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Quantum parameters of the geometric Langlands theory

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Abstract

Fix a smooth, complete algebraic curve X over an algebraically closed field k of characteristic zero. To a reductive group G over k, we associate an algebraic stack Par_G of quantum parameters for the geometric Langlands theory. Then we construct a family of (quasi-)twistings parametrized by Par_G, whose module categories give rise to twisted $\mathcal{D}$-modules on Bun_G as well as quasi-coherent sheaves on the DG stack LocSys_G.

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1 Introduction

1.1 The geometric Langlands conjecture

1.1.1. The goal of the Langlands program can be broadly described as to establish a correspondence between automorphic forms attached to a reductive group G and Galois representations valued in the Langlands dual group $\check{G}$.

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1.1.2. In the (global, unramified) geometric theory, we fix a smooth, connected, projective curve X over an algebraically closed field k. For simplicity, let G be a reductive group over k (where "reductive" is meant to imply "connected"). Then automorphic functions correspond to certain sheaves on the stack Bun_G parametrizing G-bundles over X, and the role of Galois representations is played by local systems on X valued in $\check{G}$, the Langlands dual group defined over a coefficient field E.

If we further specialize to the case where k is of characteristic zero, then it is possible to take $E = k$ and study the de Rham $\check{G}$-local systems on X. The latter also form a moduli stack over k, denoted by $\mathrm{LocSys}_{\check{G}}$.

1.1.3. Unlike Bun_G, the stack $\mathrm{LocSys}_{\check{G}}$ is not smooth. Furthermore, it is a DG algebraic stack in general and the correct formulation of the geometric Langlands conjecture has to take into account its DG nature.

After Arinkin and Gaitsgory [1], one conjectures an equivalence of DG categories:

$$\mathbb{L}_G : \mathcal{D}\text{-Mod}(\mathrm{Bun}_G) \xrightarrow{\sim} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}). \tag{1.1}$$

Here, the left-hand-side is the DG category of $\mathcal{D}$-modules on Bun_G. The right-hand-side is the DG category of ind-coherent sheaves on $\mathrm{LocSys}_{\check{G}}$ whose singular support is contained in the global nilpotent cone. This DG category is an enlargement of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$, and the appearance of singular support is the geometric incarnation of Arthur parameters.

1.2 What do we mean by "quantum"?

1.2.1. The quantum geometric Langlands theory seeks to simultaneously deform both sides of (1.1) in a way to make them look more symmetric. The main idea, due to Drinfeld and expounded on by Stoyanovsky [26] and Gaitsgory [14], is to consider the DG category of *twisted* $\mathcal{D}$-modules on Bun_G.

1.2.2. To explain this approach, let us temporarily assume that G is simple. Write $\mathcal{L}_{G,\det}$ for the determinant line bundle over Bun_G associated to the adjoint representation.

To every value $c \in k$ one can associate the DG category $\mathcal{D}\text{-Mod}^c(\mathrm{Bun}_G)$ of $\mathcal{D}$-modules over Bun_G twisted by the $(\frac{c-h^\vee}{2h^\vee})$th power of $\mathcal{L}_{G,\det}$, where $h^\vee$ denotes the dual Coxeter number of G.

Let $r = 1, 2$, or 3 be the maximal multiplicity of arrows in the Dynkin diagram of G. One expects an equivalence of DG categories:

$$\mathbb{L}_G^{(c)} : \mathcal{D}\text{-Mod}^c(\mathrm{Bun}_G) \xrightarrow{\sim} \mathcal{D}\text{-Mod}^{-\frac{1}{rc}}(\mathrm{Bun}_{\check{G}}) \tag{1.2}$$

The equivalence $\mathbb{L}_G^{(c)}$ should vary "continuously" in c, and "degenerate" to (1.1) as c tends to zero.[1] For a survey on the conjecture (1.2), see [23].

[1] Indeed, the left-hand-side of (1.1) should more naturally be the DG category of $\mathcal{L}_{G,\det}^{-\frac{1}{2}}$-twisted $\mathcal{D}$-modules, otherwise known as $\mathcal{D}$-modules at the *critical level*. The two DG categories are equivalent by the existence of the Pfaffian.

1.2.3. We remark that the conjecture (1.2) is made prior to the formulation of (1.1). It must also be corrected by a renormalization procedure, analogous to the replacement of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$ by $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$.

The renormalized DG categories $\mathcal{D}\text{-Mod}^{c}_{\mathrm{ren}}(\mathrm{Bun}_G)$ have different behaviors depending on the rationality and positivity of c, and we do not know how to fit them in a family.

1.2.4. In the present article, we fulfill a more modest goal: we make precise the degeneration of $\mathcal{D}\text{-Mod}^{c}(\mathrm{Bun}_G)$ to $\mathrm{QCoh}(\mathrm{LocSys}_G)$ by constructing a quasi-coherent sheaf of DG categories over a space of quantum parameters, but we do not take into account the renormalization mentioned above.

1.3 What's in this article?

1.3.1. Let us admit right away that when G is simple, the space of quantum parameters is just a copy of $\mathbb{P}^1$, and when the genus of the curve X is at least 2, the stack LocSys_G is classical. In this case, the $\mathbb{P}^1$-family of DG categories has already been constructed by Stoyanovsky [26], making use of the line bundle $\mathcal{L}_{G,\det}$.

1.3.2. In the present article, we construct the space of quantum parameters and an analogous degeneration for a reductive group G. However, our construction proceeds along totally different lines from [26]. This departure in point of view is motivated by the following considerations:

(a) In the study of the Langlands correspondence for G, an instrumental role is played by its Levi subgroups M. The relationship between G and M is codified by the constant term functors (and their adjoints, the Eisenstein series functors). Even for simple G, the constant term functor carries $\mathcal{D}\text{-Mod}^{c}(\mathrm{Bun}_G)$ to a twisted category of $\mathcal{D}$-modules on Bun_M which does *not* arise from the determinant line bundle (see [13, Sects. 3.3–3.4] for example).
It is desirable, therefore, to include these additional twists into the space of quantum parameters for M. Our construction achieves this in a natural way. For a reductive group G, our space of quantum parameters consists of a pair $(\mathfrak{g}^{\kappa}, E)$, where $\mathfrak{g}^{\kappa}$ is a generalized symmetric bilinear form on the Lie algebra $\mathfrak{g}$ of G, and E is an *additional parameter* which depends on the center of G as well as the curve X.
(b) The DG nature of LocSys_G requires us to consider generalizations of rings of twisted differential operators (TDOs) whose underlying $\mathcal{O}$-modules are chain complexes. It is *a priori* unclear how to even define such gadgets, because chain complexes interact poorly with explicit formulas. To circumvent this, we make a geometric construction using the recent theory of derived formal moduli problems developed by Lurie, Gaitsgory, and Rozenblyum.
More precisely, [17] introduces a theory of *twistings* which gives the derived generalization of a ring of TDOs. (We call the latter *classical twistings*). We introduce the notion of a *quasi-twisting* which incorporates commutative degenerations of twistings.

1.3.3. Driven by these considerations, we give a construction which completely dispenses of the line bundle $\mathcal{L}_{G,\det}$ and contains more information as soon as the center of G is nontrivial. The key steps in this construction are summarized by the following chart:[2]

$$\left\{\begin{array}{c}\text{quantum}\\ \text{parameter } (\mathfrak{g}^\kappa, E)\end{array}\right\} \rightsquigarrow \left\{\begin{array}{c}\text{Lie-}*\text{ algebra}\\ \widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)} \text{ over } X\end{array}\right\}$$
$$\rightsquigarrow \left\{\begin{array}{c}\text{classical quasi-twisting}\\ \tilde{\mathcal{T}}_G^{(\kappa,E)} \text{ over } \mathrm{Bun}_{G,\infty x}\end{array}\right\} \rightsquigarrow \left\{\begin{array}{c}\text{quasi-twisting}\\ \mathcal{T}_G^{(\kappa,E)} \text{ over } \mathrm{Bun}_G\end{array}\right\}.$$

The family of DG categories ultimately arises as the module category of $\mathcal{T}_G^{(\kappa,E)}$, as we vary the quantum parameter $(\mathfrak{g}^\kappa, E)$. The family of quasi-twistings $\mathcal{T}_G^{(\kappa,E)}$ is the central object defined and studied in this article.

1.4 Organization of this article

1.4.1. We start in Sect. 2 with the definition of Par_G, the space of quantum parameters. It is a fiber bundle over a compactification of $\mathrm{Sym}^2(\mathfrak{g}^*)^G$, with fibers being vector stacks describing the "additional parameters."

The aforementioned compactification of $\mathrm{Sym}^2(\mathfrak{g}^*)^G$ is simply the space of G-invariant Lagrangian subspaces of $\mathfrak{g} \oplus \mathfrak{g}^*$, where a G-invariant symmetric bilinear form embeds as its graph. The level "at ∞" is understood as the Lagrangian subspace $\mathfrak{g}^\infty := 0 \oplus \mathfrak{g}^*$.

1.4.2 The main idea

Let us take a k-point in Par_G, which is a Lagrangian subspace $\mathfrak{g}^\kappa \subset \mathfrak{g} \oplus \mathfrak{g}^*$ together with an additional parameter E (see Sect. 2.4.1 where it is defined). Using the theory of Lie-$*$ algebras developed in [3], we construct a central extension

$$0 \to \mathcal{O}_{\mathrm{Bun}_{G,\infty x}} \to \widehat{\mathcal{L}}^{(\kappa,E)} \to \mathcal{L}^\kappa \to 0 \tag{1.3}$$

of Lie algebroids over the scheme $\mathrm{Bun}_{G,\infty x}$ parametrizing G-bundles trivialized over the formal neighborhood D_x of a fixed closed point $x \in X$. We refer to central extensions of Lie algebroids as *classical quasi-twistings*.

For $\mathfrak{g}^\kappa$ arising from a symmetric bilinear form, the reduced universal envelope of (1.3):

$$\mathrm{U}_{\mathrm{red}}(\widehat{\mathcal{L}}^{(\kappa,E)}) := \mathrm{U}(\widehat{\mathcal{L}}^{(\kappa,E)})/(1-\mathbf{1})$$

defines a TDO over $\mathrm{Bun}_{G,\infty x}$. At $(\mathfrak{g}^\kappa, E) = (\mathfrak{g}^\infty, 0)$, the algebra $\mathrm{U}_{\mathrm{red}}(\widehat{\mathcal{L}}^{(\infty,0)})$ becomes commutative, and identifies with the ring of functions on the ind-scheme

[2] For objects that depend on $\mathfrak{g}^\kappa$ (resp. $(\mathfrak{g}^\kappa, E)$), we only retain the character κ (resp. (κ, E)) in the notation.

$\mathrm{LocSys}_{G,\infty x}(X - \{x\})$ parametrizing a point $(\mathcal{P}_T, \eta) \in \mathrm{Bun}_{G,\infty x}$ together with a connection ∇ over $\mathcal{P}_T|_{X-\{x\}}$.

To obtain a central extension of Lie algebroids over Bun_G, we "descend" (1.3) along the torsor $\mathrm{Bun}_{G,\infty x} \to \mathrm{Bun}_G$.

1.4.3 The main challenge

There is, however, a caveat in what it means to "descend" the classical quasi-twisting (1.3). We need a procedure that simultaneously does the following:

(a) For $\mathfrak{g}^\kappa$ arising from a symmetric bilinear form, it performs the strong quotient of a ring of TDOs, in the sense of [2];
(b) For $\mathfrak{g}^\kappa = \mathfrak{g}^\infty$, it transforms (the ring of functions over) $\mathrm{LocSys}_{G,\infty x}(X - \{x\})$ into the DG stack LocSys_G, a procedure usually understood as symplectic reduction.

It turns out that one needs to form what we call the *quotient* of a classical quasi-twisting. In general (and in the way we will apply it), this notion belongs to the DG world, i.e., the quotient of a classical quasi-twisting may cease to be classical.

1.4.4. A (non-classical) *quasi-twisting* over a finite type scheme Y is defined as a $\widehat{\mathbb{G}}_m$-gerbe in the ∞-category of formal moduli problems under Y. They make up the geometric theory of central extensions of Lie algebroids over Y, and are studied in Sect. 3. The theory of quasi-twistings is made possible by the machinery of formal groupoids and formal moduli problems, as developed in [18].

The quotient of quasi-twistings fits into the general paradigm of taking the quotient of an inf-scheme by a group inf-scheme. The latter procedure is rather elaborate, as it mixes prestack quotient with formal groupoid quotient. This is the content of Sect. 4.

1.4.5. Finally, we need to deal with the technical annoyance that the theory of [18] is built for prestacks locally (almost) of finite type, whereas $\mathrm{Bun}_{G,\infty x}$ is of infinite type. Hence the actual quotient process has to be performed in two steps, one classical and one geometric, along the torsors:

$$\mathrm{Bun}_{G,\infty x}^{(\leq\theta)} \to \mathrm{Bun}_{G,nx}^{(\leq\theta)} \to \mathrm{Bun}_G^{(\leq\theta)},$$

where $\mathrm{Bun}_G^{(\leq\theta)}$ is a Harder-Narasimhan truncation of Bun_G and n is sufficiently large so that $\mathrm{Bun}_{G,nx}^{(\leq\theta)}$ is a scheme (of finite type). For this reason, we need to prove a number of results communicating between the classical and derived worlds in Sect. 3 and Sect. 4. It is the author's hope that an extension of [18] to ∞-dimensional algebraic geometry will render this trick obsolete.

1.4.6 The main results

In Sect. 5, we perform the main construction of the quasi-twisting $\mathcal{T}_G^{(\kappa,E)}$ over Bun_G and check that it gives rise to DG categories of twisted $\mathcal{D}$-modules when $\mathfrak{g}^\kappa$ is the graph of a bilinear form and $E = 0$.

Finally, in Sect. 6, we show that the DG category of modules over $\mathcal{T}_G^{(\infty,0)}$ is equivalent to $\mathrm{QCoh}(\mathrm{LocSys}_G)$. We end the article with remarks on the "meaning" of certain additional parameters at level ∞.

1.5 Quantum versus metaplectic parameters

1.5.1. There is another approach of deforming the DG category $\mathcal{D}\text{-Mod}(\mathrm{Bun}_G)$[3] under the name "metaplectic geometric Langlands program" (see [16], for example). We briefly explain the relation between metaplectic and quantum parameters.

For simplicity, let us focus on the points $(\mathfrak{g}^\kappa, E)$ of Par_G where $\mathfrak{g}^\kappa$ arises from a symmetric bilinear form. Such quantum parameters form an open substack isomorphic to $\mathrm{Sym}^2(\mathfrak{g}^*)^G \times \mathbf{Ext}^1(\mathfrak{z}_G \otimes \mathcal{O}_X, \omega_X)$, and the quasi-twistings on Bun_G they produce are in fact twistings.

1.5.2. Metaplectic parameters give rise to *gerbes*, as opposed to twistings, on Bun_G. Having chosen $\mathcal{D}$-modules as our sheaf-theoretic context, a *gerbe* on a prestack $\mathcal{Y}$ refers to a map from $\mathcal{Y}_{\mathrm{dR}}$ to $\mathrm{B}^2\,\mathbb{G}_m$. Note that a gerbe on Bun_G is sufficient to form the DG category of twisted $\mathcal{D}$-modules, but the additional data included in a twisting equip this DG category with a forgetful functor to $\mathrm{QCoh}(\mathrm{Bun}_G)$.

Unlike the metapletic geometric Langlands program, which has incarnations in various sheaf-theoretic contexts, the quantum geometric Langlands program is limited to the case of $\mathcal{D}$-modules. (However, it seems that the restriction $\mathrm{char}(k) = 0$ is not necessary, in light of the recent work of Travkin [28]).

1.5.3. By analogy with the ℓ-adic context, gerbes are supposed to be "topological" gadgets. However, the existence of the exponential local system on $\mathbb{A}^1$ shows that the above definition of a gerbe is too naïve. In order to retain only topological information, we ought to adjust the definition of a gerbe slightly, as a (2-)torsor over the groupoid of *regular singular* local systems. However, we will ignore this subtlety for now.

1.5.4. Let Gr_G denote the affine Grassmannian associated to G, regarded as a factorization prestack over the Ran space of X. Conjecturally, the spaces of quantum, respectively metaplectic, parameters have the following intrinsic meanings: they are the moduli spaces of factorization twistings, respectively gerbes, on Gr_G. The corresponding objects on Bun_G arise from their descent along the canonical map $\mathrm{Gr}_G \to \mathrm{Bun}_G$.

Furthermore, there is a fiber sequence of Picard groupoids, relating factorization line bundles, twistings, and gerbes on the affine Grassmannian:

$$\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_G) \to \mathbf{Tw}^{\mathrm{fact}}(\mathrm{Gr}_G) \to \mathbf{Ge}^{\mathrm{fact}}(\mathrm{Gr}_G). \tag{1.4}$$

The three items of this fiber sequence stem from apparently different sources:

1.5.5. Since the first preprint of the present paper appeared in 2017, several new developments have contributed to a better understanding of these parameters. Let us briefly

[3] Or in the context of curves over $\overline{\mathbb{F}}_p$, the category of ℓ-adic sheaves on Bun_G.

Algebro-geometric	Differential-geometric	Topological
$\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_G)$	$\mathbf{Tw}^{\mathrm{fact}}(\mathrm{Gr}_G)$	$\mathbf{Ge}^{\mathrm{fact}}(\mathrm{Gr}_G)$
K-theoretic parameters	Quantum parameters	Metaplectic parameters

report on them. The first one is a precise relationship between the K-theoretic parameters, first studied by Brylinski–Deligne [6], and factorization line bundles [15][27]. The second is a precise formulation of "topological" gerbes in the de Rham context and the classification of factorization de Rham gerbes on Gr_G [29]. In the ℓ-adic context, the analogous classification theorem now has two proofs (see [29] and the new version of [16]).

Finally, it is pointed out by an anonymous referee that the space of quantum parameters defined in this paper can be further enlarged to include the "semi-classical" degeneration of the geometric Langlands theory (from $\mathcal{D}$-$\mathrm{Mod}^c(\mathrm{Bun}_G)$, as well as $\mathrm{QCoh}(\mathrm{LocSys}_G)$ to the DG category of quasi-coherent sheaves on the cotangent stack $\mathrm{T}^*\,\mathrm{Bun}_G$). The semi-classical limit has featured in the works of Donagi–Pantev [9] (over $\mathbb{C}$) and Bezrukavnikov–Braverman [5] (in characteristic p).

Notations

Throughout this article, we work over an algebraically closed ground field k of characteristic zero. We write X for a smooth, connected, projective curve and G a reductive group over k (where "reductive" is meant to imply connected). The Lie algebra of G is denoted by $\mathfrak{g}$. Notations particular to each section will be explained as they appear.

2 The space of quantum parameters

In this section, we define the smooth algebraic stack Par_G of quantum parameters for the geometric Langlands theory. We will define a natural isomorphism $\mathrm{Par}_G \xrightarrow{\sim} \mathrm{Par}_{\check{G}}$, and explain how Par_G behaves when we change G into the Levi quotient M of a parabolic of G.

2.1 The base scheme of Par_G

2.1.1. The space of quantum parameters Par_G will be an algebraic vector stack over a smooth projective scheme. We begin by defining the base scheme of Par_G, which will be a compactification of the vector scheme of G-invariant symmetric bilinear on $\mathfrak{g}$. Its existence is based on the following fact.

Lemma 2.1 *Let (V, ω) be a symplectic vector space. The presheaf which sends an affine scheme S to the set of Lagrangian subbundles of $V \otimes \mathcal{O}_S$ is representable by a connected, smooth, projective scheme.*

Proof Let $n := \dim(V)/2$ which is an integer. The presheaf of Lagrangian subbundles of $V \otimes \mathcal{O}_S$ is a closed subfunctor of that of n-dimensional subbundles of $V \otimes \mathcal{O}_S$. The latter presheaf is represented by the Grassmannian $\mathrm{Gr}(n, V)$. Hence the former is represented by a projective scheme, to be denoted $\mathrm{Gr}_{\mathrm{Lag}}(V)$. The smoothness of $\mathrm{Gr}_{\mathrm{Lag}}(V)$ follows from a standard calculation of its cotangent complex (details omitted).

To show that $\mathrm{Gr}_{\mathrm{Lag}}(V)$ is connected, we observe that the symplectic group $\mathrm{Sp}(V)$ acts on $\mathrm{Gr}_{\mathrm{Lag}}(V)$. For a fixed k-point L of $\mathrm{Gr}_{\mathrm{Lag}}(V)$, the map $\mathrm{Sp}(V) \to \mathrm{Gr}_{\mathrm{Lag}}(V)$ induced from acting on L is surjective on k-points. Since $\mathrm{Sp}(V)$ is connected, so is $\mathrm{Gr}_{\mathrm{Lag}}(V)$. □

2.1.2. Consider the symplectic form on $\mathfrak{g} \oplus \mathfrak{g}^*$ defined by the pairing:

$$\langle \xi \oplus \varphi, \xi' \oplus \varphi' \rangle := \varphi(\xi') - \varphi'(\xi). \tag{2.1}$$

Let $\mathrm{Gr}_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*)$ denote the scheme parametrizing Lagrangian subspaces of $\mathfrak{g} \oplus \mathfrak{g}^*$. (It represents the presheaf in Lemma 2.1). The reductive group G acts on $\mathfrak{g} \oplus \mathfrak{g}^*$ via the direct sum of the adjoint and coadjoint actions. This action preserves the symplectic form (2.1). Hence, we obtain a G-action on $\mathrm{Gr}_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*)$. Thanks to the hypothesis $\mathrm{char}(k) = 0$, the group G is linearly reductive. Hence the G-fixed point scheme:

$$\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*) \subset \mathrm{Gr}_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*)$$

remains smooth, by the classical theorem of Iversen [19, Proposition 1.3]. We will denote an S-point of $\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*)$ by $\mathfrak{g}^\kappa$, regarded as a Lagrangian subbundle of $(\mathfrak{g} \oplus \mathfrak{g}^*) \otimes \mathcal{O}_S$ stable under the G-action.

