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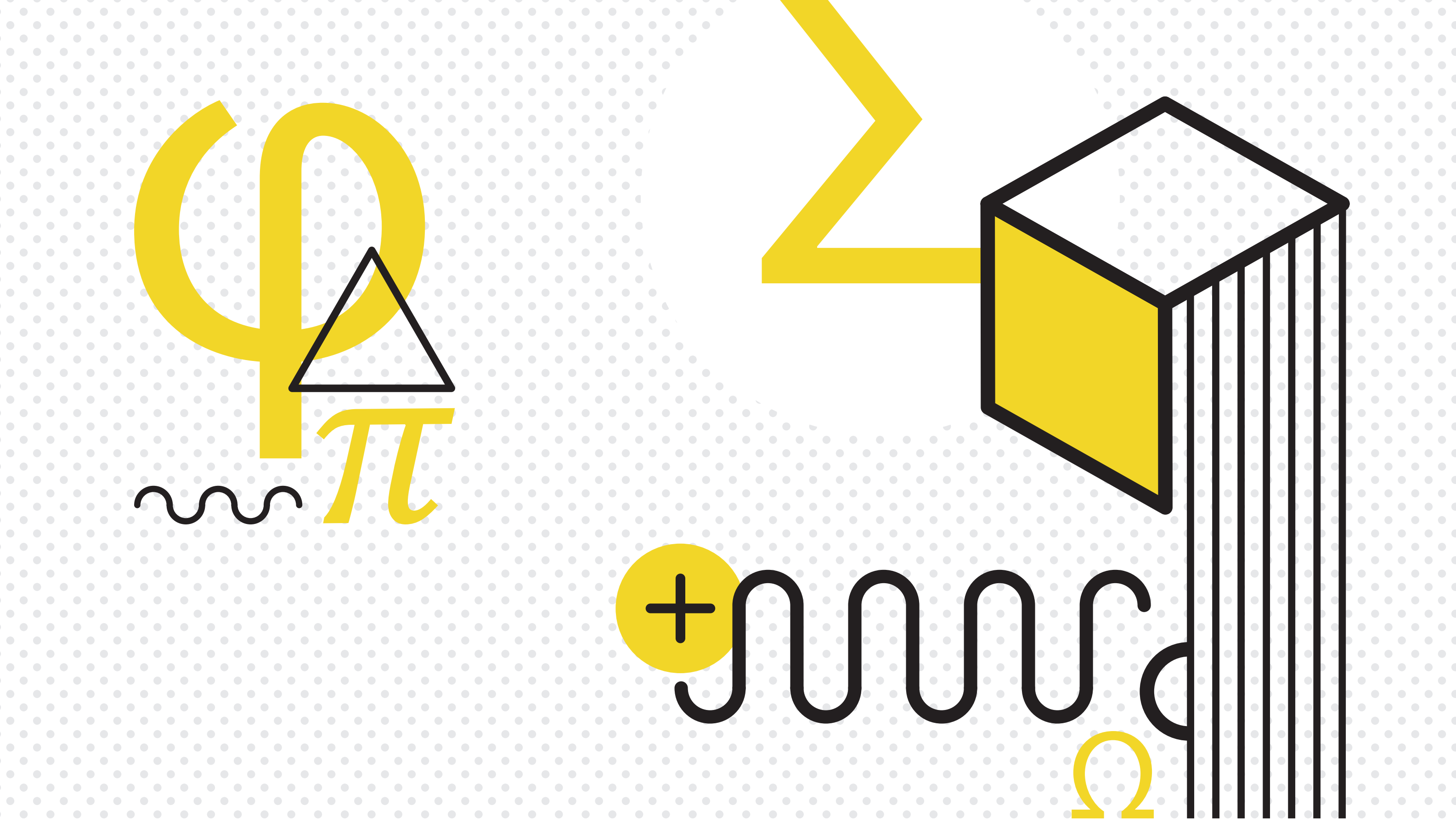

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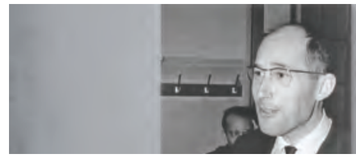

December 2017

FEATURED

We close out 2017 with a celebration of Wolf Prize recipient Charles Fefferman and memories of geometer Marcel Berger. Gamow's liquid model for the atomic nucleus returns in recent research and applications. If hiring, you don't want to miss "Reducing Bias in Faculty Searches." David Peifer provides the story of Dorothea Rockburne's encounter with Max Dehn at the legendary Black Mountain College. —***Frank Morgan, Editor-in-Chief***

GRADUATE STUDENT SECTION

COMMENTARY

COMMENTARY, CONT'D.

ALSO IN THIS ISSUE

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FROM THE AMS SECRETARY

IN EVERY ISSUE

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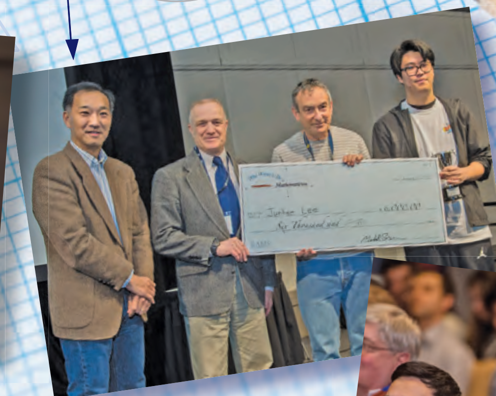
American Mathematical Society

Who Wants to Be a Mathematician

$x^2+x+1=0$

High School Students Compete for $10,000



Photos by Kate Awtrey, Atlanta Convention Photography

2018 Championship Contest

Students from the US, UK, and Canada

Saturday, January 13

1:00–2:30 pm

San Diego Convention Center 6C (Upper Level)

Mathemati-Con #JMM2018

The 2018 *Who Wants to Be a Mathematician* Championship is part of Mathemati–Con events at the meeting, open to the public on Saturday from 9:00 am to 4:00 pm, featuring the 2018 Mathematical Art Exhibition, James Tanton, Math Circles, Matt Parker, Math Wrangle, the Porter Lecture by Moon Duchin, and more.

Photo by Mihaela Bodlovic

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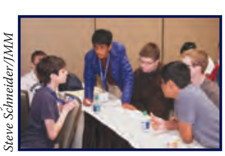

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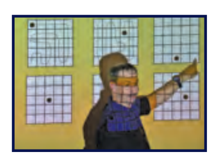

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Ad Honorem Charles Fefferman

Communicated by Stephen Kennedy and Steven J. Miller

Charles Fefferman circa 2005.

Antonio Córdoba

Introduction

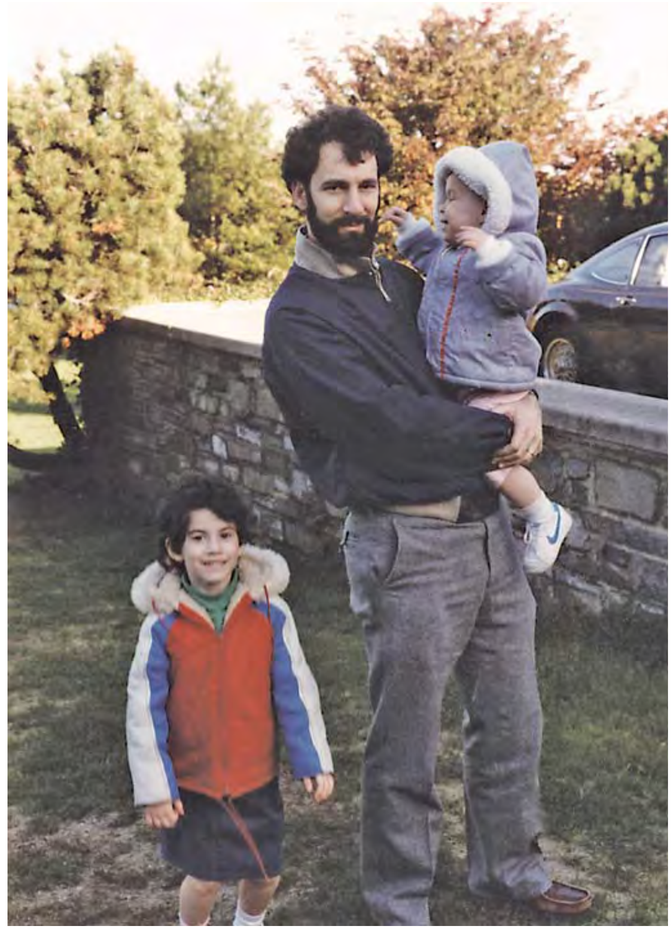
The prestigious Wolf Prize of 2017 has been awarded to Charles Fefferman, *ex aequo* with Richard Schoen. Charles Fefferman (Charlie) is a mathematician of the first rank whose outstanding findings, both classical and revolutionary, have inspired further research by many others. He is one of the most accomplished and versatile mathematicians of all time, having so far contributed with fundamental results to harmonic analysis, linear PDEs, several complex variables, conformal geometry, quantum mechanics, fluid mechanics, and Whitney's theory, together with more sporadic incursions into other subjects such as neural networks, financial mathematics, and crystallography.

"Problems seem to select me!"

I have requested the help of a distinguished group of his friends and collaborators to provide reflections on Fefferman's contributions to their respective fields. Before reading their remarks, it will be interesting to hear from Charlie himself: "Problems seem to select me! It's just so exciting. A problem sort of chooses you, and you can't stop thinking about it. At first, you try something, and it doesn't work. You get clobbered! You try something else and get clobbered again! Eventually you get some insights and things begin to come together. Everything starts to move. Everyday things can look different. It's very exciting. Eventually you manage to solve it all, and that's a great feeling!"

Born on April 18, 1949, Fefferman was a child prodigy who at the age of seventeen graduated from the University of Maryland, where he received a joint bachelor's degree in mathematics and physics. In 1969 he gained his PhD at Princeton University under the supervision of Eli Stein. In 1971, at the University of Chicago, he became the youngest full professor at any US college or university, a fact that merited his appearance in *Time* and *Newsweek* magazines in that same year.

Charlie returned to Princeton University in the fall of 1974, where since then he has pursued his mathematical career. In 1975 he and his wife, Julie, got married and went on to have two daughters, Nina and Lainie. Nina is a computational biologist who applies mathematical models to complex biological systems, while Lainie is a composer and holds a PhD in musical composition from Princeton University. Charlie has a brother, Robert, who is also a mathematician and professor at the University of Chicago. Julie has this to say about her husband: "When Charlie was young, he fell in love: (1) with painting and art and (2) with math. He says that for a while the two were tied, but painting was never in the lead. Eventually he realized that he was much better at math. After knowing him for thirty-five years, if I had to account for his choice, I'd say that his passion for the beauty of math was what overtook everything else. I think that for him it's almost an addiction to the art of beautiful mathematics…Whenever I ask him to try to explain his work to me, his eyes sparkle and his voice and gestures are infused with an animation that is not present at any other time."



Charlie with daughters Nina (in red) and Lainie around 1983.

I met Charlie at the University of Chicago during the academic year 1971–1972. He was then a recently appointed full professor and I was a first-year graduate student. The Calderón-Zygmund seminar was probably the place where we first got acquainted. At the end of that academic year Charlie agreed to be my thesis advisor. Let me add that for me it was a fantastic experience; we are of the same age, and at that time we became close friends. We played ping-pong together in the Eckhard Hall basement and had long conversations about science, art, movies, music, politics, and, of course, mathematics.

I had the privilege of being his first graduate student, thereby initiating a set that now contains more than twenty elements. There is no doubt in my mind that the

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Charlie Fefferman with the ICMAT fluids team, Madrid circa 2011, from left to right: Javier Gómez-Serrano, Angel Castro, Charles Fefferman, Diego Córdoba, Francisco Gancedo

opportunity of enjoying Charlie's advice and friendship is an experience we all will treasure.

Elias Stein

Fefferman's Early Work: The Epic Years (1969–1974)

I want to write about the first five years of Charlie Fefferman's major work, a brief period in which his many innovations transformed our views of several subjects in an intensive series of achievements unique in the history of modern mathematics.

The Dissertation

I first met Charlie in the fall of 1967. He had started as a graduate student at Princeton the year before, but I was on leave that year. Our first real contact was after he had whizzed through his qualifying exam and said he wanted to write his thesis with me.

At that time the "Calderón–Zygmund paradigm," as it was later to be called, had already proved very successful in freeing parts of harmonic analysis from its heavy reliance on complex methods and leading it to the open vistas in higher dimensions, with new challenges and possibilities.

One question that particularly fascinated me was that of developing further methods for proving L^p estimates, in particular where the critical limiting exponent for p might be greater than 1, unlike the standard singular integrals. A problem of this kind arose in the Littlewood–Paley theory of square functions, and I proposed it to Charlie, eager to see what this promising eighteen-year-old could do.

To explain what was involved, there were first the more standard square functions, the g-function, and its nontangential version, the Lusin area integral S, for which it was known that the mappings $g \mapsto g(f), g \mapsto S(f)$ were of weak-type 1 and bounded on $L^p, 1 < p < \infty$. Another more arcane ("tangential") variant arose in some problems, the function g_λ^*, defined by

(1)
$$g_\lambda^*(f)(x)^2 = \int_{\mathbb{R}_+^{n+1}} |\nabla u(x-y,t)|^2 \left(\frac{t}{|y|+t}\right)^{n\lambda} t^{1-n} dy dt.$$

Here $u(x,t)$ is the Poisson integral of a function f on $\mathbb{R}^n$. In fact g_λ^* dominated both S and g, and it could be proved that $f \to g_\lambda^*(f)$ was bounded on L^p if $1 < \lambda < 2$ and $p > 2/\lambda$; moreover, this failed when $p = 2/\lambda$. It seemed that the right assertion was that $f \mapsto g_\lambda^*(f)$ was of weak-type p if $p > 1$, but I had no real idea how to proceed, and this clearly required a new approach.

In just a short few weeks Charlie came back with the proof. I was surprised not only by the speed with which he had accomplished it but by the strength of his mathematics. I will not describe his idea of the proof but only say that it provided him a guide to solving the next problem—one that I proposed to him soon thereafter.

This problem concerned highly oscillating singular integrals. An example was the transformation $T : f \to f * K$ where the kernel K (a distribution) is given by

$$(2) \qquad K(x) = \frac{e^{\frac{i}{|x|}}}{|x|^n}, \text{ when } 0 < |x| \leq 1$$

and vanishes for $|x| > 1$. It was known that the resulting T was bounded on $L^p, 1 < p < \infty$, but the hoped-for weak-type 1 result seemed out of reach.

Charlie set himself to work on this, and again, within a short few weeks he proved the desired assertion. His ideas were as follows.

To begin with he found the right restatement of the problem in its general form. It concerned a distribution kernel K of compact support (represented by a function $K(x)$, when $x \neq 0$) which for a fixed parameter θ, with $0 \leq \theta < 1$, satisfied the following two conditions:

(i) $\hat{K}(\xi) = O(|\xi|^{-\theta\frac{n}{2}})$ as $|\xi| \to \infty$,
(ii) $\int_{|x| \gtrsim |y|^{1-\theta}} |K(x-y) - K(x)| dx \leq A$.

The standard Calderón–Zygmund kernels correspond to the case $\theta = 0$, while the highly oscillatory ones correspond to $0 < \theta < 1$, with $\theta = \frac{1}{2}$ for the particular case (2) above.

With this incisive first step, he next decomposed an arbitrary f in L^1, for fixed $\alpha > 0$, in the standard way, writing $f = g + \Sigma_j b_j$, with g in L^2, $\|g\|_{L^2} \leq \alpha$, and the b_j supported on disjoint cubes, while

$$\frac{1}{|Q_j|} \int_{Q_j} |b_j| dx \approx \alpha \quad \text{and} \quad \int_{Q_j} b_j dx = 0.$$

So $T(g)$ could be disposed with by the L^2 theory. Coming to the bad part, we let B_j be the ball surrounding Q_j (having the same center but twice the diameter). Here the main idea was to introduce the balls B_j^*, with the same center, but with

$$\text{diam } B_j^* \approx (\text{diam } B_j)^{1-\theta}.$$

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Charlie circa 1976 at Princeton.

(Only balls with diam $B_j \leq 1$ are significant here.) The contribution of $T(b_j)$ outside B_j^* can be handled by (ii), as in the standard situation.

There remain the critical contribution of $T(b_j)$ in B_j^*/B_j and the corresponding estimates of $T(\tilde{b}) = \sum T(\tilde{b_j})$ (where $\tilde{b_j}$ is a cleverly chosen replacement of b_j). At this stage he had prepared everything to go back to L^2 estimates, and he succeeded because he was able to combine directly two facts in this setting, namely,

$$\|T(\tilde{b})\|_{L^2} \lesssim \|(1-\Delta)^{-n\frac{\varrho}{4}}\tilde{b}\|_{L^2} \quad \text{and}$$

$$\|(1-\Delta)^{-n\frac{\varrho}{4}}\tilde{b}\|_{L^2} \lesssim \alpha\|b\|_{L^1}.$$

The result in its elegance and power, achieved so quickly, amazed me, as it would have anyone. Later his approach became a subject of much wider interest, as it was adapted to various other problems by S. Chanillo, M. Christ, Rubio de Francia, and others.

Because of these two striking successes, I had little hesitation in urging Charlie to attack a major problem, one which had concerned me for a number of years—that of the Bochner–Riesz means. To give some background:

In 1936 S. Bochner had introduced the operators B_R^δ defined by

$$B_R^\delta(\hat{f})(\xi) = \begin{cases} (1-\frac{|\xi|^2}{R^2})^\delta \hat{f}(\xi) & \text{for } |\xi| < R, \\ 0 & \text{for } |\xi| \geq R. \end{cases}$$

These were later dubbed the Bochner–Riesz means (of order δ). For $n = 1, \delta = 0$, these were the classical partial-sum operators that were closely related to the Hilbert transform, and thus one had L^p-norm control for $1 < p < \infty$. Bochner had pointed out that for $n > 1$, the order $\delta = \frac{n-1}{2}$ was the critical index in the sense that if $\delta > \frac{n-1}{2}$, then $B_R(f) = f * K_R$, where the kernels K_R are good approximations to the identity, while for $\delta \leq \frac{n-1}{2}$ things are quite different and depend critically on the oscillatory nature of the K_R. By 1960 it was known that for each $p, 1 < p < \infty$, one had L^p control for some $\delta = \delta(p) < \frac{n-1}{2}$ (also convergence a.e. as $R \to \infty$), but these conclusions were far from optimal. In fact, one was led to expect that the B_R^δ when $\delta = 0$ were bounded in L^p for $\frac{2n}{n+1} < p < \frac{2n}{n-1}$ (in two dimensions $\frac{4}{3} < p < 4$), together with the wider expectation that the B_R^δ were bounded outside that range when $\delta > n|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}$. Charlie's challenge was to go decisively beyond what was then known in a quest to achieve these ultimate goals.

In urging him to undertake this clearly difficult and uncertain effort, I could not be of much help, except for one thing I had observed the year before: when $n > 1$ and p is sufficiently close to 1, the Fourier transform of an L^p function can be restricted to the unit sphere S^{n-1} of $\mathbb{R}^n$. More precisely, one had, in the present day terminology, the (L^p, L^q) restriction phenomenon

$$\left(\int_{S^{n-1}} |\hat{f}(\xi)|^q d\sigma(\xi)\right)^{\frac{1}{q}} \lesssim \|f\|_{L^p(\mathbb{R}^n)} \tag{3}$$

when $q = 2$, for p close to 1. The obvious suggestion was to explore the possible implications of this to the BR means.

Here Charlie succeeded marvelously with a remarkable conclusion: whenever the (L^p, L^2) restriction phenomenon held, the optimal bounds for B_R^δ with the same p would follow as a consequence.

After this accomplishment, we had some brief discussions on trying to extend the restriction phenomenon. By dualizing the problem we succeeded in obtaining the optimal conclusion in two dimensions: the (L^p, L^q) restriction holds for $p = \frac{4}{3} - \epsilon$, and $q = \frac{4}{3} + \epsilon'$, more precisely for $3q = p'$, with $1 \leq p < \frac{4}{3}$. All the above results are in Charlie's dissertation, published in 1970.

Shortly thereafter, two exciting developments followed. First, L. Carleson and P. Sjölin showed that in two dimensions, whenever $\delta > 0$, one had boundedness of BR means for $\frac{4}{3} \leq p \leq 4$. This they did in part by adapting the philosophy for the restriction theorem in two dimensions. However gratifying this result was, it left open the pressing question: how to obtain the L^p boundedness for the BR means of order $\delta = 0$ in the range $\frac{4}{3} < p < 4$. In other words, how to deal with the disc multiplier.

Eli Stein (center) with the Fefferman brothers at the conference to celebrate Stein's sixtieth birthday at Princeton in 1991. Charlie is on the left, Robert on the right.

Charlie's answer was an unexpected shock. His remarkable counterexample showed that the hoped-for result fails, except for the obvious case $p = 2$. It dramatically transformed our view of the subject. This development and a number of Charlie's other achievements are covered in Terry Tao's section, so I will not say more about this direction of his work.

The H^1-BMO Duality

In the fall of 1970, after spending his first postdoctoral year in Princeton, Charlie took up a position at the University of Chicago. There he met Antoni Zygmund, who almost immediately put a question to him: what is the Poisson integral characterization of BMO? This question led Charlie to think more about BMO and ultimately to formulate and prove the famous aforementioned duality. (A charming reminiscence of his thinking about the issues involved can be found in his contribution to the collection *All That Math: Portraits of Mathematicians as Young Readers.*)

Let me briefly recall some facts about H^1 and BMO. The classical Hardy space $H^p, p > 0$, arose at the intersection of complex analysis and Fourier analysis. Rephrased in the setting of the upper half-plane $\mathbb{R}^2_+$ with its boundary $\mathbb{R}$, it consisted of those functions F on $\mathbb{R}$ that arose as boundary values of the holomorphic functions F in $\mathbb{R}^2_+$ that are uniformly bounded in L^p on all lines in $\mathbb{R}^2_+$ parallel to $\mathbb{R}$. Formally, each such F can be represented as $F = f + iH(f)$, where f is real-valued and H is the Hilbert transform. Thus if $1 < p < \infty$, classical H^p was essentially real $L^p(\mathbb{R})$. However, when $p \le 1$, things were quite different, and H^p had all sorts of interesting properties either not valid for L^1 or totally missing for the trivial space L^p when $p < 1$.

By 1960 it was clearly time to see whether there was anything like real Hardy spaces in $\mathbb{R}^n$, for $n > 1$. Here G. Weiss and I had some initial success. We defined the space $H^1(\mathbb{R}^n)$ to consist of those L^1 functions for which, in addition, their Riesz transforms, $R_j(f), j = 1, \dots, n$, also belonged to L^1. It turned out that this space had a number of interesting features going beyond L^1, in particular a maximal characterization involving Poisson integrals in terms of L^1 (and not merely weak-L^1). A further hint that $H^1(\mathbb{R}^n)$ was the appropriate substitute for $L^1(\mathbb{R}^u)$ came several years later, when it was seen that classical singular integrals, broadly speaking, took $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ and in fact preserved $H^1(\mathbb{R}^n)$.

Let us come to the other branch of the tree: BMO. That space (functions of bounded mean oscillation) appeared first in 1961 in the work of F. John and L. Nirenberg. A function f on $\mathbb{R}^n$ was said to be in BMO if

$$\sup_Q \frac{1}{|Q|} \int_Q |f - f_Q| dx \le A < \infty. \tag{4}$$

Here $f_Q = \frac{1}{|Q|} \int_Q f dx$, and the Q range over all cubes Q in $\mathbb{R}^n$.

John and Nirenberg proved the remarkable fact that (4) implies an analogous inequality with $|f - f_Q|$ replaced by $|f - f_Q|^p$, for any $p < \infty$, and in fact replaced by $e^{c|f - f_Q|}$ for an appropriate positive c.

This kind of result had immediate consequences: in the work of John for rotation and strain of mappings and in J. Moser's elegant proof of the DiGiorgi–Nash estimates. Then in 1966 the relevance of BMO to harmonic analysis—and the possibility that it might serve as the appropriate substitute for L^∞—became a little clearer when J. Peetre, S. Spanne, and I (quite independently) observed that the standard singular integrals took L^∞ to BMO and, even better, that BMO was stable under these transformations.

With this we return to the question Zygmund asked Charlie: to characterize BMO by Poisson integrals. Charlie's answer was given in terms of three interrelated assertions, each remarkable in its own right:

(i) The dual space of H^1 is BMO.

(ii) A function f is in BMO if and only if

$$f = f_0 + \sum_{j=1}^{n} R_j(f_j), \text{ where } f_0, f_1, \dots, f_n \text{ are all in } L^\infty.$$

(iii) f is in BMO if and only if $t|\nabla u(x,t)|^2 dxdt$ is a Carleson measure.

The assertion (iii) was the answer Zygmund sought but did not expect! Here $u(x,t), x \in \mathbb{R}^n, t > 0$, is the Poisson integral of f, and the statement (iii) was that

$$\sup_B \frac{1}{|B|} \int_{T(B)} t|\nabla u(x,t)|^2 dxdt \leq A < \infty,$$

where $T(B) = \{x, t : |x-y| < r-t, 0 < t < r\}$ is the "tent" over the ball $B \subset \mathbb{R}^n$, centered at y of radius r, and B ranges over all balls. It is noteworthy that measures of this kind arose for $n = 1$ in Carleson's work in 1962 on the corona conjecture.

Charlie's discovery reawakened interest in Hardy spaces and BMO, and a flurry of activity followed immediately. Working together we developed a systematic theory of Hardy spaces, which in particular freed it from its reliance on harmonic functions. This showed that an element of H^p could be characterized in a variety of different but equivalent ways:

(i) in terms of maximal characterizations by general smooth approximations of the identity (not just restricted to Poisson integrals),
(ii) similarly, in terms of general square functions, and
(iii) in terms of Riesz transforms and their higher analogues.

A consequence of this was the confirmation of the status of H^1 as the rightful substitute for L^1 (and H^p for $L^p, p < 1$) in the theory of singular integrals; not only was it stable under the action of these operators, but appropriate maximal and square functions had L^1 control, and there was also a useful Calderón–Zygmund decomposition in this context.

Another interesting by-product was the function $f^{\#}$ defined by

$$f^{\#}(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f - f_Q| dx.$$

In analogy with the John–Nirenberg theorem one had that $f^{\#} \in L^p$ implied that $f \in L^p$, if $p < \infty$, and this was useful in resolving the problem of complex interpolation between H^1 or BMO and $L^p, 1 < p < \infty$.

A significant further development started with Charlie's observation that the duality of H^1 and BMO can be restated by saying each $f \in H^1$ has an atomic decomposition. This meant that any such f could be written as $\sum_j \lambda_j a_j$, where each atom, a_j, was supported on a cube Q_j, with $|a_j(x)| \leq \frac{1}{|Q_j|}$, $\int a_j(x)dx = 0$, and the constants λ_j satisfied $\sum_j |\lambda_j| < \infty$.

In the hands of R. Coifman, C. Herz, R. Latter, G. Weiss, and others, the idea of atomic decomposition was greatly developed, encompassing $H^p, p < 1$, and ultimately becoming the preferred starting point for various extensions of the theory.

The Mapping Theorem and Bergman Kernel

In the following years Charlie made a number of other notable advances. Any list would have to include his version of Carleson's theorem about the convergence of Fourier series, his solution with R. Beals of an outstanding local solvability problem for linear partial differential equations, and his work involving the mapping theorem and the Bergman kernel—to which I'll limit myself.

While still at the University of Chicago, Charlie was told by R. Narasimhan of an outstanding problem in several complex variables: Suppose Ω_1 and Ω_2 are a pair of bounded domains in $\mathbb{C}^n, n > 1$, each having a smooth (C^∞) boundary and each being strongly pseudoconvex. Suppose there is a bijective biholomorphism Φ from Ω_1 to Ω_2. Then does Φ extend to a smooth diffeomorphism of $\bar{\Omega}_1$ to $\bar{\Omega}_2$?

It must be stressed that the case $n \geq 2$ differs essentially from the classical case $n = 1$. First, there is no adequate theory of conformal mappings to help when $n \geq 2$. In particular, two domains may be such that $\bar{\Omega}_1$ and $\bar{\Omega}_2$ are close (in C^∞) without there existing a biholomorphism between them. Moreover, pseudoconvexity must enter the picture, one way or another. After explaining the problem, Narasimhan said to Charlie, "You must do that."

…as inspired and natural in conception as it was difficult and painful in execution…

Charlie was immediately taken with the problem. He remembered a course he attended as a graduate student on boundary behavior of holomorphic functions and having been intrigued by the Bergman kernel and resulting metric. The facts about these are as follows.

The Bergman kernel $K_\Omega(z,w)$ of domain Ω is determined by the fact that $f \mapsto \int_\Omega K_\Omega(z,w)f(w)dw$ is the orthogonal projection of $L^2(\Omega, dw)$ onto the subspace of L^2 holomorphic functions on Ω. A remarkable feature was that if the Hermitian (Riemannian) metric $ds^2 = \sum \frac{\partial}{\partial \bar{z}_j \partial z_k}(\log K_\Omega(z,z)) d\bar{z}_j, dz_k$ is attached to Ω, then a biholomorphism $\Phi : \Omega_1 \to \Omega_2$ induces an isometry in these metrics.

Charlie's idea was as inspired and natural in conception as it was difficult and painful in execution. In brief, what had to be done was to carry out the following plan: after fixing corresponding points $P_1 \in \Omega_1, P_2 = \Phi(P_1)$, follow any geodesic on Ω_1 starting at P_1 and the corresponding geodesic Ω_2 to infinite time when both reach the boundary at points $Q_1 \in \partial\Omega_1$ and $Q_2 \in \partial\Omega_2$. The mapping $Q_1 \mapsto Q_2$ should then yield the correspondence of the boundaries and thus give the C^∞ extension of Φ.

The first step therefore was analyzing the behavior of the Bergman metric, which required a clearer understanding of the Bergman kernel, a question of great interest in its own right. The description of K_Ω he obtained was as follows. Assuming that Ω is a bounded domain with C^∞ boundary which is strongly pseudoconvex, then

$$K_\Omega(z,w) = \frac{A(z,w)}{(Q(z,w))^{n+1}} + B(z,w)\log Q(z,w). \tag{5}$$

Here $Q(z,w)$ is the holomorphic part of the second-order Taylor expansion of a defining function ρ of the domain Ω; A and B are smooth functions, with

***Studio Interior*, painted by Charlie circa 1968.**

$A(z,z) \neq 0$. It should be noted that in the case Ω is the unit ball, we can take $\rho(z) = 1 - |z|^2, Q(z,w) = 1 - z\bar{w}$, and $A(z,w) \equiv c_n, B(z,w) \equiv 0$. Moreover, it should be stressed that no such general result has been proved (or even formulated!) if one drops the assumption of *strong* pseudoconvexity.

To prove (5) required that at each boundary point $w \in \partial\Omega$ one osculate to a high degree a version of a complex ball and transplant its explicit Bergman kernel to Ω. This gave a first-order approximation to K_Ω, which then had to be followed by a highly intricate iterative procedure to ultimately obtain (5).

Once one has (5) and the Bergman metric is controlled, one can proceed to the main lemma: suppose $X(t, P_0, \xi)$ is the point of the geodesic starting at P_0 in the unit direction ξ and at time t. Assume that for some ξ_0 the nonnegative time portion on the geodesic $X(t, P_o, \xi_0)$ does not lie in a compact set. Then

(i) $\lim_{t\to\infty} X(t, P_0, \xi_0)$ converges to a boundary point of Ω.
(ii) The same is true for all ξ near ξ_0, and the resulting mapping of $\xi \mapsto X(\infty, P_0, \xi)$ is a local diffeomorphism.
(iii) All boundary points can be reached this way.

The proof of the lemma requires again a highly involved argument, because, among other things, it is imperative to overcome the obstacle of the log term in (5). When finally all this is done the theorem is:

Let Ω_1 and Ω_2 be a pair of bounded domains with C^∞ boundaries that are strongly pseudoconvex. Suppose there is a biholomorphism between them. Then this extends to a C^∞ diffeomorphism between $\bar{\Omega}_1$ and $\bar{\Omega}_2$.

After his paper (1974) various alternate approaches, simplifications, or extensions were obtained by others, including: L. Boutet de Monvel and J. Sjöstrand (1976); S. Bell and E. Ligocka (1980); L. Nirenberg, S. Webster, and P. Yang (1980); S. Bell and D. Catlin (1982); and F. Forsternic (1992). Still, more than forty years later, Charlie's achievement stands as a milestone in the development of several complex variables.

Terence Tao

Connecting Fourier Analysis and Geometry

During the early 1970s Fefferman made a number of important and fundamental contributions in the theory of oscillatory singular integrals, the study of which can be motivated by the classical problem of determining the nature of convergence of Fourier series and Fourier integrals and which has since had remarkable connections and applications to many other fields of mathematics, including partial differential equations, analytic number theory, and geometric measure theory.

To motivate the subject, let us first work in the simplest setting of Fourier series on the unit circle $\mathbb{R}/\mathbb{Z}$. Given any absolutely integrable function $f : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$, one can form the Fourier coefficients $\hat{f}(n)$ for any integer n by the formula

$$\hat{f}(n) := \int_{\mathbb{R}/\mathbb{Z}} f(x) e^{-2\pi i n x}\, dx,$$

and then the Fourier inversion formula asserts that one should have the identity

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}. \tag{6}$$

If f is sufficiently regular, then there is no difficulty in interpreting and proving this identity; for instance, if f is continuously twice differentiable, one can show that the Fourier coefficients $\hat{f}$ are absolutely summable and that the series appearing on the right-hand side of (6) converges uniformly to f. If f is instead assumed to be square-integrable, then the Fourier coefficients need not be absolutely summable any more, but it follows easily from Plancherel's theorem that the series in (6) still converges unconditionally to f within the Hilbert space $L^2(\mathbb{R}/\mathbb{Z})$.

The situation becomes more subtle if one weakens the regularity hypotheses on f or asks for stronger notions of convergence. Suppose for instance that f is an arbitrary p^{th}-power integrable function for some $1 < p < \infty$ not

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equal to 2; thus $f \in L^p(\mathbb{R}/\mathbb{Z})$. As it turns out, the series on the right-hand side of (6) will not, in general, be unconditionally convergent or absolutely convergent in $L^p(\mathbb{R}/\mathbb{Z})$; however, the partial sums

$$S_N(f)(x) := \sum_{n=-N}^{N} \hat{f}(n) e^{2\pi i n x}$$

will still converge in $L^p(\mathbb{R}/\mathbb{Z})$ norm to f. Establishing this fact is equivalent (by the uniform boundedness principle) to demonstrating that operators S_N are uniformly bounded in $L^p(\mathbb{R}/\mathbb{Z})$. The operators S_N can be explicitly described as an integral operator:

$$S_N f(x) = \int_{\mathbb{R}/\mathbb{Z}} f(x-t) \frac{\sin((N+\frac{1}{2})t)}{\sin(t/2)}\, dt.$$

The kernel $\frac{\sin((N+\frac{1}{2})t)}{\sin(t/2)}$ has a numerator which oscillates rapidly when N is large and a denominator that goes to zero as t goes to zero. As such it is a simple example of an oscillatory singular integral operator (though in this case, it is technically not singular because the numerator also vanishes at $t = 0$, though its derivative is quite large at that point). There are by now many techniques to establish the required boundedness for $1 < p < \infty$; for instance, one can use the Calderón–Zygmund theory of singular integral operators. On the other hand, boundedness (and hence convergence) in L^p norm is known to fail when $p = 1$ or $p = \infty$.

A more difficult question is whether the partial sums $S_N(f)$ converge *pointwise almost everywhere* to f. (One can construct examples to show that pointwise everywhere convergence can fail, even if f is assumed to be continuous.) This turns out to basically be equivalent to establishing the boundedness (or more precisely, weak-type boundedness) in L^p of not just each individual operator S_N, but the more complicated *Carleson maximal operator*

$$Cf(x) := \sup_{N>0} |S_N(f)(x)|.$$

Controlling this operator is substantially more difficult than controlling a single S_N, but this was famously achieved in 1966 by Lennart Carleson (for $p = 2$) and then by Richard Hunt (for general $1 < p < \infty$). On the other hand, a famous construction of Kolmogorov produces an absolutely integrable function f whose partial Fourier series $S_N(f)$ diverges pointwise almost everywhere (or even everywhere, if one is more careful in the construction).

Fefferman made multiple contributions to these questions and their higher-dimensional analogues; we shall restrict our attention here to just two of his most well-known and influential works in this area. Firstly, for higher-dimensional Fourier series, if the circle $\mathbb{R}/\mathbb{Z}$ is replaced by a torus $(\mathbb{R}/\mathbb{Z})^d$ for some $d \geq 2$, Fefferman studied the *ball multiplier*

$$S_N(f)(x) := \sum_{n \in \mathbb{Z}^d : |n| \leq N} \hat{f}(n) e^{2\pi i x \cdot n}\, dx,$$

where $|n|$ denotes the Euclidean norm of n; thus S_N sums the terms of the Fourier series whose frequency n lies in the ball of radius N. On the one hand, Fefferman observed that if one replaced the ball with a cube, then a modification of the one-dimensional theory allowed one to establish L^p boundedness and convergence of these multipliers for all $1 < p < \infty$. On the other hand, he showed the surprising fact that the ball multipliers were *not* bounded uniformly on L^p for any $p \neq 2$, which implied in particular that one could construct functions in $L^p((\mathbb{R}/\mathbb{Z})^d)$ whose partial Fourier series $S_N(f)$ did not converge in L^p norm! We can describe a modern version of Fefferman's remarkable construction here in the two-dimensional case $d = 2$. This construction revealed an important connection between such Fourier-analytic problems and questions in

Charlie with his wife, Julie, in front of their home in Princeton circa 1980.

an important connection between such Fourier-analytic problems and questions in geometric measure theory

geometric measure theory and in particular the *Kakeya needle problem*, which asked one to find the minimal area subset of the plane in which one could rotate a unit line segment (or "needle") by a full rotation. In 1919 Abram Besicovitch gave a construction that implied that such a rotation could be carried out in a set of arbitrarily small measure. A modification of this construction produces, for any given $\varepsilon > 0$, a family of rectangles $R_1, \dots, R_n$ which overlapped heavily in the sense that the measure of the union of the R_i was less than ϵ times the sum of the individual measures, but such that if one translated each rectangle R_i along its long axis by (say) three times its length, the resulting rectangles $\tilde{R}_1, \dots, \tilde{R}_n$ were disjoint. On the other hand, after suitable rescaling of this family of rectangles and taking N large enough, it is possible to construct a wave packet f_i supported on each shifted rectangle $\tilde{R}_i$ such that the two-dimensional ball multiplier S_N, when applied to f_i, became large on the rectangle R_i. By testing S_N on a suitable linear combination of these wave packets f_i, one can demonstrate the unboundedness of S_N for any $p > 2$, and a duality argument then handles the remaining case $p < 2$.

The fundamental connection between higher-dimensional Fourier analysis and Kakeya-type questions has guided much of the subsequent work in the area; it is now standard practice to control oscillatory integral operators (such as S_N) by first decomposing the functions involved into wave packets such as the functions f_i mentioned above, apply the operators to each wave packet individually, and use a combination of Fourier analytic methods and geometric analysis to control the superposition of these operators. A recent triumph of these sets of techniques has been the resolution last year of the Vinogradov main conjecture in analytic number theory by Bourgain, Demeter, and Guth, which gives near-optimal bounds on the mean values of exponential sums such as

$$\sum_{n=1}^{N} e^{2\pi i(\alpha_1 n + \alpha_2 n^2 + \dots + \alpha_k n^k)}$$

as one varies the frequencies $\alpha_1, \dots, \alpha_k$ and which has important consequences in analytic number theory.

In 1973 Fefferman revisited Carleson's theorem on the pointwise convergence of Fourier series and gave a striking new proof that was a key impetus for the modern field of *time-frequency analysis*. Whereas Carleson's argument focused on carefully decomposing the function f, Fefferman's strategy proceeded by instead decomposing the operator C. The first step was to linearize this operator by replacing it with the infinite family of operators

$$\begin{aligned} S_{N(\cdot)}(f)(x) &:= S_{N(x)} f(x) \\ &= \int_{\mathbb{R}/\mathbb{Z}} f(x-t) \frac{\sin((N(x)+\frac{1}{2})t)}{\sin(t/2)}\, dt, \end{aligned}$$

where N ranged over all measurable functions $N : \mathbb{R}/\mathbb{Z} \to \mathbb{Z}$ from the circle to the integers. As no regularity hypotheses are placed on the function N, this integral operator is very rough and seemingly hopeless to attack by conventional harmonic analysis methods. However, guided by key examples of these functions N, Fefferman realized that the graph of the function N (viewed as a subset of *phase space* $\mathbb{R}/\mathbb{Z} \times \mathbb{Z}$) could be used as a sort of road map to efficiently decompose the operator $S_{N(\cdot)}$ into components indexed by phase space rectangles (or tiles), which could then be organized into trees and then forests, the contributions of which could be estimated by a combination of clever combinatorial arguments and the almost orthogonality of various pieces of the operator. This highly original argument took some time to be properly integrated with the rest of the subject, but through the work of Michael Lacey, Christoph Thiele, and others, starting in the late 1990s, the techniques of Carleson and Fefferman were unified with more traditional tools from Calderón–Zygmund theory to obtain a systematic set of methods in time frequency analysis to control highly singular operators that had previously been out of reach, such as the bilinear Hilbert transform of Calderón, as well as many variants of relevance to ergodic theory or to scattering theory.

Selected Honors and Distinctions of Charles Fefferman

Raphael Salem Prize 1971
American Academy of Arts and Sciences 1972
Alan T. Waterman Award 1976
Fields Medal 1978
National Academy of Sciences 1979
Doctor *Honoris causa*: University of Maryland 1979
American Philosophical Society 1985
Doctor *Honoris causa*: Universidad Autónoma de Madrid 1990
Bergman Prize 1992
Bôcher Prize 2008[a]
Wolf Prize 2017

[a] See article in April 2008 *Notices*, www.ams.org/notices/200804/tx080400499p.pdf.

Louis Nirenberg

Charlie Fefferman on PDEs

As is well known, Charlie Fefferman was a child prodigy: when he was in fourth grade he read, and understood, math books going up to calculus. He received his PhD at Princeton at age twenty. His adviser was Eli Stein, also a wonderful mathematician and mentor.

I first met Charlie around 1972, shortly after he was made full professor at the University of Chicago at age twenty-two—the youngest full professor anywhere. I still remember listening with great pleasure to his invited

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Louis Nirenberg (seated), Charlie, and Elias Stein at a celebration of Nirenberg's Abel Prize, Courant Institute, New York, circa 2015.

address at the International Congress in Vancouver in 1974 on recent progress in classical Fourier analysis.

Charlie has made fundamental contributions in an enormously wide range of subjects: harmonic analysis—he's a world master; analysis on complex manifolds; microlocal analysis and linear partial differential (and pseudodifferential) operators; quantum mechanics; fluid flow, Euler, and Navier-Stokes equations; and generalizing Whitney's extension theorem.

I will confine myself to a few topics connected with partial differential equations and related things. In 1971 Charlie proved that the space of functions BMO is the dual to the Hardy space H^1.

F. Treves and I presented a condition, P, which we conjectured would be necessary and sufficient for local solvability of linear PDEs of principal type. We proved this in the case that the real and imaginary parts of the leading coefficients are real analytic. Shortly afterwards Charlie and R. Beals proved the sufficiency of P in the nonanalytic case.

Charlie, in collaboration with A. Córdoba and D. H. Phong, wrote a series of deep papers connected with microlocal analysis and PDEs. In 1985 Charlie wrote a beautiful expository paper on "the uncertainty principle" describing some of their work. It contains a wealth of deep results; I recommend that all students studying PDE read it. If a function u is mainly concentrated in a box Q and its Fourier transform is concentrated in a box Q', one says that u is microlocalized in $Q \sharp Q'$. The uncertainty principle says, essentially, that $|Q| \cdot |Q'| \geq 1$. They study more complicated regions B_a (than the cubes) and decompose L^2 functions into a sum of components microlocalized using the B_a. The decomposition is used to diagonalize pseudodifferential operators modulo small errors. There are applications to solvability, to fundamental solutions, and to Schrödinger equations. Connections with Egorov's theorem and quantum theory are given. Using this a pseudodifferential operator is reduced to a multiplier. Their work also connects with symplectic geometry. It is simply striking.

…why states of matter, such as molecules, form at suitable temperatures…

Charlie has written many deep papers on mathematical physics. A long one, in 1986, studies the quantum mechanics of N electrons and M nuclei. He tries to understand why states of matter, such as molecules, form at suitable temperatures. He obtains very striking results, but the general problem is still open. Things depend on the lowest eigenvalue of a Hamiltonian. With L. Seco he wrote a series of interesting papers. They obtain asymptotic formulas for the ground state of a nonrelativistic atom. Very refined estimates for the eigenvalues and eigenfunctions of an ODE are obtained.

Charlie also wrote a number of interesting papers in fluid dynamics with various co-authors. With Donnelly he wrote a lovely paper on nodal sets for real eigenfunctions F satisfying $\Delta F + aF = 0$ on a compact connected Riemannian manifold. Near the nodal set N (where F vanishes) they prove that F vanishes on N to at most order $c|a|^{1/2}$. The paper contains a variety of results, including behavior near zeros of holomorphic functions in higher dimensions.

I would like to add that I consider Charlie one of the deepest and most brilliant mathematicians I have ever met. He is also an excellent speaker, and his papers are a pleasure to read. He and his wife, Julie, have been close friends all these years.

Joseph J. Kohn

Fefferman's Contributions to the Theory of Several Complex Variables

Charles L. Fefferman has made numerous fundamental contributions to the theory of several complex variables (SCV). His published papers have had a major impact on the field. His expository writing, his lectures, and his discussions have provided students, colleagues, and collaborators with much inspiration and deep insights into the subject. He obtained many seminal results in SCV covering a large range of topics; it is not possible to give a coherent description of all these within the limits of this article. Here I will briefly describe some of the highlights of his groundbreaking work.

Fefferman's first contribution to complex analysis was a remarkable achievement: He proved boundary regularity of holomorphic mappings using the Bergman metric. His proof is highly original, it is a tour-de-force. It is described in more detail by Stein above.

One of the principal themes in SCV is the study of domains of holomorphy in $\mathbb{C}^n$. Let $D \subset \mathbb{C}^n$ be a domain with smooth boundary ∂D. Then it is a domain of holomorphy if and only if it is pseudoconvex, which means that the Levi form is nonnegative. Great progress has been made in the study of the analysis and geometry associated with strictly pseudoconvex domains (that is, when the Levi form is positive definite), much of it due to Fefferman's groundbreaking research. In "The Bergman kernel and biholomorphic mappings of pseudoconvex domains," Fefferman proves a fundamental result: if $F : D_1 \to D_2$ is a biholomorphic map between two strictly pseudoconvex domains, then it is smooth up to the boundary of D_1. This result is a generalization of the Riemann mapping theorem; it enables one to attach invariants to the boundary points, providing a powerful hold on the classification problem. Fefferman's proof of this result involves a profound analysis of the Bergman kernel and metric; it is sketched in Stein's section.

Fefferman continues his study of strictly pseudoconvex domains in "Monge-Ampère equations, the Bergman kernel, and geometry of pseudoconvex domains." Here he studies their geometry, in particular the behavior of chains—curves on the boundary that are preserved by biholomorphic maps; they are analogous to geodesics in Riemannian geometry. To compute the Bergman metric and its associated Hamiltonian, Fefferman considers the Dirichlet problem for the following Monge-Ampère equation:

$$J(u) = (-1)^n \det \begin{pmatrix} u & u_{\bar{k}} \\ u_j & u_{j\bar{k}} \end{pmatrix}_{i \le j,k \le n} = 1$$

in D with $u = 0$ on ∂D. Then the Bergman kernel is replaced by $C_n(u(z))^{-(n+1)}$ with u as above. The difficulty here is that it is not known whether such a u exists. Nevertheless, Fefferman finds a method of finding formal approximate solutions, and with these, using the smoothness of biholomorphic mappings, he actually computes the metric ds^2 and its associated Hamiltonian. He uses this to show some surprising behavior of chains.

Fefferman's study of these problems continues in the monumental paper "Parabolic invariant theory in complex analysis." His analysis starts with an analogy between Riemannian manifolds M and strictly pseudoconvex domains D. An isometry of M in normal coordinates is a rotation in $O(n)$. Thus the problem of determining whether two Riemnnian manifolds are isometric is reduced to finite dimensions. In Fefferman's setting the analogue of $O(n)$ is the group H^+ of all linear fractional transformations of $\mathbb{C}^n$ which preserve ∂D_0, where D_0 is an approximation of D. If $\Phi : D \to \tilde{D}$ is a biholomorphism, then the Moser normal form at $p \in \partial D$ corresponds to the Moser normal form at $\Phi(p)$ by an element of H^+. So, analogously to the Riemannian case, the problem of deciding whether two strictly convex domains are biholomorphic is reduced to finite dimensions. This leads to certain invariant polynomials related to the Moser normal form. The Bergman kernel on D is analogous to a heatlike kernel, $K_t(x,y)$, on M. This K_t has an asymptotic expansion, and $K_t(x,x) = c_n t^{-\frac{n}{2}} \cdot \{1 + \sum_{k+1} \gamma_k(x) t^k\}$, where the γ_k are invariants determined by the Riemannian metric. Analogously for $D = \{\psi > 0\}, K_D(z,z) = \phi(z)/\psi^{n+1} + \tilde{\phi}(z)\log\psi$. The Taylor expansions of $\tilde{\phi}$ and ϕ modulo $O(\psi^{n+1})$ are uniquely determined by the Taylor expansion of ∂D. To carry out the analogy between the heat kernel and the Bergman kernel, Fefferman finds a function which is the analogue of t. This analogue is an approximate formal solution of the Monge-Ampère equation $J(u) = 1$. To complete the analysis Fefferman must overcome enormous difficulties which arise from the fact that while Weyl's analysis is based on the semisimple group $O(n)$, Fefferman has to deal with the group H^+, which is not semisimple. This is not only an impressive technical feat but it introduces seminal original methods to the subject.

