepsilon_0) \to \mathbb{R}$ are $\mathcal{C}^{k+1}$ functions, where $D$ is an open and bounded interval of $(0,\infty)$, and $\mathbb{S}^1 \equiv \mathbb{R}/(2\pi)$.

We point out that taking $\alpha = 2\pi$ system (5.1) becomes continuous. So the averaging theory developed in this section also applies to continuous differential systems.

For $i = 1,2,\ldots,k$, we define the *averaging function* $f_i : D \to \mathbb{R}$ of order $i$ as

$$f_i(\rho) = \frac{y_i^+(\alpha,\rho) - y_i^-(\alpha - 2\pi,\rho)}{i!}, \tag{5.3}$$

where $y_i^\pm : \mathbb{S}^1 \times D \to \mathbb{R}$, for $i = 1,2,\ldots,k-1$, are defined recurrently as

$$\begin{aligned} y_i^\pm(\theta,\rho) = \quad & i! \int_0^\theta \Bigg( F_i^\pm(\phi,\rho) + \sum_{l=1}^{i} \sum_{S_l} \frac{1}{b_1!\, b_2! 2!^{b_2} \cdots b_l! l!^{b_l}} \cdot \\ & \partial^L F_{i-l}^\pm(\phi,\rho) \prod_{j=1}^{l} y_j^\pm(\phi,\rho)^{b_j} \Bigg) d\phi, \end{aligned} \tag{5.4}$$

where $S_l$ is the set of all $l$-tuples of non–negative integers $(b_1, b_2, \cdots, b_l)$ satisfying $b_1 + 2b_2 + \cdots + lb_l = l$, and $L = b_1 + b_2 + \cdots + b_l$.

As we shall see the averaging functions $f_i$ control the existence of isolated periodic solutions of system (5.1). These functions are obtained directly from $y_i^\pm$ using (5.4). We provide the explicit formulae of $y_i^\pm$ up to order 7 in Appendix C.

### 5.1.2 Bifurcation of limit cycles from the uniform isochronous centers of degree 3

As already previously stated in this work, a *small limit cycle* is one which bifurcates from a center equilibrium point, and a *medium limit cycle* is one which bifurcates from a periodic orbit surrounding a center.

We study the largest number of small and medium limit cycles for the uniform isochronous cubic centers, when they are perturbed either inside the class of all continuous cubic polynomial differential systems, or inside the class of all discontinuous differential systems formed by two cubic differential systems separated by the straight line $y = 0$.

Given a perturbed differential system, the next result provides a method to write it under the form (5.2) for the continuous case, that is, by setting $\alpha = 2\pi$ in system (5.1).

**Theorem 5.1.** *Consider the unperturbed system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$, where $P, Q : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions, and assume that this system has a continuous family of period solutions $\{\Gamma_h\} \subset \{(x, y) : \mathcal{H}(x, y) = h, h_1 < h < h_2\}$, where $\mathcal{H}$ is a first integral of the system. For a given first integral $H$ assume that $xQ(x, y) - yP(x, y) \neq 0$ for all $(x, y)$ in the period annulus formed by the ovals $\{\Gamma_h\}$. Let $\rho : (\sqrt{h_1}, \sqrt{h_2}) \times [0, 2\pi) \to [0, \infty)$ be a continuous function such that*

$$H(\rho(R, \theta) \cos\theta, \rho(R, \theta) \sin\theta) = R^2$$

*for all $R \in (\sqrt{h_1}, \sqrt{h_2})$ and all $\theta \in [0, 2\pi)$. Then the differential equation which describes the dependence between the square root of the energy $R = \sqrt{h}$ and the angle $\theta$ for the perturbed system $\dot{x} = P(x, y) + \varepsilon p(x, y)$, $\dot{y} = Q(x, y) + \varepsilon q(x, y)$, where $p, q : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions is*

$$\frac{dR}{d\theta} = \varepsilon \frac{\mu(x^2 + y^2)(Qp - Pq)}{2R(Qx - Py)} + \mathcal{O}(\varepsilon^2) \tag{5.5}$$

*where $\mu = \mu(x, y)$ is the integrating factor corresponding to the first integral $H$ of the unperturbed system and $x = \rho(R, \theta) \cos\theta$, $y = \rho(R, \theta) \sin\theta$.*

For more details see [11].

We also need the next results. The first one can be found in Proposition 1 of [39] and the latter in [36].

**Proposition 5.2.** *Let $f_0, \dots, f_n$ be analytic functions defined on an open interval $I \subset \mathbb{R}$. If $f_0, \dots, f_n$ are linearly independent then there exists $s_1, \dots, s_n \in I$ and $\lambda_0, \dots, \lambda_n \in \mathbb{R}$ such that for every $j \in \{1, \dots, n\}$ we have $\displaystyle\sum_{i=0}^{n} \lambda_i f_i(s_j) = 0$.*

We say that the functions $(f_0, \dots, f_n)$ defined on the interval $I$ form an *Extended Chebyshev system* or ET-system on $I$, if and only if, any nontrivial linear combination of these functions has at most $n$ zeros counting their multiplicities and this number is reached. The functions $(f_0, \dots, f_n)$ are an *Extended Complete Chebyshev system* or an ECT-system on $I$ if and only if for any $k \in \{0, 1, \dots, n\}$, $(f_0, \dots, f_k)$ form an ET-system.

**Theorem 5.3.** *Let $f_0, \dots, f_n$ be analytic functions defined on an open interval $I \subset \mathbb{R}$. Then $(f_0, \dots, f_n)$ is an ECT-system on I if and only if for each $k \in \{0, 1, \dots, n\}$ and all $y \in I$ the Wronskian*

$$W(f_0, \dots, f_k)(y) = \begin{vmatrix} f_0(y) & f_1(y) & \cdots & f_k(y) \\ f_0'(y) & f_1'(y) & \cdots & f_k'(y) \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(k)}(y) & f_1^{(k)}(y) & \cdots & f_k^{(k)}(y) \end{vmatrix}$$

*is different from zero.*

In order to study the bifurcation of limit cycles in the planar uniform cubic centers we take into account Proposition 3.2 due to Collins [18], presented in chapter 3 of this work. According to this proposition, a planar cubic differential system with a uniform isochronous center at the origin can be reduced to either of the following two forms

$$\begin{aligned} \dot{x} &= -y(1-x^2), \\ \dot{y} &= \quad x(1+y^2), \end{aligned} \tag{5.6}$$

$$\begin{aligned} \dot{x} &= -y + x^2 + Ax^2y, \\ \dot{y} &= \quad x + xy + Axy^2. \end{aligned} \tag{5.7}$$

with $A \in \mathbb{R}$. For now on we shall call (5.6) and (5.7) as *Collins first form* and *Collins second form*, respectively.

## 5.2 Main results

### 5.2.1 Result on averaging theory

Our result on the periodic solutions of (5.1) is the following.

**Theorem 5.4.** *Assume that, for some $\ell \in \{1, 2, \ldots, k\}$, $f_i = 0$ for $i = 1, 2, \ldots, \ell - 1$ and $f_\ell \neq 0$. If there exists $\rho^* \in D$ such that $f_\ell(\rho^*) = 0$ and $f'_\ell(\rho^*) \neq 0$, then for $|\varepsilon| > 0$ sufficiently small there exists a $2\pi$–periodic solution $r(\theta, \varepsilon)$ of (5.1) such that $r(0, \varepsilon) \to \rho^*$ when $\varepsilon \to 0$.*

Theorem 5.4 is proved in section 5.3.1. This result has been submitted for publication, see [34].

### 5.2.2 Bifurcation of limit cycles from the uniform isochronous centers of degree 3

We consider the following continuous systems

$$\begin{aligned} \dot{x} &= -y + xf(x,y) + \sum_{i=1}^{6} \varepsilon^i p_i(x,y), \\ \dot{y} &= \quad x + yf(x,y) + \sum_{i=1}^{6} \varepsilon^i q_i(x,y), \end{aligned} \tag{5.8}$$

where $f(x, y)$ is as in Theorem 3.1, and the system

$$\dot{x} = -y + x^2y + \varepsilon p_K(x,y), \quad \dot{y} = \quad x + xy^2 + \varepsilon q_K(x,y), \tag{5.9}$$

where

$$\begin{aligned} p_j &= \alpha_1^j x + \alpha_2^j y + \alpha_3^j x^2 + \alpha_4^j xy + \alpha_5^j y^2 + \alpha_6^j x^3 + \alpha_7^j x^2y + \alpha_8^j xy^2 + \alpha_9^j y^3, \\ q_j &= \beta_1^j x + \beta_2^j y + \beta_3^j x^2 + \beta_4^j xy + \beta_5^j y^2 + \beta_6^j x^3 + \beta_7^j x^2y + \beta_8^j xy^2 + \beta_9^j y^3, \end{aligned}$$

$$p_K = \alpha_0 + p_1, \qquad q_K = \beta_0 + q_1.$$

Moreover we consider the discontinuous systems

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathcal{X}(x,y) = \begin{cases} X_1(x,y) & \text{if} \quad y > 0; \\ X_2(x,y) & \text{if} \quad y < 0. \end{cases} \tag{5.10}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathcal{Y}(x,y) = \begin{cases} Y_1(x,y) & \text{if} \quad y > 0; \\ Y_2(x,y) & \text{if} \quad y < 0. \end{cases} \tag{5.11}$$

where

$$X_1(x,y) = \begin{pmatrix} -y + xf(x,y) + \sum_{i=1}^{6} \varepsilon^i p_i(x,y) \\ x + yf(x,y) + \sum_{i=1}^{6} \varepsilon^i q_i(x,y) \end{pmatrix},$$

$$X_2(x,y) = \begin{pmatrix} -y + xf(x,y) + \sum_{i=1}^{6} \varepsilon^i u_i(x,y) \\ x + yf(x,y) + \sum_{i=1}^{6} \varepsilon^i v_i(x,y) \end{pmatrix},$$

$$Y_1(x,y) = \begin{pmatrix} -y + x^2y + \varepsilon p_K(x,y) \\ x + xy^2 + \varepsilon q_K(x,y) \end{pmatrix},$$

$$Y_2(x,y) = \begin{pmatrix} -y + x^2y + \varepsilon u_K(x,y) \\ x + xy^2 + \varepsilon v_K(x,y) \end{pmatrix},$$

$$u_j = \gamma_1^j x + \gamma_2^j y + \gamma_3^j x^2 + \gamma_4^j xy + \gamma_5^j y^2 + \gamma_6^j x^3 + \gamma_7^j x^2 y + \gamma_8^j xy^2 + \gamma_9^j y^3,$$
$$v_j = \delta_1^j x + \delta_2^j y + \delta_3^j x^2 + \delta_4^j xy + \delta_5^j y^2 + \delta_6^j x^3 + \delta_7^j x^2 y + \delta_8^j xy^2 + \delta_9^j y^3,$$
$$u_K = \gamma_0 + u_1, \qquad v_K = \delta_0 + v_1.$$

In what follows we state our results.

**Theorem 5.5.** *For $|\varepsilon| \neq 0$ sufficiently small the maximum number of small limit cycles of the differential system* (5.8) *is 3 using the averaging theory of order 6, and this number can be reached.*

Theorem 5.5 is proved in section 5.3.2 of this chapter.

**Theorem 5.6.** *For $|\varepsilon| \neq 0$ sufficiently small the maximum number of medium limit cycles of the differential system* (5.9) *is 3 using the first order averaging theory and this number can be reached.*

Theorem 5.6 is proved in section 5.3.3.

**Theorem 5.7.** *For $|\varepsilon| \neq 0$ sufficiently small the maximum number of small limit cycles of the discontinuous differential system* (5.10) *is 5 using the averaging method of order 6 and this number can be reached.*

Theorem 5.7 is proved in section 5.3.4.

**Theorem 5.8.** *For $|\varepsilon| \neq 0$ sufficiently small the maximum number of medium limit cycles of the discontinuous differential system* (5.11) *is 7 using the averaging method of first order and this number can be reached.*

Theorem 5.8 is proved in section 5.3.5.

Theorems 5.5 and 5.6 extend previous results presented in [25]. In that work the authors proved the existence of one or two limit cycles in some subfamilies of uniform isochronous cubic centers. Moreover Theorem 5.8 extends the work done in [40] on the number of medium limit cycles which can bifurcate from a family of uniform isochronous quadratic centers perturbed by discontinuous differential systems with the straight line of discontinuity $y = 0$, to the uniform isochronous cubic centers given by the Collins first form.

These results have been published in [31].

# 5.3 Proofs of the results

## 5.3.1 Proof of Theorem 5.4

The proof of Theorem 5.4 is based on the following lemma.

**Lemma 5.9** (Fundamental Lemma). *Let $r^{\pm}(\cdot,\rho,\varepsilon) : [0,\theta_\rho) \to \mathbb{R}^k$ be the solution of $r' = F^{\pm}(\theta, r, \varepsilon)$ with $r^{\pm}(0,\rho,\varepsilon) = \rho$. If $\theta_\rho > T$, then*

$$r^{\pm}(\theta,\rho,\varepsilon) = \rho + \sum_{i=1}^{k} \varepsilon^i \frac{y_i^{\pm}(\theta,\rho)}{i!} + \mathcal{O}_{k+1}(\varepsilon),$$

*where $y_i^{\pm}(t,z)$ for $i = 1,2,\dots,k$ are defined in* (5.4).

The proof of Lemma 5.9 can be found in [43].

Now we prove Theorem 5.4. First of all we have to show that there exists $\varepsilon_0$ sufficiently small such that for each $\rho \in \overline{D}$ and for every $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ the solutions $r^{\pm}(\theta,\rho,\varepsilon)$ are defined for every $\theta \in [0,T]$. Indeed, by the *Existence and Uniqueness Theorem* of solutions (see, for example, Theorem 1.2.4 of [53]), $r^{\pm}(\theta,\rho,\varepsilon)$ is defined for all $0 \le \theta \le \inf\left(T, d/M^{\pm}(\varepsilon)\right)$, for each $r$ with $|r-\rho| < d$ and for every $\rho \in \overline{D}$, where

$$M^{\pm}(\varepsilon) \ge \left| \sum_{i=1}^{k} \varepsilon^i F_i^{\pm}(\theta, r) + \varepsilon^{k+1} R^{\pm}(\theta, r, \varepsilon) \right|.$$

Clearly $\varepsilon$ can be taken sufficiently small in order that $\inf\left(T, d/M^{\pm}(\varepsilon)\right) = T$ for all $\rho \in \overline{D}$. Moreover, since the vector fields $F^{\pm}(\theta, r, \varepsilon)$ are $T$–periodic, the solutions $r^{\pm}(\theta,\rho,\varepsilon)$ can be extended for $\theta \in \mathbb{R}$.

We denote

$$f(\rho,\varepsilon) = r^{+}(\alpha,\rho,\varepsilon) - r^{-}(\alpha - T,\rho,\varepsilon).$$

It is easy to see that system (5.1) for $\varepsilon = \bar{\varepsilon} \in (-\varepsilon_0, \varepsilon_0)$ has a periodic solution passing through $\bar{\rho} \in D$ if and only if $f(\bar{\rho}, \bar{\varepsilon}) = 0$.

From Lemma 5.9 we have that

$$\begin{aligned} f(\rho,\varepsilon) = & \sum_{i=1}^{k} \varepsilon^i \frac{y_i^{+}(\alpha,\rho) - y_i^{-}(\alpha - T,\rho)}{i!} + \mathcal{O}_{k+1}(\varepsilon) \\ = & \sum_{i=1}^{k} \varepsilon^i f_i(\rho) + \mathcal{O}_{k+1}(\varepsilon) \end{aligned}$$

where the function $f_i$ is the one defined in (5.3) for $i = 1, 2, \cdots, k$. From hypothesis

$$f(\rho, \varepsilon) = \varepsilon^r f_r(\rho) + \cdots + \varepsilon^k f_k(\rho) + \mathcal{O}_{k+1}(\varepsilon).$$

Since $f_r(\rho^*) = 0$ and $f_r'(\rho^*) \neq 0$, the *implicit function theorem* applied to the function $\mathcal{F}(\rho, \varepsilon) = f(\rho, \varepsilon)/\varepsilon^r$ guarantees the existence of a differentiable function $\rho(\varepsilon)$ such that $\rho(0) = \rho^*$ and $f(\rho(\varepsilon), \varepsilon) = \varepsilon^r \mathcal{F}(\rho(\varepsilon\ ), \varepsilon) = 0$ for every $|\varepsilon| \neq 0$ sufficiently small. Then the proof of the theorem follows.

### 5.3.2 Proof of Theorem 5.5

We use the Collins first and second forms to prove Theorem 5.5. Due to the fact that system (5.1) becomes continuous by taking $\alpha = 2\pi$, the averaging theory developed in subsection 5.1.1 also applies to continuous differential systems.

We shall use the averaging theory given in Theorem 5.4 up to order 6 to study the limit cycles for Collins first and second forms. In order to calculate the respective averaging functions $f_i$, for $i = 1, \ldots, 6$, we used the formulae of $y_i$, for $i = 1, \ldots, 6$ presented in Appendix C (for simplicity we omit the "$\pm$" in the notations of the functions $f_i$ and $y_i$ because we are in the continuous case).

**Collins first form**

Consider system (5.8) with $f(x, y) = xy$, that is, the unperturbed system is the Collins first form.

$$\begin{aligned} \dot{x} &= -y + x^2 y + \sum_{i=1}^{6} \varepsilon^i p_i(x, y), \\ \dot{y} &= \ \ x + xy^2 + \sum_{i=1}^{6} \varepsilon^i q_i(x, y). \end{aligned} \tag{5.12}$$

In order to analyze the Hopf bifurcation for system (5.12), applying Theorem 5.4, we introduce a small parameter $\varepsilon$ doing the change of coordinates $x = \varepsilon X,\ y = \varepsilon Y$. After that we perform the polar change of coordinates $X = r\cos\theta, Y = r\sin\theta$, and by doing a Taylor expansion truncated at the $6^{th}$ order in $\varepsilon$ we obtain an expression for $dr/d\theta$ similar to (5.2) with $\alpha = 2\pi$. The explicit expression is quite large so we omit it.

System (5.12) is a polynomial system. The functions $F_i(\theta, r)$, $i = 1, \ldots, 6$ and $R(\theta, r, \varepsilon)$ (we omit the "$\pm$" in the notations of the functions $F_i$ and $R$ because we are in the continuous case) of system (5.12) are analytic. Moreover these functions are $2\pi-$periodic because the variable $\theta$ appears through sines and cosines. Hence the assumptions of Theorem 5.4 are satisfied. We take the open interval $D$ of Theorem 5.4 as $D = \{r : 0 < r < 1\}$ because the Collins first form has the period annulus of the center in the band $-1 < x < 1$.

Applying Theorem 5.4 we obtain the averaging function of first order

$$f_1(r) = \pi r(\alpha_1^1 + \beta_2^1).$$

Clearly $f_1(r)$ has no solution in $D$. Thus there is no small limit cycle which bifurcates from the uniform isochronous center at the origin by the averaging method of first order.

Setting $\beta_2^1 = -\alpha_1^1$ we obtain $f_1(r) = 0$. So we can apply the averaging theory of second order using Theorem 5.4, obtaining the averaging function of second order.

$$f_2(r) = \pi r(\alpha_1^2 + \beta_2^2).$$

Since $f_2(r)$ has no solution in $D$, there is no small limit cycle which bifurcates from the uniform isochronous center at the origin applying the averaging method of second order. Doing $\beta_2^2 = -\alpha_1^2$ we get $f_2(r) = 0$, and then we can apply the averaging method of third order obtaining

$$f_3(r) = r(A_3 r^2 + A_1),$$

where

$$A_3 = \frac{\pi}{4}(4\alpha_1^1 + 3\alpha_6^1 + \alpha_8^1 + \beta_7^1 + 3\beta_9^1), \quad A_1 = \pi(\alpha_1^3 + \beta_2^3).$$

The rank of the Jacobian matrix of the function $\mathcal{A} = (A_1, A_3)$ with respect to the variables $\alpha_1^1, \alpha_1^3, \alpha_6^1, \alpha_8^1, \beta_2^3, \beta_7^1, \beta_9^1$ is maximal. Then the coefficients $A_1$ and $A_3$ are linearly independent in their variables.

The averaging function $f_3(r)$ has one solution in $D$ if $0 < -A_1/A_3 < 1$. Hence applying Theorem 5.4 it is proved that at most 1 small limit cycle can bifurcate from the uniform isochronous center at the origin and this number can be reached.

In order to apply the averaging method of order 4, we need to have $f_3(r) = 0$ so we set $\beta_2^3 = -\alpha_1^3$ and $\beta_7^1 = -(4\alpha_1^1 + 3\alpha_6^1 + \alpha_8^1 + 3\beta_9^1)$. The resulting averaging function of fourth order is

$$f_4(r) = r(B_3 r^2 + B_1),$$

where

$$\begin{aligned} B_3 =& \frac{\pi}{4}(4\alpha_1^1\alpha_2^1 + 2\alpha_1^1\alpha_7^1 + 2\alpha_1^1\beta_8^1 + 3\beta_1^1\beta_9^1 + \alpha_2^1\alpha_8^1 + 3\alpha_2^1\beta_9^1 - 2\alpha_3^1\beta_3^1 \\ & + \alpha_3^1\alpha_4^1 - \beta_3^1\beta_4^1 + \alpha_4^1\alpha_5^1 - \beta_4^1\beta_5^1 + 2\alpha_5^1\beta_5^1 + \beta_1^1\alpha_8^1 + 4\alpha_1^2 + 3\alpha_6^2 \\ & + \beta_7^2 + \alpha_8^2 + 3\beta_9^2), \\ B_1 =& \pi(\alpha_1^4 + \beta_2^4). \end{aligned}$$

The rank of the Jacobian matrix of the function $\mathcal{B} = (B_1, B_3)$ with respect to its variables is maximal, thus $B_1, B_3$ are linearly independent in their variables.

Therefore $f_4(r)$ has one solution in $D$ if $0 < -B_1/B_3 < 1$. Hence we can show that at most 1 small limit cycle can bifurcate from the uniform isochronous center and this number can be reached, applying Theorem 5.4. Solving $B_1 = 0$ for $\beta_2^4$ and $B_3 = 0$ for $\beta_7^2$, we obtain $f_4(r) = 0$ so we can apply the averaging theory of order 5, and its corresponding averaging function is

$$f_5(r) = r(C_5 r^4 + C_3 r^2 + C_1),$$

where

$$\begin{aligned} C_5 =& \frac{\pi}{4}(2\alpha_1^1 + 2\alpha_6^1 + \alpha_8^1 + \beta_9^1), \\ C_3 =& \frac{\pi}{4}(4\alpha_1^1(\alpha_2^1)^2 + 2\alpha_1^1\alpha_2^1\alpha_7^1 + 2\alpha_1^1\alpha_2^1\beta_8^1 + 2\alpha_1^1(\alpha_3^1)^2 - \alpha_1^1\alpha_3^1\beta_4^1 \\ & + \beta_1^1\beta_3^1\beta_4^1 + 2\alpha_1^1\alpha_3^1\alpha_5^1 - 2\alpha_1^1\beta_3^1\beta_5^1 + \alpha_1^1(\alpha_4^1)^2 - \alpha_1^1(\beta_4^1)^2 - \alpha_1^1\beta_3^1\alpha_4^1 \end{aligned}$$

$$
\begin{aligned}
&+\alpha_1^1\alpha_4^1\beta_5^1-2\alpha_1^1(\beta_5^1)^2+\alpha_1^1\beta_4^1\alpha_5^1+4\alpha_1^1\alpha_2^2+2\alpha_1^1\alpha_7^2+2\alpha_1^1\beta_8^2\\
&+3\beta_1^1\beta_9^2+(\alpha_2^1)^2\alpha_8^1+3(\alpha_2^1)^2\beta_9^1+3\beta_1^1\alpha_2^1\beta_9^1+\alpha_2^1\alpha_3^1\alpha_4^1+2\alpha_2^1\alpha_4^1\alpha_5^1\\
&-\alpha_2^1\beta_4^1\beta_5^1+4\alpha_2^1\alpha_5^1\beta_5^1+\beta_1^1\alpha_2^1\alpha_8^1+4\alpha_2^1\alpha_1^2+\alpha_2^1\alpha_8^2+3\alpha_2^1\beta_9^2\\
&+2\beta_1^1\alpha_3^1\beta_3^1-2\alpha_3^1\beta_3^2+\alpha_3^1\alpha_4^2-\beta_3^1\beta_4^2+\beta_1^1\alpha_4^1\alpha_5^1+\alpha_4^1\alpha_3^2-\beta_4^1\beta_3^2\\
&+\alpha_4^1\alpha_5^2-\beta_4^1\beta_5^2+2\beta_1^1\alpha_5^1\beta_5^2+\alpha_5^1\alpha_4^2-\beta_5^1\beta_4^2+2\alpha_5^1\beta_5^2+2\alpha_7^1\alpha_1^2\\
&+\alpha_8^1\beta_1^2+\alpha_8^1\alpha_2^2+3\beta_9^1\beta_1^2+2\beta_8^1\alpha_1^2+3\beta_9^1\alpha_2^2-2\beta_3^1\alpha_3^2+2\beta_5^1\alpha_5^2\\
&+\beta_1^1\alpha_8^2+4\alpha_1^3+3\alpha_6^3+\beta_7^3+\alpha_8^3+3\beta_9^3),\\
C_1=&\pi(\alpha_1^5+\beta_2^5),
\end{aligned}
$$

and since the rank of the Jacobian matrix of the function $\mathcal{C}=(C_1,C_3,C_5)$ with respect to its variables is maximal, $C_i$, $i=1,3,5$ are linearly independent in their variables.

The averaging function of fifth order $f_5(r)$ can have at most 2 solutions in $D$, see Proposition 5.2. Thus applying Theorem 5.4 it is proved that at most 2 small limit cycles can bifurcate from the uniform isochronous center at the origin and this number can be reached, using the averaging method of order 5.