2.1.3. Let $\mathrm{Sym}^2(\mathfrak{g}^*)^G$ denote the vector space of G-invariant symmetric bilinear forms on $\mathfrak{g}$, regarded as a vector scheme. There is a morphism of schemes:

$$\mathrm{Sym}^2(\mathfrak{g}^*)^G \to \mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*) \tag{2.2}$$

sending a form κ, viewed as a linear map $\kappa : \mathfrak{g} \to \mathfrak{g}^*$, to its graph $\mathfrak{g}^\kappa$. The morphism (2.2) is an open immersion, whose image consists of those subbundles $\mathfrak{g}^\kappa \subset (\mathfrak{g} \oplus \mathfrak{g}^*) \otimes \mathcal{O}_S$ for which the projection to $\mathfrak{g} \otimes \mathcal{O}_S$ is an isomorphism.

2.1.4. We will use the following notations for special points of $\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*)$:

(a) $\mathfrak{g}^\infty$ denotes the k-point $\mathfrak{g}^*$ of $\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*)$;
(b) $\mathfrak{g}^{\mathrm{crit}}$ is the graph of the *critical* form $\mathrm{crit} := -\frac{1}{2}\,\mathrm{Kil}$, where Kil is the Killing form of $\mathfrak{g}$.
(c) for every S-point $\mathfrak{g}^\kappa$ of $\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*)$, the notation $\mathfrak{g}^{\kappa - \mathrm{crit}}$ denotes the Lagrangian subbundle of $(\mathfrak{g} \oplus \mathfrak{g}^*) \otimes \mathcal{O}_S$ defined by the property:

$$\xi \oplus \varphi \in \mathfrak{g}^\kappa \iff \xi \oplus (\varphi - \mathrm{crit}(\xi)) \in \mathfrak{g}^{\kappa - \mathrm{crit}}.$$

Remark 2.2 Note that if $\kappa \in \mathrm{Sym}^2(\mathfrak{g}^*)^G$, then $\mathfrak{g}^{\kappa-\mathrm{crit}}$ is the graph of $\kappa - \mathrm{crit}$, so the above notation is unambiguous; we also have $\mathfrak{g}^{\infty-\mathrm{crit}} = \mathfrak{g}^{\infty}$.

Remark 2.3 More generally, one may replace $\mathfrak{g}^{\kappa-\mathrm{crit}}$ in the above construction by $\mathfrak{g}^{\kappa+\kappa_0}$ for any $\kappa_0 \in \mathrm{Sym}^2(\mathfrak{g}^*)^G$. This construction defines an action of $\mathrm{Sym}^2(\mathfrak{g}^*)^G$ on $\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*)$ that extends the addition on $\mathrm{Sym}^2(\mathfrak{g}^*)^G$.

2.2 Decomposition into simple factors

2.2.1. Let $\mathfrak{g} = \mathfrak{z} \oplus \sum_i \mathfrak{g}_i$ be the decomposition of $\mathfrak{g}$ into its center $\mathfrak{z}$ and simple factors $\mathfrak{g}_i$. In this subsection, we study how $\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*)$ interacts with this direct sum decomposition. Combined with some knowledge of this space for a simple group, we will be able to describe $\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*)$ much more explicitly. First, we begin with a lemma on the level of k-points.

Lemma 2.4 *Any Lagrangian, G-invariant subspace $L \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}^*$ takes the form $L = L_{\mathfrak{z}} \oplus \sum_i L_i$ where:*

(a) $L_{\mathfrak{z}}$ is a Lagrangian subspace of $\mathfrak{z} \oplus \mathfrak{z}^*$;
(b) each L_i is a Lagrangian, G-invariant subspace of $\mathfrak{g}_i \oplus \mathfrak{g}_i^*$.

Proof The decomposition of $\mathfrak{g}$ induces a decomposition $\mathfrak{g} \oplus \mathfrak{g}^* = (\mathfrak{z} \oplus \mathfrak{z}^*) \oplus \sum_i (\mathfrak{g}_i \oplus \mathfrak{g}_i^*)$ where the summands are mutually orthogonal with respect to the symplectic form (2.1). We may also decompose $L = L_{\mathfrak{z}} \oplus \sum_j L_j$, where $L_{\mathfrak{z}}$ is the G-fixed subspace and each L_j is irreducible. Obviously, the embedding $L \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}^*$ sends $L_{\mathfrak{z}}$ into $\mathfrak{z} \oplus \mathfrak{z}^*$ as an isotropic subspace.

We *claim* that each embedding $L_j \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}^*$ factors through $\mathfrak{g}_i \oplus \mathfrak{g}_i^*$ for a unique i. In other words, the composition $L_j \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}^* \twoheadrightarrow \mathfrak{g}_i \oplus \mathfrak{g}_i^*$ must vanish for all but one i. Suppose, to the contrary, we have $i \neq i'$ such that both

$$L_j \to \mathfrak{g}_i \oplus \mathfrak{g}_i^*, \quad \text{and} \quad L_j \to \mathfrak{g}_{i'} \oplus \mathfrak{g}_{i'}^*$$

are nonzero. Without loss of generality, we may assume that the projections onto the first factors $L_j \to \mathfrak{g}_i$, $L_j \to \mathfrak{g}_{i'}$ are nonzero. Hence we have

(a) $L_j \cong \mathfrak{g}_i \cong \mathfrak{g}_{i'}$ as G-representations; and
(b) the image of L_j under the projection $\mathfrak{g} \oplus \mathfrak{g}^* \twoheadrightarrow \mathfrak{g}_i \oplus \mathfrak{g}_{i'}$ is a G-invariant subspace with nonzero projection onto both factors.

The second statement implies that this image is the entire space $\mathfrak{g}_i \oplus \mathfrak{g}_{i'}$, contradicting the equality $\dim(L_j) = \dim(\mathfrak{g}_i)$ from the first statement. This prove the claim.

Now, suppose $j \neq j'$ and both embeddings $L_j, L_{j'} \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}^*$ factor through the same $\mathfrak{g}_i \oplus \mathfrak{g}_i^*$. This is obviously impossible since $L_j \oplus L_{j'} \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}^*$ would factor through an isomorphism $L_j \oplus L_{j'} \xrightarrow{\sim} \mathfrak{g}_i \oplus \mathfrak{g}_i^*$, so it is *not* isotropic. We conclude that there is a bijection between the sets $\{L_j\}$ and $\{\mathfrak{g}_i \oplus \mathfrak{g}_i^*\}$ such that each $L_j \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}^*$ factors through the corresponding item $\mathfrak{g}_i \oplus \mathfrak{g}_i^*$.

Finally, since each L_j is an isotropic subspace of $\mathfrak{g}_i \oplus \mathfrak{g}_i^*$, we have:

$$\dim(\mathfrak{g}) = \dim(L_\mathfrak{z}) + \sum_j \dim(L_j) \leq \dim(\mathfrak{z}) + \sum_i \dim(\mathfrak{g}_i) = \dim(\mathfrak{g}).$$

Hence the equality is achieved, and each L_j (resp. $L_\mathfrak{z}$) is a Lagrangian subspace of $\mathfrak{g}_i \oplus \mathfrak{g}_i^*$ (resp. $\mathfrak{z} \oplus \mathfrak{z}^*$). □

Corollary 2.5 *Let L be a Lagrangian, G-invariant subspace of $\mathfrak{g} \oplus \mathfrak{g}^*$. Then there is a (non-canonical) isomorphism $L \xrightarrow{\sim} \mathfrak{g}$ of G-representations.* □

Note that we have an obvious morphism:

$$\mathrm{Gr}_{\mathrm{Lag}}(\mathfrak{z} \oplus \mathfrak{z}^*) \times \prod_i \mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g}_i \oplus \mathfrak{g}_i^*) \to \mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*) \tag{2.3}$$

sending a series of vector bundles $\mathfrak{z}^\kappa$, $\{\mathfrak{g}_i^\kappa\}$ over S to their direct sum $\mathfrak{z}^\kappa \oplus \sum_i \mathfrak{g}_i^\kappa$, which is a subbundle of $(\mathfrak{g} \oplus \mathfrak{g}^*) \otimes \mathcal{O}_S$.

Corollary 2.6 *The morphism (2.3) is an isomorphism.*

Proof Indeed, (2.3) is a proper morphism between smooth schemes. Lemma 2.4 shows that it is bijective on k-points, so in particular quasi-finite, and therefore finite (by properness). A finite morphism of degree 1 between smooth schemes is an isomorphism. □

2.2.2. To proceed furthermore, let us note that any G-invariant symmetric bilinear form κ_i on $\mathfrak{g}_i$ defines an isomorphism $\mathbb{A}^1 \xrightarrow{\sim} \mathrm{Sym}^2(\mathfrak{g}_i^*)^G$, sending c to the form $c\kappa_i$. This isomorphism extends to a map:

$$\mathbb{P}^1 \to \mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g}_i \oplus \mathfrak{g}_i^*), \quad c \rightsquigarrow \mathfrak{g}^{c\kappa_i}. \tag{2.4}$$

In fact, an argument analogous to the proof of Corollary 2.6 shows that (2.4) is an isomorphism. Combining with the isomorphism (2.3), we see that $\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*)$ is *non-canonically* isomorphic to the product of a Lagrangian Grassmannian with finitely many copies of $\mathbb{P}^1$, one for each simple factor of $\mathfrak{g}$.

2.3 Reduction to $Z(G)$

2.3.1. We will now work towards the definition of Par_G, which is a vector stack over $\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*)$. The fibers of this vector stack are the so-called additional parameters. They will only come into play when the center $Z(G)$ is nontrivial. In this subsection, we focus on the central component of $\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*)$ with respect to the product decomposition (2.3).

2.3.2. Consider the projection map (whose existence owes to Corollary 2.6):

$$\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*) \to \mathrm{Gr}_{\mathrm{Lag}}(\mathfrak{z} \oplus \mathfrak{z}^*) \tag{2.5}$$

Note that $\mathfrak{z}$ is identified with the subspace of G-invariants of $\mathfrak{g}$. Although $\mathfrak{z}^*$ is more naturally the space of G-coinvariants of $\mathfrak{g}^*$, we will identify it with the invariants $(\mathfrak{g}^*)^G$ via the isomorphism $(\mathfrak{g}^*)^G \hookrightarrow \mathfrak{g}^* \twoheadrightarrow \mathfrak{z}^*$.

More intrinsically, the morphism (2.5) is defined on S-points by:

$$\mathfrak{g}^\kappa \rightsquigarrow (\mathfrak{g}^\kappa)^G := \mathfrak{g}^\kappa \cap ((\mathfrak{z} \oplus \mathfrak{z}^*) \otimes \mathcal{O}_S).$$

where $(\mathfrak{z} \oplus \mathfrak{z}^*) \otimes \mathcal{O}_S$ is regarded as a submodule of $(\mathfrak{g} \oplus \mathfrak{g}^*) \otimes \mathcal{O}_S$. In particular, $(\mathfrak{g}^\kappa)^G$ may be viewed as a submodule of $\mathfrak{g}^\kappa$.

Remark 2.7 We refer to $(\mathfrak{g}^\kappa)^G$ as the *G-invariants* of $\mathfrak{g}^\kappa$. The same terminology is used in the sequel when we replace G by a different group H and $\mathfrak{g}^\kappa$ by an H-invariant subspace of $V \oplus V^*$, where V is any H-representation for which the composition $(V^*)^H \hookrightarrow V^* \twoheadrightarrow (V^H)^*$ is an isomorphism.

Remark 2.8 Since crit vanishes on $\mathfrak{z}$, the submodules $(\mathfrak{g}^{\kappa-\mathrm{crit}})^G$, $(\mathfrak{g}^\kappa)^G \subset (\mathfrak{z} \oplus \mathfrak{z}^*) \otimes \mathcal{O}_S$ are equal for any $\mathfrak{g}^\kappa$.

2.3.3. Since the embedding $\mathfrak{z} \hookrightarrow \mathfrak{g}$ canonically splits with kernel $\mathfrak{g}_{\mathrm{s.s.}} := [\mathfrak{g}, \mathfrak{g}]$, there is a surjection $(\mathfrak{g} \oplus \mathfrak{g}^*) \otimes \mathcal{O}_S \twoheadrightarrow (\mathfrak{z} \oplus \mathfrak{z}^*) \otimes \mathcal{O}_S$. Under this surjection, the image of $\mathfrak{g}^\kappa$ is identified with $(\mathfrak{g}^\kappa)^G$, and the composition $(\mathfrak{g}^\kappa)^G \hookrightarrow \mathfrak{g}^\kappa \twoheadrightarrow (\mathfrak{g}^\kappa)^G$ is the identity. In other words,

Lemma 2.9 *The morphism $(\mathfrak{g}^\kappa)^G \hookrightarrow \mathfrak{g}^\kappa$ canonically splits.* □

We denote the complement of $(\mathfrak{g}^\kappa)^G$ in $\mathfrak{g}^\kappa$ by $\mathfrak{g}^\kappa_{\mathrm{s.s.}}$. The decomposition:

$$\mathfrak{g}^\kappa \cong (\mathfrak{g}^\kappa)^G \oplus \mathfrak{g}^\kappa_{\mathrm{s.s.}}$$

mimics the decomposition of $\mathfrak{g}$ into its center and its semisimple part.

2.4 Definition of Par_G

2.4.1. We are now ready to define the stack Par_G of quantum parameters. For an affine shceme S, the groupoid $\mathrm{Maps}(S, \mathrm{Par}_G)$ consists of pairs $(\mathfrak{g}^\kappa, E)$, where $\mathfrak{g}^\kappa$ is an S-point of $\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*)$, and E is an extension of $\mathcal{O}_\mathcal{X}$-modules:

$$0 \to \omega_{\mathcal{X}/S} \to E \to (\mathfrak{g}^\kappa)^G \boxtimes \mathcal{O}_X \to 0. \tag{2.6}$$

Here, $\mathcal{X} := S \times X$, and $\omega_{\mathcal{X}/S} \cong \mathcal{O}_S \boxtimes \omega_X$ is the relative dualizing sheaf.

In other words, Par_G is a fiber bundle over $\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*)$, whose fiber at a k-point $\mathfrak{g}^\kappa$ is the vector stack $\mathbf{Ext}((\mathfrak{g}^\kappa)^G \boxtimes \mathcal{O}_X, \omega_X)$ of extensions over X. We think of $\mathfrak{g}^\kappa$ as a generalized symmetric bilinear form on $\mathfrak{g}$ and E as an *additional* parameter.

Remark 2.10 The substack of Par_G corresponding to the points $(\mathfrak{g}^\kappa, E)$ where $\mathfrak{g}^\kappa$ arises from a bilinear form conjecturally parametrizes *factorization twistings* on the affine Grassmannian Gr_G, subject to a certain regularity condition (see Sect. 1.5). Hence, one may view Par_G as a (partial) compactification of the stack of factorization twistings. We hope to address this conjecture in a forthcoming work.

2.5 Langlands duality of Par_G

2.5.1. We now fix a maximal torus $T \hookrightarrow G$. Let $\check{G}$ denote the Langlands dual group of (G, T). Namely, it is a pinned reductive group over k whose root datum is dual to that of (G, T). In particular, $\check{G}$ comes with a maximal torus $\check{T} \subset \check{G}$ dual to T.

2.5.2. Let $W := N_G(T)/T$ denote the Weyl group of (G, T). It acts on $\mathfrak{t} \oplus \mathfrak{t}^*$ in the standard way. There is a symplectic isomorphism:

$$\mathfrak{t} \oplus \mathfrak{t}^* \xrightarrow{\sim} \check{\mathfrak{t}} \oplus \check{\mathfrak{t}}^*, \quad \xi \oplus \varphi \rightsquigarrow \varphi \oplus (-\xi) \tag{2.7}$$

defined using the canonical identifications $\mathfrak{t}^* \xrightarrow{\sim} \check{\mathfrak{t}}$ and $\mathfrak{t} \xrightarrow{\sim} \check{\mathfrak{t}}^*$. Furthermore, (2.7) intertwines the W and $\check{W}$ actions (again, under the canonical identification $W \xrightarrow{\sim} \check{W}$).

Remark 2.11 The sign (2.7) is needed to match up the symplectic forms. On the other hand, the conjectural quantum Langlands correspondence is an equivalence between a positively twisted category of $\mathcal{D}$-modules on Bun_G and a negatively twisted category of $\mathcal{D}$-modules on $\mathrm{Bun}_{\check{G}}$. This change of signs is reflected in the identification (2.7).

2.5.3. Let $\mathrm{Gr}^W_{\mathrm{Lag}}(\mathfrak{t} \oplus \mathfrak{t}^*)$ denote the scheme parametrizing W-invariant, Lagrangian subspaces of $\mathfrak{t} \oplus \mathfrak{t}^*$. It is connected, smooth, and projective, thanks to Lemma 2.1 and the fact that W is a finite group. The isomorphism (2.7) induces an isomorphism:

$$\mathrm{Gr}^W_{\mathrm{Lag}}(\mathfrak{t} \oplus \mathfrak{t}^*) \xrightarrow{\sim} \mathrm{Gr}^{\check{W}}_{\mathrm{Lag}}(\check{\mathfrak{t}} \oplus \check{\mathfrak{t}}^*). \tag{2.8}$$

We denote the image of $\mathfrak{t}^\kappa$ under (2.8) by $\check{\mathfrak{t}}^{\check{\kappa}}$, and view it as the dual of the generalized bilinear form $\mathfrak{t}^\kappa$. Note that $\mathrm{Sym}^2(\mathfrak{t}^*)^W$ is not preserved under the duality (2.8).

2.5.4. We define a morphism (the "naïve reduction")

$$\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*) \to \mathrm{Gr}^W_{\mathrm{Lag}}(\mathfrak{t} \oplus \mathfrak{t}^*) \tag{2.9}$$

by sending an S-point $\mathfrak{g}^\kappa$ to $(\mathfrak{g}^\kappa)^T$, the T-invariants of $\mathfrak{g}^\kappa$. An argument similar to the one in §2.3.2 shows that we have a well-defined map $\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*) \to \mathrm{Gr}_{\mathrm{Lag}}(\mathfrak{t} \oplus \mathfrak{t}^*)$; it is clear that the image lies in the W-fixed locus.

Lemma 2.12 *The morphism (2.9) is an isomorphism.*

Proof Indeed, a decomposition of $\mathfrak{g} = \mathfrak{z} \oplus \sum_i \mathfrak{g}_i$ into simple factors induces a decomposition $\mathfrak{t} = \mathfrak{z} \oplus \sum_i \mathfrak{t}_i$, where each $\mathfrak{t}_i$ is the maximal torus of the factor $\mathfrak{g}_i$. Note that $\mathfrak{t}_i$ is irreducible as a W-representation. An analogue of Corollary 2.6 asserts an isomorphism $\mathrm{Gr}^W_{\mathrm{Lag}}(\mathfrak{t} \oplus \mathfrak{t}^*) \xrightarrow{\sim} \mathrm{Gr}_{\mathrm{Lag}}(\mathfrak{z} \oplus \mathfrak{z}^*) \times \prod_i \mathrm{Gr}^W_{\mathrm{Lag}}(\mathfrak{t}_i \oplus \mathfrak{t}_i^*)$, making the following diagram commute:

$$\begin{array}{ccc} \mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*) & \xrightarrow{\quad (2.9) \quad} & \mathrm{Gr}^W_{\mathrm{Lag}}(\mathfrak{t} \oplus \mathfrak{t}^*) \\ \downarrow\wr & & \downarrow\wr \\ \mathrm{Gr}_{\mathrm{Lag}}(\mathfrak{z} \oplus \mathfrak{z}^*) \times \prod_i \mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g}_i \oplus \mathfrak{g}_i^*) & \longrightarrow & \mathrm{Gr}_{\mathrm{Lag}}(\mathfrak{z} \oplus \mathfrak{z}^*) \times \prod_i \mathrm{Gr}^W_{\mathrm{Lag}}(\mathfrak{t}_i \oplus \mathfrak{t}_i^*). \end{array}$$

Note that the bottom arrow is an isomorphism since the choice of a G-invariant, symmetric bilinear form on $\mathfrak{g}_i$ (hence a W-invariant form on $\mathfrak{t}_i$) identifies both $\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g}_i \oplus \mathfrak{g}_i^*)$ and $\mathrm{Gr}^W_{\mathrm{Lag}}(\mathfrak{t}_i \oplus \mathfrak{t}_i^*)$ with $\mathbb{P}^1$ (see Sect. 2.2.2). □

Remark 2.13 Using T, we may also rewrite (2.5) as the two-step procedure of first taking T-invariants and then taking W-invariants:

$$(\mathfrak{g}^\kappa)^G \xrightarrow{\sim} ((\mathfrak{g}^\kappa)^T)^W.$$

This isomorphism again follows from the description of fibers of $\mathfrak{g}^\kappa$ in Lemma 2.4.