Among Fefferman's many important contributions to the study of strictly pseudoconvex domains is his 1985 joint paper with Harold Donnelly, "Fixed point formula for the Bergman kernel." They prove the following. Let Ω be a strictly pseudoconvex domain and $K(z,w)$ the associated Bergman kernel. Suppose that $\gamma : \Omega \to \Omega$ is a holomorphic automorphism having no fixed points on $\partial\Omega$. Then the fixed point set of γ consists of a finite number of points $p_1, p_2, \dots, p_k$. Denote by γ_{*j} the holomorphic differential of γ at the point p_j. Let I denote the identity endomorphism and $J_{\gamma(z)}$ the holomorphic Jacobian of γ. Then

$$\int_\Omega \overline{K(z,\gamma z)} J_\gamma(z) dz = \sum_j \frac{(-1)^n}{det(I - \gamma_{*j})} .$$

This formula is then applied to study circle actions on Ω.

Fefferman has made several contributions to geometry which are relevant to the understanding of CR structures, as in the 2003 joint paper with Kengo Hirachi, "Ambient metric construction of Q-curvature in conformal and CR geometries." All the work described above concerns

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the study of strictly pseudoconvex domains and CR manifolds. He has also made major contributions to the weakly convex case, which are described briefly below.

The local analysis of strictly pseudoconvex domains and CR manifolds is driven by the approximation of D by D_0 mentioned above. In the general pseudoconvex case there is no analogous method. I have been privileged to collaborate with Fefferman in working on some of these problems. In a beautiful expository paper, "Kohn's microlocalization of $\bar{\partial}$ problems," Fefferman gives the background for our results. Our work deals with Hölder estimates for the operators $\bar{\partial}$ and $\bar{\partial}_b$, the associated Laplacians, and projection operators. The starting point for the analysis of $\bar{\partial}_b$ on ∂D and on CR manifolds is the local subelliptic estimate

$$\|\varphi\|_\varepsilon^2 \leq C(\|\bar{\partial}_b\varphi\|^2 + \|\bar{\partial}_b^*\varphi\|^2),$$

where the φ are $(0,1)$-forms in $C_0^\infty(U)$ in an open set U. For the analysis of $\bar{\partial}$ the starting point is an analogous estimate on $C_0^\infty(U \cap \overline{D})$ with appropriate boundary conditions on $U \cap \partial D$. These estimates imply C^∞ regularity. The problem is to prove Hölder regularity. This regularity is proved in the special case where the Levi form is locally diagonalizable, which means that the Levi form can be expressed in terms of local tangential $(1,0)$ vector fields $L_1, \dots, L_{n-1}$ by $c_{ij} = \mathcal{L}(L_i, \bar{L}_j)$, and it is diagonalizable if there exist Ls so that $c_{ij} = \lambda_i \delta_{ij}$. Diagonalizability clearly holds when $n = 2$, but in case $n > 2$ it is a severe restriction. This work and an extension to the three-dimensional case is contained in a collection of joint papers (one including M. Machedon) published in 1988–1990. In the general nondiagonalizable case it remains a difficult problem to prove Hölder estimates. The first obstacle is that in these cases the subelliptic estimates do not fit the CR structure, and thus different and much stronger estimates are needed for a starting point. In Fefferman's remarkable paper "The uncertainty principle" and in subsequent work with D. H. Phong, powerful new methods of proving a priori estimates in PDE are established. These give insights into the type of problems discussed here and, hopefully, will lead to a solution.

Sun-Yung Alice Chang and C. Robin Graham

Fefferman's Work in Conformal Geometry

As Charlie's colleagues and collaborators, we feel privileged to have had the opportunity to work with him. His contributions to conformal geometry are centered around the ambient metric construction. We begin with a discussion of some background leading up to its discovery.

The conformal ambient metric grew out of Charlie's work in several complex variables, in particular in his efforts to understand the asymptotic expansion of the Bergman kernel. As Charlie had previously shown, the restriction to the diagonal of the Bergman kernel K of a smooth, bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ can be written in the form

$$K(z,z) = \varphi(z)\rho(z)^{-n-1} + \psi(z)\log\rho(z),$$

where ρ is a smooth defining function for $\partial\Omega$ and φ, $\psi \in C^\infty(\overline{\Omega})$. This expansion can be viewed by analogy to the asymptotic expansion of the heat kernel of a Riemannian manifold restricted to the diagonal. The heat kernel can be expanded in powers of the time variable t, and the coefficients in the expansion are local scalar invariants of Riemannian metrics, which can be constructed as contractions of tensor products of covariant derivatives of the curvature tensor. The Bergman kernel on the diagonal is determined locally by the boundary up to a smooth function, so it was natural to try to find an analogous expansion for φ to order $n+1$ and for ψ to infinite order. But several problems immediately arose in contemplating carrying this out. One was that Ω is not canonically a product near $\partial\Omega$, so there was no obvious analogue of t, nor was there an obvious way to formulate an expansion in such a way that the coefficients would be geometric invariants of the boundary. But even if one could surmount these difficulties, the most glaring problem was the fact that it was not known how to construct general scalar invariants of CR structures, the geometric structures induced on nondegenerate hypersurfaces by the complex structure on the background $\mathbb{C}^n$.

Charlie resolved these difficulties in his groundbreaking paper "Parabolic invariant theory in complex analysis." His solution was to construct a Lorentz signature, asymptotically Kähler-Einstein metric $\widetilde{g}$ on $\mathbb{C}^* \times \Omega$, where $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$, via a formal solution of a degenerate complex Monge-Ampère equation. This metric $\widetilde{g}$ is invariant under

C. Robin Graham, Charlie Fefferman, and Sun-Yung Alice Chang in 2009 at a conference in Charlie's honor.

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rotations and homogeneous under dilations in $\mathbb{C}^*$ and is invariantly associated to the CR geometry on $\partial\Omega$. The formal solution to the Monge-Ampère equation plays the role of the time variable t, and scalar CR invariants can be constructed using the curvature tensor and the Levi-Civita connection of $\tilde{g}$ in a manner roughly analogous to the case of Riemannian geometry. There are more details of this solution in Joseph Kohn's section.

The conformal ambient metric, introduced in joint work with C. R. Graham, "Conformal invariants," and also denoted $\tilde{g}$, is an analogue in a different setting: it is determined by the datum of a conformal class $(M, [g])$ of Riemannian metrics on a manifold M of dimension $n \geq 3$. Metrics in the conformal class are sections of a ray subbundle $\mathcal{G}$ of the bundle of symmetric 2-tensors on M, and $\tilde{g}$ is a Lorentz signature metric on the ambient space $\tilde{\mathcal{G}} = \mathcal{G} \times \mathbb{R}$ determined asymptotically along $\mathcal{G} \cong \mathcal{G} \times \{0\}$. The model is the sphere S^n, whose group of conformal motions is the Lorentz group $O(n+1,1)$ of linear transformations of $\mathbb{R}^{n+2}$ preserving a quadratic form of signature $(n+1,1)$. The ray bundle $\mathcal{G}$ can be identified with the forward pointing half of the null cone of the quadratic form. The ambient metric for the sphere is just the Minkowski metric on $\mathbb{R}^{n+2}$, which is clearly preserved by the conformal motions in $O(n+1,1)$ in their linear action on $\mathbb{R}^{n+2}$. For a general conformal class of metrics $(M, [g])$, the ambient metric is a Lorentz signature metric on $\tilde{\mathcal{G}}$ determined asymptotically along $\mathcal{G} \times \{0\}$ by the following three conditions:

(i) $\tilde{g}$ is homogeneous of degree two with respect to natural dilations on $\tilde{\mathcal{G}}$,
(ii) $\iota^* \tilde{g} = \mathbf{g_0}$,
(iii) $\mathrm{Ric}(\tilde{g})$ vanishes asymptotically at $\mathcal{G} \times \{0\}$.

Here $\mathbf{g_0}$ is a tautological symmetric 2-tensor on $\mathcal{G}$ determined by the conformal class, and $\iota : \mathcal{G} \to \mathcal{G} \times \{0\} \subset \mathcal{G} \times \mathbb{R}$ is the inclusion. In (iii) the asymptotic order of vanishing is infinite if n is odd and is $n/2 - 1$ if n is even. The ambient metric $\tilde{g}$ is uniquely determined up to diffeomorphism by these conditions: to infinite order if n is odd and to order $n/2$ if n is even.

The conformal ambient metric was inspired by Charlie's construction in several complex variables, but the relationship is closer than mere analogy: the Kähler–Lorentz metric in the several complex variables construction can be viewed as a special case of a conformal ambient metric. One takes the conformal class to be the so-called "Fefferman metric" (there are too many of these!) on $\partial\Omega \times S^1$, which Charlie had constructed earlier. Although one usually thinks of conformal geometry as simpler than CR geometry, by this construction the class of CR structures induced on nondegenerate boundaries of domains in $\mathbb{C}^n$ can be viewed as a subclass of the class of conformal structures on even-dimensional manifolds.

There is a second, equivalent formulation of ambient metrics, namely, as Poincaré metrics. The model here is hyperbolic space. Recall that S^n can be viewed as the boundary at infinity of hyperbolic space, which arises as the restriction of the Minkowski metric on $\mathbb{R}^{n+2}$ to one sheet of the hyperboloid arising as the -1-level set of the Lorentz signature quadratic form. This construction generalizes to the case of a general conformal manifold as the "conformal infinity"; the ambient metric can be restricted to a natural hypersurface in $\tilde{\mathcal{G}}$, and the resulting Poincaré metric has asymptotically constant negative Ricci curvature.

The ambient metric opened up new arenas for study in geometric analysis.

These constructions have been enormously influential; the ambient and Poincaré metrics are now viewed as fundamental in conformal geometry and beyond. As described above, one of Charlie's original motivations for the construction in CR geometry was to construct and characterize scalar invariants of CR structures to describe the asymptotic expansion of the Bergman kernel. The ambient metric enables construction of scalar conformal invariants as Weyl invariants, constructed as linear combinations of complete contractions of covariant derivatives of the curvature tensor of the ambient metric. Determining the extent to which all invariants arise by this construction involved developing a new parabolic invariant theory; this was carried out in joint work with Graham and by Graham with Bailey and Eastwood.

The ambient metric opened up new arenas for study in geometric analysis. Conformally invariant powers of the Laplacian were constructed in terms of the ambient metric, leading to Branson's construction of Q-curvature. Q-curvature is a higher-dimensional version of scalar curvature in dimension 2. Branson's original definition proceeded by analytic continuation in the dimension, and Q curvature was originally regarded as rather mysterious. Charlie, in separate papers with Graham and Hirachi in the first years of this century, helped illuminate its nature. Q curvature enters into the Gauss–Bonnet integrand in higher dimensions (albeit in a more complicated way than the scalar curvature in dimension 2). In dimension 4 and modulo the part which is pointwise conformally invariant, the Gauss–Bonnet integrand has fully nonlinear structure under conformal change of metric. The analytic study of such partial differential equations has become a central topic in conformal geometry.

The fundamental idea underlying the ambient/Poincaré metric is to study geometry in dimension n by passing to a different but essentially equivalent description in dimension $n+1$ or $n+2$. The AdS/CFT correspondence in physics, a major development since its introduction by Maldacena in 1997, is based on the same idea. In fact, the Poincaré metric construction amounts to the geometry underlying the AdS/CFT duality between conformal field theories on a boundary at infinity and supergravity in the bulk. This synergy between geometry and physics has stimulated both fields and continues to be a source of exciting developments today.

Diego Córdoba

Fefferman on Fluid Dynamics

The search for singularities in incompressible fluids has become a major challenge in the area of nonlinear and nonlocal partial differential equations. In particular the existence, or absence, of finite-time singularities with finite energy remains an open problem for 3D incompressible Euler equations in the whole domain $\mathbb{R}^3$ or in the periodic $\mathbb{T}^3$ setting. The local existence in time of classical solutions is well known, and in dimension two these classical solutions exist for all time. A result of Beale, Kato, and Majda asserts that if a singularity forms at time T, then the vorticity $\omega(x,t)$ grows so rapidly that

$$\int_0^T sup_x|\omega(x,t)|dt = \infty. \tag{7}$$

Notice that in dimension two the vorticity ω remains bounded, since it is transported by the flow; this is a main difference with the situation in dimension three.

Fefferman's interest in fluid dynamics started at the beginning of the 1990s in a collaboration with Constantin and Majda. They showed that if the velocity remains bounded up to the time of singularity formation, then the vorticity direction $\frac{\omega(x,t)}{|\omega(x,t)|}$ cannot remain uniformly Lipschitz continuous up to that time. A similar result was proven in the presence of viscosity. This result is extremely useful, since vortex lines are transported by the 3D incompressible Euler flow. Recall that a vortex line in a fluid is an arc on an integral curve of the vorticity $\omega(x,t)$ for fixed t. In numerical simulations of 3D Euler solutions, one routinely sees that vortex tubes (tubular neighborhoods arising as a union of vortex lines) grow longer and thinner while bending and twisting. In particular, if the thickness of a piece of a vortex tube becomes zero in finite time, then one has a singular solution of 3D Euler. I collaborated with Charlie to prove that given a criterion only at the level of the velocity field then a vortex tube cannot reach zero thickness in finite time, unless it bends and twists so violently that no part of it forms a "smooth" tube anymore. If additionally we add the assumption that there is a uniform collapse, in the sense that the maximum and the minimum thickness are comparable, then we can obtain a lower bound on the rate of decay of the thickness of the tube.

A key ingredient in the success of proving the formation of singularities is to identify a scenario in which there is a clear mechanism developing fast vorticity growth, and accurate numerics plays a crucial role in this search. Below we describe two plausible singular scenarios discovered first by numerical simulations which led later to a rigorous proof: splash singularities for water waves and shift stability for the dynamics of the interface between two immiscible incompressible fluids in a porous medium.

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Charlie with Antonio Córdoba and Antonio's son, Diego Córdoba, both PhD students of Fefferman.

Splash and Splat Singularities

In the case of the incompressible Euler equations with a free boundary, Charlie, in collaboration with Castro, Gancedo, Gómez-Serrano, and me, established the formation in finite time of splash and splat singularities for the water wave problem. A splash singularity appears when the free boundary remains smooth but self-intersects at a point; a splat singularity is when it self-intersects along an arc. The difference with the fixed domain case is that the pressure is constant at the boundary, and the boundary moves with the velocity of the fluid. Therefore finding the domain of the fluid is part of the problem. The main idea of the proof is to choose a conformal map ϕ which separates the point of collapse, such that its singular points (where ϕ cannot be inverted) are located outside the domain of the fluid. This map transforms the splash into a closed curve whose chord-arc is well defined. We can select an initial velocity that immediately separates the point of collapse because the equations are reversible in time and we can solve backwards in time. In order to return to the original domain and obtain solutions to the free boundary incompressible Euler equations, it is necessary to invert the map ϕ. In the presence of viscosity the proof has to be modified since the equations are no longer reversible; we use again the conformal map ϕ that separates the self-intersecting points of the splash curve. But instead of showing local existence backwards in time in the transformed domain, we prove local existence forward in time and show that the solutions depend stably on the initial conditions. We apply a perturbative argument to prove a splash, but not a splat, singularity for Navier–Stokes.

In recent work Fefferman, in collaboration with Ionescu and Lie, showed that the presence of a second fluid prevents the formation in finite time of both splash and splat singularities. The condition that the densities of the fluids are positive is used to show a critical L^∞ bound for the measure of the vorticity in the boundary which prevents self-intersections of the boundary between the fluids.

Shift of Stability and Breakdown

Charlie has also worked on the dynamics of the interface between two incompressible viscous fluids with different densities in a porous medium, which is modeled by Darcy's law. This problem is also known as the Muskat problem. If the heavier fluid is below the light fluid the system is well posed in a Sobolev space H^k, but if the interface is not a graph the system is unstable. Together with collaborators Castro, Gancedo, and López-Fernández we have proven that there exists a nonempty open set of initial data in the stable regime (heavier fluid below the light fluid for which the interface is initially in H^4), such that the solution of the Muskat problem becomes immediately real-analytic and then passes to the unstable regime in finite time. Moreover, the Cauchy–Kowalewski theorem shows that a real-analytic Muskat solution continues to exist for a short time after the turnover. The interface becomes more and more unstable as the turnover progresses. In fact, there exist interfaces of the Muskat problem such that after turnover their smoothness breaks down. The proof follows by a rigorous analysis of the full nonlinear problem and establishing analytic continuation of Muskat solutions to the time-varying strip of analyticity.

Bo'az Klartag

Fefferman's Work on the Whitney Extension Problem

Given an arbitrary set $E \subseteq \mathbb{R}^n$ and a function $f : E \to \mathbb{R}$ and an $m \geq 1$, does there exist a C^m-smooth function $F : \mathbb{R}^n \to \mathbb{R}$ that extends f (i.e., $F|_E = f$)? A problem attributed to Hassler Whitney seeks to find plausible necessary and sufficient conditions for the feasibility of such a C^m-extension.

In the 1930s Whitney proved that the function f extends to a C^m-smooth function on the entire real line if and only if for any accumulation point x of the set E, the difference quotient of order m,

$$[x_0, \dots, x_m](f) := m! \cdot \sum_{i=0}^{m} \left(\prod_{j \neq i} \frac{1}{x_i - x_j} \right) f(x_i),$$

converges to a finite limit as the distinct points $x_0, \dots, x_m \in E$ tend to the point x. This finite limit will be the m^{th}-derivative at x of any C^m-smooth function that extends f. In fact, $[x_0, \dots, x_m](f)$ equals the m^{th}-derivative of the Lagrange interpolation polynomial P of degree m with $P(x_i) = f(x_i)$ for all i.

The extension problem in $\mathbb{R}^n$ for $n \geq 2$ is more difficult, as there could be many potential candidates for the m^{th}-order Taylor polynomial of the extension function.

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Bo'az Klartag and Charlie at Princeton circa 2015.

We first discuss the case of $C^{m,1}$-functions, which are C^m-smooth functions $F : \mathbb{R}^n \to \mathbb{R}$ for which the norm

$$\|F\|_{C^{m,1}} := \sup_{x \in \mathbb{R}^n} \max_{|\alpha| \leq m} |\partial^\alpha F(x)| + \sup_{x \neq y \in \mathbb{R}^n} \max_{|\alpha| = m} \frac{|\partial^\alpha F(x) - \partial^\alpha F(y)|}{|x - y|} \tag{8}$$

is finite. Here, the multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$ is a vector of nonnegative integers, $\partial^\alpha F(x) = (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_n})^{\alpha_n} F(x)$ and $|\alpha| = \sum_i \alpha_i$. In a seminal work from the early 2000s, Charlie Fefferman proved the following finiteness principle, which had been conjectured earlier by Brudnyi and Shvartsman:

Theorem 1. *Let $n \geq 1, m \geq 0$, and let $E \subseteq \mathbb{R}^n$ and $f : E \to \mathbb{R}$ be arbitrary. Assume that there exists $M > 0$ with the following property: For any finite subset $S \subseteq E$ of size at most $k(m,n) < \infty$, the function $f|_S$ extends to an auxiliary $C^{m,1}$-function $F_S : \mathbb{R}^n \to \mathbb{R}$ with $\|F_S\|_{C^{m,1}} \leq M$. Then there exists a $C^{m,1}$-function $F : \mathbb{R}^n \to \mathbb{R}$ with $F|_E = f$. Moreover, $\|F\|_{C^{m,1}} \leq CM$ for some constant $C = C(m,n)$.*

Thus in order to tell whether f extends to a $C^{m,1}$-function on the entire $\mathbb{R}^n$, it suffices to consider subsets of E with at most $k(m,n)$ elements. The optimal finiteness constant $k(m,n)$ is not known in general.

Merely a year or two following his proof of Theorem 1, Fefferman was able to resolve Whitney's C^m-extension problem in another remarkable work. The reader is referred to Fefferman's 2009 *Bulletin* article on the topic for a precise formulation of his C^m-theorem, which involves the notion of Glaeser refinement of certain fiber bundles over an arbitrary set $E \subseteq \mathbb{R}^n$, where the fiber of the bundle at a base point $x \in E$ consists of potential Taylor polynomials at x of a C^m-extension function. This form of Fefferman's solution is closely related to an earlier work by Bierstone, Milman, and Pawłucki dealing with a subanalytic set E. Glaeser himself settled the case $m = 1$ in the 1950s.

The next question is whether one can turn Fefferman's constructive proof into an actual algorithm for a nearly optimal C^m-interpolation of data. From the viewpoint of computer science, the problem may be formulated as follows. We are given a large, finite subset $E \subset \mathbb{R}^n$ and a function $f : E \to \mathbb{R}$. For $m \geq 1$ define

$$\|f\|_{C^m(E)} = \inf\{\|F\|_{C^m(\mathbb{R}^n)};\; F : \mathbb{R}^n \to \mathbb{R} \text{ is } C^m\text{-smooth with } F|_E = f\},$$

where $\|F\|_{C^m(\mathbb{R}^n)} = \|F\|_{C^{m-1,1}}$ for a C^m-smooth function $F : \mathbb{R}^n \to \mathbb{R}$. Our goal is to compute, using an (idealized) digital computer, a function $F : \mathbb{R}^n \to \mathbb{R}$ with $F|_E = f$ and $\|F\|_{C^m(\mathbb{R}^n)} \leq C\|f\|_{C^m(E)}$, for $C > 0$ being a constant depending only on m and n. In a joint work with Klartag, Fefferman has transformed his proof of Theorem 1 into an algorithm with the following interface:

(i) First, the user enters the entire data set (the coordinates of the points of E and the values of f) into the computer. The computer then works for a while, performing at most $CN \log N$ operations where $N = \#(E)$, after which it signals that it is ready to accept further input.

(ii) Then, whenever the user enters the coordinates of a point $x \in \mathbb{R}^n$, the computer rapidly responds, using only $C \log N$ operations with the value $F(x)$.

The function F that is computed by the algorithm is guaranteed to be an extension of f, with $\|F\|_{C^m(\mathbb{R}^n)} \leq C\|f\|_{C^m(E)}$. The storage required by the algorithm is bounded by CN. Of course, the handling of real numbers in the digital computer is only up to finite precision; the algorithm returns its answers in the given precision of the digital computer. The reader might wonder why $CN \log N$ operations suffice when Theorem 1 requires an inspection of all subsets of E of size $k(m,n)$. As it turns out, there exists a list of no more than CN subsets, each of size no greater than C, that matter, as proven by Fefferman. Extensions of his result exist for smooth selection and for interpolating multidimensional functions with noisy data. Particularly notable is the extension to Sobolev spaces; in a series of papers Fefferman, Israel, and Luli proved the existence of an extension and described an algorithm to construct it. That algorithm shares features with the C^m-algorithm described above: the preprocessing time is $CN \log N$, and the query time is only $C \log N$.

We move on to discuss some of Fefferman's work on the $(1+\varepsilon)$-version of the quantitative Whitney problem, where one is given a small $\varepsilon > 0$ and is seeking a C^m-function $F : \mathbb{R}^n \to \mathbb{R}$ extending the given function $f : E \to \mathbb{R}$ such that

$$\|F\|_{C^m(\mathbb{R}^n)} \leq (1+\varepsilon) \cdot \|f\|_{C^m(E)}. \tag{9}$$

The $C^m(\mathbb{R}^n)$-norm is usually considered only up to a multiplicative constant; some authors define the C^m and $C^{m,1}$-norms a bit differently, replacing the sum in (8) by a maximum. Nevertheless, from the point of view of data interpolation, it makes sense to fix a reasonable C^m-norm and ask for an efficient algorithm that produces a function F satisfying (9), given a finite set $E \subseteq \mathbb{R}^n$ and function values $f : E \to \mathbb{R}$.

Fefferman devised a polynomial-time algorithm for carrying out this task for all m and n. In the case of $C^2(\mathbb{R}^2)$, the preprocessing time is at most $C(\varepsilon)N \log N$, the query work is $C \log(N/\varepsilon)$, and the storage $C(\varepsilon)N$. Moreover, Fefferman has also come up with a $(1+\varepsilon)$-version of Whitney's extension theorem, whose proof uses the notion of a "gentle partition of unity." For simplicity, we restrict our attention to the central case of a finite set E in the following formulation of his theorem:

Theorem 2. *Let $m, n \geq 1$ and $\varepsilon > 0$ be given, and let $E \subseteq \mathbb{R}^n$ be a finite set. Let $(P_x)_{x \in E}$ be a family of polynomials of degree m in n real variables.*

Assume that for any $S \subseteq E$ with $\#(S) \leq k^\#(m,n,\varepsilon)$ there exists an auxiliary C^m-smooth function $F_S : \mathbb{R}^n \to \mathbb{R}$ with $\|F_S\|_{C^m(\mathbb{R}^n)} \leq 1$, such that for any $x \in S$, the m^{th}-order Taylor polynomial of F_S at the point x, denoted by $J_x(F_S)$, satisfies $J_x(F_S) = P_x$. Then there exists a C^m-smooth function $F : \mathbb{R}^n \to \mathbb{R}$ with $J_x(F) = P_x$ for all $x \in E$ such that $\|F\|_{C^m(\mathbb{R}^n)} \leq 1 + \varepsilon$.

We could go on and on. Only space limitations prevent us from describing additional Whitney-related theorems that were published by Fefferman in recent years. We would like to conclude by wishing Charlie and the mathematical community many more beautiful results on the subject in years to come!

Postscript: In June 2017 Fefferman and Shvartsman announced their proof of a finiteness principle for Lipschitz selection in an arbitrary metric space. That is, assume that we are given a metric space X and a compact, convex set $K_x \subseteq \mathbb{R}^d$ associated with any point $x \in X$. Assume that for any subset $S \subseteq X$ with $\#(S) \leq 2^d$ there exists a 1-Lipschitz map $F_S : S \to \mathbb{R}^d$ with $F_S(x) \in K(x)$ for all $x \in S$. Then there exists a C-Lipschitz map $F : X \to \mathbb{R}^d$ with $F(x) \in K(x)$ for all $x \in X$, where C is a constant depending only on d.

Jürg Fröhlich, Luis Seco, and Michael Weinstein

Charlie's Romance with Quantum Theory

Our memories of Charlie Fefferman go back to the middle of the seventies when Jürg and Charlie arrived at Princeton. Barry Simon, then at Princeton, had announced that he had a "secret weapon" that might well enable him to construct models of local relativistic quantum fields in four-dimensional Minkowski space—a dream of many mathematical physicists working in constructive quantum field theory, *yet to come true.* It turned out that Barry's secret weapon was Charlie! He was convinced that if Charlie were willing to put his mind to problems in relativistic quantum field theory success in solving

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them would soon be within reach. Alas, Barry's hopes did not materialize. Luckily we all recovered from this mishap and went on to solve other interesting, albeit somewhat less challenging, problems in quantum theory. For example, Charlie and his former student Antonio Córdoba developed the so-called wave-packet transform, an analytical tool built upon Heisenberg's uncertainty relations.

There followed a long period when Jürg's and Charlie's trajectories did not intersect. In the mid-eighties, after Jürg had moved back to ETH in Zürich, he heard of an exciting discovery: Charlie had solved the problem of understanding why, at appropriate temperatures and densities, atomic nuclei and electrons form gases of neutral atoms or molecules. His result appeared under the title "The atomic and molecular nature of matter" in the first issue of the first volume of a new journal that A. Córdoba had founded, *Rev. Mat. Iberoamericana*. The abstract of the paper reads: "The purpose of this article is to show that electrons and protons, interacting by Coulomb forces and governed by quantum statistical mechanics at suitable temperature and density, form a gas of Hydrogen atoms or molecules."

…his distinctive style in analysis: at once powerful, bare-hands, and elegant

At around that time, Luis arrived at Princeton as a graduate student. He was closely familiar with Charlie's previous work on harmonic analysis, but he had no idea what an electron was. One of the unwritten rules of the Princeton mathematics department is that it does not offer graduate courses on subjects where there is already a good book. As a consequence, professors teach courses on whatever topic they just spent their time working on, and it is not unusual that some of their colleagues attend the lectures.

In September 1984 Michael introduced himself to Charlie as they were riding on a Fine Hall elevator and said he was new at Princeton. Charlie introduced himself to Michael and said he had been there forever. He gave Michael his coordinates and encouraged him to stop by to chat. That semester Michael followed Charlie's course, advertised in the catalogue as Fourier Analysis on Groups, which turned out to be about Charlie's ongoing work on stability of matter, treated quantum-mechanically, first established by Dyson and Lenard and later by Lieb and Thirring and others. His course was also attended by another assistant professor, Rafael de la Llave, who was Charlie's collaborator in his work "Relativistic stability of matter." It was through these wonderful lectures that Charlie introduced Michael to deep mathematical problems in quantum theory and to his distinctive style in analysis: at once powerful, bare-hands, and elegant. And then there was Luis, taking his first course with Charlie after arriving at Princeton.

Here is what stability of matter means: Consider a system in physical space $\mathbb{R}^3$ consisting of M static nuclei at positions y_j with atomic numbers (charges) Z_j, $j = 1, \dots, M$, and N electrons at positions x_k, $k = 1, \dots, N$, interacting through Coulomb forces. The Hamiltonian operator of such a system is given by

$$H_{Z,M,N} = \sum_{k=1}^{N} (-\Delta_{x_k}) + V_{Z,M,N}. \tag{10}$$

The Laplacian terms represent the kinetic energy of the electrons (their mass being set to $1/2$), while the second term on the right side is the Coulomb potential energy, which is given by the multiplication operator

$$V_{Z,M,N} = \sum_{j<k} \frac{1}{|x_j - x_k|} + \sum_{j<k} \frac{Z_j Z_k}{|y_j - y_k|} - \sum_{j,k} \frac{Z_k}{|x_j - y_k|}. \tag{11}$$

The Hamiltonian $H_{Z,M,N}$ acts on the Hilbert space $\mathcal{H}_N$ given by the N-fold antisymmetric tensor product of the one-electron Hilbert space, $L^2(\mathbb{R}^3, \mathrm{d}^3x) \otimes \mathbb{C}^2$, where $L^2(\mathbb{R}^3, \mathrm{d}^3x)$ is the space of orbital wave functions of an electron and $\mathbb{C}^2$ is the space of its spin states.

One also considers the pseudo-relativistic version of the Hamiltonian, with $H_{Z,M,N}$ now given by

$$H_{Z,M,N} = \sum_{k=1}^{N} (-\Delta_{x_k})^{1/2} + \alpha \cdot V_{Z,M,N}. \tag{12}$$

In this operator the fine structure constant $\alpha \simeq 1/137$ appears as a fundamental parameter, and (12) comes with other mathematical licenses that turn the original physical problem into this form, keeping the essential feature that, in relativity theory, kinetic energy is proportional to momentum in the high-energy limit.

Stability of matter is the property that

$$\langle H_{Z,M,N}\psi, \psi\rangle \geq -C_Z \cdot (M + N) \tag{13}$$

in the nonrelativistic case (10), and

$$\langle H_{Z,M,N}\psi, \psi\rangle \geq 0 \tag{14}$$

for the relativistic Hamiltonian (12), where ψ is an arbitrary wave function in $\mathcal{H}_N$ of norm 1. In the latter case, the critical role of the fine structure constant becomes clear after one realizes that in (12), the Coulomb potential energy and the relativistic kinetic energy scale in the same way, namely, as inverse lengths. Thus, either (14) holds or $\inf\langle H_{Z,M,N}\psi, \psi\rangle = -\infty$. If the bounds (13), (14), respectively, did not hold true one would conclude that matter must collapse and, in the process, release a devastating amount of energy.

Charlie's first landmark achievement in quantum theory was his work with Rafael de la Llave on stability of matter for Coulomb systems described by (10) and their pseudorelativistic cousins, as described by (12). Needless to say, Luis understood nothing of all this, except for one thing: in the relativistic case, Fefferman's philosophy was that, since the scaling properties of the operator in (12) make the problem hard, one had better exploit these properties to one's advantage by rewriting the kinetic energy and the Coulomb potential in a convenient form, as follows:

(15)
$$\frac{1}{|x|} = \frac{1}{\pi} \int\int_{R>0, z\in \mathbb{R}^3} \begin{Bmatrix} 1 & \text{if both } x, 0 \in B(z,R), \\ 0 & \text{otherwise} \end{Bmatrix} \frac{dz\, dR}{R^5},$$

with a similar expression for the Laplacian term. This expression is, in some sense, obvious, since the left and right sides scale in the same way (as an inverse length) and both are translation invariant. The revolutionary insight here is that the left-hand side is hard, but the right-hand side is easy: all it requires is counting the number of electrons or nuclei in a ball. Equation (15) turned out to be a game-changing element, with the strong flavor of microlocal analysis. This attracted Luis to make a move that would seem absurd to an outsider: to do a thesis on mathematical physics. Luis's move began a fruitful collaboration that would last a decade, involved very smart discussions, on-going oversight by de la Llave, and produced a long list of results on Coulomb systems. Their work concerned single atoms—to be precise, atoms with large atomic number Z, setting $M = 1$ in (10). The ground-state energy of such atoms is given by

$$E(Z,N) = \min_{\psi\in\mathcal{H}_N, \|\psi\|=1} \langle H_{Z,N}\psi, \psi\rangle .$$

They adopted the asymptotic perspective that $Z \to \infty$. While, in nature, Z does not get to be much bigger than 100, the Hamiltonian acts on functions that can easily depend on several hundred variables. Much of the mathematical complexity therefore comes from dimensionality considerations that will recreate the flavor of the semiclassical limit, WKB theory, spectral asymptotics, and the lattice-point problem of number theory.

Their first achievement was a proof of asymptotic neutrality of large ions, proving that the number $N(Z)$ of electrons that renders $E(Z,N)$ minimal,

$$E(Z) = E(Z, N(Z)) = \min_N E(Z,N),$$

satisfies the asymptotic expression

$$N(Z) = Z + O(Z^{0.84}).$$

The key ingredient was to perform a deep, rigorous benchmarking of the atomic Hamiltonian with its semiclassical version, the well-known Thomas–Fermi theory. This strategy was repeated in several different papers—in one notable example to obtain asymptotic estimates for the total spin of an atom—but its climax was reached in 1991 when Fefferman and Seco announced their proof of the Dirac–Schwinger conjecture, a long-standing open problem on the altar of mathematical physics:

(16)
$$E(Z) = -c_{TF}Z^{7/3} + c_S Z^2 + c_{DS}Z^{5/3} + o(Z^{5/3}).$$

Each term above is loaded with history, and the last term also signified its last chapter. Its derivation is deep and long, involving interesting elements of number theory and computer assistance. That number theory becomes relevant in the semiclassical limits is easy to understand: the lattice-point problem is encountered in the spectral asymptotics of the Laplacian. Without this fact, the term proportional to $Z^{5/3}$ in (16) would be multiplied by an oscillatory term depending on Z. This paints a number-theoretical perspective on the periodic table of elements.

As these papers on atomic physics were being finished, in the first half of the nineties, Charlie announced that he would visit ETH. He told Jürg that he would like to meet him and find out what he was up to. Jürg told Charlie that he had a PhD student[1] whom he had encouraged to work on a problem that had interested him for well over a decade:

Consider the quantum-mechanical description of a physical system consisting of N nonrelativistic electrons with spin-$\frac{1}{2}$ moving in an arbitrary external magnetic field $\vec{B} = \mathrm{curl}\vec{A}$, where $\vec{A}$ is the electromagnetic vector potential (with $\vec{\nabla}\cdot\vec{A} = 0$), and in the electro-static field of M nuclei with atomic numbers $Z_j \le Z < \infty$, $j = 1, 2, \dots, M$. Let $H_{Z,M,N}(\vec{A})$ denote the Hamiltonian of this system, which is a self-adjoint operator acting on the Hilbert space $\mathcal{H}_N$. The magnetic moment of the electron is proportional to its spin operator. The so-called gyro-magnetic factor, g, of the electron is the factor of proportionality between its spin and its magnetic moment; it has the value $g = 2$, as correctly predicted by Dirac's equation for a relativistic electron. It had already been shown by Barry Simon et al. that if g were smaller than 2 the system would be *stable*, in the sense that the corresponding Hamiltonian satisfies the lower bound

(17)
$$H_{Z,M,N}(\vec{A}) > -CZ^2N,$$

uniformly in the vector potential $\vec{A}$, for a g-dependent finite constant $C > 0$. Furthermore, if $g > 2$ the system is *unstable*, even for a single electron, which is easy to verify. Nature wants us to understand whether the system is stable for $g = 2$!

E. H. Lieb, M. Loss, and H.-T. Yau, partly in collaboration with Jürg, proved in 1985 that for the physical value of the fine-structure constant α, there is a critical value, Z_{crit}, of the atomic number of nuclei with the property that, for $Z > Z_{\text{crit}}$, the system is *unstable* even for only a single electron, quite a subtle result. However, at the time of Charlie's visit in Zurich, it was unknown whether, for the physical values of the g-factor and the fine-structure constant, i.e., for $g = 2$ and $\alpha \simeq \frac{1}{137}$, *stability of matter in magnetic fields* holds, i.e., whether there exists a strictly positive Z_{crit} such that, for $Z_j \le Z < Z_{\text{crit}}$, $j = 1, \dots, M$, the system is stable, in the sense that

(18)
$$H_{Z,M,N}(\vec{A}) > -C_Z(N+M),$$

for a finite constant C_Z, uniformly in $\vec{A}$, for an arbitrary number of electrons and nuclei. To make some relevant steps in the direction of establishing (18) was the project Jürg had proposed to his PhD student. The problem was well known to some of Jürg's friends at Princeton. When Charlie arrived in Zurich, Jürg had not advanced far in unraveling the mathematical subtleties hiding beneath (18)

[1] *Maxi Seifert—nowadays an expert in the field of mathematical finance, the area that Luis also moved into after quantum mechanics.*

yet. But he knew that some of Charlie's techniques might be helpful and he explained the rather poor state of their understanding. Somewhat to his surprise, Charlie had never heard nor thought of this problem, but found the question worth thinking about. *Two weeks after Charlie had returned to the US, he sent Jürg copies of hand-written notes containing a proof of* (18) *for* $Z = 1$, assuming that α is small enough. Jürg was deeply impressed by the ideas, which he found very beautiful, and even more by the enormous speed of Charlie's work. Needless to say, for Jürg and his student, this development was a mixed blessing.

Shortly after Charlie had found his proof of "Stability of matter in magnetic fields," Lieb, Loss, and Solovej discovered a considerably shorter and more elegant proof of this result, using one of Charlie's ideas. They proved the following variant of (18):

$$H_{Z,M,N} > C_Z N^{1/3} M^{2/3}, \tag{19}$$

provided $\alpha < 0.06$ and $Z \cdot \alpha^2 < 0.04$. Thus, the problem was settled!

Jürg also got more than his share of pleasure in studying systems of electrons and nuclei interacting with the electromagnetic field. For example, in a collaboration with Charlie, Gian Michele Graf, and a student, they showed that systems of electrons and nuclei interacting with the *quantized electromagnetic field* are stable, provided an ultraviolet cutoff is imposed on the electromagnetic field.

Charlie spent the academic year 2007–08 at Columbia University, during which he gave the Eilenberg Lectures on his seminal work on Whitney problems (described in Klartag's contribution). During this period, Charlie's and Michael's families befriended each other, and the conversations between Charlie and Michael ranged widely. Eventually they focused on mathematical aspects of an important problem in condensed matter physics: the properties of graphene and its artificial analogues.

Around that time Michael heard the physicists Horst Stoermer and Philip Kim lecture on the remarkable physical properties of graphene. Graphene is a nearly perfect two-dimensional material, a single atomic layer of carbon atoms arranged in a honeycomb structure. Just about every lecture on graphene by experimentalists and theorists begins with figures of its iconic *Dirac cones*, conical singularities in the band structure; see, for example, Figure 1. This structure is beautiful for its simplicity (generic band structures of crystals can be very messy) and profound in its physical implications.

To understand Dirac cones one considers the Schrödinger operator $H^\lambda = -\Delta + \lambda^2 V(\mathbf{x})$, where V is a honeycomb lattice potential. That is, V is real-valued and periodic with respect to the equilateral triangular lattice Λ. Furthermore, with respect to some origin of coordinates in $\mathbb{R}^2$, V is inversion symmetric (even) and $120°$ rotationally invariant. An important example, corresponding to the single-electron model of graphene, is obtained by taking V to be a sum of translates of a fixed atomic potential well, V_0, centered at the sites of a regular honeycomb structure. For any periodic potential,

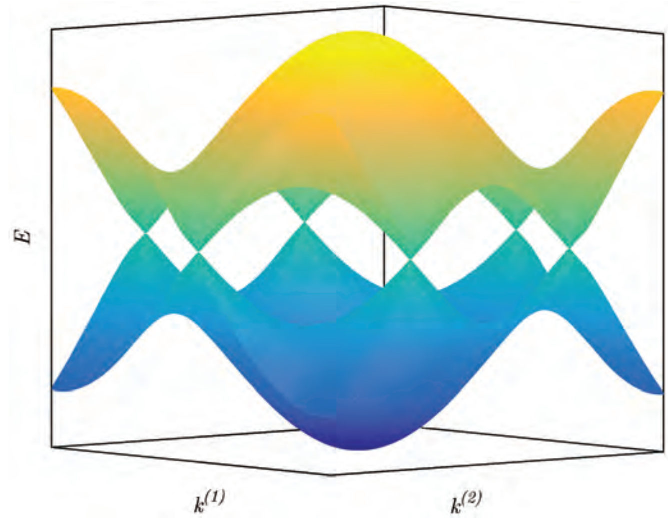

Figure 1. Dispersion surfaces of Wallace's celebrated 2-band tight-binding model with its conical singularities.

the L^2-spectrum is the union over *quasi-momenta* $\mathbf{k}$ in the *Brillouin zone* of the discrete (Floquet–Bloch) spectra of Hamiltonians $H^\lambda(\mathbf{k}) = -(\nabla + i\mathbf{k})^2 + \lambda^2 V(\mathbf{x})$ on $L^2(\mathbb{R}^2/\Lambda)$:

$$E_1^\lambda(\mathbf{k}) \le E_2^\lambda(\mathbf{k}) \le \cdots \le E_b^\lambda(\mathbf{k}) \le \cdots .$$

The maps $\mathbf{k} \mapsto E_b^\lambda(\mathbf{k})$ are Lipschitz continuous, and their graphs are called dispersion surfaces. Dirac points are energy-quasimomentum pairs $(E_D^\lambda, \mathbf{K}_\star)$ in a neighborhood of which two successive dispersion surfaces, $E_-^\lambda(\mathbf{k})$ and $E_+^\lambda(\mathbf{k})$, have the following behavior for $\mathbf{k}$ near $\mathbf{K}_\star$:

$$E_\pm^\lambda(\mathbf{k}) - E_D^\lambda \;=\; \pm\, v_F^\lambda\, |\mathbf{k} - \mathbf{K}_\star|\,(\,1 \,+\, \mathcal{O}(|\mathbf{k} - \mathbf{K}_\star|)\,)\,,$$

where $v_F^\lambda > 0$. The dispersion surfaces typically displayed by physicists are not of a Schrödinger operator on the continuum but rather the 2-band tight-binding (discrete) model of P. R. Wallace from his 1947 pioneering study of graphite; see Figure 1. This model is the workhorse for condensed matter modeling of graphene.

As their conversations meandered between math and non-math, Fefferman and Weinstein noted that all honeycomb Schrödinger operators with small (low contrast) potentials have Dirac points and Wallace's tight-binding model, likewise, has Dirac points. They wondered whether Dirac points exist for arbitrary, or at least generic, potentials of this class. In a joint 2012 paper in the *Journal of the American Mathematical Society* they proved that they do. Furthermore, for $V(\mathbf{x})$ equal to a sum of atomic potential wells centered on the honeycomb structure we have that $\left(E_\pm^\lambda(\mathbf{k}) - E_D^\lambda \right)/v_F^\lambda$ converges onto the two dispersion surfaces of Wallace's tight-binding model (Figure 1) as λ tends to ∞ uniformly in $\mathbf{k}$ varying in the Brillouin zone, where $v_F^\lambda \approx e^{-c\lambda} > 0$ depends on the atomic well.

They further explored the role that Dirac points play in the origin of robust (topologically protected) *edge states*, which are localized transverse to and plane-wave-like parallel to line defects through the honeycomb structure.

Their remarkable stability and propagation properties are the hallmarks of topological insulators. Can one construct such states analytically? These questions led to a sequence of recent papers with J. P. Lee-Thorp; there remains a great deal to explore and understand in this very active area motivated by questions in condensed matter physics, photonics, and other fields.

These results are not the end of the story but are a promising beginning. During the past twenty-five years, many interesting results and novel analytical methods have been discovered. But some of the most challenging problems of quantum theory, e.g., the construction of a physically interesting four-dimensional, local, relativistic quantum field theory, in which Charlie's participation would undoubtedly have made a difference, remain open. We can always hope that Charlie will expand his engagement with the mathematics of quantum theory and fulfill Barry Simon's hope to become his secret weapon.

Fefferman's PhD Students:

Antonio Córdoba	The University of Chicago	1974
Elena Prestini	The University of Maryland	1976
Bernard Marshall	Princeton University	1977
Roberto Moriyon	Princeton University	1979
Adrian Nachman	Princeton University	1980
Antonio Sánchez Calle	Princeton University	1983
Barnwell Hughes	Princeton University	1986
Matei Machedon	Princeton University	1986
Joseph Gregg	Princeton University	1987
Luis Seco	Princeton University	1989
Alberto Parmeggiani	Princeton University	1993
Jarosław Wróblewski	Princeton University	1993
Alejandro Andreotti	Princeton University	1994
Diego Córdoba	Princeton University	1998
Ronald Howard	Princeton University	1999
Alan Ho	Princeton University	2001
Jorge Silva	Princeton University	2001
Rami Shakarchi	Princeton University	2002
Jose Rodrigo Diez	Princeton University	2004
Spyros Alexakis	Princeton University	2005
Garving Luli	Princeton University	2010
Arie Israel	Princeton University	2011

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All remaining photos and Figure 1 courtesy of Julie Fefferman.

CALL FOR NOMINATIONS

D. R. FULKERSON PRIZE

The Fulkerson Prize Committee invites nominations for the Delbert Ray Fulkerson Prize, sponsored jointly by the Mathematical Optimization Society (MOS) and the American Mathematical Society. Up to three awards of $1,500 each are presented at each (triennial) International Symposium of the MOS. The Fulkerson Prize is for outstanding papers in the area of discrete mathematics. The prize will be awarded at the 23rd International Symposium on Mathematical Programming to be held in Bordeaux, France on July 1–6, 2018.

Eligible papers should represent the final publication of the main result(s) and should have been published in a recognized journal or in a comparable, well-refereed volume intended to publish final publications only, during the six calendar years preceding the year of the Symposium (thus, from January 2012 through December 2017). The prizes will be given for single papers, not series of papers or books, and in the event of joint authorship the prize will be divided.

The term "discrete mathematics" is interpreted broadly and is intended to include graph theory, networks, mathematical programming, applied combinatorics, applications of discrete mathematics to computer science, and related subjects. While research work in these areas is usually not far removed from practical applications, the judging of papers will only be based on their mathematical quality and significance. Further information about the Fulkerson Prize can be found at **www.mathopt.org/?nav=fulkerson** and at **www.ams.org/prizes/fulkerson-prize.html**.

The Fulkerson Prize Committee consists of Maria Chudnovsky (Princeton University), Friedrich Eisenbrand (EPFL), Chair, and Martin Grötschel (Berlin-Brandenburgische Akademie der Wissenschaften). Please send your nominations (including reference to the nominated article and an evaluation of the work) by **February 15, 2018**, to the chair of the committee. Electronic submissions to Professor Eisenbrand at **Friedrich.Eisenbrand@epfl.ch** are preferred.

Friedrich Eisenbrand
EPFL, Station 8
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An Old Problem Resurfaces Nonlocally: Gamow's Liquid Drops Inspire Today's Research and Applications

Rustum Choksi, Cyrill B. Muratov, and Ihsan Topaloglu

ABSTRACT. Gamow's simple liquid drop model of an atomic nucleus recently captured the attention of mathematicians and has inspired numerous advances and open questions in geometric and variational analysis.

"A mathematician friend of mine, the late S. Banach, once told me, 'The good mathematicians see analogies between theorems or theories; the very best see analogies between analogies.' This ability to see analogies between models for physical theories Gamow possessed to an almost uncanny degree."