In order to apply the averaging theory of order 6 we solve $C_1=0$ for $\beta_2^5$, $C_3=0$ for $\beta_7^3$ and $C_5$ for $\beta_9^1$, resulting that $f_5(r)=0$. Calculating the averaging function of sixth order we have

$$f_6(r)=r(D_5r^4+D_4r^3+D_3r^2+D_1),$$

where

$$
\begin{aligned}
D_5=&-\frac{1}{384}\pi(45\alpha_1^1\alpha_2^1+192\alpha_6^1\alpha_2^1-112\alpha_3^1\alpha_4^1-112\alpha_4^1\alpha_5^1-192\alpha_1^1\alpha_7^1\\
&+96\alpha_1^1\alpha_9^1+288\alpha_6^1\alpha_9^1+96\alpha_8^1\alpha_9^1-192\alpha_1^2-192\alpha_6^2-96\alpha_8^2\\
&+192\alpha_6^1\beta_1^1+288\alpha_3^1\beta_3^1+64\alpha_5^1\beta_3^1-16\beta_3^1\beta_4^1+320\alpha_3^1\beta_5^1+96\alpha_5^1\beta_5^1\\
&+237\alpha_1^1\beta_1^1-16\beta_4^1\beta_5^1+288\alpha_1^1\beta_6^1+288\alpha_6^1\beta_6^1+96\alpha_8^1\beta_6^1-96\beta_9^2),\\
D_4=&-\frac{1}{8}\alpha_1^1\pi(\alpha_2^1+\beta_1^1)(\alpha_4^1+\beta_3^1+2\beta_5^1),\\
D_3=&-\frac{1}{512}\pi(108\alpha_2^1(\alpha_1^1)^3+36\beta_1^1(\alpha_1^1)^3-384\alpha_3^1\alpha_4^1(\alpha_1^1)^2\\
&+72\alpha_1^2(\alpha_1^1)^2+256\alpha_3^1\beta_3^1(\alpha_1^1)^2+128\beta_3^1\beta_4^1(\alpha_1^1)^2-256\alpha_5^1\beta_5^1(\alpha_1^1)^2\\
&+384\beta_4^1\beta_5^1(\alpha_1^1)^2+319(\alpha_2^1)^3\alpha_1^1-27(\beta_1^1)^3\alpha_1^1-256\alpha_2^1(\alpha_3^1)^2\alpha_1^1\\
&-256\alpha_2^1(\alpha_4^1)^2\alpha_1^1+9\alpha_2^1(\beta_1^1)^2\alpha_1^1+128\alpha_2^1(\beta_4^1)^2\alpha_1^1-128\beta_1^1(\beta_4^1)^2\alpha_1^1\\
&-128\alpha_4^1\alpha_5^1(\alpha_1^1)^2+512\alpha_2^1(\beta_5^1)^2\alpha_1^1-512\alpha_2^1\alpha_3^1\alpha_5^1\alpha_1^1-256\alpha_5^1\alpha_3^2\alpha_1^1\\
&+572\alpha_2^1\alpha_2^2\alpha_1^1-256\alpha_7^1\alpha_2^2\alpha_1^1-512\alpha_3^1\alpha_3^2\alpha_1^1-256(\alpha_2^1)^2\alpha_7^1\alpha_1^1\\
&-256\alpha_4^1\alpha_4^2\alpha_1^1-256\alpha_3^1\alpha_5^2\alpha_1^1-256\alpha_2^1\alpha_7^2\alpha_1^1+256\alpha_2^3\alpha_1^1-256\alpha_7^3\alpha_1^1\\
&+867(\alpha_2^1)^2\beta_1^1\alpha_1^1+256(\alpha_3^1)^2\beta_1^1\alpha_1^1+828\alpha_2^2\beta_1^1\alpha_1^1+128\alpha_2^1\alpha_4^1\beta_3^1\alpha_1^1\\
&+128\alpha_4^2\beta_3^1\alpha_1^1-128\alpha_4^1\beta_1^1\beta_3^1\alpha_1^1+128\alpha_2^1\alpha_3^1\beta_4^1\alpha_1^1-256\alpha_2^1\alpha_5^1\beta_4^1\alpha_1^1\\
&+128\alpha_3^2\beta_4^1\alpha_1^1-128\alpha_5^2\beta_4^1\alpha_1^1-128\alpha_3^1\beta_1^1\beta_4^1\alpha_1^1-256\alpha_2^1\alpha_4^1\beta_5^1\alpha_1^1\\
&-128\alpha_4^2\beta_5^1\alpha_1^1+256\alpha_2^1\beta_3^1\beta_5^1\alpha_1^1-256\beta_1^1\beta_3^1\beta_5^1\alpha_1^1-256(\alpha_2^1)^2\beta_8^1\alpha_1^1\\
&-256\alpha_2^2\beta_8^1\alpha_1^1+828\alpha_2^1\beta_1^2\alpha_1^1+60\beta_1^1\beta_1^2\alpha_1^1+128\alpha_4^1\beta_3^2\alpha_1^1\\
&+256\beta_5^1\beta_3^2\alpha_1^1+128\alpha_3^1\beta_4^2\alpha_1^1-128\alpha_5^1\beta_4^2\alpha_1^1+256\beta_4^1\beta_4^2\alpha_1^1
\end{aligned}
$$

$$
\begin{aligned}
&- 128\alpha_4^1\beta_5^2\alpha_1^1 + 256\beta_3^1\beta_5^2\alpha_1^1 + 512\beta_5^1\beta_5^2\alpha_1^1 - 256\alpha_2^1\beta_8^2\alpha_1^1 \\
&+ 768\beta_1^3\alpha_1^1 - 256\beta_8^3\alpha_1^1 + 30\alpha_1^2(\beta_1^1)^2 + 128\alpha_1^2(\beta_4^1)^2 \\
&+ 256\alpha_1^2(\beta_5^1)^2 - 128(\alpha_2^1)^2\alpha_3^1\alpha_4^1 - 384(\alpha_2^1)^2\alpha_4^1\alpha_5^1 + 768(\alpha_2^1)^3\alpha_6^1 \\
&+ 256(\alpha_2^1)^3\alpha_8^1 - 482(\alpha_2^1)^2\alpha_1^2 - 256(\alpha_3^1)^2\alpha_1^2 - 128(\alpha_4^1)^2\alpha_1^2 \\
&- 256\alpha_3^1\alpha_5^1\alpha_1^2 - 256\alpha_2^1\alpha_7^1\alpha_1^2 - 128\alpha_3^1\alpha_4^1\alpha_2^2 - 256\alpha_4^1\alpha_5^1\alpha_2^2 \\
&+ 1536\alpha_2^1\alpha_6^1\alpha_2^2 + 512\alpha_2^1\alpha_8^1\alpha_2^2 - 512\alpha_1^2\alpha_2^2 - 128\alpha_2^1\alpha_4^1\alpha_3^2 \\
&- 128\alpha_2^1\alpha_3^1\alpha_4^2 - 256\alpha_2^1\alpha_5^1\alpha_4^2 - 128\alpha_3^2\alpha_4^2 - 256\alpha_2^1\alpha_4^1\alpha_5^2 \\
&- 128\alpha_4^2\alpha_5^2 - 256\alpha_1^2\alpha_7^2 - 128(\alpha_2^1)^2\alpha_8^2 - 128\alpha_2^2\alpha_8^2 - 512\alpha_2^1\alpha_1^3 \\
&- 256\alpha_7^1\alpha_1^3 + 768\alpha_6^1\alpha_2^3 + 256\alpha_8^1\alpha_2^3 - 128\alpha_4^1\alpha_3^3 - 128\alpha_3^1\alpha_4^3 \\
&- 128\alpha_5^1\alpha_4^3 - 128\alpha_4^1\alpha_5^3 - 128\alpha_2^1\alpha_8^3 - 512\alpha_1^4 - 384\alpha_6^4 - 128\alpha_8^4 \\
&- 256\alpha_2^1\alpha_4^1\alpha_5^1\beta_1^1 + 768(\alpha_2^1)^2\alpha_6^1\beta_1^1 + 256(\alpha_2^1)^2\alpha_8^1\beta_1^1 + 60\alpha_2^1\alpha_1^2\beta_1^1 \\
&+ 768\alpha_6^1\alpha_2^2\beta_1^1 + 256\alpha_8^1\alpha_2^2\beta_1^1 - 128\alpha_5^1\alpha_4^2\beta_1^1 - 128\alpha_4^1\alpha_5^2\beta_1^1 \\
&- 128\alpha_2^1\alpha_8^2\beta_1^1 - 128\alpha_8^3\beta_1^1 + 256\alpha_3^1(\beta_1^1)^2\beta_3^1 + 128\alpha_4^1\alpha_1^2\beta_3^1 \\
&+ 256\alpha_3^3\beta_3^1 - 256\alpha_3^2\beta_1^1\beta_3^1 + 128\alpha_3^1\alpha_1^2\beta_4^1 - 128\alpha_5^1\alpha_1^2\beta_4^1 \\
&+ 128(\beta_1^1)^2\beta_3^1\beta_4^1 - 768(\alpha_2^1)^2\alpha_5^1\beta_5^1 - 128\alpha_4^1\alpha_1^2\beta_5^1 - 512\alpha_5^1\alpha_2^2\beta_5^1 \\
&- 512\alpha_2^1\alpha_5^2\beta_5^1 - 256\alpha_5^3\beta_5^1 - 512\alpha_2^1\alpha_5^1\beta_1^1\beta_5^1 - 256\alpha_5^2\beta_1^1\beta_5^1 \\
&+ 256\alpha_1^2\beta_3^1\beta_5^1 + 128(\alpha_2^1)^2\beta_4^1\beta_5^1 + 128\alpha_2^2\beta_4^1\beta_5^1 - 256\alpha_2^1\alpha_1^2\beta_8^1 \\
&- 256\alpha_1^3\beta_8^1 - 128\alpha_4^1\alpha_5^1\beta_1^2 + 768\alpha_2^1\alpha_6^1\beta_1^2 + 256\alpha_2^1\alpha_8^1\beta_1^2 - 128\alpha_8^2\beta_1^2 \\
&- 256\alpha_3^1\beta_3^1\beta_1^2 - 128\beta_3^1\beta_4^1\beta_1^2 - 256\alpha_5^1\beta_5^1\beta_1^2 + 256\alpha_3^2\beta_3^2 - 256\alpha_3^1\beta_1^1\beta_3^2 \\
&- 128\beta_1^1\beta_4^1\beta_3^2 - 128\beta_1^1\beta_3^1\beta_4^2 + 128\alpha_2^1\beta_5^1\beta_4^2 + 128\beta_3^2\beta_4^2 - 512\alpha_2^1\alpha_5^1\beta_5^2 \\
&- 256\alpha_5^2\beta_5^2 - 256\alpha_5^1\beta_1^1\beta_5^2 + 128\alpha_2^1\beta_4^1\beta_5^2 + 128\beta_4^2\beta_5^2 - 256\alpha_1^2\beta_8^2 \\
&- 384(\alpha_2^1)^2\beta_9^2 - 384\alpha_2^2\beta_9^2 - 384\alpha_2^1\beta_1^1\beta_9^2 - 384\beta_1^2\beta_9^2 + 768\alpha_6^1\beta_1^3 \\
&+ 256\alpha_8^1\beta_1^3 + 256\alpha_3^1\beta_3^3 + 128\beta_4^1\beta_3^3 + 128\beta_3^1\beta_4^3 + 128\beta_5^1\beta_4^3 \\
&- 256\alpha_5^1\beta_5^3 + 128\beta_4^1\beta_5^3 - 384\alpha_2^1\beta_9^3 - 384\beta_1^1\beta_9^3 - 128\beta_7^4 - 384\beta_9^4),
\end{aligned}
$$

$$D_1 =\pi(\alpha_1^6 + \beta_2^6),$$

Therefore $f_6(r)$ can have 3 solutions in $D$ according to Proposition 5.2. By Theorem 5.3 $(r, r^3, r^4, r^5)$ is an ECT-system because $W_1(z) = z$, $W_2(z) = 2z^3$, $W_3(z) = 6z^5$, $W_4(z) = 48z^7$ are nonzero in $D$, where $W_j(z), j = 1, 2, 3$ denotes the Wronskian of the first $j$ functions in $(r, r^3, r^4, r^5)$. Moreover $D_1$, $D_3$, $D_4$ and $D_5$ are linearly independent functions. In fact only $D_5$ presents the coefficients $\alpha_9^1$ and $\alpha_6^2$, only $D_3$ has the coefficient $\alpha_2^2$, and $D_1$ is the only one with the coefficients $\alpha_1^6$ and $\beta_2^6$. We claim that $D_4$ is also linearly independent of the other coefficients. Suppose that this is false. Then there exist real numbers $k, l, m$ not all zero such that $D_4 = kD_1 + lD_3 + mD_5$. But $D_1$ is the only one with the variables $\alpha_1^6$ and $\beta_2^6$, so in order to $D_4$ does not present these variables we must set $k = 0$. Since the other two functions $D_3$ and $D_5$ also have variables which uniquely appears in their respective expressions, the same argument holds so $l = m = 0$. But then $D_4 \equiv 0$, which is a contradiction. Therefore $D_1$, $D_3$, $D_4$ and $D_5$ are linearly independent functions.

Hence applying the averaging theory of order 6 we can show that at most 3 small limit cycles can bifurcate from the uniform isochronous center at the origin and this number can be reached.

Now we perform similar calculations to the Collins second form.

**Collins second form**

Consider system (5.8) with $f(x, y) = x + Axy$.

$$\begin{aligned} \dot{x} &= -y + x^2 + Ax^2y + \sum_{i=1}^{6} \varepsilon^i p_i(x, y), \\ \dot{y} &= \quad x + xy + Axy^2 + \sum_{i=1}^{6} \varepsilon^i q_i(x, y), \end{aligned} \tag{5.13}$$

where $A \in \mathbb{R}\backslash\{0\}$, since for $A = 0$ system (5.13) is a quadratic system, which has been exhaustively studied.

Similarly to the previous procedures applied in the Collins first form, in order to analyze the Hopf bifurcation for system (5.13), applying Theorem 5.4, we introduce a small parameter $\varepsilon$ doing the change of coordinates $x = \varepsilon X,\ y = \varepsilon Y$. After that we perform the polar change of coordinates $X = r\cos\theta, Y = r\sin\theta$, and by doing a Taylor expansion truncated at the $6^{th}$ order in $\varepsilon$ we obtain an expression for $dr/d\theta$ similar to (5.2) with $\alpha = 2\pi$. Using the same arguments as in the proof of the Collins first form the differential equation $dr/d\theta = \ldots$ satisfies the assumptions of Theorem 5.4. We take $D = \{r : 0 < r < r_0 < 1\}$, where the unperturbed system has periodic solutions passing through the point $(r < r_0, \theta = 0)$.

Applying Theorem 5.4 we obtain the averaging function of first order

$$f_1(r) = \pi r(\alpha_1^1 + \beta_2^1).$$

Clearly $f_1(r)$ has no solution in $D$. Setting $\beta_2^1 = -\alpha_1^1$ we obtain $f_1(r) = 0$. So we can apply the averaging theory of order 2 using Theorem 5.4, obtaining

$$f_2(r) = \pi r(\alpha_1^2 + \beta_2^2)$$

Again $f_2(r)$ has no solution in $D$. Doing $\beta_2^2 = -\alpha_1^2$ we get $f_2(r) = 0$. Then we can apply the averaging method of third order

$$f_3(r) = r(A_3 r^2 + A_1),$$

where

$$\begin{aligned} A_3 =& \frac{\pi}{4}(4A\alpha_1^1 + \alpha_4^1 + 3\alpha_6^1 + \alpha_8^1 - 3\beta_3^1 - \beta_5^1 + \beta_7^1 + 3\beta_9^1), \\ A_1 =& \pi(\alpha_1^3 + \beta_2^3). \end{aligned}$$

Thus $f_3(r)$ can have one solution in $D$ if $0 < -A_1/A_3 < r_0$. In order to apply the averaging method of forth order, we need to have $f_3(r) = 0$. We set $\beta_2^3 = -\alpha_1^3$ and $\beta_7^1 = -(4A\alpha_1^1 + \alpha_4^1 + 3\alpha_6^1 + \alpha_8^1 - 3\beta_3^1 - \beta_5^1 + 3\beta_9^1)$. The resulting averaging function of fourth order is

$$f_4(r) = r(B_3 r^2 + B_1),$$

where

$$
\begin{aligned}
B_3 =&\frac{\pi}{4}(4A\alpha_1^1\alpha_2^1 + 4A\alpha_1^2 + 3\alpha_1^1\alpha_3^1 + 3\beta_1^1\beta_3^1 - 3\alpha_1^1\beta_4^1 + 3\alpha_1^1\alpha_5^1 + 2\alpha_1^1\alpha_7^1 \\
&+ 2\alpha_1^1\beta_8^1 + 3\beta_1^1\beta_9^1 + \alpha_2^1\alpha_4^1 - \alpha_2^1\beta_5^1 + \alpha_2^1\alpha_8^1 + 3\alpha_2^1\beta_9^1 - 2\alpha_3^1\beta_3^1 \\
&+ \alpha_3^1\alpha_4^1 - \beta_3^1\beta_4^1 + \alpha_4^1\alpha_5^1 - \beta_4^1\beta_5^1 + 2\alpha_5^1\beta_5^1 + \beta_1^1\alpha_8^1 - 3\beta_3^2 + \alpha_4^2 \\
&- \beta_5^2 + 3\alpha_6^2 + \beta_7^2 + \alpha_8^2 + 3\beta_9^2), \\
B_1 =&\pi(\alpha_1^4 + \beta_2^4).
\end{aligned}
$$

Then $f_4(r)$ has one solution in $D$ if $0 < -B_1/B_3 < r_0$. Solving $B_1 = 0$ for $\beta_2^4$, and $B_3 = 0$ for $\beta_7^2$, we obtain $f_4(r) = 0$, and we can apply the averaging theory of order 5. Its corresponding averaging function is

$$f_5(r) = r(C_5 r^4 + C_3 r^2 + C_1),$$

where

$$
\begin{aligned}
C_5 =&\frac{\pi}{24}(12A^2\alpha_1^1 + 18A\alpha_1^1 - 17A\beta_3^1 + 7A\alpha_4^1 - 19A\beta_5^1 + 12A\alpha_6^1 + 6A\alpha_8^1 \\
&+ 6A\beta_9^1 - 12\beta_3^1 + 6\alpha_4^1 - 6\beta_5^1 + 18\alpha_6^1 + 12\alpha_8^1 + 18\beta_9^1), \\
C_3 =&\frac{\pi}{4}(4A\alpha_1^1(\alpha_2^1)^2 + 4A\alpha_1^1\alpha_2^2 + 4A\alpha_2^1\alpha_1^2 + 4A\alpha_1^3 - 3(\alpha_1^1)^2\beta_3^1 - 3(\beta_1^1)^2\beta_3^1 \\
&+ 3(\alpha_1^1)^2\alpha_4^1 - 3(\alpha_1^1)^2\beta_5^1 + 3\beta_1^1\alpha_1^1\beta_4^1 + 3\alpha_1^1\alpha_2^1\alpha_3^1 - 3\alpha_1^1\alpha_2^1\beta_4^1 + \beta_7^3 \\
&+ 6\alpha_1^1\alpha_2^1\alpha_5^1 + 2\alpha_1^1\alpha_2^1\alpha_7^1 + 2\alpha_1^1\alpha_2^1\beta_8^1 + 2\alpha_1^1(\alpha_3^1)^2 - 3\beta_1^1\alpha_1^1\alpha_3^1 + \alpha_8^3 \\
&- \alpha_1^1\alpha_3^1\beta_4^1 + \beta_1^1\beta_3^1\beta_4^1 + 2\alpha_1^1\alpha_3^1\alpha_5^1 - 2\alpha_1^1\beta_3^1\beta_5^1 + \alpha_1^1(\alpha_4^1)^2 - \alpha_1^1(\beta_4^1)^2 \\
&- \alpha_1^1\beta_3^1\alpha_4^1 + \alpha_1^1\alpha_4^1\beta_5^1 - 2\alpha_1^1(\beta_5^1)^2 + \alpha_1^1\beta_4^1\alpha_5^1 + \alpha_8^1\alpha_2^2 + 2\beta_5^1\alpha_5^2 + 3\beta_9^3 \\
&- 3\alpha_1^1\beta_4^2 + 3\alpha_1^1\alpha_5^2 + 2\alpha_1^1\alpha_7^2 + 2\alpha_1^1\beta_8^2 + 3\beta_1^1\beta_9^2 + (\alpha_2^1)^2\alpha_4^1 - (\alpha_2^1)^2\beta_5^1 \\
&+ (\alpha_2^1)^2\alpha_8^1 + 3(\alpha_2^1)^2\beta_9^1 + 3\beta_1^1\alpha_2^1\beta_9^1 + \alpha_2^1\alpha_3^1\alpha_4^1 + 2\alpha_2^1\alpha_4^1\alpha_5^1 + \beta_1^1\alpha_8^2 \\
&- \alpha_2^1\beta_4^1\beta_5^1 + 4\alpha_2^1\alpha_5^1\beta_5^1 + \beta_1^1\alpha_2^1\alpha_8^1 + \alpha_2^1\alpha_4^2 - \alpha_2^1\beta_5^2 + \alpha_2^1\alpha_8^2 + 3\alpha_2^1\beta_9^2 \\
&+ 2\beta_1^1\alpha_3^1\beta_3^1 + 3\alpha_3^1\alpha_1^2 + 3\beta_3^1\beta_1^2 - 2\alpha_3^1\beta_3^2 + \alpha_3^1\alpha_4^2 - \beta_3^1\beta_4^2 + \beta_1^1\alpha_4^1\alpha_5^1 \\
&+ \alpha_4^1\alpha_2^2 + \alpha_4^1\alpha_3^2 - \beta_4^1\beta_3^2 + \alpha_4^1\alpha_5^2 - \beta_4^1\beta_5^2 + 2\beta_1^1\alpha_5^1\beta_5^1 - 3\beta_3^3 + 3\beta_1^1\beta_3^2 \\
&+ 3\alpha_5^1\alpha_1^2 + \alpha_5^1\alpha_4^2 - \beta_5^1\beta_4^2 + 2\alpha_5^1\beta_5^2 + 2\alpha_7^1\alpha_1^2 + \alpha_8^1\beta_1^2 + 3\alpha_1^1\alpha_3^2 + \alpha_4^3 \\
&+ 3\beta_9^1\beta_1^2 - 3\beta_4^1\alpha_1^2 + 2\beta_8^1\alpha_1^2 - \beta_5^1\alpha_2^2 + 3\beta_9^1\alpha_2^2 - 2\beta_3^1\alpha_3^2 - \beta_5^3 + 3\alpha_6^3), \\
C_1 =&\pi(\alpha_1^5 + \beta_2^5),
\end{aligned}
$$

and since the rank of the Jacobian matrix of the function $\mathcal{C} = (C_1, C_3, C_5)$ with respect to its variables is maximal, $C_i$, $i = 1, 3, 5$ are linearly independent in their variables The averaging function $f_5(r)$ has at most 2 solutions in $D$, see Proposition 5.2. In order to apply the averaging method of order 6 we solve $C_1 = 0$ for $\beta_2^5$, $C_3 = 0$ for $\beta_7^3$, and $C_5 = 0$ for $\beta_9^1$, resulting $f_5(r) = 0$. We remark that these expressions only hold for $A \neq -3$. The results for $A = -3$ are presented later on. Calculating the averaging function of sixth order we obtain

$$f_6(r) = r(D_5 r^4 + D_3 r^2 + D_1).$$

The expressions of $D_i$ for $i = 1, 3, 5$ are very long so we present them in Appendix D.

Therefore $f_6(r)$ has at most 2 solutions in $D$. Using the same arguments than in the proof of the Collins first form for $f_6(r)$ we can show that at most 2 small limit cycles can bifurcate from the uniform isochronous center at the origin and this number can be reached.

Now we analyze the bifurcation of small limit cycles for the center of (5.13) in the case $A = -3$. We remark that until the averaging method of order 5 the respective averaging functions for this special case can be obtained by plugging $A = -3$ in the equations of the general case, so we do not explicit them. Hence we solve $C_1 = 0$ for $\beta_2^5$, $C_3 = 0$ for $\beta_7^3$, and $C_5 = 0$ for $\alpha_8^1$, and we get $f_5(r) = 0$ when $A = -3$. Calculating the averaging function of sixth order we obtain

$$f_6(r) = r(\mathcal{D}_5 r^4 + \mathcal{D}_4 r^3 + \mathcal{D}_3 r^2 + \mathcal{D}_1).$$

The explicit expressions of $\mathcal{D}_j$ for $j = 1, 3, 4, 5$ are very long so we present them in Appendix E.

Therefore $f_6(r)$ has at most 3 solutions in $D$ according to Proposition 5.2. Using similar arguments as those applied in the proof of the Collins first form for $f_6(r)$ it is proved that at most 3 small limit cycles can bifurcate from the uniform isochronous center at the origin and this number can be reached.

This completes the proof of Theorem 5.5.

### 5.3.3 Proof of Theorem 5.6

A first integral $H$ and its corresponding integrating factor $\mu$ for system (5.6) are $H(x, y) = (x^2 + y^2)/(1 - x^2)$ and $\mu = -2/(x^2 - 1)^2$. When $h \in (0, 1)$ then $H(x, y) = h$ are periodic solutions around the center $(0, 0)$ contained in the open disc of radius 1 centered at the origin. For proving Theorem 5.6 we shall use Theorem 5.1. Therefore applying the notation of Theorem 5.1 we have $h_1 = 0$, $h_2 = 1$ and $\rho(R, \theta) = R/(R^2 \cos^2\theta + 1)$ for all $0 < R < 1$ and $\theta \in [0, 2\pi)$. Then all the hypotheses of Theorem 5.1 are satisfied for system (5.6). Using Theorem 5.1 we transform the perturbed differential system (5.9) into the form

$$\frac{dR}{d\theta} = \varepsilon \frac{\sum_{i=0}^{5} M_i(\theta, \alpha, \beta) R^i}{1 + R^2 \cos^2\theta} + O(\varepsilon^2) \tag{5.14}$$

where

$$\begin{aligned}
M_0(\theta, \alpha, \beta) =& -\sqrt{1 + R^2 \cos^2\theta}(\alpha_0 \cos\theta + \beta_0 \sin\theta),\\
M_1(\theta, \alpha, \beta) =& -\alpha_1 \cos^2\theta - (\alpha_2 + \beta_1)\cos\theta \sin\theta - \beta_2 \sin^2\theta,\\
M_2(\theta, \alpha, \beta) =&(-1/4\sqrt{2})\sqrt{2 + R^2 + R^2\cos(2\theta)}((7\alpha_0 + 3\alpha_3 + \alpha_5\\
&+ \beta_4)\cos\theta + (\alpha_0 + \alpha_3 - \alpha_5 - \beta_4)\cos(3\theta) + 2(\alpha_4\\
&+ \beta_0 + \beta_3 + \beta_5 + (\alpha_4 + \beta_0 + \beta_3 - \beta_5)\cos(2\theta))\sin\theta),\\
M_3(\theta, \alpha, \beta) =& -(2\alpha_1 + \alpha_6)\cos^4\theta - (2\alpha_2 + \alpha_7 + \beta_1 + \beta_6)\cos^3\theta \sin\theta\\
&- (\alpha_1 + \alpha_8 + \beta_2 + \beta_7)\cos^2\theta \sin^2\theta - (\alpha_2 + \alpha_9 + \beta_8)\\
&\cos\theta \sin^3\theta - \beta_9 \sin^4\theta,\\
M_4(\theta, \alpha, \beta) =&(-1/2\sqrt{2})\cos\theta\sqrt{2 + R^2 + R^2\cos(2\theta)}(\alpha_0 + \alpha_3 + \alpha_5\\
&+ (\alpha_0 + \alpha_3 - \alpha_5)\cos(2\theta) + \alpha_4 \sin(2\theta)),
\end{aligned}$$

$$\begin{aligned}M_5(\theta,\alpha,\beta) =&(-1/4)\cos\theta((3(\alpha_1+\alpha_6)+\alpha_8)\cos\theta+(\alpha_1+\alpha_6-\alpha_8)\\&\cos 3\theta+2(\alpha_2+\alpha_7+\alpha_9+(\alpha_2+\alpha_7-\alpha_9)\cos 2\theta)\sin\theta),\end{aligned}$$

where $\alpha=(\alpha_0,\ldots,\alpha_9)$ and $\beta=(\beta_0,\ldots,\beta_9)$.

We must study the zeros of the averaging function $f:(0,1)\to\mathbb{R}$ defined by

$$f(R)=\int_0^{2\pi}\frac{\sum_{i=0}^5 M_i(\theta,\alpha,\beta)R^i}{1+R^2\cos^2\theta}d\theta.$$

By computing the previous integral, we obtain

$$\begin{aligned}f(R)=&\ \pi(\alpha_6-\alpha_1-3\alpha_8-\beta_2-\beta_7+3\beta_9)g_0-\pi(\alpha_1+\alpha_6+\alpha_8)g_1\\&+2\pi(\alpha_8-\beta_9)g_2+2\pi(\alpha_6-\alpha_8-\beta_7+\beta_9)g_3,\end{aligned}\tag{5.15}$$

where

$$g_0=R,\quad g_1=R^3,\quad g_2=R\sqrt{1+R^2},\quad g_3=(1-\sqrt{1+R^2})/R.$$

In order to find the maximum number of simple zeros of the function $f$ we need to prove that the four functions $g_i:(0,1)\to\mathbb{R}$, $i\in\{0,\ldots,3\}$ given in (5.15) are an ECT-system and according to Theorem 5.3 this is the case if each Wronskian $W_j(g_0,\ldots,g_j)\neq 0,\ j\in\{0,\ldots,3\}$. More precisely

$$\begin{aligned}W_0 =&R,\qquad W_1=2R^3,\qquad W_2=-2R^6/(1+R^2)^{3/2}\\W_3 =&12R^2(8+12R^2+4R^4-8(1+R^2)^{3/2}-R^4\sqrt{1+R^2})/(1+R^2)^{7/2}.\end{aligned}$$

For $R\in(0,1)$ we have that all the Wronskians above are nonzero. Moreover the rank of the Jacobian matrix of the coefficients of $g_i$, $i=0,\ldots,3$ in $f(R)$ in the variables $\alpha_1,\alpha_6,\alpha_8,\beta_2,\beta_7,\beta_9$ is maximal. Thus applying Theorems 5.3 and 5.4, we conclude that wit is proved that at most 3 medium limit cycles can bifurcate from the periodic solutions surrounding the uniform isochronous cubic center of the Collins first form and this number can be reached. This completes the proof of Theorem 5.6.

### 5.3.4 Proof of Theorem 5.7

We use the Collins first and second forms to prove Theorem 5.7. We were able to apply up to the averaging theory of order 6 using Theorem 5.4, and in order to calculate the respective averaging functions $f_i$, for $i=1,\ldots,6$ we used the formulae of $y_i^{\pm}$ from Appendix C.

**Collins first form**

Consider the planar cubic polynomial differential system (5.10) with $f(x,y)=xy$. In order to analyze the Hopf bifurcation for this system, applying Theorem 5.4, we set $\alpha=\pi$ and we introduce a small parameter $\varepsilon$ doing the change of coordinates $x=\varepsilon X$, $y=\varepsilon Y$. After that we perform the polar change of coordinates $X=r\cos\theta$, $Y=r\sin\theta$ and by doing a Taylor expansion truncated at the $6^{th}$ order in $\varepsilon$ we obtain an expression for $dr/d\theta$ similar to (5.1), with $\alpha=\pi$. The explicit expression is very large so we omit it.

The differential system (5.10) with $f(x,y)=xy$ is a polynomial system, therefore the corresponding functions $F_i^{\pm}(\theta,r)$ and $R_i^{\pm}(\theta,r,\varepsilon)$, for $i=1,\ldots,4$ are analytic. Moreover, since the variable $\theta$ appears through sines and cosines, system (5.10) with $f(x,y) = xy$ is $2\pi-$periodic when it it is written under the form $dr/d\theta$. We shall have $D$ of Theorem 5.4 as $D = \{r : 0 < r < 1\}$.