2.5.5. We will consider a slight variant of the isomorphism (2.9) which takes into account the critical shift (the "critically-shifted reduction"):

$$\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*) \xrightarrow{\sim} \mathrm{Gr}^W_{\mathrm{Lag}}(\mathfrak{t} \oplus \mathfrak{t}^*), \quad \mathfrak{g}^\kappa \rightsquigarrow (\mathfrak{g}^{\kappa - \mathrm{crit}})^T. \tag{2.10}$$

There is an isomorphism between $\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*)$ and the corresponding space for $\check{G}$, making the following diagram commute:

$$\begin{array}{ccc} \mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*) & \xrightarrow{\sim} & \mathrm{Gr}^{\check{G}}_{\mathrm{Lag}}(\check{\mathfrak{g}} \oplus \check{\mathfrak{g}}^*) \\ \downarrow (2.10) & & \downarrow (2.10) \text{ for } \check{G} \\ \mathrm{Gr}^W_{\mathrm{Lag}}(\mathfrak{t} \oplus \mathfrak{t}^*) & \xrightarrow[\sim]{(2.8)} & \mathrm{Gr}^{\check{W}}_{\mathrm{Lag}}(\check{\mathfrak{t}} \oplus \check{\mathfrak{t}}^*) \end{array}$$

We denote the image of $\mathfrak{g}^\kappa$ in $\mathrm{Gr}^{\check{G}}_{\mathrm{Lag}}(\check{\mathfrak{g}} \oplus \check{\mathfrak{g}}^*)$ by $\check{\mathfrak{g}}^{\check{\kappa}}$. The generalized bilinear forms $\mathfrak{g}^\kappa$ and $\check{\mathfrak{g}}^{\check{\kappa}}$ are supposed to be intertwined by the geometric Langlands correspondence. They have a built-in critical shift.

2.5.6. Using the identification $(\mathfrak{g}^{\kappa - \mathrm{crit}})^G \cong (\mathfrak{g}^\kappa)^G$ (see Remark 2.8), we see that an extension E of $(\mathfrak{g}^\kappa)^G \boxtimes \mathcal{O}_X$ by $\omega_{\mathcal{X}/S}$ (see (2.6)) is equivalent to an extension $\check{E}$ of $(\check{\mathfrak{g}}^{\check{\kappa}}) \boxtimes \mathcal{O}_X$ by $\omega_{\mathcal{X}/S}$. Indeed, the following $\mathcal{O}_S$-modules are all isomorphic:

$$(\mathfrak{g}^\kappa)^G \xrightarrow{\sim} (\mathfrak{g}^{\kappa - \mathrm{crit}})^G \cong (\check{\mathfrak{g}}^{\check{\kappa} - \mathrm{crit}})^{\check{G}} \xleftarrow{\sim} (\check{\mathfrak{g}}^{\check{\kappa}})^{\check{G}},$$

where the middle isomorphism comes from the identification of $(\mathfrak{g}^{\kappa-\mathrm{crit}})^T$ and $(\check{\mathfrak{g}}^{\check{\kappa}-\mathrm{crit}})^{\check{T}}$ under (2.8). This observation implies:

Lemma 2.14 *There is a canonical isomorphism of algebraic stacks:*

$$\mathrm{Par}_G \xrightarrow{\sim} \mathrm{Par}_{\check{G}}, \quad (\mathfrak{g}^{\kappa}, E) \rightsquigarrow (\mathfrak{g}^{\check{\kappa}}, \check{E}). \tag{2.11}$$

We refer to (2.11) as the *Langlands duality* for the space of quantum parameters Par_G.

Example 2.15 Suppose G is simple, and we fix a k-valued parameter $(\mathfrak{g}^{\kappa}, 0)$ of Par_G corresponding to some bilinear form κ on $\mathfrak{g}$. Then $\kappa = \lambda \cdot \mathrm{Kil}_G$ for some $\lambda \in k$. Write $\lambda = (c - h^\vee)/2h^\vee$ for some $c \in k$, where $h^\vee$ denotes the dual Coxeter number of G. Under the isomorphism (2.11), $(\mathfrak{g}^{\kappa}, 0)$ corresponds to the parameter $(\check{\mathfrak{g}}^{\check{\kappa}}, 0)$.

Assume $c \neq 0$. Then we claim that $\check{\mathfrak{g}}^{\check{\kappa}}$ arises from the bilinear form $\check{\kappa}$ defined by the formulae:

$$\check{\kappa} = \check{\lambda} \cdot \mathrm{Kil}_{\check{G}}, \quad \check{\lambda} = (-\frac{1}{rc} - h)/2h, \tag{2.12}$$

where $r = 1, 2$ or 3 denotes the maximal multiplicity of arrows in the Dynkin diagram of G.[4] Indeed, to see that $\check{\mathfrak{g}}^{\check{\kappa}}$ is given by the formulas (2.12), one first notes that $(1/2h^\vee) \cdot \mathrm{Kil}_G$ is the "minimal" W-invariant bilinear form min_G on $\mathfrak{t}$, defined by the property that the short coroot has self-pairing 2. Hence, κ is equal to $c \cdot \mathrm{min}_G + \mathrm{crit}_G$. Likewise, $\check{\kappa}$ is equal to $-\frac{1}{rc} \cdot \mathrm{min}_{\check{G}} + \mathrm{crit}_{\check{G}}$. We then appeal to the fact that r is the ratio of the self-pairing of long and short roots of G (under any W-invariant symmetric bilinear form).

2.6 Parabolics and anomalies

2.6.1. We now explain how to incorporate, via an additional parameter, the anomaly term that appears in the study of constant term functors (see [13, Sect. 3.3–3.4]). In *op.cit.*, the anomaly term is introduced to compare the constant term functor on $\mathcal{D}$-modules on Bun_G with the BRST reduction functor on the representation category of Kac–Moody Lie algebra associated to $\mathfrak{g}$.

The appropriately defined constant term functor for $\mathcal{D}$-modules does *not* go from $\mathcal{D}\text{-Mod}^{\kappa}(\mathrm{Bun}_G)$ to $\mathcal{D}\text{-Mod}^{\kappa-\mathrm{crit}}(\mathrm{Bun}_T)$ (shift by the critical level of G), but rather to the latter category twisted with a specific line bundle on Bun_T, namely the Tate line bundle.

The observation relevant for us is that this line bundle on Bun_T can be viewed as being attached to a quantum parameter for the reductive group T, in the form of an additional parameter in the sense of §2.4.1. Thus, the constant term morphism

[4] These are the numerics which appear in the typical formulation of the quantum Langlands correspondence for simple groups, see [23, Sect. 2] for example. Note that the critical shift is often omitted as the determinant line bundle on Bun_G admits a square root.

$\mathrm{Par}_G \to \mathrm{Par}_T$ we shall presently build takes $(\mathfrak{g}^\kappa, E)$ to $((\mathfrak{g}^{\kappa-\mathrm{crit}})^T, E_{G\to T})$, where the second term $E_{G\to T}$ accounts for the anomaly term.

2.6.2. In this subsection, we fix two additional pieces of structure:

(a) a Borel subgroup $B \subset G$ containing T;
(b) a *theta characteristic* on the curve X, i.e., a line bundle θ together with an isomorphism $\theta^{\otimes 2} \xrightarrow{\sim} \omega_X$.

The term *standard parabolic* refers to a parabolic subgroup $P \subset G$ containing B.

2.6.3. Let P be a standard parabolic. Denote by M its Levi quotient, which is a reductive group. The canonical map from T to M realizes T as a maximal torus of M. The Weyl group W_M of (M, T) can be identified with a subgroup of W.

Since $\mathfrak{z} \cong \mathfrak{t}^W$ and $\mathfrak{z}_M \cong \mathfrak{t}^{W_M}$, there is a canonical embedding $\mathfrak{z} \hookrightarrow \mathfrak{z}_M$. We claim that this embedding is canonically split. Indeed, this is because the composition $Z_0(G) \hookrightarrow G \twoheadrightarrow G/[G, G]$ is an isogeny, so it gives rise to the projection $\mathfrak{z}_M \to \mathfrak{z}$. It follows that we have a canonical map from the W_M-invariants of $\mathfrak{t} \oplus \mathfrak{t}^*$ to its W-invariants:

$$\mathfrak{z}_M \oplus \mathfrak{z}_M^* \to \mathfrak{z}_G \oplus \mathfrak{z}_G^*. \tag{2.13}$$

In particular, given any Lagrangian, W-invariant subbundle $\mathfrak{t}^\kappa \subset (\mathfrak{t} \oplus \mathfrak{t}^*) \otimes \mathcal{O}_S$, we have a morphism of $\mathcal{O}_S$-modules:

$$(\mathfrak{t}^\kappa)^{W_M} \to (\mathfrak{t}^\kappa)^W. \tag{2.14}$$

This morphism is compatible with (2.13) in the sense they intertwine the inclusion of $(\mathfrak{t}^\kappa)^{W_M}$ into $\mathfrak{z}_M \oplus \mathfrak{z}_M^*$ (resp. of $(\mathfrak{t}^\kappa)^W$ into $\mathfrak{z}_G \oplus \mathfrak{z}_G^*$).

2.6.4. There is a *reduction morphism* ("critically-shifted reduction" for M):

$$\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*) \to \mathrm{Gr}^M_{\mathrm{Lag}}(\mathfrak{m} \oplus \mathfrak{m}^*), \tag{2.15}$$

defined by the composition:

$$\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*) \xrightarrow{\sim} \mathrm{Gr}^W_{\mathrm{Lag}}(\mathfrak{t} \oplus \mathfrak{t}^*) \hookrightarrow \mathrm{Gr}^{W_M}_{\mathrm{Lag}}(\mathfrak{t} \oplus \mathfrak{t}^*) \xleftarrow{\sim} \mathrm{Gr}^M_{\mathrm{Lag}}(\mathfrak{m} \oplus \mathfrak{m}^*)$$

where the isomorphisms are supplied by the critically-shifted reductions (2.10) for G, respectively M. In other words, the image of $\mathfrak{g}^\kappa$ under (2.15) is an S-point $\mathfrak{m}^\kappa$ such that $(\mathfrak{m}^{\kappa-\mathrm{crit}})^T$ and $(\mathfrak{g}^{\kappa-\mathrm{crit}})^T$ are canonically isomorphic as subbundles of $(\mathfrak{t} \oplus \mathfrak{t}^*) \otimes \mathcal{O}_S$.[5] The morphism (2.15) includes (2.10) as a special case.

2.6.5. Let $Z_0(M)$ denote the neutral component of the center of M. Write $2\check{\rho}_M$ for the character of $Z_0(M)$ determined by the representation $\det(\mathfrak{n}_P)$, where $\mathfrak{n}_P$ is the Lie algebra of the unipotent part of P. Let $\check{Z}_0(M)$ denote the dual torus of $Z_0(M)$. We

[5] Here, $\mathfrak{m}^{\kappa-\mathrm{crit}}$ is defined with reference to the critical form on $\mathfrak{m}$ (as opposed to $\mathfrak{g}$).

use $\omega_X^{\check{\rho}_M}$ to denote the $\check{Z}_0(M)$-bundle on X induced from θ under $2\check{\rho}_M$ (regarded as a cocharacter of $\check{Z}_0(M)$). Then the Atiyah bundle of $\omega_X^{\check{\rho}_M}$ fits into an exact sequence:

$$0 \to \mathfrak{z}_M^* \otimes \mathcal{O}_X \to \mathrm{At}(\omega_X^{\check{\rho}_M}) \to \mathcal{T}_X \to 0.$$

Its monoidal dual gives rise to an extension of $\mathcal{O}_{\mathcal{X}}$-modules for every S (recall the notation $\mathcal{X} := S \times X$):

$$0 \to \omega_{\mathcal{X}/S} \to \mathcal{O}_S \boxtimes \mathrm{At}(\omega_X^{\check{\rho}_M})^* \to (\mathfrak{z}_M \otimes \mathcal{O}_S) \boxtimes \mathcal{O}_X \to 0. \tag{2.16}$$

For each S-point $\mathfrak{m}^\kappa$ of $\mathrm{Gr}^M_{\mathrm{Lag}}(\mathfrak{m} \oplus \mathfrak{m}^*)$, we let $E^+_{G\to M}$ denote the extension of $(\mathfrak{m}^\kappa)^M$ induced from (2.16) along the canonical map, pulled back along $\mathcal{X} \to S$:

$$(\mathfrak{m}^\kappa)^M \hookrightarrow (\mathfrak{z}_M \oplus \mathfrak{z}_M^*) \otimes \mathcal{O}_S \twoheadrightarrow \mathfrak{z}_M \otimes \mathcal{O}_S.$$

The additional parameter $E^+_{G\to M}$ is the *anomaly term* at level $\mathfrak{m}^\kappa$.

2.6.6. The reduction morphism for quantum parameters is defined by ("constant term morphism" for the space of quantum parameters):

$$\mathrm{CT}_P : \mathrm{Par}_G \to \mathrm{Par}_M, \quad (\mathfrak{g}^\kappa, E) \rightsquigarrow (\mathfrak{m}^\kappa, E_{G\to M}) \tag{2.17}$$

where $\mathfrak{m}^\kappa$ is the image of $\mathfrak{g}^\kappa$ under (2.15), and $E_{G\to M}$ is the Baer sum of the following two extensions of $(\mathfrak{m}^\kappa)^M$:

(a) an extension induced from E (which is an extension of $(\mathfrak{g}^\kappa)^G$) via the map:

$$(\mathfrak{m}^\kappa)^M \xrightarrow{\sim} (\mathfrak{m}^{\kappa-\mathrm{crit}})^M \to (\mathfrak{g}^{\kappa-\mathrm{crit}})^G \xrightarrow{\sim} (\mathfrak{g}^\kappa)^G,$$

where the map in the middle comes from (2.14) for $\mathfrak{t}^\kappa := (\mathfrak{m}^{\kappa-\mathrm{crit}})^T \cong (\mathfrak{g}^{\kappa-\mathrm{crit}})^T$;

(b) the anomaly term $E^+_{G\to M}$ at level $\mathfrak{m}^\kappa$.

Remark 2.16 The image of $(\mathfrak{g}^\infty, E)$ under CT_P agrees with $(\mathfrak{m}^\infty, E)$. In other words, the anomaly term $E^+_{G\to M}$ vanishes at level ∞.

In particular, we see that CT_P is *incompatible* with Langlands duality for quantum parameters, i.e., if we let $\check{M}$ be the Langlands dual of M viewed as the Levi quotient of a parabolic subgroup $\check{P} \subset \check{G}$, the following diagram does *not* commute:

$$\begin{array}{ccc} \mathrm{Par}_G & \xrightarrow{(2.11)} & \mathrm{Par}_{\check{G}} \\ {\scriptstyle \mathrm{CT}_P}\downarrow & & \downarrow{\scriptstyle \mathrm{CT}_{\check{P}}} \\ \mathrm{Par}_M & \xrightarrow{(2.11)} & \mathrm{Par}_{\check{M}} \end{array}$$

It is not clear how this phenomenon is reflected in the conjectural quantum geometric Langlands correspondence. However, it seems related to the fact that the compatibility of the Langlands duality functor and the constant term functor involves an autoequivalence of the target category $\mathcal{D}\text{-Mod}^{\check{\kappa}}(\mathrm{Bun}_{\check{M}})$ (for $\check{\mathfrak{g}}^{\check{\kappa}} = \check{\mathfrak{g}}^\infty$, see [1, Conjecture 13.2.9]).

Remark 2.17 For $P = B$ and $M = T$, the character $2\check{\rho}$ is the sum of positive roots, and splittings of (2.16) form a $\mathfrak{t}^* \otimes \omega_X$-torsor $\mathrm{Conn}(\omega_X^{\check{\rho}})$, which is also known as the space of *Miura opers* (see [10]).

2.7 Structures on $\mathfrak{g}^\kappa$

2.7.1. We finish this section with a description of some structures on the vector bundle $\mathfrak{g}^\kappa$ functorially attached to an S-point of $\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*)$.

2.7.2. There is an $\mathcal{O}_S$-bilinear Lie bracket:

$$[-,-] : \mathfrak{g}^\kappa \underset{\mathcal{O}_S}{\otimes} \mathfrak{g}^\kappa \to \mathfrak{g}^\kappa \tag{2.18}$$

defined by the formula (on the ambient bundle $(\mathfrak{g} \oplus \mathfrak{g}^*) \otimes \mathcal{O}_S$):

$$[(\xi \oplus \varphi) \otimes \mathbf{1}, (\xi' \oplus \varphi') \otimes \mathbf{1}] := ([\xi, \xi'] \oplus \mathrm{Coad}_\xi(\varphi')) \otimes \mathbf{1}.$$

One checks immediately that the image lies in $\mathfrak{g}^\kappa$ and the required identities hold. Note that (2.18) factors through the embedding $\mathfrak{g}^\kappa_{\mathrm{s.s.}} \hookrightarrow \mathfrak{g}^\kappa$.

2.7.3. There is an $\mathcal{O}_S$-bilinear symmetric pairing:

$$(-,-) : \mathfrak{g}^\kappa \underset{\mathcal{O}_S}{\otimes} \mathfrak{g}^\kappa \to \mathcal{O}_S \tag{2.19}$$

defined by the formula:

$$((\xi \oplus \varphi) \otimes \mathbf{1}, (\xi' \oplus \varphi') \otimes \mathbf{1}) := \varphi'(\xi) \cdot \mathbf{1}.$$

The pairing (2.19) gives rise to a canonical central extension of the loop algebra $\mathfrak{g}^\kappa(\!(t)\!)$:

$$0 \to \mathcal{O}_S \to \widehat{\mathfrak{g}}^\kappa \to \mathfrak{g}^\kappa(\!(t)\!) \to 0$$

whose cocycle is given by the residue pairing $\mathrm{Res}(-, d-)$. This is the prototype of a *generalized* Kac-Moody extension. We will return to it in Sect. 5 (in the setting of Lie-$*$ algebras).

Example 2.18 For the k-point $\mathfrak{g}^\infty$ of $\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*)$, the Lie bracket (2.18) is zero. The pairing (2.19) is also zero. Hence $\widehat{\mathfrak{g}}^\infty$ is the abelian Lie algebra $\mathcal{O}_S \oplus \mathfrak{g}^\infty(\!(t)\!)$.

2.7.4. Fixing an S-point $(\mathfrak{g}^\kappa, E)$ of Par_G, there is an extension of $\mathcal{O}_\mathcal{X}$-modules:

$$0 \to \omega_{\mathcal{X}/S} \to \widehat{\mathfrak{g}}^{(\kappa,E)} \to \mathfrak{g}^\kappa \boxtimes \mathcal{O}_X \to 0. \tag{2.20}$$

induced from (2.6) along $\mathfrak{g}^\kappa \otimes \mathcal{O}_S \to (\mathfrak{g}^\kappa)^G \otimes \mathcal{O}_X$. In other words, $\widehat{\mathfrak{g}}^{(\kappa,E)}$ is the direct sum of E and $\mathfrak{g}^\kappa_{\mathrm{s.s.}} \boxtimes \mathcal{O}_X$, corresponding to the decomposition $\mathfrak{g}^\kappa \xrightarrow{\sim} \widehat{\mathfrak{g}}^\kappa \oplus \mathfrak{g}^\kappa_{\mathrm{s.s.}}$.

3 Quasi-twistings

In this section, we make sense of a central extension of Lie algebroids in the DG setting; such objects are called *quasi-twistings*. A dynamic theory of Lie algebroids in such generality has been built by Gaitsgory and Rozenblyum [18], and our results in Sects. 3 and 4 are no more than a modest extension of their theory.

Notations

We work over a fixed affine scheme S smooth[6] over k. Some of the notions in this section involve the interplay between classical and derived algebraic geometry. For the latter, we use the theory of ∞-category as developped in [20, 21] and the theory of derived algebraic geometry modeled on commutative DG algebras, using [18] as our main reference.

By a *scheme*, we shall mean a classical scheme (as opposed to a DG scheme). On the other hand, a *prestack* means a presheaf on affine DG schemes valued in ∞-groupoids. More specialized notations involving derived formal moduli problems will be explained in Sect. 3.3.

3.1 The classical notion

3.1.1. Let Y be a scheme over S. A *Lie algebroid* over Y (relative to S) is an $\mathcal{O}_Y$-module $\mathcal{L}$ together with an $\mathcal{O}_S$-linear Lie bracket $[-,-]$ and an $\mathcal{O}_Y$-module map $\sigma : \mathcal{L} \to \mathcal{T}_{Y/S}$ such that the following properties are satisfied:

(a) $[l_1, f \cdot l_2] = \sigma(l_1)(f) \cdot l_2 + f[l_1, l_2]$;
(b) σ intertwines $[-,-]$ with the canonical Lie bracket on $\mathcal{T}_{Y/S}$.

The morphism σ is called the *anchor map* of $\mathcal{L}$. The category of Lie algebroids over Y is denoted by $\mathrm{LieAlgd}_{/S}(Y)$. A *Picard algebroid* is a central extension of the tangent Lie algebroid $\mathcal{T}_{Y/S}$ by $\mathcal{O}_Y$; they are equivalent to a ring of twisted differential operators (TDOs) over Y (see [2]).

Definition 3.1 A *classical quasi-twisting* $\mathcal{T}^{\mathrm{cl}}$ over Y (relative to S) is a central extension:

$$0 \to \mathcal{O}_Y \to \widehat{\mathcal{L}} \to \mathcal{L} \to 0 \tag{3.1}$$

of Lie algebroids.