—Stanislaw Ulam

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Gamow's Liquid Drop Model

The 1920s may arguably be called the Golden Age of quantum mechanics. An explosive development of this emerging field of physics drew scores of aspiring researchers, one of whom was the young Russian theoretical physicist George Gamow (see Figure 1, page 1276 and sidebar on page 1278.) His name shot to fame when in 1928, at the age of twenty-four, he published a paper in which he explained the phenomenon of alpha-decay as a quantum tunneling effect. In the words of Hans Bethe, the results of this paper constituted "the first successful application of quantum theory to nuclear phenomena." In the paper, Gamow also acknowledged a little help from Russian mathematician N. Kochin, admitting himself that he was "not good in mathematics."

In 1928 Gamow [1] made another important discovery that has become forever linked with his name. During his stay in Copenhagen with Niels Bohr in the fall of that year, Gamow conceived of what is now known as the liquid drop model of the atomic nucleus. This simple model, which was soon refined and further developed by Heisenberg, von Weizsäcker, and Bohr after the discovery of neutrons in 1932, treats the collection of protons and neutrons inside an atomic nucleus as an incompressible, uniformly charged fluid. With only a few fitting parameters and an assumption that the nuclei are spherical, this model was able to accurately predict the mass defect curve—the loss

Figure 1. George Gamow (portrait, 1932). At the age of twenty-four, George Gamow introduced the liquid drop model that successfully explained the basic characteristics of the atomic nuclei and predicted nuclear fission as a result of an instability of large spherical nuclei with respect to nonspherical distortions.

of mass or energy when nucleons come together to bind in the nucleus as a function of the number of nucleons. Thus, indirectly, the model also predicted the spherical shape of most nuclei. The model's ultimate triumph came from explaining the phenomenon of nuclear fission in terms of an instability of large spherical nuclei with respect to nonspherical distortions (Bohr and Wheeler, 1939). The model has also been extensively used in astrophysics to describe exotic phases of nuclear matter at ultrahigh densities found in neutron stars (Baym, Bethe, and Pethick, 1971).

In the modern rendition of Gamow's liquid drop model the attractive short-range nuclear force gives rise to excess surface energy due to lower nucleon density near the nucleus boundary, while the presence of positively charged protons gives rise to a repulsive Coulombic force. Since the Coulomb energy of a proton in a nucleus is much smaller than its average kinetic energy determined by strong nuclear forces, to a good approximation the spatial distribution of charge in a nucleus is uniform. Therefore, mathematically the energy of a nucleus within the model may be written (up to shape-independent bulk terms and after a suitable nondimensionalization) as

$$E(\Omega) := \mathrm{Per}(\Omega) + \frac{1}{8\pi}\int_\Omega\int_\Omega \frac{1}{|x-y|}\,dx\,dy, \tag{1}$$

where the nucleus $\Omega \subset \mathbb{R}^3$ is a measurable set with fixed volume $|\Omega| = m$. We refer to m as "mass," which is a parameter proportional to the number of nucleons in a nucleus. $\mathrm{Per}(\Omega)$ is the perimeter of the set Ω, i.e., a suitably generalized notion of the surface measure of $\partial\Omega$.[1] The ground state of a nucleus with a given number of nucleons is then the minimizer of E, i.e., the set Ω that achieves the least energy,

$$e(m) := \inf\{E(\Omega) : |\Omega| = m\}, \tag{2}$$

for a given mass m.

a marriage (or rather divorce) of two older geometric problems

Ultimately, the purpose of this liquid drop model is to predict 1) the shape of nuclei, 2) the nonexistence of arbitrarily large nuclei, and 3) the existence of a nucleus with the least energy per nucleon (the element having the greatest nuclear binding energy). It is precisely the competition between the forces which try to minimize the surface area of the nucleus and those which try to spread the nuclear charges apart that makes answering these questions nontrivial.

Gamow's liquid drop problem is a *marriage* (or rather *divorce*) of two older geometric problems:

(1) the *Classical Isoperimetric Problem* of minimizing the perimeter of a body with fixed mass m; and
(2) the *Problem of the Equilibrium Figure* of a self-gravitating fluid body of mass m.

For the first problem, whose roots go back to antiquity, Schwarz demonstrated the minimizing property of balls in 1884 for piecewise-smooth sets in three dimensions. The complete solution was given in 1958 by De Giorgi, who showed that the unique minimizer of the perimeter functional among all measurable sets with fixed mass is given by a ball. Starting with Newton (1687), the second problem attracted the attention of many celebrated mathematicians. Assuming zero angular momentum, the total potential energy of a self-gravitating fluid body, represented by a measurable set $\Omega \subset \mathbb{R}^3$, is given, up to a constant, by

$$-\int_\Omega\int_\Omega \frac{1}{|x-y|}\,dx\,dy, \qquad |\Omega| = m, \tag{3}$$

where $-|x-y|^{-1}$ is the potential resulting from the gravitational attraction between two point masses at positions x and y in the fluid. Lyapunov (1886) made the first mathematical breakthrough by establishing that every regular minimizer of (3) is a ball. Poincaré commented on Lyapunov's proof in 1887 and went on to make the problem famous in his 1902 treatise *Figures d'Equilibre d'une Masse Fluide*. Almost twenty years later, Carleman (1919) showed that balls are indeed minimizers. Yet it was not until the work of Lieb (1977) that a complete solution based on strict Riesz rearrangement inequality

[1] *See "WHAT IS…Perimeter?" in the October 2017* Notices.

became available. Steiner symmetrization plays a central role in the analysis of both problems.

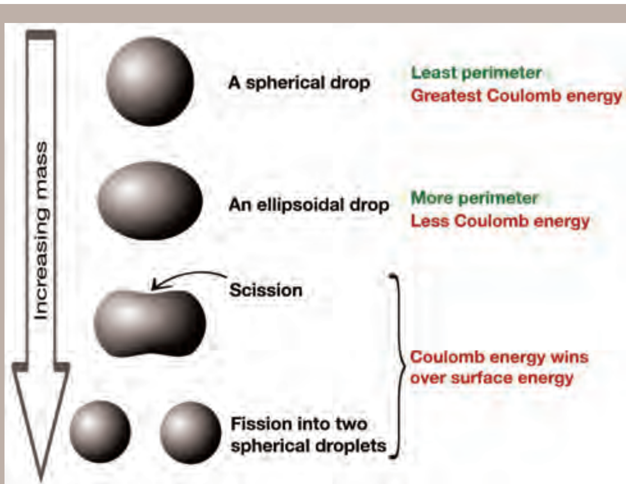

Figure 2. In Gamow's liquid drop problem, the two terms in the energy are in direct competition: the perimeter term favors a single ball, while the Coulomb term favors splitting. For small values of m the perimeter wins completely, whereas for large values of m the Coulomb term dominates, resulting in fission.

Thus, up to translations, the ball of mass m is the unique minimizer in the two problems above. Gamow's liquid drop model, on the other hand, puts them in direct opposition, with balls being best for the perimeter term but worst for the Coulomb term. The relative strength of the two terms is controlled by the magnitude of m. One can readily see the effect of the size of m as to which term dominates from a simple scaling argument. Namely, by looking at $\lambda\Omega$ for any $\lambda > 0$ we see that

$$\lambda^{-3}E(\lambda\Omega) = \lambda^{-1}\operatorname{Per}(\Omega) + \frac{\lambda^2}{8\pi}\int_\Omega\int_\Omega \frac{1}{|x-y|}\,dxdy.$$

Thus the energy per nucleon $\lambda^{-3}E(\lambda\Omega)$ is dominated by the perimeter term for small values of $\lambda > 0$ and by the Coulomb term for large values of $\lambda > 0$ (see Figure 2). Therefore, we would be led to expect that the unique solution (up to translations) of problem (2) is given by a ball when m is small, whereas this problem does not have a solution when m is large. Also, for fixed Ω there is a unique value of λ at which the energy per nucleon (the negative of the binding energy) is minimum.

If one is to believe that Gamow's liquid drop model correctly captures the ground state configurations of the atomic nuclei, then nature tells us that the solution of the liquid drop problem is always given by a ball. In this case an explicit calculation gives the value of $m = \frac{40\pi}{3}(2^{1/3} + 2^{-1/3} - 1) \approx 44.1$, beyond which minimizers no longer exist. This critical value is such that one ball of mass m has exactly the same energy as two balls with mass $\frac{1}{2}m$ infinitely far apart (Frank and Lieb, 2015). Under the same assumption, the minimum energy per unit mass is attained for $m = 10\pi \approx 31.4$.

Although Gamow's liquid drop model has been very well known among physicists, it is surprising that it did not receive much attention in the mathematics community until recently. In 2010 the liquid drop problem resurfaced in an asymptotic study by Choksi and Peletier [2], [3] of the Ohta-Kawasaki functional arising in a completely unrelated field of polymer physics. In fact, it is clear that the liquid drop problem represents a prime example of a problem of pattern formation from energetic competitions, which is an area driving much work in modern calculus of variations.

Global Minimizers of the Liquid Drop Problem

Global existence for minimizers of $E(\Omega)$ with $|\Omega| = m$ on unbounded domains is nontrivial because of the lack of compactness of minimizing sequences. What we know so far regarding the problem associated with $e(m)$ can be summarized by the following theorem. Here we consider the minimization problem (2) over all sets of finite perimeter (Caccioppoli sets).

Theorem 1. *There exist constants $0 < m_0 \leq m_1 \leq m_2$ such that:*

(1) *If $m \leq m_1$, then there* **exists** *a minimizer.*
(2) *If $m \leq m_0$, then the* **unique** *minimizer is a* **ball**.
(3) *If $m > m_2$, then there is* **no minimizer**.

The result in this theorem, in this form, was first provided by Knüpfer and Muratov [6] in 2014. The existence of solutions for certain values of m was already noted by Choksi and Peletier [2] in 2010 where the strict concavity of $e(m)$ for small values of m was also established. Knüpfer and Muratov took a more direct approach and proved that for sufficiently small mass every minimizing sequence can be replaced by another minimizing sequence consisting of sets with uniformly bounded diameter. This uniform bound provided the necessary compactness for the minimizing sequence, and with the lower semicontinuity of $E(\Omega)$ with respect to the L^1-topology the existence of minimizers follows from the direct method of the calculus of variations. They also showed that minimizers of $E(\Omega)$ are so-called "quasi-minimizers" of the perimeter functional and hence have smooth boundaries. Implementing the sharp quantitative isoperimetric inequality, proved by Fusco, Maggi, and Pratelli (2008), combined with the regularity, Knüpfer and Muratov proved that balls are, up to translations, unique minimizers of $E(\Omega)$ for sufficiently small mass. In order to prove that the problem does not have a solution for large masses, the authors first showed that minimizers are connected. However, they also showed that any connected set with sufficiently large mass can be divided into two large pieces by a suitable plane. Moving these pieces far apart from each other, they compared the energy of the transformed disconnected set with the original configuration and concluded the proof via a contradiction argument.

George Gamow

George Gamow (1904–1968) was a Russian-American theoretical physicist who introduced some of the most influential ideas to twentieth-century science. Born in 1904 in Odessa, Ukraine, Gamow graduated from Leningrad University in 1926, where together with fellow students L. Landau and D. Ivanenko he founded the group of the Three Musketeers to discuss the latest topics in theoretical physics, in addition to fooling around with friends and generally having a good time.

In the summer of 1928, Gamow undertook his first trip abroad on a four-month fellowship to the University of Göttingen, where he met many of the pioneers of quantum theory. Importantly, the trip also gave Gamow an opportunity to visit N. Bohr in Copenhagen and E. Rutherford at Cambridge, which was instrumental for the development of his quantum mechanical ideas in connection with nuclear phenomena.

> *I shall never forget the first time he appeared in Göttingen—how could anyone who has ever met Gamow forget his first meeting with him—a Slav giant, fair-haired and speaking a very picturesque German; in fact he was picturesque in everything, even in his physics.*
>
> —L. Rosenfeld

Gamow returned to the Soviet Union as a sort of celebrity. Yet the political atmosphere in the country had been rapidly changing, and Gamow felt it personally when in 1931 he was denied an exit visa to go to an international nuclear physics congress in Rome. From that time on, Gamow became obsessed with the desire to escape the confines of the Soviet Union, which in his mind had begun to interfere with the free movement of people and ideas across state borders. After an unsuccessful attempt to cross the Black Sea from Crimea into Turkey with his wife on a kayak, Gamow finally managed to obtain permission to leave the Soviet Union in 1933 for a conference, a trip from which he never returned.[2] In 1934, Gamow settled at George Washington University in the United States, where he spent the next twenty-two years, the most productive period of his scientific career. He then moved to the University of Colorado, Boulder, where he worked until his death in 1968.

Gamow never received a Nobel Prize in physics, although his greatest achievements included the Big Bang theory, the theory of stellar structure and evolution, and a key insight into the nature of the genetic code. To the general public, however, Gamow is best known as the author of a popular science series, *The Adventures of Mr. Tompkins*, explaining the fundamental concepts of modern physics to millions of readers.[a]

[a]This issue's BookShelf (page 1327) features Gamow's popular text, *One Two Three…Infinity: Facts and Speculations of Science.*

Julin (2014) independently proved that for m sufficiently small the ball of mass m is the unique minimizer for $e(m)$, up to translations. His approach is to use a stronger version of the quantitative isoperimetric inequality. This version measures the difference between perimeters of a set and a ball of the same mass in terms of the oscillations of the boundary quantified by the L^2-norm of the difference of generalized outward normal vectors.

Independently, Lu and Otto (2014) also proved that for sufficiently large m, the energy $E(\Omega)$ does not admit a minimizer. By the subadditivity of $e(m)$,

$$e(m) \le e(m') + e(m - m'), \qquad 0 < m' < m,$$

they obtain an upper bound of the form

$$e(m) \le Cm, \qquad m > m_0.$$

In order to refine the lower bound of this scaling estimate, Lu and Otto prove a density estimate which states that a minimizer of $E(\Omega)$ cannot be thinner than order 1 in m, and combining this with the connectedness of minimizers, they obtain that for large values of m,

$$\int_\Omega \int_\Omega \frac{1}{|x-y|}\, dx\, dy \ge C\, m \log m.$$

Together with the upper bound this estimate yields a contradiction when m is sufficiently large. More recently, via a very simple averaging technique applied to a cutting argument, Frank, Killip, and Nam (2016) proved that if $m > 32\pi \approx 100.53$, then the problem does not admit a minimizer.

[2]*The other members of the Three Musketeers group did not fare so well: L. Landau was arrested in 1938 and nearly died after one year in detention; D. Ivanenko was sentenced in 1935 to a three-year term in labor camps, substituted with a two-year exile in Tomsk after serving one year in Gulag; M. Bronshtein, another key member of the Leningrad group, was executed by NKVD in 1938.*

The Liquid Drop Model on the Torus and the Ohta-Kawasaki Functional

In the astrophysics context, one often considers the liquid drop model on the three-dimensional flat torus $\mathbb{T}^3_\ell$ of sidelength ℓ. Specifically, for $\Omega \subset \mathbb{T}^3_\ell$ with $|\Omega| = m$ one considers the energy

$$E_\ell(\Omega) := \operatorname{Per}(\Omega, \mathbb{T}^3_\ell) + \frac{1}{2}\int_{\mathbb{T}^3_\ell} (\chi_\Omega - \rho)(-\Delta)^{-1}(\chi_\Omega - \rho)\, dx, \tag{4}$$

where $\operatorname{Per}(\Omega, \mathbb{T}^3_\ell)$ denotes the perimeter of Ω relative to $\mathbb{T}^3_\ell$, χ_Ω is the characteristic function of Ω, and $\rho = m/\ell^3$ is the density of the neutralizing background charge supplied by electrons. While this problem is well known in the studies of the structure of nuclear matter in neutron stars, it only recently received attention in the mathematics literature as the sharp interface version of the Ohta-Kawasaki functional

$$\mathcal{E}_\ell(u) := \int_{\mathbb{T}^3_\ell} \left(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{4\varepsilon}(1-u^2)^2 + \frac{1}{2}(u-\bar{u})(-\Delta)^{-1}(u-\bar{u}) \right) dx, \qquad \frac{1}{\ell^3}\int_{\mathbb{T}^3_\ell} u\, dx = \bar{u}, \tag{5}$$

which is a diffuse interface version of the energy in (4) with the asymmetry parameter $\bar{u} \in (-1, 1)$ fixed and $\varepsilon \ll 1$. The energy in (5) was introduced in 1986 by Ohta and Kawasaki as a simple model for self-assembly of diblock copolymer melts, even if its nuclear physics analog goes back all the way to the classical 1935 paper of von Weizsäcker. It can also be considered as a tool for numerical studies of (4).

Unlike the liquid drop problem posed on all space, the energy E_ℓ always admits a minimizer, and a question closely related to the liquid drop problem is what happens to those minimizers when $\ell \to \infty$ and/or $m \to 0$. Here the finiteness of the domain forces minimizers to always balance the effects of the perimeter and the Coulomb terms. In fact, numerically the putative global minimizers of the Ohta-Kawasaki functional appear to be periodic, with the interfaces within each intrinsic periodic cell resembling a constant mean curvature surface; e.g., a lamella, sphere, cylinder, double-gyroid, etc., as in Figure 3. Thus numerics suggest a separation of the effects of the perimeter and the nonlocal term: the latter sets an intrinsic periodicity, while the former dictates the interface structure within the period cell. This separation has been exploited in a number of papers by Ren and Wei (2007, 2008, 2009) and by Cristoferi (2016) for constructing critical points of variants of (4) perturbatively. To gain further insights into this separation, an asymptotic analysis was presented by Choksi and Peletier [2], [3] and by Knüpfer, Muratov, and Novaga [4] for the droplet regimes wherein $\rho \to 0$. Here, Gamow's liquid drop model (on all $\mathbb{R}^3$) emerged as the leading-order energy describing the shape of a single droplet, ignoring the next-order effects of the neighboring droplets.

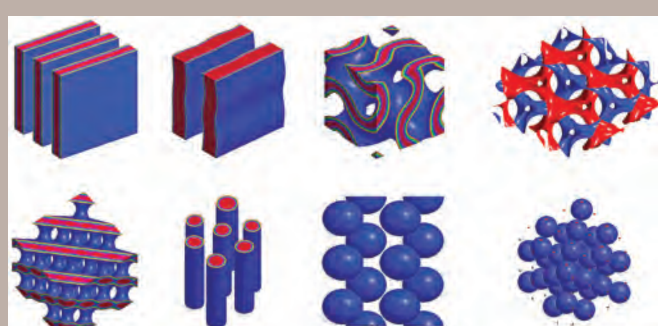

Figure 3. Level sets of the computed local minimizers of Ohta-Kawasaki energy $\mathcal{E}_\ell$ in (5) as the value of $\bar{u}$ is varied.

While one may conjecture that it is possible to choose $m_0 = m_1 = m_2$, based on physical evidence, to date it remains open to prove or disprove whether any (or all) of the constants m_i, $i = 0, 1, 2$, may be pairwise equal. One result in that direction was obtained by Knüpfer, Muratov, and Novaga [4] in 2016, who show that the set

$$\mathcal{I} = \{m > 0 \colon e(m) \text{ is attained}\}$$

is a closed (hence compact) subset of $(0, \infty)$.

They establish this by proving Lipschitz continuity of $e(m)$ on compact subsets of $(0, \infty)$ combined with the BV-compactness of sequences of *generalized minimizers*. Here a generalized minimizer of $E(\Omega)$ refers to a finite collection of sets of masses m_j adding up to m, where each set is a minimizer for $e(m_j)$. Around the same time, Frank and Lieb (2015) obtained the same result, using a diameter bound of the form $\operatorname{diam}\Omega \le Cm$ for minimizers of $E(\Omega)$, along with a concentration-compactness lemma,

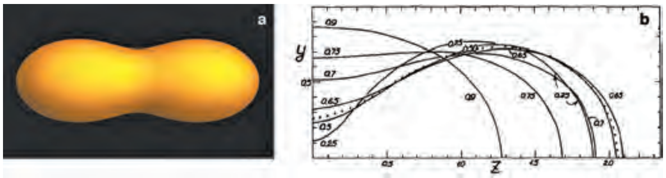

Figure 4. Dumbbell-shaped critical points of the liquid drop model for several values of the fissility parameter $\alpha = m/(40\pi)$ obtained from the numerical solution of (6) by Strutinsky, Lyashchenko, and Popov (1963): a three-dimensional schematic (a), profile cross-sections in the first quadrant (b).

particularly well-suited to the liquid drop problem, for a sequence of sets of uniformly bounded perimeter. Together with strict concavity of $e(m)$ for $m \leq 4\pi$, these results also yield the existence of a minimizer in this range of masses.

Finally, referring back to the third purpose of the liquid drop model, existence of the tightest bound nucleus was established in 2016 by Knüpfer, Muratov, and Novaga [4], who show that there exists $m^* \in \mathcal{I}$ such that

$$f^* := \inf_{m \in \mathcal{I}} f(m) = f(m^*), \qquad f(m) := \frac{e(m)}{m}.$$

The proof of this fact follows from Lipschitz continuity of the energy-per-particle $f(m)$ on the set $[m_0, \infty)$ combined with compactness of the set $\mathcal{I}$. In fact, the analysis further shows that the quantity f^* appears as the leading order term in the asymptotic behavior of uniform energy density of minimizers of $E_\ell(\Omega)$ as mass $m \to 0$ and domain size $\ell \to \infty$. Frank and Lieb (2015) also prove the same existence result for $f(m)$ by relaxing the mass constraint $|\Omega| = m$ and considering $\inf_{0<|\Omega|\leq m}(E(\Omega)/|\Omega|)$.

Critical Points and Local Minimizers of the Liquid Drop Problem

One of the main physical motivations in the studies of Gamow's liquid drop model was to compute the minimal energy required for nuclear fission. Bohr and Wheeler (1939), following the suggestion of Meitner and Frisch (1939), were the first to analyze the critical points of the energy in (1). In particular, they wrote down the associated Euler-Lagrange equation

$$2\mathcal{H}(x) + \frac{1}{4\pi}\int_\Omega \frac{1}{|x-y|}\,dy = \lambda, \tag{6}$$

where $\mathcal{H}(x)$ denotes the mean curvature of $\partial\Omega$ (positive if Ω is convex) and λ is a Lagrange multiplier. Clearly, a ball of mass m is always a solution of (6). However, as was shown numerically in a series of papers from the 1940s through the 1970s, many types of nonspherical solutions to (6) are possible, including ovoid, dumbbell, pear-shaped, and toroid nuclei. In fact, when the ball of mass m passes the stability threshold of $m = 40\pi \approx 125.66$, it bifurcates into a family of spheroids, first identified by Bohr and Wheeler (1939), using formal bifurcation analysis near the distortion instability. Numerical continuation studies of those solutions into the subcritical mass region were carried out by several authors, but, surprisingly, only one well-converged numerical study, by Strutinsky, Lyashchenko, and Popov (1963), appears to exist to date; see Figure 4.

Although mathematically it is not in the literature whether suitable nonspherical distortions of balls are critical points, using Lyapunov-Schmidt type reduction methods Ren and Wei (2011, 2014) obtained existence of solutions to the Euler-Lagrange equation which are almost of the shape of single and double tori. However, these more exotic patterns, as shown by Ren and Wei (2017), are highly unstable in the sense of the second variation of the energy. Earlier, oval-shaped solutions in a two-dimensional version of the problem and spherical shell solutions in three dimensions were constructed by Ren and Wei (2009), with some numerical studies of spherical shell and toroidal solutions previously carried out by Wong (1973).

Obtaining quantitative information about (local) minimizers through stability is the approach taken by Bonacini and Cristoferi (2014), based on the explicit calculations of the second variation of $E(\Omega)$ obtained by Muratov (2002) and Choksi and Sternberg (2007). They generalize the result of Acerbi, Fusco, and Morini (2013) for bounded domains to the unbounded setting of the liquid drop problem and prove that strictly stable critical points, i.e., sets for which the first variation of $E(\Omega)$ vanishes and the second variation of $E(\Omega)$ is strictly positive-definite, are local minimizers. That is, these patterns minimize $E(\Omega)$ with respect to competitors close in the L^1-topology. Applying this result to the ball of mass m, they conclude that the ball is an isolated local minimizer if $m < 40\pi$. Finally, very recently Julin (2016) proved that every local minimizer with sufficiently small mass and perimeter is a ball, using the characterization of compact almost constant mean curvature surfaces by Ciraolo and Maggi (2015).

At the heart of these criticality and stability results lies the regularity of the boundaries of local minimizers. As was first proved by Sternberg and Topaloglu (2011), local minimizers of $E_\ell(\Omega)$ have $C^{3,\alpha}$ boundaries by using classical geometric measure theoretic arguments from regularity of minimal surfaces. More recently, Julin and Pisante (2015) demonstrated that the boundaries of local minimizers are in fact of class C^∞, and Julin (2016) established analyticity of the boundary of regular critical points. Since the regularity arguments are local arguments, they immediately apply to local (and global) minimizers of the liquid drop problem on the whole of $\mathbb{R}^n$. Furthermore, any critical point of the energy in the L^1-topology is what is known as a Λ-minimizer of the perimeter and hence is also regular.

The simplicity of Gamow's liquid drop model conceals the true richness.

Figure 5. Examples of critical points of the liquid drop model on the sphere. In the first two panels, axisymmetric configurations with three and four interfaces are shown. The third panel is the result of a hybrid numerical simulation by Shahriari, Ruuth, and Choksi, where a spiraling pattern achieves a stable low energy state.

Extensions and Related Problems

The simplicity of Gamow's liquid drop model conceals the true richness of behaviors exhibited by the solutions of this geometric variational problem. At the same time, it is clear that the main ingredients of Gamow's model, namely, the competition between the short-range attractive and long-range repulsive forces, are not unique to that model and should be expected to produce interesting behaviors in a variety of contexts, both mathematically and physically.

The simplest natural mathematical generalization of Gamow's liquid drop model is to extend it to other space dimensions. Considering $\Omega \subset \mathbb{R}^n$ with $n > 3$ amounts to replacing the Newtonian kernel in (1) with a Riesz kernel $|x-y|^{-\alpha}$, where $\alpha = n-2$. For this problem, it was shown by Knüpfer and Muratov [5], [6] that minimizers still exist for all sufficiently small masses and that minimizers for yet smaller masses are balls when $n \leq 7$. Julin (2014) extended this result to all dimensions by removing the reliance on the classical regularity of minimal surfaces. Yet, the arguments leading to nonexistence no longer apply, and it is an open problem whether or not minimizers to the liquid drop problems, which must then be nonspherical as it is always better to split one ball into two identical balls far apart for $m \gg 1$, exist for arbitrarily large masses when $n \geq 4$. The absence of such a nonexistence result also prevents proving existence of minimizers with optimal energy per unit mass.

On the other hand, Gamow's liquid drop problem becomes trivial for $n < 3$, since in that case the fundamental solution of Laplace's equation is no longer positive, and minimizers of the liquid drop problem do not exist for any value of $m > 0$. The problem is still meaningful if posed instead on a compact Riemannian manifold, and results for global minimizers on a two-dimensional flat torus or a two-dimensional sphere are available; see Sternberg and Topaloglu (2011) and Topaloglu (2013). Locally minimizing axisymmetric critical patterns on the sphere as those in Figure 5 have also been investigated by Choksi, Topaloglu, and Tsogtgerel (2015). The one-dimensional periodic case is considerably simpler and was solved completely by Ren and Wei (2003). Another alternative is to consider the problem on the whole of $\mathbb{R}^n$ but replace the Newtonian potential with a general Riesz kernel with $0 < \alpha < n$. In fact, the case $n = 2$ and $\alpha = 1$ arises in the physical modeling of charge condensation in high-T_c superconductors. In this case existence of minimizers with small masses was again obtained by Knüpfer and Muratov [5], [6], where minimality of balls of small masses was shown for $n = 2$ and for all $3 \leq n \leq 7$ and $\alpha < n-1$. The latter result was extended to $n \geq 8$ by Bonacini and Cristoferi (2014). The case of $n \geq 3$ and $n-1 \leq \alpha < n$ was settled by Figalli, Fusco, Maggi, Millot, and Morini (2015), who also considered a nonlocal generalization of the perimeter functional to fractional perimeter. We note that the latter has been the subject of considerable attention recently, starting with the work of Caffarelli, Roquejoffre, and Savin (2010), and gives rise to nonlocal minimal surfaces. In contrast with Gamow's model, however, the fractional perimeter represents a nonlocal generalization of the *attractive* potential that keeps the "liquid drop" together, and, therefore, it alone does not produce the phenomenology of Gamow's liquid drop problem. Back to the liquid drop problem with a Riesz kernel, a nonexistence result for large masses is available for $\alpha < 2$ (Knüpfer and Muratov [5], [6]). Again, it is an open problem whether or not large nonspherical minimizers could exist for $n \geq 3$ and $\alpha \geq 2$. Lastly, in the special case $n = 2$ and α sufficiently small, a complete solution of the liquid drop problem with a Riesz potential was given by Knüpfer and Muratov [5] in 2013. The obtained solution agrees with the physical conjecture about the structure of the minimizers of the classical Gamow's liquid drop problem mentioned earlier.

Moving on to further extensions, we note that Coulombic repulsive forces naturally arise in a number of other physical contexts. Perhaps the best known example goes back to Lord Rayleigh (1882), who considered equilibrium shapes of charged liquid drops that are perfect conductors. The associated energy functional may be written in a nondimensional form as

$$E(\Omega) := \mathrm{Per}(\Omega) + \frac{\lambda}{8\pi} \inf_{\mu \in \mathcal{P}(\Omega)} \int_\Omega \int_\Omega \frac{d\mu(x)d\mu(y)}{|x-y|}, \tag{7}$$

where $\Omega \subset \mathbb{R}^3$ with $|\Omega| = m$, as before, and the infimum is taken over all probability measures supported on Ω. One would naturally be led to believe that the minimization problem for (7) would behave quite similarly to the Gamow liquid drop problem. Yet, Goldman, Novaga, and Ruffini (2015) showed that, surprisingly, the minimization problem for (7) does not admit minimizers for any $m > 0$ and $\lambda > 0$. Even worse, the problem does not exhibit *local* minimizers in any reasonable sense, indicating that the energy in (7) does not provide a complete physical picture as a model of charged liquid drops, as shown by Muratov and Novaga (2016). The reason for this behavior is a kind of incompatibility between the sets seen by the perimeter, which have finite two-dimensional Hausdorff measure of the measure theoretic boundary, and sets of positive capacity, which may have positive Hausdorff measure of dimension $1 < d < 2$. This phenomenon also gives rise to singular solutions of the associated Euler–Lagrange equation, such as the famous *Taylor cone* with

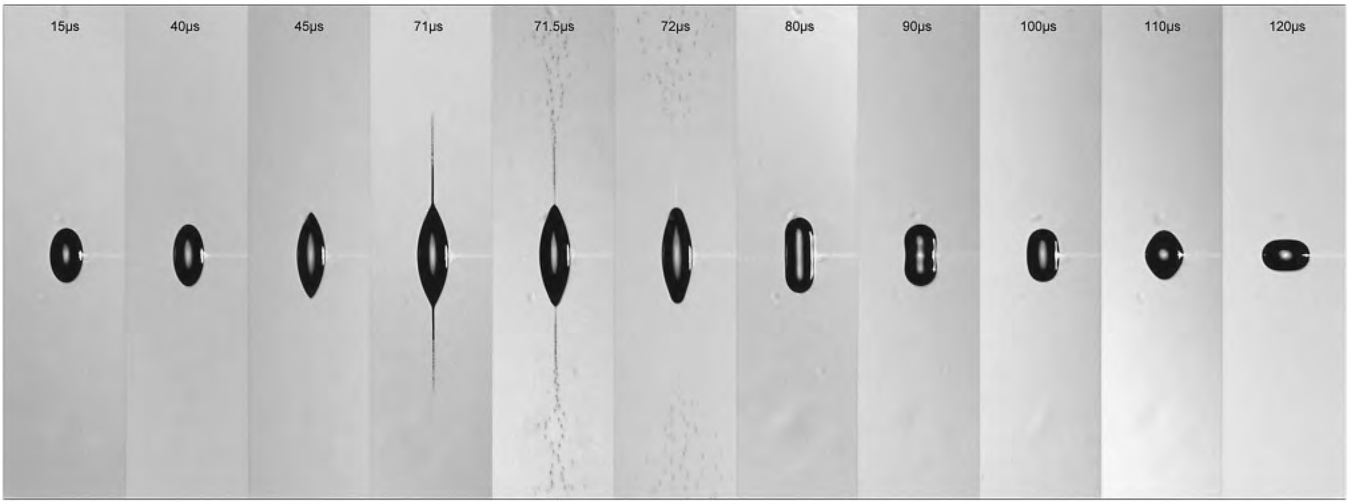

Figure 6. A sequence of snapshots of a levitating charged liquid drop reaching the threshold of the charge-driven instability as the liquid in the drop evaporates.

an opening half-angle of $\approx 49.3°$, obtained by Taylor (1962). Further regularity assumptions on the set, such as convexity of Ω, yield existence, as was shown by Goldman, Novaga, and Ruffini (2016). It is also clear that the model in (7) is quite sensitive to the space dimensionality. In fact, a complete solution of a variant of the minimization problem for (7) in which the set Ω is flat, i.e., $\Omega = D \times \{0\}$, where $D \subset \mathbb{R}^2$, with $|D| = m$, is a *closed* set and the perimeter is replaced by the one-dimensional Hausdorff measure of ∂D, has been obtained by Muratov, Novaga, and Ruffini (2017), who prove that the minimizer with "mass" m and "charge" $\sqrt{\lambda}$ exists if and only if $\lambda \leq 4m/\pi$ and is given by a disk. Note that here a single disk remains a minimizer against splitting into two equal disks for all $\lambda < 4m\sqrt{2}/\pi$. Thus, the scenario of existence failure is different from the one hypothesized for Gamow's liquid drop model: instead of splitting into two disks, a single disk breaks into one big disk with a certain amount of charge and many tiny disks that carry the excess charge to infinity to reduce energy for $\lambda > 4m/\pi$. We note that in physical experiments the failure of existence of minimizers in the three-dimensional charged liquid drop model manifests itself spectacularly via thin jets of electrified liquid emitted from the droplet as its smoothness is lost, as in Figure 6. Smoothness is then restored as a significant portion of charge escapes the drop via the jet.

Conclusion

Gamow's liquid drop model is indeed a very simple model for a highly complex system. However, its beauty is really in tune with the famous quote attributed to Albert Einstein: *Everything should be made as simple as possible, but not simpler.* While it was initially posed to describe nuclear structures, the fact that it encapsulates a rather ubiquitous competition of short- and long-range effects is behind a *universality*, with the liquid drop model's phenomenology shared by many other systems operating at very different length scales: from femtometer nuclear scale to nanoscale in condensed matter systems, to centimeter scale for fluids and autocatalytic reaction-diffusion systems, all the way to cosmological scales. It has generated much recent attention by mathematicians and continues to this day to entice and challenge us with its beguilingly simple yet rich structure.

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Figure 3 simulations by J. F. Williams, discussed in Choksi, Peletier, and Williams (2009).

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Rustum Choksi focuses his mathematical research on the calculus of variations and its applications to pattern formation. Since he was a child, he has been a passionate lover of classical music, from chamber music to opera. He also devotes too much time and money to collecting minerals, in particular agates with their alluring and infinite-dimensional pattern morphology.

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Associate Editor, Mathematical Reviews
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Marcel Berger Remembered

Claude LeBrun, Editor and Translator

EDITOR'S NOTE. Claude LeBrun has kindly assembled this memorial for Marcel Berger. This month also marks the conference "Riemannian Geometry Past, Present and Future: An Homage to Marcel Berger" at IHÉS.

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Gérard Besson

Marcel Berger,[1] one of the world's leading differential geometers and a corresponding member of the French Academy of Sciences for half a century, passed away on October 15, 2016, at the age of eighty-nine.

Marcel Berger's contributions to geometry were both broad and deep. The classification of Riemannian holonomy groups provided by his thesis has had a lasting impact on areas ranging from theoretical physics to algebraic geometry. His 1960 proof that a complete oriented even-dimensional manifold with strictly quarter-pinched positive curvature must be a topological sphere is the

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[1] *Pronounced bare-ZHAY.*

Marcel Berger, during his directorship (1985–1994) of the IHÉS, in Bures-sur-Yvette, France.

direct ancestor of a vast sector of subsequent research in global Riemannian geometry. Through his many students and collaborators, he created a school which carried the torch of differential geometry into a new era. Under his leadership, a group of mathematicians collaborating under the pseudonym of Arthur L. Besse produced several landmark books [1], [19], [20] that inspired a ferment of research activity in connection with key topics of interest: the study of Riemannian manifolds whose geodesics are all closed, links between volume and injectivity radius [11], and the theory of Einstein manifolds [10]. He has also left us an impressive series of popular and pedagogical books, as exemplified by his massive treatise *A Panoramic View of Riemannian Geometry* [14]. His efforts in this last vein also included a campaign to increase the understanding of the work of Mikhail Gromov.

Born in Paris on April 14, 1927, Berger was a student at the École Normale Supérieure from 1948 until 1953, when he was awarded a research assistantship by the CNRS[2] after distinguishing himself in a national competition. In 1954, under the supervision of André Lichnerowicz, he then defended his doctoral thesis, which proved a landmark classification of the holonomy groups of Riemannian manifolds.

After spending the academic year 1956–57 at MIT, Berger was promoted to the rank of research scientist by the CNRS. In 1958 he then accepted a junior appointment at the University of Strasbourg, where he was eventually promoted to full professor in 1962, shortly after his return from a year in Berkeley. After moving to a professorship in Nice for two years, he then moved again in 1966, this time to the University of Paris, which at that time was still a single, unified institution. When the university was broken into smaller units in the wake of the 1968 student riots, Berger was attached to the University of Paris VII, at Jussieu, and was promoted to the rank of research director by the CNRS. He served as president of the French Mathematical Society (SMF) during the period of 1979–80 and in this capacity helped oversee the foundation of the International Center for Mathematical Workshops (CIRM) in Luminy. In 1985 he then became director of the Institute for Higher Scientific Studies (IHÉS) in Bures-sur-Yvette, a position in which he served until 1993, when he was succeeded by his former student Jean-Pierre Bourguignon.

The mathematical community will remain eternally indebted to Marcel Berger for his marvelous contributions to our mathematical heritage, which have substantially transformed the field of differential geometry.

[2] *The Centre national de la recherche scientifique is the French government's main agency for the support of fundamental scientific research. Unlike the American NSF, the CNRS supports a large network of scientists on a permanent basis.*

Misha Gromov

Inspired by Marcel Berger: Recollections

(1955–1995) There is no need for me to say anything about Marcel's famous $1/4$-*pinching theorem*, his discovery of collapse with bounded curvature, his contributions and leadership in the field of spectral geometry, or his classification of *special holonomy groups*—everybody in our universe[3] is aware of these.

However, it is perhaps worth mentioning that the last was cited as a source of inspiration by Jim Simons when the *Légion d'Honneur* was bestowed on him by the French ambassador in New York.

(1972) **Systolic Inequalities and Calibrations.** In a couple of articles from this period, Marcel proposed a new conceptual (systolic) rendition of Loewner's early results on surfaces. With characteristic modesty, he attributed the idea to René Thom.

In his evaluation of systoles of particular manifolds, I believe Marcel made the very first use ever, in the context of quaternionic spaces, of the method of calibrations and proved that

The quaternionic projective subspaces $\mathbb{HP}^k \subset \mathbb{HP}^n$ *are volume minimizing among all* $4k$*-cycles not homologous to zero in* $\mathbb{HP}^n$.

(1978) **Blaschke Conjecture for Spheres**. *If a Riemannian manifold* X *homeomorphic to* S^n *has diameter equal to its injectivity radius, then* X *is isometric to a standard "round" (constant-curvature) sphere.*

Marcel's dazzlingly simple proof of this (which involves a contribution by Jerry Kazdan, who proved a key analytic inequality suggested by Marcel), hardly takes half a page.

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[3] *Here, "our universe" means the "community of differential geometers."*

(1980) **Isoembolic Inequality**. *Among all n-dimensional manifolds with a given injectivity radius, the round sphere has the least volume.*

(2000?) **Buying Cheese in Paris**. One day, while a guest at Marcel's apartment in Paris, I tried a piece of cheese whose flavor far excelled anything I had ever tasted before. When I asked Marcel where this cheese came from, he could not hide his surprise at the impudence of my question.

Of course—he said—*it is from Quatrehomme. Do you think we would serve our guests cheese from Dubois*?

Please,—I said—*what is the name of this cheese, and where is this boutique? I want to go there and buy this cheese myself.*

Marcel literally froze, with his mouth open.

Yourself? Are you kidding? Do you imagine that you will just GO *to* Quatrehomme *and* BUY *Pont-l'Évêque*??!!

Why not? What could be complicated about buying cheese?

Of course, it is simple. But getting GOOD *cheese takes time—about a year, maybe, a half-year if you really know how to go about it.*

By this time, it was my own mouth that was agape. Taking pity on my ignorance, Marcel then explained,

A respectable cheese has a narrow time window of 3–4 days during which it is good to eat. Only the proprietor of a boutique knows which of his cheeses are ripe. If you are a serious customer, he will tell you which cheeses you should buy on a particular day.

To gain his respect you have to come to the shop regularly, at least once a week, and buy a couple kilograms of several cheeses. Besides—and this is crucial—you should show some understanding of cheese.

Eventually, he will accept you, and then you will start getting cheeses like the one you have just tasted.

Ourselves, concluded Marcel—

We do it together. Five families! One of us, always the same man, has gone to Quatrehomme *for five years. Simple, and it works!*

Jim Simons

I very much admired Marcel Berger. He was an exceptional mathematician and a gracious individual. His death is a sad loss to the community.

My thesis in 1961 was a direct result of his remarkable work on holonomy in irreducible Riemannian manifolds, in which he enumerated all possible holonomy groups. These all turned out to be transitive on the unit sphere, which led to the question of how one could prove this directly without recourse to his list. Such a proof was provided by my thesis.

Berger also proved the first pinching theorems in Riemannian geometry, proving that an even-dimensional compact Riemannian manifold whose sectional curvature is strictly between $\frac{1}{4}$ and 1 is necessarily homeomorphic to the sphere and later showing that, if we don't insist that the inequalities be strict, the only exceptions we've now allowed in are just the symmetric spaces of rank 1. This was seminal work. I showed it to my first student, Jeff Cheeger, and it inspired him to write an excellent thesis. Jeff then went on to develop these ideas further, initially alone and later in collaboration with Detlef Gromoll. This illustrates the way that Berger's ideas quickly led to an important body of work, relating curvature to the topology of Riemannian manifolds.

This was seminal work.

Jim Simons is chairman at Renaissance Technologies and research professor of mathematics at Stony Brook University.

Jeff Cheeger

Marcel Berger was one of the leading Riemannian geometers of the mid-twentiethth century and for many years the dean of French differential geometry. His beautiful contributions to metric geometry, especially those from 1955 to 1980, provided a high standard for the period and played a major role in setting the stage for the subject's next explosive phase of development. A few of the highlights were the even-dimensional sphere theorem with (sharp) 1/4-pinching, associated results such as the minimal diameter rigidity theorem (arguably, the first global rigidity theorem), the sphere theorem with pinching slightly below 1/4 (via a compactness/contradiction argument), and his comparison theorem, known as "Rauch II"; for details, see [4], [6], [12], [7].

With hindsight, Berger's lemma, concerning pairs of points at maximum distance, to some extent foreshadowed the Grove–Shiohama–Petersen theory of critical points of distance functions, which has had a number of very remarkable applications.

Berger's example of a collapsing sequence of positively curved metrics on S^3 is now viewed as the first nontrivial example of collapse with bounded curvature. It is very illuminating and remains required reading for anyone interested in learning the theory of collapse with bounded curvature. To fully appreciate its significance, one must be aware that at the time, Riemannian geometry was very short of "meaningful" examples. The point for Berger was that it demonstrated that the known lower bound estimates on the injectivity radius in the odd-dimensional simply connected case no longer held for pinching $< 1/9$.

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Berger in Paris in 1968.

Berger's sharp upper bound for the injectivity radius in terms of the volume [9], [11] is remarkable in that curvature does not enter. Closely related work with Jerry Kazdan was used in their proof [18] of the Blaschke conjecture: Manifolds for which the cut locus of every point consists of a single point are isometric to standard spheres.

As a graduate student interested in learning Riemannian geometry, I was enormously impacted by Berger's work. It was inspiring. I became excited by the idea that one could make progress in geometry just by drawing simple pictures, staring at them, waiting for the light to go on, and at the appropriate point bringing in some ordinary differential equations and perhaps some topology. A bit later, when I was a postdoc at Berkeley, I was taken aback when Dennis Sullivan said, "Don't you think that it's just a little bit naive?"

From my personal perspective, Riemannian geometry was virtually a perfect subject. It had received tremendous impetus from the geometrical tools introduced by Rauch, Berger, Klingenberg, and Toponogov, and a bit later by Gromoll and Meyer. Yet it was still in a very underdeveloped state. One could get started without having to know all that much. Analysis was not playing the central role that it came to play after the revolutionary impact of the work of Yau. That was just as well, since at the time, I didn't know much analysis. During the first few years after my degree, almost all of my work was in metric geometry, much of it in collaboration with Detlef Gromoll. My book with David Ebin, *Comparision Theorems in Riemannian Geometry*, contained an exposition of many of the key results up to 1975, including, of course, various theorems of Berger.

Berger's fundamental classification of the candidates for holonomy groups of Riemannian manifolds, a tour de force, was accomplished [2], [3] during the period 1953–55. An unexplained consequence was the observation that if the holonomy group of M^n acts irreducibly, then either it acts transitively on the unit sphere or M^n is a symmetric space of rank ≥ 2. In his thesis, Jim Simons gave an algebraic proof of this fact, one which avoided the classification. Later, Jim became my teacher and directed me to the study of comparison theorems. He said, "There isn't too much else going on in geometry right now." (And this just slightly before his own spectacular work on minimal varieties and Chern–Simons invariants!)

Berger visited Berkeley for the "big" Global Analysis Summer Institute, which lasted for three weeks during the summer of 1968. I was just finishing up my postdoc, and at some point I was told that the great man would like to meet with me to discuss geometry. Excited but a bit nervous, I asked if I could bring along my mathematical big brother, Detlef. Naturally, it turned out that Marcel (as I will now call him) was kind and friendly.

A few summers later, at Marcel's invitation, Detlef and I spent several weeks in Paris. He was surrounded by a large and impressive group of students, including Jean-Pierre Bourguignon, with whom we later became good friends. During the visit we asked Marcel if he could recommend a restaurant. He sent us to a place with the amusing name Le Sanglier Bleu, where we had a great meal and a super good time.

It turned out that Marcel had become interested in the spectrum of the Laplacian, especially the question of what geometric information might be contained in the asymptotic expansion of the trace of the heat kernel. The coefficients in this expansion are local invariants involving the curvature and successively more of its covariant derivatives. Attention had been drawn to them by the visionary paper of McKean and Singer,[4] in which they proposed the heat equation method for proving the Atiyah–Singer index theorem for the operator $d + \delta : \Lambda^{even} \to \Lambda^{odd}$, whose index is the Euler characteristic, and expressed the hope that a "fantastic cancellation" in the coefficients would lead to the Chern–Gauss–Bonnet integrand.

Something significant which came out of the Laplacian project was Marcel's book with Gauduchon and Mazet, *Le spectre d'une variété riemannienne* [17]. During our visit, he told me, "My less strong students, I put them on the

[4]*Curvature and the eigenvalues of the Laplacian,* J. Diff. Geometry ***1** (1967), 43–69.*

spectre." This was puzzling, since it seemed to me that he didn't have any "less strong" students. Also, I was vaguely aware of the Ray–Singer conjecture on the equality of Reidemeister torsion and analytic torsion, which struck me as very exciting, though I could only understand it in a superficial way. I knew that I needed to learn more about the heat kernel, and *Le spectre d'une variété Riemannienne* was the ideal place to start. I decided to make an intensive study of this book, which worked out very well.

an aversion to taking the credit for his own mathematical contributions

In the period 1980–2000, I encountered Marcel many times during stays at IHÉS, where for most of that time he was the director. As usual, he was nice and friendly, but at some point it struck me that his personality had a side which seemed slightly "quirky," though not quite in a way on which I could put my finger. One thing I did eventually realize was that he had an aversion to taking the credit for his own mathematical contributions.

Marcel wrote a large number of excellent expository works on different aspects of geometry. A number of them were at an "elementary" level, though perhaps from a somewhat advanced standpoint. Most notable for me were the monumental *A Panoramic View of Riemannian Geometry* [14] and the shorter *Riemannian Geometry during the Second Half of the Twentieth Century* [13]. While he was preparing these, we had quite a few email exchanges. I was impressed by his strong desire to get the history right, no matter how much back and forth it took, and once again by his aversion to explicitly awarding himself the credit for his own work. On several occasions I was forced to insist, "But, Marcel, it was *you* who did that!"

Marcel was an outstanding mathematician and a unique character. I miss him.

Jean-Pierre Bourguignon

I am indebted to Marcel Berger for a multitude of things that have decisively influenced my mathematical life. These reflect not only his unique place in the rich landscape of late-twentieth-century French mathematics but also his unique personal qualities: his generosity, his geometric vision, his international network of friendships, his modesty, his high standards, and his strong commitment to public service.