We obtain each $y_i^+$ and $y_i^-$, $i = 1.\dots,4$ using the formulae in Appendix C, respectively for $X_1$ and $X_2$ of system (5.10) with $f(x,y) = xy$, after the changes described in the previous paragraphs of this section. Then we calculate the averaging functions $f_i, i = 1.\dots,6$ using equation (5.3). Therefore, by Theorem 5.4 we have the averaging function of first order

$$f_1(r) = \pi r(\alpha_1^1 + \beta_2^1 + \gamma_1^1 + \delta_2^1).$$

Clearly $f_1(r)$ has no solution in $D$. Thus there is no small limit cycles bifurcating from the uniform isochronous center at the origin by the averaging theory of first order. Now setting $\gamma_1^1 = -(\alpha_1^1 + \beta_2^1 + \delta_2^1)$ we obtain $f_1(r) = 0$. So we can apply the averaging theory of order 2, obtaining

$$f_2(r) = r(A_2 r + A_1),$$

where

$$\begin{aligned}
A_2 =& \frac{2}{3}(\alpha_4^1 - \gamma_4^1 + \beta_3^1 + 2\beta_5^1 - \delta_3^1 - 2\delta_5^1),\\
A_1 =& \frac{\pi}{4}(\alpha_1^1\alpha_2^1 + 2\alpha_1^2 + 2\pi(\alpha_1^1)^2 - \alpha_1^1\gamma_2^1 + 2\gamma_1^2 - \alpha_1^1\beta_1^1 + \alpha_2^1\beta_2^1 + 4\pi\alpha_1^1\beta_2^1\\
&- \gamma_2^1\beta_2^1 - \beta_1^1\beta_2^1 + 2\pi(\beta_2^1)^2 + 2\beta_2^2 + \alpha_1^1\delta_1^1 + \beta_2^1\delta_1^1 + 2\delta_2^2).
\end{aligned}$$

Thus $f_2(r)$ has one solution in $D$ if $0 < -A_1/A_2 < 1$. Therefore applying Theorem 5.4 it is proved that at most 1 small limit cycle can bifurcate from the uniform isochronous center at the origin and this number can be reached. To apply the averaging method of third order we need that $f_2(r) = 0$. Thus we solve $A_1 = 0$ for $\gamma_4^1$ and $A_2 = 0$ for $\gamma_1^2$ from these coefficients. Calculating the next averaging function we have

$$f_3(r) = r(B_3 r^2 + B_2 r + B_1),$$

where

$$\begin{aligned}
B_3 =& \frac{1}{8}\pi(-4\beta_2^1 + 3\alpha_6^1 + \beta_7^1 + \alpha_8^1 + 3\beta_9^1 - 4\delta_2^1 + \delta_7^1 + 3\delta_9^1 + 3\gamma_6^1 + \gamma_8^1),\\
B_2 =& \frac{2}{9}(\alpha_1^1\alpha_3^1 - 3\beta_1^1\beta_3^1 + 6\pi\alpha_1^1\beta_3^1 - \alpha_1^1\beta_4^1 + 6\pi\alpha_1^1\alpha_4^1 + 2\alpha_1^1\alpha_5^1 + 12\pi\alpha_1^1\beta_5^1\\
&- \alpha_1^1\delta_4^1 + \alpha_1^1\gamma_3^1 + 2\alpha_1^1\gamma_5^1 + 6\pi\beta_2^1\beta_3^1 + 3\alpha_2^1\alpha_4^1 - 4\beta_2^1\beta_4^1 + 6\alpha_2^1\beta_5^1\\
&+ 12\pi\beta_2^1\beta_5^1 - \beta_2^1\delta_4^1 - 5\beta_2^1\alpha_3^1 + 6\pi\beta_2^1\alpha_4^1 - 3\alpha_4^1\gamma_2^1 + 2\beta_2^1\alpha_5^1 + 3\beta_3^2\\
&+ 3\alpha_4^2 + 6\beta_5^2 + 3\delta_1^1\delta_3^1 + 3\delta_2^1\delta_4^1 - 3\beta_3^1\gamma_2^1 - 6\beta_5^1\gamma_2^1 + 3\delta_3^1\gamma_2^1 + \beta_2^1\gamma_3^1\\
&+ 6\delta_2^1\gamma_3^1 + 2\beta_2^1\gamma_5^1 - 3\delta_3^2 - 3\gamma_4^2 - 6\delta_5^2),\\
B_1 =& \frac{1}{16}\pi(10\pi^2(\alpha_1^1)^3 - 8\pi\beta_1^1(\alpha_1^1)^2 + 30\pi^2(\alpha_1^1)^2\beta_2^1 - 4(\alpha_1^1)^2\beta_2^1\\
&+ 8\pi(\alpha_1^1)^2\alpha_2^1 + 3(\beta_1^1)^2\beta_2^1 + 4\pi(\alpha_1^1)^2\delta_1^1 - 4(\alpha_1^1)^2\delta_2^1 - 4\pi(\alpha_1^1)^2\gamma_2^1\\
&+ 3(\beta_1^1)^2\alpha_1^1 - 16\pi\beta_1^1\alpha_1^1\beta_2^1 - 2\beta_1^1\alpha_1^1\delta_1^1 + 3\alpha_1^1(\alpha_2^1)^2 + 30\pi^2\alpha_1^1(\beta_2^1)^2\\
&- 4\alpha_1^1(\beta_2^1)^2 - 8\pi\beta_1^1(\beta_2^1)^2 - 2\beta_1^1\alpha_1^1\alpha_2^1 + 16\pi\alpha_1^1\alpha_2^1\beta_2^1 + 2\alpha_1^1\alpha_2^1\delta_1^1\\
&+ 8\pi\alpha_1^1\beta_2^1\delta_1^1 - 2\beta_1^1\beta_2^1\delta_1^1 - 8\alpha_1^1\beta_2^1\delta_2^1 - 2\alpha_1^1\alpha_2^1\gamma_2^1 - 4\alpha_1^1\beta_1^2
\end{aligned}$$

$$
\begin{aligned}
&+ 16\pi\alpha_1^1\alpha_1^2 + 4\alpha_1^1\alpha_2^2 + 16\pi\alpha_1^1\beta_2^2 - 4\beta_1^1\beta_2^2 - \alpha_1^1(\delta_1^1)^2 \\
&- 4\alpha_1^1(\delta_2^1)^2 - \alpha_1^1(\gamma_2^1)^2 + 2\beta_1^1\alpha_1^1\gamma_2^1 - 8\pi\alpha_1^1\beta_2^1\gamma_2^1 - 2\alpha_1^1\delta_1^1\gamma_2^1 \\
&+ 4\alpha_1^1\delta_1^2 - 4\alpha_1^1\gamma_2^2 + 10\pi^2(\beta_2^1)^3 + 3(\alpha_2^1)^2\beta_2^1 + 4\pi(\beta_2^1)^2\delta_1^1 \\
&- 4(\beta_2^1)^2\delta_2^1 - 2\beta_1^1\alpha_2^1\beta_2^1 + 8\pi\alpha_2^1(\beta_2^1)^2 + 2\alpha_2^1\beta_2^1\delta_1^1 + 4\alpha_2^1\alpha_1^2 \\
&- 4\beta_2^1\beta_1^2 + 4\alpha_2^1\beta_2^2 + 16\pi\beta_2^1\beta_2^2 - \beta_2^1(\delta_1^1)^2 - 4\beta_2^1(\delta_2^1)^2 \\
&- 2\alpha_2^1\beta_2^1\gamma_2^1 + 4\beta_2^1\delta_1^2 - 4\beta_1^1\alpha_1^2 + 16\pi\beta_2^1\alpha_1^2 + 4\alpha_1^2\delta_1^1 \\
&- 4\alpha_1^2\gamma_2^1 + 4\beta_2^1\alpha_2^2 + 4\beta_2^2\delta_1^1 + 8\alpha_1^3 + 8\beta_2^3 - \beta_2^1(\gamma_2^1)^2 + 2\beta_1^1\beta_2^1\gamma_2^1 \\
&- 4\pi(\beta_2^1)^2\gamma_2^1 - 2\beta_2^1\delta_1^1\gamma_2^1 - 4\beta_2^2\gamma_2^1 - 4\beta_2^1\gamma_2^2 + 8\gamma_1^3 + 8\delta_2^3),
\end{aligned}
$$

and since the rank of the Jacobian matrix of the function $\mathcal{B} = (B_1, B_2, B_3)$ with respect to its variables is maximal, $B_i$, $i = 1, \ldots, 3$ are linearly independent in their variables.

The averaging function $f_3(r)$ can have at most 2 solutions in $D$, see Proposition 5.2. Thus using Theorem 5.4 we conclude that at most 2 small limit cycles can bifurcate from the uniform isochronous center at the origin and this number can be reached. In order to apply the averaging theory of order 4 we need that $f_3(r) = 0$, so we vanish its coefficients $B_1$, $B_2$ and $B_3$ by conveniently isolating $\delta_2^3$, $\delta_5^2$ and $\delta_9^1$ from these coefficients. The resulting averaging function of order 4 is

$$
f_4(r) = r(C_4r^3 + C_3r^2 + C_2r + C_1),
$$

where

$$
\begin{aligned}
C_4 =& \frac{4}{15}(\alpha_4^1 - 4\beta_3^1 - 6\beta_5^1), \\
C_3 =& \frac{1}{144}(128(\alpha_4^1)^2 + 256\beta_3^1\alpha_4^1 + 512\beta_5^1\alpha_4^1 + 36\alpha_3^1\pi\alpha_4^1 + 36\alpha_5^1\pi\alpha_4^1 \\
&- 144\pi^2(\beta_2^1)^2 + 128(\beta_3^1)^2 + 512(\beta_5^1)^2 - 54\alpha_6^1\pi\beta_1^1 + 18\alpha_8^1\pi\beta_1^1 \\
&+ 72\pi\beta_1^1\beta_2^1 - 144\alpha_1^1\pi^2\beta_2^1 + 108\alpha_6^1\pi^2\beta_2^1 + 36\alpha_8^1\pi^2\beta_2^1 - 216\alpha_2^1\pi\beta_2^1 \\
&- 63\alpha_7^1\pi\beta_2^1 + 27\alpha_9^1\pi\beta_2^1 - 72\alpha_3^1\pi\beta_3^1 - 36\pi\beta_3^1\beta_4^1 + 512\beta_3^1\beta_5^1 \\
&- 36\pi\beta_4^1\beta_5^1 + 72\alpha_5^1\pi\beta_5^1 - 99\pi\beta_2^1\beta_6^1 - 99\alpha_1^1\pi\beta_6^1 - 18\pi\beta_1^1\beta_7^1 \\
&+ 36\pi^2\beta_2^1\beta_7^1 + 36\alpha_1^1\pi^2\beta_7^1 + 18\alpha_2^1\pi\beta_7^1 - 81\pi\beta_2^1\beta_8^1 - 9\alpha_1^1\pi\beta_8^1 \\
&+ 54\pi\beta_1^1\beta_9^1 + 108\pi^2\beta_2^1\beta_9^1 + 108\alpha_1^1\pi^2\beta_9^1 + 162\alpha_2^1\pi\beta_9^1 - 144\pi\beta_2^2 \\
&+ 36\pi\beta_7^2 + 108\pi\beta_9^2 + 108\alpha_1^1\alpha_6^1\pi^2 + 36\alpha_1^1\alpha_8^1\pi^2 + 54\alpha_2^1\alpha_6^1\pi \\
&+ 9\alpha_1^1\alpha_7^1\pi + 54\alpha_2^1\alpha_8^1\pi + 27\alpha_1^1\alpha_9^1\pi + 108\alpha_6^2\pi + 36\alpha_8^2\pi), \\
C_2 =& \frac{1}{108}(32\alpha_4^1(\alpha_1^1)^2 + 45\pi^2\beta_3^1(\alpha_1^1)^2 - 16\beta_3^1(\alpha_1^1)^2 - 36\pi\beta_4^1(\alpha_1^1)^2 \\
&+ 90\pi^2\beta_5^1(\alpha_1^1)^2 - 32\beta_5^1(\alpha_1^1)^2 + 45\alpha_4^1\pi^2(\alpha_1^1)^2 + 36\alpha_3^1\pi(\alpha_1^1)^2 \\
&+ 48\alpha_2^1\alpha_3^1\alpha_1^1 + 192\alpha_2^1\alpha_5^1\alpha_1^1 + 48\alpha_3^2\alpha_1^1 + 96\alpha_5^2\alpha_1^1 - 48\alpha_3^1\beta_1^1\alpha_1^1 \\
&- 54\alpha_4^1\pi\beta_1^1\alpha_1^1 - 176\alpha_4^1\beta_2^1\alpha_1^1 + 90\alpha_4^1\pi^2\beta_2^1\alpha_1^1 - 144\alpha_3^1\pi\beta_2^1\alpha_1^1 \\
&- 162\pi\beta_1^1\beta_3^1\alpha_1^1 + 90\pi^2\beta_2^1\beta_3^1\alpha_1^1 + 112\beta_2^1\beta_3^1\alpha_1^1 + 54\alpha_2^1\pi\beta_3^1\alpha_1^1 \\
&+ 48\beta_1^1\beta_4^1\alpha_1^1 - 180\pi\beta_2^1\beta_4^1\alpha_1^1 - 108\pi\beta_1^1\beta_5^1\alpha_1^1 + 180\pi^2\beta_2^1\beta_5^1\alpha_1^1 \\
&+ 324\alpha_2^1\pi\beta_5^1\alpha_1^1 + 108\pi\beta_3^2\alpha_1^1 - 48\beta_4^2\alpha_1^1 + 216\pi\beta_5^1\alpha_1^1 \\
&+ 162\alpha_2^1\alpha_4^1\pi\alpha_1^1 - 256\beta_2^1\beta_5^1\alpha_1^1 - 48\alpha_2^1\beta_4^1\alpha_1^1 + 144\alpha_5^1\pi\beta_2^1\alpha_1^1
\end{aligned}
$$

$$
\begin{aligned}
&+ 108\alpha_4^2\pi\alpha_1^1 - 64\alpha_4^1(\beta_2^1)^2 + 45\alpha_4^1\pi^2(\beta_2^1)^2 - 180\alpha_3^1\pi(\beta_2^1)^2 \\
&+ 144(\alpha_2^1)^2\alpha_4^1 + 48\alpha_3^1\alpha_1^2 + 96\alpha_5^1\alpha_1^2 + 144\alpha_4^1\alpha_2^2 + 144\alpha_2^1\alpha_4^2 \\
&+ 144\alpha_4^3 - 240\alpha_2^1\alpha_3^1\beta_2^1 + 192\alpha_2^1\alpha_5^1\beta_2^1 - 240\alpha_3^2\beta_2^1 + 96\alpha_5^2\beta_2^1 \\
&+ 240\alpha_3^1\beta_1^1\beta_2^1 - 54\alpha_4^1\pi\beta_1^1\beta_2^1 + 162\alpha_2^1\alpha_4^1\pi\beta_2^1 + 108\alpha_4^2\pi\beta_2^1 \\
&+ 45\pi^2(\beta_2^1)^2\beta_3^1 + 128(\beta_2^1)^2\beta_3^1 - 162\pi\beta_1^1\beta_2^1\beta_3^1 + 54\alpha_2^1\pi\beta_2^1\beta_3^1 \\
&- 144\pi(\beta_2^1)^2\beta_4^1 - 48\alpha_1^2\beta_4^1 - 192\alpha_2^1\beta_2^1\beta_4^1 + 192\beta_1^1\beta_2^1\beta_4^1 \\
&+ 288(\alpha_2^1)^2\beta_5^1 + 108\alpha_1^2\pi\beta_3^1 + 144(\beta_1^1)^2\beta_3^1 + 72\alpha_5^1\pi(\beta_2^1)^2 \\
&+ 72\alpha_5^1\pi(\alpha_1^1)^2 + 90\pi^2(\beta_2^1)^2\beta_5^1 + 64(\beta_2^1)^2\beta_5^1 \\
&+ 216\alpha_1^2\pi\beta_5^1 - 144\beta_3^1\beta_1^2 - 240\alpha_3^1\beta_2^1 + 96\alpha_5^1\beta_2^2 + 108\pi\beta_3^1\beta_2^2 \\
&- 192\beta_4^1\beta_2^2 + 216\pi\beta_5^1\beta_2^2 + 108\alpha_4^1\pi\beta_2^2 - 144\beta_1^1\beta_3^2 + 108\pi\beta_2^1\beta_3^2 \\
&- 192\beta_2^1\beta_4^2 + 288\alpha_2^1\beta_5^2 + 216\pi\beta_2^1\beta_5^2 + 144\beta_3^3 + 288\beta_5^3 \\
&+ 108\alpha_4^1\alpha_1^2\pi - 108\pi\beta_1^1\beta_2^1\beta_5^1 + 324\alpha_2^1\pi\beta_2^1\beta_5^1 + 288\alpha_2^2\beta_5^1),
\end{aligned}
$$

$$
\begin{aligned}
C_1 =&\frac{1}{192}\pi(\pi^3(\alpha_1^1)^4 + 12\pi(\alpha_1^1)^4 + 36\alpha_2^1(\alpha_1^1)^3 - 12\pi^2\beta_1^1(\alpha_1^1)^3 - 36\beta_1^1(\alpha_1^1)^3 \\
&+ 4\pi^3\beta_2^1(\alpha_1^1)^3 + 12\alpha_2^1\pi^2(\alpha_1^1)^3 + 48\pi(\beta_1^1)^2(\alpha_1^1)^2 + 6\pi^3(\beta_2^1)^2(\alpha_1^1)^2 \\
&- 24\pi(\beta_2^1)^2(\alpha_1^1)^2 + 72\alpha_1^2(\alpha_1^1)^2 - 48\alpha_2^1\pi\beta_1^1(\alpha_1^1)^2 - 36\alpha_2^1\beta_2^1(\alpha_1^1)^2 \\
&- 36\pi^2\beta_1^1\beta_2^1(\alpha_1^1)^2 + 36\beta_1^1\beta_2^1(\alpha_1^1)^2 + 36\alpha_2^1\pi^2\beta_2^1(\alpha_1^1)^2 - 48\pi\beta_1^2(\alpha_1^1)^2 \\
&+ 24\pi^2\beta_2^2(\alpha_1^1)^2 - 24\beta_2^2(\alpha_1^1)^2 + 24\alpha_1^2\pi^2(\alpha_1^1)^2 + 48(\alpha_2^1)^2\pi(\alpha_1^1)^2 \\
&+ 48\alpha_2^2\pi(\alpha_1^1)^2 + 60(\alpha_2^1)^3\alpha_1^1 - 60(\beta_1^1)^3\alpha_1^1 + 4\pi^3(\beta_2^1)^3\alpha_1^1 \\
&+ 36\alpha_2^1(\beta_1^1)^2\alpha_1^1 - 36\alpha_2^1(\beta_2^1)^2\alpha_1^1 - 36\pi^2\beta_1^1(\beta_2^1)^2\alpha_1^1 + 36\beta_1^1(\beta_2^1)^2\alpha_1^1 \\
&+ 36\alpha_2^1\pi^2(\beta_2^1)^2\alpha_1^1 + 144\alpha_2^1\alpha_2^2\alpha_1^1 + 96\alpha_2^3\alpha_1^1 - 36(\alpha_2^1)^2\beta_1^1\alpha_1^1 \\
&- 48\alpha_2^2\beta_1^1\alpha_1^1 - 96\alpha_1^2\pi\beta_1^1\alpha_1^1 + 96\pi(\beta_1^1)^2\beta_2^1\alpha_1^1 - 48\alpha_1^2\beta_2^1\alpha_1^1 \\
&- 96\alpha_2^1\pi\beta_1^1\beta_2^1\alpha_1^1 + 48\alpha_1^2\pi^2\beta_2^1\alpha_1^1 + 96(\alpha_2^1)^2\pi\beta_2^1\alpha_1^1 + 96\alpha_2^2\pi\beta_2^1\alpha_1^1 \\
&- 48\alpha_2^1\beta_1^2\alpha_1^1 + 144\beta_1^1\beta_1^2\alpha_1^1 - 96\pi\beta_2^1\beta_1^2\alpha_1^1 - 96\pi\beta_1^1\beta_2^2\alpha_1^1 \\
&+ 48\pi^2\beta_2^1\beta_2^2\alpha_1^1 - 48(\beta_2^1)\beta_2^2\alpha_1^1 + 96\alpha_2^1\pi\beta_2^1\alpha_1^1 - 96\beta_1^3\alpha_1^1 \\
&+ 96\pi\beta_2^3\alpha_1^1 + 96\alpha_2^1\alpha_1^2\pi\alpha_1^1 + 96\alpha_1^3\pi\alpha_1^1 + \pi^3(\beta_2^1)^4 + 12\pi(\beta_2^1)^4 \\
&- 12\pi^2\beta_1^1(\beta_2^1)^3 - 36\beta_1^1(\beta_2^1)^3 + 12\alpha_2^1\pi^2(\beta_2^1)^3 + 72\alpha_1^2(\beta_1^1)^2 \\
&+ 48\pi(\beta_1^1)^2(\beta_2^1)^2 - 24\alpha_1^2(\beta_2^1)^2 - 48\alpha_2^1\pi\beta_1^1(\beta_2^1)^2 + 24\alpha_1^2\pi^2(\beta_2^1)^2 \\
&+ 48(\alpha_2^1)^2\pi(\beta_2^1)^2 + 48\alpha_2^2\pi(\beta_2^1)^2 + 48\pi(\beta_2^2)^2 + 72(\alpha_2^1)^2\alpha_1^2 \\
&+ 96\alpha_1^2\alpha_2^2 + 96\alpha_2^1\alpha_1^3 + 192\alpha_1^4 - 48\alpha_2^1\alpha_1^2\beta_1^1 + 36\alpha_2^1(\beta_2^1)^3 \\
&- 96\alpha_1^3\beta_1^1 + 60(\alpha_2^1)^3\beta_2^1 - 60(\beta_1^1)^3\beta_2^1 + 36\alpha_2^1(\beta_1^1)^2\beta_2^1 + 144\alpha_2^1\alpha_2^2\beta_2^1 \\
&+ 96\alpha_2^3\beta_2^1 - 36(\alpha_2^1)^2\beta_1^1\beta_2^1 - 48\alpha_2^2\beta_1^1\beta_2^1 - 96\alpha_1^2\pi\beta_1^1\beta_2^1 + 96\alpha_2^1\alpha_1^2\pi\beta_2^1 \\
&+ 96\alpha_1^3\pi\beta_2^1 - 48\pi(\beta_2^1)^2\beta_1^2 - 48\alpha_2^1\beta_2^1\beta_1^2 + 144\beta_1^1\beta_2^1\beta_1^2 + 72(\alpha_2^1)^2\beta_2^2 \\
&- 96\alpha_1^2\beta_1^2 + 72(\beta_1^1)^2\beta_2^2 + 24\pi^2(\beta_2^1)^2\beta_1^2 + 72(\beta_2^1)^2\beta_2^2 + 96\alpha_2^2\beta_2^2 \\
&- 48\alpha_2^1\beta_1^1\beta_2^2 - 96\pi\beta_1^1\beta_2^1\beta_2^2 + 96\alpha_2^1\pi\beta_2^1\beta_2^2 - 96\beta_1^2\beta_2^2 + 96\alpha_1^2\pi\beta_2^2 \\
&- 96\beta_2^1\beta_1^3 + 96\alpha_2^1\beta_2^3 - 96\beta_1^1\beta_2^3 + 96\pi\beta_2^1\beta_2^3 + 192\beta_2^4 + 48(\alpha_1^2)^2\pi).
\end{aligned}
$$

Using similar arguments as in the study of the previous averaging functions, we conclude that $f_4(r)$ can have at most 3 solutions in $D$, so at most 3 small limit cycles can bifurcate from the uniform isochronous center at the origin and this number can be reached. In order to apply the averaging method of order 5 we must have that $f_4(r) = 0$. Thus we solve $C_1 = 0$, $C_2 = 0$, $C_3 = 0$ and $C_4$=0 isolating $\beta_2^4$, $\beta_5^3$, $\beta_9^2$ and $\beta_5^1$ respectively. Now we can apply the averaging theory of order 5, and its averaging function is

$$f_5(r) = r(D_5r^4 + D_4r^3 + D_3r^2 + D_2r + D_1),$$

where again we do not provide the explicit expressions of $D_j$ for $j = 1, \ldots, 5$. Hence $f_5(r)$ has at most 4 solutions in $D$. Doing analogous arguments as in the proof of Theorem 5.5 to prove that $D_j$ for $j = 1, \ldots, 5$ are linearly independent functions and to prove that $(r^5, r^4, r^3, R^2, r)$ is an ECT-system (see Theorem 5.3), we prove that at most 4 small limit cycles can bifurcate from the uniform isochronous center at the origin using the averaging theory of order 5, and this number can be reached.

To apply the averaging theory of order 6 we solve $D_1 = 0$ for $\delta_2^5$, $D_2 = 0$ for $\delta_5^4$, $D_3 = 0$ for $\delta_9^3$, $D_4$ for $\delta_3^2$, and $D_5 = 0$ for $\gamma_6^1$, so we get $f_5(r) = 0$. Calculating the averaging function of order 6 we obtain

$$f_6(r) = r(E_6r^5 + E_5r^4 + E_4r^3 + E_3r^2 + E_2r + E_1).$$

We do not provide the expressions of $E_i$ for $i = 1, \ldots, 6$ because they are too long. Thus $f_6(r)$ has at most 5 solutions in $D$. Doing analogous arguments than in the proof of Theorem 5.5 we can show that at most 5 small limit cycles can bifurcate from the uniform isochronous center at the origin using the averaging theory of order 6, and this number can be reached.

**Collins second form**

Similarly to the previous arguments used in the Collins first form case, we apply Theorem 5.4 to study the Hopf bifurcation for system (5.10) with $f(x, y) = x + Axy$, for $A \in \mathbb{R}$. We set $\alpha = \pi$ and we introduce a small parameter $\varepsilon$ by doing the change of coordinates $x = \varepsilon X$, $y = \varepsilon Y$ and then we perform the standard polar change of coordinates $X = r\cos\theta$, $Y = r\sin\theta$. Doing a Taylor expansion truncated at the $6^{th}$ order in $\varepsilon$ we obtain an expression for $dr/d\theta$ under the form (5.1), with $\alpha = \pi$. The explicit expression is very large so we omit it.

We shall have the open interval $D$ of Theorem 5.4 as $D = \{r : 0 < r < r_0 < 1\}$, where the unperturbed system has periodic solutions passing through the point $(r < r_0, \theta = 0)$. Moreover since system (5.10) with $f(x, y) = x + Axy$ is a polynomial differential system, the corresponding functions $F_i^{\pm}(\theta, r)$ and $R_i^{\pm}(\theta, r, \varepsilon)$, $i = 1, \ldots, 4$ are analytic. Finally, the variable $\theta$ appears through sines and cosines in system (5.10) with $f(x, y) = x + Axy$ when it is written under the form $dr/d\theta$, and therefore it is $2\pi-$periodic.