[6] Most of the materials in Sects. 3 and 4 should extend to any base affine scheme S over k. The reason we choose not to work in this generality is because the theory of ind-coherent sheaves in [18] is built in an absolute setting whereas we would need a notion of ind-coherent sheaves for an S-scheme Y which is "quasi-coherent along S." Since our ultimate goal is to construct a quasi-coherent sheaf of categories (which are fppf-local objects, see [12, Appendix A]) on the smooth algebraic stack Par_G, it is enough to limit our attention to smooth test schemes $S \to \mathrm{Par}_G$.

We say that $\mathcal{T}^{\mathrm{cl}}$ is *based* at the Lie algebroid $\mathcal{L}$. Classical quasi-twistings with a fixed base $\mathcal{L}$ form a k-linear, strictly commutative Picard groupoid under the operation of Baer sum. We denote it by $\mathrm{QTw}^{\mathrm{cl}}_{/S}(Y/\mathcal{L})$. The following is obvious:

Lemma 3.2 *A classical quasi-twisting $\mathcal{T}^{\mathrm{cl}}$ is a Picard algebroid if and only if the anchor map of $\mathcal{L}$ is an isomorphism.* □

3.1.2. Given a classical quasi-twisting $\mathcal{T}^{\mathrm{cl}}$, the *(reduced) universal envelope* of $\mathcal{T}^{\mathrm{cl}}$ is defined to be the $\mathcal{O}_Y$-algebra:

$$\mathrm{U}(\mathcal{T}^{\mathrm{cl}}) := \mathrm{U}(\widehat{\mathcal{L}})/(1-\mathbf{1}),$$

where $\mathrm{U}(\widehat{\mathcal{L}})$ is the universal enveloping algebra of $\widehat{\mathcal{L}}$, and $\mathbf{1}$ denotes the image of the unit in $\mathcal{O}_Y$. A *module* over $\mathcal{T}^{\mathrm{cl}}$ is a $\mathrm{U}(\mathcal{T}^{\mathrm{cl}})$-module, or equivalently, a module over the Lie algebroid $\widehat{\mathcal{L}}$ on which $\mathbf{1}$ acts by the identity.

3.2 Some ∞-dimensional geometry

3.2.1. Suppose Y is a scheme over S but *not* locally of finite type. The above notion of Lie algebroids is not very amenable to study. We will occasionally encounter some ∞-type schemes, for which we need the notion of a Lie algebroid "on Tate module".

Let R be a (discrete) ring over k. The notion of Tate R-modules is developed in [7]. We briefly recall the definitions.

3.2.2. An *elementary Tate R-module* is a topological R-module isomorphic to $P \oplus Q^*$, where P and Q are discrete, projective R-modules.[7] A *Tate R-module* is topological R-module isomorphic to a direct summand of some elementary Tate R-module. There are two important types of submodules of a Tate R-module M:

(a) a *lattice* is an open submodule L^+ with the property that L^+/U is finitely generated for any open submodule $U \hookrightarrow L^+$.
(b) a *co-lattice* is a submodule L^- such that for some lattice L^+, both $L^+ \cap L^-$ and $M/(L^+ + L^-)$ are finitely generated.

Example 3.3 Clearly, every profinite R-module is an elementary Tate R-module. The Laurent series ring $R((t))$ is also an elementary Tate module (but *not* profinite).

3.2.3. Given a map of (discrete) rings $R \to R'$, the *pullback* of a Tate R-module M is defined by

$$M \underset{R}{\widehat{\otimes}} R' := \varprojlim (M/U) \underset{R}{\otimes} R'$$

where U ranges over open submodules of M.

[7] The topology on Q^* is generated by opens of the form $U^{\perp}$ where U is a finite generated R-submodule of Q.

Tate R-modules are local objects for the flat topology (see [7, Theorem 3.3]). In particular, we may define a Tate $\mathcal{O}_Y$-module $\mathcal{F}$ over a scheme Y (or more generally, an algebraic stack) as a compatible system of Tate $\mathcal{O}_Z$-modules $\mathcal{F}\big|_Z$ for every affine scheme Z mapping to Y.

3.2.4. Let Y be a scheme over S. Then Y is *placid* if Zariski locally there is a presentation $Y \xrightarrow{\sim} \varprojlim Y_i$, where each Y_i is a scheme of finite type, and the connecting morphisms $Y_j \to Y_i$ are smooth surjections. We call a placid scheme Y *pro-smooth*, if we can furthermore choose each Y_i to be smooth.

If Y is a pro-smooth placid scheme, then the tangent sheaf $\mathcal{T}_{Y/S}$ is naturally a Tate $\mathcal{O}_Y$-module. Indeed, locally on Y there is an isomorphism:

$$\mathcal{T}_{Y/S} \xrightarrow{\sim} \varprojlim \pi_i^* \mathcal{T}_{Y_i/S},$$

where $\pi_i : Y \to Y_i$ is the canonical map.

3.2.5. Suppose Y is a pro-smooth placid scheme. We define a *Lie algebroid on Tate module* over Y as a Tate $\mathcal{O}_Y$-module $\mathcal{L}$ together with a *continuous* $\mathcal{O}_Y$-linear map $\sigma : \mathcal{L} \to \mathcal{T}_{Y/S}$, such that as a *plain* $\mathcal{O}_Y$-module, $\mathcal{L}$ has the structure of a Lie algebroid with σ as its anchor map.

Example 3.4 The tangent sheaf $\mathcal{T}_{Y/S}$ has the structure of a Lie algebroid on Tate module.

A *classical quasi-twisting on Tate modules* $\mathcal{T}^{\mathrm{cl}}$ over Y is a central extension (3.1) of Lie algebroids on Tate modules where all the morphisms are continuous.

Remark 3.5 The above notion is very naïve, as it does not indicate how the Lie bracket interacts with the topology on $\mathcal{L}$. However, it suffices for our purpose since in the construction of $\mathcal{T}_G^{(\kappa,E)}$ in Sect. 5, the first quotient step will reduce the classical quasi-twisting on Tate modules $\tilde{\mathcal{T}}_G^{(\kappa,E)}$ into a discrete, classical quasi-twisting over $\mathrm{Bun}_{G,nx}^{(\leq\theta)}$.

Remark 3.6 We will frequently refer to a classical quasi-twisting on Tate modules simply as a classical quasi-twisting, as the Tate structures should be clear from the context.

3.3 Formal groupoids

3.3.1. In this subsection, we review the theory of derived formal moduli problems. Let Vect denote the derived ∞-category of chain complexes of k-vector spaces. It has a natural symmetric monoidal structure which commutes with colimits in both variables. As such, it may be viewed as a commutative algebra object in the ∞-category of presentable stable ∞-categories equipped with the Lurie tensor product.

By a *DG category*, we mean a module object over Vect in this symmetric monoidal ∞-category. We use the notation $\mathrm{DGCat}_{\mathrm{cont}}$ to denote the ∞-category of DG categories

(whose functors are *continuous*, i.e., colimit-preserving). The ∞-category $\mathrm{DGCat}_{\mathrm{cont}}$ inherits a symmetric monoidal structure.

3.3.2. We use the notation $\mathrm{PStk}_{\text{laft-def}\,/S}$ to mean the ∞-category of prestacks locally almost of finite type ("laft") over S which admit deformation theory (see [18, III.1]). A simplicial object $\mathcal{R}^\bullet$ of $\mathrm{PStk}_{\text{laft-def}\,/S}$ is called a *groupoid* (relative to S) if the following conditions are satisfied:

(a) for every $n \geq 2$, the map $\mathcal{R}^n \to \mathcal{R}^1 \underset{\mathcal{Y}}{\times} \cdots \underset{\mathcal{Y}}{\times} \mathcal{R}^1$ induced by products of the maps $[1] \to [n]$ sending $0 \rightsquigarrow i$, $1 \rightsquigarrow i+1$, is an isomorphism;
(b) the map $\mathcal{R}^2 \to \mathcal{R}^1 \underset{\mathcal{Y}}{\times} \mathcal{R}^1$ induced by the product of the maps $[1] \to [2]$ sending

$$0 \rightsquigarrow 0,\ \ 1 \rightsquigarrow 1 \text{ and } 0 \rightsquigarrow 0,\ \ 1 \rightsquigarrow 2$$

is an isomorphism.

Furthermore, $\mathcal{R}^\bullet$ is a *formal groupoid* if all morphisms in $\mathcal{R}^\bullet$ are nil-isomorphisms, i.e., they induce isomorphisms on the reduced prestacks. We denote the ∞-category of formal groupoids (relative to S) by $\mathrm{FGpd}_{/S}$. There is a functor

$$\mathrm{FGpd}_{/S} \to \mathrm{PStk}_{\text{laft-def}\,/S}, \quad \mathcal{R}^\bullet \rightsquigarrow \mathcal{R}^0, \tag{3.2}$$

whose fiber at $\mathcal{Y}$ is denoted by $\mathrm{FGpd}_{/S}(\mathcal{Y})$ and is referred to as the ∞-category of formal groupoids *actin on* $\mathcal{Y}$.

Example 3.7 Completion along the main diagonals $\mathcal{Y} \to \mathcal{Y} \underset{S}{\times} \cdots \underset{S}{\times} \mathcal{Y}$ organizes into a formal groupoid $\mathcal{R}^\bullet := (\mathcal{Y}^\bullet)_{\widehat{\mathcal{Y}}}$ acting on $\mathcal{Y}$. This is the final object of $\mathrm{FGpd}_{/S}(\mathcal{Y})$ and is called the *infinitesimal groupoid* acting on $\mathcal{Y}$.

3.3.3. The functor (3.2) is a Cartesian fibration of ∞-categories. The Cartesian arrows in $\mathrm{FGpd}_{/S}$ are maps $\mathcal{R}^\bullet \to \mathcal{T}^\bullet$ such that the induced morphism

$$\mathcal{R}^\bullet \to \mathcal{T}^\bullet \underset{(\mathcal{Z}^\bullet)_{\widehat{\mathcal{Z}}}}{\times} (\mathcal{Y}^\bullet)_{\widehat{\mathcal{Y}}}, \quad \text{where } \mathcal{Y} := \mathcal{R}^0 \text{ and } \mathcal{Z} := \mathcal{T}^0$$

is an isomorphism.

3.4 Formal moduli problems

3.4.1. Let $\mathrm{FMod}_{/S}$ denote the ∞-category of morphisms $\mathcal{Y} \to \mathcal{Y}^\flat$ in $\mathrm{PStk}_{\text{laft-def}\,/S}$ which are nil-isomorphisms.[8] In particular, $\mathrm{FMod}_{/S}$ is a full subcategory of the functor category $\mathrm{Fun}(\Delta^1, \mathrm{PStk}_{\text{laft-def}\,/S})$. Its objects are called *formal moduli problems* (relative to S). We have a functor

[8] Caution: our notation $\mathrm{FMod}_{/S}$ is different from [18, IV.1, §1], where the analogous notation means formal moduli problems *over* a fixed laft prestack.

$$\mathrm{FMod}_{/S} \to \mathrm{PStk}_{\text{laft-def}\,/S}, \quad (\mathcal{Y} \to \mathcal{Y}^\flat) \rightsquigarrow \mathcal{Y}, \tag{3.3}$$

whose fiber at $\mathcal{Y} \in \mathrm{PStk}_{\text{laft-def}\,/S}$ is by definition the ∞-category of formal moduli problems *under* $\mathcal{Y}$, and is denoted by $\mathrm{FMod}_{/S}(\mathcal{Y})$.

3.4.2. The functor (3.3) is a Cartesian fibration of ∞-categories, whose Cartesian arrows are commutative diagrams on the left whose induced square on the right is Cartesian:

$$\begin{array}{ccc} \mathcal{Y} & \longrightarrow & \mathcal{Z} \\ \downarrow & & \downarrow \\ \mathcal{Y}^\flat & \to & \mathcal{Z}^\flat \end{array} \qquad \begin{array}{ccc} \mathcal{Y}^\flat & \longrightarrow & \mathcal{Z}^\flat \\ \downarrow & & \downarrow \\ \mathcal{Y}_{\mathrm{dR}} & \to & \mathcal{Z}_{\mathrm{dR}} \end{array}$$

Applying straightening to (3.3), we obtain a *pullback* functor for every morphism $f : \mathcal{Y} \to \mathcal{Z}$ in $\mathrm{PStk}_{\text{laft-def}\,/S}$:

$$f^!_{\mathrm{FMod}} : \mathrm{FMod}_{/S}(\mathcal{Z}) \to \mathrm{FMod}_{/S}(\mathcal{Y}), \quad f^!_{\mathrm{FMod}}\mathcal{Z}^\flat := \mathcal{Z}^\flat \underset{\mathcal{Z}_{\mathrm{dR}}}{\times} \mathcal{Y}_{\mathrm{dR}}.$$

3.4.3. The Čech nerve construction defines a functor $\Omega : \mathrm{FMod}_{/S} \to \mathrm{FGpd}_{/S}$ of ∞-categories over $\mathrm{PStk}_{\text{laft-def}\,/S}$. The main result in [18, Sect. IV.1] (which has its origin in Lurie's theory of formal moduli problems) can be summarized as follows:

Theorem 3.8 (Lurie-Gaitsgory-Rozenblyum) *The functor Ω is an equivalence.*

Proof Indeed, [18, Sect. IV.1, Theorem 2.3.2] shows that Ω is an equivalence when restricted to the fiber at each $\mathcal{Y} \in \mathrm{PStk}_{\text{laft-def}\,/S}$. The above formulation follows because Ω also preserves Cartesian arrows (and we appeal to [20, Corollary 2.4.4.4]). □

We denote the functor inverse to Ω by $\mathrm{B} : \mathrm{FGpd}_{/S} \to \mathrm{FMod}_{/S}$. Their restrictions to the fiber at $\mathcal{Y} \in \mathrm{PStk}_{\text{laft-def}\,/S}$ are denoted by $\Omega_{\mathcal{Y}}$ and $\mathrm{B}_{\mathcal{Y}}$.

Example 3.9 (de Rham prestack) Let $\mathcal{Y}_{\mathrm{dR}/S}$ denote the fiber product $\mathcal{Y}_{\mathrm{dR}} \underset{S_{\mathrm{dR}}}{\times} S$ which is the terminal object of $\mathrm{FMod}_{/S}(\mathcal{Y})$. Then $\mathcal{Y}_{\mathrm{dR}/S}$ corresponds to the infinitesimal groupoid $(\mathcal{Y})^\bullet_{\widehat{\mathcal{Y}}}$ (Example 3.7) under the equivalence $\mathrm{FGpd}_{/S}(\mathcal{Y}) \xrightarrow{\sim} \mathrm{FMod}_{/S}(\mathcal{Y})$.

In particular, given any group object $H \in \mathrm{PStk}_{\text{laft-def}\,/S}$, there is a canonical short exact sequence of group prestacks:

$$1 \to H_{\{\widehat{1}\}} \to H \to H_{\mathrm{dR}/S} \to 1 \tag{3.4}$$

Corollary 3.10 *The prestack $\mathrm{B}_{\mathcal{Y}}(\mathcal{R}^\bullet)$ is identified with the quotient of $\mathcal{R}^\bullet$ in $\mathrm{PStk}_{\text{laft-def}\,/S}$.*

Proof We need to show that $\mathrm{B}_{\mathcal{Y}}(\mathcal{R}^\bullet)$ identifies with $\underset{\Delta^{\mathrm{op}}}{\mathrm{colim}}\, \mathcal{R}^\bullet$, where the colimit is taken in $\mathrm{PStk}_{\text{laft-def}\,/S}$. This follows from the fact that $\mathrm{Maps}(\mathrm{B}_{\mathcal{Y}}(\mathcal{R}^\bullet), \mathcal{Z})$ identifies with the mapping space from $\mathcal{Y} \to \mathrm{B}_{\mathcal{Y}}(\mathcal{R}^\bullet)$ to $\mathcal{Z} \to \mathcal{Z}$ in $\mathrm{FMod}_{/S}$, which by Theorem 3.8 identifies with $\mathrm{Maps}(\mathcal{R}^\bullet, \mathcal{Z})$. □

3.4.4. However, we point out that the quotient of $\mathcal{R}^\bullet$ in $\mathrm{PStk}_{\text{laft-def}\,/S}$ may not agree with that in $\mathrm{PStk}_{/S}$, which is one of the main technical complications for us.

Example 3.11 Let $S = \mathrm{pt}$ and we omit the subscript $/S$ from the notations. The Čech nerve of the object $\mathrm{pt} \to \mathbb{A}^1_{\widehat{\{0\}}}$ in FMod is the formal groupoid $\mathcal{R}^\bullet := \mathrm{pt} \underset{\mathbb{A}^1}{\times} \cdots \underset{\mathbb{A}^1}{\times} \mathrm{pt}$. The quotient $\underset{\Delta^{\mathrm{op}}}{\mathrm{colim}}\, \mathcal{R}^\bullet$ taken in PStk does *not* agree with $\mathbb{A}^1_{\widehat{\{0\}}}$. Indeed, since colimits in PStk are computed pointwise, we have an equivalence:

$$\mathrm{Maps}(\mathrm{Spec}(k[\varepsilon]/(\varepsilon^2)), \underset{\Delta^{\mathrm{op}}}{\mathrm{colim}}\, \mathcal{R}^\bullet) \cong \underset{\Delta^{\mathrm{op}}}{\mathrm{colim}}\, \mathrm{Maps}(\mathrm{Spec}(k[\varepsilon]/(\varepsilon^2)), \mathcal{R}^\bullet). \tag{3.5}$$

On the other hand, morphisms from a classical scheme to a DG scheme factors through its classical subscheme. Since the classical subscheme of each $\mathcal{R}^n$ is a point, the colimit (3.5) yields a point (as an ∞-groupoid). However, the formal scheme $\mathbb{A}^1_{\widehat{\{0\}}}$ receives nontrivial maps from $\mathrm{Spec}(k[\varepsilon]/(\varepsilon^2))$.

3.4.5. We note one case where $\mathrm{B}_{\mathcal{Y}}(\mathcal{R}^\bullet)$ agrees with the quotient in $\mathrm{PStk}_{/S}$.

Lemma 3.12 *Suppose the morphisms $\mathcal{R}^1 \rightrightarrows \mathcal{Y}$ are formally smooth. Then the canonical map $\underset{\Delta^{\mathrm{op}}}{\mathrm{colim}}\, \mathcal{R}^\bullet \to \mathrm{B}_{\mathcal{Y}}(\mathcal{R}^\bullet)$, where the colimit is taken in $\mathrm{PStk}_{/S}$, is an isomorphism.*

Recall that a morphism $\mathcal{X} \to \mathcal{Y}$ of prestacks is called *formally smooth* if for every affine DG scheme T over $\mathcal{Y}$, and a nilpotent embedding $T \hookrightarrow T'$, the map

$$\mathrm{Maps}(T', \mathcal{Y}) \to \mathrm{Maps}(T, \mathcal{Y})$$

is surjective on π_0 (see [18, III.1, §7.3]). Let $\mathcal{T}^*_{\mathcal{X}/\mathcal{Y}}\big|_x$ denote the cotangent complex at a T-point $x : T \to \mathcal{X}$. It is proved in *op.cit.* that if $\mathcal{X} \to \mathcal{Y}$ admits (relative) deformation theory, then formal smoothness is equivalent to

$$\mathrm{Maps}(\mathcal{T}^*_{\mathcal{X}/\mathcal{Y}}\big|_x, \mathcal{F}) \in \mathrm{Vect}^{\leq 0}, \tag{3.6}$$

where $\mathcal{F} \in \mathrm{QCoh}(T)^\heartsuit$ and T is any *affine* DG scheme with a morphism $x : T \to \mathcal{X}$.[9]

Proof of Lemma 3.12 The authors of [18] give the following description of $\mathrm{B}_{\mathcal{Y}}(\mathcal{R}^\bullet)$. Let U be an affine DG scheme. Then $\mathrm{Maps}(U, \mathrm{B}_{\mathcal{Y}}(\mathcal{R}^\bullet))$ is the space of the following data:

(a) a formal moduli problem $\widetilde{U}$ *over* U;

[9] We use the notation $\mathrm{QCoh}(Y)$ to denote the DG category of complexes of $\mathcal{O}_Y$-modules. In contrast, the abelian category of $\mathcal{O}_Y$-modules is denoted by $\mathrm{QCoh}(Y)^\heartsuit$, understood as the heart of a natural t-structure on $\mathrm{QCoh}(Y)$.

(b) a morphism from the Čech nerve of $\widetilde{U} \to U$ to $\mathcal{R}^\bullet$, such that the following diagram is Cartesian for each of the vertical arrows:

$$\begin{array}{ccc} \widetilde{U} \underset{U}{\times} \widetilde{U} & \to & \mathcal{R}^1 \\ \downarrow\downarrow & & \downarrow\downarrow \\ \widetilde{U} & \to & \mathcal{R}^0 \end{array}$$

On the other hand, $\mathrm{Maps}(U, \underset{\Delta^{\mathrm{op}}}{\mathrm{colim}}\, \mathcal{R}^\bullet)$ classifies the above data satisfying the *condition* that $\widetilde{U} \to U$ admits a section. Now, since $\widetilde{U} \to U$ is a nil-isomorphism, we obtain a section over U^{red}. A lift of this section to U exists if the morphism $\widetilde{U} \to U$ is formally smooth.