Having given up previous academic positions in Strasbourg and Nice, Marcel Berger became a professor in Paris just in time to experience the tumult of the 1968 student riots. Perceiving that society was undergoing a transformational shift in relations between generations, he concluded that it was imperative that he overcome his natural shyness in order to establish a seminar and develop strong working bonds with a group of research students. One consequence was that he agreed in 1969 to become my thesis adviser, taking over a role previously fulfilled by Gustave Choquet. This turn of events was in part made possible by the fact that I had attended his graduate course on the spectrum of a Riemannian manifold, notes on which were at that time becoming famous under the acronym BGM (for Berger–Gauduchon–Mazet). But I came to Riemannian geometry largely by default after failing to convince any leading Parisian expert on classical mechanics to supervise my project of solving the Euler equations by applying Arnold's strategy to the group of diffeomorphisms—a strategy that David Ebin and Jerrold Marsden in any case successfully developed soon afterwards. This left me to master Riemannian geometry from scratch, having to learn almost everything.

The next year, however, because I was Berger's only protégé to currently hold a CNRS *attaché de recherche* position, Marcel bestowed upon me the amazing gift of a significant block of his time each week. Every Tuesday he would spend the whole day trying to tell me everything he knew about geometry—and that, of course, was vast: curvature and topology, the geometry of geodesics, holonomy, Kähler geometry, the Calabi conjecture, etc. The topics ran the gamut of geometric research, and the approach he took was holistic.

Marcel waxed enthusiastic when I suggested that David Ebin might spend a year in Paris teaching us global analysis. He had previously collaborated with David during a visit to Berkeley, and Marcel now clearly foresaw the remarkably transformative role that nonlinear partial differential equations would surely play in the development of geometry in the years to come.

Marcel insisted that his Wednesday seminar (which, per local tradition, also bore his name) should be a real working seminar, with a year-long theme providing the focus for the lectures. While he would always have the last word when it came to the choice of topics, he consistently gave us all the opportunity to present our most recent results. We learned a huge amount, because Marcel's ambition was for everyone to become acquainted with every topic necessary for a deeper understanding of geometry. He nonetheless gave his students plenty of leeway to develop their own research programs. For example, he in no way objected to my own decision to dedicate my *thèse d'État* to the development of a "stratification of the space of Riemannian structures."

Berger developed a network of friendships with international mathematical leaders and was always eager to see his students benefit from this network. His strong ties with Wilhelm Klingenberg gave us the benefit of participating in the "Small Arbeitstagung in Differential Geometry," which provided us with our first real international exposure, since, paradoxically enough, Bonn was more international than Paris at the time. Then, in June 1972, a visit to Paris by Jim Simons, who had re-proved Berger's classification of Riemannian holonomy from a new viewpoint, afforded me the opportunity to spend

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the academic year 1972–1973 in Stony Brook, which had become the Mecca of differential geometry, with no fewer than fourteen specialists on its faculty. Spending a year in the US had a decisive impact on my own career; it not only resulted in my first joint work with S. T. Yau, who was there in his first professorial position, but also allowed me to attain a more global perspective on my own work. This visit was followed by a summer spent at Stanford at the invitation of Robert Osserman, again on Berger's recommendation. These contacts then led to several other opportunities. For example, participation in the *Differential Geometrie im Grossen* meetings at Oberwolfach, organized by S.-S. Chern and Klingenberg, became the source of many unforgettable experiences and gave me privileged access to Chern because of the latter's considerable respect for Berger. The good relations Marcel developed with Isadore Singer during a stay at MIT opened other doors for me and led to inspiring discussions of the relationship between geometry and physics. Similarly, Berger's close ties with Shingo Murakami led to my first visit to Osaka in 1979, followed by many other trips to Japan; Berger's work is highly influential in Japan and has provided the starting point for many papers by leading Japanese mathematicians. And the list of course goes on, as Berger's international contacts put me in contact with such figures as Eugenio Calabi, Manfredo do Carmo, James Eells, Jerry Kazdan, and Shoshichi Kobayashi, to name only a few.

Besse was intended as a direct challenge to Bourbaki.

The "Berger school" eventually chose to cloak itself under the collective pseudonym of Arthur L. Besse after a foundational "Round Table" workshop in the village of Besse-en-Chandesse. Besse's last name was of course provided by the village, while his first and middle names of Arthur and Lancelot came from the medieval legends of the Knights of the Round Table.[5] In a very real sense, Besse was intended as a direct challenge to Bourbaki: Besse's philosophy emphasizes the need to mobilize many different specialized bodies of knowledge to attack deep questions, in complete contrast to the model provided by Bourbaki's *Éléments de mathématiques*. Indeed, the approach adopted by Besse was rather similar to the one previously advocated by André Weil in his *Publications de l'Université de Nancago*,[6] much to the discomfiture of some of Weil's Bourbaki collaborators. In any case, Besse's approach soon justified itself by bearing tangible fruit. It also became evident that this exciting and successful venture would not have been possible without Berger's leadership and his generous, open personality.

[5] *While nearly every child in the English-speaking world seems to have heard tales of King Arthur, Lancelot, and the Knights of the Round Table, these characters are perhaps best known in France to those with something of a scholarly interest in the medieval French romances that are the main sources of Arthurian legend.*

[6] *Nancago refers to the two cities where Weil taught at the time: Nancy and Chicago.*

Berger (right) talks with Hermann Karcher at Oberwolfach in 1982. Also present: Maung Min-Oo (left) and Shoshichi Kobayashi (middle).

Of course, much more could be said. Marcel's conviction that Mikhail Gromov's ideas would completely transform differential geometry was quickly borne out by events, and the powerful impact of this new perspective on French mathematics became particularly evident after Gromov decided to settle at the IHÉS. Berger went on to become a tireless and persuasive public advocate of Gromov's ideas and vision.[7]

But here I would really have to enter into yet another chapter of Berger's life, namely, his directorship of the Institut des Hautes Études Scientifiques (IHÉS). Although I eventually became Berger's successor there, I am actually less qualified to discuss this period than certain other contributors to this collection of reminiscences, and I will therefore leave it to them to comment on this important part of Marcel's life and career. Paradoxically, much of my prior knowledge regarding the running of the institute was gleaned from conversations with Marcel's predecessor, Nicolaas Kuiper, during the period of his own mandate. Since the duties of the director of the IHÉS do not include the selection of his successor, I next discussed my views concerning the institute's future, not with Marcel, but rather with the institute's permanent professors. Only when my term as director was about to start and upon my return from an extended stay at MSRI did Marcel share with me some of his vision and experience concerning the institute.

Marcel's absence presents us all with a profound loss. We will miss his frequent exhortations to think geometrically, as well as his gentle manner and modest attitude, which will remain a model for us all.

[7] *Berger's masterful survey of Gromov's work, "La géométrie selon Gromov," a lecture delivered at the IHÉS in 2000 in celebration of Gromov's Balzan Prize, was recorded by François Tisseyre and is available as a free video at* `https://vimeo.com/188585117`.

Dennis Sullivan

Marcel Berger and his wife, Odile, provided a warm family atmosphere for the scientists at IHÉS during the years of his directorship of the Institut des Hautes Études Scientifiques. I fondly remember Odile welcoming us to their apartment overlooking the Champ-de-Mars while noticing Marcel's patient tolerance of my American manners. These personal contacts introduced me to many aspects of the French tradition—not only to such mathematical matters as the history of differential geometry in France but also to some of the charming eccentricities of everyday life in that country.

Marcel was an able leader of the IHÉS. On his watch, important matters were mainly decided by consensus, but I remember one occasion when he felt compelled to override the deliberations of the Scientific Committee, vetoing a proposal on very reasonable, common-sense grounds.

Much later, in mathematical activities involving colleagues here at Stony Brook, I came to truly appreciate the deep and beautiful contribution Marcel made to the subject of special holonomy. These results, elegantly reformulated in the thesis of Jim Simons, have led to a flourishing key area of current research in mathematics and physics. They contribute to the unifying theme underlying activities at our Simons Center for Geometry and Physics and also to certain international collaborations funded by the Simons Foundation.

Jacques Lafontaine

Since many people have already written so eloquently about Marcel Berger's mathematical work, I will not try to expand further on that theme, but will instead confine myself to a few remarks about Marcel's personality.

He strongly felt that lectures should be accessible to beginners.

Marcel enjoyed attending mathematical lectures, and when he did so he always asked many questions. But these questions often focused on elementary points and could sometimes seem so naïve that someone might be tempted to blurt out, "Marcel, how is it possible that you don't know that?"

However, he eventually confessed to me that he often asked questions whose answer he knew perfectly well, either just to slow down the lecturer or to draw attention to some neglected important point. He strongly felt that lectures should be accessible to beginners, at least whenever possible.

Of course, Marcel's former students all eventually became aware of the game he was playing. Nonetheless, when asking such questions he could manage to appear so bewildered that we often caught ourselves being taken in by the performance. He wasn't just an outstanding mathematician—he was an amazingly good actor!

Now that he has left us, these feigned expressions of bewilderment remain vividly alive in my memory and somehow encapsulate the essential benevolence of his attitude towards everyone around him. I have met mathematicians whose technical skills were more highly praised or highly prized, but I have never met a mathematician who had so thoroughly earned the unalloyed admiration and affection of his students.

Having now told you something about Marcel, I will conclude by telling you the story of his frequently misunderstood collaborator

Arthur L. Besse (1975–?): Mathematical Knight of the Auvergne

Marcel was literally obsessed by certain questions that he judged to be natural and important: Riemannian analogs of Kac's celebrated problem on "hearing the shape of a drum," isosystolic inequalities, and classifying Riemannian manifolds whose geodesics are all closed and of the same length. In the early 1970s, one seemed to know surprisingly little about this last question, despite the availability of some key evidence:

- the series of examples given by the rank-1 symmetric spaces,
- the existence of metrics on spheres that enjoy this property but are geometrically different from the obvious "round" example, and
- the fact that any manifold admitting such a metric must have the cohomology ring of a rank-1 symmetric space.

Marcel thus judged the time to be ripe for a systematic attack on the problem and therefore, in May 1975, organized a workshop in the village of Besse-en-Chandesse,[8] located in the highlands of the Massif Central region of south-central France. The meeting took place at a hotel named Les Mouflons, after the wild mountain sheep that wander the surrounding Volcans d'Auvergne Natural Park, under a sky full of towering cumulo-nimbus clouds. Both of these features of the area were later mentioned by the pseudonymous Arthur L. Besse in the cryptic introduction to his first mathematical publication. Even though the locale was reasonably easy to reach from Paris or Grenoble, this tranquil, bucolic setting seemed a world away from the participants' daily routine, and we felt a certain quiet pride in being able to show the foreigners in our midst around this picturesque village, with its remarkable natural and architectural environment. While some foreign participants initially seemed surprised to be

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[8] *Traditionally simply called Besse, the village is now part of a larger municipality called Besse-et-Saint-Anastaise.*

spending so much of their time at the table, the food was outstanding, and the delightful local cheeses seemed to help put our discussions on a fast-track to mathematical discovery.

In view of the meeting's overwhelming success, Marcel soon urged the participants to cast their newfound understanding in permanent form by jointly authoring a book. A consensus emerged that the group should write under a collective pseudonym, thereby following (in this respect only) the earlier model provided by Bourbaki. The last name of Besse was chosen in homage to the site of our fruitful meeting. Because the CNRS had designated the workshop as a "round table" for funding purposes, it was natural to invoke the legend of Camelot to provide the first name of Arthur. To decrease the probability of a lawsuit being brought by some real-life Arthur Besse, it was then decided that the name should be further embellished by the addition of an American-style middle initial, abbreviating the undisclosed middle name of Lancelot. Such was the birth of Arthur L. Besse.

The writing had a catalytic effect.

As soon as the meeting ended, Arthur and his collaborators got to work. The book, entitled *Manifolds All of Whose Geodesics Are Closed* [19], was published by Springer in 1978. The writing had a catalytic effect; as the book took shape, significant new results were obtained by members of the group, and these were then added to the book as appendices under the names of the individual authors. In one such case, the British author and his French-teacher wife submitted an appendix carefully written in beautiful French, only to be met with implacable demands from the publisher that it be translated into English! However, in the end, we did manage to wrangle one major concession: the book's introduction was published in French and takes the form of a cryptic letter bearing the signature of Arthur L. Besse, ostensibly writing from his estate at Le Faux.[9] Besse's letter begins with some brief musings on a visit he received four decades earlier from a mathematically inclined general[10] who had described both his grandiose vision of the whole of mathematics and his plans for a treatise on differential geometry that had somehow never come to fruition. The practical joke then continues with a wry mixture of fact and fiction concerning French mathematics, Besse's beloved home, and the contents of the book.

But Arthur Besse didn't stop there. Pleased with the success accorded to his first foray into collaborative writing, he then went on to write a second book, *Einstein Manifolds* [20], which differs markedly from its predecessor in both style and scope; while the first book might be called a monograph, the second could better be termed a treatise. Preparations for this new effort obliged Besse to leave his native Auvergne for the adjoining Rouergue region, where a meeting at Espalion, 100 miles south of Besse-en-Chandesse, resulted in a broad outline of the new book in September 1977. The intellectual pleasures of our mathematical discussions were once again reinforced by those of the local gastronomy, and the weather, far milder than that at our previous meeting, allowed for a memorable excursion to the medieval pilgrimage site of Conques, some fifty kilometers to the west.

The author's royalties from the first book helped underwrite the working meetings that made the second one possible. Thicker, more classic, and less rococco than its predecessor, this second book appeared in 1987, exactly at a moment when a conjunction of scientific developments highlighted its many deep connections to the mainstream of differential geometric research. It continues to be a key reference even now and has become Arthur Besse's best known and most widely cited work.

And what of a third book? Well, it was never in the cards for a variety of reasons. In the 1970s Besse's collaborators were largely concentrated at Jussieu and in the surrounding neighborhoods of Paris; for example, the École Polytechnique was still located in Paris, on the nearby rue Descartes. But little by little, the group dispersed to the suburbs and to other regions of France. The international model of publish or perish also began to take hold in France, making it far more difficult to convince mathematicians (and especially young ones) to work quasi-anonymously, focusing on the beauty of mathematics rather than on their own personal glory.

For all these reasons, Arthur Besse eventually became content to simply lend his name to a seminar in differential geometry, first at Jussieu and later at the École Polytechnique's new location in Palaiseau. However, Arthur continued to play a role in organizing conferences from time to time. These activities have left a durable mark on mathematics in the form of various publications bearing the Besse imprimatur [1], [15], [21], [22].

One of Arthur Besse's last public acts was to organize a conference at the CIRM in Luminy, honoring both the twentieth anniversary of *Einstein Manifolds* and the eightieth birthday of Besse's true father, Marcel Berger. Speaking on behalf of Springer-Verlag, Arthur's old friend Catriona Byrne made the surprise announcement that a second edition of *Einstein Manifolds* had been published and then proceeded to provide each conference participant with a free copy. But while the conference center, located amidst the limestone cliffs that line the coast south and east of Marseille, was both functional and pleasant, it also served as a sign of the times, reminding some of us that those heady days of intimate meetings at gastronomical hotels had, alas, now become a thing of the past.

As for Arthur, it seems that he has now definitively retired to a life of leisure. But his beneficent presence will long remain with us, because the royalties generated

[9] *The name of this real place, located roughly a kilometer southwest of Besse-en-Chandesse, means* The False *or* The Forgery *in French.*

[10] *General Nicolas Bourbaki and Arthur L. Besse share an important commonality that is rarely noted. Bourbaki's founding conference was held July 10–17, 1935, in Besse-en-Chandesse. Yes, really!*

by his books have been donated in perpetuity to the European Mathematical Society for the sole purpose of providing financial support to young mathematicians in developing countries.

Jerry Kazdan

Marcel Berger was a world leader in differential geometry. He had inspiring geometric vision, asking fertile questions that give insight to techniques involving both geometry and analysis.

He had inspiring geometric vision.

In addition to his own fundamental research, his seminars and projects attracted a remarkable group of younger mathematicians, primarily in France but also worldwide. Some of the fruits are collected in books, both his own and by his group, "Arthur Besse."

He welcomed younger mathematicians. Of one early trip I made to Paris, I warmly remember a lunch in his apartment. He and his wife, Odile, were splendid hosts, warm and interesting.

In both 1992 and 1994 he visited our department at the University of Pennsylvania. His lectures and conversations were widely appreciated by everyone, including our graduate students.

My main direct mathematical interaction with Marcel concerned *Wiedersehen manifolds*. Intuitively, these are compact Riemannian manifolds (M^n, g) without boundary with the property that at any point (think of the north pole of the standard round sphere) if a number of people beginning in different directions walk along geodesics at the same constant speed, then after a certain amount of time they all meet again (think of the south pole). The conjecture was that this is only possible on the standard round sphere. It had been proved in dimension 2.

Berger had the vision that a key ingredient in the proof of the higher-dimensional cases would involve a sharp isoperimetric inequality. At a point p of M consider a small disk of radius r. As r gets larger, the disk eventually begins to overlap itself (think of a small disk on the surface of a two-dimensional torus in $\mathbb{R}^3$ and gradually increase the radius of this disk). This largest radius is called the *injectivity radius* at p. The largest radius ρ that works for all points p on (M^n, g) is called the injectivity radius of (M^n, g). For the standard round sphere of radius c in $\mathbb{R}^{n+1}$ it is the geodesic distance between antipodal points: $\rho = \pi c$.

Berger conjectured:

Isoperimetric Inequality. *If (M^n, g) has injectivity radius ρ, then its volume is at least as large as the volume of the round sphere with the same injectivity radius. Equality holds if and only if the manifold is isometric to this round sphere.*

By a beautiful geometric argument, Berger reduced the proof to a quite specific sharp analytic inequality. I learned of this from a conversation with Jean-Pierre Bourguignon at a 1977 geometry conference in Berlin. I was later pleased to be able to prove [18], [19] the desired inequality.

My contribution primarily relies on convexity inequalities. It would be useful to find a more conceptual (and less technical) proof.

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Marie-Louise Michelsohn

Over the years, I have always had immense fondness and respect for Marcel Berger. I was truly saddened to learn of his death, and it is not an exaggeration to say that I miss him.

I would like to communicate a side of him that many may not know by an anecdote that has stayed with me these many years and that I think of frequently.

About three decades ago Marcel and I were corresponding mathematically, which at that time was done by postal service and hand-written letters. In his first letters, he addressed me as *vous*, as did essentially all French male mathematicians when addressing their female colleagues. In a later letter, I finally requested that he address me as *tu*, but he then replied that this would be "contre mon éducation." I replied that to use *vous* for women was to exclude them from the *confrérie* and so was not at all positive. It should be remembered here that Marcel's upbringing was particularly conservative. Nonetheless, he understood right away, and until his death I was therefore always *tu*.

This is a small illustration of the extraordinary character of Marcel Berger. I am greatly saddened by his death.

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Pierre Pansu

One day in October 1979, at Misha Gromov's first lecture at Paris VII, Marcel Berger asked the audience if anyone would volunteer to take notes on the lecture series, with the idea that these notes might eventually form the basis for a book. I raised my hand. In many ways, this simple, spontaneous act determined much of my future: I would become a geometer, I would benefit from Gromov's influence, I would participate in collective book projects, and I would receive a great deal of personal attention from Marcel Berger.

Berger's fascination with Gromov's achievements meshed perfectly with his more general enthusiasm for geometry. Nevertheless, he was also fond of saying that every deep result in geometry should entail some serious input from analysis. This precept has sometimes served as a counterweight to the style of thinking that I learned from Gromov.

Berger considered collective authorship to be a perfect training ground for young mathematicians. Fortunately,

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he was a model co-author. Although hardly a model pupil as a child—he was allegedly expelled from 11 schools!—he was the only one in our "class" of twenty co-authors to consistently turn in his "homework" (book chapters) on time. His chapters sometimes contained key original results, and this enhanced the value of our work, because he would never have bothered to publish these results under his own name.

Berger exercised considerable care in checking (and improving) various proofs in my thesis. Although he claimed to simply be doing his duty, this was an incredible favor. During the decades that followed, his attention from afar continued to provide me with a key source of cheer and encouragement.

I never regretted raising my hand.

David Ebin

Berger and I first met in 1968 at an AMS Summer Institute in Berkeley. I had heard of him and was interested in meeting him. I saw someone I thought might be Berger chatting with other mathematicians, and so I asked, "Are you Berger?" He replied, "Yes, but there are two." Since I knew only his first initial, I then tried to be more precise, asking, "M. Berger?" And again he said,"Yes, but there are two." Eventually, I found out that the other one was Mel Berger,[11] who was also at the Summer Institute.

Berger was interested in variations of structures on Riemannian manifolds due to a change in the Riemannian metric. I had recently written my thesis on the space of Riemannian metrics on a manifold and on the natural action of the diffeomorphism group of the manifold on this space. The thesis gave a convenient way to distinguish between changes in the metric that were due to diffeomorphisms and those that represented actual changes in the geometry. This turned out to be what Berger was looking for, so we discussed the matter and soon produced a joint paper [16].

Three years later, Berger and Laurent Schwartz arranged for me to give a semester of lectures at Paris VII and the École Polytechnique. They also found lodging for my wife and me in the IHÉS Residence, and it was most enjoyable. My wife, who is an amateur violinist, played in a string quartet at the Château de l'Ermitage, in nearby Gif-sur-Yvette. The Bergers also invited us to dinner twice—first with their family, and later to what might today be called a mathematician's power dinner, with the Cartans and the Thoms.

I next saw Berger at another AMS Summer Institute, in Stanford. There he gave a series of lectures on the work of Colin de Verdière. It used the heat equation to get information about geodesics and frequently involved the fundamental solution of the heat equation, FSHE, which Berger referred to as *fish*. After that, Berger came to visit our department in Stony Brook several times. During his last visit, he gave a course on history of mathematics, perhaps because he had largely stopped doing his own research by that time.

To me, Berger was a kind, soft-spoken man of fine, strong good will. I will miss him, and I think we all will.

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[11] *Melvyn Berger's surname is pronounced* Burger.

Karsten Grove

I feel enormously privileged and fortunate that, at an early stage of my career, two of my idols, M. Berger and W. Klingenberg, both chose to take me under their wing. I first met Berger at a small workshop in Bonn in the fall of 1971, where Berger, Klingenberg, their students, and their visitors accounted for the vast majority of the participants. Fortunately, I later had the pleasure of seeing Berger many more times, not only at Oberwolfach but also in Paris, at Luminy, and, most memorably, at the remarkable *Table Ronde* workshop held at Besse-en-Chandesse in 1975.

The topic of the Besse conference was Riemannian manifolds with periodic geodesic flow, and stellar presentations by Berger and his school concerning such manifolds provided the main focus of the activity there. This was the germ of what later evolved into Arthur Besse's first book, *Manifolds All of Whose Geodesics Are Closed* [19], which to this day covers much of what is known on the topic. This was later followed by two more Besse books, *Einstein Manifolds* [20] and *Géométrie riemannienne en dimension* 4 [1].

Among the many profound and diverse contributions Berger made to Riemannian geometry, those that most inspired and affected my own work were the ones concerning manifolds with positive sectional curvature.

Prior to 1960, the only known (simply connected) manifolds of positive curvature were the rank-one symmetric spaces, i.e., the standard spheres and projective spaces. Moreover, the celebrated Rauch–Berger–Klingenberg $1/4$-*Pinching Theorem* shows that, up to homeomorphism,[12] these are the only such manifolds with curvature varying between 1 and 4; and if the manifold is not topologically a sphere, Berger moreover proved that it must actually be *isometric* to a rank-one symmetric space. The fact that these spaces all have diameter at least $\pi/2$ helped inspire my own proof, with Shiohama and Gromoll, of the so-called *Diameter Sphere Theorem*. In conjunction with a 2001 result of Wilking, this says that a complete Riemannian manifold with sectional curvature ≥ 1 and diameter $\geq \pi/2$ is finitely covered by either a topological sphere or a compact rank-one symmetric space.

In the early 1960s, Berger found two additional examples [5] of positively curved manifolds by studying a class of spaces called normal homogeneous manifolds. Since then, many other manifolds of positive sectional curvature have been found, with some new construction being discovered every decade or so. A common feature of all known constructions is that they produce examples with

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[12] *A 2009 paper by Brendle and Schoen shows that the conclusion even holds up to diffeomorphism; thus, an exotic sphere can never admit a* $1/4$*-pinched metric.*

relatively large groups of symmetries, and a great deal of subsequent work has therefore focused on classifying positively curved manifolds with large isometry groups. One of the basic tools used in this area is also due to Berger, who proved [8] that any Killing field (vector field generating a one-parameter group of isometries) on an even-dimensional positively curved manifold must have a zero; similarly, in odd dimensions, two Killing fields will be dependent at some point. Yet another important tool in this context, often called the *Cheeger deformation*, has its roots in a deformation introduced by Berger in his analysis of what are now called *Berger spheres*.

As a supportive colleague with a positive and friendly outlook, Berger will be sorely missed. But as an exceptional mathematician with a tremendous impact, he will never be forgotten.

EDITOR'S NOTE. See also Berger's article on Gromov in the February 2000 *Notices*, "Encounter with a Geometer," www.ams.org/notices/200002/fea-berger.pdf and "WHAT IS...a Systole?" in the March 2008 *Notices*, www.ams.org/notices/200803tx080300374p.pdf.

Berger's major surveys [13], [14] are highly recommended.

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Photo of Marcel Berger in the Bourguignon section ©Mathematisches Forschungsinstitut Oberwolfach Photo Collection; photo by Dirk Ferus, Berlin.

Ciprian Manolescu Interview

Conducted by Alexander Diaz-Lopez

Ciprian Manolescu is professor of mathematics at the University of California, Los Angeles, working in topology and geometry. He is a winner of the Morgan Prize and the European Mathematical Society Prize, a Fellow of the American Mathematical Society, and an invited speaker at the 2018 International Congress of Mathematicians. He is the only person to achieve three perfect scores at the International Mathematical Olympiad.

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Diaz-Lopez: *When did you know you wanted to be a mathematician?*

Manolescu: As a kid I liked math puzzles and games. I remember being particularly fond of a book by Martin Gardner (in Romanian translation). In middle school, I started participating in math olympiads, and by then I figured I wanted to do mathematics in some form when I grew up. At the time I did not know any research mathematicians personally, but I read a couple of books popularizing mathematical research and news articles about Wiles' proof of Fermat's Last Theorem. It all sounded very exciting.

Diaz-Lopez: *Who encouraged or inspired you?*

Manolescu: My parents have been extremely supportive, and I was fortunate to have good teachers—especially my high school math teacher Stefan Alexe, who lent me a large part of his collection of problem books to prepare for the olympiads. Clearly, however, my greatest mathematical debt is to my college and PhD advisor, Peter Kronheimer. We had weekly meetings for several years, and I learned a great deal of mathematics from him.

Diaz-Lopez: *How would you describe your work to a graduate student?*

Manolescu: I work in topology, where the main motivating problem is the classification of smooth shapes (manifolds). There is a theorem that a complete classification is not possible in dimensions four or higher, because we run into group-theoretic difficulties. However, one can focus on manifolds with a reasonable fundamental group, and then the problem becomes tractable in dimensions five or higher. Dimensions three and four are the interesting ones. In dimension four, most questions are still open, but (starting with the work of Simon Donaldson in the 1980s) there has been steady progress on this topic, using gauge theory. Mathematical gauge theory is the study of a certain kind of partial differential equations (Yang–Mills, Seiberg-Witten) that originated in theoretical physics. My research focus is on Floer homology, which is a way of capturing the gauge-theoretic information for a three-dimensional manifold. The ultimate goal is to understand four-dimensional manifolds, and Floer homology tells us something about what happens when we cut the four-dimensional mani-

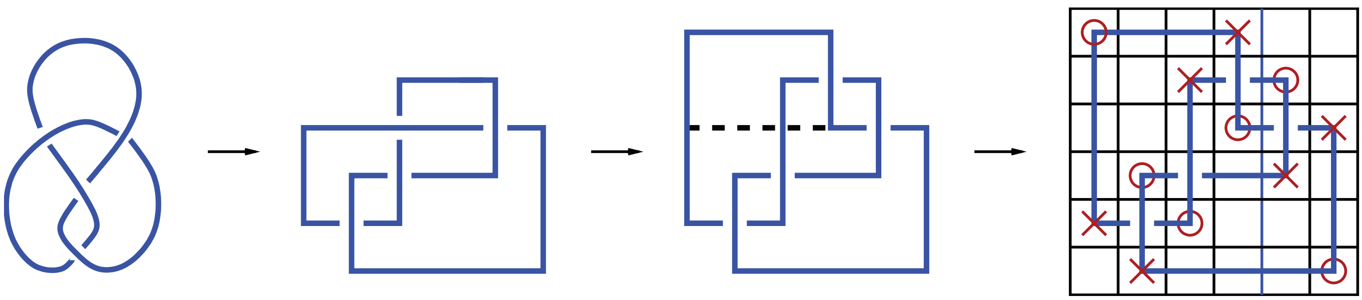

Figure 1: The unknot detection algorithm based on knot Floer homology starts by representing the knot by a grid diagram: an n-by-n grid with one O marking and one X marking in each row and in each column, such that if we join the Os to the Xs by vertical and horizontal arcs (with the vertical arcs on the top), we obtain the knot. The figure above shows how to transform a planar projection of the figure-eight knot into a grid diagram.

fold along a three-dimensional one. Floer homology also has applications to three dimensions per se, and to knot theory. For example, with my collaborators Peter Ozsváth and Sucharit Sarkar, we used Floer homology to describe a new algorithm (see Figure 1) for detecting when a tangled-up piece of string can be unknotted (without breaking it).

Diaz-Lopez: *What theorem are you most proud of and what was the most important idea that led to this breakthrough?*

Manolescu: The disproof of the high-dimensional triangulation conjecture. The question of whether all manifolds can be triangulated was raised by Kneser in 1924. The work of Andrew Casson (based on that of Mike Freedman) in the 1980s provided counterexamples in dimension four. In dimensions five or higher, the question had been reduced (by David Galewski, Ronald Stern, and Takao Matumoto) to a different problem, about the three-dimensional homology cobordism group. Thus, one could hope to use the techniques of gauge theory to solve it. I knew about the problem for a while, but I only started thinking about it seriously in 2012, and it came to my attention in a roundabout way. I had an idea about how to construct a version of an invariant of Casson's, and I was asking other mathematicians about what kind of applications one could hope for from such a construction. Several mentioned the triangulation problem, as something that Casson's invariant helped solve in dimension four but not in higher dimensions. It soon became clear that the version I had in mind would be of no help with this question. Nevertheless, I then became curious about what kind of invariant one needs in order to solve the problem. I remembered a different invariant from the literature, due to Kim Frøyshov, which came closer to what was needed. After some thought, I realized that if one incorporated into the picture a further symmetry of the Seiberg-Witten equations, based on the non-abelian group Pin(2), then one could solve the triangulation problem.

Diaz-Lopez: *What advice do you have for graduate students?*

Manolescu: Choose a good advisor, by talking to the older students and asking them about their experiences; also, find out what kind of jobs the past students of each potential advisor got after graduation. Pick a subfield of mathematics that you enjoy and try to specialize early in grad school, so that you have time to write a good thesis (and maybe a few other papers). Start attending the research seminars in your area as soon as you can—even if you don't understand the talks, just getting an idea of what people are working on is helpful. At the same time, after you decide on your field, do keep learning mathematics in other areas. This will help you in the future, as many fruitful projects come from making connections between different parts of mathematics. With regard to writing your thesis, keep in mind that it is normal for it to take a long time; the transition from solving homework problems to doing research is difficult for everyone.

Diaz-Lopez: *All mathematicians feel discouraged occasionally. How do you deal with discouragement?*

Manolescu: There are a few types of disappointments that appear regularly in a mathematician's life. The first is when you cannot solve the problem you're working on. It is then good to remember that this is expected—perhaps the problem just cannot be solved with current techniques; the miracle is when you solve it! The second type is when you think you solved a problem, you start writing it up, and then you run into a technical difficulty, which might invalidate the whole thing. This happens to me quite often, and it is disconcerting to think that all the previous work I did on a project could be in vain. However, so far I've almost always managed to solve this kind of issue, after some more work. By now I know that if something looks like a technical but not a fundamental difficulty, it probably is—so I should just be pushing harder to overcome it.

Diaz-Lopez: *You have won several honors and awards. Which one has been the most meaningful and why?*

Manolescu: The Prize of the European Mathematical Society, which I received in 2012. It came as a nice surprise, at a time when I was getting bogged down in some long and technical research projects. The prize gave me the courage and confidence to think about some other fun problems on the side, including the triangulation conjecture—which I disproved eight months later.

Diaz-Lopez: *Is being the only person to achieve three perfect scores at the International Math Olympiad a close second?*

Manolescu: This was certainly important for me in high school, and it helped with my career path, since it enabled me to get a scholarship to study at Harvard. I should say that my performance involved a good amount of luck. I prepared a lot for the contests, but it is not the case that I could have solved any Olympiad problem. For example, I did not solve all the problems on the Romanian selection tests for the IMO. It just happened that I managed to see the solutions to the IMO problems, the three times that I participated.

Diaz-Lopez: *If you could recommend one book to graduate students, what would it be?*

Manolescu: *Milnor's Morse Theory* is the book that turned me towards geometry and topology. It is short, clear, and beautifully written. It not only covers Morse theory (which lies at the basis of the study of manifolds), but also touches upon several other useful topics, from the Lefschetz theorem in algebraic geometry to Bott periodicity. Chapter 2 is the place from where I first learned the basics of differential geometry.

Image Credits

Photo of Ciprian Manolescu courtesy of Reed Hutchinson.

Figure 1 based on Figure 7 of "An introduction to knot Floer homology," *Physics and Mathematics of Link Homology* **680** (2016), Amer. Math. Soc. and CRM.

Alexander Diaz-Lopez

ABOUT THE INTERVIEWER

Alexander Diaz-Lopez, having earned his PhD at the University of Notre Dame, is now assistant professor at Villanova University. Diaz-Lopez was the first graduate student member of the *Notices* Editorial Board.

WHAT IS…

a Scheme?

Tom Gannon

Communicated by Cesar E. Silva

Schemes pop up everywhere in mathematics. But getting a grasp on this ubiquitous concept is not so easy. And the bare definition might leave you wondering why a scheme is defined the way it is. The following interpretation introduces important elements in the definition of a scheme and helped me understand the basic idea. I hope it can help you too.

The idea of a scheme borrows from the idea of a manifold.

Schemes have played a fundamental role in algebraic geometry ever since they were introduced by Alexander Grothendieck [Gr]. The idea of a scheme borrows from the idea of a manifold in differential topology. Recall that a manifold is a topological space locally diffeomorphic to an open subset of $\mathbb{R}^n$, subject to some additional topological considerations to avoid pathologies. Now, if you have a function defined on an open neighborhood of a point in $\mathbb{R}^n$, you can take the derivative of that function at that point. So a manifold is a topological space that locally looks like "the place to do differential calculus." In a similar way, a scheme is a topological space that locally looks like "the place to do algebra/find zeros of functions." But that's a pretty long name, so let's call "the place to do algebra/find zeros of functions" an *affine scheme* instead. (Note for advanced readers: There's actually more to it than just a topological space. There is a *sheaf* of functions associated to the space as well, but we won't go into that here.)

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But what exactly is an affine scheme? Affine schemes are the full manifestation of a simple idea: If A is a commutative ring, we should view the elements of that ring as functions instead of points. If you're thinking about rings like $\mathbb{Q}(i)$ or $\mathbb{Z}/6\mathbb{Z}$, this can be a little weird, but if you do the construction by first imagining rings like $\mathbb{R}[x,y]$ it might be more clear. But the beautiful part is, this idea works for any ring, meaning that any ring can be viewed as functions on some space.

For the remainder of this article, fix a commutative ring A. You might ask, "What space could this ring A be a function on?" To motivate this, let's do an example from basic high school algebra. If you want to see where the polynomial $p(x) := x^3 + x^2 + x$ vanishes, your natural instinct would be to factor: $p(x) = x(x^2 + x + 1)$. Then if $p(x) = 0$, you know that either $x = 0$ or $x^2+x+1 = 0$. Put in slightly more abstract terms, if the function $p(x)$ vanishes at some point, then either the function x vanishes at that point or the function $(x^2 + x + 1)$ vanishes at that point. This is an important property we'd like our topological space to satisfy:

The "Vanishing" Property: If $f, g \in A$ and the product fg vanishes at a point p in the topological space, then either f vanishes at p or g vanishes at p.

This definition looks suspiciously close to the definition of a prime ideal of a ring. Recall that a prime ideal $\mathfrak{p}$ of a ring A is an ideal where if $f, g \in A$ and $fg \in \mathfrak{p}$, then either $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$. So if we assume that the "points" of our special topological space are prime ideals of the ring A, then we have an obvious choice for what it might mean for a function $f \in A$ to vanish at the point $\mathfrak{p}$. We'll say f *vanishes* at $\mathfrak{p}$ if $f \in \mathfrak{p}$ (or equivalently, it is zero in the ring $A/\mathfrak{p}$).

Given a ring A, define the *spectrum* of A, written $Spec(A)$ to be the set of prime ideals of A. (Again, for advanced readers: We've just defined a scheme in the category of sets, but you can put additional structure on your spectrum so that it becomes an object in the category of *locally ringed spaces*.) Now I've promised a topology on this space, and any good topology in a subject relating to "vanishing" should have the topology related to vanishing. And luckily for us, it does. We define a set of points (which are prime ideals, but we're thinking of them as "points") to be closed if it is of the form $V(S) = \{\mathfrak{q} \in Spec(A) : S \subset \mathfrak{q}\}$, for some set of "functions" $S \subset A$ (meaning, "every element of S vanishes at $\mathfrak{q}$.") Checking this forms a topology on $Spec(A)$ is not too hard and is a good exercise.

This is a good definition for affine schemes (which, recall, was meant to be related to determining where functions vanish) because in some sense the definition is "maximal." That is, any set of elements in the ring A that can be the set of functions vanishing at a point is a prime ideal (since it has the "Vanishing" Property). Moreover, functions distinguish the points in the topological space. That is, if two points $\mathfrak{p}, \mathfrak{q}$ in the space are distinct, then there is a function that vanishes on one of the points, but not on the other. Seeing why this is true is a good check on your understanding of the concept. However, given two points $\mathfrak{p}, \mathfrak{q} \in Spec(A)$, it isn't necessarily the case that you can find a function $f \in A$ vanishing at $\mathfrak{p}$ but not at $\mathfrak{q}$. This is a good exercise to check your understanding of the material, and an example is also provided below.

Any ring can be viewed as functions on some space.

Example: Consider the ring $A := \mathbb{C}[x, y]$. What are some points in $Spec(A)$? Certainly if you choose some $a, b \in \mathbb{C}$, then $\mathfrak{p} := (x - a, y - b)$ is a maximal ideal, since the quotient field $A/\mathfrak{p}$ is $\mathbb{C}$, a field, and thus it is a prime ideal. A good way to think about points of the form $(x - a, y - b)$ is to think of the points of the form $(a, b) \in \mathbb{C}^2$ (equivalently, the places on the xy plane where $x - a = 0$ and $y - b = 0$). The function $p(x, y) := x^4 + x^3y - xy - y^2 \in A$ vanishes at $\mathfrak{p}$ since

$$p(x) = r(x, y)(x - 2) + (-x - y)(y - 8) \in \mathfrak{p}$$

where

$$r(x, y) = x^3 + 2x^2 + 4x + x^2y + 2xy + 4y.$$

Also, $p(x, y)$ vanishes in our normal sense because

$$p(2, 8) = 2^4 + 2^6 - 2^4 - 2^6 = 0.$$

So we can conclude that vanishing in our new sense generalizes vanishing in the old sense.

Now the question becomes, what else can we get from this scheme construction? One quick payoff from the definition of affine schemes is that we can talk about functions vanishing at places that aren't points in the traditional sense. For example, consider the polynomial $q(x, y) := x^3 - y$. It is an irreducible polynomial by Eisenstein's Criterion, so the ideal $\mathfrak{q} := (x^3 - y)$ is another point in $Spec(A)$. This has an obvious interpetation in $\mathbb{C}^2$. It's the curve $y = x^3$. And more amazingly, notice that $p(x, y) := x^4 + x^3y - xy - y^2 = (x^3 - y)(x + y)$, so $p(x, y) \in \mathfrak{q}$. We can interpret it to mean the function $p(x, y)$ vanishes on the entire curve. That is very pretty.

This is about half of the definition of an affine scheme. Continuing with the "rings are functions on some natural space" interpretation, we can develop a *sheaf* on the space $Spec(A)$ so that we can, for example, talk about the function $\frac{x^2+y^2}{x-3}$ on the places where $x - 3$ doesn't vanish. Sheaves are beyond the scope of this short piece, which is intended only to give you a good picture of what the topological space of an affine scheme looks like. And I hope it has also given you a new interpretation of rings!

Further Reading

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EDITOR'S NOTE. This article is a revision of Gannon's post "The 'Idea' of a Scheme" on the AMS Graduate Student Blog blogs.ams.org/mathgradblog. A related article "WHAT IS...an Anabelian Scheme?" by Kirsten Wickelgren appeared in the March 2016 issue of the *Notices*, www.ams.org/notices/201603/rnoti-p285z.pdf.

ABOUT THE AUTHOR

When not doing mathematics, Tom Gannon can often be found in the kitchen cooking new recipes, or can be found in the kitchen cooking old recipes.

Tom Gannon

Photo Credit

Photo of Tom Gannon courtesy of Tom Gannon.

AMS Blogs
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Graduate Student Blog
by and for math graduate students

The AMS Graduate Student Blog, by and for graduate students, includes puzzles and a variety of interesting columns, such as this one from August 2017: blogs.ams.org/mathgradblog.

Donaldson Turns 60

by Jacob Gross

Simon Donaldson

This month [August] marks [Simon Donaldson's] 60th birthday. A conference[1] on symplectic geometry will be held at the Isaac Newton Institute, Cambridge, to celebrate this occasion and various advances in symplectic geometry. Below [in this blog post] we mention some major contributions of Donaldson to the ancient science of geometry; in particular, to gauge theory and to Kähler–Einstein metrics. ...

Photo Credits

Photo of Simon Donaldson by Bob Giglione/Simon Center for Geometry and Physics.
Photo of Jacob Gross courtesy of Jacob Gross.

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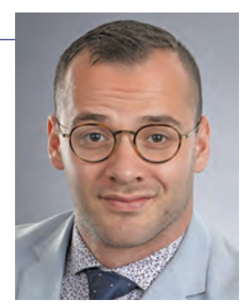
Jacob Gross is a graduate student in geometry at Oxford University, working under the supervision of Dominic Joyce FRS, and supported by the Simons Collaboration on Special Holonomy in Geometry, Analysis, and Physics. His email address is jacob.a.gross@gmail.com.



Jacob Gross

[1] www.newton.ac.uk/event/sygw05

COMMUNICATION

Reducing Bias in Faculty Searches

Elizabeth A. Burroughs

Communicated by Stephen Kennedy

Historically, women and individuals from marginalized populations have been underserved by the discipline of mathematics, with the result that the knowledge and talents of these individuals have not been available to enrich the discipline. Across the mathematical community, there is a growing awareness about the need and opportunity to foster diversity. Individuals with diverse backgrounds can provide a department with new research perspectives or methods that enhance the research work of faculty with similar interests, and may attract more diverse students and collaborators. These contributions to diversity can fulfill underlying goals of a department in reaching more students and generating new mathematical knowledge.

What Is Implicit Bias?

Implicit bias is the phenomenon where a stereotype influences a person's decisions without his or her awareness. Because the stereotype operates without a person's awareness, it takes conscious effort to overcome this implicit bias [2]. There is a wide base of research literature documenting implicit bias and studying its effects on hiring, including in academia. Social-psychology research has identified implicit bias based on a variety of stereotypes: gender bias, ethnic or racial bias, motherhood bias, sexuality bias, and disability bias have all been documented.

What Montana State Does

There are two foci in Montana State University's plan[1] to increase diversity in faculty hires:

(1) recognizing that everyone holds unintentional biases that can influence our evaluation of applicants; and

(2) purposefully focusing on recruitment to create a pool of applicants who are well-qualified and from diverse backgrounds.

The university focuses on search committee training as a tool to give practical strategies to address both aspects of this plan. Other programming across campus, such as guest speakers and campus community discussions, have provided a forum to discuss the underlying reasons that having diversity on campus can be valuable.

Search Committee Training

The training consists of a thorough handbook and a one-hour training, conducted by a faculty member and an HR representative, to make committee members conscious of biases and understand ways to work to overcome them in evaluation and decision-making. The overarching message of this training is that an awareness of bias reduces its effect. As committee members, simply reminding ourselves that we are prone to a bias can reduce the effect of the bias.

Everyone holds unintentional biases that can influence our evaluation.

Search committees are encouraged to consider diversity as an asset and use it as one of many criteria in evaluation. The materials remind committees that anyone can demonstrate a commitment to creating an inclusive workplace, and infusing diversity into the search process can help to hire a faculty member who values and respects difference, no matter his or her identity. There is no expecta-

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[1] *In 2012 Montana State received an NSF ADVANCE award (no. 1208831) to increase the recruitment, retention, and advancement of women in STEM disciplines across campus. See the October 7, 2016* Chronicle of Higher Education. *In 2015, Maria Klawe [1] issued a call in the* Notices *to talk about gender equality in mathematics.*

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tion that searches will hire under-qualified applicants just to satisfy a diversity goal. The search training directly addresses potential fears of search committees that their consideration of diversity will somehow prevent them from selecting the "best" candidate by some objective measure, such as the number of publications or grant dollars. Many factors usually prevent such a candidate from being deemed the best—for example, if a candidate's area of expertise overlaps too much with existing faculty members' research, or if it doesn't match a department's existing research program enough. The training provides the advice that in cases where a department lacks diversity, a candidate who adds diversity may be the best candidate overall for the program because of the benefit that diversity itself affords.

… awareness of bias reduces its effect.

The final advice is that because stereotypes are mental shortcuts, one way to combat the bias from stereotypes is to take time. Rushing through a review of materials is more likely to activate stereotypes, so allow each application a careful review. In academic life, there are many competing demands, and it can be easy to try to minimize time on committee work. But search committee members must make a commitment to investing the time it takes to complete a search that won't rely on quick judgments. Committee members can focus on whom to include rather than whom to exclude, and be sure to articulate a reason to advance or exclude someone. Making that decision public gives other committee members the opportunity to weigh in and serve as a check that the decision is not based in bias.

Recruitment

The key to being able to hire diverse candidates is getting those diverse candidates to apply for the open position. The strategies we use are focusing on the wording of the advertisement, advertising in venues that appeal to diverse candidates, and personally reaching out to encourage candidates to apply.

The wording of the announcement should be carefully considered, with a focus on broad descriptions of areas rather than a narrow focus. This is based on research from psychology: "On average, men apply for a job when they meet only 60% of the qualifications. Women apply only when they meet 90% of them" [3]. In our department, we have focused on broad descriptions of research areas. We also include our desire for diversity in the position advertisement, for example, by including language such as, "We hope to attract applicants who are committed to helping students from diverse backgrounds succeed."

The materials also caution against making assumptions about a candidate's motivation or interest in an opening. Making inferences like, "She lives in a big city so she won't be interested in our position," or, "She will have offers at more prestigious universities so wouldn't really consider an offer from us," only undermine our efforts to recruit for diversity. Our obligation is to recruit broadly into the search pool, and then let the candidates decide if they want the position.

We advertise in publications and websites that target women and minorities. In the mathematical sciences, available options are the Association for Women in Mathematics, the Benjamin Banneker Association, TODOS, the National Association of Mathematicians, the Society for the Advancement of Chicanos/Hispanics and Native Americans in Science, and the Caucus for Women in Statistics.

We are also encouraged to use personal invitations to encourage applications. Montana State challenges each search committee member to personally contact ten potential applicants with a "we'd love to review your materials" phone call or email.

Equity Advocate

We do not rely exclusively on departmental committee members to combat the effects of bias. The university has initiated a program called the Equity Advocate program. Equity Advocates are faculty members from across campus who have attended additional training to learn about recognizing and preventing bias. These individuals are available to serve on search committees in any department. We have had two Equity Advocates who are members of our department, but we have also had Equity Advocates from other departments serve on our search committees. Though every member of the search committee attends the search committee training, which includes the materials about bias and acting against bias, it is useful to have someone whose responsibility on the committee is to recognize potential bias.

… one way to combat the bias from stereotypes is to take time.

Equity Advocates can often speak up when a departmental colleague might not. Imagine being one of the few women in a department and serving on a search committee. These three underlying assumptions are unfair and not necessarily true:

(i) because you are a woman it is your job to recognize bias,
(ii) it is your responsibility to alert the committee to bias, and
(iii) decisions you make are not based on bias.

There's also an additional mental tax that comes from not only fulfilling the usual obligations of the committee, but also having to be attuned to everyone's choices and statements and screen them for potential bias. This is an unreasonable burden on women who serve on committees—an already high-workload assignment. The practice of committee training for all, with the expectation that all members of a committee will be alert for their own bias and actively work against it, combined with the use of

Equity Advocates, whose only role can be to be alert for potential bias, results in a stronger committee structure.

Family Advocate Meetings during Interview

One component of the interview visits that has become standard on our campus is a visit with the University Family Advocate. The Family Advocate is a tenured faculty member who has been appointed to serve the university by disseminating information about resources and policies affecting families on campus.