We obtain each $y_i^+$ and $y_i^-, i = 1\ldots, 4$ using the formulae provided in Appendix C respectively for $X_1$ and $X_2$ of system (5.10) with $f(x, y) = x + Axy$, after the changes described before. Then we calculate the averaging functions $f_i, i = 1\ldots, 6$ using equation (5.3). Hence, by Theorem 5.4 we have the averaging function of first order

$$f_1(r) = \frac{1}{2}\pi r(\alpha_1^1 + \beta_2^1 + \delta_2^1 + \gamma_1^1).$$

Therefore $f_1(r)$ has no solution in $D$. Setting $\gamma_1^1 = -(\alpha_1^1 + \beta_2^1 + \delta_2^1)$ we have $f_1(r) = 0$. So we can apply the averaging theory of order 2 obtaining

$$f_2(r) = r(A_2r + A_1),$$

where

$$
\begin{aligned}
A_2 =& \frac{2}{3}(-3\beta_2^1 + \beta_3^1 + \alpha_4^1 + 2\beta_5^1 + 3\delta_2^1 - \delta_3^1 - 2\delta_5^1 - \gamma_4^1), \\
A_1 =& \frac{\pi}{4}(2\pi(\alpha_1^1)^2 + \alpha_1^1(-\beta_1^1 + \alpha_2^1 + 4\pi\beta_2^1 + \delta_1^1 - \gamma_2^1) - \beta_1^1\beta_2^1 + 2\pi(\beta_2^1)^2 \\
& + \alpha_2^1\beta_2^1 + \beta_2^1\delta_1^1 + 2\alpha_1^2 + 2\beta_2^2 - \beta_2^1\gamma_2^1 + 2\gamma_1^2 + 2\delta_2^2).
\end{aligned}
$$

Thus $f_2(r)$ can have one solution in $D$ if $0 < -A_1/A_2 < r_0$. Therefore applying Theorem 5.4 we can show that at most 1 small limit cycle can bifurcate from the uniform isochronous center at the origin and this number can be reached. To apply the averaging theory of order 3 we solve $A_1 = 0$ and $A_2 = 0$ isolating $\gamma_4^1$ and $\gamma_1^2$ respectively. Calculating the next averaging function we have

$$
f_3(r) = r(B_3 r^2 + B_2 r + B_1),
$$

where

$$
\begin{aligned}
B_3 =& \frac{\pi}{8}(-4A\beta_2^1 - 4A\delta_2^1 - 3\beta_2^1 - 2\beta_3^1 + 2\alpha_4^1 + \beta_5^1 + 3\alpha_6^1 + \beta_7^1 + \alpha_8^1 + 3\beta_9^1 \\
& + 3\delta_2^1 - 4\delta_3^1 - 3\delta_5^1 + \delta_7^1 + 3\delta_9^1 + 3\gamma_6^1 + \gamma_8^1), \\
B_2 =& + \frac{2}{9}(9\beta_1^1\beta_2^1 - 18\pi\alpha_1^1\beta_2^1 + \alpha_1^1\alpha_3^1 - 3\beta_1^1\beta_3^1 + 6\pi\alpha_1^1\beta_3^1 - \alpha_1^1\beta_4^1 \\
& + 6\pi\alpha_1^1\alpha_4^1 + 2\alpha_1^1\alpha_5^1 + 12\pi\alpha_1^1\beta_5^1 - \alpha_1^1\delta_4^1 + \alpha_1^1\gamma_3^1 + 2\alpha_1^1\gamma_5^1 - 18\pi(\beta_2^1)^2 \\
& + 6\pi\beta_2^1\beta_3^1 - 4\beta_2^1\beta_4^1 + 3\alpha_2^1\alpha_4^1 + 12\pi\beta_2^1\beta_5^1 + 6\alpha_2^1\beta_5^1 - \beta_2^1\delta_4^1 - 5\beta_2^1\alpha_3^1 \\
& + 6\pi\beta_2^1\alpha_4^1 - 3\alpha_4^1\gamma_2^1 + 2\beta_2^1\alpha_5^1 - 9\beta_2^2 + 3\beta_3^2 + 3\alpha_4^2 + 6\beta_5^2 - 9\delta_1^1\delta_2^1 \\
& + 3\delta_1^1\delta_3^1 + 3\delta_2^1\delta_4^1 + 9\beta_2^1\gamma_2^1 - 3\beta_3^1\gamma_2^1 - 6\beta_5^1\gamma_2^1 + 3\delta_3^1\gamma_2^1 - 9\alpha_2^1\beta_2^1 \\
& + \beta_2^1\gamma_3^1 + 6\delta_2^1\gamma_3^1 + 2\beta_2^1\gamma_5^1 + 9\delta_2^2 - 3\gamma_3^2 - 3\gamma_4^2 - 6\delta_5^2), \\
B_1 =& + \frac{\pi}{16}(10\pi^2(\alpha_1^1)^3 - 8\pi\beta_1^1(\alpha_1^1)^2 + 30\pi^2(\alpha_1^1)^2\beta_2^1 - 4(\alpha_1^1)^2\beta_2^1 \\
& + 8\pi(\alpha_1^1)^2\alpha_2^1 + 3(\beta_1^1)^2\beta_2^1 + 4\pi(\alpha_1^1)^2\delta_1^1 - 4(\alpha_1^1)^2\delta_2^1 - 4\pi(\alpha_1^1)^2\gamma_2^1 \\
& - 16\pi\beta_1^1\alpha_1^1\beta_2^1 - 2\beta_1^1\alpha_1^1\delta_1^1 + 3\alpha_1^1(\alpha_2^1)^2 + 30\pi^2\alpha_1^1(\beta_2^1)^2 - 4\alpha_1^1(\beta_2^1)^2 \\
& - 8\pi\beta_1^1(\beta_2^1)^2 - 2\beta_1^1\alpha_1^1\alpha_2^1 + 16\pi\alpha_1^1\alpha_2^1\beta_2^1 + 2\alpha_1^1\alpha_2^1\delta_1^1 + 8\pi\alpha_1^1\beta_2^1\delta_1^1 \\
& - 2\beta_1^1\beta_2^1\delta_1^1 - 8\alpha_1^1\beta_2^1\delta_2^1 - 2\alpha_1^1\alpha_2^1\gamma_2^1 - 4\alpha_1^1\beta_1^2 + 16\pi\alpha_1^1\alpha_1^2 + 4\alpha_1^1\alpha_2^2 \\
& + 16\pi\alpha_1^1\beta_2^2 - 4\beta_1^1\beta_2^2 - \alpha_1^1(\delta_1^1)^2 - 4\alpha_1^1(\delta_2^1)^2 - \alpha_1^1(\gamma_2^1)^2 \\
& + 2\beta_1^1\alpha_1^1\gamma_2^1 - 8\pi\alpha_1^1\beta_2^1\gamma_2^1 - 2\alpha_1^1\delta_1^1\gamma_2^1 + 4\alpha_1^1\beta_1^3 - 4\alpha_1^1\gamma_2^2 + 10\pi^2(\beta_2^1)^3 \\
& + 3(\alpha_2^1)^2\beta_2^1 + 4\pi(\beta_2^1)^2\delta_1^1 - 4(\beta_2^1)^2\delta_2^1 - 2\beta_1^1\alpha_2^1\beta_2^1 + 8\pi\alpha_2^1(\beta_2^1)^2 \\
& + 2\alpha_2^1\beta_2^1\delta_1^1 + 4\alpha_2^1\alpha_1^2 - 4\beta_2^1\beta_1^2 + 4\alpha_2^1\beta_2^2 + 16\pi\beta_2^1\beta_2^2 - \beta_2^1(\delta_1^1)^2 \\
& - 4\beta_2^1(\delta_2^1)^2 - 2\alpha_2^1\beta_2^1\gamma_2^1 + 4\beta_2^1\delta_1^2 - 4\beta_1^1\alpha_1^2 + 16\pi\beta_2^1\alpha_1^2 \\
& + 4\alpha_1^2\delta_1^1 - 4\alpha_1^2\gamma_2^1 + 4\beta_2^1\alpha_2^2 + 4\beta_2^2\delta_1^1 + 8\alpha_1^3 + 8\beta_2^3 + 3(\beta_1^1)^2\alpha_1^1 \\
& - \beta_2^1(\gamma_2^1)^2 + 2\beta_1^1\beta_2^1\gamma_2^1 - 4\pi(\beta_2^1)^2\gamma_2^1 - 2\beta_2^1\delta_1^1\gamma_2^1 - 4\beta_2^2\gamma_2^1 \\
& - 4\beta_2^1\gamma_2^2 + 8\gamma_1^3 + 8\delta_2^3).
\end{aligned}
$$

Then $f_3(r)$ has at most 2 solutions in $D$. Thus applying Theorem 5.4 it is proved that at most 2 small limit cycles can bifurcate from the uniform isochronous center at the origin and this number can be reached. To apply the averaging method of order 4 we solve $B_1 = 0$, $B_2 = 0$ and $B_3 = 0$ isolating $\delta_2^3$, $\delta_5^2$, $\delta_9^1$ respectively. The next averaging function is

$$f_4(r) = r(C_4 r^3 + C_3 r^2 + C_2 r + C_1).$$

We do not provide the expressions of $C_j$ for $j = 1, \ldots, 4$ because they are too long.

Of course $f_4(r)$ has at most 3 solutions in $D$, that is, applying the averaging theory of order 4 we can show that at most 3 small limit cycles can bifurcate from the uniform isochronous center at the origin and this number can be reached. To apply the averaging method of order 5 we solve $C_1 = 0$, $C_2 = 0$, $C_3 = 0$ and $C_4 = 0$ isolating $\beta_2^4$, $\beta_5^3$, $\beta_9^2$ and $\beta_9^1$ respectively. The next averaging function is

$$f_5(r) = r(D_5 r^4 + D_4 r^3 + D_3 r^2 + D_2 r + D_1),$$

where again we do not give the expressions of $D_j$ for $j = 1, \ldots, 5$. Hence $f_5(r)$ has at most 4 solutions in $D$. Using analogous arguments than in the proof of Theorem 5.5 and applying Theorem 5.4 we can show that at most 4 small limit cycles can bifurcate from the uniform isochronous center at the origin and this number can be reached.

In order to apply the averaging theory of order 6 we solve $D_1 = 0$ for $\delta_2^5$, $D_2 = 0$ for $\delta_5^4$, $D_3 = 0$ for $\delta_9^3$, $D_4$ for $\delta_9^2$, and $D_5 = 0$ for $\gamma_6^1$, so we get $f_5(r) = 0$. Calculating the averaging function of order 6 we obtain

$$f_6(r) = r(E_6 r^5 + E_5 r^4 + E_4 r^3 + E_3 r^2 + E_2 r + E_1).$$

We do not provide the expressions of $E_i$ for $i = 1, \ldots, 6$ because they are too long. Thus $f_6(r)$ has at most 5 solutions in $D$. Doing analogous arguments than in the proof of Theorem 5.5 and applying Theorem 5.4 it follows that at most 5 small limit cycles can bifurcate from the uniform isochronous center at the origin using the averaging theory of order 6, and this number can be reached.

This ends the proof of Theorem 5.7.

### 5.3.5 Proof of Theorem 5.8

We proceed as in the proof of Theorem 5.6 in section 5.3.3 since the unperturbed system (5.6) is the same. Hence a first integral $H$, its corresponding integrating factor $\mu$, and a function $\rho$ satisfying the hypotheses of Theorem 5.1 are $H(x, y) = (x^2 + y^2)/(1 - x^2)$, $\mu = -2/(x^2 - 1)^2$, and $\rho(R, \theta) = R/(R^2 \cos^2\theta + 1)$ for all $0 < R < 1$ and $\theta \in [0, 2\pi)$.

Applying Theorem 5.1 we transform the perturbed differential system (5.11) into the form

$$\frac{dR}{d\theta} = \begin{cases} \varepsilon \dfrac{\sum_{i=0}^{5} M_i(\theta, \alpha, \beta) R^i}{1 + R^2 \cos^2\theta} + O(\varepsilon^2) & \text{if} \quad y > 0, \\ \varepsilon \dfrac{\sum_{i=0}^{5} N_i(\theta, \gamma, \delta) R^i}{1 + R^2 \cos^2\theta} + O(\varepsilon^2) & \text{if} \quad y < 0, \end{cases} \tag{5.16}$$

where the functions $M_i(\theta, \alpha, \beta)$ coincide with those given in system (5.14), $N_i(\theta, \gamma, \delta) = M_i(\theta, \gamma, \delta)$ for $i = 0, \ldots, 5$, with $\gamma = (\gamma_0, \ldots, \gamma_9)$, $\delta = (\delta_0, \ldots, \delta_9)$.

The discontinuous differential system (5.16) is under the assumptions of Theorem 5.1. Hence we must study the zeros of the averaged function $f:(0,1)\to\mathbb{R}$

$$f(R)=\int_0^{\pi}\frac{\sum_{i=0}^{5}M_i(\theta,\alpha,\beta)R^i}{1+R^2\cos^2\theta}d\theta+\int_{\pi}^{2\pi}\frac{\sum_{i=0}^{5}N_i(\theta,\gamma,\delta)R^i}{1+R^2\cos^2\theta}d\theta$$

We compute these integrals obtaining

$$\begin{aligned}f(R)=&\pi(\alpha_6-\alpha_8-\beta_7+\beta_9+\gamma_6-\gamma_8-\delta_7+\delta_9)g_0+\pi/2(\alpha_6\\&-\alpha_1-3\alpha_8-\beta_2-\beta_7+3\beta_9-\gamma_1+\gamma_6-3\gamma_8-\delta_2-\delta_7\\&+3\delta_9)g_1-\pi/2(\alpha_1+\alpha_6+\alpha_8+\gamma_1+\gamma_6+\gamma_8)g_2+(\beta_5\\&-\alpha_4-\beta_0-\beta_3+\gamma_4+\delta_0+\delta_3-\delta_5)g_3+\pi(\alpha_8-\beta_9+\gamma_8\\&-\delta_9)g_4+(\gamma_4-\alpha_4)g_5+(\alpha_4-\beta_0+\beta_3-\beta_5-\gamma_4+\delta_0\\&-\delta_3+\delta_5)g_6+(\alpha_4-2\beta_5-\gamma_4+2\delta_5)g_7,\end{aligned}\tag{5.17}$$

where

$$\begin{aligned}g_0=&(1-\sqrt{1+R^2})/R, && g_1=R, && g_2=R^3,\\g_3=&\sqrt{1+R^2}, && g_4=R\sqrt{1+R^2}, && g_5=R^2\sqrt{1+R^2},\\g_6=&(\operatorname{arcsinh}R)/R, && g_7=R\operatorname{arcsinh}R.\end{aligned}$$

In order to find the maximum number of simple zeros of function $f$ we need to prove that the eight functions $g_i:(0,1)\to\mathbb{R}$, $i\in\{0,\dots,7\}$ given in (5.17) form an ECT-system and according to Theorem 5.3 this is the case if each Wronskian $W_j(g_0,\dots,g_j)\neq 0$, $j\in\{0,\dots,7\}$. More precisely

$$\begin{aligned}W_0=&(1-K)/R,\quad W_1=(2K-2-R^2)/(RK),\\W_2=&2K^{-3}(1-6K^2+8K^3-3K^4),\\W_3=&6R^{-3}K^{-7}(8-8K+4R^6K+R^4(16-7K)+4R^2(6-5K)),\\W_4=&-36R^{-2}K^{-10}(4R^6K+R^2(76-56K)+R^4(40-17K)\\&-40(K-1)),\\W_5=&1080R^{-5}K^{-15}(24(K-1)+R^2(R^2K(3R^2-5)+4(4K-7))),\\W_6=&25920R^{-7}K^{-20}(64(1-K)+R^2(R^2K(6R^2-17)+32(7-6K))\\&+105R^3\operatorname{arcsinh}R),\\W_7=&1244160R^{-8}K^{-26}(4R^8-515R^4-12R^6-256(K-1)+R^2(896K\\&-243)+105RK(2R^2-5)\operatorname{arcsinh}R),\end{aligned}$$

where $K=\sqrt{1+R^2}$. For $0<R<1$ we have that all the Wronskians above are nonzero. Moreover the rank of the Jacobian matrix of the coefficients of $g_i$ for $i\in\{0,\dots,7\}$ in (5.17) in the variables $\alpha_1,\alpha_4,\alpha_6,\alpha_8$, $\beta_0,\beta_2,\beta_3,\beta_5,\beta_7,\beta_9$, $\gamma_1,\gamma_4,\gamma_6,\gamma_8$, $\delta_0,\delta_2,\delta_3,\delta_5,\delta_7,\delta_9$ is 8. Hence applying the averaging theory of first order and Theorem 5.3 it is proved that at most 7 medium limit cycles can bifurcate from the periodic solutions of the cubic uniform isochronous center of the Collins first form and this number can be reached. This completes the proof of Theorem 5.8.

# Chapter 6

# Limit cycles bifurcating from continuous and discontinuous perturbations of uniform isochronous centers of degree 4

In this chapter, we apply the averaging theory developed in chapter 5 to provide lower bounds for the maximum number of limit cycles that bifurcate from the origin of quartic polynomial differential systems of the form $\dot{x} = -y + xp(x, y), \quad \dot{y} = x + yp(x, y)$, with $p(x, y)$ a polynomial of degree 3 without constant term, when they are perturbed, either inside the class of all continuous quartic polynomial differential systems, or inside the class of all discontinuous piecewise quartic polynomial differential systems with two zones separated by the straight line $y = 0$.

## 6.1 Background

According to Proposition 2.1 (see section 2.2) any planar polynomial differential system of degree $n$ with a uniform isochronous center can be written into the form

$$\dot{x} = -y + x\, p(x, y), \quad \dot{y} = x + y\, p(x, y), \tag{6.1}$$

where $p(x, y)$ is a polynomial in $x$ and $y$ of degree $n - 1$ and $p(0, 0) = 0$.

Let $H_c(n)$ denote the maximum number of limit cycles that bifurcate from the origin of system (6.1), when it is perturbed inside the class of all continuous polynomial differential systems of degree $n$, and $H_d(n)$ denotes the maximum number of limit cycles that bifurcate from the origin of system (6.1), when it is perturbed inside the class of all discontinuous piecewise polynomial differential systems of degree $n$ with two zones separated by the straight line $y = 0$. We provide lower bounds for $H_c(4)$ and $H_d(4)$ in both cases when the origin is either a uniform isochronous center, or a weak focus. The method used for obtaining these lower bounds is based on the averaging theory.

In order to prove our results we also need the *Descartes Theorem* about the number of zeros of a real polynomial, see [8].

**Theorem 6.1** (Descartes theorem)**.** *Consider the real polynomial $r(x) = a_{i_1}x^{i_1} + a_{i_2}x^{i_2} + \ldots + a_{i_r}x^{i_r}$ with $0 = i_1 < i_2 < \ldots < i_r$ and $a_{i_j} \neq 0$ real constants for $j \in \{1, 2, \ldots, r\}$. When $a_{i_j}a_{i_{j+1}} < 0$, we say that $a_{i_j}$ and $a_{i_{j+1}}$ have a variation of sign. If the number of variations of signs is $m$, then $r(x)$ has at most $m$ positive real roots. Moreover, it is always possible to choose the coefficients of $r(x)$ in such a way that $r(x)$ has exactly $r-1$ positive real roots.*

## 6.2 Main results

We consider the following family of continuous differential systems

$$
\begin{aligned}
\dot{x} &= -y + xp(x,y) + \sum_{i=1}^{4} \varepsilon^i p_i(x,y),\\
\dot{y} &= \quad x + yp(x,y) + \sum_{i=1}^{4} \varepsilon^i q_i(x,y),
\end{aligned}
\tag{6.2}
$$

where

$$
\begin{aligned}
p_j =&\alpha_0^j + \alpha_1^j x + \alpha_2^j y + \alpha_3^j x^2 + \alpha_4^j xy + \alpha_5^j y^2 + \alpha_6^j x^3 + \alpha_7^j x^2 y + \alpha_8^j xy^2 + \alpha_9^j y^3\\
&+ \alpha_{10}^j x^4 + \alpha_{11}^j x^3 y + \alpha_{12}^j x^2 y^2 + \alpha_{13}^j xy^3 + \alpha_{14}^j y^4,\\
q_j =&\beta_0^j + \beta_1^j x + \beta_2^j y + \beta_3^j x^2 + \beta_4^j xy + \beta_5^j y^2 + \beta_6^j x^3 + \beta_7^j x^2 y + \beta_8^j xy^2 + \beta_9^j y^3\\
&+ \beta_{10}^j x^4 + \beta_{11}^j x^3 y + \beta_{12}^j x^2 y^2 + \beta_{13}^j xy^3 + \beta_{14}^j y^4,
\end{aligned}
$$

and of the discontinuous differential systems

$$
\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathcal{X}(x,y) = \begin{cases} X_1(x,y) & \text{if} \quad y > 0,\\ X_2(x,y) & \text{if} \quad y < 0, \end{cases}
\tag{6.3}
$$

where

$$
X_1(x,y) = \begin{pmatrix} -y + xp(x,y) + \sum_{i=1}^{k} \varepsilon^i p_i(x,y) \\ x + yp(x,y) + \sum_{i=1}^{k} \varepsilon^i q_i(x,y) \end{pmatrix},
$$

$$
X_2(x,y) = \begin{pmatrix} -y + xp(x,y) + \sum_{i=1}^{k} \varepsilon^i u_i(x,y) \\ x + yp(x,y) + \sum_{i=1}^{k} \varepsilon^i v_i(x,y) \end{pmatrix},
$$

$$
\begin{aligned}
u_j =&\gamma_0^j + \gamma_1^j x + \gamma_2^j y + \gamma_3^j x^2 + \gamma_4^j xy + \gamma_5^j y^2 + \gamma_6^j x^3 + \gamma_7^j x^2 y + \gamma_8^j xy^2 + \gamma_9^j y^3\\
&+ \gamma_{10}^j x^4 + \gamma_{11}^j x^3 y + \gamma_{12}^j x^2 y^2 + \gamma_{13}^j xy^3 + \gamma_{14}^j y^4,\\
v_j =&\delta_0^j + \delta_1^j x + \delta_2^j y + \delta_3^j x^2 + \delta_4^j xy + \delta_5^j y^2 + \delta_6^j x^3 + \delta_7^j x^2 y + \delta_8^j xy^2 + \delta_9^j y^3\\
&+ \alpha_{10}^j x^4 + \delta_{11}^j x^3 y + \delta_{12}^j x^2 y^2 + \delta_{13}^j xy^3 + \delta_{14}^j y^4,
\end{aligned}
$$

with $k = 4$ or $k = 7$ depending on the order of the averaging theory that we can compute. For the continuous and the discontinuous cases we have to consider either

$$
p(x,y) = t_{10}x + t_{01}y + t_{20}x^2 + t_{11}xy + t_{02}y^2 + t_{30}x^3 + t_{21}x^2y + t_{12}xy^2 + t_{03}y^3,
\tag{6.4}
$$

with $t_{ij} \in \mathbb{R}$, $i + j = 1, 2, 3$, $t_{30}^2 + t_{21}^2 + t_{12}^2 + t_{03}^2 \neq 0$, or

$$p(x, y) = t_{10}x + t_{11}xy + t_{30}x^3 + t_{12}xy^2, \tag{6.5}$$

with $t_{30}^2 + t_{12}^2 \neq 0$, or

$$p(x, y) = t_{30}x^3 + t_{21}x^2y + t_{12}xy^2 + t_{03}y^3. \tag{6.6}$$

We remark that the polynomials $p(x, y)$ in (6.5) and (6.6) are used to study the cases of quartic polynomial differential systems with a uniform isochronous center at the origin, having a non–homogeneous nonlinear part (see Theorem 4.1) or a homogeneous nonlinear part, respectively. On the other hand, since (6.4) is a general cubic polynomial in $x$ and $y$ without constant term, it is used to study the bifurcation of limit cycles in both cases when the origin can be either a uniform isochronous center or a weak focus.

In the following we state our results.

**Theorem 6.2.** *Using averaging theory of order 4 we obtain, for* $|\varepsilon| \neq 0$ *sufficiently small,* $H_d(4) \geq 6$ *for the differential system* (6.3) *with* $p(x, y)$ *of the form* (6.4) (*i.e. system* (6.3) *has a weak focus or a uniform isochronous center at the origin*).

Theorem 6.2 is proved in section 6.3.1.

**Theorem 6.3.** *Using averaging theory of order 4 we obtain, for* $|\varepsilon| \neq 0$ *sufficiently small,* $H_d(4) \geq 5$ *for the differential system* (6.3) *with* $p(x, y)$ *either of the form* (6.5) *or* (6.6) (*i.e. system* (6.3) *has a uniform isochronous center at the origin*).

Theorem 6.3 is proved in section 6.3.2.

**Theorem 6.4.** *Using the averaging theory of order 7 we obtain, for* $|\varepsilon| \neq 0$ *sufficiently small,* $H_d(4) \geq 6$ *for the differential system* (6.3) *with* $p(x, y)$ *of the form* (6.5) *and* $\alpha_0^j = \beta_0^j = \gamma_0^j = \delta_0^j = 0$, $j = 1, \ldots, 7$.

Theorem 6.4 is proved in section 6.3.3.

**Theorem 6.5.** *Using the averaging theory of order 4 we obtain, for* $|\varepsilon| \neq 0$ *sufficiently small,* $H_c(4) \geq 2$ *for the differential system* (6.2) *with* $p(x, y)$ *of the form* (6.4).

**Theorem 6.6.** *Using the averaging theory of order 4 we obtain, for* $|\varepsilon| \neq 0$ *sufficiently small,* $H_c(4) \geq 1$ *for the differential system* (6.2) *with* $p(x, y)$ *either of the form* (6.5) *or* (6.6).

Theorems 6.5 and 6.6 are proved in section 6.3.4.

These results have been submitted for publication, see [34].

We remark that all these results were obtained for Hopf bifurcation, that is, we studied the number of small limit cycles that can bifurcate from the uniform isochronous center.

We also remark that to prove Theorems 6.2 and 6.3 (respectively Theorems 6.5 and 6.6) we shall use the averaging theory of order 4 for discontinuous (respectively continuous) differential systems, together with a rescaling of the variables. In these proofs we can see, using Descartes Theorem (see Theorem 6.1 in this work), that the lower bounds which appear in the theorems are actually upper bounds for the averaging theory of order 4. From Theorems 6.2 and 6.3 (respectively Theorems 6.5 and 6.6) it follows that if applying the averaging theory of order 4 to the differential system (6.3) (respectively (6.2)) we obtain 6 (respectively 2) limit cycles, the origin of the differential system (6.3) (respectively (6.2)) is a weak focus.

All calculations were performed with the assistance of the software *Mathematica*.

## 6.3 Proofs of the results

### 6.3.1 Proof of Theorem 6.2

Consider system (6.3) with $p(x,y)$ of the general form (6.4). In order to analyze the Hopf bifurcation for this system, applying Theorem 5.4, we set $\alpha=\pi$ and we introduce a small parameter $\varepsilon$ doing the change of coordinates $x=\varepsilon X,\ y=\varepsilon Y$. After that we perform the polar change of coordinates $X=r\cos\theta, Y=r\sin\theta$, and by doing a Taylor expansion truncated at the $4^{th}$ order in $\varepsilon$ we obtain an expression for $dr/d\theta$ of the form (5.1), with $\alpha=\pi$. The explicit expression is quite large so we omit it.

The differential system (6.3) is a polynomial system, so the corresponding functions $F_i^{\pm}(\theta,r)$ and $R_i^{\pm}(\theta,r,\varepsilon)$, $i=1,\ldots,4$ are analytic. Moreover, since the variable $\theta$ appears through sines and cosines, system (6.3) in the form $dr/d\theta$ is $2\pi-$periodic. It suffices to apply Theorem 5.4 to take the open interval $D=\{r: 0<r<r_0\}$, where the unperturbed system has periodic solutions passing through the points $(0,r)$ with $0<r<r_0$.

We obtain each $y_i^+$ and $y_i^-, i=1.\ldots,4$ using the formulae provided in Appendix C respectively for $X_1$ and $X_2$ of system (6.3), after the changes described in the first paragraph of this section. Then we calculate the averaging functions $f_i, i=1.\ldots,4$ using equation (5.3). Hence, by Theorem 5.4 we have the averaging function of first order

$$f_1(r)=A_1r+A_0,$$

where

$$\begin{aligned}A_1=&\frac{1}{2}\pi(3t_{01}(\alpha_0^1+\gamma_0^1)+\alpha_1^1+\beta_2^1+\gamma_1^1+\delta_2^1-3t_{10}(\beta_0^1+\delta_0^1)),\\ A_0=&2(\beta_2^1\alpha_0^1+(\alpha_0^1)^2t_{01}-\beta_0^1(\alpha_0^1t_{10}+\beta_1^1)-\gamma_0^1\delta_2^1-(\gamma_0^1)^2t_{01}+\delta_0^1(\gamma_0^1t_{10}+\delta_1^1)\\ &+\beta_0^2-\delta_0^2).\end{aligned}$$

The rank of the Jacobian matrix of the function $\mathcal{A}=(A_0,A_1)$ with respect to the variables $t_{01},t_{10},\alpha_0^1,\alpha_1^1,\beta_0^1,\beta_1^1,\beta_2^1,\gamma_0^1,\gamma_1^1,\delta_0^1,\delta_1^1,\delta_2^1$ is maximal. Then the coefficients $A_0$ and $A_1$ are linearly independent in their variables.