Now, let T be affine DG scheme equipped with a map $\tilde{u} : T \to \widetilde{U}$. The Cartesian diagrams:

$$\begin{array}{ccc} \widetilde{U} \underset{U}{\times} \widetilde{U} & \to & \widetilde{U} \\ \downarrow & & \downarrow \\ \widetilde{U} & \to & U \end{array} \qquad \begin{array}{ccc} \widetilde{U} \underset{U}{\times} \widetilde{U} & \to & \mathcal{R}^1 \\ \downarrow & & \downarrow \\ \widetilde{U} & \to & \mathcal{Y} \end{array}$$

show that $\mathcal{T}^*_{\widetilde{U}/U}\big|_{\tilde{u}}$ is isomorphic to $\mathcal{T}^*_{\widetilde{U} \underset{U}{\times} \widetilde{U}/\widetilde{U}}\big|_{(\tilde{u},\tilde{u})}$, which is in turn isomorphic to $\mathcal{T}^*_{\mathcal{R}^1/\mathcal{Y}}\big|_{r^1}$ where r^1 is the composition $T \xrightarrow{(\tilde{u},\tilde{u})} \widetilde{U} \underset{U}{\times} \widetilde{U} \to \mathcal{R}^1$. Hence the formal smoothness of $\mathcal{R}^1$ over $\mathcal{Y}$ implies that of $\widetilde{U}$ over U. □

3.4.6. In particular, let $\mathfrak{h}$ be a (classical) Lie algebra over $\mathcal{O}_S$, such that $\exp(\mathfrak{h})$ acts on some $\mathcal{Y} \in \mathrm{PStk}_{\text{laft-def}\,/S}$. Then the groupoid $\mathcal{Y} \underset{S}{\times} \exp(\mathfrak{h}) \rightrightarrows \mathcal{Y}$ is formally smooth, so its quotient may be formed in $\mathrm{PStk}_{/S}$. We have two particular instances of this example:

(a) Taking $\mathcal{Y} = \mathrm{pt}$, we see that $\mathrm{B}\exp(\mathfrak{h})$ is the prestack quotient $\mathrm{pt}\,/\exp(\mathfrak{h})$;
(b) Let H be a group scheme. Then the prestack quotient $H/\exp(\mathfrak{h})$ identifies with $H_{\mathrm{dR}/S}$.

3.5 Modules over a formal moduli problem

3.5.1. Recall that for an affine DG scheme Y almost of finite type over S, the DG category $\mathrm{IndCoh}(Y)$ is the ind-completion of the full subcategory $\mathrm{Coh}(Y) \hookrightarrow \mathrm{QCoh}(Y)$. There is a symmetric monoidal functor:

$$\Upsilon_{Y/S} : \mathrm{QCoh}(Y) \to \mathrm{IndCoh}(Y), \quad \mathcal{F} \rightsquigarrow \mathcal{F} \otimes \omega_{Y/S}, \tag{3.7}$$

which is an equivalence of DG categories if $Y \to S$ is smooth ([18, II.3]). The basic functoriality of ind-coherent sheaves is the (derived) !-pullback functor. It is well-defined for any morphism $f : X \to Y$ of affine DG schemes almost of finite type over

S and (3.7) intertwines it with the (derived) pullback functor f^* on quasi-coherent sheaves.

3.5.2. For a laft prestack $\mathcal{Y}$, the DG category $\mathrm{IndCoh}(\mathcal{Y})$ is defined as the limit of $\mathrm{IndCoh}(T)$ over all affine DG schemes T equipped with a map to $\mathcal{Y}$ (with transition functors given by !-pullback). The formalism of Kan extension allows us to regard $\mathrm{IndCoh}(-)$ as a functor:

$$\mathrm{IndCoh} : \mathrm{PStk}_{\mathrm{laft}\,/S} \to \mathrm{DGCat}_{\mathrm{cont}}\,.$$

In particular, a morphism $f : \mathcal{X} \to \mathcal{Y}$ of laft prestacks gives rise to the functor of !-pullback: $f^! : \mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{IndCoh}(\mathcal{X})$.

3.5.3. Note that if $f : \mathcal{X} \to \mathcal{Y}$ is an inf-schematic nil-isomorphism, then the functor $f^!$ is conservative ([18, III.3, Proposition 3.1.2]). It furthermore has a left adjoint f_*^{IndCoh} and the pair $(f_*^{\mathrm{IndCoh}}, f^!)$ is monadic. One deduces from this a descent property (see Proposition 3.3.3 of *op.cit.*):

Proposition 3.13 *Let $\mathcal{X}^\bullet_{\mathcal{Y}}$ be the Čech nerve of an inf-schematic nil-isomorphism $f : \mathcal{X} \to \mathcal{Y}$. Then the canonical functor:*

$$\mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{Tot}(\mathrm{IndCoh}(\mathcal{X}^\bullet_{\mathcal{Y}})) \tag{3.8}$$

is an equivalence. □

3.5.4. The DG category of *modules* over an object $\mathcal{Y}^\flat \in \mathrm{FMod}_{/S}(\mathcal{Y})$ is defined as $\mathrm{IndCoh}(\mathcal{Y}^\flat)$. Note that $\mathrm{IndCoh}(\mathcal{Y}^\flat)$ is a module object over $\mathrm{QCoh}(S)$. By the above discussion, there is a conservative functor **oblv** : $\mathrm{IndCoh}(\mathcal{Y}^\flat) \to \mathrm{IndCoh}(\mathcal{Y})$ given by !-pullback along $\mathcal{Y} \to \mathcal{Y}^\flat$. Furthermore, Proposition 3.13 provides an equivalence of categories:

$$\mathrm{IndCoh}(\mathcal{Y}^\flat) \xrightarrow{\sim} \mathrm{Tot}(\mathrm{IndCoh}(\mathcal{R}^\bullet)). \tag{3.9}$$

whenever $\mathcal{Y}^\flat = \mathrm{B}_{\mathcal{Y}}(\mathcal{R}^\bullet)$.

3.5.5. Given $\mathcal{Y}^\flat \in (\mathrm{PStk}_{\text{laft-def}})_{\mathcal{Y}//S}$, we can associate the relative tangent complex $\mathbb{T}_{\mathcal{Y}/\mathcal{Y}^\flat}$ which is in general an object of $\mathrm{IndCoh}(\mathcal{Y})$. (Since the cotangent complex naturally lives in the pro-category of quasi-coherent sheaves, the tangent complex is naturally an ind-coherent sheaf by a version of Serre duality, see [18, III.1, §4.4] for details). The following result is [18, IV.4, Theorem 9.1.5]:

Theorem 3.14 *Suppose Y is a finite type scheme over S. We have a fully faithful functor:*

$$\mathrm{LieAlgd}_{/S}(Y) \hookrightarrow \mathrm{FGpd}_{/S}(Y), \tag{3.10}$$

whose essential image consists of those formal groupoids $\mathcal{R}^\bullet$ such that $\mathbb{T}_{Y/\mathrm{B}_Y(\mathcal{R}^\bullet)}$ lies in the essential image of $\mathrm{QCoh}(Y)^\heartsuit$ under $\Upsilon_{Y/S}$. □

Composing (3.10) with B_Y, we obtain a fully faithful functor

$$\mathrm{LieAlgd}_{/S}(Y) \hookrightarrow \mathrm{FMod}_{/S}(Y), \tag{3.11}$$

whose essential image consists of those formal moduli problems $\mathcal{Y}^\flat \in \mathrm{FMod}_{/S}(Y)$ such that $\mathbb{T}_{Y/\mathcal{Y}^\flat}$ lies in $\Upsilon_Y(\mathrm{QCoh}(Y)^\heartsuit)$. Furthermore, given a smooth morphism $\pi : Y' \to Y$ of finite type schemes over S, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{LieAlgd}_{/S}(Y) & \xrightarrow{\pi^!_{\mathrm{LieAlgd}}} & \mathrm{LieAlgd}_{/S}(Y') \\ \downarrow{\scriptstyle (3.11)} & & \downarrow{\scriptstyle (3.11)} \\ \mathrm{FMod}_{/S}(Y) & \xrightarrow{\pi^!_{\mathrm{FMod}}} & \mathrm{FMod}_{/S}(Y') \end{array} \tag{3.12}$$

where $\pi^!_{\mathrm{LieAlgd}}$ is the pullback of Lie algebroids (as defined in [2]), and $\pi^!_{\mathrm{FMod}}$ is the functor described in §3.4.1.

In what follows, we will frequently use the fact that $\pi^!_{\mathrm{LieAlgd}}(\mathcal{L})$ has underlying $\mathcal{O}_{Y'}$-module given by $\pi^*\mathcal{L} \underset{\pi^*\mathcal{T}_{Y/S}}{\times} \mathcal{T}_{Y'/S}$.

Notation 3.15 We shall refer to the image $\mathcal{Y}^\flat$ of a Lie algebroid $\mathcal{L}$ under (3.11) as the formal moduli problem *associated* to $\mathcal{L}$, and denote it by $\mathcal{Y}^\flat := \mathcal{L}_{\mathbf{F}}$.

Note that when $Y \to S$ is smooth, $\mathrm{IndCoh}(\mathcal{Y}^\flat)$ is identified with the DG category of complexes of (quasi-coherent) $\mathcal{L}$-modules.

3.6 Quasi-twistings

3.6.1. Let $\mathcal{Y} \in \mathrm{PStk}_{\text{laft-def}\,/S}$. We use $\widehat{\mathbb{G}}_m$ to denote the formal completion of $\mathbb{G}_m$ at identity. It is a group formal scheme.

Definition 3.16 A *quasi-twisting* $\mathcal{T}$ over $\mathcal{Y}$ consists of the following data:

(a) an object $\mathcal{Y}^\flat \in \mathrm{FMod}_{/S}(\mathcal{Y})$;
(b) a $\widehat{\mathbb{G}}_m$-gerbe $\widehat{\mathcal{Y}}^\flat$ over $\mathcal{Y}^\flat$;
(c) a trivialization of the pullback of $\widehat{\mathcal{Y}}^\flat$ along $\mathcal{Y} \to \mathcal{Y}^\flat$.

We say that $\mathcal{T}$ is *based* at the formal moduli problem $\mathcal{Y}^\flat$.

Remark 3.17 For an abelian group prestack A over S, the notion of an A-gerbe here is taken in the naïve sense: the prestack $\mathrm{B}^2 A$ classifies A-gerbes (on an affine S-scheme) that are globally nonempty, and an A-gerbe on a prestack $\mathcal{Y}$ is an object of

$$\mathrm{Ge}_A(\mathcal{Y}) := \lim_{T \to \mathcal{Y}} \mathrm{Maps}(T, \mathrm{B}^2 A),$$

where T ranges through affine S-schemes mapping to $\mathcal{Y}$. (Informally, an A-gerbe is a torsor for the classifying prestack $\mathrm{B}\,A$). We will later show that using étale locally

trivial $\widehat{\mathbb{G}}_m$-gerbes in the definition of a quasi-twisting produces the same class of objects.

Remark 3.18 Alternatively, one can think of a quasi-twisting $\mathcal{T}$ as consisting of two formal moduli problems $\widehat{\mathcal{Y}}^\flat \to \mathcal{Y}^\flat$ under $\mathcal{Y}$, equipped with the structure of a $\widehat{\mathbb{G}}_m$-gerbe.

3.6.2. The ∞-groupoid of quasi-twistings $\mathcal{T}$ based at $\mathcal{Y}^\flat$ can be defined as a fiber of ∞-groupoids:

$$\mathrm{QTw}_{/S}(\mathcal{Y}/\mathcal{Y}^\flat) := \mathrm{Fib}(\mathrm{Ge}_{\widehat{\mathbb{G}}_m}(\mathcal{Y}^\flat) \to \mathrm{Ge}_{\widehat{\mathbb{G}}_m}(\mathcal{Y})).$$

More generally, we use $\mathrm{QTw}^A_{/S}(\mathcal{Y}/\mathcal{Y}^\flat)$ to denote an analogously defined category, with the abelian group prestack A acting as the structure group instead of $\widehat{\mathbb{G}}_m$.

3.6.3. We now show that quasi-twistings can be defined using different structure groups. The same results about twistings are obtained in [17].

Lemma 3.19 *The functor of inducing an A-gerbe from an $A_{\{\widehat{1}\}}$-gerbe gives rise to an equivalence of categories* $\mathrm{QTw}^{A_{\{\widehat{1}\}}}_{/S}(\mathcal{Y}/\mathcal{Y}^\flat) \xrightarrow{\sim} \mathrm{QTw}^A_{/S}(\mathcal{Y}/\mathcal{Y}^\flat)$.

Proof In light of the exact sequence (3.4), an inverse functor exists if the induced $A_{\mathrm{dR}/S}$-gerbe of any object in $\mathrm{QTw}^A_{/S}(\mathcal{Y}/\mathcal{Y}^\flat)$ is canonically trivialized. Indeed, let $\widehat{\mathcal{Y}}^\flat_{A_{\mathrm{dR}/S}}$ be the $A_{\mathrm{dR}/S}$-gerbe over $\mathcal{Y}^\flat$ induced from some A-gerbe $\widehat{\mathcal{Y}}^\flat_A$. Clearly, there is an identification between $\widehat{\mathcal{Y}}^\flat_{A_{\mathrm{dR}/S}}$ and the formal completion of $\widehat{\mathcal{Y}}^\flat_A$ inside $\mathcal{Y}^\flat$, i.e., $\widehat{\mathcal{Y}}^\flat_{A_{\mathrm{dR}}} \xrightarrow{\sim} (\widehat{\mathcal{Y}}^\flat_A)_{\mathrm{dR}/S} \underset{\mathcal{Y}_{\mathrm{dR}/S}}{\times} \mathcal{Y}^\flat$ (c.f. Example 3.9).

Therefore, a section of the $A_{\mathrm{dR}/S}$-gerbe $\widehat{\mathcal{Y}}^\flat_{A_{\mathrm{dR}/S}}$ amounts to filling in the dotted arrow

$$\begin{array}{ccc} \widehat{\mathcal{Y}}^\flat_{A_{\mathrm{dR}/S}} & \longrightarrow & (\widehat{\mathcal{Y}}^\flat_A)_{\mathrm{dR}/S} \\ \downarrow & \nearrow & \downarrow \\ \mathcal{Y}^\flat & \longrightarrow & \mathcal{Y}_{\mathrm{dR}/S} \end{array}$$

making the lower-right triangle commute. However, the structure of a quasi-twisting on $\widehat{\mathcal{Y}}^\flat_A$ supplies a section $\mathcal{Y} \to \widehat{\mathcal{Y}}^\flat_A$ over $\mathcal{Y}^\flat$. Hence we obtain a map $\mathcal{Y}^\flat \to \mathcal{Y}_{\mathrm{dR}/S} \to (\widehat{\mathcal{Y}}^\flat_A)_{\mathrm{dR}/S}$ over $\mathcal{Y}_{\mathrm{dR}/S}$. □

It follows from Lemma 3.19 that the following functors are equivalences:

$$\mathrm{QTw}^{\mathbb{G}_m}_{/S}(\mathcal{Y}/\mathcal{Y}^\flat) \xleftarrow{\sim} \mathrm{QTw}_{/S}(\mathcal{Y}/\mathcal{Y}^\flat) \xrightarrow{\sim} \mathrm{QTw}^{\widehat{\mathbb{G}}_a}_{/S}(\mathcal{Y}/\mathcal{Y}^\flat) \xrightarrow{\sim} \mathrm{QTw}^{\mathbb{G}_a}_{/S}(\mathcal{Y}/\mathcal{Y}^\flat). \tag{3.13}$$

Let $\mathrm{QTw}^{\text{ét}}_{/S}(\mathcal{Y}/\mathcal{Y}^\flat)$ denote the ∞-groupoid of étale locally trivial $\widehat{\mathbb{G}}_m$-gerbes over $\mathcal{Y}^\flat$, equipped with a section over $\mathcal{Y}$.

Corollary 3.20 *The tautological functor* $\mathrm{QTw}_{/S}(\mathcal{Y}/\mathcal{Y}^\flat) \to \mathrm{QTw}^{\text{ét}}_{/S}(\mathcal{Y}/\mathcal{Y}^\flat)$ *is an equivalence.*

Proof We use the $\mathbb{G}_a$-incarnation of quasi-twistings, as well as their counterparts defined by étale locally trivial gerbes (see Lemma 3.19). For an affine S-scheme T, there holds

$$\mathrm{H}^1_{\text{ét}}(T, \mathbb{G}_a) = 0, \quad \mathrm{H}^2_{\text{ét}}(T, \mathbb{G}_a) = 0.$$

Let $\mathrm{B}^2_{\text{ét}}\,\mathbb{G}_a$ denote the étale sheafification of $\mathrm{B}^2\,\mathbb{G}_a$. Thus, it classifies étale locally trivial $\mathbb{G}_a$-gerbes. The above vanishing statements show that the canonical map $\mathrm{B}^2\,\mathbb{G}_a \to \mathrm{B}^2_{\text{ét}}\,\mathbb{G}_a$ is an isomorphism. It follows that the corresponding notions of quasi-twistings are also equivalent. □

3.7 Modules over a quasi-twisting

3.7.1. We continue to assume that $\mathcal{Y} \in \mathrm{PStk}_{\text{laft-def}\,/S}$ and $\mathcal{T}$ is a quasi-twisting over $\mathcal{Y}$. Our goal now is to define $\mathcal{T}$-Mod as a DG category tensored over $\mathrm{QCoh}(S)$ (i.e., it is a module object over $\mathrm{QCoh}(S)$, see Sect. 3.3.1). We first proceed more generally and define ind-coherent sheaves "twisted" by a $\widehat{\mathbb{G}}_m$-gerbe.

The discussion below applies also to $\mathbb{G}_m$-gerbes, where alternative definitions of the twisted category exist (for example, the category denoted $D^b(\widetilde{\mathcal{Y}})_1$ of [5, Sect. 2.1]). In fact, these notions agree after inducing a $\widehat{\mathbb{G}}_m$-gerbe along the map of structure groups $\widehat{\mathbb{G}}_m \to \mathbb{G}_m$. We choose to present the construction in terms of $\widehat{\mathbb{G}}_m$-gerbes since our theory uses only nil-isomorphisms.

3.7.2. Let $\mathcal{Z} \in \mathrm{PStk}_{\text{laft-def}\,/S}$, and $\widehat{\mathcal{Z}}$ be a $\widehat{\mathbb{G}}_m$-gerbe over $\mathcal{Z}$. Consider the canonical action of $\mathrm{B}\,\mathbb{G}_m$ on Vect, which induces an action of $\mathrm{B}\,\widehat{\mathbb{G}}_m$ (see [4, Sects. 1–2] for notions pertaining to group actions on DG categories. Informally, the $\mathrm{B}\,\mathbb{G}_m$-action on Vect is given by tensoring a vector space with a line). Formally, Vect can be regarded as a co-module object in $\mathrm{DGCat}_{\text{cont}}$ over the co-algebra $(\mathrm{IndCoh}(\mathrm{B}\,\widehat{\mathbb{G}}_m), m^!)$, where m is the multiplication map on $\mathrm{B}\,\widehat{\mathbb{G}}_m$. The co-action

$$\mathrm{Vect} \to \mathrm{Vect} \otimes \mathrm{IndCoh}(\mathrm{B}\,\widehat{\mathbb{G}}_m) \xrightarrow{\sim} \mathrm{IndCoh}(\mathrm{B}\,\widehat{\mathbb{G}}_m)$$

is specified by $\chi \in \mathrm{IndCoh}(\mathrm{B}\,\widehat{\mathbb{G}}_m)$, the character sheaf induced from the map $\mathrm{B}\,\widehat{\mathbb{G}}_m \to \mathrm{B}\,\mathbb{G}_m$.

Note that $\mathrm{IndCoh}(\widehat{\mathcal{Z}})$ admits a $\mathrm{B}\,\widehat{\mathbb{G}}_m$-action, so the product $\mathrm{IndCoh}(\widehat{\mathcal{Z}}) \otimes \mathrm{Vect}$ is again acted on by $\mathrm{B}\,\widehat{\mathbb{G}}_m$. The corresponding co-simplicial system $\{\mathrm{IndCoh}(\widehat{\mathcal{Z}} \times \mathrm{B}\,\widehat{\mathbb{G}}_m^{\times n})\}_{[n]\in\Delta}$ has the following first few terms:

$$\cdots \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow\\ \longleftarrow}\; \mathrm{IndCoh}(\widehat{\mathcal{Z}} \times \mathrm{B}\,\widehat{\mathbb{G}}_m^{\times 2}) \;\substack{\xleftarrow{(\mathrm{act}\times 1)^!}\\ \xleftarrow{(1\times m)^!}\\ \xleftarrow[\mathrm{pr}_{12}^! \otimes \chi]{}}\; \mathrm{IndCoh}(\widehat{\mathcal{Z}} \times \mathrm{B}\,\widehat{\mathbb{G}}_m) \;\substack{\xleftarrow{\mathrm{act}^!}\\ \xleftarrow[\mathrm{pr}_1^! \otimes \chi]{}}\; \mathrm{IndCoh}(\widehat{\mathcal{Z}}). \tag{3.14}$$

We define the DG category $\mathrm{IndCoh}(\mathcal{Z})_{\widehat{\mathcal{Z}}}$ of $\widehat{\mathcal{Z}}$-*twisted* ind-coherent sheaves on $\mathcal{Z}$ as the totalization of the above co-simplicial system. One sees immediately that $\mathrm{IndCoh}(\mathcal{Z})_{\widehat{\mathcal{Z}}}$ is tensored over $\mathrm{QCoh}(S)$.