The Family Advocate has responsibilities for work with current employees, but also participates in the recruitment process. She is completely separate from the search committee, offers no feedback to the department or to the hiring committee, and is able to hold confidential meetings with every candidate. The meetings have a standard format: they last twenty to thirty minutes, and they open with the Family Advocate describing the programs on campus that support work-life balance, such as on-campus preschool, employee wellness benefits, and stop-the-clock tenure policies. There is always time for open question and answer, and it is often here that the candidate asks questions about dual-career partner accommodations or asks questions about the campus climate.

Our department has found these visits are useful as a recruiting tool, but also as an honest reflection of our department's values. As a group, we intend to support our faculty candidates, and it is a relief to know that they can have open and frank conversations early in the process and get advice and answers from someone knowledgeable on campus.

I have occasionally served as a Family Advocate for candidates in searches from other departments or colleges, and I've seen how much candidates value this time. I've had conversations about schools for children, about opportunities for partner employment on campus and in the region, and about quality of life in general in Bozeman.

Dual-Career Arrangements

Most everyone who has been involved in an academic search is aware of the two-body problem, where a candidate has a partner or spouse who is also looking for an academic position. At Montana State, we are particularly sensitive that while we live in a vibrant community with an appealing natural setting, we are in a small city in a rural state bordered by other rural states. We begin all searches with the recognition that nearly all faculty candidates will come with a spouse or partner, and that person will also need to find meaningful employment. We acknowledge that we can't leave the partners to fend for themselves in finding a new job, as might be reasonable in an institution in a larger city. We also recognize that in the case that the partner is an academic, we have the opportunity to secure a dual-career couple. MSU has found that we can identify a two-body opportunity rather than problem.

The mechanics of how this works vary. Sometimes there is a tenure-track opening in a suitable department; other times we are authorized to pursue a target-of-opportunity hire, where a tenure-track position is authorized to accommodate the partner. It is comparatively simpler to arrange for a non-tenure-track appointment. It is cooperation and effort at the deans' level, with partnerships across colleges, that make the system work. It doesn't always work, but it often does.

From a search committee member and department head perspective, the sooner the candidate declares the desire for an academic position for a partner, the easier it is to arrange and more likely it is to succeed. Early declaration perhaps flies in the face of advice many candidates have historically been given by their advisors and mentors, and many candidates often wait until the relative last minute to seek an accommodation. We rely on our advertising and the Family Advocate program to assure candidates of our sincere desire to accommodate two-body opportunities.

Changes at the Departmental Level

These initiatives have had an effect at MSU. Since 2012, the university has hired 72 faculty members in STEM departments. Of those 72 positions, 36 have been filled by women. Our own department has made twelve hires at the assistant professor level since 2013, and six of those new faculty members are women, in contrast to only three women among the thirteen tenured faculty.

We've seen results beyond simply quantifying new hires at the department level as well. One difference I have noticed is that the search committee training gives the committee permission to discuss potential sources of bias. Having the search materials gives us a starting place for these conversations about bias. It has also given us a venue for discussions about diversity and the advantages that having students and colleagues with diverse perspectives brings to our department. In the past, it has been simple to lament our lack of diversity and blame the lack of diversity within mathematics as a whole; now, we have knowledge and strategies that can help us to achieve a more diverse faculty.

There is the secondary effect that we have an increased awareness of the effects of bias in teaching. We had a department-wide discussion of how gendered perceptions of mathematicians affect individuals in our department. The immediate result was an awareness of how many women receive anonymous comments on course evaluations that refer to their appearance or contain sexual innuendos. It was a bit sad for everyone to realize that the women were not surprised to have had these experiences, and that the men were surprised that women have these experiences. This collective recognition of the gender bias in instruments has led to a discussion of how the department might deemphasize the use of this type of instrument, and focus on better evaluation through more time and care. Of course, the broader issue of how to address the biased expectations that students bring to the classroom won't be fixed by changing how we measure teaching effectiveness. Our department and the individuals in it continue to wrestle with issues of diversity as we consider our responsibility to reduce the effects of bias.

Note: MSU maintains a two-page summary of search tips as well as the faculty-search toolkit, which is available by request to the email address on the page `www.montana.edu/nsfadvance`. Both are influenced by the work of Fine and Handlesman [4]. The documents on the webpage are freely provided for your use after registration. You can also purchase the toolkit from Wisconsin.

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Elizabeth A. Burroughs

OPINION

Four Goals for Instructors Using Inquiry-Based Learning

Chris Rasmussen, Karen Marrongelle, Oh Nam Kwon, and Angie Hodge

Communicated by Benjamin Braun

Note: The opinions expressed here are not necessarily those of Notices.
Responses on the Notices *webpage are invited.*

In a recent *Notices* article [1], Ernst, Hodge, and Yoshinobu describe two pillars of inquiry-based learning (IBL): deep engagement in mathematics and opportunities to collaborate [2]. These two pillars identify key aspects of what students do in an IBL classroom. What, however, is the *instructor's* role in a successful IBL classroom? This question leads to a third pillar of IBL: instructor interest in and use of student thinking. In a stereotypical math classroom, instructor interest is often expressed with prompts such as: Does anyone have a question? What is the inverse of this linear transformation? How do you simplify this expression? These types of questions tend to have an expected response and lead to what is referred to as an Initiation-Response-Evaluation pattern of instructor and student talk. The instructor initiates a discussion by asking a question, a student responds, and the instructor indicates whether the response is correct or not (or asks a follow-up question directed toward the expected response). While such talk patterns can be useful, they do not usually promote the pillars of deep engagement in the mathematics and collaboration that characterize student experiences with IBL.

> *What is the instructor's role in a successful IBL classroom?*

In this article, we discuss ways in which instructors can achieve four research-based goals in IBL classrooms. These goals and corresponding instructor actions are grounded in research conducted in multiple inquiry-based classrooms, including differential equations and linear algebra [3], [4], [5]:

(1) get students to share their thinking,
(2) help students to orient to and engage in others' thinking,
(3) help students deepen their thinking, and
(4) build on and extend student ideas.

As anyone who has tried to implement IBL can attest, realizing these goals takes time, patience, occasional failures, and the willingness to try out new patterns of interaction with students (often called "talk moves"). While these four goals are not an exclusive list of instructor responsibilities, they serve as a solid foundation for successful teaching in IBL classrooms.

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#1: Get Students to Share Their Thinking

In an IBL classroom, students spend their time actively working on problems (often with their peers) that are novel and/or challenging to them. Instructors strive to bring forth student ideas and make them public, even if the answers are incorrect or the students are tentative about their work. Creating a classroom environment where students feel safe to express their thinking is a challenge and not always easy to achieve, but research grounded in practice has revealed various prompts that can help realize this goal:

- "Dave, I know you haven't finished the problem, but tell us your initial thinking."
- "Take your time; we're not in a rush."
- "Can you say more about that?"
- "That's an important point. Keisha, can you say that again so that everyone can hear?"
- "Learning from mistakes is an important part of doing mathematics. Who can share with us an initial approach that didn't work out?"

These and similar talk moves (Figure 1) are productive for eliciting student thinking because they clearly communicate the expectation that students share their ideas rather than be completely correct.

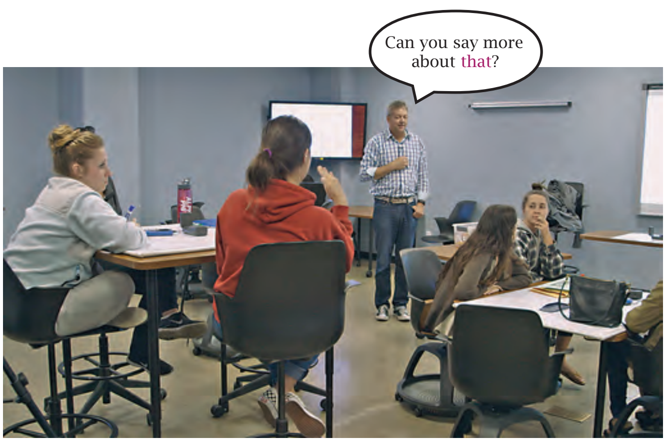

Figure 1. "Talk moves" elicit student thinking.

But getting students to talk is, of course, just the start. More often than not, students' initial thinking is muddled or difficult to interpret. Rather than evaluating the response or asking a leading follow-up question, an instructor might choose to repeat back what the student said, with or without an interpretation or rephrasing, and then check back with the speaker for accuracy. This type of talk move is referred to as *revoicing* and is a highly effective tool for initiating classroom discussions, in part because it conveys no evaluation of students' initial contributions.

#2: Help Students Orient To and Engage With Others' Thinking

Once one or more students have shared their initial thinking, the role of the IBL instructor is to promote cross-talk among students where they attend to and make sense of other students' thinking. Instructor prompts that can help achieve this goal often make use of questions that we do not normally use in everyday conversation, yet talk moves such as the ones below can be quite effective in promoting student discourse:

- "Who can repeat in their own words what Juan just said?"
- "Do you agree with what Darrel just said?"
- "Say that again please so that everyone can hear."
- "Can someone rephrase Diane's explanation in their own words?"
- "So, Debbie, is Minsoo saying that…?"
- "Jesika, what do you think about what Ding just said?"

These types of talk moves encourage other students to revoice the thinking of their classmates. Sometimes student revoicing leads to a different conclusion or justification and thus creates an opportunity for a broader debate among multiple students. A student misinterpretation may come to the surface, encouraging the original speaker to use more precise mathematical language or explain their idea in a different way. All of these outcomes are mathematically productive.

#3: Help Students Deepen Their Thinking

Even if students express their thoughts and listen to others' ideas, the discussion can still fail to be mathematically productive if it does not include solid and sustained reasoning. Most students are not skilled at pushing to deepen their own reasoning or that of their classmates. Therefore, a key role of the instructor is to continuously and skillfully press students for reasoning, explanation, and justification. Instructors can help achieve this by using the following prompts:

- "How can we check to make sure this is correct?"
- "What is the reasoning that allows you to make that conclusion?"
- "Do you agree or disagree and why?"
- "How does that relate to what we learned yesterday?"
- "Can anyone come up with a different way to explain that?"
- "OK, I hear what you are saying, but what about this counterexample?"
- "We now have three different solutions and not all three can be correct. Work with your partner(s) to decide, with justification, whether each solution is correct or not."

As these questions suggest, IBL instructors must take a proactive role in pressing students to develop their ideas and to explain and justify their thinking, navigating between students' ways of reasoning and conventions of the broader mathematical community (Figure 2).

#4: Build On and Extend Students' Ideas

IBL classroom discussions are opportunities for students to report on the progress they have made working (individually, in pairs, or in small groups) on challenging problems. Such occasions are also opportunities for the instructor to build on student work to advance their mathematical agenda, to make connections to more formal or conventional mathematics, and/or to help students see and appreciate that they are actually *doing* mathematics. Using the following instructor moves are some ways in which instructors can build on and extend student work:

Figure 2. After being prompted to consider a classmate's claim, students work in small groups to decide if they agree or disagree and why.

- Restating what students said or did in more conventional or formal terms.
- Introducing a new but related concept, definition, representation, or procedure that extends what students did.
- Restating a student's explanation and attributing authorship to the student or students, i.e., creating the sense that mathematics is arising out of students' own work.
- Restating student ideas in ways that connect to established mathematical culture.

Benefits of such instructor moves include highlighting how student ideas fit into the larger mathematical trajectory; providing scaffolding for students to clarify, to elaborate, and to extend their mathematical positions; representing mathematics as something that can develop from students' own work; and supporting students' enculturation into the discipline of mathematics.

Conclusion

Effectively supporting students' deep engagement in mathematics and peer-to-peer collaboration is an omnipresent challenge for instructors. The four goals discussed here provide some insight into what an instructor can do to realize the student-focused pillars of deep engagement in mathematics and collaboration. Just as a stool with three legs is sturdier than one with only two, IBL classrooms are on a stronger foundation when all three pillars are taken into account: the two focused on the activity of students and the one focused on the activity of the instructor.

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Photo of Angie Hodge courtesy of Stan Yoshinobu.

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Angie Hodge

香 港 大 學
THE UNIVERSITY OF HONG KONG

Founded in 1911, The University of Hong Kong is committed to the highest international standards of excellence in teaching and research, and has been at the international forefront of academic scholarship for many years. The University has a comprehensive range of study programmes and research disciplines spread across 10 faculties and over 140 academic departments and institutes/centres. There are 28,000 undergraduate and postgraduate students who are recruited globally, and more than 2,000 members of academic and academic-related staff coming from multi-cultural backgrounds, many of whom are internationally renowned.

Tenure-Track Associate Professor/Assistant Professor in the Research Division of Mathematical and Statistical Science, Faculty of Science (several posts)
(Ref.: 201701228)

The Faculty of Science will be 80 years old in 2019 and to celebrate this anniversary we are recruiting a cohort of 80th Anniversary professors. This is an unprecedented expansion focused on realizing the Faculty's vision to be pre-eminent in Hong Kong, leading in Asia and highly competitive globally. We are aiming to expand by about 20% over the next year.

The Faculty of Science is one of the larger faculties in HKU with about 130 academic staff in five departments (Chemistry, Earth Sciences, Mathematics, Physics, and Statistics and Actuarial Science) and one school (Biological Sciences) and an off-site marine facility, The Swire Institute of Marine Science (SWIMS). The Faculty provides a supportive and friendly environment and is embarking on a programme of recruitment to invest in areas of acknowledged strength and to invest in areas of internationally competitive activity. Information about the Faculty can be obtained at http://www.scifac.hku.hk/. In addition, the Faculty is committed to gender equality and is looking to promote women in science as such the Faculty will particularly welcome applications from women who are currently underrepresented at these levels. The Faculty will also be happy to consider applications from individuals with their own funding who wish to explore proleptic positions.

To better support research, the Faculty has established a new research division structure and there are six new research divisions:

- Chemistry
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- Mathematical and Statistical Science
- Molecular and Cell Biology
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Applicants should submit a cover letter, a completed application form, together with an up-to-date C.V., which should include hyperlinks to google scholar profiles and ORCID, a detailed publication list since 2014 with journal impact factor and citations indicated for each publication, a research proposal and a statement on teaching philosophy to

https://web.science.hku.hk/job/201701228.html

Application form (341/1111) can be downloaded at http://www.hku.hk/apptunit/form-ext.doc. Enquiries about the posts should be sent to Head, Department of Mathematics (email: ntw@maths.hku.hk), or Head, Department of Statistics & Actuarial Science (email: gyin@hku.hk).

Further particulars can be obtained at http://jobs.hku.hk/. **Closes December 31, 2017**. Interviews for the positions are expected to be held from March 5 to 9, 2018.

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Dorothea Rockburne and Max Dehn at Black Mountain College

David Peifer

Communicated by Thomas Garrity

ABSTRACT. The artist Dorothea Rockburne was inspired by the mathematician Max Dehn while a student at Black Mountain College.

Figure 1. Dorothea Rockburne is a distinguished artist with a fascination for mathematics. Rockburne in 2014, photo by Zia O'Hara, (on left) and in her New York studio in 2009 (on right).

Dorothea Rockburne is a New York based artist whose paintings are inspired by her fascination with mathematics. Over her more than fifty years as a successful exhibiting artist, Rockburne has won many awards for her work, including the 1999 American Academy of Arts and Letters Jimmy Ernst Lifetime Achievement Award in Art and the 2009 National Academy Museum and School of Fine Arts Lifetime Achievement Award. A major retrospective, "Dorothea Rockburne: In My Mind's Eye," held in 2012 at the Parrish Art Museum, exhibited works spanning her career, including works from her Pascal series and paintings inspired by her reading of Poincaré. A multi-gallery exhibition at the Museum of Modern Art in 2014, *Dorothea Rockburne: Drawing Which Makes Itself*, included some of her earliest works from her Set Theory series.

the ability of mathematics to shed light on the underlying principles of nature

The catalyst of Rockburne's interest in mathematics came during the first two years of her undergraduate education. In 1950, at age 17, Rockburne left her home in Montreal, Canada, to study painting at Black Mountain College (BMC), an experimental liberal arts college with an emphasis on the arts (see sidebar).

In 1945, the BMC faculty was joined by the German refugee mathematician Max Dehn. Dehn taught mathematics, philosophy, Greek, and Latin. He and his wife lived on the top floor in a small house on campus. Like most of the faculty, the Dehns ate their meals in the dining hall and participated in the social life on campus. On Sundays, when meals were not provided, Dehn and his wife frequently invited students to their home for dinner. Dehn was well known for taking faculty, students, and visitors on long hikes into the surrounding mountains in search of local wildflowers.

Before arriving at BMC, Rockburne already had a solid education in painting. While she was an elementary student, she won a merit scholarship to the Ecole des Beaux-Arts in Montreal, for Saturday classes. Rockburne always had a deep respect and fascination with nature. She recalls

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this wonder for nature being sparked one day during an outdoor painting class on Mount Royal, in Montreal:

> Although my teacher praised my painting, as I stood and looked first at my work and then at the landscape, I observed and questioned, "Nature does it better. How?" More specifically, I asked myself, what are nature's organizing principles? These questions probably arose because of my father's Algonquin traditions: "Nature is sacred and should be understood, not violated." This moment of inquiry would later directly lead to my ongoing passion for mathematics as the basis for the understanding of nature.

During high school, Rockburne won another scholarship to take classes at the Montreal Museum School. Gordon Webber introduced Rockburne to artistic theories from the Bauhaus and Russian constructivism. Rockburne recalls that this was "a different kind of art thinking and was a large part of what drew me to BMC."

In her first week at BMC, Rockburne tried taking a mathematics class with Dehn, but quickly found she was not prepared and dropped out. She says that Dehn must have seen something in her mathematical abilities, because a few days later he offered to teach her mathematics as it is seen in nature. He suggested that she accompany him on his morning hike. Rockburne was greatly influenced by her time with Dehn and this would ultimately shape her career as an artist. Rockburne fondly recalls her time with Dehn:

> Dehn considered my lack of math instruction an advantage. I wasn't mathphobic, and, as he so happily put it, I hadn't been poisoned. Dehn had a lively, disciplined, but fearless mind. His enthusiasm for everything was infectious. When I told him I was having some difficulties with assignments, he said, "What you need is to understand the principles of math as they occur in nature. Why don't you join me every day on my early morning hike, and I will teach you mathematics for artists through nature?"

In Dehn Rockburne found a man who, like her father, had a great respect for nature, someone who understood nature and who would show her how mathematics could be used to unlock its guiding principles. She recalls a steep hike, about a mile uphill behind the Dehns' home, where they walked to a small waterfall. Dehn discussed the golden ratio and "showed [her] that every kind of growth has recognizable mathematical properties." Dehn recommended that she read several books, including *Science and Method* by Henri Poincaré, *The Laws of Thought* by Henry Parker Manning, *The Fourth Dimension* by C. Howard Hilton, and *Flatland* by Edwin A. Abbott. In later years, Rockburne would go back to these books for inspiration.

Figure 2. Max Dehn greatly influenced Dorothea Rockburne (1952) at Black Mountain College.

During Rockburne's second year at BMC, Dehn passed away. Rockburne married Carroll Williams, an apprentice teacher and former student at BMC. She participated in John Cage's Theatre Piece No. 1, sometimes called "The First Happening." She attended the international pottery seminar lead by the Japanese potter Shoji Hamada. She danced with Merce Cunningham. In 1954, the family moved to Manhattan. Unfortunately, the marriage did not last long. Soon Rockburne was a young single mother working hard to bring up her daughter in New York City.

not afraid to pick up a real mathematics text to see what she could get out of it

Rockburne stayed in touch with many BMC artists, students, and faculty who had moved to the New York area. Encouraged by her BMC friends, in 1957 Rockburne had an exhibition of her work at the Nonagon Gallery in New York. The show got good reviews and she sold several works. Rockburne herself was encouraged, but she felt she needed more time to develop. She recalls, "I knew that these paintings did not emotionally or creatively correspond to my inner need or voice. … I returned to my studio for ten years to work, study, and form a voice." By keeping her expenses low, she managed to support herself and care for her daughter. For a short while she worked at the Metropolitan Museum of Art and assisted in cataloguing the museum's collection of Egyptian antiquities. Most of the time she worked as a waitress. "To stay sane," she went back and reread the books that Dehn had suggested. She also began her own investigations into mathematics such as reading books on non-Euclidean geometry and set theory. She was not afraid to pick up a real mathematics text to see what she could get out of it. She read these books with the goal to "use them as a learning foundation from which to conceive art." She was fascinated with the history of mathematics and said, "By understanding the history of mathematics, I learned of an exquisite emotional beauty of thought. This in turn gave me greater access to an understanding of a more universal creative process.…" Dehn had always stressed the importance of history. If Dehn had talked with Rockburne about hyperbolic geometry, which he used

Max Dehn

After escaping the Nazis, Max and his wife Toni Dehn eventually settled at BMC in the mountains of North Carolina.

Max Dehn received his doctorate under David Hilbert in Göttingen in 1900, the same year that Hilbert gave his famous lecture outlining 23 problems for the twentieth century. The next year Dehn was the first to solve one of the problems on Hilbert's list. He solved the third problem, showing that the Archimedean axiom is logically needed to prove that tetrahedra of equal base and height have equal volume. Dehn went on to do ground-breaking work in topology and the theory of infinite groups. In 1907, Dehn coauthored with Poul Heegaard the first comprehensive articles on topology, including the first complete proof of the classification of 2-manifolds. Dehn's work on 3-manifolds included inventing a way to construct new 3-manifolds from existing ones. This process is now called *Dehn surgery* and is still important in topology today. Dehn used his surgery to construct an infinite set of examples of Poincaré homology spheres. Before this, only Poincaré's single example was known. Dehn was the first to formulate the word and conjugacy problems for finitely presented groups. In an amazing paper from 1911, using hyperbolic geometry, Dehn was able to solve these problems for surface groups. The word and conjugacy problems have continued to play an important role in the study of infinite group theory through present day. In a 1914 paper, Dehn was the first to prove that the right and left trefoil knots are not equivalent. Later in his career he worked on the mapping class groups, finding a finite set of generators now known as Dehn twists. Dehn was an intuitive geometer at heart. Many of the mathematical ideas attributed to him, such as Dehn diagrams, Dehn surgery, and Dehn twists, are in fact simple diagrams that are easy to visualize. Their beauty is that these simple diagrams provide a way to understand more general and complicated situations.

Dehn seems to have been admired and respected by everyone he met. He had grown up with an interest in the arts and music as well as science and mathematics. He learned to read and speak several languages, including Greek, Latin, French, Norwegian, and English. He had a lifelong interest in philosophy, in particular, ancient Greek and modern German philosophy. He had a great love for hiking and during his life developed an extensive knowledge of botany and the flora of the forest. He had, what we would call today, an interdisciplinary approach. André Weil, in his autobiography, *The Apprenticeship of a Mathematician*, compared Max Dehn to Socrates and said, "...for such a man, truth is all one, and mathematics is but one of the mirrors in which it is reflected. … [Dehn was a] humanistic mathematician, who saw mathematics as one chapter—certainly not the least important—in the history of the human mind."

Dehn had been forced by the Nazis to leave his position at the University of Frankfurt in 1935. Three years later, he was arrested the day after Kristallnacht. Dehn and his wife went into hiding and eventually escaped to Norway, where Dehn took a position at the Technical University in Trondheim. After the Germans invaded Norway, the Dehns again had to go into hiding and eventually escaped to Sweden. There they began their long travels to the US through Siberia and Japan and across the Pacific until eventually settling at BMC.[1]

so frequently in his work, or if he talked to her about his views on set theory, axiomatic geometry, topology, and the general abstraction of mathematics, details are lost to time. It seems clear that at least Dehn had sparked her interest in some of these ideas. In particular, as a combinatorial topologist, he could not have helped but to provide her with some of his insights concerning the shape of space.

In the early 1970s Rockburne did a series of works based on her reading of Cantor and set theory, which propelled her into the forefront of the art world. These works, including "Set," "Intersection," and "Null Set," were shown in her first two solo exhibits, as well as group exhibitions in New York, Minneapolis, and Eindhoven in The Netherlands. The Museum of Modern Art purchased "A, C and D (from Group/And)," a wall-size installation of paper and chipboard with natural pigments, rolled out on the wall and floor, and held together by nails. In March of 1972, Artforum featured her work "Domain of The Variable," on its cover and included an article and an interview with Rockburne.

Rockburne's earliest works were identified with the minimalists. The 1971 work "Scalar" (Figure 3) is inspired by her study of linear algebra. It is made of several 30"x40" sheets of chipboard that had been saturated in crude oil, stacked in her loft, and then left to sit for a summer. These sheets were then attached to the wall by nails in an over-

[1] *See the 2002* Notices *article on "Max Dehn, Kurt Gödel, and the Trans-Siberian Escape Route"* `www.ams.org/notices/200209/fea-dawson.pdf`.

Figure 3. Rockburne's 1971 "Scalar," inspired by linear algebra, had an intriguing three-dimensional aspect. Museum of Modern Art.

lapping array. The whole work measures approximately 10'x6'. In the art world, it is praised as an early example of experimentation with materials. Rockburne says she used crude oil because she was poor at the time and couldn't afford paints. What she really loved about the oil was how it worked its way through the board and how the image on one sheet was superimposed on the sheets above and below. It was this three-dimensional aspect that intrigued her. By sitting for a summer, the oil had transformed the sheets. Since the sheets had been stacked, they showed a dynamic of this slow transformation of oil soaking down through the sheets and imprinting on each other. The placement of the sheets on the wall provides the viewer with Rockburne's interpretation of this dynamic system.

Rockburne became good friends with Sol LeWitt and other minimalists. She had several works in group exhibitions featuring minimalist artists. However, from the start, Rockburne did not feel that her art fit within the framework of minimalism. "Like them, my work was not decorative nor did I make decisions for aesthetic reasons. My art-making decisions were based on the principles of topology, a complex study of continuous space. … This is a different spatial concept from the minimalist practice of working on a flat grid." Rockburne had studied the ways in which artists throughout history created space in their works. She had studied ancient Egyptian art, where small fragments can contain incredibly large scale. Early on she had learned of the Renaissance artists and their use of geometry, vanishing points, and the golden mean. She understood the significance of the rebellious Dada artists and how these ideas, mixed with Freud's theories of the subconscious, went on to influence surrealism and abstract expressionism giving a more dreamlike sense of space. Rockburne understood how the cubists had defined a new fragmented canvas that allowed for views from varying angles and staggered moments of time. With her work, she wanted to define a new way to view space that incorporated and drew from these past perspectives, but also would go beyond to new dimensions.

Rockburne's 1973 "Drawing Which Makes Itself: Neighborhood" (Figure 4) is a wonderful example of her ideas to change the way space and time are viewed in a work of art. This work consists of a large rectangular sheet of translucent vellum that has creases folded from corner to corner. This is placed horizontally at the exact center of the wall. There are several lines drawn with different sized colored pencils scattered on the wall around and underneath the vellum. These lines clearly indicate the position of the vellum if it were carefully folded and unfolded to move geometrically along the wall. While the placement of the lines seems to follows some mathematical laws, their position does not look symmetric or mechanical. The color and size of the lines seems to indicate a rhythm and passing of time. Standing back, one can almost see the sheet of vellum beautifully dancing across the wall and leaving only a trace of its passing.

Rockburne's use of lines on the wall around the vellum clearly indicate a break from the tradition of drawing and painting made on a rectangular flat surface. Her works move across the wall and roll out onto the floor. These drawings have broken free of the rectangular frames that hold bounded most other drawings. In this way, her work has opened up a new dimension. It is also interesting to note that in this work she uses translucent vellum. This provides the ability to see through and around the material. Just as in other works where she rolls or folds paper, she is showing both sides of the paper. This gives the viewer a multi-layered perspective.

It is interesting to note what Rockburne means by the title "Neighborhood." As a mathematician, familiar with Rockburne's references to topology, I would be inclined to think of a topological neighborhood. However, this is not the case. The title comes from her reading of *Flatland* and the mention that one-dimensional Linelanders could never pass each other. Thus, neighborhoods for them are like marriages. This gives an even better insight into Rockburne's thinking. *Flatland* is about trying to visualize

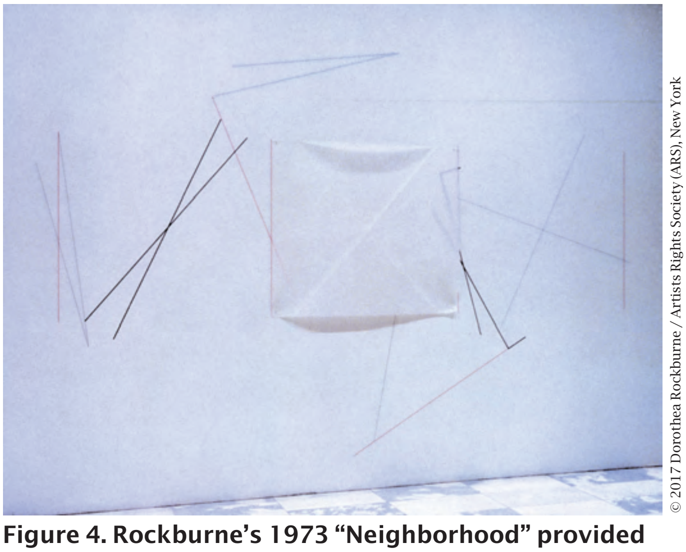

Figure 4. Rockburne's 1973 "Neighborhood" provided a new way to view space and time. Museum of Modern Art.

Figure 5. Rockburne's Pascal works, such as "Pascal, State of Grace" (left, 1986–87), were inspired by the philosophy of Blaise Pascal and "Visitation" (right, 1528–29) by Jacopo da Pontormo. Private collection. Pieve di San Michele Arcangelo, Carmignano.

different dimensions, just as Rockburne is trying to do in her work. Also in *Flatland* women are line segments. Once I made this connection, I could not help but see this work as a woman (a 1-dimensional being), dancing in time through *Flatland* (a 2-dimensional world), with an order induced by simple mathematical folds, translations and rotations in space (a 3-dimensional universe). The analogy goes even further if you consider that Abbot's book is a satire on 19th century society. The women in Flatland are line segments and thus at the bottom of the social stratus, whereas, in 1970 Rockburne is a woman struggling to be part of a male-dominated profession. This work becomes a self-portrait. Whatever Rockburne intended us to feel and think when we see this work, there is no question that she is extending the way in which space and time are represented in paintings and drawings.

Art historians have found it is hard to classify Rockburne's work. This is partly because, in her 50-year career, she has never stayed with one style. Her earliest work is almost monochromatic. Her later work is full of rich colors, including deep blues and bright golds inspired by her study of Renaissance paintings. She has paintings that measure as small as 4"x6" and others that cover walls over 40 feet high. She has painted on chipboard, paper, vellum, parchment, and copper. Her painting techniques include classic styles of oil painting and frescoes, but also include unique applications of watercolors or crude oil. In general art historians call her a post-modern painter.

In 1988, Rockburne first showed her "Pascal" works (Figure 5), inspired by the philosophy of Blaise Pascal. Here she uses stretched canvas on rectangular (usually golden rectangles) and triangular frames. These are then overlapped to create a painting that is no longer rectangular, but polygonal. The canvases are painted to suggest overlapping and transparency, in a way that is not always consistent with reality. This causes a kind of folding and turning of space that makes it appear that the paintings are looking back at themselves. She uses deep and brilliant colors in these works, a bold step from her earlier use of natural tones or pure white paper. Many of the deep blues and reds and the gold are borrowed from specific paintings by Renaissance masters. Gold leaf was used in Renaissance paintings because of its wonderful ability to reflect natural light and to a create special brilliance, as in halos. In many of these older works there is subtle geometry in the composition, sometimes from the gestures or lines of sight of people in the paintings. Other times shading and colors create planes and simple geometric shapes. The lines and planes in Rockburne's works are inspired by the subtle geometry in these masterpieces. The 1530 painting "Visitation," by Pontormo, was an inspiration for many of the Pascal works.

Conclusion. In some ways Rockburne's career is completely different from Dehn's. She is an artist and Dehn was a mathematician. Her career was only beginning in 1950 while Dehn's was coming to an end. However, in many ways the lifelong endeavors of these individuals run parallel. Both have shown a deep intuitive sense of geometry. Both have worked to improve the understanding of geometry and space in their own disciplines. Dehn's mathematics shows an astonishing geometric insight. His early works, including his solution to Hilbert's third problem, answered deep philosophic questions about Euclidean geometry. His solution to the word problem for surface groups showed a fundamental connection between group theory and hyperbolic geometry. Dehn's work on three manifolds and knot theory was decades ahead of its time. Dehn's geometric insight has had consequences reaching back to Euclid and forward to recent work in geometric group theory and low dimensional topology.

Rockburne's works show similar geometric insight, but applied to paintings and drawings. Like Dehn, Rockburne is deeply interested in history and realizes the importance of studying the works of the past. Her education was in the classics, as well as the most avant-garde of contemporary art. She has a deep understanding of space, shape, and perspective. She has the mind of a geometer. Her early works in the 1970s were multi-dimensional, using the whole room to communicate with the viewers and draw them into the experience. Her works play with geometric shapes, time, and space in ways that other artists had never done before. It is hard to know exactly what specific mathematical details Rockburne learned from her mentor Max Dehn, almost seventy years ago. However, it is clear that theirs was the meeting of two great geometric minds.

COMMUNICATION

Black Mountain College

Opened in 1933 and located in a remote area of the North Carolina mountains near Asheville, Black Mountain College (BMC) was an experimental liberal arts college with an emphasis on the arts. Although BMC only existed for 24 years, the college faculty and students have had a tremendous influence on contemporary painting, dance, music, poetry, ceramics, performance art, and architecture.

The curriculum at Black Mountain College was a mix of John Dewey's philosophy on education and theories from the Bauhaus, a German school of design. The production of art—paintings, theater, writing, weaving, furniture making—served as a central core in the students' education. The college was run under democratic principles with no administration, no grades, and no accreditation. Faculty owned the campus. Decisions were made by consensus. Campus upkeep was done by the community of faculty and students. This included building barns, homes, and at one point, a large Studies Building. In 1933 the BMC founders hired Josef and Anni Albers, who had been teaching at the Bauhaus before it was closed by the Nazis. Partly due to the Albers' influence, the college soon began to attract refugee painters, composers, and other artists from Germany and across Europe. Over the years, the college became a magnet for avant-garde European and American artists. The list of distinguished painters who were faculty or students at BMC includes African-American Jacob Lawrence, Willem de Kooning and Robert Motherwell of the "New York School," and pre-pop artist Robert Rauschenberg. A sample of artists from other disciplines includes composer-choreographer duo John Cage and Merce Cunningham, sculptors Ruth Asawa and Kenneth Snelson, and poets Robert Creeley and M. C. (Mary Caroline) Richards.

The Studies Building on the Lake Eden campus of Black Mountain College.

The campus also attracted notable faculty in the sciences. Both Nathan Rosen and Peter Bergmann taught at BMC for a year after doing postdoctorial work with Albert Einstein.

Einstein visited the campus one day in 1941 and remained an "honorary member" of the advisory board for several years. Natasha Goldowski Renner, a metallurgist, had been one of the top-ranking woman in the Manhattan Project before teaching at BMC for six years. In 1949, Buckminster Fuller built his first successful geodesic dome on the campus.

Two of Dehn's BMC students went on to get PhDs in mathematics and statistics. Several mathematicians visited Dehn at BMC, including Emil Artin, Wilhelm Magnus, Carl Siegel, and André Weil.

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Image Credits

Figures 1, 3, 4, and 5 (left) courtesy of Dorothea Rockburne.

Figure 2 (left by Trude Guermonprez; right by Marie Tavroges Stilkind) courtesy of the Western Regional Archives, State Archives of North Carolina.

Figure 5 (right) "Visitation" from Scala / Art Resource, NY.

Photo of Max and Toni Dehn by Trude Guermonprez, courtesy of the Western Regional Archives, State Archives of North Carolina.

Photo of BMC Studies Building by Hazel Larsen Archer, "The Path to the Studies Building, Black Mountain College," ca. late 1940s–early 1950s, courtesy of Estate of Hazel Larsen Archer and Black Mountain College Museum + Arts Center.

Author photo by Brenda Carleton.

ABOUT THE AUTHOR

David Peifer has been a board member at the Black Mountain College Museum and Art Center for the past nine years. He enjoys hiking and mountain biking with his family and friends.

David Peifer

Autobiography of Wentsun Wu (1919–2017)

Wentsun Wu won the 2006 Shaw Prize in Mathematical Sciences for his contributions to the new interdisciplinary field of mathematics mechanization.

EDITOR'S NOTE. The Shaw Prize Foundation kindly provided this autobiography of Wu from their website `shawprize.org`, which recognizes Wu's "outstanding contributions [to] topology, automatic reasoning, machine proof, algebraic geometry, Chinese mathematics history, game theory," including the "Wu formula," "Wu Class," and "Wu method." According to the prize citation, "Wu introduced a powerful mechanical method that transforms a problem in elementary geometry into an algebraic statement which lends itself to effective computation. This method of Wu completely revolutionized the field, effectively provoking a paradigm shift."

I was born in Shanghai, China, on May 12, 1919. I received my BS degree in mathematics from Jiaotung University, Shanghai, in 1940 during the war against Japan (1937–1945). On graduating from university, because of the war, I had to teach for years in junior middle schools bringing to a halt my further learning of mathematics.

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In 1946 I met the great geometer Chern Shiing-Shen. Chern is particularly renowned for the introduction of CHERN Classes and CHERN numbers of unitary bundles which are of extreme importance among the various kinds of characteristic classes of fiber bundles. At that time Chern was in charge of the newly established Institute of Mathematics belonging to Academia Sinica. This meeting with Chern was decisive for the future of my career in mathematics.

Chern admitted me as one of the young students in his institute, all learning algebraic topology under his guidance. One year later I brought out my first paper about a simple proof of the product formula of sphere bundles discovered by H. Whitney for which his original proof was extremely complicated and had never been published.

In 1946 I also passed the national examination for sending students abroad, and in 1947 I was sent to study mathematics in France as part of a Sino-France Exchange Program. I went to Strasbourg to study under Professor Ch. Ehresmann. In 1949 I passed my doctor thesis and then went to Paris to study under Professor H. Cartan. During my stay in Strasbourg I made the acquaintance of R. Thom who was also a student of Cartan but who while at Strasbourg, had much contact with Ehresmann too. The collaboration was a very fruitful one. In 1950 Thom discovered the topological invariance of Stiefel–Whitney classes, while I, with the aid of Cartan, discovered the classes and formulas now bearing my name.

In 1951 I returned to China, and in 1953 became a researcher in the Chinese Academy of Sciences (CAS) where I remain to the present day. From 1953 onwards I made a somewhat systematic investigation of classical topological but non-homotopic problems which were being ignored at that time owing to the rapid development of homotopy theory. I introduced the notion of imbedding classes, and established a theory of imbedding, immersion, and isotopy of polyhedra in Euclidean spaces which was published in book form later in 1965. In 1965, I was awarded one of the three national first prizes for natural sciences for my work on characteristic classes and imbedding classes.

During the cultural revolution I was sent to a factory manufacturing computers. I was initially struck by the power of the computer. I was also devoted to the study of Chinese ancient mathematics and began to understand what Chinese ancient mathematics really was. I was greatly struck by the depth and powerfulness of its thought and its methods. It was under such influence that I investigated the possibility of proving geometry theorems in a mechanical way. In 1977 I ultimately succeeded in developing a method of proving mechanical geometry theorems. This method has been applied to prove or even discover hundreds of non-trivial difficult theorems in elementary geometries on a computer in a simplistic way and was henceforth called WU's method in the literature. The discovery of WU's method marks the second turning point in my scientific life, the first one being my meeting with Chern. Since that time I have completely changed my direction of research and concentrated my efforts on extending the method in various directions, both theoretical and practical, aiming at what I have called "mechanization of mathematics."

Among the honors I have received for my research we may cite:

In 1991 I received the mathematics award from and became a member of the Academy of Sciences for the Developing World (previously called The Third World Academy of Sciences). In 1997 I received the Herbrand Award on automated deduction for my mechanical geometry theorem-proving. In 2001 I was awarded the first State Supreme Science and Technology Award of the Chinese government in recognition of my achievements in mathematics research, both in pure mathematics and in mathematics mechanization.

Finally, Mumford and I together were named as winners of the 2006 Shaw Prize in Mathematical Sciences for our research in pure mathematics, especially with regard to computer applications to mathematics which represents a new role model for mathematicians of the future.

12 September 2006, Hong Kong.

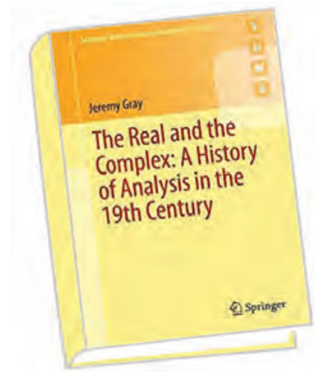
Photo Credit

Photo of Wentsun Wu courtesy of the Shaw Prize Foundation.

BOOK REVIEW

The Real and the Complex

A Review by Judith V. Grabiner



The Real and the Complex: A History of Analysis in the 19th Century
Jeremy Gray
Springer, 2015
Paperback, 350 pages
ISBN–13: 978-3-319-23714-5
eBook ISBN: 978-3-319-23715-2

Most basic formulas and techniques in what is often called advanced calculus were developed in the eighteenth century. More careful definitions and more rigorous proofs, new results that are too subtle to do without such proofs, and new concepts arising from these subtleties are products of the nineteenth century. The nineteenth century is also the source of the basics of complex analysis, from the Cauchy–Riemann equations to Riemannian surfaces and beyond. If you want to see how this all came about, you want to read *The Real and the Complex* by Jeremy Gray.

In the eighteenth century, most mathematicians acted as though symbolism and its heuristic power gave complete insight into the general structure of mathematical ideas. They moved back and forth between the real and the complex, the finite and the infinite. The Bernoullis, Taylor, Maclaurin, d'Alembert, Euler, Lagrange, and Laplace studied functions by means of their power-series representations, solved both ordinary and partial differential equations, and developed methods of integration. Integrals were defined as the inverse of derivatives. General statements about functions were accepted even if there were counterexamples, as long as the "exceptions" occurred only at a finite number of identifiable isolated points. Most mathematicians of the eighteenth century (Lagrange was an exception) regarded questions like "what is a limit?" or "how can you talk about infinity or the infinitely small?" as perhaps of philosophical interest but not essential to mathematical progress. Lagrange understandably, though mistakenly, believed that he had given the calculus a rigorous foundation based on what he thought of as the algebra of power series.

takes us on quite a ride through the ideas, results, and difficulties

But things turned out not to be so simple, and what happened next is the subject of Jeremy Gray's thorough, detailed, and fascinating book.[1] Gray's 29 chapters take us on quite a ride through the ideas, results, and difficulties of the major figures of nineteenth-century analysis. As the century began, analysts may have expected to succeed by continuing the calculational approach. But as they turned to Fourier series, elliptic integrals, foundations of calculus, and complex functions, earlier approaches did not suffice, and eighteenth-century problems and methods gave rise to nineteenth-century theories and standards of proof. We will hit some high points, hoping that readers will turn to the full text to experience the wealth and originality of the reasoning and ideas of the mathematicians of the nineteenth century.

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[1] *Gray intends this book as the second in a four-part series on nineteenth-century mathematics, all arising from courses he has taught to advanced undergraduates. The first,* Worlds Out of Nothing: A Course in the History of Geometry *(Springer, 2011) has appeared; the last two are to cover differential equations and algebra.*

Augustin-Louis Cauchy (1789–1857) began to treat real-variable calculus with a new level of rigor. *The Real and the Complex* lets readers follow the way that Cauchy moved in various directions in complex analysis as well.

Joseph Fourier, in his *Analytical Theory of Heat* of 1822, operated like an eighteenth-century mathematician when he modeled how heat flows with a differential equation: its solution was an infinite series of sine and cosine functions. Fourier said that all functions could be written as such a series and said also that such a series converges everywhere to the function it represents. Much later important work in analysis arose from figuring out when his statements are true and when they are not.

Beginning around 1790, Adrien-Marie Legendre studied elliptic integrals, another topic in real analysis with roots in applied mathematics. He might have wanted to evaluate them exactly, but found that he could not, though he successfully reduced them to three canonical forms. Later, Abel and Jacobi, each inverting Legendre-style elliptic integrals by treating the upper endpoints as independent complex variables, discovered elliptic functions. The early theory of elliptic functions remained formal; only later were they brought into a general theory of complex functions, notably by Joseph Liouville's theorem on doubly periodic functions and Cauchy's proof of it. So complex analysis eventually made sense out of elliptic functions and elliptic integrals. In this episode, as in others throughout the book, Gray emphasizes how the real and the complex interacted in unexpected and unpredictable ways.

Gray emphasizes how the real and the complex interacted in unexpected and unpredictable ways.

In the 1820s, Cauchy began to treat real-variable calculus with a new level of rigor. He discredited Lagrange's foundations by exhibiting a function that did not equal its Taylor series. He insisted that formulas hold "only under certain conditions and for certain values of the quantities they contain." He made the Newtonian concept of limit precise and reasoned with epsilons (borrowing many inequality techniques from Lagrange) in key proofs about derivatives and in his systematic treatment of the convergence of series. Instead of taking the existence of the integral for granted, Cauchy defined the definite integral as the limit of sums and gave a proof, assuming the function to be continuous (and, implicitly, uniformly continuous), that the limit existed. All this began the rigorization of the calculus, though much remained to be done. For instance, Cauchy thought he had proved that an infinite series of continuous functions was itself continuous. Abel found an "exception" (a Fourier series), but exactly what went wrong was not worked out until much later.

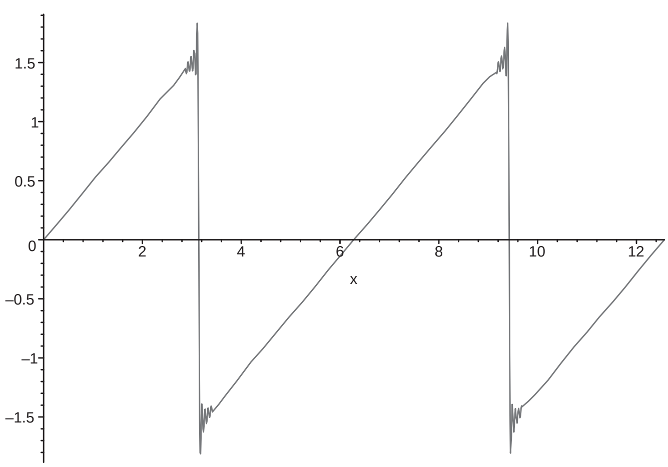

Figure 1. This graph was generated by the computer algebra system Maple and shows the first hundred terms of Abel's function *sin x – (1/2) sin 2x + (1/3) sin 3x* –… . Gray remarks on "how unhappy Maple gets at the points where the Abel function will be discontinuous." This is one of a number of such instructive Maple graphs of functions with unusual properties in Gray's book.

Cauchy also recognized that complex functions could be a separate object of study. Gray lets us follow the way Cauchy moved in various directions, finding the Cauchy integral theorem early in the century, though not yet realizing its fundamental importance. But by the 1840s and 1850s, after many fits and starts, Cauchy had worked out many key properties of complex functions, including what are now called the Cauchy–Riemann equations and their relationship to differentiability, contour integrals, Taylor-series expansions of complex functions, poles, and Laurent series.

Bernhard Riemann (1826–1866) emphasized a conceptual rather than computational approach and transformed the theory of real and complex functions.

At this point in the narrative, Gray has demonstrated that much had changed in analysis. He reflects on what these changes teach us in a brilliant review chapter, his Chapter 12. These reflections enable readers to share in the experiences of leading mathematicians struggling with problems and choices, occasional false trails, as well as unexpected successes, never sure about where their approaches would lead. Gray wants to make absolutely clear that this is what life is like when you do mathematics. If you read nothing else in Gray's book, do read this chapter,[2] which makes this point using historical examples as well as I have ever seen it done. Similarly, the concluding sections of many of his detail-packed chapters successfully convey the essence of what the chapter covered.

After a chapter on potential functions and Green's theorem, Gray turns to the work of Dirichlet, who brought a new standard of rigor to much earlier work, in particular about the convergence of Fourier series. Rather than assume that every function has a convergent Fourier series that equals it, Dirichlet asked under what conditions this was true. He found integral formulas for partial sums of the series and conditions for which they do converge. Gray illustrates Dirichlet's reputation for rigor by quoting a letter from Jacobi to Alexander von Humboldt: "If Gauss says he has proved something, it seems very probable to me; if Cauchy says so, it is about as likely as not; if Dirichlet says so, it is certain."

Riemann, especially by emphasizing a conceptual rather than computational approach, transformed the theory of real and complex functions. As Gray puts it, "Riemann offered an approach to the study of functions that did not depend on any particular expression for it, and accordingly played down the role of long complicated manipulation of such expressions." His introduction of trigonometric series, even if the coefficients are not coefficients of a Fourier series, made the study of nondifferentiable functions possible. Defining analytic functions independently of the particular analytic expression for them by using the Cauchy–Riemann equations let Riemann relate them to harmonic functions. Riemann outlined a proof that two simply connected planar regions can be mapped conformally onto one another by complex analytic maps, now known as the Riemann mapping theorem. He showed how the geometry of complex curves illuminated the study of elliptic functions. And he defined the Riemann integral and enlarged the set of functions that can be integrated.