Clearly $f_1(r)$ has at most one solution in $D$. Thus applying Theorem 5.4 it is proved that at most 1 limit cycle can bifurcate from the origin of system (6.3) with $p(x,y)$ of the form (6.4), using the averaging theory of first order. Solving $A_1$ for $\alpha_1^1$ and $A_0$ for $\delta_0^2$ we have $f_1(r)=0$, and we can apply the averaging theory of order 2. Its corresponding averaging function is

$$f_2(r)=B_3r^3+B_2r^2+B_1r+B_0,$$

where

$$
\begin{aligned}
B_3 =& 2\pi(t_{02}+t_{20}),\\
B_2 =& \frac{1}{3}(-4)(3t_{01}(2\alpha_0^1 t_{10}-\alpha_2^1+4\gamma_0^1 t_{10}+\gamma_2^1)-8t_{02}(\alpha_0^1-\gamma_0^1)-\alpha_0^1 t_{20}-\alpha_4^1-\beta_3^1\\
&-2\beta_5^1+6\gamma_1^1 t_{10}+\gamma_0^1 t_{20}+\gamma_4^1+\delta_3^1+2\delta_5^1+3t_{01}^2(\beta_0^1-\delta_0^1)-3\beta_0^1 t_{10}^2\\
&-15\delta_0^1 t_{10}^2+3\beta_2^1 t_{10}+3\delta_2^1 t_{10}+4\beta_0^1 t_{11}-4\delta_0^1 t_{11}),\\
B_1 =& \frac{1}{4}\pi(-8\alpha_3^1\beta_0^1+8\alpha_0^1\beta_5^1-3t_{01}(t_{10}(-\alpha_0^1\gamma_0^1+15\beta_0^1\delta_0^1+8(\alpha_0^1)^2+8(\beta_0^1)^2\\
&+(\gamma_0^1)^2-7(\delta_0^1)^2)+3\alpha_2^1\gamma_0^1-5\beta_0^1\delta_2^1-4\alpha_0^1\alpha_2^1-4\beta_0^1\beta_2^1-5\beta_0^1\gamma_1^1\\
&-\beta_1^1\gamma_0^1+5\gamma_1^1\delta_0^1+\gamma_0^1\delta_1^1-7\gamma_0^1\gamma_2^1+\delta_0^1\delta_2^1-4\alpha_0^2-4\gamma_0^2)\\
&+3t_{01}^2(8\alpha_0^1\beta_0^1+\gamma_0^1(15\beta_0^1-7\delta_0^1))+16t_{02}((\alpha_0^1)^2+(\gamma_0^1)^2)+\alpha_0^1\gamma_1^1 t_{10}\\
&-3\alpha_2^1\gamma_1^1-3\alpha_2^1\delta_2^1+\beta_1^1\delta_2^1+24\alpha_0^1\beta_0^1 t_{10}^2-3\alpha_0^1\delta_0^1 t_{10}^2\\
&-24\alpha_0^1\beta_2^1 t_{10}+9\alpha_2^1\delta_0^1 t_{10}+\alpha_0^1\delta_2^1 t_{10}-16\alpha_0^1\beta_0^1 t_{11}+4\alpha_0^1\alpha_4^1-4\beta_0^1\beta_4^1\\
&+\beta_1^1\gamma_1^1-8\gamma_3^1\delta_0^1-\gamma_1^1\delta_1^1+3\gamma_2^1\delta_2^1+8\gamma_0^1\delta_5^1+3\gamma_0^1\delta_0^1 t_{10}^2\\
&-9\gamma_2^1\delta_0^1 t_{10}-\gamma_0^1\delta_2^1 t_{10}-\gamma_0^1\gamma_1^1 t_{10}-16\gamma_0^1\delta_0^1 t_{11}+3\gamma_1^1\gamma_2^1+4\gamma_0^1\gamma_4^1\\
&-\delta_1^1\delta_2^1-4\delta_0^1\delta_4^1+4\alpha_1^2+4\beta_2^2+4\gamma_1^2+4\delta_2^2-3\beta_1^1\delta_0^1 t_{10}\\
&+24\beta_0^1\beta_1^1 t_{10}+3\delta_0^1\delta_1^1 t_{10}-24\beta_0^2 t_{10}+16(\beta_0^1)^2 t_{20}+16(\delta_0^1)^2 t_{20}),\\
B_0 =& -4(-\alpha_0^1\beta_2^1\delta_1^1+\alpha_0^1\beta_1^1\beta_2^1+\alpha_0^1\beta_0^1\beta_4^1-(\alpha_0^1)^2\beta_5^1+t_{01}((\alpha_0^1)^2(\beta_1^1-\delta_1^1)\\
&+\alpha_0^1(3\alpha_2^1\gamma_0^1+2\beta_0^1(\gamma_1^1+\delta_2^1))+t_{10}(-6\beta_0^1\alpha_0^1\delta_0^1-(\alpha_0^1)^2\gamma_0^1+(\alpha_0^1)^3\\
&-6\gamma_0^1(3\beta_0^1\delta_0^1+(\beta_0^1)^2-(\delta_0^1)^2))+\gamma_0^1(6\beta_0^1\delta_2^1+3\beta_0^1\beta_2^1+6\beta_0^1\gamma_1^1\\
&-4\gamma_1^1\delta_0^1+3\gamma_0^1\gamma_2^1-\delta_0^1\delta_2^1+3\alpha_0^2+3\gamma_0^2))+3\gamma_0^1 t_{01}^2(2\alpha_0^1\beta_0^1\\
&+\gamma_0^1(3\beta_0^1-\delta_0^1))+t_{02}((\gamma_0^1)^3-(\alpha_0^1)^3)+\alpha_0^1\beta_0^1\gamma_0^1 t_{10}^2-\alpha_0^1\beta_2^1\gamma_0^1 t_{10}\\
&+\beta_0^1(\delta_2^1)^2+\alpha_0^1\alpha_2^1\gamma_1^1+\beta_0^1\beta_1^1\delta_1^1+\alpha_0^1\alpha_2^1\delta_2^1+\beta_0^1\beta_2^1\delta_2^1\\
&+\beta_1^1\beta_0^2+\beta_0^1\beta_1^2-\alpha_0^1\beta_2^2-(\alpha_0^1)^2\beta_0^1 t_{10}^2+\alpha_0^1\beta_0^1\delta_1^1 t_{10}-2\alpha_0^1\beta_0^1\beta_1^1 t_{10}\\
&+(\alpha_0^1)^2\beta_2^1 t_{10}-3\alpha_0^1\alpha_2^1\delta_0^1 t_{10}+\alpha_0^1\beta_0^2 t_{10}+(\alpha_0^1)^2\beta_0^1 t_{11}-\alpha_0^1(\beta_0^1)^2 t_{20}\\
&-\beta_0^1(\beta_1^1)^2-(\beta_0^1)^2\beta_3^1+2\beta_0^1\gamma_1^1\delta_2^1+\beta_0^1(\gamma_1^1)^2+\beta_0^1\beta_2^1\gamma_1^1-(\gamma_1^1)^2\delta_0^1\\
&-\gamma_1^1\delta_2^1\delta_0^1-\gamma_0^1\delta_4^1\delta_0^1+\gamma_0^1\gamma_2^1\delta_2^1+(\gamma_0^1)^2\delta_5^1+\gamma_1^1\alpha_0^2-\delta_1^1\beta_0^2\\
&-\delta_0^1\delta_1^2+\gamma_1^1\gamma_0^2+\gamma_0^1\delta_2^2-6\beta_0^1\gamma_1^1\delta_0^1 t_{10}-2(\beta_0^1)^2\gamma_1^1 t_{10}\\
&+\beta_0^1\beta_1^1\gamma_0^1 t_{10}+4\gamma_1^1(\delta_0^1)^2 t_{10}-3\gamma_0^1\gamma_2^1\delta_0^1 t_{10}-\gamma_0^1\beta_0^2 t_{10}-(\gamma_0^1)^2\delta_0^1 t_{11}\\
&+\gamma_0^1(\delta_0^1)^2 t_{20}+\delta_3^1(\delta_0^1)^2+\gamma_0^1\gamma_1^1\gamma_2^1+\delta_2^1\alpha_0^2-3\delta_0^1\alpha_0^2 t_{10}\\
&+\delta_2^1\gamma_0^2-3\delta_0^1\gamma_0^2 t_{10}-\beta_0^3+\delta_0^3+9\beta_0^1(\delta_0^1)^2 t_{10}^2+6(\beta_0^1)^2\delta_0^1 t_{10}^2\\
&-3(\delta_0^1)^3 t_{10}^2-3\beta_0^1\beta_2^1\delta_0^1 t_{10}-6\beta_0^1\delta_2^1\delta_0^1 t_{10}-2(\beta_0^1)^2\delta_2^1 t_{10}+\delta_2^1(\delta_0^1)^2 t_{10}),
\end{aligned}
$$

and since the rank of the Jacobian matrix of the function $\mathcal{B}=(B_0,B_1,B_2,B_3)$ with respect to its variables is maximal, $B_i$, $i=0,\dots,3$ are linearly independent in their variables.

Hence $f_2(r)$ has at most 3 solutions in $D$, see Theorem 6.1. Applying Theorem 5.4 it is proved that at most 3 limit cycles can bifurcate from the origin of system (6.3) with $p(x,y)$ of the form (6.4), using the averaging theory of order 2. Solving $B_3$ for $t_{02}$, $B_2$ for $\alpha_4^1$, $B_1$ for $\beta_2^2$ and $B_0$ for $\delta_0^3$ we obtain $f_2(r)=0$, and we can apply the averaging theory of order 3, which corresponding averaging function is of the form

$$rf_3(r) = C_4r^4 + C_3r^3 + C_2r^2 + C_1r + C_0,$$

and $C_i$ for $i = 0, \ldots, 4$ are linearly independent in their variables, because the rank of the Jacobian matrix of the function $\mathcal{C} = (C_0, \ldots, C_4)$ with respect to its variables is maximal. We do not explicitly provide their expressions, since they are very long. Therefore $f_3(r)$ has at most 4 solutions in $D$, by Theorem 6.1. Applying Theorem 5.4 it is proved that at most 4 limit cycles can bifurcate from the origin of system (6.3) with $p(x, y)$ of the form (6.4) using the averaging theory of order 3. By conveniently choosing variables to cancel the coefficients $C_i, i = 0, \ldots, 4$ we have $f_3(r) = 0$. Hence we apply the averaging theory of order 4 to obtain the averaging function of order 4

$$rf_4(r) = D_6r^6 + D_5r^5 + D_4r^4 + D_3r^3 + D_2r^2 + D_1r + D_0.$$

Since the rank of the Jacobian matrix of the function $\mathcal{D} = (D_0, \ldots, D_6)$ with respect to its variables is maximal, the coefficients $D_i$, $i = 0, \ldots, 6$ are linearly independent in their variables. Their expressions are very long so we do not provide them here. As a result of these calculations, it follows that $f_4(r)$ has at most 6 solutions in $D$ by Theorem 6.1. Applying Theorem 5.4 we conclude that at most 6 limit cycles can bifurcate from the origin of system (6.3) with $p(x, y)$ of the form (6.4), using the averaging theory of order 4. This result is a lower bound for $H_d(4)$, hence Theorem 6.2 is proved.

### 6.3.2 Proof of Theorem 6.3

First we consider the systems of the form (6.3) with $p(x, y)$ of the form (6.5). According to Theorem 4.1, the corresponding unperturbed system has a uniform isochronous center at the origin. In order to study the Hopf bifurcation for this case, we apply the results obtained in the proof of Theorem 6.2, by conveniently vanishing the coefficients of (6.4), used in that proof. More precisely, we take $t_{01} = t_{20} = t_{02} = t_{21} = t_{03} = 0$.

We also consider the systems of the form (6.3), with $p(x, y)$ of the form (6.6), whose corresponding unperturbed system also has a uniform isochronous center at the origin, see Theorem 2.7. Again, we use the results obtained in the proof of Theorem 6.2, vanishing the appropriate coefficients of (6.4), that is, we take $t_{01} = t_{10} = t_{20} = t_{11} = t_{02} = 0$.

Considering the above restrictions to the coefficients of $p(x, y)$ we obtain the averaging functions $f_i, i = 1, \ldots, 4$ and since they are similar to those calculated in the proof of Theorem 6.2 we do not explicitly present them here. It is interesting to observe that the same number of limit cycles in each averaging order was obtained with $p(x, y)$ of the form (6.5) and (6.6).

The following table summarizes the results obtained in this proof and in the proof of Theorem 6.2.

It follows that if system (6.3) has 6 limit cycles up to the averaging theory of order 4, then it must have a weak focus at the origin.

### 6.3.3 Proof of Theorem 6.4

Consider system (6.3) with $p(x, y)$ of the form (6.5) and take $\alpha_0^j = \beta_0^j = \gamma_0^j = \delta_0^j = 0$, for $j = 1, \ldots, 7$. In this case the corresponding unperturbed system has a uniform isochronous

| Averaging order | # limit cycles | |
|---|---|---|
| | Theorem 6.2 | Theorem 6.3 with $p(x,y)$ given by (6.5) or (6.6) |
| 1 | 1 | 1 |
| 2 | 3 | 2 |
| 3 | 4 | 4 |
| 4 | 6 | 5 |

Table 6.1: Number of limit cycles for discontinuous differential systems (6.3).

center at the origin, see Theorem 4.1. In order to analyze the Hopf bifurcation for this case, applying Theorem 5.4, we set $\alpha = \pi$ and we introduce a small parameter $\varepsilon$ doing the rescaling $x = \varepsilon X,\ y = \varepsilon Y$. After that doing the polar change of coordinates $X = r\cos\theta, Y = r\sin\theta$ and a Taylor expansion truncated at the $7^{th}$ order in $\varepsilon$ we obtain an expression for $dr/d\theta$ of the form (5.1), with $\alpha = \pi$. The explicit expression is quite large so we omit it. All hypotheses for applying Theorem 5.4 to this case are satisfied using similar arguments to those presented for the proof of Theorem 6.2.

We obtain each $y_i^+$ and $y_i^-, i = 1\dots,7$ using the formulae provided in Appendix C respectively for $X_1$ and $X_2$ of system (6.3), after the changes previously described. Then we calculate the averaging functions $f_i, i = 1\dots,7$ using equation (5.3). We remark that, up to the averaging theory of order 4, the results in this case can be easily obtained from those already calculated in the proof of Theorem 6.3, taking into account the condition $\alpha_0^j = \beta_0^j = \gamma_0^j = \delta_0^j = 0,\ j = 1,\dots,7$, so we do not explicitly present the averaging functions from order 1 to 3 here. Starting from the averaged function of order 4 we have

$$f_4(r) = R_4r^4 + R_3r^3 + R_2r^2 + R_1r,$$

and $R_i$ for $i = 1,\dots,4$ are linearly independent in their variables, since the rank of the Jacobian matrix of the function $\mathcal{R} = (R_1,\dots,R_4)$ with respect to its variables is maximal. We do not explicitly provide their expressions, because they are very long. Therefore $f_4(r)$ has at most 3 solutions in $D$, by Theorem 6.1. Applying Theorem 5.4 it is proved that at most 3 limit cycles can bifurcate from the origin of system (6.3) with $p(x,y)$ of the form (6.5), and $\alpha_0^j = \beta_0^j = \gamma_0^j = \delta_0^j = 0,\ j = 1,\dots,7$ using the averaging theory of order 4.

The next averaging functions are calculated in a similar way, so we obtain

$$f_5(r) = S_5r^5 + S_4r^4 + S_3r^3 + S_2r^2 + S_1r,$$

and $S_i$ for $i = 1,\dots,5$ are linearly independent in their variables,

$$f_6(r) = T_6r^6 + T_5r^5 + T_4r^4 + T_3r^3 + T_2r^2 + T_1r,$$

and $T_j$ for $j = 1,\dots,6$ are linearly independent in their variables,

$$f_7(r) = U_7r^7 + U_6r^6 + U_5r^5 + U_4r^4 + U_3r^3 + U_2r^2 + U_1r,$$

and $U_k$ for $k = 1,\dots,7$ are linearly independent in their variables. The expressions of $S_i$, $i = 1,\dots,5$, $T_j$, $j = 1,\dots,6$ and $U_k$, $k = 1,\dots,7$ are very long so we do not provide them here.

Thus $f_5(r)$, $f_6(r)$ and $f_7(r)$ has at most 4, 5 and 6 solutions in $D$, respectively, see Theorem 6.1. Applying Theorem 5.4 we conclude that at most 4, 5, and 6 limit cycles can bifurcate from the origin of system (6.3) with $p(x,y)$ of the form (6.5), and $\alpha_0^j = \beta_0^j = \gamma_0^j = \delta_0^j = 0$, $j = 1, \ldots, 7$ using the averaging theory of order 5, 6 and 7, respectively. Therefore Theorem 6.4 is proved.

The following table summarizes our results for this case

| Averaging order | # limit cycles |
|---|---|
| 1 | 0 |
| 2 | 1 |
| 3 | 2 |
| 4 | 3 |
| 5 | 4 |
| 6 | 5 |
| 7 | 6 |

Table 6.2: Limit cycles for quartic discontinuous differential systems with a uniform isochronous center at the origin.

### 6.3.4 Proof of Theorems 6.5 and 6.6

System (5.1) becomes continuous by taking $\alpha = 2\pi$ and therefore the averaging theory developed in chapter 5 also applies to continuous differential systems.

First, consider the continuous differential system (6.2) with $p(x,y)$ of the form (6.4). In order to study the limit cycles for this system we only need the expressions of $y_i^+, i = 1\ldots, 4$, which were already calculated for studying the previous cases. Hence, the averaging functions $f_i, i = 1\ldots, 4$ can be obtained by the same algorithm used for the discontinuous differential systems, by taking $\alpha = 2\pi$.

The unperturbed continuous differential system corresponding to the perturbed system (6.2), with either $p(x,y)$ of the form (6.5) or (6.6) has a uniform isochronous center at the origin, according to Theorems 4.1 and 2.7, respectively. We apply the same arguments as in the previous paragraph, by taking $\alpha = 2\pi$ and using the expressions of $y_i^+, i = 1\ldots, 4$ calculated in the proof of Theorem 6.3 to obtain the averaging functions $f_i, i = 1\ldots, 4$ for this case. We remark that the same number of limit cycles was obtained in both cases where $p(x,y)$ is either of the form (6.5) or (6.6), in each averaging order studied.

Since the calculations and arguments are quite similar to those used in the previous proofs, we omit the explicit expressions of the averaging functions. We summarize our results in the following table

We remark that from this proof, it follows that system (6.2) with $p(x,y)$ of the form (6.4) has a weak focus at the origin provided that it has 2 limit cycles up to the averaging theory of order 4.

| Averaging order | # limit cycles | |
|---|---|---|
| | general case | Uniform center |
| 1 | 0 | 0 |
| 2 | 1 | 0 |
| 3 | 1 | 1 |
| 4 | 2 | 1 |

Table 6.3: Number of limit cycles for continuous differential systems (6.2).

# Chapter 7

# Application of the averaging theory in a concrete planar polynomial differential system of degree 4

In this chapter, we apply the averaging theory to study the bifurcation of limit cycles in a concrete planar polynomial differential system of degree 4 with a uniform isochronous center at the origin. More precisely, we study the limit cycles that bifurcate from the periodic solutions of the differential system $\dot{x} = -y + xy(x^2 + y^2)$, $\dot{y} = x + y^2(x^2 + y^2)$ when it is perturbed inside the class of all quartic polynomial differential systems. Using the averaging theory of first order we show that at least 8 limit cycles can bifurcate from the period annulus of the considered center.

## 7.1 Background

Peng and Feng studied in [48] the following quartic polynomial differential system with a uniform isochronous center at the origin

$$\dot{x} = -y + xy(x^2 + y^2), \quad \dot{y} = x + y^2(x^2 + y^2). \tag{7.1}$$

They show that under any quartic homogeneous polynomial perturbations, at most 2 limit cycles bifurcate from the period annulus of system (7.1) using averaging theory of first order, and this upper bound can be reached. In addition these authors prove that for the family of perturbed quartic polynomial differential systems

$$\begin{aligned} \dot{x} =& - y + xy(x^2 + y^2) + \varepsilon(a_{10}x + a_{01}y + a_{11}xy + a_{21}x^2y + a_{03}y^3 \\ & + a_{40}x^4 + a_{31}x^3y + a_{22}x^2y^2 + a_{13}xy^3 + a_{04}y^4), \\ \dot{y} =& x + y^2(x^2 + y^2) + \varepsilon(b_{10}x + b_{01}y + b_{20}x^2 + b_{02}y^2 + b_{30}x^3 \\ & + b_{12}xy^2 + b_{40}x^4 + b_{31}x^3y + b_{22}x^2y^2 + b_{13}xy^3 + b_{04}y^4), \end{aligned} \tag{7.2}$$

there are at most 3 limit cycles bifurcating from the period annulus of (7.1) using averaging theory of first order, and this upper bound is sharp. We remark that the perturbed system (7.2) studied by Peng and Feng do not consider all the quartic polynomial differential systems because they omit the coefficients $a_{00}, a_{20}, a_{02}, a_{30}, a_{12}, b_{00}, b_{11}, b_{21}, b_{03}$ as we shall present in the next section.

## 7.2 Main results

we consider the polynomial differential systems

$$\begin{aligned} \dot{x} &= -y + xy(x^2+y^2) + \varepsilon \sum_{i=0}^{4} p_i(x,y), \\ \dot{y} &= \ \ x + y^2(x^2+y^2) + \varepsilon \sum_{i=0}^{4} q_i(x,y), \end{aligned} \tag{7.3}$$

where $p_i = \sum_{j+k=i} a_{jk}x^j y^k$ and $q_i = \sum_{j+k=i} b_{jk}x^j y^k$ are real homogeneous polynomials of degree $i$.

In what follows we state our result.

**Theorem 7.1.** *For $|\varepsilon| \neq 0$ sufficiently small there are quartic polynomial differential systems* (7.3) *having at least 8 limit cycles bifurcating from the periodic orbits of the uniform isochronous center* (7.1)*.*

Note that Theorem 7.1 improves the result of Peng and Feng in 5 additional limit cycles. The proof of Theorem 7.1 is presented in section 7.3. This result has been submitted for publication, see [33].

All calculations were performed with the assistance of the software *Mathematica*.

## 7.3 Proof of Theorem 7.1

By Theorem 2.7 it follows that system (7.1) has a uniform isochronous center at the origin. A first integral $H$ and its corresponding integrating factor $\mu$ for system (7.1) are

$$H(x,y) = \frac{1}{3(x^2+y^2)^{3/2}} - \frac{x}{(x^2+y^2)^{1/2}}, \quad \mu(x,y) = \frac{1}{(x^2+y^2)^{5/2}},$$

respectively. When $h \in (1,+\infty)$ then $H(x,y) = h$ are periodic solutions around the center $(0,0)$. For proving Theorem 7.1 we shall use Theorem 5.1. We choose

$$\rho(R,\theta) = \frac{1}{(R^2+3\cos\theta)^{1/3}},$$

then $H(\rho\cos\theta, \rho\sin\theta) = R^2/3$ for all $R > \sqrt{3}$ and $\theta \in [0,2\pi)$. Therefore all the hypotheses of Theorem 5.1 are satisfied for system (7.1). Using Theorem 5.1 we transform the perturbed differential system (7.3) into the form

$$\frac{dR}{d\theta} = \varepsilon \left( \frac{3}{2R} \frac{Qp - Pq}{\rho^5} \right)\bigg|_{x=\rho\cos\theta, y=\rho\sin\theta} + O(\varepsilon^2), \tag{7.4}$$

where

$$Qp - Pq = A + B,$$

with

$$
\begin{aligned}
A =& a_{00}x + b_{00}y + (a_{02} + b_{11})xy^2 + a_{20}x^3 + (a_{00} + b_{03})y^4 \\
& - b_{00}xy^3 + (a_{00} + a_{12} + b_{21})x^2y^2 - b_{00}x^3y + a_{30}x^4 + a_{02}y^6 \\
& + (a_{02} + a_{20} - b_{11})x^2y^4 + (a_{20} - b_{11})x^4y^2 + (a_{12} - b_{03})xy^6 \\
& (a_{12} + a_{30} - b_{03} - b_{21})x^3y^4 + (a_{30} - b_{21})x^5y^2, \\
B =& a_{10}x^2 + (a_{01} + b_{10})xy + b_{01}y^2 + (a_{11} + b_{20})x^2y + b_{02}y^3 \\
& + (a_{21} + b_{30})x^3y + (a_{03} + b_{12})xy^3 + a_{40}x^5 + (a_{31} + b_{40} - b_{10})x^4y \\
& + (a_{22} + a_{10} + b_{31} - b_{01})x^3y^2 + (a_{13} + a_{01} + b_{22} - b_{10})x^2y^3 \\
& + (a_{04} + a_{10} + b_{13} - b_{01})xy^4 + (a_{01} + b_{04})y^5 - b_{20}x^5y \\
& + (a_{11} - b_{20} - b_{02})x^3y^3 + (a_{11} - b_{02})xy^5 - b_{30}x^6y \\
& + (a_{21} - b_{30} - b_{12})x^4y^3 + (a_{21} + a_{03} - b_{12})x^2y^5 + a_{03}y^7 \\
& - b_{40}x^7y + (a_{40} - b_{31})x^6y^2 + (a_{31} - b_{40} - b_{22})x^5y^3 \\
& + (a_{40} + a_{22} - b_{31} - b_{13})x^4y^4 + (a_{31} + a_{13} - b_{22} - b_{04})x^3y^5.
\end{aligned}
$$

The coefficients $\{a_{ij}, b_{ij}\}_{i,j\in\{0,\ldots,4\}}$ which appear in $A$ and $B$ are different. The expression $B$ corresponds to the perturbed system (7.2) studied in [48]. The authors of [48] obtained for this system the following averaging function

$$
\begin{aligned}
g_B(R) = \frac{3}{4R}\Bigg[&\left(M_4 - \frac{3M_1 + 4M_2 + 8M_3}{36}\right)R^2 - \frac{M_1 + 2M_2}{82}R^6 - \frac{2M_1}{729}R^{10} \\
&+ \left(\frac{2M_1}{729}R^{12} + \frac{2M_2}{81}R^8 + \frac{2M_3}{9}R^4 - 2(M_1 + M_2 + M_3)\right)\frac{1}{\sqrt{R^4 - 9}}\Bigg],
\end{aligned}
\tag{7.5}
$$

where

$$
\begin{aligned}
M_1 =& a_{22} - a_{40} - a_{04} + b_{31} - b_{13}, \\
M_2 =& -2a_{22} + a_{40} + 3a_{04} - b_{31} + 2b_{13}, \\
M_3 =& a_{22} - 3a_{04} - b_{13}, \\
M_4 =& a_{10} + b_{01}.
\end{aligned}
\tag{7.6}
$$

Peng and Feng prove that the function $g_B(R)$ has at most 3 zeros in $R \in (\sqrt{3}, +\infty)$, and using the averaging theory of first order they show that the maximum number of limit cycles of system (7.2) emerging from the period annulus of the unperturbed system (7.1) is 3.

In this work we extend the results presented in [48] by calculating the part of the averaging function of system (7.3) corresponding to the expression $A$. In this way we perturbed the center (7.1) inside the whole class of quartic polynomial differential systems. We note that (7.4) is continuous and bounded for $\theta \in (0, 2\pi)$ and $R \in (\sqrt{3}, +\infty)$ therefore the integral of (7.4) is the sum of the integrals of its parts $A$ and $B$. Then from the expression (7.4) we have

$$
\frac{dR}{d\theta} = \varepsilon\left(\frac{3}{2R}\frac{A}{\rho^5}\right)\Bigg|_{\substack{x=\rho\cos\theta \\ y=\rho\sin\theta}} + \varepsilon\left(\frac{3}{2R}\frac{B}{\rho^5}\right)\Bigg|_{\substack{x=\rho\cos\theta \\ y=\rho\sin\theta}} + O(\varepsilon^2).
$$

We obtain the averaging function $f(R) = g_A(R) + g_B^*(R)$ where

$$
\begin{aligned}
g_A(R) =& a_{00}g_0(R) + a_{02}g_1(R) + a_{12}g_2(R) + a_{20}g_3(R) + a_{30}g_4(R) \\
&+ b_{03}g_5(R) + b_{11}g_6(R) + b_{21}g_7(R), \\
g_B^*(R) =& \sum_{j=1}^{4} M_i \, g_{Mj}(R),
\end{aligned}
$$

and $g_B^*(R)$ is the function (7.5) rearranged in a convenient way, with $M_j$, $j = 1, \ldots, 4$ given in (7.6). The expressions of $g_i(R)$, $i = 0, \ldots, 7$ and of $g_{Mj}(R)$, $j = 1, \ldots, 4$ are shown in sections F.1 and F.1 of Appendix F, respectively.

Out of the 12 functions $G_i = g_i : (\sqrt{3}, +\infty) \to \mathbb{R}$, $i \in \{0, \ldots, 7\}$, $G_{i+7} = g_{Mi} : (\sqrt{3}, +\infty) \to \mathbb{R}$, $i \in \{1, \ldots, 4\}$ we have that 9 are linearly independent. Indeed, by proceeding to the calculation of the Taylor expansions in the variable $R$ around $R = 2$ until the $15^{th}$ power of $R$ for the 12 functions, which are too long and therefore they are not presented here, and by using the software *Mathematica* we conclude that the rank of the matrix $12 \times 16$, where in the $k$ row there are the 16 coefficients of $R^0, R^1, \ldots, R^{15}$ of the Taylor expansion of $G_k$, $k \in \{0, \ldots, 11\}$, is 9.

By Proposition 5.2 since there are 9 linearly independent functions among the 12 previously described, then there exists a linear combination of them with at least 8 zeros, because all the coefficients of the 12 functions are linearly independent, as it is easy to check, and hence the coefficients of the 9 linearly independent functions, which are a subset of the 12 functions, are also linearly independent. Thus there exist $R_1, R_2, \ldots, R_8 \in (\sqrt{3}, +\infty)$ and coefficients $a_{ij}, b_{ij} \in \mathbb{R}$, $i, j \in \{0, \ldots, 4\}$ such that $f(R_k) = 0$, $k \in \{1, \ldots, 8\}$.

In summary, there are quartic polynomial differential systems (7.3) having at least 8 limit cycles bifurcating from the period orbits of the uniform isochronous center (7.1).

Note that in Theorem 7.1 we study medium limit cycles, i.e. limit cycles bifurcating from the periodic orbits surrounding the uniform isochronous center of the differential system (7.1), whereas in chapter 6 we have studied the small limit cycles of all quartic uniform isochronous centers, i.e. the limit cycles bifurcating from the center equilibrium point.

# Appendix A

# Poincaré Compactification

Consider $\mathcal{X}$ a planar polynomial vector field of degree $n$. The *Poincaré compactified vector field* $p(\mathcal{X})$ *corresponding to* $\mathcal{X}$ is an analytic vector field induced on $\mathbb{S}^2$ as follows, for further details see for instance [28], or chapter 5 of [22]. Let $\mathbb{S}^2 = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$ (the so called *Poincaré sphere*) and $T_y\mathbb{S}^2$ be the tangent space to $\mathbb{S}^2$ at the point $y$. Moreover, consider the central projection $f : T_{(0,0,1)}\mathbb{S}^2 \to \mathbb{S}^2$. This map defines 2 copies of $\mathcal{X}$, one in the northern hemisphere and the other in the southern one. Denote by $\mathcal{X}'$ the vector field $Df \circ \mathcal{X}$ defined on $\mathbb{S}^2$ except on its equator $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$. Note that $\mathbb{S}^1$ is identified to the *infinity* of $\mathbb{R}^2$. Then $p(\mathcal{X})$ is the only analytic extension of $y_3^{n-1}\mathcal{X}'$ to $\mathbb{S}^2$. On $\mathbb{S}^2 \backslash \mathbb{S}^1$ there are two symmetric copies of $\mathcal{X}$, and studying the behavior of $p(\mathcal{X})$ around $\mathbb{S}^1$, we obtain the behavior of $\mathcal{X}$ at infinity. The projection of the closed northern hemisphere of $\mathbb{S}^2$ on $y_3 = 0$ under $(y_1, y_2, y_3) \longmapsto (y_1, y_2)$ is known as the *Poincaré disc*, and it is denoted by $\mathbb{D}^2$. One important property of the Poincaré compactification is that $\mathbb{S}^1$ is invariant under the flow of $p(\mathcal{X})$.

Since $\mathbb{S}^2$ is a differentiable manifold we consider the six local charts $U_i = \{y \in \mathbb{S}^2 : y_i > 0\}$, and $V_i = \{y \in \mathbb{S}^2 : y_i < 0\}$ where $i = 1, 2, 3$ for computing the expression for $p(\mathcal{X})$. The diffeomorphisms $F_i : U_i \to \mathbb{R}^2$ and $G_i : V_i \to \mathbb{R}^2$ for $i = 1, 2, 3$ are the inverses of the central projections from the planes tangent at the points $(1, 0, 0)$, $(-1, 0, 0)$, $(0, 1, 0)$, $(0, -1, 0)$, $(0, 0, 1)$, and $(0, 0, -1)$ respectively. We denote by $(u, v)$ the value of $F_i(y)$ or $G_i(y)$ for any $i = 1, 2, 3$. Note that $(u, v)$ represents different things according to the local charts under consideration.

In the local chart $(U_1, F_1)$, $p(\mathcal{X})$ is written as

$$\dot{u} = v^n \left[ -uP\left(\frac{1}{v}, \frac{u}{v}\right) + Q\left(\frac{1}{v}, \frac{u}{v}\right) \right], \quad \dot{v} = -v^{n+1} P\left(\frac{1}{v}, \frac{u}{v}\right),$$

and the expression for $p(\mathcal{X})$ in the local chart $(U_2, F_2)$ is

$$\dot{u} = v^n \left[ P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right) \right], \quad \dot{v} = -v^{n+1} Q\left(\frac{u}{v}, \frac{1}{v}\right),$$

and finally for $(U_3, F_3)$ it is

$$\dot{u} = P(u, v), \quad \dot{v} = Q(u, v).$$

The expression for $p(\mathcal{X})$ in each chart $(V_i, G_i)$ is the same as in the chart $(U_i, F_i)$, multiplied by $(-1)^{n-1}$, $i = 1, 2, 3$. The points of $\mathbb{S}^1$ in any chart have $v = 0$. Therefore we have a polynomial vector field in each local chart.