3.7.3. Since the functors associated to each face map $[n] \to [m]$ all admit left adjoints, we obtain:

$$\mathrm{IndCoh}(\mathcal{Z})_{\widehat{\mathcal{Z}}} = \lim_{[n]\in\Delta} \mathrm{IndCoh}(\widehat{\mathcal{Z}} \times \mathrm{B}\,\widehat{\mathbb{G}}_m^{\times n}) \xrightarrow{\sim} \operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} \mathrm{IndCoh}(\widehat{\mathcal{Z}} \times \mathrm{B}\,\widehat{\mathbb{G}}_m^{\times n}),$$

where we use the left adjoints to form the colimit. Here, the colimit is taken in $\mathrm{DGCat}_{\mathrm{cont}}$ (the forgetful functor from $\mathrm{DGCat}_{\mathrm{cont}}$ to plain ∞-categories does not commute with colimits).

Remark 3.21 Note that any (global) trivialization of the gerbe $\widehat{\mathcal{Z}} \to \mathcal{Z}$ gives rise to an equivalence $\mathrm{IndCoh}(\mathcal{Z})_{\widehat{\mathcal{Z}}} \xrightarrow{\sim} \mathrm{IndCoh}(\mathcal{Z})$.

Remark 3.22 In [16, Sect. 1.7], a definition of a twisted presheaf of DG categories is given. We relate their definition to ours. For the presheaf over $\mathcal{Z}$:

$$\mathrm{IndCoh}_{/\mathcal{Z}} : (\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Z}})^{\mathrm{op}} \ni S \rightsquigarrow \mathrm{IndCoh}(S)$$

and a $\widehat{\mathbb{G}}_m$-gerbe $\widehat{\mathcal{Z}}$, the *twisted sheaf of DG categories* $(\mathrm{IndCoh}_{/\mathcal{Z}})_{\widehat{\mathcal{Z}}}$ is defined by

(a) specifying its values on the category $\mathrm{Split}(\widehat{\mathcal{Z}})$ of affine DG schemes $S \to \mathcal{Z}$ equipped with a lift to $\widehat{\mathcal{Z}}$, using the canonical $\mathrm{Maps}(S, \mathrm{B}\,\widehat{\mathbb{G}}_m)$-action on $\mathrm{IndCoh}(S)$; and then
(b) applying h-descent[10] along the basis $\mathrm{Split}(\widehat{\mathcal{Z}}) \to \mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Z}}$ to obtain a sheaf (in the h-topology) over $\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Z}}$, denoted by $(\mathrm{IndCoh}_{/\mathcal{Z}})_{\widehat{\mathcal{Z}}}$.

Thus we may calculate the global section $\Gamma(\mathcal{Z}, (\mathrm{IndCoh}_{/\mathcal{Z}})_{\widehat{\mathcal{Z}}})$ by the covering $\widehat{\mathcal{Z}} \to \mathcal{Z}$. The resulting co-simplicial system is identified with (3.14). Hence the definition of $\widehat{\mathcal{Z}}$-twisted ind-coherent sheaves in [16, Sect. 1.7] (adjusted to the h-topology) agrees with ours.

3.7.4. Let $\mathcal{T}$ be a quasi-twisting over $\mathcal{Y}$, represented by the $\widehat{\mathbb{G}}_m$-gerbe $\widehat{\mathcal{Y}}^\flat \to \mathcal{Y}^\flat$. We denote by $\widehat{\mathcal{Y}}$ the $\widehat{\mathbb{G}}_m$-gerbe over $\mathcal{Y}$ pulled back along $\mathcal{Y} \to \mathcal{Y}^\flat$; it is equipped with a canonical trivialization.

We define the DG category of $\mathcal{T}$-modules by: $\mathcal{T}\text{-Mod} := \mathrm{IndCoh}(\mathcal{Y}^\flat)_{\widehat{\mathcal{Y}}^\flat}$. There is a canonical functor:

$$\mathbf{oblv}_{\mathcal{T}} : \mathcal{T}\text{-Mod} \to \mathrm{IndCoh}(\mathcal{Y})_{\widehat{\mathcal{Y}}} \xrightarrow{\sim} \mathrm{IndCoh}(\mathcal{Y}),$$

since $\widehat{\mathcal{Y}}^\flat$ is trivialized over $\mathcal{Y}$, and Remark 3.21 identifies the corresponding twisted category with $\mathrm{IndCoh}(\mathcal{Y})$.

[10] The authors of [16] work with the étale topology instead.

Proposition 3.23 *The functor* $\mathbf{oblv}_{\mathcal{T}}$ *admits a left adjoint* $\mathbf{ind}_{\mathcal{T}}$*, and the pair of functors* $(\mathbf{ind}_{\mathcal{T}}, \mathbf{oblv}_{\mathcal{T}})$ *is monadic.*

Proof The functor $\mathbf{oblv}_{\mathcal{T}}$ is by definition the totalization of the !-pullback functors:

$$(\pi^{(n)})^! : \mathrm{IndCoh}(\widehat{\mathcal{Y}}^\flat \times \mathrm{B}\,\widehat{\mathbb{G}}_m^{\times n}) \to \mathrm{IndCoh}(\widehat{\mathcal{Y}} \times \mathrm{B}\,\widehat{\mathbb{G}}_m^{\times n}),$$

where $\pi^{(n)}$ denotes the morphism $\widehat{\mathcal{Y}} \times \mathrm{B}\,\widehat{\mathbb{G}}_m^{\times n} \to \widehat{\mathcal{Y}}^\flat \times \mathrm{B}\,\widehat{\mathbb{G}}_m^{\times n}$. Each $(\pi^{(n)})^!$ admits a left adjoint $\pi^{(n)}_{*,\mathrm{IndCoh}}$. Furthermore, the diagram induced from an arbitrary face map:

$$\begin{array}{ccc} \mathrm{IndCoh}(\widehat{\mathcal{Y}} \times \mathrm{B}\,\widehat{\mathbb{G}}_m^{\times n}) & \longrightarrow & \mathrm{IndCoh}(\widehat{\mathcal{Y}} \times \mathrm{B}\,\widehat{\mathbb{G}}_m^{\times m}) \\ \downarrow \pi^{(n)}_{*,\mathrm{IndCoh}} & & \downarrow \pi^{(m)}_{*,\mathrm{IndCoh}} \\ \mathrm{IndCoh}(\widehat{\mathcal{Y}}^\flat \times \mathrm{B}\,\widehat{\mathbb{G}}_m^{\times n}) & \to & \mathrm{IndCoh}(\widehat{\mathcal{Y}}^\flat \times \mathrm{B}\,\widehat{\mathbb{G}}_m^{\times m}) \end{array}$$

which *a priori* commutes up to a natural transformation, actually commutes. Hence $\mathbf{oblv}_{\mathcal{T}}$ admits a left adjoint $\mathbf{ind}_{\mathcal{T}} := \mathrm{Tot}(\pi^{(n)}_{*,\mathrm{IndCoh}})$. We now prove:

(a) $\mathbf{oblv}_{\mathcal{T}}$ is conservative; this is because all other arrows in the following commutative diagram:

$$\begin{array}{ccc} \mathrm{IndCoh}(\mathcal{Y}^\flat)_{\widehat{\mathcal{Y}}^\flat} & \xrightarrow{\mathbf{oblv}_{\mathcal{T}}} & \mathrm{IndCoh}(\mathcal{Y})_{\widehat{\mathcal{Y}}} \\ \downarrow \mathrm{ev}^0 & & \downarrow \mathrm{ev}^0 \\ \mathrm{IndCoh}(\mathcal{Y}^\flat) & \xrightarrow{(\pi^{(0)})^!} & \mathrm{IndCoh}(\mathcal{Y}) \end{array}$$

are conservative, hence so is $\mathbf{oblv}_{\mathcal{T}}$.

(b) $\mathbf{oblv}_{\mathcal{T}}$ preserves colimits; this is obvious as we work in $\mathrm{DGCat}_{\mathrm{cont}}$.

It follows that that the pair $(\mathbf{ind}_{\mathcal{T}}, \mathbf{oblv}_{\mathcal{T}})$ is monadic, by the Barr-Beck-Lurie theorem. □

3.7.5. Using Proposition 3.23, we may regard $\mathrm{U}(\mathcal{T}) := \mathbf{oblv}_{\mathcal{T}} \circ \mathbf{ind}_{\mathcal{T}}$ as an algebra object in $\mathrm{End}(\mathrm{IndCoh}(\mathcal{Y}))$, and the DG category $\mathcal{T}$-Mod identifies with that of $\mathrm{U}(\mathcal{T})$-module objects in $\mathrm{IndCoh}(\mathcal{Y})$. We call $\mathrm{U}(\mathcal{T})$ the *universal envelope* of $\mathcal{T}$.

3.8 Comparison with the classical notion

3.8.1. Suppose Y is a (classical) scheme of finite type over S. Let $\mathcal{L}$ be a classical Lie algebroid over Y and $\mathcal{Y}^\flat \in \mathrm{FMod}_{/S}(Y)$ be the formal moduli problem associated to $\mathcal{L}$, under the embedding (3.11). The goal of this subsection is to show that quasi-twistings based at $\mathcal{Y}^\flat$ are equivalent to classical quasi-twistings based at $\mathcal{L}$.

3.8.2. Given a formal moduli problem $\widehat{\mathcal{Y}}^\flat \to \mathcal{Y}^\flat$ such that $\mathbb{T}_{Y/\widehat{\mathcal{Y}}^\flat} \in \Upsilon_Y(\mathrm{QCoh}(Y)^\heartsuit)$, one can functorially assign a classical Lie algebroid $\widehat{\mathcal{L}}$ equipped with a map $\widehat{\mathcal{L}} \to \mathcal{L}$. Furthermore, a morphism $\widehat{\mathcal{Y}}^\flat \times \mathrm{B}\,\widehat{\mathbb{G}}_m \to \widehat{\mathcal{Y}}^\flat$ in $\mathrm{FMod}_{/S}(Y)$ induces a map

$$\widehat{\mathcal{L}} \oplus \mathcal{O}_Y \to \widehat{\mathcal{L}}, \quad (l, f) \rightsquigarrow l + f\mathbf{1} \tag{3.15}$$

where $\mathbf{1}$ is the image of $(0, f)$ in $\widehat{\mathcal{L}}$. If the morphism $\widehat{\mathcal{Y}}^\flat \times \mathrm{B}\,\widehat{\mathbb{G}}_m \to \widehat{\mathcal{Y}}^\flat$ realizes $\widehat{\mathcal{Y}}^\flat$ as a $\widehat{\mathbb{G}}_m$-gerbe over $\mathcal{Y}^\flat$, then we see that $\mathcal{O}_Y \to \widehat{\mathcal{L}}$, $f \rightsquigarrow f\mathbf{1}$ is the kernel of the canonical map $\widehat{\mathcal{L}} \to \mathcal{L}$. The fact that (3.15) preserves Lie bracket then implies $\mathcal{O}_Y$ is central inside $\widehat{\mathcal{L}}$. In other words, the map $\widehat{\mathcal{L}} \to \mathcal{L}$ is a central extension of classical Lie algebroids.

3.8.3. Now, given any object in $\mathrm{QTw}_{/S}(Y/\mathcal{Y}^\flat)$, we claim that the corresponding formal moduli problem $\widehat{\mathcal{Y}}^\flat$ satisfies the property that $\mathbb{T}_{Y/\widehat{\mathcal{Y}}^\flat}$ lies in $\Upsilon_Y(\mathrm{QCoh}(Y)^\heartsuit)$. Indeed, we have a canonical triangle in $\mathrm{IndCoh}(Y)$:

$$\omega_Y \cong \mathbb{T}_{\widehat{\mathcal{Y}}^\flat/\mathcal{Y}^\flat}\big|_Y \to \mathbb{T}_{Y/\widehat{\mathcal{Y}}^\flat} \to \mathbb{T}_{Y/\mathcal{Y}^\flat}$$

and the outer terms lie in the essential image of $\mathrm{QCoh}(Y)^\heartsuit$. Hence the previous discussion shows that we have a functor:

$$\mathrm{QTw}_{/S}(Y/\mathcal{Y}^\flat) \to \mathrm{QTw}^{\mathrm{cl}}_{/S}(Y/\mathcal{L}). \tag{3.16}$$

Proposition 3.24 *The functor (3.16) is an equivalence of categories.*

In particular, the ∞-category $\mathrm{QTw}_{/S}(Y/\mathcal{Y}^\flat)$ is an ordinary category.

Proof We explicitly construct the functor inverse to (3.16). Namely, given a central extension $\widehat{\mathcal{L}}$ of $\mathcal{L}$, we need to equip its corresponding formal moduli problem $\widehat{\mathcal{Y}}^\flat$ with the structure of a $\widehat{\mathbb{G}}_m$-gerbe over $\mathcal{Y}^\flat$. As before, the action map $\widehat{\mathcal{Y}}^\flat \times \mathrm{B}\,\widehat{\mathbb{G}}_m \to \widehat{\mathcal{Y}}^\flat$ arises from the morphism of classical Lie algebroids over Y:

$$\widehat{\mathcal{L}} \oplus \mathcal{O}_Y \to \widehat{\mathcal{L}}, \quad (l, f) \rightsquigarrow l + f\mathbf{1}.$$

The morphism induced by action and projection $\widehat{\mathcal{Y}}^\flat \times \mathrm{B}\,\widehat{\mathbb{G}}_m \to \widehat{\mathcal{Y}}^\flat \underset{\mathcal{Y}^\flat}{\times} \widehat{\mathcal{Y}}^\flat$ is an isomorphism since the same holds for the corresponding map of classical Lie algebroids:

$$\widehat{\mathcal{L}} \oplus \mathcal{O}_Y \to \widehat{\mathcal{L}} \underset{\mathcal{L}}{\times} \widehat{\mathcal{L}}, \quad (l, f) \rightsquigarrow (l + f\mathbf{1}, l).$$

It remains to show that $\widehat{\mathcal{Y}}^\flat \to \mathcal{Y}^\flat$ admits a section over any affine DG scheme T mapping to $\mathcal{Y}^\flat$. We shall deduce the existence of this section from the following claim:

Claim 3.25 The morphism $\widehat{\mathcal{Y}}^\flat \to \mathcal{Y}^\flat$ is formally smooth.

Indeed, let T be any affine DG scheme with a morphism $\widehat{y} : T \to \widehat{\mathcal{Y}}^\flat$. By the criterion of formal smoothness (3.6), we ought to show $\mathrm{Maps}(\mathcal{T}^*_{\widehat{\mathcal{Y}}^\flat/\mathcal{Y}^\flat}\big|_{\widehat{y}}, \mathcal{F}) \in \mathrm{Vect}^{\leq 0}$ for all $\mathcal{F} \in \mathrm{QCoh}(T)^\heartsuit$. The Cartesian square:

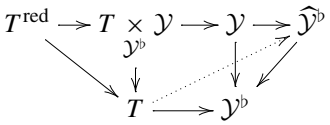
$$\begin{array}{ccc} T & \xrightarrow{(\widehat{y},\widehat{y})} & \widehat{\mathcal{Y}}^\flat \underset{\mathcal{Y}^\flat}{\times} \widehat{\mathcal{Y}}^\flat \to \widehat{\mathcal{Y}}^\flat \\ & & \downarrow \qquad\quad \downarrow \\ & & \widehat{\mathcal{Y}}^\flat \longrightarrow \mathcal{Y}^\flat \end{array}$$

together with the isomorphism above gives:

$$\mathcal{T}^*_{\widehat{\mathcal{Y}}^\flat/\mathcal{Y}^\flat}\big|_{\widehat{y}} \xrightarrow{\sim} \mathcal{T}^*_{\widehat{\mathcal{Y}}^\flat \underset{\mathcal{Y}^\flat}{\times} \widehat{\mathcal{Y}}^\flat/\widehat{\mathcal{Y}}^\flat}\big|_{(\widehat{y},\widehat{y})} \xrightarrow{\sim} \mathcal{T}^*_{\widehat{\mathcal{Y}}^\flat \times \mathrm{B}\,\widehat{\mathbb{G}}_m/\widehat{\mathcal{Y}}^\flat}\big|_{(\widehat{y},1)} \xrightarrow{\sim} \mathcal{O}_T[-1].$$

One deduces from this the required degree estimate.

Using the claim, we will construct a section of $\widehat{\mathcal{Y}}^\flat \to \mathcal{Y}^\flat$ over $T \to \mathcal{Y}^\flat$ as follows. First consider the fiber product $T \underset{\mathcal{Y}^\flat}{\times} \mathcal{Y}$, which is equipped with a nil-isomorphism to T. We obtain a solid commutative diagram:



Formal smoothness now implies the existence of the dotted arrow. □

Remark 3.26 By letting $\mathcal{L} = \mathcal{T}_{Y/S}$ be the tangent Lie algebroid, we obtain from Proposition 3.24 the fact that Picard algebroids identify with twistings on classical schemes locally of finite type. The same result is established in [17, Sect. 6.5] using a computation involving de Rham cohomology.

4 How to take quotient of a Lie algebroid?

This section is devoted to the study of quotients of Lie algebroids, in both classical and DG settings. The set-up involves an H-torsor $Y \to Z$ and a Lie algebroid $\mathcal{L}$ over Y. With additional data on $\mathcal{L}$, there exists a *quotient Lie algebroid* over Z. The quotient procedure we shall describe takes as input a map $\eta : \mathfrak{k} \otimes \mathcal{O}_Y \to \mathcal{L}$, where $\mathfrak{k}$ is an arbitrary Lie algebra. It generalizes two existing notions—*weak* and *strong* quotients—both considered by Beilinson and Bernstein [2]. For technical reasons involving ∞-type schemes, we shall construct two quotient functors:

(a) $\mathbf{Q}^{(\mathfrak{k},H)}_{\mathrm{inj}}$, which is a classical procedure that works in the case where η is injective;
(b) $\mathbf{Q}^{(H,H^\flat)}$, which is its geometric counterpart for Y locally of finite type,

and we check that they agree in overlapping cases. A geometric procedure that works in full generality should exist as soon as the theory in [18] is extended to ∞-type situations.

Throughout this section, we work over an affine scheme S smooth over k.

4.1 ($\mathfrak{k}$, H)-Lie algebroids

4.1.1. We describe the necessary data for taking quotients of Lie algebroids.

Definition 4.1 A *classical action pair* $(\mathfrak{k}, H)$ consists of a flat affine group scheme H over S, an $\mathcal{O}_S$-linear Lie algebra $\mathfrak{k}$ acted on by H, as well as a morphism of Lie algebras:

$$\mathfrak{k} \to \mathfrak{h} := \mathrm{Lie}(H) \tag{4.1}$$

with the following properties:

(a) (4.1) is H-equivariant, where $\mathfrak{h}$ is equipped with the adjoint H-action;
(b) the $\mathfrak{k}$-action on itself induced from (4.1) is the adjoint action.

Remark 4.2 This datum is superficially similar to that of a Harish-Chandra pair, but they serve very different purposes.

Example 4.3 Fix an S-point $\mathfrak{g}^\kappa$ of $\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*)$ (see Sect. 2). Then we have a classical action pair $(\mathfrak{g}^\kappa[\![t]\!], S \times G[\![t]\!])$, where the morphism (4.1) is induced from the projection $\mathfrak{g}^\kappa \to \mathfrak{g} \otimes \mathcal{O}_S$. All classical action pairs considered in this paper are variants of $(\mathfrak{g}^\kappa[\![t]\!], S \times G[\![t]\!])$. Note that the group scheme $S \times G[\![t]\!]$ is *not* of finite type.

4.1.2. The notion of a morphism $(\mathfrak{k}^0, H^0) \to (\mathfrak{k}, H)$ of classical action pairs is obvious. We say that $(\mathfrak{k}^0, H^0)$ is a *normal* subpair if $\mathfrak{k}^0 \hookrightarrow \mathfrak{k}$ is an ideal, $H^0 \hookrightarrow H$ is a normal subgroup, the H-action stabilizes $\mathfrak{k}^0$, and H^0 acts trivially on $\mathfrak{k}/\mathfrak{k}^0$. This definition means precisely that a normal subpair fits into an *exact sequence* (in the obvious sense):

$$1 \to (\mathfrak{k}^0, H^0) \to (\mathfrak{k}, H) \to (\mathfrak{k}_0, H_0) \to 1. \tag{4.2}$$

4.1.3. Let Y be a classical scheme over S equipped with an H-action. Recall that every H-equivariant $\mathcal{O}_Y$-module $\mathcal{F}$ admits an $\mathfrak{h}$-action by derivations. Specializing to $\mathcal{O}_Y$ itself, we obtain a canonical map:

$$\mathfrak{h} \otimes \mathcal{O}_Y \to \mathcal{T}_{Y/S}. \tag{4.3}$$

On the other hand, the $\mathcal{O}_Y$-module $\mathcal{T}_{Y/S}$ admits a canonical H-equivariance structure, given by pushforward of tangent vectors.