Weierstrass had a different, more algebraic, approach to complex function theory, choosing to emphasize the representation of complex analytic functions by convergent power series rather than the equivalent property of complex differentiability.[3] The power-series approach let Weierstrass develop methods that apply to functions of one or several variables. He treated hyperelliptic and Abelian functions; made the concept of analytic continuation, previously known to Riemann, central to his own theory; and distinguished between poles and essential singularities. His lectures at the University of Berlin widely disseminated his conclusions about continuity, convergence, uniform convergence, and term-by-term differentiability for both real and complex functions. In particular, he showed that there are continuous real functions that are nowhere differentiable.

Gray next provides a chapter describing nineteenth-century work on uniform convergence, culminating in Weierstrass's definitive and detailed account. Then, in later chapters, Gray discusses topics for which some questions are answered in the nineteenth century while related questions are answered only in the twentieth. Dirichlet and Riemann had called attention to functions that were not integrable or were defined by series that do not converge uniformly or that exhibit unusual relation-

[2]*He names this chapter "Revision," using the British term for review. Similarly, he describes the book's appendix reviewing potential theory as "a revision of the theory."*

[3]*Recall that Lagrange had tried to base his theory of real functions on a detailed study of their real power series in his* Théorie des fonctions analytiques *of 1797.*

ships between their continuity and differentiability. Heine, Cantor, Hankel, Du Bois-Reymond, H. J. S. Smith, Darboux, and, especially, Weierstrass treated a variety of examples. No longer could it be assumed that functions are continuous and differentiable, except perhaps at isolated points, or that differentiable functions had to have integrable derivatives, and Bolzano, Riemann, Weierstrass, Hankel, Schwarz, and Dini began to consider functions in general and to ask which functions have the important properties of integrability and continuity. Gray describes how Weierstrass related implicit functions to holomorphic functions and how Dini made clear the distinction between the real and complex cases and also treated the several-variable case. Again, there were further developments in the twentieth century. Gray also addresses the completeness of the real numbers and the constructions of the reals from the rationals given by Weierstrass, Dedekind, Heine, and Cantor. Weierstrass realized that the Riemann integral was not sufficient, though he did not know how to make it so, and Gray later shows how such considerations led to Lebesgue's theory of integration.

Gray's last three chapters explicitly address how problems posed by nineteenth-century approaches moved toward the more modern and abstract ideas characteristic of the twentieth century, a process that resembles the way analysis had become both deeper and broader between the eighteenth century and the nineteenth. One chapter describes Lebesgue's theory of integration. Another treats Cantor's set theory and the foundations of mathematics. The final chapter is about topology. The reader may be eager for more forays into the twentieth century, but Gray is true to his goal to let us see nineteenth-century analysis in its own terms and not as a prelude to the more abstract ideas of the modern period. And he concludes with the statement that "the mathematical analysis of the last century also needs its historians." Would that they do as good a job for the twentieth century as Gray has done for the nineteenth.

Gray shows, rather than tells, providing a great deal of close-up mathematical detail.

As this brief overview may indicate, Gray's book is not light reading. But it is a very good book indeed. Any mathematician who would like to know how analysis developed should find it interesting. Those who teach analysis can gain much insight from seeing the blind alleys as well as the great ideas, since watching how the mathematics unfolded helps one appreciate what can be really hard and what types of confusion can occur even to the most able student. Seeing past contrasting approaches to the same topic, like those of Riemann and Weierstrass on complex functions, both humanizes the subject and enhances understanding. And Gray really knows his material, both the original sources and the best of the current literature. He scrupulously gives credit to historians whose research helped inform his own. His conclusions about controversial questions are generally well supported and judicious. He shows, rather than tells, as much as he can, providing a great deal of close-up mathematical detail. The writing style is clear and unpretentious, as well as marked by immense learning.

Karl Weierstrass (1815–1897) took a more algebraic approach to complex function theory than did Riemann. His lectures at the University of Berlin widely disseminated much of his work.

There are many ways one could write the history of nineteenth-century analysis. Gray emphasizes the internal development of the mathematics, though he does include some biographical information about the lives of the mathematicians and their mutual relationships, especially when he deems it crucial to explain how the mathematics developed. Other historians might have said much more about how nineteenth-century society and culture affected mathematicians' careers, choice of problems, the teaching of mathematics, and the needs of the sciences, or might have discussed in general the nineteenth-century professionalization of mathematics, increase in specialization, and journals and societies dedicated solely to mathematics. It would have been worth mentioning the work of Sofya Kovalevskaya and noting the institutional barriers that prevented talented women from entering higher mathematics. Of course both approaches mentioned here are valuable, and Gray's choice is designed for his intended audience.

Both my paperback review copy and the corresponding eBook version have more than the typical number of typographical errors. For instance, Cauchy's function that does not converge to its Taylor series is not $e^{-1/x}$; the exponent should be $-1/x^2$. But most are simply punctuation marks or words that are left out—in one case the word "not"—and a second printing could correct these; a careful reader can usually navigate around them. Other matters worth noting: Cauchy did not, in defining the definite integral in terms of sums, restrict the mode of division of the interval of integration to equal subdivisions. And, especially since Gray makes a point of recommending accessible and instructive English-language sources, the Bradley–Sandifer translation of Cauchy's *Cours d'analyse* ought to be added to the 12-page multilingual and comprehensive bibliography.

I strongly recommend *The Real and the Complex* to readers of the *Notices*. As a result of Gray's vision and scholarship, any mathematically informed reader can now watch the giants of the nineteenth century, most notably Legendre, Fourier, Cauchy, Gauss, Dirichlet, Riemann, Weierstrass, Dedekind, and Cantor, developing much of the mathematics we use today—not flying straight as an arrow to their target, not possessing all our modern concepts, but doing the best they could so that, as Gray says, they "found their ways, imperfectly, from what they knew to what they wanted to know."

ACKNOWLEDGMENT. I dedicate this review to the memory of Dr. Uta C. Merzbach (1933–2017), inspiring mentor, valued colleague, and beloved friend. Her long-anticipated study of Dirichlet, with the editorial assistance of Judy Green and Jeanne LaDuke, is forthcoming from Birkhäuser.

Image Credits

Illustration of Augustin Cauchy courtesy of Library of Congress Prints and Photographs Division. From an illustration in: *Das Neunzehnte Jahrhundert in Bildnissen* / Karl Werckmeister, ed. Berlin : Kunstverlag der photographische Gesellschaft, 1901, vol. V, no. 581

Figure 1 by Jeremy Gray.

Photo of Bernhard Riemann courtesy of Smithsonian Libraries Scientific Identity website.

Photo of Karl Weierstrass courtesy of Smithsonian Libraries Scientific Identity website.

Photo of Judith V. Grabiner courtesy of Pitzer College.


Judith V. Grabiner

ABOUT THE REVIEWER

Judith V. Grabiner, author of *The Origins of Cauchy's Rigorous Calculus* and "The role of mathematics in liberal arts education" (in M. Matthews, ed., *International Handbook of Research in History, Philosophy and Science Teaching*), received the Beckenbach Prize from the Mathematical Association of America for her book *A Historian Looks Back: The Calculus As Algebra and Selected Writings*.

BOOKSHELF

New and Noteworthy Titles on Our Bookshelf
December 2017

One Two Three…Infinity: Facts and Speculations of Science, by George Gamow (Dover, 1988)

This month's *Notices* carries an article (see page 1275) about George Gamow's liquid model for the atomic nucleus, which has arisen in recent research and applications. Born in Odessa, Ukraine, in 1904, Gamow was a theoretical physicist and cosmologist with wide interests ranging from radioactive decay, to "big bang" cosmology, to genetics. He may be best known among mathematicians for his now-classic book highlighted here, *One Two Three…Infinity*. Originally published in 1947, the book has charmed readers for decades by presenting an appealing set of topics from mathematics and science in simple, concrete terms that zoom in on the essential ideas. The book has a leisurely feel that makes the reader feel like a participant in a tour led by a welcoming, enthusiastic, and highly knowledgeable guide. The opening chapter discusses numbers and counting and even explains the notion of transfinite numbers. Subsequent chapters discuss space, time, and Einstein's theories of relativity, and the micro- and macro-cosmos. The book includes over 100 pen-and-ink illustrations drawn by Gamow himself, which illustrate the ideas with humor and clarity and give a window into the mind of a man in love with ideas. Also known for his series of popular science books featuring the inquisitive Mr. Tompkins, Gamow received the UNESCO Kalinga Prize for the popularization of science in 1956.

The Universal Computer: The Road from Leibniz to Turing, by Martin Davis (AK Peters/CRC Press, 2011)

Next year, Martin Davis will turn 90 years old. A legendary figure in mathematical logic, he is best known for resolving Hilbert's Tenth Problem, together with Yuri Matiyasevich, Hilary Putnam, and Julia Robinson. Davis also did some of the world's first computer programming, working in the 1950s on the ORDVAC computer, which had a central memory consisting of 40 vacuum tubes. A master expositor, Davis received the AMS Steele Prize for Exposition as well as the Chauvenet and Ford Prizes of the Mathematical Association of America. His popular book *The Universal Computer* originally appeared to wide acclaim in 2000 and was re-issued in 2012, in a special edition to honor the centenary of Alan Turing. The tale of how computers developed has been told many times and in many forms—and often with heavy emphasis on the computer as an engineering feat. Davis takes a completely different tack by tracing the origin of computers in developments in logic, starting with the ideas of Leibniz, who, as Davis puts it, "dreamed of machines capable of carrying out calculations, freeing the mind for creative thought." After an opening chapter on "Leibniz's Dream," Davis discusses the lives and work of Boole, Frege, Cantor, Hilbert, Gödel, and Turing. The book closes with a discussion of the first computers that were built as well as a look towards the future. "When a distinguished expert offers a popular exposition of his subject, we greet the effort with keen anticipation," wrote Brian Blank in a review that appeared in the May 2001 issue of the *Notices*. "That is all the more true when the writer is as skilled as Martin Davis. It is a pleasure to report that in this case our anticipation is richly rewarded."

Suggestions for the BookShelf can be sent to `notices-booklist@ams.org`.

We try to feature items of broad interest. Appearance of a book in the Notices *BookShelf does not represent an endorsement by the* Notices *or by the AMS. For more, visit the AMS Reviews webpage* `www.ams.org/news/math-in-the-media/reviews`.

LETTERS TO THE EDITOR

Article about "Wide Influence" of L. E. Dickson

I read, and am fascinated by, the article on "The 'Wide Influence' of Leonard Eugene Dickson," by Della Dumbaugh and Amy Shell-Gellasch in the August 2017 *Notices*. It describes Dickson's wide influence as reflected in three of his students graduating in the same year: A. Adrian Albert, Mina Rees, and Ko-Chuen Yang.

All three are fascinating subjects by themselves. Albert is a well-known, notable algebraist of the Chicago school and an administrator of the highest order. Rees is the subject of a biography written by Shell-Gellasch. Yang and his student Hua Luogeng were fine contributors to the development of the Chinese school of modern mathematics. I am particularly interested in Yang, because he taught at Southwest Associated Universities, which had a seven-year tenure during the Sino-Japanse war and which was attended by both my parents as well as others, such as Yang's son, the physics Nobel Laureate Chen-Ning (Frank) Yang. (By the way, Southwest Associated Universities had three presidents, whose disparate personalities were often the subject of good-natured humor and banter.)

Admittedly, I am an admirer also of Leonard Eugene Dickson, whose seven-volume *Mathematical Papers* still sit on the shelves of my personal library, which I have for some time vowed to downsize, along with the rest of my hoard in the house.

I applaud the authors and the *Notices* for including the Chinese name (in Unicode) of Ko-Chuen Yang, for disambiguation, because Chinese mathematicians often published with names in Anglicized forms. I know six or seven Chinese transcription systems, including the now-standard Hanyu Pinyin and others that preceded it (such as Wade-Giles and Gwoyeu Romatzyh)—Shiing-Shen Chern's name belongs to the latter system. People used whatever official system was then in force in official dictionaries for these transcriptions. (The doubling of "i" in "Shiing" denotes third tone; the seemingly extraneous "r" in "Chern" denotes second tone—these were concocted before the introduction of diacritical marks in computerized typesetting and typography.) Besides, in the old Wade-Giles system, there were mispronunciations, or dialectical influences, so that the system was many-to-one for the same Chinese characters. So, I believe supplying the Chinese names of Chinese scientists helps a lot to disambiguate.

The full space the article introduced in the Chinese name of 楊 克 純 can be considered an honest (though unorthodox) attempt to highlight his family name, but it is really unnecessary, as the correct form is to put his family name first (before the given name), which is the standard Chinese appellation—no spacing inserted. I would also like to see Hua Luogeng's Chinese name (华罗庚) used as well.

I thank the authors for writing such an inspiring article, which I have thoroughly enjoyed.

—Stanley Liu
East Setauket, New York

(Received July 27, 2017)

Updating Math Genealogy

In our article on "The 'Wide Influence' of Leonard Dickson" in the *Notices* (August 2017), Amy Shell-Gellasch and I noted that Dickson had 67 PhD students in his forty-year career at the University of Chicago. Peter May inquired about this number, since the Mathematical Genealogy Project listed only 53 students for Dickson. Following the guidelines for submitting new students to the Mathematical Genealogy website, I added 15 names to Dickson's list and removed one name, Marion Stark, who was not his PhD student. I want to extend a generous thanks to Peter May for calling this discrepancy to our attention and to the Mathematical Genealogy Project for their excellent service to the mathematical community. I encourage other members of our community to submit updates to this valuable database.

—Della Dumbaugh
University of Richmond
`ddumbaugh@richmond.edu`

(Received August 25, 2017)

Peripatetic Accent Marks

I would like to reassure the readers that the accent marks in my name are firmly rooted on the letters "a" and they have no business wandering over to the "o" as in the September 2017 *Notices* (p. 922 column 2 and p. 923).

—János Kollár
Department of Mathematics
Princeton University
`kollar@math.princeton.edu`

(Received September 11, 2017)

EDITOR'S NOTE. Yes, sorry, it is a shame to spoil such nice symmetry.

THE BACK PAGE

"All mathematicians live in two different worlds. They live in a crystalline world of perfect platonic forms. An ice palace. But they also live in the common world where things are transient, ambiguous, subject to vicissitudes. Mathematicians go backward and forward from one world to another. They're adults in the crystalline world, infants in the real one."—*Sylvain Cappell*

Concept and artwork by Anna Pun and Richard Stanley.

Robert McCann reports that his young son Ian was looking for words beginning with "lem." Robert responded, "Lemon, lemma, and lemniscate, although I don't know what that is." Ian asked, "Daddy, are there mathematicians better than you who know what a lemniscate is?"

What crazy things happen to you? Readers are invited to submit original short amusing stories, math jokes, cartoons, and other material to: **`noti-backpage@ams.org`**.

Mathematics Opportunities

Listings for upcoming math opportunities to appear in Notices *may be submitted to* notices@ams.org .

AMS–AAAS Mass Media Summer Fellowships

The American Mathematical Society provides support each year for a graduate student in the mathematical sciences to participate in the American Association for the Advancement of Science (AAAS) Mass Media Science and Engineering Fellows Program. This summer fellowship program pairs graduate students with major media outlets nationwide, where they will research, write, and report on science news and use their skills to bring technical subjects to the general public.

The principal goal of the program is to increase the public's understanding of science and technology by strengthening the connection between scientists and journalists to improve coverage of science-related issues in the media. Past AMS-sponsored fellows have held positions at National Public Radio, *Scientific American*, Voice of America, *The Oregonian*, *NOVA*, the *Chicago Tribune*, and the *Milwaukee Journal Sentinel*.

Fellows receive a weekly stipend of US$500, plus travel expenses, to work for ten weeks during the summer as reporters, researchers, and production assistants in newsrooms across the country. They observe and participate in the process by which events and ideas become news, improve their ability to communicate about complex technical subjects in a manner understandable to the public, and increase their understanding of editorial decision making and of how information is effectively disseminated. Each fellow attends an orientation and evaluation session in Washington, DC, and begins the internship in mid-June. Fellows submit interim and final reports to AAAS. A wrap-up session is held at the end of the summer.

Mathematical sciences faculty are urged to make their graduate students aware of this program. Further information about the fellowship program and application procedures is available online at www.aaas.org/programs/education/MassMedia; or applicants may contact Rebekah Corlew, Program Director, AAAS Mass Media Science and Engineering Fellows Program, 1200 New York Avenue, NW, Washington, DC 20005; email rcorlew@aaas.org . Further information is also available at www.ams.org/programs/ams-fellowships/media-fellow/massmediafellow and through the AMS Washington Office, 1527 Eighteenth Street, NW, Washington, DC 20036; telephone 202-588-1100; email amsdc@ams.org. The deadline for applications is **January 15, 2018**.

—*AMS Washington Office*

*NSF Program in Computational and Data-Enabled Science and Engineering in Mathematical and Statistical Sciences

The Program in Computational and Data-Enabled Science and Engineering in Mathematical and Statistical Sciences (CDS&E-MSS) of the National Science Foundation accepts proposals that confront and embrace the host of mathematical and statistical challenges presented to the scientific and engineering communities by the ever-expanding role of computational modeling and simulation on the one hand and the explosion in production of digital and observational data on the other. The CDS&E-MSS program supports fundamental research in mathematics and statistics whose primary emphasis is on meeting these challenges. The window for submission is **November 25–December 11, 2017**. See www.nsf.gov/funding/pgm_summ.jsp?pims_id=504687&org=DMS&from=home.

—*NSF announcement*

**The most up-to-date listing of NSF funding opportunities from the Division of Mathematical Sciences can be found online at:* www.nsf.gov/dms *and for the Directorate of Education and Human Resources at* www.nsf.gov/dir/index.jsp?org=ehr. *To receive periodic updates, subscribe to the DMSNEWS listserv by following the directions at* www.nsf.gov/mps/dms/about.jsp.

NEWS

National Defense Science and Engineering Graduate Fellowships

To help increase the number of US citizens trained in disciplines of military importance in science and engineering, the Department of Defense awards National Defense Science and Engineering Graduate Fellowships to individuals who have demonstrated ability and special aptitude for advanced training in science and engineering. The fellowships are awarded for a period of up to four years for study and research leading to doctoral degrees in any of fifteen scientific disciplines. Application forms are available online at `www.ndsegfellowships.org/application` and are due **December 31, 2017**.

—From an NDSEG announcement

CRM Intensive Research Programs

The Centre de Recerca Matemàtica (CRM) will hold two Intensive Research Programs in the spring of 2018. An advanced course in "Recent Progress in Mathematical Biology" will be held April 3–6, 2018, with a conference to be held June 4–8, 2018. A course in "Discrete, Combinatorial, and Computational Geometry" will be held from April 16–June 8, 2018. See `www.crm.cat/en/Host/SciEvents/IRP/Pages/CurrentIRP.aspx`.

—From a CRM announcement

STaR Fellowship Program

The Service, Teaching, and Research (STaR) Program of the Association of Mathematics Teacher Educators (AMTE) supports the development of early-career mathematics educators, including their induction into the professional community of university-based teacher educators and researchers in mathematics education. Senior and mid-career mathematics education faculty organize and facilitate STaR events, serving as mentors to Fellows. Applications are due **December 1, 2017**; see `www.amte.net/star/apply`.

—From an AMTE announcement

MSRI Summer Research for Women in Mathematics

MSRI invites applications for participation in the Summer Research for Women in Mathematics program (June 11, 2018–August 3, 2018).

Groups of two to six women with partial results on an established project may submit an application to the program. Each member of the group must have a PhD in mathematics or advanced graduate standing.

The deadline for application will be **February 1, 2018**.

Financial Support for travel and local expenses is available. For more information, please see `www.msri.org/programs/329`.

—From an MSRI announcement

Mathematics People

Tikhomirov Awarded CMS Doctoral Prize

Konstantin Tikhomirov

KONSTANTIN TIKHOMIROV of Princeton University has been awarded the 2017 Doctoral Prize of the Canadian Mathematical Society (CMS) for his "outstanding contributions to several open problems in probability theory, convex geometry, functional analysis, and random matrix theory." According to the prize citation, in his doctoral studies, Tikhomirov investigated "a series of open problems in diverse areas of mathematics. He has written at least ten papers related to asymptotic geometric analysis, random matrices, probability theory, and convex geometry. In particular, he worked on the problem of estimating the distance between an n-dimensional polytope with a fixed number of vertices and the Euclidian ball. In this case he solved the exact dependence between the dimension and the number of vertices.

"Tikhomirov also considered problems like understanding better the limit of the smallest singular value of some families of random matrices, as well as when the convex hull of a random walk includes the origin. The impact of his work will have many ramifications for its innovation and its ability to be applied in other situations."

Tikhomirov received his PhD from the University of Alberta in 2016. The Doctoral Prize is awarded annually to a doctoral student from a Canadian university who has demonstrated exceptional performance in mathematical research.

—From a CMS announcement

Prizes of the Mathematical Society of Japan

The Mathematical Society of Japan (MSJ) has awarded its Prizes for Excellent Applied Mathematicians for 2017. TOMOYA MEMMOCHI of the University of Tokyo was honored for the study of L^∞-error estimates for the finite element approximation of parabolic problems on domains with smooth boundaries. AKITO SUZUKI of Shinshu University was recognized for a spectral and scattering theoretic proof of the weak limit theorem for quantum walks. SHUYA CHIBA of Kumamoto University was honored for his work on 2-factors containing perfect matchings in bipartite graphs and directed 2-factors in digraphs. JUNYA NISHIGUCHI of Kyoto University was recognized for his work entitled "How Should We Understand the Time-Delay Structure in Dynamics?" YUTO MIYATAKE of Nagoya University received the prize for his work on finite difference schemes preserving multiple invariants for evolutionary partial differential equations.

—From an MSJ announcement

2017 Davidson Fellows Chosen

Arjun Ramani

Felix Wang

Two high school students whose projects involved the mathematical sciences have been named 2017 Davidson Fellows. ARJUN RAMANI, eighteen, of West Lafayette, Indiana, was awarded a US$25,000 scholarship for his project "Fast Sampling of Stochastic Kronecker Graphs." Ramani also won third place in this year's Regeneron Science talent search. He tells the *Notices*: "I recently created a rap song on GarageBand that I have posted online on Soundcloud. I have played competitive tennis since the age of nine. I became interested in statistics because of my love of NBA basketball at a young age."

FELIX WANG, eighteen, of Newton, Massachusetts, was awarded a US$25,000 scholarship for his project "Functional Equations in Complex Analysis and Number Theory."

The Davidson Fellows program, a project of the Davidson Institute for Talent Development, awards scholarships to students eighteen years of age or younger who have created significant projects that have the potential to benefit society in the fields of science, technology, mathematics, literature, music, and philosophy.

—From a Davidson Fellows announcement

NDSEG Fellowships Awarded

Twenty-one young scholars whose work involves the mathematical sciences have been awarded National Defense Science and Engineering Graduate (NDSEG) Fellowships by the Department of Defense (DoD) for 2017. The fellowships are sponsored by the United States Army, Navy, and Air Force. As a means of increasing the number of US citizens trained in disciplines of military importance in science and engineering, DoD awards fellowships to individuals who have demonstrated ability and special aptitude for advanced training in science and engineering.

Following are the names of the fellows, their research fields, their institutions, and the offices that awarded the fellowships:

- Dallas Albritton, mathematics, University of Minnesota-Twin Cities, Army Research Office (ARO)
- Brandon Bohrer, computer and computational sciences, Carnegie Mellon University, Air Force Research Laboratory (AFRL)
- Steven Brill, computer and computational sciences, Stanford University, AFRL
- Victoria Chayes, mathematics, Rutgers University, AFRL
- Caleb Chuck, computer and computational sciences, University of Texas at Austin, Office of Naval Research (ONR)
- Tanner Fiez, computer and computational sciences, University of Washington, ARO
- Yakir Forman, mathematics, Yale University, AFRL
- Jesse Freeman, mathematics, Massachusetts Institute of Technology, ONR
- Pranav Gokhale, computer and computational sciences, University of Chicago, AFRL
- Benjamin Gunby, mathematics, Harvard University, ARO
- Tianshe He, mathematics, University of North Carolina at Chapel Hill, ONR
- Siddhartha Jayanti, computer and computational sciences, Massachusetts Institute of Technology, ARO
- Samuel McLaren, mathematics, University of Arizona, AFRL
- Matthew McMillan, mathematics, University of California Los Angeles, ARO
- David Mildebrath, computer and computational sciences, Rice University, ARO
- Brandon Shapiro, mathematics, Cornell University, AFRL
- Mackenzie Simper, mathematics, Stanford University, ONR
- Samuel Stewart, mathematics, University of Minnesota-Twin Cities, AFRL
- William Thomason, computer and computational sciences, Cornell University, ONR
- Brian Thompson, computer and computational sciences, ARO
- Marvin Zhang, computer and computational sciences, University of California Berkeley, ARO

—From a DoD announcement

CME Group–MSRI Prize Awarded

Robert B. Wilson of the Stanford University Graduate School of Business has been awarded the CME Group–MSRI Prize in Innovative Quantitative Applications. His research and teaching focus on market design, pricing, negotiation, and related topics concerning industrial organization and information economics. He is an expert on game theory and its applications.

—From a CME–MSRI announcement

Malliaris and Shelah Awarded Hausdorff Medal

Maryanthe Malliaris of the University of Chicago and Saharon Shelah of the Hebrew University of Jerusalem and Rutgers University have been awarded the third Hausdorff Medal for their joint paper "Cofinality Spectrum Theorems in Model Theory, Set Theory, and General Topology," published in the *Journal of the American Mathematical Society* **29** (2016). The Hausdorff Medal is awarded every two years by the European Set Theory Society (ESTS) for the most influential work in set theory published in the preceding five years.

—From an ESTS announcement

Royal Society of Canada Fellows

The Royal Society of Canada (RSC) has elected two mathematical scientists to its 2017 class of new fellows in mathematical sciences. They are Yoshua Bengio of the University of Montreal and Robert Jerrard of the University of Toronto.

—From an RSC announcement

Photo Credits

Photo of Konstantin Tikhomirov courtesy of Konstantin Tikhomirov

Photos of Arjun Ramani and Felix Wang courtesy of Davidson Institute for Talent Development.

Inside the AMS

Project NExT 2017 Fellows Chosen

Six mathematicians have been selected as AMS Project NExT Fellows for 2017. Their names and affiliations follow.

- Edgar A. Bering, Temple University
- Sara Clifton, University of Illinois at Urbana—Champaign
- Patrick Devlin, Yale University
- Brian Hwang, Cornell University
- Pamela Beth Pyzza, Ohio Wesleyan University
- Amelia Tebbe, Indiana University

Project NExT (New Experiences in Teaching) is a professional development program for new and recent PhDs in the mathematical sciences (including pure and applied mathematics, statistics, operations research, and mathematics education). It addresses all aspects of an academic career: improving the teaching and learning of mathematics, engaging in research and scholarship, and participating in professional activities. It also provides the participants with a network of peers and mentors as they assume these responsibilities. The AMS provides funding for a number of the Fellowships.

—From an MAA announcement

AMS Department Chairs Workshop

The annual workshop for department chairs will be held a day before the start of the Joint Mathematics Meetings in San Diego, California, on Tuesday, January 9, 2018, from 8:00 am to 6:30 pm at the Marriott Marquis San Diego Marina hotel.

Workshop leaders will be Malcolm Adams, former head, Department of Mathematics, University of Georgia; Krista Maxson, vice president of academic affairs, University of Science and Arts of Oklahoma and former chair of the Department of Mathematical Sciences at Shawnee State University; Irina Mitrea, chair, Department of Mathematics, Temple University; and Douglas Mupasiri, head, Department of Mathematics, University of Northern Iowa.

What makes a chair different from any other engaged faculty member in the department? This one-day workshop will examine the chair's role in leading a department. The day will be structured to include and encourage networking and sharing of ideas among participants. There will be four sessions:

(1) View from the top. What responsibilities, duties, and expectations do deans, provosts, and other chief academic officers have for their chairs?

(2) Improving students' experience. Possible topics include curriculum and research opportunities, student recruitment and diversity, program assessment, career counseling, and also personnel issues such as faculty development and incentives and the increasing numbers of non-tenure-track faculty.

(3) Outreach and communication: Building effective internal partnerships. Possible topics include collaborations with other departments, working with university offices such as honors programs, government relations offices, career and internship offices, development office, and the dean and upper administration.

(4) Outreach and communication: Building effective external partnerships. Possible topics include collaborations with local businesses, local school systems, and other regional or national efforts.

The workshop registration fee of US$150 is in addition to and separate from the Joint Meetings registration. Those interested in attending can register at `bit.ly/2xtP45U` by **December 20, 2017**. For further information, please contact the AMS Washington Office at 202-588-1100 or `alb@ams.org`.

—Anita L. Benjamin
AMS Washington Office

Free Grant-Writing Workshop Offered

The American Mathematical Society, in conjunction with the National Science Foundation Directorate for Education and Human Resources (NSF-EHR), is pleased to offer a free workshop entitled "Writing a Competitive Grant Proposal to NSF-EHR." This grant-writing workshop will be held on Monday, January 8, 2018, from 3:00 pm to 6:00 pm at the Marriott Marquis San Diego Marina hotel.

Workshop Goals:

- To familiarize participants with current direction/priorities in EHR

- To familiarize participants with key EHR education research and development programs
- To consider common issues of competitive proposals
- To prepare participants to write a competitive proposal

Topics covered will include discussion of key programs in EHR; the merit review process and merit review criteria; discussion of scenarios—short passages drawn from proposals in the EHR portfolio designed to stimulate discussion about strengths and weaknesses of a proposal; and opportunities to discuss possible proposal ideas with program officers.

This free workshop is open to all interested participants by registering by **December 20, 2017**, at `bit.ly/2fz1R0a`. For further information, please contact the AMS Washington Office at 202-588-1100 or `alb@ams.org`.

—Anita L. Benjamin
AMS Washington Office

Twenty Years Ago in the Notices

The Football Player and the Infinite Series
by Harold P. Boas

`www.ams.org/notices/199711/boas.pdf`

The star football player—soccer player in American lingo—referred to in the article title played in the 1908 Olympics on Denmark's silver-medal soccer team. But this player was also an outstanding mathematician. When he defended his thesis, "the air buzzed with anticipation as the football team crowded excitedly into the lecture hall." The title of the dissertation was *Contributions to the Theory of Dirichlet Series*, and the candidate's name was Harald Bohr. This article discusses Bohr's thesis work involving Dirichlet series and the Riemann zeta function.

The article author, Harold P. Boas, an outstanding expositor and recipient of the Bergman Prize, became editor of the *Notices* three years after this article was published.

For Your Information

Departments Coordinate Job Offer Deadlines

For the past eighteen years, the American Mathematical Society has led the effort to gain broad endorsement for the following proposal:

That mathematics departments and institutes agree not to require a response prior to a certain date (usually around February 1 of a given year) to an offer of a postdoctoral position that begins in the fall of that year.

This proposal is linked to an agreement made by the National Science Foundation (NSF) that the recipients of the NSF Mathematical Sciences Postdoctoral Fellowships would be notified of their awards, at the latest, by the end of January.

This agreement ensures that our young colleagues entering the postdoctoral job market have as much information as possible about their options before making a decision. It also allows departmental hiring committees adequate time to review application files and make informed decisions. From our perspective, this agreement has worked well and has made the process more orderly. There have been very few negative comments. Last year, one hundred seventy-seven (177) mathematical sciences departments and four (4) mathematics institutes endorsed the agreement.

Therefore we propose that mathematics departments again collectively enter into the same agreement for the upcoming cycle of recruiting, with the deadline set for **Monday, January 29, 2018**. The NSF's Division of Mathematical Sciences has already agreed that it will complete its review of applications by January 19, 2018, at the latest, and that all applicants will be notified electronically at that time.

The American Mathematical Society facilitated the process by sending an e-mail message to all doctoral-granting mathematics and applied mathematics departments and mathematics institutes. The list of departments and institutes endorsing this agreement was widely announced on the AMS website beginning November 1, 2017, and is updated weekly until mid-January.

We ask that you view this year's formal agreement at `www.ams.org/employment/postdoc-offers.html` along with **this year's list** of adhering departments.

Important: To streamline this year's process for all involved, we ask that you notify T. Christine Stevens at the AMS (`Postdoc-deadline@ams.org`) **if and only if**:

(1) your department is not listed and you would like to be listed as part of the agreement;

or

(2) your department is listed and you would like to withdraw from the agreement and be removed from the list.

Please feel free to e-mail us with questions and concerns. Thank you for consideration of the proposal.

—Catherine A. Roberts
AMS Executive Director
T. Christine Stevens
AMS Associate Executive Director

The EDGE Program Turns 20

EDGE@20

The EDGE Program (Enhancing Diversity in Graduate Education) will celebrate its twentieth anniversary in 2018. Since 1998, the program's four-week summer session, which initially alternated between Bryn Mawr and Spelman Colleges, has been held at colleges and universities around the country. The current codirectors are Ami Radunskaya of Pomona College and Raegan Higgins of Texas Tech University. With a mission to prepare a diverse group of women students for the academic and cultural demands of graduate programs in the mathematical sciences, the program has hosted 241 students to date, 78 of whom have received PhDs. Many more have earned master's degrees, are currently pursuing advanced degrees, or have pursued careers in academia, government, industry, K–12 education, finance, and entrepreneurship. EDGE alumnae have received NSF, Fulbright, Alliance, and Congressional Fellowships, chaired their departments, become deans, designed outreach programs, started collaborative research groups, and made countless positive changes in the world.

In 2013, the nonprofit Sylvia Bozeman and Rhonda Hughes EDGE Foundation was established to ensure the financial future of the EDGE Program (`www.edgefoundation.net`). Throughout 2018, a fundraising campaign, EDGE@20, as well as special sessions at the Joint Mathematics Meetings in San Diego and celebrations around the country, will mark the program's twenty-year milestone.

Visit EDGE at `facebook@edge4women` or at `www.edgeforwomen.org`.

—EDGE Program announcement

Photo Credit

EDGE@20 photo by Maia Averett, Mills College.

American Mathematical Society

WILL YOU BE ATTENDING

the Joint Mathematics Meetings in San Diego?

Visit the AMS Membership Booth **to learn more about the benefits** of membership!

The AMS is excited to announce a NEW benefit available to all individual members, **FREE SHIPPING!** In addition to receiving a discount on books purchased through the online bookstore and at meetings, members are also entitled to receive free shipping on their purchases. Join or renew your membership and receive a complimentary gift!

AMS Members, get your professional portrait taken at the Membership Booth!

The AMS Membership Department has arranged for a photographer to take your professional portrait and have it emailed to you in just a few minutes! You can upload this photo to your MathSciNet® Author Profile page, use it on your University website, submit it as the professional photograph for your book publication, or use it as your profile picture in email and on social platforms.

Availability:

Thursday, January 11th 2018, 9:30 am–12:30 pm and Friday, January 12th 2018, 2:30 pm–5:30 pm.

Schedule your appointment at:

amermathsoc.simplybook.me

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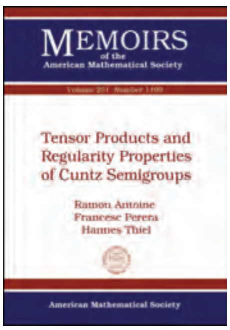
Algebra and Algebraic Geometry



Tensor Products and Regularity Properties of Cuntz Semigroups

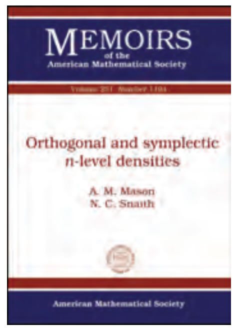
Ramon Antoine, *Universitat Autónoma de Barcelona, Spain*, **Francesc Perera**, *Universitat Autónoma de Barcelona, Spain*, and **Hannes Thiel**, *Universität Münster, Germany*

This item will also be of interest to those working in number theory.

Contents: Introduction; Pre-completed Cuntz semigroups; Completed Cuntz semigroups; Additional axioms; Structure of Cu-semigroups; Bimorphisms and tensor products; Cu-semirings and Cu-semimodules; Structure of Cu-semirings; Concluding remarks and open problems; Appendix A. Monoidal and enriched categories; Appendix B. Partially ordered monoids, groups and rings; Bibliography; Index of terms; Index of symbols.

Memoirs of the American Mathematical Society, Volume 251, Number 1199

December 2017, 191 pages, Softcover, ISBN: 978-1-4704-2797-9, 2010 *Mathematics Subject Classification:* 06B35, 06F05, 15A69, 46L05; 06B30, 06F25, 13J25, 16W80, 16Y60, 18B35, 18D20, 19K14, 46L06, 46M15, 54F05, **Individual member US$47**, List US$78, Institutional member US$62, Order code MEMO/251/1199



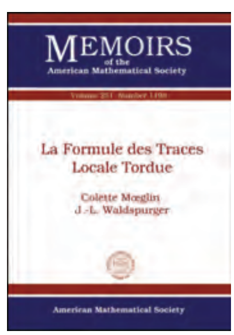
Orthogonal and Symplectic n-level Densities

A. M. Mason, *University of Oxford, United Kingdom*, and **N. C. Snaith**, *University of Bristol, United Kingdom*

Contents: Introduction; Eigenvalue statistics of orthogonal matrices; Eigenvalue statistics of symplectic matrices; L-functions; Zero statistics of elliptic curve L-functions; Zero statistics of quadratic Dirichlet L-functions; n-level densities with restricted support ; Example calculations; Bibliography.

Memoirs of the American Mathematical Society, Volume 251, Number 1194

December 2017, 93 pages, Softcover, ISBN: 978-1-4704-2685-9, 2010 *Mathematics Subject Classification:* 11M50; 15B52, 11M26, 11G05, 11M06, 15B10, **Individual member US$47**, List US$78, Institutional member US$62, Order code MEMO/251/1194



La Formule des Traces Locale Tordue

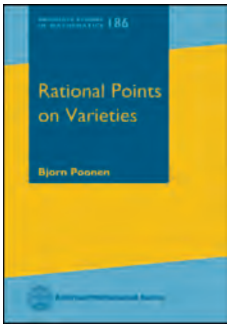
Colette Mœglin, *Institut de Mathématiques de Jussieu-CNRS, France*, and **J.-L. Waldspurger**, *Institut de Mathématiques de Jussieu-CNRS, France*

This item will also be of interest to those working in number theory.

Contents: La formule des traces locale tordue ; La formule des traces locale tordue sous forme symétrique; Index des notations, par ordre alphabétique et par chapitre; Bibliography.

Memoirs of the American Mathematical Society, Volume 251, Number 1198

December 2017, 180 pages, Softcover, ISBN: 978-1-4704-2771-9, **Individual member US$47**, List US$78, Institutional member US$62, Order code MEMO/251/1198



Rational Points on Varieties

Bjorn Poonen, *Massachusetts Institute of Technology, Cambridge, MA*

This book is motivated by the problem of determining the set of rational points on a variety, but its true goal is to equip readers with a broad range of tools essential for current research in algebraic geometry and number theory. The book is unconventional in that it provides concise accounts of many topics instead of a comprehensive account of just one—this is intentionally designed to bring readers up to speed rapidly. Among the topics included are Brauer groups, faithfully flat descent, algebraic groups, torsors, étale and fppf cohomology, the Weil conjectures, and the Brauer-Manin and descent obstructions. A final chapter applies all these to study the arithmetic of surfaces.

The down-to-earth explanations and the over 100 exercises make the book suitable for use as a graduate-level textbook, but even experts will appreciate having a single source covering many aspects of geometry over an unrestricted ground field and containing some material that cannot be found elsewhere.

The origins of arithmetic (or Diophantine) geometry can be traced back to antiquity, and it remains a lively and wide research domain up to our days. The book by Bjorn Poonen, a leading expert in the field, opens doors to this vast field for many readers with different experiences and backgrounds. It leads through various algebraic geometric constructions towards its central subject: obstructions to existence of rational points.

—***Yuri Manin, Max-Planck-Institute, Bonn***

It is clear that my mathematical life would have been very different if a book like this had been around at the time I was a student.

—***Hendrik Lenstra, University Leiden***

Understanding rational points on arbitrary algebraic varieties is the ultimate challenge. We have conjectures but few results. Poonen's book, with its mixture of basic constructions and openings into current research, will attract new generations to the Queen of Mathematics.

—***Jean-Louis Colliot-Thelene, Université Paris-Sud***

A beautiful subject, handled by a master.

—***Joseph Silverman, Brown University***

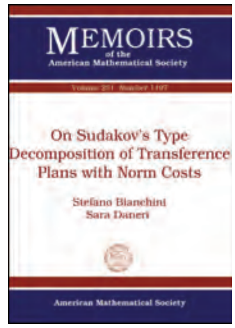
This item will also be of interest to those working in number theory.

Contents: Fields; Varieties over arbitrary fields; Properties of morphisms; Faithfully flat descent; Algebraic groups; Étale and fppf cohomology; The Weil conjecture; Cohomological obstructions to rational points; Surfaces; Universes; Other kinds of fields; Properties under base extension; Bibliography; Index.

Graduate Studies in Mathematics, Volume 186

December 2017, 337 pages, Hardcover, ISBN: 978-1-4704-3773-2, LC 2017022803, 2010 *Mathematics Subject Classification:* 14G05; 11G35, **AMS members US$66.40**, List US$83, Order code GSM/186

Analysis



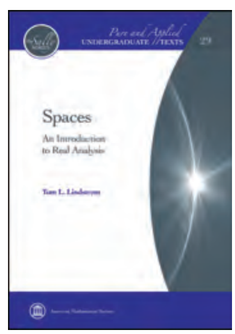
On Sudakov's Type Decomposition of Transference Plans with Norm Costs

Stefano Bianchini, *SISSA, Trieste, Italy*, and **Sara Daneri**, *Friedrich-Alexander-Universität Erlangen-Nürnberg, Germany*

Contents: Introduction; General notations and definitions; Directed locally affine partitions on cone-Lipschitz foliations; Proof of Theorem 1.1; From $\tilde{\mathbf{C}}^k$-fibrations to linearly ordered $\tilde{\mathbf{C}}^k$-Lipschitz foliations; Proof of Theorems 1.2–1.6; Appendix A. Minimality of equivalence relations; Chapter B. Notation; Chapter C. Index of definitions; Bibliography.

Memoirs of the American Mathematical Society, Volume 251, Number 1197

December 2017, 112 pages, Softcover, ISBN: 978-1-4704-2766-5, 2010 *Mathematics Subject Classification:* 28A50, 49Q20, **Individual member US$47**, List US$78, Institutional member US$62, Order code MEMO/251/1197



Spaces

An Introduction to Real Analysis

Tom L. Lindstrøm, *University of Oslo, Norway*

Spaces is a modern introduction to real analysis at the advanced undergraduate level. It is forward-looking in the sense that it first and foremost aims to provide students with the concepts and techniques they need in order to follow more advanced courses in mathematical analysis and neighboring fields. The only prerequisites are a solid understanding of calculus and linear algebra. Two introductory chapters will help students with the transition from computation-based calculus to theory-based analysis.

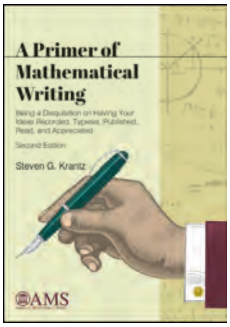
The main topics covered are metric spaces, spaces of continuous functions, normed spaces, differentiation in normed spaces, measure and integration theory, and Fourier series. Although some of the topics are more advanced than what is usually found in books of this level, care is taken to present the material in a way that is suitable for the intended audience: concepts are carefully introduced and motivated, and proofs are presented in full detail. Applications to differential equations and Fourier analysis are used to illustrate the power of the theory, and exercises of all levels from routine to real challenges help students develop their skills and understanding. The text has been tested in classes at the University of Oslo over a number of years.

Contents: Introduction – Mainly to the students; Preliminaries: Proofs, sets, and functions; The foundation of calculus; Metric spaces; Spaces of continuous functions; Normed spaces and linear operators; Differential calculus in normed spaces; Measure and integration; Constructing measures; Fourier series; Bibliography; Index.

Pure and Applied Undergraduate Texts, Volume 29

January 2018, 369 pages, Hardcover, ISBN: 978-1-4704-4062-6, LC 2017022199, 2010 *Mathematics Subject Classification:* 26-01, 28-01, 42-01, 46-01, 54E35, 26E15, **AMS members US$71.20**, List US$89, Order code AMSTEXT/29

General Interest



A Primer of Mathematical Writing

Being a Disquisition on Having Your Ideas Recorded, Typeset, Published, Read, and Appreciated Second Edition

Steven G. Krantz, *Washington University, St. Louis, MO*

This is the second edition of a book originally published in 1997. Today the internet virtually consumes all of our lives (especially the lives of writers). As both readers and writers, we are all aware of blogs, chat rooms, and preprint servers. There are now electronic-only journals and print-on-demand books, Open Access journals and joint research projects such as MathOverflow—not to mention a host of other new realities. It truly is a brave new world, one that can be overwhelming and confusing. The truly new feature of this second edition is an extensive discussion of technological developments. Similar to the first edition, Krantz's frank and straightforward approach makes this book particularly suitable as a textbook for an undergraduate course.

Reviews and Endorsements of the First Edition:

Krantz, a prolific and distinguished mathematical author, discourses engagingly (yet seriously) on the art and etiquette of virtually all types of writing an academic mathematician is likely to encounter … Grammatical points, stylistic and typesetting issues, and the correct and effective use of mathematical notation are handled deftly and with good humor … [Hopefully] senior faculty will consider it mandatory reading for graduate students and even upper-division undergraduates. An enjoyable way to learn some fundamentals of good mathematical writing. Highly recommended.

—***CHOICE***

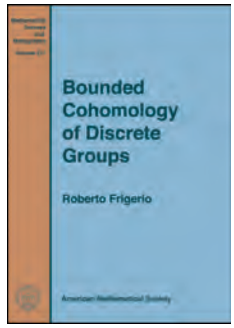
Well written in a lively style and will be found useful by anybody who is aware of the power and significance of writing in the mathematical profession.

—***European Mathematical Society Newsletter***

Contents: The basics; Topics specific to the writing of mathematics; Exposition; Other types of writing; Books; Writing with a computer; The world of high-tech publishing; Closing thoughts; Bibliography; Index.

December 2017, 243 pages, Softcover, ISBN: 978-1-4704-3658-2, 2010 *Mathematics Subject Classification:* 00-01, 00-02, **AMS members US$36**, List US$45, Order code MBK/112

Geometry and Topology



Bounded Cohomology of Discrete Groups

Roberto Frigerio, *University of Pisa, Italy*

The theory of bounded cohomology, introduced by Gromov in the late 1980s, has had powerful applications in geometric group theory and the geometry and topology of manifolds, and has been the topic of active research continuing to this day. This monograph provides a unified, self-contained introduction to the theory and its applications, making it accessible to a student who has completed a first course in algebraic topology and manifold theory. The book can be used as a source for research projects for master's students, as a thorough introduction to the field for graduate students, and as a valuable landmark text for researchers, providing both the details of the theory of bounded cohomology and links of the theory to other closely related areas.

The first part of the book is devoted to settling the fundamental definitions of the theory, and to proving some of the (by now classical) results on low-dimensional bounded cohomology and on bounded cohomology of topological spaces. The second part describes applications of the theory to the study of the simplicial volume of manifolds, to the classification of circle actions, to the analysis of maximal representations of surface groups, and to the study of flat vector bundles with a particular emphasis on the possible use of bounded cohomology in relation with the Chern conjecture. Each chapter ends with a discussion of further reading that puts the presented results in a broader context.

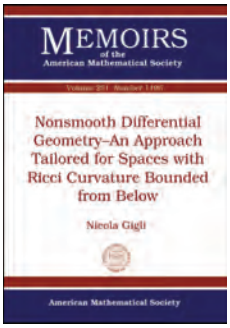
The author manages a near perfect equilibrium between necessary technicalities (always well motivated) and geometric intuition, leading the readers from the first simple definition to the most striking applications of the theory in 13 very pleasant chapters. This book can serve as an ideal textbook for a graduate topics course on the subject and become the much-needed standard reference on Gromov's beautiful theory.

—***Michelle Bucher***

Contents: (Bounded) cohomology of groups; (Bounded) cohomology of groups in low degree; Amenability; (Bounded) group cohomology via resolutions; Bounded cohomology of topological spaces; ℓ^1-homology and duality; Simplicial volume; The proportionality principle; Additivity of the simplicial volume; Group actions on the circle; The Euler class of sphere bundles; Milnor-Wood inequalities and maximal representations; The bounded Euler class in higher dimensions and the Chern conjecture; Index; List of symbols; Bibliography.

Mathematical Surveys and Monographs, Volume 227

December 2017, 191 pages, Hardcover, ISBN: 978-1-4704-4146-3, LC 2017023394, 2010 *Mathematics Subject Classification:* 18G60, 20J06, 55N10, 57N65; 37E10, 37C85, 53C23, 57M07, 57R20, **AMS members US$92.80**, List US$116, Order code SURV/227



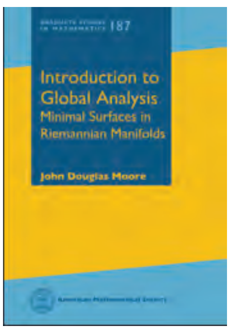
Nonsmooth Differential Geometry–An Approach Tailored for Spaces with Ricci Curvature Bounded from Below

Nicola Gigli, *SISSA, Trieste, Italy*

Contents: Introduction; The machinery of $L^p(\mathfrak{m})$-normed modules; First order differential structure of general metric measure spaces; Second order differential structure of $\mathsf{RCD}(K,\infty)$ spaces; Bibliography.