We define *finite* (respectively, *infinite*) singular points of $\mathcal{X}$ or $p(\mathcal{X})$ the singular points of $p(\mathcal{X})$ which lie in $\mathbb{S}^2 \backslash \mathbb{S}^1$ (respectively $\mathbb{S}^1$). We note that if $y \in \mathbb{S}^1$ is an infinite singular point, then $-y$ is also a singular point. Since the local behavior near $-y$ is the local behavior near $y$ multiplied by $(-1)^{n-1}$, it follows that the orientation of the orbits changes when the degree is even.

The unique singular points at infinity which cannot be contained into the charts $U_1 \cup V_1$ are the origins of $U_2$ and $V_2$. Then, when we study the infinite singular points on the charts $U_2 \cup V_2$, we only have to verify if the origin of these charts are singularities.

# Appendix B

# Topological equivalence

Two polynomial vector fields $X$ and $Y$ on $\mathbb{R}^2$ are *topologically equivalent* if there exists a homeomorphism on $\mathbb{S}^2$ which preserves the infinity $\mathbb{S}^1$ carrying orbits of the flow induced by $p(X)$ into orbits of the flow induced by $p(Y)$, preserving or reversing simultaneously the sense of all orbits.

A *separatrix* of $p(X)$ is an orbit which is either a singular point, or a limit cycle, or a trajectory which lies in the boundary of a hyperbolic sector at a finite or infinity singular point.

We denote by $\mathrm{Sep}(p(X))$ the set formed by all separatrices of $p(X)$. The set $\mathrm{Sep}(p(X))$ is closed, see [47]. Each open connected component of $\mathbb{S}^2 \setminus \mathrm{Sep}(p(X))$ is called a *canonical region* of $p(X)$. A *separatrix configuration* is a union of $\mathrm{Sep}(p(X))$ plus one representative solution chosen from each canonical region. Moreover, $\mathrm{Sep}(p(X))$ and $\mathrm{Sep}(p(Y))$ are *equivalent* if there exists a homeomorphism in $\mathbb{S}^2$ preserving the infinity $\mathbb{S}^1$ carrying orbits of $\mathrm{Sep}(p(X))$ into orbits of $\mathrm{Sep}(p(Y))$, preserving or reversing simultaneously the sense of all orbits.

The next result is due to Neumann [47] and characterizes the topologically equivalence between two Poincaré compactified vector fields.

**Theorem B.1.** *Let $X$ and $Y$ be two polynomial vector fields in $\mathbb{R}^2$. If $p(X)$ and $p(Y)$ have finitely many separatrices, then $p(X)$ and $p(Y)$ are topologically equivalent if and only if their separatrix configurations are equivalent.*

Theorem B.1 implies that, to obtain the global phase portrait of a polynomial vector field $p(X)$ with finitely many separatrices, we need to determine the separatrices of $p(X)$ and one orbit in each canonical region.

Using the arguments of the proof of Theorem B.1 the next result follows.

**Theorem B.2.** *Let $X$ and $Y$ be two polynomial vector fields in $\mathbb{R}^2$. If $p(X)$ and $p(Y)$ have the infinity filled of singular points and finitely many separatrices in $\mathbb{R}^2$, then $p(X)$ and $p(Y)$ are topologically equivalent if and only if their separatrix configurations are equivalent.*

According to Theorem B.2, in order to have the global phase portrait of a polynomial vector field $X$ with the infinity filled of singular points and finitely many separatrices in $\mathbb{R}^2$, we need to determine the separatrices of $p(X)$ and one orbit in each canonical region.

# Appendix C

# Expressions of $\mathbf{y_i}(\theta,\rho)$, for $\mathbf{i}=\mathbf{1},\ldots,\mathbf{7}$

We present the expressions of $y_i(\theta,\rho)$, for $i=1,\ldots,7$.

$$
\begin{aligned}
y_1^{\pm}(\theta,\rho) &= \int_0^{\theta} F_1^{\pm}(\phi,\rho)d\phi,\\
y_2^{\pm}(\theta,\rho) &= \int_0^{\theta} \Big(2F_2^{\pm}(\phi,\rho)+2\partial F_1^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)\Big)d\phi,\\
y_3^{\pm}(\theta,\rho) &= \int_0^{\theta} \Big(6F_3^{\pm}(\phi,\rho)+6\partial F_2^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)\\
&\quad+3\partial^2F_1^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)^2+3\partial F_1^{\pm}(\phi,\rho)\,y_2^{\pm}(\phi,\rho)\Big)d\phi,\\
y_4^{\pm}(\theta,\rho) &= \int_0^{\theta} \Big(24F_4^{\pm}(\phi,\rho)+24\partial F_3^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)\\
&\quad+12\partial^2F_2^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)^2+12\partial F_2^{\pm}(\phi,\rho)y_2^{\pm}(\phi,\rho)\\
&\quad+12\partial^2F_1^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)y_2^{\pm}(\phi,\rho)\\
&\quad+4\partial^3F_1^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)^3+4\partial F_1^{\pm}(\phi,\rho)y_3^{\pm}(\phi,\rho)\Big)d\phi,\\
y_5^{\pm}(\theta,\rho) &= \int_0^{\theta} \Big(120F_5^{\pm}(\phi,\rho)+120\partial F_4^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)\\
&\quad+60\partial^2F_3^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)^2+60\partial F_3^{\pm}(\phi,\rho)y_2^{\pm}(\phi,\rho)\\
&\quad+60\partial^2F_2^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)y_2^{\pm}(\phi,\rho)+20\partial^3F_2^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)^3\\
&\quad+20\partial F_2^{\pm}(\phi,\rho)y_3^{\pm}(\phi,\rho)+20\partial^2F_1^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)y_3^{\pm}(\phi,\rho)\\
&\quad+15\partial^2F_1^{\pm}(\phi,\rho)y_2^{\pm}(\phi,\rho)^2+30\partial^3F_1^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)^2y_2^{\pm}(\phi,\rho)\\
&\quad+5\partial^4F_1^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)^4+5\partial F_1^{\pm}(\phi,\rho)y_4^{\pm}(\phi,\rho)\Big)d\phi,\\
y_6^{\pm}(\theta,\rho) &= \int_0^{\theta} \Big(720F_6^{\pm}(\phi,\rho)+720\partial F_5^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)\\
&\quad+360\partial^2F_4^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)^2+360\partial F_4^{\pm}(\phi,\rho)y_2^{\pm}(\phi,\rho)\\
&\quad+120\partial^3F_3^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)^3+360\partial^2F_3^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)y_2^{\pm}(\phi,\rho)\\
&\quad+120\partial F_3^{\pm}(\phi,\rho)y_3^{\pm}(\phi,\rho)+30\partial^4F_2^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)^4\\
&\quad+180\partial^3F_2^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)^2y_2^{\pm}(\phi,\rho)+120\partial^2F_2^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)y_3^{\pm}(\phi,\rho)
\end{aligned}
$$

$$
\begin{aligned}
&+ 90\partial^2 F_2^{\pm}(\phi,\rho) y_2^{\pm}(\phi,\rho)^2 + 30\partial F_2^{\pm}(\phi,\rho) y_4^{\pm}(\phi,\rho) \\
&+ 60\partial^4 F_1^{\pm}(\phi,\rho) y_1^{\pm}(\phi,\rho)^3 y_2^{\pm}(\phi,\rho) + 60\partial^3 F_1^{\pm}(\phi,\rho) y_1^{\pm}(\phi,\rho)^2 y_3^{\pm}(\phi,\rho) \\
&+ 90\partial^3 F_1^{\pm}(\phi,\rho) y_1^{\pm}(\phi,\rho) y_2^{\pm}(\phi,\rho)^2 + 30\partial^2 F_1^{\pm}(\phi,\rho) y_1^{\pm}(\phi,\rho) y_4^{\pm}(\phi,\rho) \\
&+ 60\partial^2 F_1^{\pm}(\phi,\rho) y_2^{\pm}(\phi,\rho) y_3^{\pm}(\phi,\rho) + 6\partial^5 F_1^{\pm}(\phi,\rho) y_1^{\pm}(\phi,\rho)^5 \\
&+ 6\partial F_1^{\pm}(\phi,\rho) y_5^{\pm}(\phi,\rho)\Big) d\phi,
\end{aligned}
$$

$$
\begin{aligned}
y_7^{\pm}(t,\rho) = \int_0^t &\big(5040 F_7^{\pm}(\phi,\rho) + 5040\partial F_6^{\pm}(\phi,\rho) y_1^{\pm}(\phi,\rho) \\
&+ 2520\partial^2 F_5^{\pm}(\phi,\rho) y_1^{\pm}(\phi,\rho)^2 + 2520\partial F_5^{\pm}(\phi,\rho) y_2^{\pm}(\phi,\rho) \\
&+ 2520\partial^2 F_4^{\pm}(\phi,\rho) y_1^{\pm}(\phi,\rho) y_2^{\pm}(\phi,\rho) + 840\partial^3 F_4^{\pm}(\phi,\rho) y_1^{\pm}(\phi,\rho)^3 \\
&+ 840\partial F_4^{\pm}(\phi,\rho) y_3^{\pm}(\phi,\rho) + 840\partial^2 F_3^{\pm}(\phi,\rho) y_1^{\pm}(\phi,\rho) y_3^{\pm}(\phi,\rho) \\
&+ 630\partial^2 F_3^{\pm}(\phi,\rho) y_2^{\pm}(\phi,\rho)^2 + 1260\partial^3 F_3^{\pm}(\phi,\rho) y_1^{\pm}(\phi,\rho)^2 y_2^{\pm}(\phi,\rho) \\
&+210\partial^4 F_3^{\pm}(\phi,\rho) y_1^{\pm}(\phi,\rho)^4 + 210\partial F_3^{\pm}(\phi,\rho) y_4^{\pm}(\phi,\rho) \\
&+ 210\partial^2 F_2^{\pm}(\phi,\rho) y_1^{\pm}(\phi,\rho) y_4^{\pm}(\phi,\rho) + 420\partial^3 F_2^{\pm}(\phi,\rho) y_1^{\pm}(\phi,\rho)^2 y_3^{\pm}(\phi,\rho) \\
&+ 420\partial^4 F_2^{\pm}(\phi,\rho) y_1^{\pm}(\phi,\rho)^3 y_2^{\pm}(\phi,\rho) + 630\partial^3 F_2^{\pm}(\phi,\rho) y_2^{\pm}(\phi,\rho)^2 y_1^{\pm}(\phi,\rho) \\
&+ 42\partial^5 F_2^{\pm}(\phi,\rho) y_1^{\pm}(\phi,\rho)^5 + 420\partial^2 F_2^{\pm}(\phi,\rho) y_2^{\pm}(\phi,\rho) y_3^{\pm}(\phi,\rho) \\
&+ 42\partial F_2^{\pm}(\phi,\rho) y_5^{\pm}(\phi,\rho) + 630\partial^3 F_2^{\pm}(\phi,\rho) y_2^{\pm}(\phi,\rho)^2 y_1^{\pm}(\phi,\rho) \\
&+ 7\partial^6 F_1^{\pm}(\phi,\rho) y_1^{\pm}(\phi,\rho)^6 + 105\partial^5 F_1^{\pm}(\phi,\rho) y_1^{\pm}(\phi,\rho)^4 y_2^{\pm}(\phi,\rho) \\
&+ 140\partial^4 F_1^{\pm}(\phi,\rho) y_1^{\pm}(\phi,\rho)^3 y_3^{\pm}(\phi,\rho) + 630\partial^4 F_1^{\pm}(\phi,\rho) y_1^{\pm}(\phi,\rho)^2 y_2^{\pm}(\phi,\rho)^2 \\
&+ 105\partial^3 F_1^{\pm}(\phi,\rho) y_1^{\pm}(\phi,\rho)^2 y_4^{\pm}(\phi,\rho) + 42\partial^2 F_1^{\pm}(\phi,\rho) y_1^{\pm}(\phi,\rho) y_5^{\pm}(\phi,\rho) \\
&+ 420\partial^3 F_1^{\pm}(\phi,\rho) y_1^{\pm}(\phi,\rho) y_2^{\pm}(\phi,\rho) y_3^{\pm}(\phi,\rho) \\
&+ 105\partial^3 F_1^{\pm}(\phi,\rho) y_2^{\pm}(\phi,\rho)^3 + 105\partial^2 F_1^{\pm}(\phi,\rho) y_2^{\pm}(\phi,\rho) y_4^{\pm}(\phi,\rho) \\
&+70\partial^2 F_1^{\pm}(\phi,\rho) y_3^{\pm}(\phi,\rho)^2 + 7\partial F_1^{\pm}(\phi,\rho) y_6^{\pm}(\phi,\rho)\big)\, d\phi.
\end{aligned}
$$

# Appendix D

# Averaging function of order 6 for the Collins second form, A $\neq$ 0. Continuous Case

We present the averaging function of order 6 for the Collins second form, in the case $A \neq 0$, discussed in the proof of Theorem 5.5.

$$f_6(r) = r(D_5 r^4 + D_3 r^2 + D_1).$$

where

$$\begin{aligned}
D_5 = &-\pi(237A^3\alpha_1^1\beta_1^1 + 45A^3\alpha_1^1\alpha_2^1 - 192A^3\alpha_1^2 + 2406A^2\alpha_1^1\beta_1^1 \\
&+ 1446A^2\alpha_1^1\alpha_2^1 - 476A^2\alpha_1^1\alpha_3^1 + 524A^2\alpha_1^1\beta_4^1 + 60A^2\alpha_1^1\alpha_5^1 \\
&+ 694A^2\beta_1^1\beta_5^1 + 288A^2\alpha_1^1\beta_6^1 - 192A^2\alpha_1^1\alpha_7^1 + 96A^2\alpha_1^1\alpha_9^1 \\
&+ 272A^2\alpha_2^1\alpha_4^1 + 454A^2\alpha_2^1\beta_5^1 + 192A^2\alpha_2^1\alpha_6^1 - 400A^2\alpha_3^1\alpha_4^1 \\
&- 256A^2\alpha_3^1\beta_5^1 + 656A^2\beta_1^1\alpha_4^1 + 16A^2\alpha_4^1\beta_4^1 - 176A^2\alpha_4^1\alpha_5^1 \\
&+ 16A^2\beta_4^1\beta_5^1 - 32A^2\alpha_5^1\beta_5^1 + 192A^2\beta_1^1\alpha_6^1 - 720A^2\alpha_1^2 + 272A^2\beta_3^2 \\
&- 112A^2\alpha_4^2 + 304A^2\beta_5^2 - 192A^2\alpha_6^2 - 96A^2\alpha_8^2 - 96A^2\beta_9^2 \\
&+ 5229A\alpha_1^1\beta_1^1 + 4509A\alpha_1^1\alpha_2^1 - 540A\alpha_1^1\alpha_3^1 + 2124A\alpha_1^1\beta_4^1 \\
&+ 1980A\alpha_1^1\alpha_5^1 + 2322A\beta_1^1\beta_5^1 + 2376A\alpha_1^1\beta_6^1 - 1080A\alpha_1^1\alpha_7^1 \\
&- 72A\alpha_1^1\beta_8^1 + 792A\alpha_1^1\alpha_9^1 + 1008A\alpha_2^1\alpha_4^1 + 1746A\alpha_2^1\beta_5^1 \\
&+ 864A\alpha_2^1\alpha_6^1 - 360A\alpha_3^1\beta_5^1 + 432A\alpha_3^1\alpha_6^1 + 176A\alpha_3^1\alpha_8^1 \\
&+ 1584A\beta_1^1\alpha_4^1 + 288A\alpha_4^1\beta_4^1 + 864A\alpha_4^1\alpha_5^1 + 216A\beta_4^1\beta_5^1 \\
&+ 720A\alpha_4^1\beta_6^1 - 48A\alpha_4^1\alpha_7^1 + 48A\alpha_4^1\beta_8^1 + 432A\alpha_4^1\alpha_9^1 \\
&+ 744A\alpha_5^1\beta_5^1 + 720A\alpha_5^1\alpha_6^1 + 648A\beta_5^1\beta_6^1 + 208A\alpha_5^1\alpha_8^1 \\
&+ 24A\beta_5^1\beta_8^1 + 576A\beta_1^1\alpha_6^1 + 144A\beta_4^1\alpha_6^1 + 288A\alpha_6^1\beta_6^1 \\
&+ 288A\alpha_6^1\alpha_9^1 - 24A\beta_5^1\alpha_7^1 - 96A\beta_1^1\alpha_8^1 + 16A\beta_4^1\alpha_8^1 + 96A\beta_6^1\alpha_8^1 \\
&+ 96A\alpha_8^1\alpha_9^1 + 216A\beta_5^1\alpha_9^1 - 432A\alpha_1^2 + 1008A\beta_3^2 - 432A\alpha_4^2 \\
&+ 1008A\beta_5^2 - 864A\alpha_6^2 - 480A\alpha_8^2 - 576A\beta_9^2 + 3456\alpha_1^1\beta_1^1 \\
&+ 1728\alpha_1^1\alpha_2^1 - 2880\alpha_1^1\alpha_3^1 + 1152\alpha_1^1\beta_4^1 + 864\alpha_1^1\alpha_5^1
\end{aligned}$$