Definition 4.4 A $(\mathfrak{k}, H)$-*Lie algebroid* on Y consists of a Lie algebroid $\mathcal{L} \in \mathrm{LieAlgd}_{/S}(Y)$, an H-equivariance structure on the underlying $\mathcal{O}_Y$-module of $\mathcal{L}$, and a morphism $\eta : \mathfrak{k} \otimes \mathcal{O}_Y \to \mathcal{L}$ of H-equivariant $\mathcal{O}_Y$-modules, subject to the following conditions:

(a) the H-equivariance structure on $\mathcal{L}$ is compatible with its Lie bracket;
(b) the anchor map σ of $\mathcal{L}$ intertwines the H-equivariance structures on $\mathcal{L}$ and $\mathcal{T}_{Y/S}$;
(c) the following diagram is commutative:

$$\begin{array}{ccccc} & & \mathcal{L} & & \\ & \overset{\eta}{\nearrow} & & \overset{\sigma}{\searrow} & \\ \mathfrak{k} \otimes \mathcal{O}_Y & & & & \mathcal{T}_{Y/S} \\ & \underset{(4.1)}{\searrow} & & \underset{(4.3)}{\nearrow} & \\ & & \mathfrak{h} \otimes \mathcal{O}_Y & & \end{array} \tag{4.4}$$

(d) η is compatible with the Lie bracket on $\mathcal{L}$ in the following sense: given $\xi \in \mathfrak{k} \otimes \mathcal{O}_Y$ and $l \in \mathcal{L}$, there holds:

$$[\eta(\xi), l] = \xi_{\mathfrak{h}} \cdot l \in \mathcal{L} \tag{4.5}$$

where $\xi_{\mathfrak{h}}$ is the image of ξ in $\mathfrak{h} \otimes \mathcal{O}_Y$ along (4.1), and $\xi_{\mathfrak{h}} \cdot l$ denotes the action of $\xi_{\mathfrak{h}}$ on l coming from the equivariance structure.

We will frequently write a $(\mathfrak{k}, H)$-Lie algebroid as $(\mathcal{L}, \eta)$, in order to emphasize the dependence on η. The category of $(\mathfrak{k}, H)$-Lie algebroids on Y is denoted by $\mathrm{LieAlgd}_{/S}^{(\mathfrak{k},H)}(Y)$. Given another scheme Y' over S acted on by H and an H-equivariant morphism $Y' \to Y$, one can form the pullback of a $(\mathfrak{k}, H)$-Lie algebroid in a way compatible with the forgetful functor to plain Lie algebroids.

4.2 Quotient of Lie algebroids

4.2.1. We describe how to form the quotient of a $(\mathfrak{k}, H)$-Lie algebroid when the morphism η is *injective*. Denote the category of such $(\mathfrak{k}, H)$-Lie algebroid by $\mathrm{LieAlgd}_{\mathrm{inj}/S}^{(\mathfrak{k},H)}(Y)$.

4.2.2. Suppose Z is a scheme over S and Y is an H-torsor over Z. Since H is affine and flat, the projection $\pi : Y \to Z$ is an affine, faithfully flat cover (in particular, fpqc). We will define a *quotient* functor:

$$\mathbf{Q}_{\mathrm{inj}}^{(\mathfrak{k},H)} : \mathrm{LieAlgd}_{\mathrm{inj}/S}^{(\mathfrak{k},H)}(Y) \to \mathrm{LieAlgd}_{/S}(Z) \tag{4.6}$$

on each $(\mathcal{L}, \eta) \in \mathrm{LieAlgd}_{\mathrm{inj}}^{(\mathfrak{k},H)}(Y/S)$ by the following procedure:

(a) ($\mathcal{O}_Z$*-module and anchor map*) We have a morphism of H-equivariant $\mathcal{O}_Y$-modules:

$$\mathcal{L}/(\mathfrak{k} \otimes \mathcal{O}_Y) \to \mathcal{T}_{Y/S}/(\mathfrak{h} \otimes \mathcal{O}_Y) \xrightarrow{\sim} \pi^* \mathcal{T}_{Z/S}$$

by (4.4). Let $\mathcal{L}_0$ denote the fpqc descent of $\mathcal{L}/(\mathfrak{k} \otimes \mathcal{O}_Y)$ to Z, so we obtain a map of $\mathcal{O}_Z$-modules $\sigma_0 : \mathcal{L}_0 \to \mathcal{T}_{Z/S}$. The image of $(\mathcal{L}, \eta)$ under $\mathbf{Q}_{\mathrm{inj}}^{(\mathfrak{k},H)}$ is supposed to have underlying $\mathcal{O}_Z$-module $\mathcal{L}_0$ and anchor map σ_0.

(b) (*Lie bracket*) Since π is affine, it suffices to define an $\mathcal{O}_S$-linear Lie bracket on $\pi^{-1}\mathcal{L}_0$. Consider the embedding:

$$\pi^{-1}\mathcal{L}_0 \hookrightarrow \pi^*\mathcal{L}_0 \xrightarrow{\sim} \mathcal{L}/(\mathfrak{k} \otimes \mathcal{O}_Y).$$

The Lie bracket on $\mathcal{L}$ will induce one on $\pi^{-1}\mathcal{L}_0$ if $[\mathfrak{k} \otimes \mathcal{O}_Y, \pi^{-1}\mathcal{L}_0] = 0$ in $\mathcal{L}$. The latter identity is guaranteed by (4.5).

We omit checking that this procedure gives rise to a well-defined functor $\mathbf{Q}_{\mathrm{inj}}^{(\mathfrak{k},H)}$.

4.2.3. Given a *flat* morphism of schemes $f : Z' \to Z$, we set $Y' := Z' \underset{Z}{\times} Y$ which is an H-torsor over Z'. The map $\tilde{f} : Y' \to Y$ is H-equivariant, and the pullback of $(\mathcal{L}, \eta) \in \mathrm{LieAlgd}_{\mathrm{inj}/S}^{(\mathfrak{k},H)}(Y)$ along $\tilde{f}$ lies in $\mathrm{LieAlgd}_{\mathrm{inj}/S}^{(\mathfrak{k},H)}(Y')$. Furthermore, $\mathbf{Q}_{\mathrm{inj}}^{(\mathfrak{k},H)}$ is compatible with pullbacks along f and $\tilde{f}$.

Remark 4.5 Since Lie algebroids are smooth local objects (see [2]) and $\mathbf{Q}_{\mathrm{inj}}^{(\mathfrak{k},H)}$ is compatible with flat pullbacks, we may generalize $\mathbf{Q}_{\mathrm{inj}}^{(\mathfrak{k},H)}$ to the case where $\mathcal{Z} := Y/H$ is representable by an algebraic stack (i.e., smooth locally a scheme).

Remark 4.6 The special case where the classical action pair is given by $(\mathfrak{h}, H)$ with (4.1) being the identity map, has been studied in [2] under the name *strong quotient*. Note that when H acts freely on Y, the map η is automatically injective.

Example 4.7 Another instance of the functor (4.6) is the *weak quotient*. This is the case where $\mathfrak{k} = 0$. The only data needed in defining a $(0, H)$-Lie algebroid are a Lie algebroid $\mathcal{L} \in \mathrm{LieAlgd}_{/S}(Y)$, together with an H-equivariance structure on the underlying $\mathcal{O}_Y$-module of $\mathcal{L}$, subject to the first two conditions in Sect. 4.4.

Suppose Y/H is representable by an algebraic stack. Then the resulting quotient $\mathbf{Q}_{\mathrm{inj}}^{(0,H)}(\mathcal{L})$ has underlying $\mathcal{O}_{Y/H}$-module the descent of (the $\mathcal{O}_Y$-module) $\mathcal{L}$ along $Y \to Y/H$.

4.2.4. We now characterize the object $\mathbf{Q}_{\mathrm{inj}}^{(\mathfrak{k},H)}(\mathcal{L}) \in \mathrm{LieAlgd}_{/S}(Z)$ by a universal property. Consider an arbitrary Lie algebroid $\mathcal{M} \in \mathrm{LieAlgd}_{/S}(Z)$. We can equip $\pi^!_{\mathrm{LieAlgd}}\mathcal{M}$ with the structure of a $(\mathfrak{k}, H)$-Lie algebroid as follows:

(a) regarding $\pi^!_{\mathrm{LieAlgd}}\mathcal{M}$ as the $\mathcal{O}_Y$-module $\pi^*\mathcal{M} \underset{\pi^*\mathcal{T}_{Z/S}}{\times} \mathcal{T}_{Y/S}$, the H-equivariance structure is a combination of the natural H-equivariance structures on $\pi^*\mathcal{M}$ and $\mathcal{T}_{Y/S}$;

(b) the morphism $\eta : \mathfrak{k} \otimes \mathcal{O}_Y \to \pi^!_{\mathrm{LieAlgd}}(\mathcal{M})$ is a combination of the *zero* map $\mathfrak{k} \otimes \mathcal{O}_Y \to \pi^*\mathcal{M}$ and the composition $\mathfrak{k} \otimes \mathcal{O}_Y \to \mathfrak{h} \otimes \mathcal{O}_Y \to \mathcal{T}_{Y/S}$.

Note that $\pi^!_{\mathrm{LieAlgd}}\mathcal{M} \in \mathrm{LieAlgd}_{/S}^{(\mathfrak{k},H)}(Y)$ does *not* belong to $\mathrm{LieAlgd}_{\mathrm{inj}/S}^{(\mathfrak{k},H)}(Y)$ in general.

Proposition 4.8 *There is a natural bijection:*

$$\mathrm{Maps}_{\mathrm{LieAlgd}_{/S}(Z)}(\mathbf{Q}_{\mathrm{inj}}^{(\mathfrak{k},H)}(\mathcal{L}), \mathcal{M}) \xrightarrow{\sim} \mathrm{Maps}_{\mathrm{LieAlgd}_{/S}^{(\mathfrak{k},H)}(Y)}(\mathcal{L}, \pi^!_{\mathrm{LieAlgd}}\mathcal{M}) \quad (4.7)$$

Proof A morphism $\mathbf{Q}_{\mathrm{inj}}^{(\mathfrak{k},H)}(\mathcal{L}) \to \mathcal{M}$ is equivalent to an H-equivariant map $\phi : \mathcal{L}/\mathfrak{k} \otimes \mathcal{O}_Y \to \pi^*\mathcal{M}$ preserving the Lie bracket on H-invariant sections. We *claim* that such datum is equivalent to a morphism $\widetilde{\phi} : \mathcal{L} \to \pi^!_{\mathrm{LieAlgd}}\mathcal{M}$ of $(\mathfrak{k}, H)$-Lie algebroids.

Indeed, given ϕ, the map $\widetilde{\phi}$ is uniquely determined by the properties that the following diagrams commute:

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\widetilde{\phi}} & \pi^!_{\mathrm{LieAlgd}}\mathcal{M} \\ \downarrow & & \downarrow \\ \mathcal{L}/\mathfrak{k}\otimes\mathcal{O}_Y & \xrightarrow{\phi} & \pi^*\mathcal{M} \end{array} \qquad \begin{array}{ccc} \mathcal{L} & \xrightarrow{\widetilde{\phi}} & \pi^!_{\mathrm{LieAlgd}}\mathcal{M} \\ & \searrow^{\sigma} & \downarrow \\ & & \mathcal{T}_{Y/S}. \end{array}$$

Furthermore, $\widetilde{\phi}$ preserves the Lie bracket on $\mathcal{L}$, because $\mathcal{L}$ is generated over $\mathcal{O}_Y$ by H-invariant sections and on such sections, the Lie bracket factors through $\mathcal{L}/\mathfrak{k}\otimes\mathcal{O}_Y$ and is preserved by ϕ. Conversely, given $\widetilde{\phi}$, the map ϕ is uniquely determined by the first commutative diagram above. □

4.2.5. Suppose we are given an exact sequence (4.2) of classical action pairs, and an object $(\mathcal{L},\eta)\in \mathrm{LieAlgd}^{(\mathfrak{k},H)}_{\mathrm{inj}/S}(Y)$. Assume also that Y/H is representable by an algebraic stack. Note that:

(a) Y/H^0 admits an H_0-action, realizing it as an H_0-torsor over Y/H (in particular, Y/H^0 is also representable by an algebraic stack);

(b) there is an induced $(\mathfrak{k}_0, H_0)$-Lie algebroid structure on $\mathbf{Q}^{(\mathfrak{k}^0,H^0)}_{\mathrm{inj}}(\mathcal{L})$, for which the structure map

$$\eta_0 : \mathfrak{k}_0\otimes\mathcal{O}_{Y/H^0}\to \mathbf{Q}^{(\mathfrak{k}^0,H^0)}_{\mathrm{inj}}(\mathcal{L})$$

is again injective, i.e., $(\mathbf{Q}^{(\mathfrak{k}^0,H^0)}_{\mathrm{inj}}(\mathcal{L}),\eta_0)\in \mathrm{LieAlgd}^{(\mathfrak{k}_0,H_0)}_{\mathrm{inj}/S}(Y/H^0)$.

We have a version of the second isomorphism theorem:

Proposition 4.9 *There is a natural isomorphism:*

$$\mathbf{Q}^{(\mathfrak{k}_0,H_0)}_{\mathrm{inj}}\circ\mathbf{Q}^{(\mathfrak{k}^0,H^0)}_{\mathrm{inj}}(\mathcal{L})\xrightarrow{\sim}\mathbf{Q}^{(\mathfrak{k},H)}_{\mathrm{inj}}(\mathcal{L}).$$

Proof As $\mathcal{O}_{Y/H^0}$-modules, the cokernel of η_0 identifies with the descent of $\mathcal{L}/\mathfrak{k}\otimes\mathcal{O}_Y$ along $Y\to Y/H^0$ since the latter map is faithfully flat. Hence the underlying $\mathcal{O}_{Y/H}$-module of $\mathbf{Q}^{(\mathfrak{k}_0,H_0)}_{\mathrm{inj}}\circ\mathbf{Q}^{(\mathfrak{k}^0,H^0)}_{\mathrm{inj}}(\mathcal{L})$ agrees with that of $\mathbf{Q}^{(\mathfrak{k},H)}_{\mathrm{inj}}(\mathcal{L})$. Identifications of the anchor maps and the Lie brackets are immediate. □

4.2.6. Suppose we have a classical quasi-twisting (3.1) over Y, where both Lie algebroids $\widehat{\mathcal{L}}$ and $\mathcal{L}$ have the structure of $(\mathfrak{k},H)$-algebroids, and $\widehat{\mathcal{L}}\to\mathcal{L}$ is a morphism of such. In particular, the structure map $\widehat{\eta}:\mathfrak{k}\otimes\mathcal{O}_Y\to\widehat{\mathcal{L}}$ is a lift of η. Hence, if $(\mathcal{L},\eta)\in\mathrm{LieAlgd}^{(\mathfrak{k},H)}_{\mathrm{inj}/S}(Y)$, then so does $(\widehat{\mathcal{L}},\widehat{\eta})$. For fixed $(\mathcal{L},\eta)$, we denote the category of classical quasi-twistings with this additional structure by $\mathrm{QTw}^{(\mathfrak{k},H)}_{/S}(Y/\mathcal{L})$.

Assuming that $\mathcal{Z}:=Y/H$ is represented by an algebraic stack. Then the quotient Lie algebroids again form a central extension:

$$0\to\mathcal{O}_{Y/H}\to\mathbf{Q}^{(\mathfrak{k},H)}_{\mathrm{inj}}(\widehat{\mathcal{L}})\to\mathbf{Q}^{(\mathfrak{k},H)}_{\mathrm{inj}}(\mathcal{L})\to 0.$$

Therefore, we may regard $\mathbf{Q}_{\mathrm{inj}}^{(\mathfrak{k},H)}$ as a functor:

$$\mathbf{Q}_{\mathrm{inj}}^{(\mathfrak{k},H)} : \mathrm{QTw}_{/S}^{(\mathfrak{k},H)}(Y/\mathcal{L}) \to \mathrm{QTw}_{/S}(\mathcal{Z}/\mathbf{Q}_{\mathrm{inj}}^{(\mathfrak{k},H)}(\mathcal{L})).$$

Remark 4.10 When Y is placid and $\mathfrak{k}$ is a topological Lie algebra over $\mathcal{O}_S$, we can adapt the above definitions to make sense of a Tate $(\mathfrak{k}, H)$-Lie algebroid $\mathcal{L}$ (c.f. §3.2.5). In particular, η will be a map out of the completed tensor product $\mathfrak{k}\widehat{\otimes}\mathcal{O}_Y \to \mathcal{L}$.

We do not discuss how to keep track of the topology in the (analogously defined) quotient $\mathbf{Q}_{\mathrm{inj}}^{(\mathfrak{k},H)}(\mathcal{L})$, since all quotients considered in this paper have the properties that Y/H is locally of finite type and $\mathbf{Q}_{\mathrm{inj}}^{(\mathfrak{k},H)}(\mathcal{L})$ should be discrete.

4.3 $(H, H^\flat)$-formal moduli problems

4.3.1. We now study the geometric version of quotient of Lie algebroids. Recall the ∞-category $\mathrm{FMod}_{/S}$ of Sect. 3.3.

Definition 4.11 We call a group object $(H, H^\flat)$ in $\mathrm{FMod}_{/S}$ a *geometric action pair* if H is a group scheme locally of finite type.

Explicitly, a geometric action pair consists of a group scheme H, a group prestack $H^\flat \in \mathrm{PStk}_{\text{laft-def}\,/S}$, and a nil-isomorphism $H \to H^\flat$ that respects the group structure.

4.3.2. We will functorially construct a geometric action pair from any classical action pair $(\mathfrak{k}, H)$, where H is locally of finite type. Indeed, there is a morphism $\exp(\mathfrak{k}) \to H$ coming from the composition $\exp(\mathfrak{k}) \to \exp(\mathfrak{h}) \to H$. Furthermore, the H-action on $\exp(\mathfrak{k})$ equips the prestack quotient $H^\flat := H/\exp(\mathfrak{k})$ with a group structure, such that $H \to H^\flat$ is a group morphism. Note that Lemma 3.12 identifies $H^\flat$ with $\mathrm{B}_H(H \times \exp(\mathfrak{k})^\bullet)$; in particular, $H^\flat \in \mathrm{PStk}_{\text{laft-def}\,/S}$, so $(H, H^\flat)$ is a geometric action pair.

Lemma 4.12 *The category of classical action pairs is identified with the full subcategory of geometric action pairs $(H, H^\flat)$, for which the tangent complex $\mathbb{T}_{H/H^\flat}$ belongs to $\Upsilon_H(\mathrm{QCoh}(H)^\heartsuit)$.*

Proof We explicitly construct the inverse functor. Given a geometric action pair $(H, H^\flat)$ for which $\mathbb{T}_{H/H^\flat} \in \Upsilon_H(\mathrm{QCoh}(H)^\heartsuit)$, we can functorially associate a classical Lie algebroid $\mathcal{L}$ over H. The following Cartesian diagrams:

$$\begin{array}{ccc} H \underset{S}{\times} H & \to & H^\flat \underset{S}{\times} H \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle \mathrm{act}} \\ H & \to & H^\flat \end{array} \qquad \begin{array}{ccc} H \underset{S}{\times} H & \to & H \underset{S}{\times} H^\flat \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle \mathrm{act}} \\ H & \to & H^\flat \end{array}$$

equip the underlying $\mathcal{O}_H$-module of $\mathcal{L}$ with right, respectively left, H-equivariance structures. Hence we may realize $\mathcal{L}$ as $\mathfrak{k} \otimes \mathcal{O}_H$ where $\mathfrak{k}$ is an $\mathcal{O}_S$-module equipped with an H-action. The Lie bracket on $\mathfrak{k}$ comes from the Lie algebroid bracket on $\mathcal{L}$. We omit checking that these data make $(\mathfrak{k}, H)$ into a classical action pair. □

4.3.3. For a geometric action pair $(H, H^\flat)$, we define $\mathrm{FMod}_{/S}^{(H,H^\flat)}$ to be the ∞-category of objects in $\mathrm{FMod}_{/S}$ equipped with an $(H, H^\flat)$-action. Explicitly, an object of $\mathrm{FMod}_{/S}^{(H,H^\flat)}$ consists of the following data:

(a) $\mathcal{Y}, \mathcal{Y}^\flat \in \mathrm{PStk}_{\text{laft-def}\,/S}$ together with a nil-isomorphism $\mathcal{Y} \to \mathcal{Y}^\flat$;
(b) an H-action on $\mathcal{Y}$, and an $H^\flat$-action on $\mathcal{Y}^\flat$, such that the morphism $\mathcal{Y} \to \mathcal{Y}^\flat$ intertwines them.

Note that there is a functor

$$\mathrm{FMod}_{/S}^{(H,H^\flat)} \to \mathrm{PStk}_{\text{laft-def}\,/S}^{H}, \quad (\mathcal{Y}, \mathcal{Y}^\flat) \rightsquigarrow \mathcal{Y} \tag{4.8}$$

where $\mathrm{PStk}_{\text{laft-def}\,/S}^{H}$ denotes the ∞-category of objects in $\mathrm{PStk}_{\text{laft-def}\,/S}$ equipped with an H-action. The fiber of (4.8) at $\mathcal{Y}$ is denoted by $\mathrm{FMod}_{/S}^{(H,H^\flat)}(\mathcal{Y})$. Informally, $\mathrm{FMod}_{/S}^{(H,H^\flat)}(\mathcal{Y})$ is the ∞-category of formal moduli problems $\mathcal{Y}^\flat$ equipped with an $H^\flat$-action that extends the H-action on $\mathcal{Y}$.