Memoirs of the American Mathematical Society, Volume 251, Number 1196

December 2017, 161 pages, Softcover, ISBN: 978-1-4704-2765-8, **Individual member US$47**, List US$78, Institutional member US$62, Order code MEMO/251/1196



Introduction to Global Analysis

Minimal Surfaces in Riemannian Manifolds

John Douglas Moore, *University of California. Santa Barbara, CA*

During the last century, global analysis was one of the main sources of interaction between geometry and topology. One might argue that the core of this subject is Morse theory, according to which the critical points of a generic smooth proper function on a manifold M determine the homology of the manifold.

Morse envisioned applying this idea to the calculus of variations, including the theory of periodic motion in classical mechanics, by approximating the space of loops on M by a finite-dimensional manifold of high dimension. Palais and Smale reformulated Morse's calculus of variations in terms of infinite-dimensional manifolds, and these infinite-dimensional manifolds were found useful for studying a wide variety of nonlinear PDEs.

This book applies infinite-dimensional manifold theory to the Morse theory of closed geodesics in a Riemannian manifold. It then describes the problems encountered when extending this theory to maps from surfaces instead of curves. It treats critical point theory for closed parametrized minimal surfaces in a compact Riemannian manifold, establishing Morse inequalities for perturbed versions of the energy function on the mapping space. It studies the bubbling which occurs when the perturbation is turned off, together with applications to the existence of closed minimal surfaces. The Morse-Sard theorem is used to develop transversality theory for both closed geodesics and closed minimal surfaces.

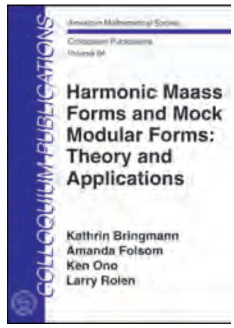
This book is based on lecture notes for graduate courses on "Topics in Differential Geometry", taught by the author over several years. The reader is assumed to have taken basic graduate courses in differential geometry and algebraic topology.

Contents: Infinite-dimensional manifolds; Morse theory of geodesics; Topology of mapping spaces; Harmonic and minimal surfaces; Generic metrics; Bibliography; Index.

Graduate Studies in Mathematics, Volume 187

January 2018, 368 pages, Hardcover, ISBN: 978-1-4704-2950-8, LC 2017028964, 2010 *Mathematics Subject Classification:* 58E12, 58E05; 46T10, 53C20, 55P62, **AMS members US$66.40**, List US$83, Order code GSM/187

Number Theory



Harmonic Maass Forms and Mock Modular Forms: Theory and Applications

Kathrin Bringmann, *University of Cologne, Germany*, **Amanda Folsom**, *Amherst College, MA*, **Ken Ono**, *Emory University, Atlanta, GA*, and **Larry Rolen**, *Trinity College, Dublin, Ireland*

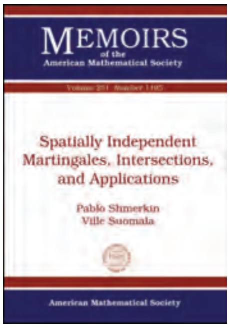
Modular forms and Jacobi forms play a central role in many areas of mathematics. Over the last 10–15 years, this theory has been extended to certain non-holomorphic functions, the so-called "harmonic Maass forms". The first glimpses of this theory appeared in Ramanujan's enigmatic last letter to G. H. Hardy written from his deathbed. Ramanujan discovered functions he called "mock theta functions" which over eighty years later were recognized as pieces of harmonic Maass forms. This book contains the essential features of the theory of harmonic Maass forms and mock modular forms, together with a wide variety of applications to algebraic number theory, combinatorics, elliptic curves, mathematical physics, quantum modular forms, and representation theory.

Contents: *Background:* Elliptic functions; Theta functions and holomorphic Jacobi forms; Classical Maass forms; *Harmonic Maass forms and mock modular forms:* The basics; Differential operators and mock modular forms; Examples of harmonic Maass forms; Hecke theory; Zwegers' thesis; Ramanujan's mock theta functions; Holomorphic projection; Meromorphic Jacobi forms; Mock modular Eichler-shimura theory; Related automorphic forms; *Applications:* Partitions and unimodal sequences; Asymptotics for coefficients of modular-type functions; Harmonic Maass forms as arithmetic and geometric generating functions; Shifted convolution L-functions; Generalized Borcherds products; Elliptic curves over $\mathbb{Q}$; Representation theory and mock modular forms; Quantum modular forms; Representations of mock theta functions; Bibliography; Index.

Colloquium Publications, Volume 64

January 2018, approximately 390 pages, Hardcover, ISBN: 978-1-4704-1944-8, 2010 *Mathematics Subject Classification:* 11F03, 11F11, 11F27, 11F30, 11F37, 11F50, **AMS members US$83.20**, List US$104, Order code COLL/64

Probability and Statistics



Spatially Independent Martingales, Intersections, and Applications

Pablo Shmerkin, *Torcuato Di Tella University, Buenos Aires, Argentina, and CONICET, Buenos Aires, Argentina*, and **Ville Suomala**, *University of Oulu, Finland*

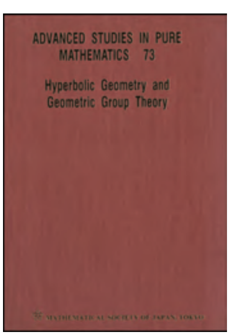
Contents: Introduction; Notation; The setting; Hölder continuity of intersections; Classes of spatially independent martingales; A geometric criterion for Hölder continuity; Affine intersections and projections; Fractal boundaries and intersections with algebraic curves; Intersections with self-similar sets and measures; Dimension of projections: applications of Theorem 4.4; Upper bounds on dimensions of intersections; Lower bounds for the dimension of intersections, and dimension conservation; Products and convolutions of spatially independent martingales; Applications to Fourier decay and restriction; Bibliography.

Memoirs of the American Mathematical Society, Volume 251, Number 1195

December 2017, 100 pages, Softcover, ISBN: 978-1-4704-2688-0, 2010 *Mathematics Subject Classification:* 28A75, 60D05; 28A78, 28A80, 42A38, 42A61, 60G46, 60G57, **Individual member US$47**, List US$78, Institutional member US$62, Order code MEMO/251/1195

New AMS-Distributed Publications

Algebra and Algebraic Geometry



Hyperbolic Geometry and Geometric Group Theory

Koji Fujiwara, *Kyoto University, Japan*, **Sadayoshi Kojima**, *Tokyo Institute of Technology, Japan*, and **Ken'ichi Ohshika**, *Osaka University, Japan*, Editors

The 7th Seasonal Institute of the Mathematical Society of Japan on Hyperbolic Geometry and Geometric Group Theory was held from July 30–August 5, 2014, at the University of Tokyo. This volume, the proceedings of the meeting, collects survey and research articles by international specialists in this fast-growing field.

This volume is recommended for researchers and graduate students interested in hyperbolic geometry, geometric group theory, and low-dimensional topology.

This item will also be of interest to those working in geometry and topology.

Published for the Mathematical Society of Japan by Kinokuniya, Tokyo, and distributed worldwide, except in Japan, by the AMS.

Advanced Studies in Pure Mathematics, Volume 73

September 2017, 280 pages, Hardcover, ISBN: 978-4-86497-042-6, 2010 *Mathematics Subject Classification:* 20F65, 57M50; 20E05, 20E08, 20F05, 20F28, 20F55, 37A20, 37F30, 37F50, 51M10, 57N16, 57S05, 57S30, 58D05, 58D27, **AMS members US$50.40**, List US$63, Order code ASPM/73

Semisimple Lie Algebras and Their Classification Over *p*-Adic Fields

Torsten Schoeneberg, *Camosun College, Victoria, BC, Canada*

This book gives a detailed structure theory for semisimple Lie algebras over arbitrary fields of characteristic 0. Starting from the well-known classification over algebraically closed fields via root systems, the author mimics the language of reductive groups, so that part of his work can be seen as an introduction to a simpler version of Borel-Tits theory. But the author also expresses his results in the language of classical (matrix) algebra as well as Galois cohomology.

In the second part of this book, the author focuses on p-adic fields and achieves a complete classification of semisimple Lie algebras over them. This classification consists essentially of a list of so-called Satake-Tits diagrams, which extend the Dynkin diagrams from the split case. Several instructive examples and historical notes supplement the text. This book can be used as the basis for a lecture on semisimple Lie algebras beyond the beginner's level and as a reference for researchers.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the U.S., Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Mémoires de la Société Mathématique de France, Number 151

October 2017, 147 pages, Softcover, ISBN: 978-2-85629-859-6, 2010 *Mathematics Subject Classification:* 17B05, 17B20, **AMS members US$41.60**, List US$52, Order code SMFMEM/151

Geometry and Topology

The Bochner Technique in Differential Geometry

Hung-Hsi Wu, *University of California, Berkeley*

This monograph is a detailed survey of an area of differential geometry surrounding the Bochner technique. This is a technique that falls under the general heading of "curvature and topology" and refers to a method initiated by Salomon Bochner in the 1940s for proving that on compact Riemannian manifolds, certain objects of geometric interest (e.g., harmonic forms, harmonic spinor fields, etc.) must satisfy additional differential equations when appropriate curvature conditions are imposed.

In 1953 K. Kodaira applied this method to prove the vanishing theorem for harmonic forms with values in a holomorphic vector bundle. This theorem, which bears his name, was the crucial step that allowed him to prove his famous imbedding theorem. Subsequently, the Bochner technique has been extended, on the one hand, to spinor fields and harmonic maps and, on the other, to harmonic functions and harmonic maps on noncompact manifolds. The last has led to the proof of rigidity properties of certain Kähler manifolds and locally symmetric spaces.

This monograph gives a self-contained and coherent account of some of these developments, assuming the basic facts about Riemannian and Kähler geometry as well as the statement of the Hodge theorem. The brief introductions to the elementary portions of spinor geometry and harmonic maps may be especially useful to beginners.

This item will also be of interest to those working in analysis.

A publication of Higher Education Press (Beijing). Distributed in North America by the American Mathematical Society.

Classical Topics in Mathematics, Volume 6

October 2017, 214 pages, Hardcover, ISBN: 978-7-04-047838-9, 2010 *Mathematics Subject Classification:* 53-XX, 58-XX, **AMS members US$47.20**, List US$59, Order code CTM/6

Number Theory

Lectures on the Theory of Elliptic Modular Functions: First Volume

Felix Klein and **Robert Fricke**
Translated by Arthur M. DuPre.

Felix Klein's famous Erlangen program made the theory of group actions into a central part of mathematics. In the spirit of this program, Klein set out to write a grand series of books which unified many different subjects of mathematics, including number theory, geometry, complex analysis, and discrete subgroups.

The first book on icosahedron and the solution of equations of the fifth degree showed closed relations between three seemingly different subjects: the symmetries of the icosahedron, the solution to fifth degree algebraic equations, and the differential equation of hypergeometric functions. It was translated into English in 1888, four years after its original German version was published in 1884. It was followed by two volumes on elliptic modular functions by Klein and Fricke and two more volumes on automorphic functions also by Klein and Fricke.

These four classic books (each volume available separately) are vast generalizations of the first volume and contain the highly original works of Poincaré and Klein on automorphic forms. They have been very influential in the development of mathematics and are now available in English for the first time. These books contain many original ideas, striking examples, explicit computations, and details which are not available anywhere else. They will be very valuable references for people at all levels and allow the reader to see the unity of mathematics through the eyes of one of the most influential mathematicians with vision, Felix Klein.

A publication of Higher Education Press (Beijing). Distributed in North America by the American Mathematical Society.

Classical Topics in Mathematics, Volume 1

October 2017, 639 pages, Hardcover, ISBN: 978-7-04-047872-3, 2010 *Mathematics Subject Classification:* 11-XX, 11F03, **AMS members US$71.20**, List US$89, Order code CTM/1

Lectures on the Theory of Elliptic Modular Functions: Second Volume

Felix Klein and **Robert Fricke**
Translated by Arthur M. DuPre.

Felix Klein's famous Erlangen program made the theory of group actions into a central part of mathematics. In the spirit of this program, Klein set out to write a grand series of books which unified many different subjects of mathematics, including number theory, geometry, complex analysis, and discrete subgroups.

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A publication of Higher Education Press (Beijing). Distributed in North America by the American Mathematical Society.

Classical Topics in Mathematics, Volume 2

October 2017, 589 pages, Hardcover, ISBN: 978-7-04-047837-2, 2010 *Mathematics Subject Classification:* 11-XX, 11F03, **AMS members US$71.20**, List US$89, Order code CTM/2

Lectures on the Theory of Automorphic Functions: First Volume

Felix Klein and **Robert Fricke**
Translated by Arthur M. DuPre.

Felix Klein's famous Erlangen program made the theory of group actions into a central part of mathematics. In the spirit of this program, Klein set out to write a grand series of books which unified many different subjects of mathematics, including number theory, geometry, complex analysis, and discrete subgroups.

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A publication of Higher Education Press (Beijing). Distributed in North America by the American Mathematical Society.

Classical Topics in Mathematics, Volume 3

October 2017, 539 pages, Hardcover, ISBN: 978-7-04-047840-2, 2010 *Mathematics Subject Classification:* 11-XX, 11F03, **AMS members US$71.20**, List US$89, Order code CTM/3

Lectures on the Theory of Automorphic Functions: Second Volume

Felix Klein and **Robert Fricke**
Translated by Arthur M. DuPre.

Felix Klein's famous Erlangen program made the theory of group actions into a central part of mathematics. In the spirit of this program, Klein set out to write a grand series of books which unified many different subjects of mathematics, including number theory, geometry, complex analysis, and discrete subgroups.

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A publication of Higher Education Press (Beijing). Distributed in North America by the American Mathematical Society.

Classical Topics in Mathematics, Volume 4

October 2017, 563 pages, Hardcover, ISBN: 978-7-04-047839-6, 2010 *Mathematics Subject Classification:* 11-XX, 11F03, **AMS members US$71.20**, List US$89, Order code CTM/4

MATHEMATICS CALENDAR

As of the January 2018 issue of *Notices*, Mathematics Calendar will be an online-only product.
The AMS invites you to submit announcements of worldwide meetings and conferences of interest to the mathematical public, including ad hoc, local, or regional meetings, and meetings and symposia devoted to specialized topics, as well as announcements of regularly scheduled meetings of national or international mathematical organizations to the interactive calendar at www.ams.org/cgi-bin/mathcal/mathcal-submit.pl.

Mathematics Calendar is available at: www.ams.org/meetings/calendar/mathcal

Any questions about or difficulties with submissions may be directed to mathcal@ams.org.

November 2017

16 – 17 **Distributional Symmetries and Independences**
Location: *Institute of Mathematics of Bordeaux, Bordeaux, France.*
URL: sites.google.com/view/symmetries-meeting/home

17 – 19 **2017 Midwest Geometry Conference**
Location: *Kansas State University, Manhattan, Kansas.*
URL: https://www.math.ksu.edu/events/conference/2017_Midwest_Geometry/2017_Midwest_Geometry.html

December 2017

4 – 6 **GAP 2017: Geometry and Physics: Curvature Flows in Complex Geometry**
Location: *Fields Institute, Toronto, Canada.*
URL: www.fields.utoronto.ca/activities/17-18/gap2017

8 – 10 **Tech Topology Conference**
Location: *Georgia Institute of Technology, Atlanta, Georgia.*
URL: ttc.gatech.edu

*9 – 11 **International Conference on Algebra, Discrete Mathematics, and Applications (ICADMA 2017)**
Location: *Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad 431004 India.*
URL: bamu.ac.in/int-conference-on-maths-2017/Home.aspx

11 – 13 **Risk Modeling, Management, and Mitigation in Health Sciences**
Location: *Centre de Recherches Mathématiques, Université de Montréal Pavillon André-Aisenstadt 2920, Chemin de la tour, 5th floor Montréal (Québec) H3T 1J4 Canada.*
URL: www.crm.umontreal.ca/2017/Sante17

12 – 14 **Mini-Workshop on Log-Correlated Random Fields**
Location: *Columbia University, New York, New York.*
URL: www.math.columbia.edu/department/probability/log17/log17.html

January 2018

3 – 5 **UCLA Topology Workshop 2018**
Location: *University of California, Los Angeles, California.*
URL: https://math.ucla.edu/~sucharit/TopologyWorkshop18

11 – 12 **Two Nonlinear Days in Perugia on the Occasion of Patrizia Pucci's Sixty-fifth Birthday**
Location: *Dipartimento di Matematica e Informatica, University of Perugia, Italy.*
URL: www.dmi.unipg.it/nonlinearperugia2018

February 2018

12 – 16 **Recent Advances in Hamiltonian Dynamics and Symplectic Topology**
Location: *Department of Mathematics 'Tullio Levi-Civita', University of Padua, Italy.*
URL: events.math.unipd.it/hamschool2018

20 – 24 **Workshop in Nonlinear PDE's and Geometric Analysis**
Location: *Universidade Federal da Paraíba, João Pessoa, Brazil.*
URL: www.mat.ufpb.br/wenlu

March 2018

8 – 10 **Latinx in the Mathematical Sciences Conference 2018**
Location: *Institute for Pure and Applied Mathematics (IPAM), UCLA, Los Angeles, California.*
URL: www.ipam.ucla.edu/lat2018

19 – 24 **Ischia Group Theory 2018**
Location: *Ischia (Naples), Italy.*
URL: www.dipmat2.unisa.it/ischiagrouptheory

April 2018

2 – June 18 **Doc-Course IEMath-Gr, IMUS, University of Cádiz and University of Málaga (Spain). April-June 2018**
Location: *IEMath-Gr (Granada), IMUS (Sevilla), Universities of Cádiz and Málaga (Cádiz, Málaga), Spain.*
URL: www.imus.us.es/DOC-COURSE18/en

9 – 11 **Algebraic Analysis in Honor of Pierre Schapira's Seventy-fifth Birthday**
Location: *Institut Henri Poincaré, Paris, France.*
URL: monge.u-bourgogne.fr/gdito/pierre75

26 – 27 **International Conference on Frontiers in Industrial and Applied Mathematics**
Location: *National Institute of Technology, Hamirpur, HP, India.*
URL: www.fiam2018.org

May 2018

25 – 28 **Geometry and Topology of 3-Manifolds Workshop**
Location: *Okinawa Institute of Science and Technology (OIST), Okinawa, Japan.*
URL: groups.oist.jp/manifolds/workshop

Classified Advertising

Positions available, items for sale, services available, and more

CALIFORNIA

Institute for Pure and Applied Mathematics University of California—Los Angeles

The Institute for Pure and Applied Mathematics (IPAM) at UCLA is seeking an Associate Director (AD), to begin a two-year appointment on August 1, 2018.

The AD is expected to be an active and established research mathematician or scientist in a related field, with experience in conference organization. The primary responsibility of the AD will be running individual programs in coordination with the organizing committees. The selected candidate will be encouraged to continue his or her personal research program within the context of the responsibilities to the institute.

For a detailed job description and application instructions, go to `www.ipam.ucla.edu/5nhZK`. Applications will receive fullest consideration if received by February 15, 2018, but we will accept applications as long as the position remains open. UCLA is an equal opportunity/affirmative action employer.

00045

INDIANA

Purdue University Department: Mathematics Position Title: Andris A. Zoltners Professor

The Department of Mathematics invites applications for an appointment at the rank of tenured full professor to fill the endowed Andris A. Zoltners Professorship in Mathematics. A PhD (or its equivalent) in mathematics or a closely related field is required. We are expecting applications from candidates with an outstanding record of research accomplishments, internationally recognized stature, credentials suitable for immediate nomination as a Distinguished Professor, and great potential for future work. We will consider applications in any area of mathematics.

Duties: Conduct research in mathematics, interact with faculty, teach undergraduate and/or graduate courses, mentor junior faculty, and participate in the governance of the Department, College, and University by serving on faculty committees.

Applications should be submitted online through `www.mathjobs.org/jobs/jobs/10781` and should include (1) the AMS cover sheet for academic employment, (2) a curriculum vitae, (3) a research statement, and (4) four letters of recommendation. Preferably one of the letters will discuss the candidate's teaching. Reference letter writers should be asked to submit their letters online through `www.mathjobs.org`.

Direct all inquiries to `mathhead@purdue.edu`. Screening of applications will begin October 30, 2017, and continue until filled. Some offers may be made before the end of January 2018.

For information about our department, see `www.math.purdue.edu/`. A background check will be required for employment in these positions.

Purdue University's Department of Mathematics is committed to advancing diversity in all areas of faculty effort, including scholarship, instruction, and engagement. Candidates should address at least one of these areas in their cover letter, indicating their past experiences, current interests or activities, and/or future goals to promote a climate that values diversity and inclusion.

Purdue University is an EOE/AA employer. All individuals, including minorities, women, individuals with disabilities, and veterans are encouraged to apply.

00043

Purdue University
Department of Mathematics
Tenure track: Assistant Professor of Mathematics

The Mathematics Department at Purdue University invites applications for up to two possible appointments in mathematics to begin August 2018. These appointments will be at the level of assistant professor. Appointments will be made based on demonstrated research and teaching qualifications. PhD (or its equivalent) in mathematics or a closely related field is required. Preference will be given to outstanding applicants in the areas of Analysis and Geometry (including stochastic analysis/probability, harmonic analysis, partial differential equations, complex analysis, and symplectic/differential geometry), Algebra (including commutative algebra, algebraic topology, algebraic geometry, automorphic forms and number theory), and Computational and Applied Mathematics (including applied, numerical, and computational analysis, the modeling of physical/biological systems, and inverse problems).

Duties: Conduct research in mathematics. Teach undergraduate and/or graduate mathematics courses to a diverse student body and supervise graduate students. Senior faculty will also mentor junior faculty and participate in the governance of the department, the College of Science, and the University by serving on faculty committees.

Applications should be submitted online through `www.mathjobs.org/jobs/jobs/10778` and should include (1) the AMS cover sheet for academic employment, (2) a curriculum vitae, (3) a research statement, and (4) four letters of recommendation, one of which discusses the candidate's teaching qualifications. Reference letter writers should be asked to submit their letters online through `www.mathjobs.org`. Direct all inquiries to `kstroud@math.purdue.edu`.

Applications are considered on a continuing basis but candidates are urged to apply by November 1, 2017. For more information about our department, see `www.math.purdue.edu/`. A background check will be required for employment in this position.

Purdue University's Department of Mathematics is committed to advancing diversity in all areas of faculty effort, including scholarship, instruction, and engagement. Candidates should address at least one of these areas in their cover letter, indicating their past experiences, current interests or activities, and/or future goals to promote a climate that values diversity and inclusion.

Purdue University is an EOE/AA employer. All individuals, including minorities, women, individuals with disabilities, and veterans are encouraged to apply.

00042

Purdue University
Department: Mathematics
Position Title: Golomb Visiting Assistant Professor and Zoltners Visiting Assistant Professor

These three-year positions intended for new and very recent PhDs will commence August 2018 and are open to mathematicians who demonstrate exceptional research promise and a strong teaching record. Ph.D. (or its equivalent) in mathematics or closely related field by August 14, 2018 is required. Applicants should have research interests in common with Purdue faculty. Duties include teaching undergraduate and graduate mathematics courses.

Applications should be submitted online through `www.mathjobs.org` and should include (1) the AMS cover sheet for academic employment, (2) a curriculum vita, (3) a research statement, and (4) three letters of recommendation, one of which discusses the candidate's teaching qualifications. Reference letter writers should be asked to submit their letters online through `www.mathjobs.org/jobs/jobs/10779`. Direct all inquiries to `math-head@purdue.edu`. Screening of applications will begin November 1, 2017 and continue until filled. Some offers will be made before the end of January 2018. For information about our department, see `www.math.purdue.edu/`. A background check will be required for employment in these positions.

Purdue University's Department of Mathematics is committed to advancing diversity in all areas of faculty effort, including scholarship, instruction, and engagement. Candidates should address at least one of these areas in their cover letter, indicating their past experiences, current interests or activities, and/or future goals to promote a climate that values diversity and inclusion.

Purdue University is an EOE/AA employer. All individuals, including minorities, women, individuals with disabilities, and veterans are encouraged to apply.

00041

KANSAS

University of Kansas
Department of Mathematics

The Department of Mathematics at the University of Kansas invites applications for a tenure-track, Assistant Professor position in Geometry expected to begin as early as August 18, 2018. Candidates must demonstrate an outstanding record of research and must be strongly committed to excellence in teaching. Requirements for the position include a PhD in mathematics and outstanding publication record in Differential Geometry and/or Algebraic Geometry, with an emphasis on Geometry.

For a complete announcement and to apply online, go to `employment.ku.edu/academic/10039BR`. At least four recommendation letters (teaching ability must be addressed in at least one letter) should be submitted electronically to `www.mathjobs.org/jobs/jobs/`, position #10761. Initial review of applications will begin November 1, 2017, and continue as long as needed to identify a qualified pool.

KU is an EO/AAE. All qualified applicants will receive consideration for employment without regard to race, color, religion, sex (including pregnancy), age, national origin, disability, genetic information or protected Veteran status.

00044

MASSACHUSETTS

Northeastern University
College of Science, Department of Mathematics and the College of Computer and Information Science
Joint Tenured/Tenure-Track Position, All Levels

The College of Computer and Information Science and the Department of Mathematics, in the College of Science, at Northeastern University invite applications for an open tenure-track/tenured faculty position at all levels, in Mathematics and Data Science, beginning in Fall 2018.

Appointments will be based on exceptional research contributions at the interface between Mathematics and Computer Science, combined with a strong commitment and demonstrated success in teaching.

Candidates will be considered from all areas in Computer and Data Science, Machine Learning, Discrete and Computational Mathematics, Probability and Statistics, and Topological Data Analysis.

Qualifications: A PhD in Computer Science, Mathematics or a closely related field to one of the above-listed areas of expertise by the start date is required. Successful candidates are expected to have or to develop an independently funded research program of international caliber and demonstrated evidence of excellent teaching in undergraduate and graduate courses. Qualified candidates should be committed to fostering diverse and inclusive environments as well as to promoting experiential learning, which are central to a Northeastern University education.

Additional Information: Northeastern University is home to 35,000 full- and part-time degree students and to the nation's premier cooperative education program. The past decade has witnessed a dramatic increase in Northeastern's international reputation for research and innovative educational programs. A heightened focus on interdisciplinary research and scholarship is driving a faculty hiring initiative at Northeastern, advancing its position amongst the nation's top research universities. The College of Computer and Information Science and the College of Science have been major participants in this initiative and will continue their efforts this year, with additional interdisciplinary searches ongoing in related areas. For more information about the College of Computer and Information Science, please visit `www.ccis.northeastern.edu`, and for the College of Science, please visit `www.northeastern.edu/cos/`.

Additional information and instructions for submitting application materials may be found at the following web site: `neu.peopleadmin.com/postings/search`.

Screening of applications begins immediately. For full consideration, application materials should be received by December 1, 2017. However, applications will be accepted until the search is completed.

Northeastern University is an Equal Opportunity, Affirmative Action Educational Institution and Employer, Title IX University. All qualified applicants will receive consideration for employment without regard to race, color, religion, sex, national origin, disability status, protected veteran status, or any other characteristic protected by the law. Northeastern University is an E-Verify Employer.

00047

NEW JERSEY

Rutgers University—New Brunswick Mathematics Department

(This is a supplement to the add posted in the October *Notices*.)

Subject to availability of funding, the Department expects to have two open positions in probability theory and stochastic analysis or mathematical finance with start date in Fall 2018: (1) three-year, non-tenure track, nonrenewable Postdoctoral Assistant Professorship and (2) non-tenure track, renewable contract Teaching Assistant Professorship.

The Postdoctoral Assistant Professorship (code MFP) has a teaching load of 2–1 and candidates for this position should have received a PhD degree, show outstanding promise of research ability in probability theory and stochastic analysis or mathematical finance as well as a capacity for effective teaching. The required application materials are as follows: (a) AMS Cover Sheet, (b) curriculum vitae (including a list of publications), (c) a research statement (not to exceed five pages in length), (d) four letters of recommendation, one of which evaluates teaching and preferably describes quantitative evidence of teaching ability, and (e) a teaching statement (not to exceed two pages in length).

The Teaching Assistant Professorship (code MFTAP) has a teaching load of 3–3 and assists the program director and staff for the Department's Mathematical Finance Master's Degree program, finmath.rutgers.edu. Candidates for this position should have received a PhD degree and show outstanding evidence of teaching ability in probability theory, stochastic analysis, and mathematical finance. The required application materials are as follows: (a) AMS Cover Sheet, (b) curriculum vitae (including a list of publications and teaching experience), (c) a summary of research (not to exceed five pages in length), (d) four letters of recommendation, at least one of which evaluates teaching thoroughly and preferably describes quantitative evidence of teaching ability, and (e) a teaching statement (not to exceed two pages in length).

All application materials for either position should be submitted electronically through the AMS website, `mathjobs.org`.

Review of applications begins December 1, 2017, and continues until any potential openings are filled.

Rutgers, the State University of New Jersey, is an Equal Opportunity / Affirmative Action Employer. Qualified applicants will be considered for employment without regard to race, creed, color, religion, sex, sexual orientation, gender identity or expression, national origin, disability status, genetic information, protected veteran status, military service or any other category protected by law. As an institution, we value diversity of background and opinion, and prohibit discrimination or harassment on the basis of any legally protected class in the areas of hiring, recruitment, promotion, transfer, demotion, training, compensation, pay, fringe benefits.

00046

UTAH

University of Utah Department of Mathematics

The Department of Mathematics at the University of Utah invites applications for the following positions:

- Full-time tenure-track or tenured appointments at the level of Assistant, Associate, or Full Professor in all areas of mathematics.
- Full-time tenure-track or tenured appointments at the level of Assistant, Associate, or Full Professor in all areas of statistics. These positions are part of a University-wide cluster hiring effort in statistics, with particular emphasis in mathematics, computer science, and bioengineering. Successful candidates will have strong interdisciplinary interests.
- Three-year Burgess, Tucker, and Wylie Assistant Professor Lecturer positions.

Please see our website at `www.math.utah.edu/positions` for information regarding available positions and application requirements. Applications must be completed through `www.mathjobs.org/jobs/Utah`. Review of complete applications for tenure-track positions will begin on October 23, 2017, and will continue until the positions are filled. Completed applications for postdoctoral positions received before January 1, 2018, will receive full consideration.

The University of Utah is an Equal Opportunity/Affirmative Action employer and educator. Minorities, women, veterans, and those with disabilities are strongly encouraged to apply. Veterans' preference is extended to qualified veterans. Reasonable disability accommodations will be provided with adequate notice. For additional information about the University's commitment to equal opportunity and access see: `www.utah.edu/nondiscrimination/`.

00029

TAIWAN

National Chengchi University Open Faculty Positions (Tenure Track)

The Department of Mathematical Sciences at National Chengchi University in Taipei, Taiwan anticipates openings for several tenure-track faculty positions. The candidate must hold a doctoral degree in (Applied) Mathematics and be able to communicate in Chinese and English. For more information, please visit `www.math.nccu.edu.tw/app/news.php?Sn=718`.

00015

MEETINGS & CONFERENCES OF THE AMS

DECEMBER TABLE OF CONTENTS

The Meetings and Conferences section of the Notices gives information on all AMS meetings and conferences approved by press time for this issue. Please refer to the page numbers cited on this page for more detailed information on each event. Invited Speakers and Special Sessions are listed as soon as they are approved by the cognizant program committee; the codes listed are needed for electronic abstract submission. For some meetings the list may be incomplete. Information in this issue may be dated.

The most up-to-date meeting and conference information can be found online at: **www.ams.org/meetings/**.

Important Information About AMS Meetings: Potential organizers, speakers, and hosts should refer to page 75 in the January 2017 issue of the *Notices* for general information regarding participation in AMS meetings and conferences.

Abstracts: Speakers should submit abstracts on the easy-to-use interactive Web form. No knowledge of LaTeX is necessary to submit an electronic form, although those who use LaTeX may submit abstracts with such coding, and all math displays and similarily coded material (such as accent marks in text) must be typeset in LaTeX. Visit **www.ams.org/cgi-bin/abstracts/abstract.pl/**. Questions about abstracts may be sent to `abs-info@ams.org`. Close attention should be paid to specified deadlines in this issue. Unfortunately, late abstracts cannot be accommodated.

MEETINGS IN THIS ISSUE

See `www.ams.org/meetings/` *for the most up-to-date information on the meetings and conferences that we offer.*

ASSOCIATE SECRETARIES OF THE AMS

Central Section: Georgia Benkart, University of Wisconsin-Madison, Department of Mathematics, 480 Lincoln Drive, Madison, WI 53706-1388; email: benkart@math.wisc.edu; telephone: 608-263-4283.

Eastern Section: Steven H. Weintraub, Department of Mathematics, Lehigh University, Bethlehem, PA 18015-3174; email: steve.weintraub@lehigh.edu; telephone: 610-758-3717.

Southeastern Section: Brian D. Boe, Department of Mathematics, University of Georgia, 220 D W Brooks Drive, Athens, GA 30602-7403, email: brian@math.uga.edu; telephone: 706-542-2547.

Western Section: Michel L. Lapidus, Department of Mathematics, University of California, Surge Bldg., Riverside, CA 92521-0135; email: lapidus@math.ucr.edu; telephone: 951-827-5910.

Meetings & Conferences of the AMS

IMPORTANT INFORMATION REGARDING MEETINGS PROGRAMS: AMS Sectional Meeting programs do not appear in the print version of the *Notices*. However, comprehensive and continually updated meeting and program information with links to the abstract for each talk can be found on the AMS website. See www.ams.org/meetings/.

Final programs for Sectional Meetings will be archived on the AMS website accessible from the stated URL .

San Diego, California

San Diego Convention Center and San Diego Marriott Hotel and Marina

January 10–13, 2018

Wednesday – Saturday

Meeting #1135

Joint Mathematics Meetings, including the 124th Annual Meeting of the AMS, 101st Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: October 2017
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: Volume 39, Issue 1

Deadlines

For organizers: Expired
For abstracts: Expired

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/national.html.

Joint Invited Addresses

Gunnar Carlsson, Stanford University, *Topological Modeling of Complex Data* (AMS-MAA Invited Address).

Moon Duchin, Tufts University, *Political Geometry: Voting districts,"compactness," and ideas about fairness* (MAA-AMS-SIAM Gerald and Judith Porter Public Lecture).

André Neves, University of Chicago, *Minimal surfaces, volume spectrum, and Morse index* (AMS-MAA Invited Address).

Jill Pipher, Brown University, *Nonsmooth boundary value problems* (AWM-AMS Noether Lecture).

AMS Invited Addresses

Federico Ardila, San Francisco State University, *Algebraic structures on polytopes.*

Robert L. Bryant, Duke University, *The Concept of Holonomy—Its History and Recent Developments* (AMS Retiring Presidential Address).

Ruth Charney, Brandeis University, *Searching for Hyperbolicity.*

Cynthia Dwork, Harvard University, *Privacy in the Land of Plenty* (AMS Josiah Willard Gibbs Lecture).

Dana Randall, Georgia Institute of Technology, *Emergent phenomena in random structures and algorithms.*

Edriss S. Titi, Texas A&M University; and The Weizmann Institute of Science, *The Navier-Stokes, Euler and Related Equations.*

Avi Wigderson, Institute for Advanced Study, *Alternate Minimization and Scaling algorithms: theory, applications and connections across mathematics and computer science* (AMS Colloquium Lectures: Lecture I).

Avi Wigderson, Institute for Advanced Study, *Proving algebraic identities* (AMS Colloquium Lectures: Lecture II).

Avi Wigderson, Institute for Advanced Study, *Proving analytic inequalities* (AMS Colloquium Lectures: Lecture III).

AMS Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as pos-

sible via the abstract submission form found at jointmathematicsmeetings.org/meetings/abstracts/abstract.pl?type=jmm.

Some sessions are cosponsored with other organizations. These are noted within the parenthesis at the end of each listing, where applicable.

A Showcase of Number Theory at Liberal Arts Colleges, **Adriana Salerno**, Bates College, and **Lola Thompson**, Oberlin College.

Accelerated Advances in Mathematical Fractional Programming, **Ram Verma**, International Publications USA, and **Alexander Zaslavski**, Israel Institute of Technology.

Advances in Applications of Differential Equations to Disease Modeling, **Libin Rong**, Oakland University, **Elissa Schwartz**, Washington State University, and **Naveen K. Vaidya**, San Diego State University.

Advances in Difference, Differential, and Dynamic Equations with Applications, **Elvan Akin**, Missouri University S&T, and **John Davis**, Baylor University.

Advances in Operator Algebras, **Marcel Bischoff**, Vanderbilt University, **Ian Charlesworth**, University of California, Los Angeles, **Brent Nelson**, University of California, Berkeley, and **Sarah Reznikoff**, Kansas State University.

Advances in Operator Theory, Operator Algebras, and Operator Semigroups, **Asuman G. Aksoy**, Claremont McKenna College, **Zair Ibragimov**, California State University, Fullerton, **Marat Markin**, California State University, Fresno, and **Ilya Spitkovsky**, New York University, Abu Dhabi.

Algebraic, Analytic, and Geometric Aspects of Integrable Systems, Painlevé Equations, and Random Matrices, **Vladimir Dragovic**, University of Texas at Dallas, **Anton Dzhamay**, University of Northern Colorado, and **Sevak Mkrtchyan**, University of Rochester.

Algebraic, Discrete, Topological and Stochastic Approaches to Modeling in Mathematical Biology, **Olcay Akman**, Illinois State University, **Timothy D. Comar**, Benedictine University, **Daniel Hrozencik**, Chicago State University, and **Raina Robeva**, Sweet Briar College.

Alternative Proofs in Mathematical Practice, **John W. Dawson, Jr.**, Pennsylvania State University, York.

Analysis of Fractional, Stochastic, and Hybrid Dynamic Systems, **John R. Graef**, University of Tennessee at Chattanooga, **Gangaram S. Ladde**, University of South Florida, and **Aghalaya S. Vatsala**, University of Louisiana at Lafayette.

Analysis of Nonlinear Partial Differential Equations and Applications, **Tarek M. Elgindi**, University of California, San Diego, and **Edriss S. Titi**, Texas A&M University and Weizmann Institute of Science.

Applied and Computational Combinatorics, **Torin Greenwood**, Georgia Institute of Technology, and **Jay Pantone**, Dartmouth College.

Arithmetic Dynamics, **Robert L. Benedetto**, Amherst College, **Benjamin Hutz**, Saint Louis University, **Jamie Juul**, Amherst College, and **Bianca Thompson**, Harvey Mudd College.

Beyond Planarity: Crossing Numbers of Graphs (a Mathematics Research Communities Session), **Axel Brandt**, Davidson College, **Garner Cochran**, University of South Carolina, and **Sarah Loeb**, College of William and Mary.

Bifurcations of Difference Equations and Discrete Dynamical Systems, **Arzu Bilgin** and **Toufik Khyat**, University of Rhode Island.

Boundaries for Groups and Spaces, **Joseph Maher**, CUNY College of Staten Island, and **Genevieve Walsh**, Tufts University.

Combinatorial Commutative Algebra and Polytopes, **Robert Davis**, Michigan State University, and **Liam Solus**, KTH Royal Institute of Technology.

Combinatorics and Geometry, **Federico Ardila**, San Francisco State University, **Anastasia Chavez**, MSRI and University of California, Davis, and **Laura Escobar**, University of Illinois Urbana Champaign.

Commutative Algebra in All Characteristics, **Neil Epstein**, George Mason University, **Karl Schwede**, University of Utah, and **Janet Vassilev**, University of New Mexico.

Computational Combinatorics and Number Theory, **Jeremy F. Alm**, Lamar University, and **David Andrews** and **Rob Hochberg**, University of Dallas.

Connections in Discrete Mathematics: Graphs, Hypergraphs, and Designs, **Amin Bahmanian**, Illinois State University, and **Theodore Molla**, University Illinois Urbana-Champaign.

Differential Geometry, **Vincent B. Bonini** and **Joseph E. Borzellino**, Cal Poly San Luis Obispo, **Bogdan D. Suceava**, California State University, Fullerton, and **Guofang Wei**, University of California, Santa Barbara.

Diophantine Approximation and Analytic Number Theory in Honor of Jeffrey Vaaler, **Shabnam Akhtari**, University of Oregon, **Lenny Fukshansky**, Claremont McKenna College, and **Clayton Petsche**, Oregon State University.

Discrete Dynamical Systems and Applications, **E. Cabral Balreira**, **Saber Elaydi**, and **Eddy Kwessi**, Trinity University.

Discrete Neural Networking and Applications, **Murat Adivar**, Fayetteville State University, **Michael A. Radin**, Rochester Institute of Technology, and **Youssef Raffoul**, University of Dayton.

Dynamical Algebraic Combinatorics, **James Propp**, University of Massachusetts, Lowell, **Tom Roby**, University of Connecticut, **Jessica Striker**, North Dakota State University, and **Nathan Williams**, University of California Santa Barbara.

Dynamical Systems with Applications to Mathematical Biology, **Guihong Fan**, Columbus State University, **Jing Li**, California State University Northridge, and **Chunhua Shan**, University of Toledo.

Dynamical Systems: Smooth, Symbolic, and Measurable (a Mathematics Research Communities Session), **Kathryn Lindsey**, Boston College, **Scott Schmieding**, Northwestern University, and **Kurt Vinhage**, University of Chicago.

Emergent Phenomena in Discrete Models, **Dana Randall**, Georgia Institute of Technology, and **Andrea Richa**, Arizona State University.

Emerging Topics in Graphs and Matrices, **Sudipta Mallik**, Northern Arizona University, **Keivan Hassani Monfared**, University of Calgary, and **Bryan Shader**, University of Wyoming.

Ergodic Theory and Dynamical Systems–to Celebrate the Work of Jane Hawkins, **Julia Barnes**, Western Carolina University, **Rachel Bayless**, Agnes Scott College, **Emily Burkhead**, Duke University, and **Lorelei Koss**, Dickinson College.

Extremal Problems in Approximations and Geometric Function Theory, **Ram Mohapatra**, University of Central Florida.

Financial Mathematics, Actuarial Sciences, and Related Fields, **Albert Cohen**, Michigan State University, **Nguyet Nguyen**, Youngstown State University, **Oana Mocioalca**, Kent State University, and **Thomas Wakefield**, Youngstown State University.

Fractional Difference Operators and Their Application, **Christopher S. Goodrich**, Creighton Preparatory School, and **Rajendra Dahal**, Coastal Carolina University.

Free Convexity and Free Analysis, **J. William Helton**, University of California, San Diego, and **Igor Klep**, University of Auckland.

Geometric Analysis, **Davi Maximo**, University of Pennsylvania, **Lu Wang**, University of Wisconsin-Madison, and **Xin Zhou**, University of California Santa Barbara.

Geometric Analysis and Geometric Flows, **David Glickenstein**, University of Arizona, and **Brett Kotschwar**, Arizona State University.

History of Mathematics, **Sloan Despeaux**, Western Carolina University, **Jemma Lorenat**, Pitzer College, **Clemency Montelle**, University of Canterbury, **Daniel Otero**, Xavier University, and **Adrian Rice**, Randolph-Macon College.

Homotopy Type Theory (a Mathematics Research Communities Session), **Simon Cho**, University of Michigan, **Liron Cohen**, Cornell University, and **Edward Morehouse**, Wesleyan University.

If You Build It They Will Come: Presentations by Scholars in the National Alliance for Doctoral Studies in the Mathematical Sciences, **David Goldberg**, Purdue University, and **Phil Kutzko**, University of Iowa.

Interactions of Inverse Problems, Signal Processing, and Imaging, **M. Zuhair Nashed**, University of Central Florida, **Willi Freeden**, University of Kaiserslautern, and **Otmar Scherzer**, University of Vienna.

Markov Chains, Markov Processes and Applications, **Alan Krinik** and **Randall J. Swift**, California State Polytechnic University.

Mathematical Analysis and Nonlinear Partial Differential Equations, **Hongjie Dong**, Brown University, **Peiyong Wang**, Wayne State University, and **Jiuyi Zhu**, Louisiana State University.

Mathematical Fluid Mechanics: Analysis and Applications, **Zachary Bradshaw** and **Aseel Farhat**, University of Virginia.

Mathematical Information in the Digital Age of Science, **Patrick Ion**, University of Michigan, **Olaf Teschke**, zbMath Berlin, and **Stephen Watt**, University of Waterloo.

Mathematical Modeling and Analysis of Infectious Diseases, **Kazuo Yamazaki**, University of Rochester.

Mathematical Modeling of Natural Resources, **Shandelle M. Henson**, Andrews University, and **Natali Hritonenko**, Prairie View A&M University.

Mathematical Modeling, Analysis and Applications in Population Biology, **Yu Jin**, University of Nebraska-Lincoln, and **Ying Zhou**, Lafayette College.

Mathematical Problems in Ocean Wave Modeling and Fluid Mechanics, **Christopher W. Curtis**, San Diego State University, and **Katie Oliveras**, Seattle University.

Mathematical Relativity and Geometric Analysis, **James Dilts** and **Michael Holst**, University of California, San Diego.

Mathematics Research from the SMALL Undergraduate Research Program, **Colin Adams**, **Frank Morgan**, and **Cesar E. Silva**, Williams College.

Mathematics of Gravitational Wave Science, **Andrew Gillette** and **Nikki Holtzer**, University of Arizona.

Mathematics of Quantum Computing and Topological Phases of Matter, **Paul Bruillard**, Pacific Northwest National Laboratory, **David Meyer**, University of California San Diego, and **Julia Plavnik**, Texas A&M University.

Metric Geometry and Topology, **Christine Escher**, Oregon State University, and **Catherine Searle**, Wichita State University.

Modeling in Differential Equations - High School, Two-Year College, Four-Year Institution, **Corban Harwood**, George Fox University, **William Skerbitz**, Wayzata High School, **Brian Winkel**, SIMIODE, and **Dina Yagodich**, Frederick Community College.

Multi-scale Modeling with PDEs in Computational Science and Engineering:Algorithms, Simulations, Analysis, and Applications, **Salim M. Haidar**, Grand Valley State University.

Network Science, **David Burstein**, Swarthmore College, **Franklin Kenter**, United States Naval Academy, and **Feng Shi**, University of North Carolina at Chapel Hill.

New Trends in Celestial Mechanics, **Richard Montgomery**, University of California Santa Cruz, and **Zhifu Xie**, University of Southern Mississippi.

Nilpotent and Solvable Geometry,I, **Michael Jablonski**, University of Oklahoma, **Megan Kerr**, Wellesley College, and **Tracy Payne**, Idaho State University.

Noncommutative Algebras and Noncommutative Invariant Theory, **Ellen Kirkman**, Wake Forest University, and **James Zhang**, University of Washington.

Nonlinear Evolution Equations of Quantum Physics and Their Topological Solutions, **Stephen Gustafson**, University of British Columbia, **Israel Michael Sigal**, University of Toronto, and **Avy Soffer**, Rutgers University.

Novel Methods of Enhancing Success in Mathematics Classes, **Ellina Grigorieva**, Texas Womans University, and **Natali Hritonenko**, Prairie View A&M University.

Open and Accessible Problems for Undergraduate Research, **Michael Dorff**, Brigham Young University, **Allison Henrich**, Seattle University, and **Nicholas Scoville**, Ursinus College.

Operators on Function Spaces in One and Several Variables, **Catherine Bénéteau**, University of South Florida, and **Matthew Fleeman** and **Constanze Liaw**, Baylor University.

Orthogonal Polynomials and Applications, **Abey Lopez-Garcia**, University of South Alabama, and **Xiang-Sheng Wang**, University of Louisiana at Lafayette.

Orthogonal Polynomials, Quantum Probability, and Stochastic Analysis, **Julius N. Esunge**, University of Mary Washington, and **Aurel I. Stan**, Ohio State University.

Quantum Link Invariants, Khovanov Homology, and Low-dimensional Manifolds, **Diana Hubbard**, University of Michigan, and **Christine Ruey Shan Lee**, University of Texas at Austin.

Quaternions, **Terrence Blackman**, Medgar Evers College, City University of New York, and **Johannes Familton** and **Chris McCarthy**, Borough of Manhattan Community College, City University of New York.

Recent Trends in Analysis of Numerical Methods of Partial Differential Equations, **Sara Pollock**, Wright State University, and **Leo Rebholz**, Clemson University.

Research by Postdocs of the Alliance for Diversity in Mathematics, **Aloysius Helminck**, University of Hawaii - Manoa, and **Michael Young**, Iowa State University.

Research from the Rocky Mountain-Great Plains Graduate Research Workshop in Combinatorics, **Michael Ferrara**, University of Colorado Denver, **Leslie Hogben**, Iowa State University, **Paul Horn**, University of Denver, and **Tyrrell McAllister**, University of Wyoming.

Research in Mathematics by Early Career Graduate Students, **Michael Bishop**, **Marat Markin**, **Khang Tran**, and **Oscar Vega**, California State University, Fresno.

Research in Mathematics by Undergraduates and Students in Post-Baccalaureate Programs, **Tamas Forgacs**, CSU Fresno, **Darren A. Narayan**, Rochester Institute of Technology, and **Mark David Ward**, Purdue University (AMS-MAA-SIAM).

Set Theory, Logic and Ramsey Theory, **Andrés Caicedo**, Mathematical Reviews, and **José Mijares**, University of Colorado, Denver (AMS-ASL).

Set-theoretic Topology (Dedicated to Jack Porter in Honor of 50 Years of Dedicated Research), **Nathan Carlson**, California Lutheran University, **Jila Niknejad**, University of Kansas, and **Lynne Yengulalp**, University of Dayton.