$$
\begin{aligned}
&+ 2016\beta_1^1\beta_5^1 + 3024\alpha_1^1\beta_6^1 - 2016\alpha_1^1\alpha_7^1 - 720\alpha_1^1\beta_8^1 + 576\alpha_2^1\alpha_4^1\\
&+ 1152\alpha_2^1\beta_5^1 + 864\alpha_2^1\alpha_6^1 - 1152\alpha_3^1\alpha_4^1 - 1152\alpha_3^1\beta_5^1\\
&- 288\alpha_3^1\alpha_6^1 + 1440\beta_1^1\alpha_4^1 + 288\alpha_4^1\beta_4^1 + 288\alpha_4^1\alpha_5^1 + 288\beta_4^1\beta_5^1\\
&+ 864\alpha_4^1\beta_6^1 - 576\alpha_4^1\alpha_7^1 - 288\alpha_4^1\beta_8^1 + 576\alpha_5^1\beta_5^1\\
&+ 864\alpha_5^1\alpha_6^1 + 1296\beta_5^1\beta_6^1 + 192\alpha_5^1\alpha_8^1 - 144\beta_5^1\beta_8^1 + 864\beta_1^1\alpha_6^1\\
&+ 288\beta_4^1\alpha_6^1 + 432\alpha_6^1\beta_6^1 - 144\alpha_6^1\alpha_7^1 - 144\alpha_6^1\beta_8^1\\
&+ 432\alpha_6^1\alpha_9^1 - 288\beta_5^1\alpha_7^1 - 48\alpha_7^1\alpha_8^1 + 144\beta_6^1\alpha_8^1 - 48\alpha_8^1\beta_8^1\\
&+ 144\alpha_8^1\alpha_9^1 + 576\beta_3^2 - 288\alpha_4^2 + 288\beta_5^2 - 864\alpha_6^2 - 576\alpha_8^2\\
&- 864\beta_9^2)/[384(A+3)],\\
D_3 = &- \pi(108A^2\alpha_2^1(\alpha_1^1)^3 - 828A\alpha_2^1(\alpha_1^1)^3 - 3456\alpha_2^1(\alpha_1^1)^3\\
&- 1152A\alpha_3^1(\alpha_1^1)^3 - 3456\alpha_3^1(\alpha_1^1)^3 - 384A\alpha_5^1(\alpha_1^1)^3\\
&- 1152\alpha_5^1(\alpha_1^1)^{+}36A^2\beta_1^1(\alpha_1^1)^3 + 2412A\beta_1^1(\alpha_1^1)^3 + 6912\beta_1^1(\alpha_1^1)^3\\
&- 1536A\alpha_2^1\alpha_4^1(\alpha_1^1)^2 - 4608\alpha_2^1\alpha_4^1(\alpha_1^1)^2 - 640A\alpha_3^1\alpha_4^1(\alpha_1^1)^2\\
&- 1920\alpha_3^1\alpha_4^1(\alpha_1^1)^2 - 128A\alpha_4^1\alpha_5^1(\alpha_1^1)^2 - 384\alpha_4^1\alpha_5^1(\alpha_1^1)^2\\
&+ 72A^2\alpha_1^2(\alpha_1^1)^2 - 2088A\alpha_1^2(\alpha_1^1)^2 - 6912\alpha_1^2(\alpha_1^1)^2 - 768A\alpha_4^2(\alpha_1^1)^2\\
&- 2304\alpha_4^2(\alpha_1^1)^2 + 1536A\alpha_4^1\beta_1^1(\alpha_1^1)^2 + 4608\alpha_4^1\beta_1^1(\alpha_1^1)^2\\
&- 128A\alpha_4^1\beta_4^1(\alpha_1^1)^2 - 384\alpha_4^1\beta_4^1(\alpha_1^1)^2 - 768A\alpha_2^1\beta_5^1(\alpha_1^1)^2\\
&- 2304\alpha_2^1\beta_5^1(\alpha_1^1)^2 - 512A\alpha_3^1\beta_5^1(\alpha_1^1)^2 - 1536\alpha_3^1\beta_5^1(\alpha_1^1)^2\\
&- 256A\alpha_5^1\beta_5^1(\alpha_1^1)^2 - 768\alpha_5^1\beta_5^1(\alpha_1^1)^2 + 1920A\beta_1^1\beta_5^1(\alpha_1^1)^2\\
&+ 5760\beta_1^1\beta_5^1(\alpha_1^1)^2 + 128A\beta_4^1\beta_5^1(\alpha_1^1)^2 + 384\beta_4^1\beta_5^1(\alpha_1^1)^2\\
&+ 384A\beta_3^2(\alpha_1^1)^2 + 1152\beta_3^2(\alpha_1^1)^2 - 384A\beta_5^2(\alpha_1^1)^2\\
&- 1152\beta_5^1(\alpha_1^1)^2 + 319A^2(\alpha_2^1)^3\alpha_1^1 + 3069A(\alpha_2^1)^3\alpha_1^1\\
&+ 2304(\alpha_2^1)^3\alpha_1^1 - 27A^2(\beta_1^1)^3\alpha_1^1 + 1071A(\beta_1^1)^3\alpha_1^1\\
&+ 3456(\beta_1^1)^3\alpha_1^1 - 256A\alpha_2^1(\alpha_3^1)^2\alpha_1^1 - 768\alpha_2^1(\alpha_3^1)^2\alpha_1^1\\
&- 384A\alpha_2^1(\alpha_4^1)^2\alpha_1^1 - 1152\alpha_2^1(\alpha_4^1)^2\alpha_1^1 + 9A^2\alpha_2^1(\beta_1^1)^2\alpha_1^1\\
&+ 27A\alpha_2^1(\beta_1^1)^2\alpha_1^1 - 1152A\alpha_3^1(\beta_1^1)^2\alpha_1^1 - 3456\alpha_3^1(\beta_1^1)^2\alpha_1^1\\
&+ 128A\alpha_2^1(\beta_4^1)^2\alpha_1^1 + 384\alpha_2^1(\beta_4^1)^2\alpha_1^1 - 128A\beta_1^1(\beta_4^1)^2\alpha_1^1\\
&- 384\beta_1^1(\beta_4^1)^2\alpha_1^1 + 512A\beta_1^1(\beta_5^1)^2\alpha_1^1 + 1536\beta_1^1(\beta_5^1)^2\alpha_1^1\\
&- 384A(\alpha_2^1)^2\alpha_3^1\alpha_1^1 - 1152(\alpha_2^1)^2\alpha_3^1\alpha_1^1 - 1152A(\alpha_2^1)^2\alpha_5^1\alpha_1^1\\
&- 3456(\alpha_2^1)^2\alpha_5^1\alpha_1^1 - 512A\alpha_2^1\alpha_3^1\alpha_5^1\alpha_1^1 - 1536\alpha_2^1\alpha_3^1\alpha_5^1\alpha_1^1\\
&- 256A(\alpha_2^1)^2\alpha_7^1\alpha_1^1 - 768(\alpha_2^1)^2\alpha_7^1\alpha_1^1 - 1920A\alpha_4^1\alpha_1^2\alpha_1^1\\
&- 5760\alpha_4^1\alpha_1^2\alpha_1^1 + 572A^2\alpha_2^1\alpha_2^2\alpha_1^1 + 5940A\alpha_2^1\alpha_2^2\alpha_1^1\\
&+ 4608\alpha_2^1\alpha_2^2\alpha_1^1 - 384A\alpha_3^1\alpha_2^2\alpha_1^1 - 1152\alpha_3^1\alpha_2^2\alpha_1^1\\
&- 768A\alpha_5^1\alpha_2^2\alpha_1^1 - 2304\alpha_5^1\alpha_2^2\alpha_1^1 - 256A\alpha_7^1\alpha_2^2\alpha_1^1\\
&- 768\alpha_7^1\alpha_2^2\alpha_1^1 - 384A\alpha_2^1\alpha_3^2\alpha_1^1 - 1152\alpha_2^1\alpha_3^2\alpha_1^1\\
&- 512A\alpha_3^1\alpha_3^2\alpha_1^1 - 1536\alpha_3^1\alpha_3^2\alpha_1^1 - 256A\alpha_5^1\alpha_3^2\alpha_1^1\\
&- 768\alpha_5^1\alpha_3^2\alpha_1^1 - 384A\alpha_4^1\alpha_4^2\alpha_1^1 - 1152\alpha_4^1\alpha_4^2\alpha_1^1\\
&- 768A\alpha_2^1\alpha_5^2\alpha_1^1 - 2304\alpha_2^1\alpha_5^2\alpha_1^1 - 256A\alpha_3^1\alpha_5^2\alpha_1^1
\end{aligned}
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\begin{aligned}
&- 768\alpha_3^1\alpha_5^2\alpha_1^1 - 256A\alpha_2^1\alpha_7^2\alpha_1^1 - 768\alpha_2^1\alpha_7^2\alpha_1^1 + 256A^2\alpha_2^3\alpha_1^1\\
&+ 2880A\alpha_2^3\alpha_1^1 + 2304\alpha_2^3\alpha_1^1 - 1152A\alpha_3^3\alpha_1^1 - 3456\alpha_3^3\alpha_1^1\\
&- 384A\alpha_5^3\alpha_1^1 - 1152\alpha_5^3\alpha_1^1 - 256A\alpha_7^3\alpha_1^1 - 768\alpha_7^3\alpha_1^1\\
&+ 867A^2(\alpha_2^1)^2\beta_1^1\alpha_1^1 + 4713A(\alpha_2^1)^2\beta_1^1\alpha_1^1 + 2304(\alpha_2^1)^2\beta_1^1\alpha_1^1\\
&+ 256A(\alpha_3^1)^2\beta_1^1\alpha_1^1 + 768(\alpha_3^1)^2\beta_1^1\alpha_1^1 + 128A(\alpha_4^1)^2\beta_1^1\alpha_1^1\\
&+ 384(\alpha_4^1)^2\beta_1^1\alpha_1^1 + 384A\alpha_2^1\alpha_3^1\beta_1^1\alpha_1^1 + 1152\alpha_2^1\alpha_3^1\beta_1^1\alpha_1^1\\
&+ 828A^2\alpha_2^2\beta_1^1\alpha_1^1 + 4596A\alpha_2^2\beta_1^1\alpha_1^1 + 2304\alpha_2^2\beta_1^1\alpha_1^1\\
&+ 1152A\alpha_3^2\beta_1^1\alpha_1^1 + 3456\alpha_3^2\beta_1^1\alpha_1^1 + 384A(\alpha_2^1)^2\beta_4^1\alpha_1^1\\
&+ 1152(\alpha_2^1)^2\beta_4^1\alpha_1^1 + 128A\alpha_2^1\alpha_3^1\beta_4^1\alpha_1^1 + 384\alpha_2^1\alpha_3^1\beta_4^1\alpha_1^1\\
&- 256A\alpha_2^1\alpha_5^1\beta_4^1\alpha_1^1 - 768\alpha_2^1\alpha_5^1\beta_4^1\alpha_1^1 + 384A\alpha_2^2\beta_4^1\alpha_1^1\\
&+ 1152\alpha_2^2\beta_4^1\alpha_1^1 + 128A\alpha_3^2\beta_4^1\alpha_1^1 + 384\alpha_3^2\beta_4^1\alpha_1^1 - 128A\alpha_5^2\beta_4^1\alpha_1^1\\
&- 384\alpha_5^2\beta_4^1\alpha_1^1 - 384A\alpha_2^1\beta_1^1\beta_4^1\alpha_1^1 - 1152\alpha_2^1\beta_1^1\beta_4^1\alpha_1^1\\
&- 128A\alpha_3^1\beta_1^1\beta_4^1\alpha_1^1 - 384\alpha_3^1\beta_1^1\beta_4^1\alpha_1^1 - 768A\alpha_2^1\alpha_4^1\beta_5^1\alpha_1^1\\
&- 2304\alpha_2^1\alpha_4^1\beta_5^1\alpha_1^1 - 1536A\alpha_1^2\beta_5^1\alpha_1^1 - 4608\alpha_1^2\beta_5^1\alpha_1^1\\
&- 384A\alpha_4^2\beta_5^1\alpha_1^1 - 1152\alpha_4^2\beta_5^1\alpha_1^1 + 512A\alpha_4^1\beta_1^1\beta_5^1\alpha_1^1\\
&+ 1536\alpha_4^1\beta_1^1\beta_5^1\alpha_1^1 - 256A(\alpha_2^1)^2\beta_8^1\alpha_1^1 - 768(\alpha_2^1)^2\beta_8^1\alpha_1^1\\
&- 256A\alpha_2^2\beta_8^1\alpha_1^1 - 768\alpha_2^2\beta_8^1\alpha_1^1 + 828A^2\alpha_2^1\beta_1^2\alpha_1^1\\
&+ 4596A\alpha_2^1\beta_1^2\alpha_1^1 + 2304\alpha_2^1\beta_1^2\alpha_1^1 + 1152A\alpha_3^1\beta_1^2\alpha_1^1\\
&+ 3456\alpha_3^1\beta_1^2\alpha_1^1 + 60A^2\beta_1^1\beta_1^2\alpha_1^1 - 2124A\beta_1^1\beta_1^2\alpha_1^1 - 6912\beta_1^1\beta_1^2\alpha_1^1\\
&+ 128A\alpha_4^1\beta_3^2\alpha_1^1 + 384\alpha_4^1\beta_3^2\alpha_1^1 + 256A\beta_5^1\beta_3^2\alpha_1^1\\
&+ 768\beta_5^1\beta_3^2\alpha_1^1 + 384A\alpha_2^1\beta_4^2\alpha_1^1 + 1152\alpha_2^1\beta_4^2\alpha_1^1\\
&+ 128A\alpha_3^1\beta_4^2\alpha_1^1 + 384\alpha_3^1\beta_4^2\alpha_1^1 - 128A\alpha_5^1\beta_4^2\alpha_1^1 - 384\alpha_5^1\beta_4^2\alpha_1^1\\
&+ 256A\beta_4^1\beta_4^2\alpha_1^1 + 768\beta_4^1\beta_4^2\alpha_1^1 - 384A\alpha_4^1\beta_5^2\alpha_1^1\\
&- 1152\alpha_4^1\beta_5^2\alpha_1^1 - 256A\alpha_2^1\beta_8^2\alpha_1^1 - 768\alpha_2^1\beta_8^2\alpha_1^1\\
&+ 768A^2\beta_1^3\alpha_1^1 + 5568A\beta_1^3\alpha_1^1 + 5760\beta_1^3\alpha_1^1\\
&- 256A\beta_8^3\alpha_1^1 - 768\beta_8^3\alpha_1^1 + 384A\alpha_4^1(\beta_1^1)^3 + 1152\alpha_4^1(\beta_1^1)^3\\
&- 256A\alpha_3^1\alpha_4^1(\beta_1^1)^2 - 768\alpha_3^1\alpha_4^1(\beta_1^1)^2 + 30A^2\alpha_1^2(\beta_1^1)^2 + 90A\alpha_1^2(\beta_1^1)^2\\
&+ 128A\alpha_1^2(\beta_4^1)^2 + 384\alpha_1^2(\beta_4^1)^2 - 256A\alpha_1^2(\beta_5^1)^2 - 768\alpha_1^2(\beta_5^1)^2\\
&+ 1408A(\alpha_2^1)^3\alpha_4^1 + 768(\alpha_2^1)^3\alpha_4^1 - 128A(\alpha_2^1)^2\alpha_3^1\alpha_4^1\\
&- 384(\alpha_2^1)^2\alpha_3^1\alpha_4^1 - 384A(\alpha_2^1)^2\alpha_4^1\alpha_5^1 - 1152(\alpha_2^1)^2\alpha_4^1\alpha_5^1\\
&+ 768A(\alpha_2^1)^3\alpha_6^1 + 1152(\alpha_2^1)^3\alpha_6^1 + 256A(\alpha_2^1)^3\alpha_8^1 + 384(\alpha_2^1)^3\alpha_8^1\\
&- 482A^2(\alpha_2^1)^2\alpha_1^2 - 1446A(\alpha_2^1)^2\alpha_1^2 - 256A(\alpha_3^1)^2\alpha_1^2\\
&- 768(\alpha_3^1)^2\alpha_1^2 - 256A(\alpha_4^1)^2\alpha_1^2 - 768(\alpha_4^1)^2\alpha_1^2\\
&- 384A\alpha_2^1\alpha_3^1\alpha_1^2 - 1152\alpha_2^1\alpha_3^1\alpha_1^2 - 768A\alpha_2^1\alpha_5^1\alpha_1^2\\
&- 2304\alpha_2^1\alpha_5^1\alpha_1^2 - 256A\alpha_3^1\alpha_5^1\alpha_1^2 - 768\alpha_3^1\alpha_5^1\alpha_1^2\\
&- 256A\alpha_2^1\alpha_7^1\alpha_1^2 - 768\alpha_2^1\alpha_7^1\alpha_1^2 + 2816A\alpha_2^1\alpha_4^1\alpha_2^2\\
&+ 1536\alpha_2^1\alpha_4^1\alpha_2^2 - 128A\alpha_3^1\alpha_4^1\alpha_2^2 - 384\alpha_3^1\alpha_4^1\alpha_2^2\\
&- 256A\alpha_4^1\alpha_5^1\alpha_2^2 - 768\alpha_4^1\alpha_5^1\alpha_2^2 + 1536A\alpha_2^1\alpha_6^1\alpha_2^2
\end{aligned}
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\begin{aligned}
&+ 2304\alpha_2^1\alpha_6^1\alpha_2^2 + 512A\alpha_2^1\alpha_8^1\alpha_2^2 + 768\alpha_2^1\alpha_8^1\alpha_2^2 \\
&- 512A^2\alpha_1^2\alpha_2^2 - 1536A\alpha_1^2\alpha_2^2 - 128A\alpha_2^1\alpha_4^1\alpha_3^2 \\
&- 384\alpha_2^1\alpha_4^1\alpha_3^2 - 384A\alpha_1^2\alpha_3^2 - 1152\alpha_1^2\alpha_3^2 - 128A(\alpha_2^1)^2\alpha_4^2 \\
&- 384(\alpha_2^1)^2\alpha_4^2 - 128A\alpha_2^1\alpha_3^1\alpha_4^2 - 384\alpha_2^1\alpha_3^1\alpha_4^2 - 256A\alpha_2^1\alpha_5^1\alpha_4^2 \\
&- 768\alpha_2^1\alpha_5^1\alpha_4^2 - 128A\alpha_2^2\alpha_4^2 - 384\alpha_2^2\alpha_4^2 - 128A\alpha_3^2\alpha_4^2 \\
&- 384\alpha_3^2\alpha_4^2 - 256A\alpha_2^1\alpha_4^1\alpha_5^2 - 768\alpha_2^1\alpha_4^1\alpha_5^2 - 384A\alpha_1^2\alpha_5^2 \\
&- 1152\alpha_1^2\alpha_5^2 - 128A\alpha_4^2\alpha_5^2 - 384\alpha_4^2\alpha_5^2 - 256A\alpha_1^2\alpha_7^2 \\
&- 768\alpha_1^2\alpha_7^2 - 128A(\alpha_2^1)^2\alpha_8^2 - 384(\alpha_2^1)^2\alpha_8^2 - 128A\alpha_2^2\alpha_8^2 \\
&- 512A^2\alpha_2^1\alpha_1^3 - 1536A\alpha_2^1\alpha_1^3 - 384A\alpha_3^1\alpha_1^3 - 1152\alpha_3^1\alpha_1^3 \\
&- 384A\alpha_5^1\alpha_1^3 - 1152\alpha_5^1\alpha_1^3 - 256A\alpha_7^1\alpha_1^3 - 768\alpha_7^1\alpha_1^3 \\
&+ 1408A\alpha_4^1\alpha_2^3 + 768\alpha_4^1\alpha_2^3 + 768A\alpha_6^1\alpha_2^3 + 1152\alpha_6^1\alpha_2^3 \\
&+ 256A\alpha_8^1\alpha_2^3 + 384\alpha_8^1\alpha_2^3 - 384A\alpha_4^1\alpha_3^3 - 1152\alpha_4^1\alpha_3^3 \\
&- 128A\alpha_2^1\alpha_4^3 - 384\alpha_2^1\alpha_4^3 - 128A\alpha_3^1\alpha_4^3 - 384\alpha_3^1\alpha_4^3 \\
&- 128A\alpha_5^1\alpha_4^3 - 384\alpha_5^1\alpha_4^3 - 128A\alpha_4^1\alpha_5^3 - 384\alpha_4^1\alpha_5^3 \\
&- 128A\alpha_2^1\alpha_8^3 - 384\alpha_2^1\alpha_8^3 - 512A^2\alpha_1^4 - 1536A\alpha_1^4 \\
&- 128A\alpha_4^4 - 384\alpha_4^4 - 384A\alpha_6^4 - 1152\alpha_6^4 - 128A\alpha_8^4 \\
&- 384\alpha_8^4 + 1536A(\alpha_2^1)^2\alpha_4^1\beta_1^1 + 1152(\alpha_2^1)^2\alpha_4^1\beta_1^1 - 384\alpha_2^2\alpha_8^2 \\
&- 256A\alpha_2^1\alpha_4^1\alpha_5^1\beta_1^1 - 768\alpha_2^1\alpha_4^1\alpha_5^1\beta_1^1 + 768A(\alpha_2^1)^2\alpha_6^1\beta_1^1 \\
&+ 1152(\alpha_2^1)^2\alpha_6^1\beta_1^1 + 256A(\alpha_2^1)^2\alpha_8^1\beta_1^1 + 384(\alpha_2^1)^2\alpha_8^1\beta_1^1 \\
&+ 60A^2\alpha_2^1\alpha_1^2\beta_1^1 + 180A\alpha_2^1\alpha_1^2\beta_1^1 + 384A\alpha_3^1\alpha_1^2\beta_1^1 + 1152\alpha_3^1\alpha_1^2\beta_1^1 \\
&+ 1536A\alpha_4^1\alpha_2^2\beta_1^1 + 1152\alpha_4^1\alpha_2^2\beta_1^1 + 768A\alpha_6^1\alpha_2^2\beta_1^1 \\
&+ 1152\alpha_6^1\alpha_2^2\beta_1^1 + 256A\alpha_8^1\alpha_2^2\beta_1^1 + 384\alpha_8^1\alpha_2^2\beta_1^1 + 256A\alpha_4^1\alpha_3^2\beta_1^1 \\
&+ 768\alpha_4^1\alpha_3^2\beta_1^1 - 128A\alpha_5^1\alpha_4^2\beta_1^1 - 384\alpha_5^1\alpha_4^2\beta_1^1 - 128A\alpha_4^1\alpha_5^2\beta_1^1 \\
&- 384\alpha_4^1\alpha_5^2\beta_1^1 - 128A\alpha_2^1\alpha_8^2\beta_1^1 - 384\alpha_2^1\alpha_8^2\beta_1^1 - 128A\alpha_8^3\beta_1^1 \\
&- 384\alpha_8^3\beta_1^1 - 128A\alpha_4^1(\beta_1^1)^2\beta_4^1 - 384\alpha_4^1(\beta_1^1)^2\beta_4^1 + 384A\alpha_2^1\alpha_1^2\beta_4^1 \\
&+ 1152\alpha_2^1\alpha_1^2\beta_4^1 + 128A\alpha_3^1\alpha_1^2\beta_4^1 + 384\alpha_3^1\alpha_1^2\beta_4^1 - 128A\alpha_5^1\alpha_1^2\beta_4^1 \\
&- 384\alpha_5^1\alpha_1^2\beta_4^1 + 384A\alpha_1^3\beta_4^1 + 1152\alpha_1^3\beta_4^1 - 384A\alpha_1^2\beta_1^1\beta_4^1 \\
&- 1152\alpha_1^2\beta_1^1\beta_4^1 + 1088A(\alpha_2^1)^3\beta_5^1 + 1536(\alpha_2^1)^3\beta_5^1 \\
&+ 2304(\beta_1^1)^3\beta_5^1 - 512A\alpha_3^1(\beta_1^1)^2\beta_5^1 - 1536\alpha_3^1(\beta_1^1)^2\beta_5^1 \\
&- 2304(\alpha_2^1)^2\alpha_5^1\beta_5^1 - 640A\alpha_4^1\alpha_1^2\beta_5^1 - 768A(\alpha_2^1)^2\alpha_5^1\beta_5^1 \\
&- 1920\alpha_4^1\alpha_1^2\beta_5^1 + 2176A\alpha_2^1\alpha_2^2\beta_5^1 + 3072\alpha_2^1\alpha_2^2\beta_5^1 \\
&- 512A\alpha_5^1\alpha_2^2\beta_5^1 - 1536\alpha_5^1\alpha_2^2\beta_5^1 - 512A\alpha_2^1\alpha_5^2\beta_5^1 - 1536\alpha_2^1\alpha_5^2\beta_5^1 \\
&+ 1088A\alpha_2^3\beta_5^1 + 1536\alpha_2^3\beta_5^1 - 512A\alpha_3^3\beta_5^1 \\
&- 1536\alpha_3^3\beta_5^1 - 256A\alpha_5^3\beta_5^1 - 768\alpha_5^3\beta_5^1 + 768A(\beta_1^1)^3\beta_5^1 \\
&+ 960A(\alpha_2^1)^2\beta_1^1\beta_5^1 + 1152(\alpha_2^1)^2\beta_1^1\beta_5^1 - 512A\alpha_2^1\alpha_5^1\beta_1^1\beta_5^1 \\
&- 1536\alpha_2^1\alpha_5^1\beta_1^1\beta_5^1 + 960A\alpha_2^2\beta_1^1\beta_5^1 + 1152\alpha_2^2\beta_1^1\beta_5^1 + 512A\alpha_3^2\beta_1^1\beta_5^1 \\
&+ 1536\alpha_3^2\beta_1^1\beta_5^1 - 256A\alpha_5^2\beta_1^1\beta_5^1 - 768\alpha_5^2\beta_1^1\beta_5^1 \\
&+ 128A(\alpha_2^1)^2\beta_4^1\beta_5^1 + 384(\alpha_2^1)^2\beta_4^1\beta_5^1 - 256A(\beta_1^1)^2\beta_4^1\beta_5^1
\end{aligned}
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\begin{aligned}
&+128A\alpha_2^2\beta_4^1\beta_5^1+384\alpha_2^2\beta_4^1\beta_5^1-256A\alpha_2^1\alpha_1^2\beta_8^1-768\alpha_2^1\alpha_1^2\beta_8^1\\
&-256A\alpha_1^3\beta_8^1-768\alpha_1^3\beta_8^1+1536A\alpha_2^1\alpha_4^1\beta_1^2+1152\alpha_2^1\alpha_4^1\beta_1^2\\
&+768\alpha_3^1\alpha_4^1\beta_1^2-128A\alpha_4^1\alpha_5^1\beta_1^2-384\alpha_4^1\alpha_5^1\beta_1^2+768A\alpha_2^1\alpha_6^1\beta_1^2\\
&-768(\beta_1^1)^2\beta_4^1\beta_5^1+1152\alpha_2^1\alpha_6^1\beta_1^2+256A\alpha_2^1\alpha_8^1\beta_1^2+384\alpha_2^1\alpha_8^1\beta_1^2\\
&-128A\alpha_8^2\beta_1^2-384\alpha_8^2\beta_1^2-768A\alpha_4^1\beta_1^1\beta_1^2+1728A\beta_5^1\beta_1^3\\
&-2304\alpha_4^1\beta_1^1\beta_1^2+128A\alpha_4^1\beta_4^1\beta_1^2+384\alpha_4^1\beta_4^1\beta_1^2+256A\alpha_3^1\alpha_4^1\beta_1^2\\
&+960A\alpha_2^1\beta_5^1\beta_1^2+1152\alpha_2^1\beta_5^1\beta_1^2+512A\alpha_3^1\beta_5^1\beta_1^2\\
&+1536\alpha_3^1\beta_5^1\beta_1^2-256A\alpha_5^1\beta_5^1\beta_1^2-768\alpha_5^1\beta_5^1\beta_1^2\\
&-1536A\beta_1^1\beta_5^1\beta_1^2-4608\beta_1^1\beta_5^1\beta_1^2+256A\beta_4^1\beta_5^1\beta_1^2\\
&+768\beta_4^1\beta_5^1\beta_1^2+384A(\beta_1^1)^2\beta_3^2+1152(\beta_1^1)^2\beta_3^2\\
&+256A\alpha_3^2\beta_3^2+768\alpha_3^2\beta_3^2-256A\alpha_3^1\beta_1^1\beta_3^2-768\alpha_3^1\beta_1^1\beta_3^2\\
&-128A\beta_1^1\beta_4^1\beta_3^2-384\beta_1^1\beta_4^1\beta_3^2-384A\beta_1^2\beta_3^2-1152\beta_1^2\beta_3^2\\
&+384A\alpha_1^2\beta_4^2+1152\alpha_1^2\beta_4^2+128A\alpha_4^1\beta_1^1\beta_4^2+384\alpha_4^1\beta_1^1\beta_4^2\\
&+128A\alpha_2^1\beta_5^1\beta_4^2+384\alpha_2^1\beta_5^1\beta_4^2+256A\beta_1^1\beta_5^1\beta_4^2\\
&+768\beta_1^1\beta_5^1\beta_4^2+128A\beta_3^2\beta_4^2+384\beta_3^2\beta_4^2+128A(\alpha_2^1)^2\beta_5^2\\
&+384(\alpha_2^1)^2\beta_5^2-512A\alpha_2^1\alpha_5^1\beta_5^2-1536\alpha_2^1\alpha_5^1\beta_5^2\\
&+128A\alpha_2^2\beta_5^2+384\alpha_2^2\beta_5^2-256A\alpha_5^2\beta_5^2-768\alpha_5^2\beta_5^2\\
&-256A\alpha_5^1\beta_1^1\beta_5^2-768\alpha_5^1\beta_1^1\beta_5^2+128A\alpha_2^1\beta_4^1\beta_5^2+384\alpha_2^1\beta_4^1\beta_5^2\\
&+128A\beta_4^2\beta_5^2+384\beta_4^2\beta_5^2-256A\alpha_1^2\beta_8^2-768\alpha_1^2\beta_8^2\\
&-384A(\alpha_2^1)^2\beta_9^2-1152(\alpha_2^1)^2\beta_9^2-384A\alpha_2^2\beta_9^2-128A\beta_7^4\\
&-1152\alpha_2^2\beta_9^2-384A\alpha_2^1\beta_1^1\beta_9^2-1152\alpha_2^1\beta_1^1\beta_9^2+384\beta_5^4\\
&-384A\beta_1^2\beta_9^2-1152\beta_1^2\beta_9^2+1920A\alpha_4^1\beta_1^3+2304\alpha_4^1\beta_1^3\\
&+768A\alpha_6^1\beta_1^3+1152\alpha_6^1\beta_1^3+256A\alpha_8^1\beta_1^3+384\alpha_8^1\beta_1^3\\
&+3456\beta_5^1\beta_1^3+256A\alpha_3^1\beta_3^3+768\alpha_3^1\beta_3^3-384A\beta_1^1\beta_3^3\\
&-1152\beta_1^1\beta_3^3+128A\beta_4^1\beta_3^3+384\beta_4^1\beta_3^3-128A\alpha_4^1\beta_4^3\\
&-384\alpha_4^1\beta_4^3-128A\beta_5^1\beta_4^3-384\beta_5^1\beta_4^3+128A\alpha_2^1\beta_5^3\\
&+384\alpha_2^1\beta_5^3-256A\alpha_5^1\beta_5^3-768\alpha_5^1\beta_5^3+128A\beta_4^1\beta_5^3\\
&+384\beta_4^1\beta_5^3-384A\alpha_2^1\beta_9^3-1152\alpha_2^1\beta_9^3-384A\beta_1^1\beta_9^3\\
&-1152\beta_1^1\beta_9^3+384A\beta_3^4+1152\beta_3^4+128A\beta_5^4-384\beta_7^4\\
&-384A\beta_9^4-1152\beta_9^4)/[512(A+3)],
\end{aligned}
$$

$$D_1 = \pi(\alpha_1^6+\beta_2^6).$$

# Appendix E

# Averaging function of order 6 for the Collins second form, A = 0. Continuous case.

We present the averaging function of order 6 for the Collins second form, in the case $A = 0$, discussed in the proof of Theorem 5.5.

$$f_6(r) = r(\mathcal{D}_5 r^4 + \mathcal{D}_4 r^3 + \mathcal{D}_3 r^2 + \mathcal{D}_1),$$

where

$$\begin{aligned}
\mathcal{D}_5 =& \frac{\pi}{192}(-1053\alpha_1^1\beta_1^1 + 459\alpha_1^1\alpha_2^1 + 678\alpha_1^1\alpha_3^1 \\
&- 1179\beta_1^1\beta_3^1 + 618\alpha_1^1\beta_4^1 + 234\alpha_1^1\alpha_5^1 - 1365\beta_1^1\beta_5^1 \\
&+ 216\alpha_1^1\beta_6^1 - 72\alpha_1^1\alpha_7^1 + 72\alpha_1^1\beta_8^1 + 72\alpha_1^1\alpha_9^1 \\
&- 144\beta_1^1\beta_9^1 - 339\alpha_2^1\beta_3^1 - 27\alpha_2^1\alpha_4^1 - 189\alpha_2^1\beta_5^1 \\
&+ 144\alpha_2^1\alpha_6^1 + 1272\alpha_3^1\beta_3^1 - 456\alpha_3^1\alpha_4^1 + 96\beta_3^1\beta_4^1 \\
&+ 1824\alpha_3^1\beta_5^1 - 216\alpha_3^1\alpha_6^1 + 276\beta_3^1\beta_6^1 + 100\beta_3^1\beta_8^1 \\
&+ 264\alpha_3^1\beta_9^1 + 285\beta_1^1\alpha_4^1 - 48\alpha_4^1\beta_4^1 - 328\alpha_4^1\alpha_5^1 \\
&+ 120\beta_4^1\beta_5^1 - 12\alpha_4^1\beta_6^1 - 20\alpha_4^1\alpha_7^1 - 44\alpha_4^1\beta_8^1 + 12\alpha_4^1\alpha_9^1 \\
&+ 24\beta_4^1\beta_9^1 + 776\beta_3^1\alpha_5^1 + 1120\alpha_5^1\beta_5^1 - 264\alpha_5^1\alpha_6^1 + 300\beta_5^1\beta_6^1 \\
&+ 140\beta_5^1\beta_8^1 + 216\alpha_5^1\beta_9^1 + 288\beta_1^1\alpha_6^1 - 72\beta_4^1\alpha_6^1 - 72\alpha_6^1\beta_6^1 \\
&- 24\alpha_6^1\alpha_7^1 - 24\alpha_6^1\beta_8^1 - 72\alpha_6^1\alpha_9^1 + 72\beta_6^1\beta_9^1 + 76\beta_3^1\alpha_7^1 \\
&+ 116\beta_5^1\alpha_7^1 + 24\alpha_7^1\beta_9^1 + 24\beta_8^1\beta_9^1 + 156\beta_3^1\alpha_9^1 + 276\beta_5^1\alpha_9^1 \\
&+ 72\alpha_9^1\beta_9^1 + 648\alpha_1^2 + 312\beta_3^2 - 120\alpha_4^2 + 408\beta_5^2 \\
&- 144\alpha_6^2 - 48\alpha_8^2), \\
\mathcal{D}_4 =& \frac{3\pi}{8}\alpha_1^1(\alpha_2^1 + \beta_1^1)(3\alpha_1^1 + \alpha_4^1 + \beta_3^1 + 2\beta_5^1), \\
\mathcal{D}_3 =& \frac{\pi}{512}(324\alpha_2^1(\alpha_1^1)^3 + 384\alpha_3^1(\alpha_1^1)^3 + 384\alpha_5^1(\alpha_1^1)^3 + 108\beta_1^1(\alpha_1^1)^3 \\
&- 384\beta_4^1(\alpha_1^1)^3 + 768\alpha_2^1\alpha_4^1(\alpha_1^1)^2 + 384\alpha_3^1\alpha_4^1(\alpha_1^1)^2 + 128\alpha_4^1\alpha_5^1(\alpha_1^1)^2 \\
&+ 216\alpha_1^2(\alpha_1^1)^2 + 384\alpha_4^2(\alpha_1^1)^2 - 384\alpha_4^1\beta_1^1(\alpha_1^1)^2 - 384\alpha_2^1\beta_3^1(\alpha_1^1)^2
\end{aligned}$$

$$
\begin{aligned}
&- 256\alpha_3^1\beta_3^1(\alpha_1^1)^2 + 768\beta_1^1\beta_3^1(\alpha_1^1)^2 - 128\beta_3^1\beta_4^1(\alpha_1^1)^2 - 768\alpha_2^1\beta_5^1(\alpha_1^1)^2 \\
&+ 256\alpha_5^1\beta_5^1(\alpha_1^1)^2 + 384\beta_1^1\beta_5^1(\alpha_1^1)^2 - 384\beta_4^1\beta_5^1(\alpha_1^1)^2 - 384\beta_3^2(\alpha_1^1)^2 \\
&- 384\beta_5^2(\alpha_1^1)^2 - 195(\alpha_2^1)^3\alpha_1^1 - 81(\beta_1^1)^3\alpha_1^1 + 256\alpha_2^1(\alpha_3^1)^2\alpha_1^1 \\
&+ 256\alpha_2^1(\alpha_4^1)^2\alpha_1^1 + 27\alpha_2^1(\beta_1^1)^2\alpha_1^1 + 384\alpha_3^1(\beta_1^1)^2\alpha_1^1 - 128\alpha_2^1(\beta_4^1)^2\alpha_1^1 \\
&+ 128\beta_1^1(\beta_4^1)^2\alpha_1^1 - 512\alpha_2^1(\beta_5^1)^2\alpha_1^1 + 384(\alpha_2^1)^2\alpha_3^1\alpha_1^1 + 1152(\alpha_2^1)^2\alpha_5^1\alpha_1^1 \\
&+ 512\alpha_2^1\alpha_3^1\alpha_5^1\alpha_1^1 + 256(\alpha_2^1)^2\alpha_7^1\alpha_1^1 + 768\alpha_4^1\alpha_1^2\alpha_1^1 - 588\alpha_2^1\alpha_2^2\alpha_1^1 \\
&+ 384\alpha_3^1\alpha_2^2\alpha_1^1 + 768\alpha_5^1\alpha_2^2\alpha_1^1 + 256\alpha_7^1\alpha_2^2\alpha_1^1 + 384\alpha_2^1\alpha_3^2\alpha_1^1 \\
&+ 512\alpha_3^1\alpha_3^2\alpha_1^1 + 256\alpha_5^1\alpha_3^2\alpha_1^1 + 256\alpha_4^1\alpha_4^2\alpha_1^1 + 768\alpha_2^1\alpha_5^2\alpha_1^1 \\
&+ 256\alpha_3^1\alpha_5^2\alpha_1^1 + 256\alpha_2^1\alpha_7^2\alpha_1^1 - 384\alpha_2^3\alpha_1^1 + 384\alpha_3^3\alpha_1^1 \\
&+ 384\alpha_5^3\alpha_1^1 + 256\alpha_7^3\alpha_1^1 + 1449(\alpha_2^1)^2\beta_1^1\alpha_1^1 - 256(\alpha_3^1)^2\beta_1^1\alpha_1^1 \\
&- 384\alpha_2^1\alpha_3^1\beta_1^1\alpha_1^1 + 1332\alpha_2^2\beta_1^1\alpha_1^1 - 384\alpha_3^2\beta_1^1\alpha_1^1 - 128\alpha_2^1\alpha_4^1\beta_3^1\alpha_1^1 \\
&- 768\alpha_1^2\beta_3^1\alpha_1^1 - 128\alpha_4^2\beta_3^1\alpha_1^1 + 128\alpha_4^1\beta_1^1\beta_3^1\alpha_1^1 - 384(\alpha_2^1)^2\beta_4^1\alpha_1^1 \\
&- 384(\beta_1^1)^2\beta_4^1\alpha_1^1 - 128\alpha_2^1\alpha_3^1\beta_4^1\alpha_1^1 + 256\alpha_2^1\alpha_5^1\beta_4^1\alpha_1^1 - 384\alpha_2^2\beta_4^1\alpha_1^1 \\
&- 128\alpha_3^2\beta_4^1\alpha_1^1 + 128\alpha_5^2\beta_4^1\alpha_1^1 + 384\alpha_2^1\beta_1^1\beta_4^1\alpha_1^1 + 128\alpha_3^1\beta_1^1\beta_4^1\alpha_1^1 \\
&+ 256\alpha_2^1\alpha_4^1\beta_5^1\alpha_1^1 - 768\alpha_1^2\beta_5^1\alpha_1^1 + 128\alpha_4^2\beta_5^1\alpha_1^1 - 256\alpha_2^1\beta_3^1\beta_5^1\alpha_1^1 \\
&+ 256\beta_1^1\beta_3^1\beta_5^1\alpha_1^1 + 256(\alpha_2^1)^2\beta_8^1\alpha_1^1 + 256\alpha_2^2\beta_8^1\alpha_1^1 + 1332\alpha_2^1\beta_1^2\alpha_1^1 \\
&- 384\alpha_3^1\beta_1^2\alpha_1^1 + 180\beta_1^1\beta_1^2\alpha_1^1 + 384\beta_4^1\beta_1^2\alpha_1^1 - 128\alpha_4^1\beta_3^2\alpha_1^1 \\
&- 256\beta_5^1\beta_3^2\alpha_1^1 - 384\alpha_2^1\beta_4^2\alpha_1^1 - 128\alpha_3^1\beta_4^2\alpha_1^1 + 128\alpha_5^1\beta_4^2\alpha_1^1 \\
&+ 384\beta_1^1\beta_4^2\alpha_1^1 - 256\beta_4^1\beta_4^2\alpha_1^1 + 128\alpha_4^1\beta_5^2\alpha_1^1 - 256\beta_3^1\beta_5^2\alpha_1^1 \\
&- 512\beta_5^1\beta_5^2\alpha_1^1 + 256\alpha_2^1\beta_8^2\alpha_1^1 + 1152\beta_1^3\alpha_1^1 - 384\beta_4^3\alpha_1^1 \\
&+ 256\beta_8^3\alpha_1^1 + 90\alpha_1^2(\beta_1^1)^2 - 128\alpha_1^2(\beta_4^1)^2 - 256\alpha_1^2(\beta_5^1)^2 \\
&- 192(\alpha_2^1)^3\alpha_4^1 + 128(\alpha_2^1)^2\alpha_3^1\alpha_4^1 + 384(\alpha_2^1)^2\alpha_4^1\alpha_5^1 - 384(\alpha_2^1)^3\alpha_6^1 \\
&- 1446(\alpha_2^1)^2\alpha_1^2 + 256(\alpha_3^1)^2\alpha_1^2 + 128(\alpha_4^1)^2\alpha_1^2 + 384\alpha_2^1\alpha_3^1\alpha_1^2 \\
&+ 768\alpha_2^1\alpha_5^1\alpha_1^2 + 256\alpha_3^1\alpha_5^1\alpha_1^2 + 256\alpha_2^1\alpha_7^1\alpha_1^2 - 384\alpha_2^1\alpha_4^1\alpha_2^2 \\
&+ 128\alpha_3^1\alpha_4^1\alpha_2^2 + 256\alpha_4^1\alpha_5^1\alpha_2^2 - 768\alpha_2^1\alpha_6^1\alpha_2^2 - 1536\alpha_1^2\alpha_2^2 \\
&+ 128\alpha_2^1\alpha_4^1\alpha_3^2 + 384\alpha_1^2\alpha_3^2 + 128(\alpha_2^1)^2\alpha_4^2 + 128\alpha_2^1\alpha_3^1\alpha_4^2 \\
&+ 256\alpha_2^1\alpha_5^1\alpha_4^2 + 128\alpha_2^2\alpha_4^2 + 128\alpha_3^2\alpha_4^2 + 256\alpha_2^1\alpha_4^1\alpha_5^2 \\
&+ 384\alpha_1^2\alpha_5^2 + 128\alpha_4^2\alpha_5^2 + 256\alpha_1^2\alpha_7^2 + 128(\alpha_2^1)^2\alpha_8^2 \\
&+ 128\alpha_2^2\alpha_8^2 - 1536\alpha_2^1\alpha_1^3 + 384\alpha_3^1\alpha_1^3 + 384\alpha_5^1\alpha_1^3 \\
&+ 256\alpha_7^1\alpha_1^3 - 192\alpha_4^1\alpha_2^3 - 384\alpha_6^1\alpha_2^3 + 128\alpha_4^1\alpha_3^3 + 128\alpha_2^1\alpha_4^3 \\
&+ 128\alpha_3^1\alpha_4^3 + 128\alpha_5^1\alpha_4^3 + 128\alpha_4^1\alpha_5^3 + 128\alpha_2^1\alpha_8^3 \\
&- 1536\alpha_1^4 + 128\alpha_4^4 + 384\alpha_6^4 + 128\alpha_8^4 - 320(\alpha_2^1)^2\alpha_4^1\beta_1^1 \\
&+ 256\alpha_2^1\alpha_4^1\alpha_5^1\beta_1^1 - 384(\alpha_2^1)^2\alpha_6^1\beta_1^1 + 180\alpha_2^1\alpha_1^2\beta_1^1 - 384\alpha_3^1\alpha_1^2\beta_1^1 \\
&- 320\alpha_4^1\alpha_2^2\beta_1^1 - 384\alpha_6^1\alpha_2^2\beta_1^1 + 128\alpha_5^1\alpha_4^2\beta_1^1 + 128\alpha_4^1\alpha_5^2\beta_1^1 \\
&+ 128\alpha_2^1\alpha_8^2\beta_1^1 + 128\alpha_8^3\beta_1^1 + 832(\alpha_2^1)^3\beta_3^1 + 384(\beta_1^1)^3\beta_3^1 \\
&- 256\alpha_3^1(\beta_1^1)^2\beta_3^1 - 128\alpha_4^1\alpha_1^2\beta_3^1 + 1664\alpha_2^1\alpha_2^2\beta_3^1 + 832\alpha_2^3\beta_3^1 \\
&- 256\alpha_3^3\beta_3^1 + 832(\alpha_2^1)^2\beta_1^1\beta_3^1 + 832\alpha_2^2\beta_1^1\beta_3^1 + 256\alpha_3^2\beta_1^1\beta_3^1 \\
&- 384\alpha_2^1\alpha_1^2\beta_4^1 - 128\alpha_3^1\alpha_1^2\beta_4^1 + 128\alpha_5^1\alpha_1^2\beta_4^1 - 384\alpha_1^3\beta_4^1
\end{aligned}
$$

$$
\begin{aligned}
&+ 384\alpha_1^2\beta_1^1\beta_4^1 - 128(\beta_1^1)^2\beta_3^1\beta_4^1 + 960(\alpha_2^1)^3\beta_5^1 + 768(\alpha_2^1)^2\alpha_5^1\beta_5^1 \\
&+ 128\alpha_4^1\alpha_1^2\beta_5^1 + 1920\alpha_2^1\alpha_2^2\beta_5^1 + 512\alpha_5^1\alpha_2^2\beta_5^1 + 512\alpha_2^1\alpha_5^2\beta_5^1 \\
&+ 960\alpha_2^3\beta_5^1 + 256\alpha_5^3\beta_5^1 + 1088(\alpha_2^1)^2\beta_1^1\beta_5^1 + 512\alpha_2^1\alpha_5^1\beta_1^1\beta_5^1 \\
&+ 1088\alpha_2^2\beta_1^1\beta_5^1 + 256\alpha_5^2\beta_1^1\beta_5^1 - 256\alpha_1^2\beta_3^1\beta_5^1 - 128(\alpha_2^1)^2\beta_4^1\beta_5^1 \\
&- 128\alpha_2^2\beta_4^1\beta_5^1 + 256\alpha_2^1\alpha_1^2\beta_8^1 + 256\alpha_1^3\beta_8^1 + 384(\alpha_2^1)^3\beta_9^1 \\
&+ 768\alpha_2^1\alpha_2^2\beta_9^1 + 384\alpha_2^3\beta_9^1 + 384(\alpha_2^1)^2\beta_1^1\beta_9^1 + 384\alpha_2^2\beta_1^1\beta_9^1 \\
&- 320\alpha_2^1\alpha_4^1\beta_1^2 + 128\alpha_4^1\alpha_5^1\beta_1^2 - 384\alpha_2^1\alpha_6^1\beta_1^2 + 128\alpha_8^2\beta_1^2 \\
&+ 832\alpha_2^1\beta_3^1\beta_1^2 + 256\alpha_3^1\beta_3^1\beta_1^2 - 768\beta_1^1\beta_3^1\beta_1^2 + 128\beta_3^1\beta_4^1\beta_1^2 \\
&+ 1088\alpha_2^1\beta_5^1\beta_1^2 + 256\alpha_5^1\beta_5^1\beta_1^2 + 384\alpha_2^1\beta_9^1\beta_1^2 - 384(\beta_1^1)^2\beta_3^2 \\
&- 256\alpha_3^2\beta_3^2 + 256\alpha_3^1\beta_1^1\beta_3^2 + 128\beta_1^1\beta_4^1\beta_3^2 + 384\beta_1^2\beta_3^2 \\
&- 384\alpha_1^2\beta_4^2 + 128\beta_1^1\beta_3^1\beta_4^2 - 128\alpha_2^1\beta_5^1\beta_4^2 - 128\beta_3^2\beta_4^2 \\
&- 128(\alpha_2^1)^2\beta_5^2 + 512\alpha_2^1\alpha_5^1\beta_5^2 - 128\alpha_2^2\beta_5^2 + 256\alpha_5^2\beta_5^2 \\
&+ 256\alpha_5^1\beta_1^1\beta_5^2 - 128\alpha_2^1\beta_4^1\beta_5^2 - 128\beta_4^2\beta_5^2 + 256\alpha_1^2\beta_8^2 \\
&+ 384(\alpha_2^1)^2\beta_9^2 + 384\alpha_2^2\beta_9^2 + 384\alpha_2^1\beta_1^1\beta_9^2 + 384\beta_1^2\beta_9^2 \\
&- 320\alpha_4^1\beta_1^3 - 384\alpha_6^1\beta_1^3 + 1216\beta_3^1\beta_1^3 + 1088\beta_5^1\beta_1^3 \\
&+ 384\beta_9^1\beta_1^3 - 256\alpha_3^1\beta_3^3 + 384\beta_1^1\beta_3^3 - 128\beta_4^1\beta_3^3 \\
&- 128\beta_3^1\beta_4^3 - 128\beta_5^1\beta_4^3 - 128\alpha_2^1\beta_5^3 + 256\alpha_5^1\beta_5^3 \\
&- 128\beta_4^1\beta_5^3 + 384\alpha_2^1\beta_9^3 + 384\beta_1^1\beta_9^3 - 384\beta_3^4 \\
&- 128\beta_5^4 + 128\beta_7^4 + 384\beta_9^4), \\
\mathcal{D}_1 =&\pi(\alpha_1^6 + \beta_2^6).
\end{aligned}
$$

# Appendix F

# Expressions of $\mathbf{g_i(R)}$ and $\mathbf{g_{Mj}(R)}$.

We present the expressions of $g_i(R)$, $i = 0, \ldots, 7$ and $g_{Mj}(R)$, $j = 1, \ldots, 4$.

## F.1 Functions $g_i(R)$, for $i = 0, \ldots, 7$

$$
\begin{aligned}
g_0 =& -3\pi((R^2+3)(-6R^{10}(R^2+3)^{2/3}+59R^6(R^2+3)^{2/3}-1440R^2\\
&(R^2+3)^{2/3}+6R^8((R^4-9)^{2/3}\sqrt[3]{R^2-3}+3(R^2+3)^{2/3})-R^4(709\\
&(R^4-9)^{2/3}\sqrt[3]{R^2-3}+177(R^2+3)^{2/3})+360(12(R^2+3)^{2/3}\\
&-61\sqrt[3]{R^2-3}(R^4-9)^{2/3}))\,{}_2F_1(-\frac{1}{2},\frac{2}{3};1;\frac{6}{R^2+3})\sqrt[3]{R^2-3}\\
&+(7320(R^4-9)^{2/3}\sqrt[3]{R^2-3}-1440(R^2+3)^{2/3}+R^2(1346\\
&(R^4-9)^{2/3}\sqrt[3]{R^2-3}+6R^8(R^2+3)^{2/3}+618(R^2+3)^{2/3}-6R^6\\
&((R^4-9)^{2/3}\sqrt[3]{R^2-3}+(R^2+3)^{2/3})-R^4(12(R^4-9)^{2/3}\\
&\sqrt[3]{R^2-3}+71(R^2+3)^{2/3})+R^2(685(R^4-9)^{2/3}\sqrt[3]{R^2-3}+59\\
&(R^2+3)^{2/3})))\,{}_2F_1(\frac{1}{2},\frac{2}{3};1;\frac{6}{R^2+3})(R^2-3)^{4/3}+(R^2+3)^{4/3}\\
&((-1346(R^4-9)^{2/3}\sqrt[3]{R^2+3}+618(R^2-3)^{2/3}+R^2(685(R^4-9)^{2/3}\\
&\sqrt[3]{R^2+3}-59(R^2-3)^{2/3}+R^2(12(R^4-9)^{2/3}\sqrt[3]{R^2+3}-71\\
&(R^2-3)^{2/3}+6R^2(-(R^4-9)^{2/3}\sqrt[3]{R^2+3}+R^2(R^2-3)^{2/3}\\
&+(R^2-3)^{2/3}))))R^2+120(61(R^4-9)^{2/3}\sqrt[3]{R^2+3}\\
&+12(R^2-3)^{2/3}))\,{}_2F_1(\frac{1}{2},\frac{2}{3};1;-\frac{6}{R^2-3})-(R^2-3)\sqrt[3]{R^2+3}(6R^{10}\\
&(R^2-3)^{2/3}-59R^6(R^2-3)^{2/3}+1440R^2(R^2-3)^{2/3}+360(61\\
&(R^4-9)^{2/3}\sqrt[3]{R^2+3}+12(R^2-3)^{2/3})+R^4(709\sqrt[3]{R^2+3}(R^4-9)^{2/3}\\
&-177(R^2-3)^{2/3})+6R^8(3(R^2-3)^{2/3}-\sqrt[3]{R^2+3}(R^4-9)^{2/3}))\\
&{}_2F_1(-\frac{1}{2},\frac{2}{3};1;-\frac{6}{R^2-3}))\Big/14560R(R^2-3)^{2/3}(R^2+3)^{2/3}(R^4-9)^{2/3};
\end{aligned}
$$

$$
\begin{aligned}
g_1 =& \pi(-(2R^4-39)(3R^2\sqrt[3]{R^2-3}+9\sqrt[3]{R^2-3}+2(R^2+3)^{2/3}\sqrt[3]{R^4-9})R^2 \\
& {}_2F_1(-\frac{2}{3},\frac{1}{2};1;\frac{6}{R^2+3})+2(R^4-12)((R^2+3)(3R^2\sqrt[3]{R^2+3} \\
& -9\sqrt[3]{R^2+3}+2(R^2-3)^{2/3}\sqrt[3]{R^4-9})\,{}_2F_1(\frac{1}{3},\frac{1}{2};1;-\frac{6}{R^2-3}) \\
& +(R^2-3)(3R^2\sqrt[3]{R^2-3}+9\sqrt[3]{R^2-3}+2(R^2+3)^{2/3}\sqrt[3]{R^4-9}) \\
& {}_2F_1(\frac{1}{3},\frac{1}{2};1;\frac{6}{R^2+3}))-R^2(2R^4-39)(3R^2\sqrt[3]{R^2+3}-9\sqrt[3]{R^2+3} \\
& +2(R^2-3)^{2/3}\sqrt[3]{R^4-9})\,{}_2F_1(-\frac{2}{3},\frac{1}{2};1;-\frac{6}{R^2-3})) \\
& \Big/ 880R\sqrt[3]{R^4-9};
\end{aligned}
$$

$$
\begin{aligned}
g_2 =& 3\pi((R^2+3)(72R^{14}(R^2+3)^{2/3}-840R^{10}(R^2+3)^{2/3}+391R^6 \\
& (R^2+3)^{2/3}+50688R^2(R^2+3)^{2/3}-72R^{12}((R^4-9)^{2/3}\sqrt[3]{R^2-3} \\
& +3(R^2+3)^{2/3})+24R^8(73(R^4-9)^{2/3}\sqrt[3]{R^2-3}+105(R^2+3)^{2/3}) \\
& -51R^4(261(R^4-9)^{2/3}\sqrt[3]{R^2-3}+23(R^2+3)^{2/3})-1728(88 \\
& (R^2+3)^{2/3}-15\sqrt[3]{R^2-3}(R^4-9)^{2/3}))\,{}_2F_1(-\frac{1}{2},\frac{2}{3};1;\frac{6}{R^2+3}) \\
& \sqrt[3]{R^2-3}+((12294(R^4-9)^{2/3}\sqrt[3]{R^2-3}-24702(R^2+3)^{2/3}+R^2 \\
& (8535(R^4-9)^{2/3}\sqrt[3]{R^2-3}+391(R^2+3)^{2/3}+R^2(-2640(R^4-9)^{2/3} \\
& \sqrt[3]{R^2-3}-1711(R^2+3)^{2/3}-24R^2(61(R^4-9)^{2/3}\sqrt[3]{R^2-3}+3R^6 \\
& (R^2+3)^{2/3}+35(R^2+3)^{2/3}-3R^4((R^4-9)^{2/3}\sqrt[3]{R^2-3}+(R^2+3)^{2/3}) \\
& -R^2(6(R^4-9)^{2/3}\sqrt[3]{R^2-3}+41(R^2+3)^{2/3})))))R^2+576(88 \\
& (R^2+3)^{2/3}-15\sqrt[3]{R^2-3}(R^4-9)^{2/3}))\,{}_2F_1(\frac{1}{2},\frac{2}{3};1;\frac{6}{R^2+3})(R^2-3)^{4/3} \\
& +\sqrt[3]{R^2+3}((50688(R^2-3)^{2/3}+R^2((391(R^2-3)^{2/3}+24R^2(73 \\
& (R^4-9)^{2/3}\sqrt[3]{R^2+3}+3R^6(R^2-3)^{2/3}-35R^2(R^2-3)^{2/3}-105 \\
& (R^2-3)^{2/3}+R^4(9(R^2-3)^{2/3}-3\sqrt[3]{R^2+3}(R^4-9)^{2/3})))R^2+51(23 \\
& (R^2-3)^{2/3}-261\sqrt[3]{R^2+3}(R^4-9)^{2/3})))R^2+1728(15(R^4-9)^{2/3} \\
& \sqrt[3]{R^2+3}+88(R^2-3)^{2/3}))\,{}_2F_1(-\frac{1}{2},\frac{2}{3};1;-\frac{6}{R^2-3})(R^2-3) \\
& -(R^2+3)^{4/3}(((-8535(R^4-9)^{2/3}\sqrt[3]{R^2+3}+391(R^2-3)^{2/3} \\
& +R^2(-2640(R^4-9)^{2/3}\sqrt[3]{R^2+3}+1711(R^2-3)^{2/3}+24R^2(61 \\
& (R^4-9)^{2/3}\sqrt[3]{R^2+3}-35(R^2-3)^{2/3}+R^2(6(R^4-9)^{2/3}\sqrt[3]{R^2+3} \\
& -41(R^2-3)^{2/3}+3R^2(-(R^4-9)^{2/3}\sqrt[3]{R^2+3}+R^2(R^2-3)^{2/3} \\
& +(R^2-3)^{2/3})))))R^2+6(2049(R^4-9)^{2/3}\sqrt[3]{R^2+3} \\
& +4117(R^2-3)^{2/3}))R^2+576(15(R^4-9)^{2/3}\sqrt[3]{R^2+3}
\end{aligned}
$$

$$+ 88(R^2-3)^{2/3}))\,{}_2F_1(\frac{1}{2},\frac{2}{3};1;-\frac{6}{R^2-3}))$$
$$\Big/ 442624R(R^2-3)^{2/3}(R^2+3)^{2/3}(R^4-9)^{2/3};$$

$$\begin{aligned}
g_3 =&\pi((-29R^2\sqrt[3]{R^2+3}+87\sqrt[3]{R^2+3}+6R^6\sqrt[3]{R^2+3}\\
&+98(R^2-3)^{2/3}\sqrt[3]{R^4-9}+R^4(4(R^2-3)^{2/3}\sqrt[3]{R^4-9}\\
&-18\sqrt[3]{R^2+3}))R^2\,{}_2F_1(-\frac{2}{3},\frac{1}{2};1;-\frac{6}{R^2-3})\\
&+(-29R^2\sqrt[3]{R^2-3}-87\sqrt[3]{R^2-3}+6R^6\sqrt[3]{R^2-3}\\
&+98(R^2+3)^{2/3}\sqrt[3]{R^4-9}+2R^4(9\sqrt[3]{R^2-3}\\
&+2(R^2+3)^{2/3}\sqrt[3]{R^4-9}))R^2\,{}_2F_1(-\frac{2}{3},\frac{1}{2};1;\frac{6}{R^2+3})\\
&-2(R^2+3)(8R^2\sqrt[3]{R^2+3}-24\sqrt[3]{R^2+3}+3R^6\sqrt[3]{R^2+3}\\
&+64(R^2-3)^{2/3}\sqrt[3]{R^4-9}+R^4(2(R^2-3)^{2/3}\sqrt[3]{R^4-9}\\
&-9\sqrt[3]{R^2+3}))\,{}_2F_1(\frac{1}{3},\frac{1}{2};1;-\frac{6}{R^2-3})-2(R^2-3)(8R^2\sqrt[3]{R^2-3}\\
&+3R^6\sqrt[3]{R^2-3}+R^4(9\sqrt[3]{R^2-3}+2(R^2+3)^{2/3}\sqrt[3]{R^4-9})\\
&+8(3\sqrt[3]{R^2-3}+8(R^2+3)^{2/3}\sqrt[3]{R^4-9}))\,{}_2F_1(\frac{1}{3},\frac{1}{2};1;\frac{6}{R^2+3}))\\
&\Big/ 880R\sqrt[3]{R^4-9};
\end{aligned}$$

$$\begin{aligned}
g_4 =&3\pi((R^2+3)(6R^8-7R^4-1056)(R^2-3)^{2/3}\,{}_2F_1(-\frac{1}{2},\frac{2}{3};1;\frac{6}{R^2+3})\\
&+(R^2+3)^{2/3}(6R^8-7R^4-1056)(R^2-3)\,{}_2F_1(-\frac{1}{2},\frac{2}{3};1;-\frac{6}{R^2-3})\\
&-(6R^8+12R^6+17R^4+58R^2-352)(R^2-3)^{5/3}\,{}_2F_1(\frac{1}{2},\frac{2}{3};1;\frac{6}{R^2+3})\\
&+(R^2+3)^{2/3}(-6R^{10}-6R^8+19R^6+7R^4+526R^2+1056)\\
&{}_2F_1(\frac{1}{2},\frac{2}{3};1;-\frac{6}{R^2-3}))\Big/ 2912R(R^4-9)^{2/3};
\end{aligned}$$

$$\begin{aligned}
g_5 =&-9\pi(-(R^2+3)(-120R^{14}(R^2+3)^{2/3}+1704R^{10}(R^2+3)^{2/3}\\
&-3641R^6(R^2+3)^{2/3}-11520R^2(R^2+3)^{2/3}+120R^{12}((R^4-9)^{2/3}\\
&\sqrt[3]{R^2-3}+3(R^2+3)^{2/3})+17280(7(R^4-9)^{2/3}\sqrt[3]{R^2-3}\\
&+2(R^2+3)^{2/3})-8R^8(403(R^4-9)^{2/3}\sqrt[3]{R^2-3}+639(R^2+3)^{2/3})\\
&+3R^4(10587(R^4-9)^{2/3}\sqrt[3]{R^2-3}+3641(R^2+3)^{2/3}))\\
&{}_2F_1(-\frac{1}{2},\frac{2}{3};1;\frac{6}{R^2+3})\sqrt[3]{R^2-3}+((35994(R^4-9)^{2/3}\sqrt[3]{R^2-3}\\
&-9858(R^2+3)^{2/3}+R^2(22585(R^4-9)^{2/3}\sqrt[3]{R^2-3}+3641(R^2+3)^{2/3}
\end{aligned}$$

$$+ R^2(-5008(R^4-9)^{2/3}\sqrt[3]{R^2-3} - 6449(R^2+3)^{2/3}$$
$$- 8R^2(343(R^4-9)^{2/3}\sqrt[3]{R^2-3} + 213(R^2+3)^{2/3}$$
$$+ 3R^2(-10(R^4-9)^{2/3}\sqrt[3]{R^2-3} + 5R^4(R^2+3)^{2/3} - 81(R^2+3)^{2/3}$$
$$\sqrt[3]{R^2-3} + (R^2+3)^{2/3}))))))R^2 + 5760(7(R^4-9)^{2/3}\sqrt[3]{R^2-3}$$
$$+ 2(R^2+3)^{2/3}))\,{}_2F_1(\frac{1}{2},\frac{2}{3};1;\frac{6}{R^2+3})(R^2-3)^{4/3}$$
$$+ \sqrt[3]{R^2+3}((11520(R^2-3)^{2/3} + R^2(-31761(R^4-9)^{2/3}\sqrt[3]{R^2+3}$$
$$+ 10923(R^2-3)^{2/3} + R^2(3641(R^2-3)^{2/3} + 8R^2(403(R^4-9)^{2/3}$$
$$\sqrt[3]{R^2+3} + 15R^6(R^2-3)^{2/3} - 213R^2(R^2-3)^{2/3} - 639(R^2-3)^{2/3}$$
$$+ 15R^4(3(R^2-3)^{2/3} - \sqrt[3]{R^2+3}(R^4-9)^{2/3})))))R^2$$
$$+ 17280(2(R^2-3)^{2/3} - 7\sqrt[3]{R^2+3}(R^4-9)^{2/3}))$$
$${}_2F_1(-\frac{1}{2},\frac{2}{3};1;-\frac{6}{R^2-3})(R^2-3) - (R^2+3)^{4/3}((35994(R^4-9)^{2/3}$$
$$\sqrt[3]{R^2+3} + 9858(R^2-3)^{2/3} + R^2(-22585(R^4-9)^{2/3}\sqrt[3]{R^2+3}$$
$$+ 3641(R^2-3)^{2/3} + R^2(-5008(R^4-9)^{2/3}\sqrt[3]{R^2+3} + 6449(R^2-3)^{2/3}$$
$$+ 8R^2(343(R^4-9)^{2/3}\sqrt[3]{R^2+3} - 213(R^2-3)^{2/3}$$
$$+ 3R^2(10(R^4-9)^{2/3}\sqrt[3]{R^2+3} - 81(R^2-3)^{2/3} + 5R^2(-(R^4-9)^{2/3}$$
$$\sqrt[3]{R^2+3} + R^2(R^2-3)^{2/3} + (R^2-3)^{2/3}))))))R^2 + 5760(2(R^2-3)^{2/3}$$
$$- 7\sqrt[3]{R^2+3}(R^4-9)^{2/3}))\,{}_2F_1(\frac{1}{2},\frac{2}{3};1;-\frac{6}{R^2-3}))$$
$$\Big/ 2213120R(R^2-3)^{2/3}(R^2+3)^{2/3}(R^4-9)^{2/3};$$

$$g_6 = \pi(2(29R^2\sqrt[3]{R^2+3} - 87\sqrt[3]{R^2+3} - 6R^6\sqrt[3]{R^2+3} + 78(R^2-3)^{2/3}$$
$$\sqrt[3]{R^4-9} + 2R^4(9\sqrt[3]{R^2+3} - 2(R^2-3)^{2/3}\sqrt[3]{R^4-9}))R^2$$
$${}_2F_1(-\frac{2}{3},\frac{1}{2};1;-\frac{6}{R^2-3}) + 2(29R^2\sqrt[3]{R^2-3} + 87\sqrt[3]{R^2-3}$$
$$- 6R^6\sqrt[3]{R^2-3} + 78(R^2+3)^{2/3}\sqrt[3]{R^4-9} - 2R^4(9\sqrt[3]{R^2-3}$$
$$+ 2(R^2+3)^{2/3}\sqrt[3]{R^4-9}))R^2\,{}_2F_1(-\frac{2}{3},\frac{1}{2};1;\frac{6}{R^2+3})$$
$$+ 4(R^2+3)(8R^2\sqrt[3]{R^2+3} + 3R^6\sqrt[3]{R^2+3} - 24(\sqrt[3]{R^2+3}$$
$$+ (R^2-3)^{2/3}\sqrt[3]{R^4-9}) + R^4(2(R^2-3)^{2/3}\sqrt[3]{R^4-9} - 9\sqrt[3]{R^2+3}))$$
$${}_2F_1(\frac{1}{3},\frac{1}{2};1;-\frac{6}{R^2-3}) + 4(R^2-3)(8R^2\sqrt[3]{R^2-3}$$
$$+ 3R^6\sqrt[3]{R^2-3} + R^4(9\sqrt[3]{R^2-3} + 2(R^2+3)^{2/3}\sqrt[3]{R^4-9})$$
$$+ 24(\sqrt[3]{R^2-3} - (R^2+3)^{2/3}\sqrt[3]{R^4-9}))\,{}_2F_1(\frac{1}{3},\frac{1}{2};1;\frac{6}{R^2+3}))$$
$$\Big/ 1760R\sqrt[3]{R^4-9};$$

$$\begin{aligned}
g_7 =& 3\pi(3(R^2+3)(2R^8-11R^4+64)(R^2-3)^{2/3}\,{}_2F_1(-\frac{1}{2},\frac{2}{3};1;\frac{6}{R^2+3}) \\
&+3(R^2+3)^{2/3}(2R^8-11R^4+64)(R^2-3)\,{}_2F_1(-\frac{1}{2},\frac{2}{3};1;-\frac{6}{R^2-3}) \\
&-(6R^8+12R^6-9R^4+6R^2+64)(R^2-3)^{5/3}\,{}_2F_1(\frac{1}{2},\frac{2}{3};1;\frac{6}{R^2+3}) \\
&-(R^2+3)^{5/3}(6R^8-12R^6-9R^4-6R^2+64)\,{}_2F_1(\frac{1}{2},\frac{2}{3};1;-\frac{6}{R^2-3})) \\
&\Big/ 2912R(R^4-9)^{2/3};
\end{aligned}$$

where ${}_2F_1(a,b,c,z)$ is the hypergeometric function which has the following series expansion

$$\sum_{k=0}^{+\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!},$$

with

$$(a)_k = \begin{cases} 1 & \text{if } k=0; \\ a(a+1)(a+2)\cdots(a+k-1) & \text{if } k>0. \end{cases}$$

## F.2 Functions $g_{Mj}(R)$, for $j = 1, \ldots, 4$

$$g_{M1} = -\frac{R}{16}-\frac{R^5}{108}-\frac{R^9}{486}-\frac{3}{2R\sqrt{R^4-9}}+\frac{R^{11}}{486\sqrt{R^4-9}};$$

$$g_{M2} = -\frac{R}{12}-\frac{R^5}{54}+\frac{\sqrt{R^4-9}}{6R}+\frac{1}{54}\sqrt{R^4-9}R^3;$$

$$g_{M3} = -\frac{R}{6}+\frac{\sqrt{R^4-9}}{6R};$$

$$g_{M4} = \frac{3R}{4}.$$

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