Special Functions and Combinatorics (in honor of Dennis Stanton's 65th birthday), **Susanna Fishel**, Arizona State University, **Mourad Ismail**, University of Central Florida, and **Vic Reiner**, University of Minnesota.

Spectral Theory, Disorder and Quantum Physics, **Rajinder Mavi** and **Jeffery Schenker**, Michigan State University.

Stochastic Processes, Stochastic Optimization and Control, Numerics and Applications, **Hongwei Mei**, University of Central Florida, **Zhixin Yang** and **Quan Yuan**, Ball State University, and **Guangliang Zhao**, GE Global Research.

Strengthening Infrastructures to Increase Capacity Around K-20 Mathematics, **Brianna Donaldson**, American Institute of Mathematics, **William Jaco** and **Michael Oehrtman**, Oklahoma State University, and **Levi Patrick**, Oklahoma State Department of Education.

Structure and Representations of Hopf Algebras: a Session in Honor of Susan Montgomery, **Siu-Hung Ng**, Louisiana State University, and **Lance Small** and **Henry Tucker**, University of California, San Diego.

Theory, Practice, and Applications of Graph Clustering, **David Gleich**, Purdue University, and **Jennifer Webster** and **Stephen J. Young**, Pacific Northwest National Laboratory.

Topological Data Analysis, **Henry Adams**, Colorado State University, **Gunnar Carlsson**, Stanford University, and **Mikael Vejdemo-Johansson**, CUNY College of Staten Island.

Topological Graph Theory: Structure and Symmetry, **Jonathan L. Gross**, Columbia University, and **Thomas W. Tucker**, Colgate University.

Visualization in Mathematics: Perspectives of Mathematicians and Mathematics Educators, **Karen Allen Keene**, North Carolina State University, and **Mile Krajcevski**, University of South Florida.

Women in Symplectic and Contact Geometry and Topology, **Bahar Acu**, Northwestern University, **Ziva Myer**, Duke University, and **Yu Pan**, Massachusetts Institute of Technology (AMS-AWM).

Columbus, Ohio

Ohio State University

March 17–18, 2018

Saturday – Sunday

Meeting #1136

Central Section
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: December 2017
Program first available on AMS website: January 31, 2018
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: Volume 39, Issue 2

Deadlines

For organizers: Expired
For abstracts: January 22, 2018

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Aaron Brown, University of Chicago, *Title to be announced.*

Tullia Dymarz, University of Wisconsin-Madison, *Title to be announced.*

June Huh, Institute for Advanced Study, *Title to be announced.*

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the ab-

stract submission form found at www.ams.org/cgi-bin/abstracts/abstract.pl.

Advances in Integral and Differential Equations (Code: SS 26A), **Jeffrey T. Neugebauer**, Eastern Kentucky University, and **Min Wang**, Rowan University.

Algebraic Coding Theory and Applications (Code: SS 27A), **Heide Gluesing-Luerssen**, University of Kentucky, **Christine A. Kelley**, University of Nebraska-Lincoln, and **Steve Szabo**, Eastern Kentucky University.

Algebraic Combinatorics: Association Schemes, Finite Geometry, and Related Topics (Code: SS 15A), **Sung Y. Song**, Iowa State University, and **Bangteng Xu**, Eastern Kentucky University.

Algebraic Curves and Their Applications (Code: SS 17A), **Artur Elezi**, American University, **Monika Polak**, Maria Curie-Sklodowska University (Poland) and University of Information Science and Technology (Mac, and **Tony Shaska**, Oakland University.

Algebraic and Combinatorial Aspects of Tropical Geometry (Code: SS 11A), **Maria Angelica Cueto**, Ohio State University, **Yoav Len**, University of Waterloo, and **Martin Ulirsch**, University of Michigan.

Algebraic, Combinatorial, and Quantum Invariants of Knots and Manifolds (Code: SS 6A), **Cody Armond**, Ohio State University, Mansfield, **Micah Chrisman**, Monmouth University, and **Heather Dye**, McKendree University.

Analytical and Computational Advances in Mathematical Biology Across Scales (Code: SS 30A), **Veronica Ciocanel** and **Alexandria Volkening**, Mathematical Biosciences Institute.

Categorical, Homological and Combinatorial Methods in Algebra (Celebrating the 80th birthday of S. K. Jain) (Code: SS 28A), **Pedro A. Guil Asensio**, University of Murcia, **Ivo Herzog**, Ohio State University, **Andre Leroy**, University of Artois, and **Ashish K. Srivastava**, Saint Louis University.

Coherent Structures in Interfacial Flows (Code: SS 14A), **Benjamin Akers** and **Jonah Reeger**, Air Force Institute of Technology.

Commutative and Combinatorial Algebra (Code: SS 18A), **Jennifer Biermann**, Hobart and William Smith Colleges, and **Kuei-Nuan Lin**, Penn State University, Greater Allegheny.

Convex Bodies in Algebraic Geometry and Representation Theory (Code: SS 20A), **Dave Anderson**, Ohio State University, and **Kiumars Kaveh**, University of Pittsburgh.

Differential Equations and Applications (Code: SS 8A), **King-Yeung Lam** and **Yuan Lou**, Ohio State University, and **Qiliang Wu**, Michigan State University.

Function Spaces, Operator Theory, and Non-Linear Differential Operators (Code: SS 21A), **David Cruz-Uribe**, University of Alabama, and **Osvaldo Mendez**, University of Texas.

Geometric Methods in Shape Analysis (Code: SS 10A), **Sebastian Kurtek** and **Tom Needham**, Ohio State University.

Graph Theory (Code: SS 5A), **John Maharry**, Ohio State University, **Yue Zhao**, University of Central Florida, and **Xiangqian Zhou**, Wright State University.

Homological Algebra (Code: SS 4A), **Ela Celikbas** and **Olgur Celikbas**, West Virginia University.

Homotopy Theory (Code: SS 29A), **Ernest Fontes**, **John E. Harper**, **Crichton Ogle**, and **Gabriel Valenzuela**, Ohio State University.

Lefschetz Properties (Code: SS 24A), **Juan Migliore**, University of Notre Dame, and **Uwe Nagel**, University of Kentucky.

Mathematical Modeling of Neuronal Networks (Code: SS 36A), **Janet Best**, Ohio State University, **Alicia Prieto Langarica**, Youngstown State University, and **Pamela B. Pyzza**, Ohio Wesleyan University.

Multiplicative Ideal Theory and Factorization (in honor of Tom Lucas retirement) (Code: SS 7A), **Evan Houston**, University of North Carolina, Charlotte, and **Alan Loper**, Ohio State University.

Noncommutative Algebra and Noncommutative Algebraic Geometry (Code: SS 16A), **Jason Gaddis**, Miami University, and **Robert Won**, Wake Forest University.

Nonlinear Evolution Equations (Code: SS 9A), **John Holmes** and **Feride Tiglay**, Ohio State University.

Nonlinear Waves and Patterns (Code: SS 19A), **Anna Ghazaryan**, Miami University, **Stephane Lafortune**, College of Charleston, and **Vahagn Manukian** and **Alin Pogan**, Miami University.

Parameter Analysis and Estimation in Applied Dynamical Systems (Code: SS 35A), **Adriana Dawes**, The Ohio State University, and **Reginald L. McGee**, Mathematical Biosciences Institute.

Probabilistic and Extremal Graph Theory (Code: SS 32A), **Louis DeBiasio** and **Tao Jiang**, Miami University.

Probability in Convexity and Convexity in Probability (Code: SS 2A), **Elizabeth Meckes**, **Mark Meckes**, and **Elisabeth Werner**, Case Western Reserve University.

Quantum Symmetries (Code: SS 3A), **David Penneys**, The Ohio State University, and **Julia Plavnik**, Texas A & M University.

Recent Advances in Approximation Theory and Operator Theory (Code: SS 1A), **Jan Lang** and **Paul Nevai**, The Ohio State University.

Recent Advances in Finite Element Methods for Partial Differential Equations (Code: SS 31A), **Ching-shan Chou**, **Yukun Li**, and **Yulong Xing**, The Ohio State University.

Recent Advances in Packing (Code: SS 23A), **Joseph W. Iverson**, University of Maryland, **John Jasper**, South Dakota State University, and **Dustin G. Mixon**, The Ohio State University.

Recent Development of Nonlinear Geometric PDEs (Code: SS 12A), **Bo Guan**, Ohio State University, **Qun Li**, Wright State University, **Xiangwen Zhang**, University of California, Irvine, and **Fangyang Zheng**, Ohio State University.

Several Complex Variables (Code: SS 13A), **Liwei Chen**, **Kenneth Koenig**, and **Liz Vivas**, Ohio State University.

Stochastic Analysis in Infinite Dimensions (Code: SS 22A), **Parisa Fatheddin**, Air Force Institute of Technology, and **Arnab Ganguly**, Louisiana State University.

Structure and Representation Theory of Finite Groups (Code: SS 33A), **Justin Lynd**, University of Louisiana at Lafayette, and **Hung Ngoc Nguyen**, University of Akron.

Symmetry in Differential Geometry (Code: SS 34A), **Samuel Lin**, Dartmouth College, **Barry Minemyer**, Bloomsburg University, and **Ben Schmidt**, Michigan State University.

The Mathematics of Phylogenetics (Code: SS 25A), **Colby Long**, Mathematical Biosciences Institute.

Topology and Geometry in Data Analysis (Code: SS 37A), **Sanjeevi Krishnan** and **Facundo Memoli**, Ohio State University.

Nashville, Tennessee

Vanderbilt University

April 14–15, 2018

Saturday – Sunday

Meeting #1138

Southeastern Section
Associate secretary: Brian D. Boe
Announcement issue of *Notices*: January 2018
Program first available on AMS website: February 22, 2018
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: Volume 39, Issue 2

Deadlines

For organizers: Expired
For abstracts: February 13, 2018

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Andrea Bertozzi, University of California Los Angeles, *Title to be announced* (Erdős Memorial Lecture).

J. M. Landsberg, Texas A & M University, *Title to be announced*.

Jennifer Morse, University of Virginia, *Title to be announced*.

Kirsten Wickelgren, Georgia Institute of Technology, *Title to be announced*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `www.ams.org/cgi-bin/abstracts/abstract.pl`.

Advances in Operator Algebras (Code: SS 7A), **Scott Atkinson**, **Dietmar Bisch**, **Vaughan Jones**, and **Jesse Peterson**, Vanderbilt University.

Algebraic Geometry, Representation Theory, and Applications (Code: SS 21A), **Shrawan Kumar**, University of North Carolina at Chapel Hill, **J. M. Landsberg**, Texas A&M University, and **Luke Oeding**, Auburn University.

Boundaries and Non-positive Curvature in Group Theory (Code: SS 15A), **Spencer Dowdall** and **Matthew Hallmark**, Vanderbilt University, and **Michael Hull**, University of Florida.

Commutative Algebra (Code: SS 8A), **Florian Enescu** and **Yongwei Yao**, Georgia State University.

Difference Equations and Applications (Code: SS 2A), **Michael A. Radin**, Rochester Institute of Technology, and **Youssef Raffoul**, University of Dayton, Ohio.

Evolution Equations and Applications (Code: SS 14A), **Marcelo Disconzi**, **Chenyun Luo**, **Giusy Mazzone**, and **Gieri Simonett**, Vanderbilt University.

Function Spaces and Operator Theory (Code: SS 9A), **Cheng Chu** and **Dechao Zheng**, Vanderbilt University.

Harmonic Analysis, Functional Analysis, and Their Applications (Code: SS 11A), **Akram Aldroubi** and **Keaton Hamm**, Vanderbilt University, **Michael Worthington**, Georgia Institute of Technology, and **Alex Powell**, Vanderbilt University.

Hermitian Geometry (Code: SS 18A), **Mehdi Lejmi**, Bronx Community College of CUNY, and **Rares Rasdeaconu** and **Ioana Suvaina**, Vanderbilt University.

Interactions between Geometry, Group Theory and Dynamics (Code: SS 13A), **Jayadev Athreya**, University of Washington, and **Caglar Uyanik** and **Grace Work**, Vanderbilt University.

Macdonald Polynomials and Related Structures (Code: SS 23A), **Jennifer Morse**, University of Virginia, and **Dan Orr** and **Mark Shimozono**, Virginia Polytechnic Institute and State University.

Mathematical Chemistry (Code: SS 10A), **Hua Wang**, Georgia Southern University.

Matroids and Related Structures (Code: SS 5A), **Carolyn Chun**, United States Naval Academy, **Deborah Chun** and **Tyler Moss**, West Virginia University Institute of Technology, and **Jakayla Robbins**, Vanderbilt University.

Partial Differential Equations and New Perspective of Variational Methods (Code: SS 16A), **Abbas Moameni**, Carleton University, **Futoshi Takahashi**, Osaka City University, and **Michinori Ishiwata**, Osaka University.

Probabilistic Models in Mathematical Physics (Code: SS 6A), **Robert Buckingham**, University of Cincinnati, **Seung-Yeop Lee**, University of South Florida, and **Karl Liechty**, DePaul University.

Quantization for Probability Distributions and Dynamical Systems (Code: SS 1A), **Mrinal Kanti Roychowdhury**, University of Texas Rio Grande Valley.

Random Discrete Structures (Code: SS 22A), **Lutz P Warnke**, Georgia Institute of Technology, and **Xavier Pérex-Giménez**, University of Nebraska-Lincoln (AMS-AAAS).

Recent Advances in Mathematical Biology (Code: SS 12A), **Glenn Webb** and **Yixiang Wu**, Vanderbilt University.

Recent Advances on Complex Bio-systems and Their Applications (Code: SS 17A), **Pengcheng Xiao**, University of Evansville.

Recent Progress and New Directions in Homotopy Theory (Code: SS 20A), **Anna Marie Bohmann**, Vanderbilt University, and **Kirsten Wickelgren**, Georgia Institute of Technology.

Selected Topics in Graph Theory (Code: SS 3A), **Songling Shan**, Vanderbilt University, and **David Chris Stephens** and **Dong Ye**, Middle Tennessee State University.

Structural Graph Theory (Code: SS 4A), **Joshua Fallon**, Louisiana State University, and **Emily Marshall**, Arcadia University.

Tensor Categories and Diagrammatic Methods (Code: SS 19A), **Marcel Bischoff**, Ohio University, and **Henry Tucker**, University of California San Diego.

Portland, Oregon

Portland State University

April 14–15, 2018

Saturday – Sunday

Meeting #1137

Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: January 2018
Program first available on AMS website: February 15, 2018
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: Volume 39, Issue 2

Deadlines

For organizers: Expired
For abstracts: February 6, 2018

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Sándor Kovács, University of Washington, Seattle, *Title to be announced*.

Elena Mantovan, California Institute of Technology, *Title to be announced*.

Dimitri Shlyakhtenko, University of California, Los Angeles, *Title to be announced*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at www.ams.org/cgi-bin/abstracts/abstract.pl.

Algebraic Geometry and its Connections (Code: SS 9A), **Sándor Kovács**, University of Washington, Seattle, and **Karl Schwede**, University of Utah, Salt Lake City.

Algebraic Topology (Code: SS 23A), **Angélica Osorno**, Reed College, and **Dev Sinha**, University of Oregon.

Algebraic and Combinatorial Structures in Knot Theory (Code: SS 3A), **Allison Henrich**, Seattle University, **Inga Johnson**, Willamette University, and **Sam Nelson**, Claremont McKenna College.

Automorphisms of Riemann Surfaces and Related Topics (Code: SS 14A), **S. Allen Broughton**, Rose-Hulman Institute of Technology, **Mariela Carvacho**, Universidad Tecnica Federico Santa Maria, **Anthony Weaver**, Bronx Community College, the City University of New York, and **Aaron Wootton**, University of Portland.

Biomathematics - Progress and Future Directions (Code: SS 4A), **Hannah Callender Highlander**, University of Portland, **Peter Hinow**, University of Wisconsin - Milwaukee, and **Deena Schmidt**, University of Nevada, Reno.

Commutative Algebra (Code: SS 5A), **Adam Boocher**, University of Utah, and **Irena Swanson**, Reed College.

Complex Analysis and Applications (Code: SS 11A), **Malik Younsi**, University of Hawaii Manoa.

Differential Geometry (Code: SS 19A), **Christine Escher**, Oregon State University, and **Catherine Searle**, Wichita State University.

Forest Modeling (Code: SS 20A), **Gatziolis Demetrios**, Pacific Northwest Research Station, US Forest Service, and **Nikolay Strigul**, Washington State University, Vancouver.

General Relativity and Geometric Analysis (Code: SS 13A), **Paul T. Allen**, Lewis & Clark College, **Jeffrey Jauregui**, Union College, and **Iva Stavrov Allen**, Lewis & Clark College.

Geometric Measure Theory and Partial Differential Equations (Code: SS 12A), **Mark Allen**, Brigham Young University, and **Spencer Becker-Kahn** and **Mariana Smit Vega Garcia**, University of Washington.

Inverse Problems (Code: SS 2A), **Hanna Makaruk**, Los Alamos National Laboratory (LANL), and **Robert Owczarek**, University of New Mexico, Albuquerque & Los Alamos.

Mock Modular and Quantum Modular Forms (Code: SS 24A), **Holly Swisher**, Oregon State University, and **Stephanie Treneer**, Western Washington University.

Modeling, Analysis, and Simulation of PDEs with Multiple Scales, Interfaces, and Coupled Phenomena (Code: SS 17A), **Malgorzata Peszynska**, Oregon State University.

Moduli Spaces (Code: SS 21A), **Renzo Cavalieri**, Colorado State University, and **Damiano Fulghesu**, Minnesota State University Moorhead.

Motivic homotopy theory (Code: SS 6A), **Daniel Dugger**, University of Oregon, and **Kyle Ormsby**, Reed College.

Noncommutative Algebraic Geometry and Related Topics (Code: SS 16A), **Jesse Levitt**, University of Southern California, **Hans Nordstrom**, University of Portland, and **Xinting Wang**, Temple University.

Nonsmooth Optimization and Applications(Dedicated to Prof. B. S. Mordukhovich on the occasion of his 70th birthday) (Code: SS 7A), **Mau Nam Nguyen**, Portland State University, **Hung M. Phan**, University of Massachusetts Lowell, and **Shawn Xianfu Wang**, University of British Columbia.

Numerical Methods for Partial Differential Equations (Code: SS 22A), **Brittany A. Erickson** and **Jeffrey S. Ovall**, Portland State University.

Pattern Formation in Crowds, Flocks, and Traffic (Code: SS 1A), **J. J. P. Veerman**, Portland State University, **Alethea Barbaro**, Case Western Reserve University, and **Bassam Bamieh**, UC Santa Barbara.

Recent Advances in Actuarial Mathematics (Code: SS 18A), **Sooie-Hoe Loke**, Central Washington University, and **Enrique Thomann**, Oregon State University.

Spectral Theory (Code: SS 8A), **Jake Fillman**, Virginia Tech, and **Milivoje Lukic**, Rice University.

Teaching and Learning in Undergraduate Mathematics (Code: SS 15A), **Natalie LF Hobson**, Sonoma State University, and **Elise Lockwood**, Oregon State University.

Wavelets, Frames, and Related Expansions (Code: SS 10A), **Marcin Bownik**, University of Oregon, and **Darrin Speegle**, Saint Louis University.

Boston, Massachusetts

Northeastern University

April 21–22, 2018

Saturday – Sunday

Meeting #1139

Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: January 2018
Program first available on AMS website: March 1, 2018
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: Volume 39, Issue 2

Deadlines

For organizers: Expired
For abstracts: February 20, 2018

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Jian Ding, University of Chicago, *Title to be announced*.

Edward Frenkel, University of California, Berkeley, *Title to be announced* (Einstein Public Lecture in Mathematics).

Valentino Tosatti, Northwestern University, *Title to be announced*.

Maryna Viazovska, École Polytechnique Fédérale de Lausanne, *Title to be announced*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `www.ams.org/cgi-bin/abstracts/abstract.pl`.

Algebraic Number Theory (Code: SS 35A), **Michael Bush**, Washington and Lee University, **Farshid Hajir**, University of Massachusetts, and **Christian Maire**, Université Bourgogne Franche-Comté.

Algebraic Statistics (Code: SS 33A), **Kaie Kubjas** and **Elina Robeva**, Massachusetts Institute of Technology.

Algebraic, Geometric, and Topological Methods in Combinatorics (Code: SS 21A), **Florian Frick**, Cornell University, and **Pablo Soberón**, Northeastern University.

Algorithmic Group Theory and Applications (Code: SS 26A), **Delaram Kahrobaei**, City University of New York, and **Antonio Tortora**, University of Salerno.

Analysis and Geometry in Non-smooth Spaces (Code: SS 5A), **Nageswari Shanmugalingam** and **Gareth Speight**, University of Cincinnati.

Arithmetic Dynamics (Code: SS 1A), **Jacqueline M. Anderson**, Bridgewater State University, **Robert Benedetto**, Amherst College, and **Joseph H. Silverman**, Brown University.

Arrangements of Hypersurfaces (Code: SS 2A), **Graham Denham**, University of Western Ontario, and **Alexander I. Suciu**, Northeastern University.

Combinatorial Aspects of Nilpotent Orbits (Code: SS 15A), **Anthony Iarrobino**, Northeastern University, **Leila Khatami**, Union College, and **Juliana Tymoczko**, Smith College.

Combinatorial Representation Theory (Code: SS 41A), **Laura Colmenarejo**, York University, **Ricky Liu**, North Carolina State University, and **Rosa Orellana**, Dartmouth College.

Connections Between Trisections of 4-manifolds and Low-dimensional Topology (Code: SS 32A), **Jeffrey Meier**, University of Georgia, and **Juanita Pinzon-Caicedó**, North Carolina State University.

Discretization in Geometry and Dynamics (Code: SS 36A), **Richard Kenyon**, **Wai Yeung Lam**, and **Richard Schwartz**, Brown University.

Dynamical systems, Geometric Structures and Special Functions (Code: SS 23A), **Alessandro Arsie**, University of Toledo, and **Oksana Bihun**, University of Colorado, Colorado Springs.

Effective Behavior in Random Environments (Code: SS 25A), **Jessica Lin**, McGill University, and **Charles Smart**, University of Chicago.

Ergodic Theory and Dynamics in Combinatorial Number Theory (Code: SS 7A), **Stanley Eigen** and **Daniel Glasscock**, Northeastern University, and **Vidhu Prasad**, University of Massachusetts, Lowell.

Extremal Graph Theory and Quantum Walks on Graphs (Code: SS 13A), **Sebastian Cioabă**, University of Delaware, **Mark Kempton**, Harvard University, **Gabor Lippner**, Northeastern University, and **Michael Tait**, Carnegie Mellon University.

Facets of Symplectic Geometry and Topology (Code: SS 3A), **Tara Holm**, Cornell University, **Jo Nelson**, Columbia University, and **Jonathan Weitsman**, Northeastern University.

Geometries Defined by Differential Forms (Code: SS 34A), **Mahir Bilen Can**, Tulane University, **Sergey Grigorian**, University of Texas Rio Grande Valley, and **Sema Salur**, University of Rochester.

Geometry and Analysis of Fluid Equations (Code: SS 28A), **Robert McOwen** and **Peter Topalov**, Northeastern University.

Geometry of Moduli Spaces (Code: SS 10A), **Ana-Marie Castravet** and **Emanuele Macrí**, Northeastern University, **Benjamin Schmidt**, University of Texas, and **Xiaolei Zhao**, Northeastern University.

Global Dynamics of Real Discrete Dynamical Systems (Code: SS 30A), **M. R. S. Kulenović** and **O. Merino**, University of Rhode Island.

Harmonic Analysis and Partial Differential Equations (Code: SS 29A), **Donatella Danielli**, Purdue University, and **Irina Mitrea**, Temple University.

Homological Commutative Algebra (Code: SS 11A), **Sean Sather-Wagstaff**, Clemson University, and **Oana Veliche**, Northeastern University.

Hopf Algebras, Tensor Categories, and Homological Algebra (Code: SS 8A), **Cris Negron**, Massachusetts Institute of Technology, **Julia Plavnik**, Texas A&M, and **Sarah Witherspoon**, Texas A&M University.

Mathematical Perspectives in Quantum Information Theory (Code: SS 24A), **Aram Harrow**, Massachusetts Institute of Technology, and **Christopher King**, Northeastern University.

Mathematical Problems of Relativistic Physics: Classical and Quantum (Code: SS 37A), **Michael Kiessling** and **A. Shadi Tahvildar-Zadeh**, Rutgers University.

Modeling of Biological Processes (Code: SS 38A), **Simone Cassani** and **Sarah Olson**, Worcester Polytechnic Institute.

New Developments in Inverse Problems and Imaging (Code: SS 9A), **Ru-Yu Lai**, University of Minnesota, and **Ting Zhou**, Northeastern University.

Noncommutative Algebra and Representation Theory (Code: SS 22A), **Van C. Nguyen**, Hood College, and **Alex Martsinkovsky** and **Gordana Todorov**, Northeastern University.

Nonlinear Reaction-Diffusion Equations and Their Applications (Code: SS 31A), **Nsoki Mavinga** and **Quinn Morris**, Swarthmore College.

Nonlinear and Stochastic Partial Differential Equations and Applications (Code: SS 19A), **Nathan Glatt-Holtz** and **Vincent Martinez**, Tulane University, and **Cecilia Mondaini**, Texas A&M University.

Numerical Methods and Applications (Code: SS 16A), **Vera Babenko**, Ithaca College.

Optimization Under Uncertainty (Code: SS 40A), **Yingdong Lu** and **Mark S. Squillante**, IBM Research.

Polytopes and Discrete Geometry (Code: SS 6A), **Gabriel Cunningham**, University of Massachusetts, Boston, **Mark Mixer**, Wentworth Institute of Technology, and **Egon Schulte**, Northeastern University.

Regularity of PDEs on Rough Domains (Code: SS 14A), **Murat Akman**, University of Connecticut, and **Max Engelstein**, Massachusetts Institute of Technology.

Relations Between the History and Pedagogy of Mathematics (Code: SS 20A), **Amy Ackerberg-Hastings**, and **David L. Roberts**, Prince George's Community College.

Singularities of Spaces and Maps (Code: SS 4A), **Terence Gaffney** and **David Massey**, Northeastern University.

The Analysis of Dispersive Equations (Code: SS 39A), **Marius Beceanu**, University at Albany, and **Andrew Lawrie**, Massachusetts Institute of Technology.

The Gaussian Free Field and Random Geometry (Code: SS 12A), **Jian Ding**, University of Chicago, and **Vadim Gorin**, Massachusetts Institute of Technology.

Topics in Qualitative Properties of Partial Differential Equations (Code: SS 27A), **Changfeng Gui**, University of Texas at San Antonio, **Changyou Wang**, Purdue University, and **Jiuyi Zhu**, Louisiana State University.

Topics in Toric Geometry (Code: SS 17A), **Ivan Martino**, Northeastern University, and **Emanuele Ventura**, Texas A&M University.

Topology of Biopolymers (Code: SS 18A), **Erica Flapan**, Pomona College, and **Helen Wong**, Carleton College.

Shanghai, People's Republic of China

Fudan University

June 11–14, 2018

Monday – Thursday

Meeting #1140

Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: April 2018
Program first available on AMS website: Not applicable
Program issue of electronic *Notices*: Not applicable
Issue of *Abstracts*: Not applicable

Deadlines

For organizers: To be announced
For abstracts: To be announced

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/internmtgs.html.

Invited Addresses

Yu-Hong Dai, Academy of Mathematics and System Sciences, *Title to be announced.*

Kenneth A. Ribet, University of California, Berkeley, *Title to be announced.*

Richard M. Schoen, University of California, Irvine, *Title to be announced.*

Sijue Wu, University of Michigan, *Title to be announced.*

Chenyang Xu, Peking University, *Title to be announced.*

Jiangong You, Nankai University, *Title to be announced.*

Newark, Delaware

University of Delaware

September 29–30, 2018

Saturday – Sunday

Meeting #1141

Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: June 2018
Program first available on AMS website: August 9, 2018
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: Volume 39, Issue 3

Deadlines

For organizers: February 28, 2018
For abstracts: July 31, 2018

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Leslie Greengard, New York University, *Title to be announced*.

Elisenda Grigsby, Boston College, *Title to be announced*.

Davesh Maulik, Massachusetts Institute of Technology, *Title to be announced*.

Fayetteville, Arkansas

University of Arkansas

October 6–7, 2018

Saturday – Sunday

Meeting #1142

Southeastern Section
Associate secretary: Brian D. Boe
Announcement issue of *Notices*: July 2018
Program first available on AMS website: August 16, 2018
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: Volume 39, Issue 3

Deadlines

For organizers: March 6, 2018
For abstracts: August 7, 2018

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Mihalis Dafermos, Princeton University, *Title to be announced*.

Jonathan Hauenstein, University of Notre Dame, *Title to be announced*.

Kathryn Mann, University of California Berkeley, *Title to be announced*.

Ann Arbor, Michigan

University of Michigan, Ann Arbor

October 20–21, 2018

Saturday – Sunday

Meeting #1143

Central Section
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: July 2018
Program first available on AMS website: August 30, 2018
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: Volume 39, Issue 4

Deadlines

For organizers: March 20, 2018
For abstracts: August 21, 2018

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Elena Fuchs, University of Illinois Urbana-Champaign, *Title to be announced*.

Andrew Putman, University of Notre Dame, *Title to be announced*.

Charles Smart, University of Chicago, *Title to be announced*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at www.ams.org/cgi-bin/abstracts/abstract.pl.

From Hyperelliptic to Superelliptic Curves (Code: SS 6A), **Tony Shaska**, Oakland University, **Nicola Tarasca**, Rutgers University, and **Yuri Zarhin**, Pennsylvania State University.

Geometry of Submanifolds, in Honor of Bang-Yen Chens 75th Birthday (Code: SS 1A), **Alfonso Carriazo**, University of Sevilla, **Ivko Dimitric**, Penn State Fayette, **Yun Myung Oh**, Andrews University, **Bogdan D. Suceava**, California State University, Fullerton, **Joeri Van der Veken**, University of Leuven, and **Luc Vrancken**, Universite de Valenciennes.

Interactions between Algebra, Machine Learning and Data Privacy (Code: SS 3A), **Jonathan Gryak**, University of

Michigan, **Kelsey Horan**, CUNY Graduate Center, **Delaram Kahrobaei**, CUNY Graduate Center and New York University, **Kayvan Najarian** and **Reza Soroushmehr**, University of Michigan, and **Alexander Wood**, CUNY Graduate Center.

Random Matrix Theory Beyond Wigner and Wishart (Code: SS 2A), **Elizabeth Meckes** and **Mark Meckes**, Case Western Reserve University, and **Mark Rudelson**, University of Michigan.

Self-similarity and Long-range Dependence in Stochastic Processes (Code: SS 4A), **Takashi Owada**, Purdue University, **Yi Shen**, University of Waterloo, and **Yizao Wang**, University of Cincinnati.

Structured Homotopy Theory (Code: SS 5A), **Thomas Fiore**, University of Michigan, Dearborn, **Po Hu** and **Dan Isaksen**, Wayne State University, and **Igor Kriz**, University of Michigan.

San Francisco, California

San Francisco State University

October 27–28, 2018

Saturday – Sunday

Meeting #1144

Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: July 2018
Program first available on AMS website: September 6, 2018
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: Volume 39, Issue 4

Deadlines

For organizers: March 27, 2018
For abstracts: August 28, 2018

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Srikanth B. Iyengar, University of Utah, *Title to be announced.*

Sarah Witherspoon, Texas A&M University, *Title to be announced.*

Abdul-Aziz Yakubu, Howard University, *Title to be announced.*

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at www.ams.org/cgi-bin/abstracts/abstract.pl.

Coupling in Probability and Related Fields (Code: SS 3A), **Sayan Banerjee**, University of North Carolina, Chapel Hill, and **Terry Soo**, University of Kansas.

Homological Aspects of Noncommutative Algebra and Geometry (Code: SS 2A), **Dan Rogalski**, University of California San Diego, **Sarah Witherspoon**, Texas A&M University, and **James Zhang**, University of Washington, Seattle.

Mathematical Biology with a focus on Modeling, Analysis, and Simulation (Code: SS 1A), **Jim Cushing**, The University of Arizona, **Saber Elaydi**, Trinity University, **Suzanne Sindi**, University of California, Merced, and **Abdul-Aziz Yakubu**, Howard University.

Baltimore, Maryland

Baltimore Convention Center, Hilton Baltimore, and Baltimore Marriott Inner Harbor Hotel

January 16–19, 2019

Wednesday – Saturday

Meeting #1145

Joint Mathematics Meetings, including the 125th Annual Meeting of the AMS, 102nd Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM)and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: October 2018
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: April 2, 2018
For abstracts: To be announced

Auburn, Alabama

Auburn University

March 15–17, 2019
Friday – Sunday

Meeting #1146
Southeastern Section
Associate secretary: Brian D. Boe
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines
For organizers: To be announced
For abstracts: To be announced

Honolulu, Hawaii

University of Hawaii at Manoa

March 22–24, 2019
Friday – Sunday

Meeting #1147
Central Section
Associate secretaries: Georgia Benkart and Michel L. Lapidus
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines
For organizers: May 15, 2018
For abstracts: January 22, 2019

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Barry Mazur, Harvard University, *Title to be announced* (Einstein Public Lecture in Mathematics).

Aaron Naber, Northwestern University, *Title to be announced.*

Deanna Needell, University of California, Los Angeles, *Title to be announced.*

Katherine Stange, University of Colorado, Boulder, *Title to be announced.*

Andrew Suk, University of Illinois at Chicago, *Title to be announced.*

Quy Nhon City, Vietnam

Quy Nhon University

June 10–13, 2019
Monday – Thursday
Associate secretary: Brain D. Boe
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines
For organizers: To be announced
For abstracts: To be announced

Binghamton, New York

Binghamton University

October 12–13, 2019
Saturday – Sunday
Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines
For organizers: March 12, 2019
For abstracts: To be announced

Gainesville, Florida

University of Florida

November 2–3, 2019
Saturday – Sunday
Southeastern Section
Associate secretary: Brian D. Boe
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines
For organizers: To be announced
For abstracts: To be announced

Denver, Colorado

Colorado Convention Center

January 15–18, 2020

Wednesday – Saturday

Joint Mathematics Meetings, including the 126th Annual Meeting of the AMS, 103rd Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM)

Associate secretary: Michel L. Lapidus

Announcement issue of *Notices*: October 2019

Program first available on AMS website: November 1, 2019

Program issue of electronic *Notices*: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: April 1, 2019

For abstracts: To be announced

Washington, District of Columbia

Walter E. Washington Convention Center

January 6–9, 2021

Wednesday – Saturday

Joint Mathematics Meetings, including the 127th Annual Meeting of the AMS, 104th Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).

Associate secretary: Brian D. Boe

Announcement issue of *Notices*: October 2020

Program first available on AMS website: November 1, 2020

Program issue of electronic *Notices*: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: April 1, 2020

For abstracts: To be announced

2017 *Notices* Issue Index

Search for specific content that appeared in the 2017 Notices.

This year-end Index is presented below in two easily searchable Parts:

Part I: ***Issue at-a-Glance***, is organized by specific Volume and Issue. This Part provides readers with a quick snapshot of the main articles featured in each specific *Notices* issue.

Part II: ***Societal Record***, is organized alphabetically under each category heading. This Part provides readers with a searchable listing of all content of record for the Society, including: elections, awards, meetings, news, opportunities, annual AMS reports, surveys, grants, fellowships, etc.

Issue at-a-Glance

January 2017
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February 2017
Volume 64, Issue 2 • pages 97–192

COMMENTARY

LETTERS TO THE EDITOR

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DEATH NOTICES/MEMORIAMS

Societal Record

AWARDS AND PRIZES

GRANTS, FELLOWSHIPS, AND OPPORTUNITIES

MEETINGS & CONFERENCES

INVITED ADDRESS SPEAKERS BY MEETING

SURVEYS AND REPORTS

2018 Joint Mathematics Meetings Advance Registration/Housing Form

JOINT MATHEMATICS MEETINGS
SAN DIEGO • JAN 10–13, 2018
AMERICAN MATHEMATICAL SOCIETY
MATHEMATICAL ASSOCIATION OF AMERICA

Name ____________________
(please print your name as you would like it to appear on your badge)

Mailing Address ____________________

____

Telephone ____________________ Fax: ____________________

In case you have an emergency at the meeting: Day #: ____________ Evening #: ____________

Email Address ____________________ Additional email address for receipt ____________

Acknowledgment of this registration and any hotel reservations will be sent to the email address(es) given here. **Check this box to receive a copy in U.S. Mail:** ☐

Affiliation for badge ____________________ (company/university) Nonmathematician guest badge name: ____________ (Note fee of US$21)

In an effort to make the JMM more environmentally friendly as well as save on printing expenses to the meeting, the JMM program books will now only be distributed to participants who ask for them. More up-to-date program and meeting information will be available on the JMM website and mobile app.
Do you want to receive a copy of the program book? Yes ☐ No ☐

☐ I DO NOT want my badge and program (if printed program is requested) to be mailed to me on 12/8/17. Materials will be mailed to the address listed above unless you check this box. Materials will not be mailed for registrations completed after November 22, or to individual commercial or artist exhibitors. Exhibiting companies may opt to have booth staff badges mailed to a company contact.

Registration Fees

Membership please ✓ all that apply. First row is eligible to register as a member.
For undergraduate students, membership in PME and KME also applies.
☐ AMS & MAA ☐ AMS but not MAA ☐ MAA but not AMS ☐ ASL ☐ CMS ☐ SIAM
Undergraduate Students Only: ☐ PME ☐ KME
Other Societies: ☐ AWM ☐ NAM ☐ YMN ☐ AMATYC

Joint Meetings	by Dec 20	at mtg	Subtotal
☐ Member AMS, MAA, ASL, CMS, or SIAM	US$ 329	US$ 433	
☐ Nonmember	US$ 522	US$ 666	
☐ Graduate Student Member (AMS, MAA ASL, CMS, or SIAM)	US$ 74	US$ 86	
☐ Graduate Student (Nonmember)	US$118	US$ 130	
☐ Undergraduate Student (Member AMS, ASL, CMS, MAA, PME, KME, or SIAM)	US$ 74	US$ 86	
☐ Undergraduate Student (Nonmember)	US$118	US$ 130	
☐ High School Student	US$ 7	US$ 14	
☐ Unemployed	US$ 74	US$ 86	
☐ Temporarily Employed	US$ 268	US$ 307	
☐ Developing Countries Special Rate	US$ 74	US$ 86	
☐ Emeritus Member of AMS or MAA	US$ 74	US$ 86	
☐ High School Teacher	US$ 74	US$ 86	
☐ Librarian	US$ 74	US$ 86	
☐ Press	US$ 0	US$ 0	
☐ Exhibitor (Commercial)	US$ 0	US$ 0	
☐ Artist Exhibitor (work in JMM Art Exhibit)	US$ 0	US$ 0	
☐ Nonmathematician Guest of registered mathematician	US$ 21	US$ 21	
			$________

AMS Short Course: *Random Growth Models (1/8-1/9)*			
☐ Member of AMS	US$ 114	US$148	
☐ Nonmember	US$ 175	US$205	
☐ Student, Unemployed, Emeritus	US$ 62	US$ 83	
			$________

MAA Minicourses (see listing in text)
I would like to attend: ☐ One Minicourse ☐ Two Minicourses
Please enroll me in MAA Minicourse(s) #______ and #______
Price: US$ 100 for each minicourse.
(For more than 2 minicourses, call or email the MMSB.) $________

Graduate School Fair Table
☐ Graduate Program Table US$125 US$125
(includes table, posterboard & electricity)
Dept. or Program to be represented (write below or email)
____________________ $________

Receptions & Banquets
☐ Graduate Student/First-Time Attendee Reception (1/10) (no charge)
☐ NAM Banquet (1/12)
#____Chicken #____Fish #____Vegan US$ 75
#____Kosher (*Additional fees apply for Kosher Meals.*) US$ 125
Total for NAM Banquet $________
☐ AMS Dinner (1/13) Regular Price #____US$ 75
Student Price #____US$ 30
(*For special dietary requests, please email mmsb@ams.org)*
Total for AMS Dinner $________

Total for Registrations and Events $________

Payment

Registration & Event Total (total from column on left) $ ________

Hotel Deposit (only if paying by check) $ ________
If you send a hotel deposit check, the deadline for this form is December 5.

Total Amount To Be Paid $ ________

Method of Payment
☐ Check. Make checks payable to the AMS. For all check payments, please keep a copy of this form for your records.
☐ Credit Card. All major credit cards accepted. For your security, we do not accept credit card numbers by email, fax, or postal mail. If the MMSB receives your registration form by any of these methods, it will contact you at the phone number provided on this form.

Signature: ____________________

☐ Purchase Order # ____________ (please enclose copy)

Other Information

Mathematical Reviews field of interest # ____________

☐ I am willing to serve as a judge for the MAA Undergraduate Student Poster Session

☐ For planning purposes for the MAA Two-year College Reception, please check if you are a faculty member at a two-year college.

☐ I am a mathematics department chair.

☐ Please do not include my name and postal address on any promotional mailing lists. (The JMM does not share email addresses.)

☐ Please do not include my name on any list of JMM participants other than the scientific program if I am, in fact, making a presentation that is part of the meeting.

☐ Please ✓ this box if you have a disability requiring special services.

Registration for the Joint Meetings is not required for the short course but it is required for the minicourses and the Employment Center. To register for the Employment Center, go to http://www.ams.org/profession/employment-services/employment-center. For questions, email: emp-info@ams.org.

Registration Deadlines

To be eligible for the complimentary hotel room lottery:	Oct. 31, 2017
In time to receive badges/programs in the mail:	Nov. 22, 2017
Hotel reservations with check deposit:	Dec. 5, 2017
Hotel reservations, changes/cancellations through the JMM website:	Dec. 6, 2017
Advance registration for the Joint Meetings, short course, minicourses, and dinner tickets:	Dec. 20, 2017
Cancel in time to receive 50% refund on advance registration, banquets, minicourses, and short course	Jan. 4, 2018*

*no refunds issued after this date.

Mailing Address/Contact:

Mathematics Meetings Service Bureau (MMSB)
P. O. Box 6887
Providence, RI 02940-6887 Fax: 401-455-4004; Email: mmsb@ams.org
Telephone: 401-455-4144 or 1-800-321-4267 x4144 or x4137

2018 Joint Mathematics Meetings Hotel Reservations – San Diego, CA

Please see the hotel information in the announcement or on the web for detailed information on each hotel. To ensure accurate assignments, please rank hotels in order of preference by writing 1, 2, 3, etc. in the column on the left and by circling the requested bed confirguration. If your requested hotel and room type is no longer available, you will be assigned a room at the next available comparable rate. Please call the MMSB for details on suite configurations, sizes, availability, etc. All reservations, including suite reservations, must be made through the MMSB to receive the JMM rates. Reservations made directly with the hotels before **December 15, 2017** may be changed to a higher rate. All rates are subject to applicable local and state taxes in effect at the time of check-in; currently 10.5% state tax, the San Diego Tourism Marketing District assessment 2% tax, and the CA Tourism fee of US$0.77 per night. **Guarantee requirements: First night deposit by check (add to payment on reverse of form) or a credit card guarantee. Please note that reservations with check deposits must be received by the MMSB by December 5, 2017.** People interested in suites should contact the MMSB directly at mmsb@ams.org or by calling 800-321-4267, ext. 4137; (401-455-4137).

☐ **Deposit enclosed (see front of form)**
☐ **Hold with my credit card**. **For your security, we do not accept credit card numbers by email, postal mail or fax.** If the MMSB receives your registration form by any of these methods, it will contact you at the phone number provided on the reverse of this form.

Date and Time of Arrival__________ **Date and Time of Departure** __________ **Number of adult guests in room**______ **Number of children** ______

Name of Other Adult Room Occupant (s) __________ Arrival: ________ Departure: ________

Housing Requests:.(example: rollaway cot, crib, nonsmoking room, low floor) __________

☐ I have disabilities as defined by the ADA that require a sleeping room that is accessible to the physically challenged. My needs are:__________

☐ I am a member of a hotel frequent-travel club and would like to receive appropriate credit. The hotel chain and card number are: __________

☐ I am not reserving a room. I am sharing with __________ , who is making the reservation.

Order of choice	Hotel	Single	Double 1 bed-2 people	Double 2 beds- 2 people	Triple 3 adults-2 beds	Quad 4 adults-2 beds	Rollaway/Cot Fee (add to special requests if reserving online)
	Marriott Marquis San Diego Marina (hdqrs)						Rollaways are available (at no charge) in king-bedded rooms only
	Bay View	US$ 222	US$ 222	US$ 222	US$ 242	UD$ 262	
	City View	US$ 207	US$ 207	US$ 207	US$ 227	UD$ 247	
	City View Student Rate	US$ 147	US$ 147	US$ 147	US$ 167	UD$ 187	
	Embassy Suites Hotel San Diego Bay	US$ 180	US$ 180	US$ 180	US$ 200	US$ 220	Rollaways are not available.
	Student Rate	US$ 160	US$ 160	US$ 160	US$ 180	US$ 200	
	Manchester Grand Hyatt San Diego	US$ 179	US$ 189	US$ 189	US$ 199	US$ 209	Rollaways are available (at no charge) in king-bedded rooms only.
	Student Rate	US$ 147	US$ 147	US$ 147	US$ 157	US$ 167	
	Omni Hotel San Diego	US$ 179	US$ 189	US$ 189	US$ 199	US$ 209	Rollaways are available (at no charge) in king-bedded rooms only
	Student Rate	US$ 147	US$ 147	US$ 147	US$ 157	US$ 167	
	Hilton Gaslamp San Diego	US$ 176	US$ 176	US$ 176	US$ 176	US$ 176	Maximum 4 people per room. Rollaways are available for US$ 20 per day in king-bedded rooms only.
	Student Rate	US$ 158	US$ 158	US$ 158	US$ 158	US$ 158	
	Hard Rock Hotel San Diego	US$ 170	US$ 170	US$ 170	US$ 190	US$ 210	Rollaways are available for US$ 30 per day in king-bedded rooms only.
	Student Rate	US$ 159	US$ 159	US$ 159	US$ 179	US$ 199	
	Best Western Plus Bayside Inn	US$ 165	US$ 165	US$ 165	US$ 175	US$ 175	Rollaways are available on request for US$ 10 per day.
	Student Rate	US$ 155	US$ 155	US$ 155	US$ 165	US$ 165	
	Solamar Hotel San Diego	US$ 165	US$ 165	US$ 165	US$ 185	US$ 205	Rollaways are available for US$ 40 per day in king-bedded rooms only.
	Student Rate	US$ 149	US$ 149	US$ 149	US$ 169	US$ 189	
	Palomar Hotel San Diego	US$ 160	US$ 160	US$ 160	US$ 185	US$ 210	Rollaways are not available.
	Student Rate	US$ 149	US$ 149	US$ 149	US$ 174	US$ 199	
	Horton Grand Hotel	US$ 159	US$ 159	US$ 159	US$ 179	US$ 199	Rollaways are not available.
	Student Rate	US$ 139	US$ 139	US$ 139	US$ 159	US$ 179	
	Porto Vista Hotel	US$ 125	US$ 125	US$ 125	US$ 145	US$ 165	Rollaways are not available.

IN THE NEXT ISSUE OF NOTICES

JANUARY 2018…

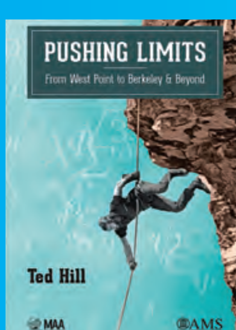
"We all have a responsibility to inspire a new generation in STEM and nurture the dreams of future mathematical leaders."
—Talithia Williams

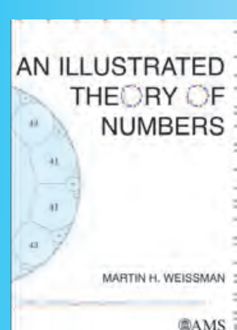
At the 2018 Joint Mathematics Meetings, **Talithia Williams** will be delivering the AMS-MAA-SIAM Hrabowski-Gates-Tapia-McBay Lecture "Mathematics for the Masses," where she discusses the new six-part PBS series *NOVA Wonders*, for which she is one of the hosts.

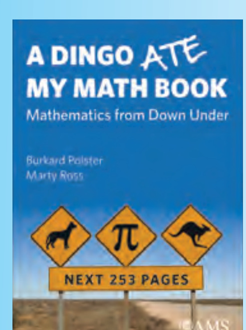
Read Talithia's introduction to her lecture in the January *Notices* **JMM 2018 Lecture Sampler**. Also included in the sampler are lecture introductions from:

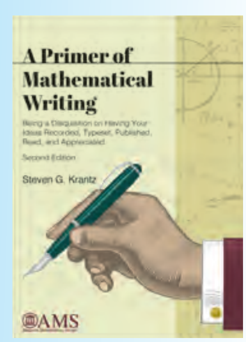
Federico Ardila; **Gunnar Carlsson**; **Ruth Charney**; **André Neves**; **Ronald E. Mickens**; **Jill Pipher**; **Dana Randall**; and **Erica Walker**.

Are you looking for holiday gift ideas at a great price?









Visit bookstore.ams.org to browse our Holiday Sale. Select titles are discounted until 12/17/17.

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