4.3.4. Suppose $(\mathfrak{k}, H)$ and $(H, H^\flat)$ are as in §4.3.2, and let Y be a scheme locally of finite type over S, equiped with an H-action. We will construct a functor:

$$\mathrm{LieAlgd}_{/S}^{(\mathfrak{k},H)}(Y) \to \mathrm{FMod}_{/S}^{(H,H^\flat)}(Y) \tag{4.9}$$

which enhances the association of formal moduli problems to Lie algebroids, in the sense that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{LieAlgd}_{/S}^{(\mathfrak{k},H)}(Y) & \xrightarrow{(4.9)} & \mathrm{FMod}_{/S}^{(H,H^\flat)}(Y) \\ \downarrow \mathrm{oblv} & & \downarrow \mathrm{oblv} \\ \mathrm{LieAlgd}_{/S}(Y) & \xrightarrow{(3.11)} & \mathrm{FMod}_{/S}(Y) \end{array}$$

To proceed, suppose $(\mathcal{L}, \eta) \in \mathrm{LieAlgd}_{/S}^{(\mathfrak{k},H)}(Y)$. We need to construct an $H^\flat$-action $\mathrm{act}^\flat$ on the formal moduli problem $\mathcal{Y}^\flat$ corresponding to $\mathcal{L}$, together with a map of simplicial prestacks:

$$\begin{array}{ccccccc} \cdots \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}\; Y \underset{S}{\times} H \underset{S}{\times} H & \substack{\xrightarrow{\mathrm{act}\times 1}\\ \xrightarrow{1\times m}\\ \xrightarrow[\mathrm{pr}_{12}]{}} & Y \underset{S}{\times} H & \substack{\xrightarrow{\mathrm{act}}\\ \xrightarrow[\mathrm{pr}_1]{}} & Y \\ \downarrow & & \downarrow & & \downarrow \\ \cdots \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathcal{Y}^\flat \underset{S}{\times} H^\flat \underset{S}{\times} H^\flat & \substack{\xrightarrow{\mathrm{act}^\flat\times 1}\\ \xrightarrow{1\times m}\\ \xrightarrow[\mathrm{pr}_{12}]{}} & \mathcal{Y}^\flat \underset{S}{\times} H^\flat & \substack{\xrightarrow{\mathrm{act}^\flat}\\ \xrightarrow[\mathrm{pr}_1]{}} & \mathcal{Y}^\flat. \end{array} \tag{4.10}$$

Since each formal moduli problem $\mathcal{Y}^\flat \underset{S}{\times} (H^\flat)^\bullet$ arises from the Lie algebroid $\mathrm{pr}_Y^* \mathcal{L} \oplus \mathrm{pr}_H^*(\mathfrak{k} \otimes \mathcal{O}_H)^{\oplus \bullet}$ over $Y \underset{S}{\times} H^\bullet$, we only need to

(a) produce a morphism

$$\alpha : \mathrm{pr}_Y^* \mathcal{L} \oplus \mathrm{pr}_H^*(\mathfrak{k} \otimes \mathcal{O}_H) \to \mathrm{act}^!_{\mathrm{LieAlgd}} \mathcal{L} \tag{4.11}$$

between Lie algebroids over $Y \underset{S}{\times} H$ (which would rise to $\mathrm{act}^\flat$, in a way compatible with the morphism act)

(b) check that the following diagram:

$$\begin{array}{ccc}
\mathrm{pr}_Y^* \mathcal{L} \oplus \mathrm{pr}_H^*(\mathfrak{k} \otimes \mathcal{O}_H)^{\oplus 2} & \xrightarrow{\mathrm{can}} & (1 \times m)^!_{\mathrm{LieAlgd}}(\mathrm{pr}_Y^* \mathcal{L} \oplus \mathrm{pr}_H^*(\mathfrak{k} \otimes \mathcal{O}_H)) \\
\downarrow{\scriptstyle \mathrm{act}^!_{\mathrm{LieAlgd}}(\alpha) \times 1} & & \downarrow{\scriptstyle (1\times m)^*_{\mathrm{LieAlgd}}(\alpha)} \\
\mathrm{act}^!_{\mathrm{LieAlgd}}(\mathcal{L}) \oplus \mathrm{pr}_H^*(\mathfrak{k} \otimes \mathcal{O}_H) & & (1 \times m)^!_{\mathrm{LieAlgd}} \mathrm{act}^!_{\mathrm{LieAlgd}}(\mathcal{L}) \\
\downarrow\wr & & \downarrow\wr \\
(\mathrm{act} \times 1)^!_{\mathrm{LieAlgd}}(\mathrm{pr}_Y^* \mathcal{L} \oplus \mathrm{pr}_H^*(\mathfrak{k} \otimes \mathcal{O}_H)) & \xrightarrow{\alpha} & (\mathrm{act} \times 1)^!_{\mathrm{LieAlgd}} \mathrm{act}^!_{\mathrm{LieAlgd}}(\mathcal{L})
\end{array} \tag{4.12}$$

of Lie algebroids over $Y \underset{S}{\times} H \underset{S}{\times} H$ is commutative. (This would affirm the commutativity of (4.10) up to 2-simplices, but the higher commutativity constraints are satisfied automatically since the corresponding ∞-categories are classical).

4.3.5. Note that as an $\mathcal{O}_{Y \underset{S}{\times} H}$-module, we have an isomorphism:

$$\mathrm{act}^!_{\mathrm{LieAlgd}}(\mathcal{L}) \xrightarrow{\sim} \mathrm{act}^* \mathcal{L} \underset{\mathrm{act}^* \mathcal{T}_{Y/S}}{\times} \mathcal{T}_{Y \underset{S}{\times} H/S}.$$

The required map α is the sum of the following components:

(a) the map $\mathrm{pr}_Y^* \mathcal{L} \to \mathrm{act}^!_{\mathrm{LieAlgd}}(\mathcal{L})$ induced from the H-equivariance structure on $\mathcal{L}$ and the composition

$$\mathrm{pr}_Y^* \mathcal{L} \xrightarrow{\mathrm{pr}_Y^* \sigma} \mathrm{pr}_Y^* \mathcal{T}_{Y/S} \hookrightarrow \mathcal{T}_{Y \underset{S}{\times} H/S},$$

where σ is the anchor map of $\mathcal{L}$;

(b) the map $\mathfrak{k} \otimes \mathcal{O}_H \to \mathrm{act}^!_{\mathrm{LieAlgd}}(\mathcal{L})$ induced from

$$\mathfrak{k} \xrightarrow{\eta} \mathrm{H}^0(Y, \mathcal{L}) \xrightarrow{\mathrm{act}^*} \mathrm{H}^0(Y \underset{S}{\times} H, \mathrm{act}^* \mathcal{L}),$$

and the composition

$$\mathfrak{k} \otimes \mathcal{O}_H \to \mathfrak{h} \otimes \mathcal{O}_H \hookrightarrow \mathcal{T}_{Y \underset{S}{\times} H/S}. \tag{4.13}$$

The following Lemma shows that the functor (4.9) is well-defined.

Lemma 4.13 *The map α is a morphism of Lie algebroids, and the diagram (4.12) commutes.*

Proof It is obvious that α is compatible with the anchor maps. To show that α preserves the Lie bracket, we check it for sections of $\mathrm{pr}_Y^*\,\mathcal{L} \oplus \mathrm{pr}_H^*(\mathfrak{k} \otimes \mathcal{O}_H)$ of the following types:

(a) $l_1, l_2 \in \mathrm{pr}_Y^{-1}\,\mathcal{L}$; this follows from the assumptions that the equivariance structure $\theta : \mathrm{pr}_Y^*\,\mathcal{L} \to \mathrm{act}^*\,\mathcal{L}$ is compatible with the Lie bracket, and σ is a map of H-equivariant sheaves;
(b) $\xi_1, \xi_2 \in \mathfrak{k}$; this is clear;
(c) $l \in \mathrm{pr}_Y^{-1}\,\mathcal{L}$ and $\xi \in \mathfrak{k}$; this is a slightly more involved calculation, which we now perform.

Write $\theta(l) = \sum_i f_i \otimes l_i$, where $f_i \in \mathcal{O}_{Y\times H}$ and $l_i \in \mathrm{act}^{-1}\,\mathcal{L}$. We need to show the vanishing of the following element in $\mathrm{act}^*\,\mathcal{L} \underset{\mathrm{act}^*\mathcal{T}_{Y/S}}{\times} \mathcal{T}_{Y\underset{S}{\times}H/S}$:

$$[\alpha(l), \alpha(\xi)] = \left[\sum_i (f_i \otimes l_i) \times \sigma(l), (1 \otimes \eta(\xi)) \times \sigma'(\xi)\right] \tag{4.14}$$

where σ' denotes the composition (4.13). Note that the $\mathcal{T}_{Y\underset{S}{\times}H/S}$-component of (4.14) vanishes tautologically, so we just need to show the vanishing of its $\mathrm{act}^*\,\mathcal{L}$-component. The latter is given (using (4.5)) by

$$\sum_i f_i \otimes [l_i, \eta(\xi)] - \sum_i \sigma'(\xi)(f_i) \otimes l_i = -\sum_i (f_i \otimes (\xi_{\mathfrak{h}} \cdot l_i) + (\xi_{\mathfrak{h}} \cdot f_i) \otimes l_i) \tag{4.15}$$

where in the second summand, $\xi_{\mathfrak{h}}$ acts on $f_i \in \mathcal{O}_{Y\underset{S}{\times}H/S}$ by derivation on the $\mathcal{O}_H$-*component*. Consider the right H-action on $Y \underset{S}{\times} H$, given by $(y, h), h' \rightsquigarrow (y, hh')$; if we equip $\mathrm{act}^*\,\mathcal{L}$ with the following H-equivariance structure:

$$\mathrm{act}^*\,\mathcal{L}\big|_{(y,h)} \xrightarrow{\sim} \mathcal{L}\big|_{yh} \xrightarrow{\theta_{(yh,h')}} \mathcal{L}\big|_{yhh'} \xrightarrow{\sim} \mathrm{act}^*\,\mathcal{L}\big|_{(y,hh')},$$

then (4.15) is the (negative of the) induced action of $\xi_{\mathfrak{h}}$ on the section $\sum_i f_i \otimes l_i = \theta(l)$ in $\mathrm{act}^*\,\mathcal{L}$. Note that $\mathrm{pr}_Y^*\,\mathcal{L}$ can also be endowed with an H-equivariance structure:

$$\mathrm{pr}^*\,\mathcal{L}\big|_{(y,h)} \xrightarrow{\sim} \mathcal{L}\big|_{y} \xrightarrow{\sim} \mathrm{pr}^*\,\mathcal{L}\big|_{(y,hh')}$$

such that θ is a map of H-equivariant $\mathcal{O}_{Y\underset{S}{\times}H}$-modules. Hence the element $\xi_{\mathfrak{h}} \cdot \theta(l)$ identifies with $\theta(\xi_{\mathfrak{h}} \cdot l)$. On the other hand, $l \in \mathrm{pr}^{-1}\,\mathcal{L}$ so $\xi_{\mathfrak{h}} \cdot l = 0$, from which we deduce the required vanishing of (4.15). Checking the commutativity of (4.12) is not difficult, and we leave it to the reader. □

4.3.6. We now characterize the image of the functor (4.9).

Proposition 4.14 *The functor (4.9) is an equivalence onto the full subcategory:*

$$\mathrm{FMod}_{/S}^{(H,H^\flat)}(Y)^{\mathrm{cl}} \hookrightarrow \mathrm{FMod}_{/S}^{(H,H^\flat)}(Y)$$

that consists of objects $\mathcal{Y}^\flat$ such that $\mathbb{T}_{Y/\mathcal{Y}^\flat}$ lies in $\Upsilon_Y(\mathrm{QCoh}(Y)^\heartsuit)$.

Proof Indeed, such a formal moduli problems $\mathcal{Y}^\flat$ arises from some Lie algebroid $\mathcal{L}$ via the functor (3.11). Given the additional data of an $(H, H^\flat)$-action, we consider the following commutative diagrams:

$$\begin{array}{ccccc} Y \underset{S}{\times} H & \xrightarrow{\quad\mathrm{act}\quad} & Y & \qquad & Y \underset{S}{\times} H & \xrightarrow{\quad\mathrm{act}\quad} & Y \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{Y}^\flat \underset{S}{\times} H \xrightarrow{i} \mathcal{Y}^\flat \underset{S}{\times} H^\flat & \xrightarrow{\mathrm{act}^\flat} & \mathcal{Y}^\flat & & Y \underset{S}{\times} H^\flat \xrightarrow{j} \mathcal{Y}^\flat \underset{S}{\times} H^\flat & \xrightarrow{\mathrm{act}^\flat} & \mathcal{Y}^\flat \end{array} \tag{4.16}$$

From these diagrams, we obtain two maps between tangent complexes:

$$\mathbb{T}_{Y\underset{S}{\times}H/\mathcal{Y}^\flat\underset{S}{\times}H} \xrightarrow{\mathrm{act}^\flat_*\circ i_*} \mathbb{T}_{Y\underset{S}{\times}H/\mathcal{Y}^\flat} \to \mathbb{T}_{Y/\mathcal{Y}^\flat}\big|_{Y\underset{S}{\times}H},$$

which gives rise to a morphism $\theta : \mathrm{pr}_Y^*\,\mathcal{L} \to \mathrm{act}^*\,\mathcal{L}$; and

$$\mathbb{T}_{Y\underset{S}{\times}H/Y\underset{S}{\times}H^\flat} \xrightarrow{\mathrm{act}^\flat_*\circ j_*} \mathbb{T}_{Y\underset{S}{\times}H/\mathcal{Y}^\flat} \to \mathbb{T}_{Y/\mathcal{Y}^\flat}\big|_{Y\underset{S}{\times}H}, \tag{4.17}$$

which gives rise to a map $\widetilde{\eta} : \mathrm{pr}_H^*(\mathfrak{k}\otimes\mathcal{O}_H) \to \mathrm{act}^*\,\mathcal{L}$; restricting to $Y \underset{S}{\times} \{1\}$, we obtain a map $\eta : \mathfrak{k}\otimes\mathcal{O}_Y \to \mathcal{L}$. The functor $\mathrm{FMod}_{/S}^{(H,H^\flat)}(Y)^{\mathrm{cl}} \to \mathrm{LieAlgd}_{/S}^{(\mathfrak{k},H)}(Y)$ inverse to (4.9) is defined by sending $\mathcal{Y}^\flat$ to the Lie algebroid $\mathcal{L}$, equipped with the $(\mathfrak{k}, H)$-structure specified by the above maps θ and η. □

4.3.7. We give an alternative description of the map α that will be used in the proof of Proposition 4.18. Consider the commutative diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\quad\mathrm{can}\quad} & Y/H \\ \downarrow & & \downarrow \\ Y \underset{S}{\times} (H^\flat/H) \xrightarrow{\widetilde{j}} \mathcal{Y}^\flat \underset{S}{\times} (H^\flat/H) & \xrightarrow{\widetilde{\mathrm{act}}^\flat} & \mathcal{Y}^\flat/H \end{array} \tag{4.18}$$

which is the "quotient" by H of the right diagram in (4.16). It produces the following map between tangent complexes:

$$\mathbb{T}_{Y/(Y\underset{S}{\times}(H^\flat/H))} \xrightarrow{\widetilde{\mathrm{act}}^\flat_*\circ\widetilde{j}_*} \mathbb{T}_{Y/(\mathcal{Y}^\flat/H)} \to \mathbb{T}_{(Y/H)/(\mathcal{Y}^\flat/H)}\big|_Y \xrightarrow{\sim} \mathbb{T}_{Y/\mathcal{Y}^\flat}. \tag{4.19}$$

We *claim* that (4.19) identifies with the restriction of (4.17) to $Y \underset{S}{\times} \{1\}$. Indeed, this follows from the fact that (4.17) is the pullback of (4.19) along $\mathrm{pr}_Y : Y \underset{S}{\times} H \to Y$, and the composition $Y \underset{S}{\times} \{1\} \hookrightarrow Y \underset{S}{\times} H \xrightarrow{\mathrm{pr}_Y} Y$ is the identity.

4.4 Quotient of formal moduli problems

4.4.1. Let $(H, H^\flat)$ be a geometric action pair (see Definition 4.11). Suppose $(\mathcal{Y}, \mathcal{Y}^\flat) \in \mathrm{FMod}_{/S}^{(H,H^\flat)}$. The *quotient* of $(\mathcal{Y}, \mathcal{Y}^\flat)$ by $(H, H^\flat)$ is defined as the quotient in the ∞-category $\mathrm{FMod}_{/S}$. In other words, it is the geometric realization of the simplicial object $(\mathcal{Y}, \mathcal{Y}^\flat) \times (H, H^\flat)^\bullet$ in $\mathrm{FMod}_{/S}^{(H,H^\flat)}$ characterizing the $(H, H^\flat)$-action on $(\mathcal{Y}, \mathcal{Y}^\flat)$.

Proposition 4.15 *The quotient of $(\mathcal{Y}, \mathcal{Y}^\flat)$ by $(H, H^\flat)$ exists.*

Proof We construct the quotient in the ∞-category $\mathrm{Fun}(\Delta^1, \mathrm{PStk}_{\text{laft-def}\,/S})$, and then check that the result belongs to the full subcategory $\mathrm{FMod}_{/S}$. Quotient in the above functor category is computed pointwise as follows:

(a) at the vertex [0], we have the prestack quotient $\mathcal{Y}/H$; it is an object of $\mathrm{PStk}_{\text{laft-def}\,/S}$ because H is a group *scheme* locally of finite type;

(b) at the vertex [1], we assert that the quotient of $\mathcal{Y}^\flat$ by $H^\flat$ exists in $\mathrm{PStk}_{\text{laft-def}\,/S}$; indeed, it is given by $\mathrm{B}_{\mathcal{Y}^\flat/H}(\mathcal{Y}^\flat \underset{S}{\overset{H}{\times}} H^\flat/H)$ where $\mathcal{Y}^\flat \underset{S}{\overset{H}{\times}} H^\flat/H$ denotes the Hecke groupoid[11] acting on the prestack quotient $\mathcal{Y}^\flat/H$:

$$\cdots \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathcal{Y}^\flat \underset{S}{\overset{H}{\times}} H^\flat \underset{S}{\overset{H}{\times}} H^\flat/H \;\substack{\xrightarrow{\mathrm{act}^\flat \times 1}\\ \xrightarrow{1\times m}\\ \xrightarrow{\mathrm{pr}_{12}}}\; \mathcal{Y}^\flat \underset{S}{\overset{H}{\times}} H^\flat/H \;\substack{\xrightarrow{\mathrm{act}^\flat}\\ \xrightarrow{\mathrm{pr}_1}}\; \mathcal{Y}^\flat/H,$$

and $\mathrm{B}_{\mathcal{Y}^\flat/H}$ is the functor from §3.4.3.

Finally, the morphism $\mathcal{Y}/H \to \mathrm{B}_{\mathcal{Y}^\flat/H}(\mathcal{Y}^\flat \underset{S}{\overset{H}{\times}} H^\flat/H)$ is a nil-isomorphism since it is the composition of nil-isomorphisms $\mathcal{Y}/H \to \mathcal{Y}^\flat/H \to \mathrm{B}_{\mathcal{Y}^\flat/H}(\mathcal{Y}^\flat \underset{S}{\overset{H}{\times}} H^\flat/H)$. □

Regarding $\mathcal{Y}$ as a fixed prestack acted on by H, we denote the resulting quotient functor by

$$\mathbf{Q}^{(H,H^\flat)} : \mathrm{FMod}_{/S}^{(H,H^\flat)}(\mathcal{Y}) \to \mathrm{FMod}_{/S}(\mathcal{Y}/H), \quad \mathcal{Y}^\flat \rightsquigarrow \mathrm{B}_{\mathcal{Y}^\flat/H}(\mathcal{Y}^\flat \underset{S}{\overset{H}{\times}} H^\flat/H). \tag{4.20}$$

[11] Suppose $\mathcal{C}$ is an ∞-category with finite products. Let $H \to K$ be a map of group objects in $\mathcal{C}$. Suppose any object in $\mathcal{C}$ with an H-action admits a quotient. Then given an object $Y \in \mathcal{C}$ with a K-action, there exists a *Hecke groupoid* $Y \underset{}{\overset{H}{\times}} K/H$ acting on Y/H whose quotient, if exists, agrees with Y/K.

4.4.2. Tautologically, the quotient $(\mathcal{Y}/H, \mathrm{B}_{\mathcal{Y}^\flat/H}(\mathcal{Y}^\flat \underset{S}{\overset{H}{\times}} H^\flat/H))$, equipped with the map from $(\mathcal{Y}, \mathcal{Y}^\flat)$, satisfies the universal property:

$$\begin{aligned}&\mathrm{Maps}_{\mathrm{FMod}_{/S}}((\mathcal{Y}/H, \mathrm{B}_{\mathcal{Y}^\flat/H}(\mathcal{Y}^\flat \underset{S}{\overset{H}{\times}} H^\flat/H)), (\mathcal{Z}, \mathcal{Z}^\flat))\\ &\xrightarrow{\sim} \mathrm{Maps}_{\mathrm{FMod}_{/S}^{(H,H^\flat)}}((\mathcal{Y}, \mathcal{Y}^\flat), (\mathcal{Z}, \mathcal{Z}^\flat)),\end{aligned}$$

where in the second expression, $(\mathcal{Z}, \mathcal{Z}^\flat)$ is equipped with the trivial $(H, H^\flat)$-action. Specializing to $\mathcal{Z} = \mathcal{Y}/H$, we see that the object $\mathbf{Q}^{(H,H^\flat)}(\mathcal{Y}^\flat) \in \mathrm{FMod}_{/S}(\mathcal{Y}/H)$ is characterized by the universal property:

$$\mathrm{Maps}_{\mathrm{FMod}_{/S}(\mathcal{Y}/H)}(\mathbf{Q}^{(H,H^\flat)}(\mathcal{Y}^\flat), \mathcal{Z}^\flat) \xrightarrow{\sim} \mathrm{Maps}_{\mathrm{FMod}_{/S}^{(H,H^\flat)}(\mathcal{Y})}(\mathcal{Y}^\flat, \pi^!_{\mathrm{FMod}}(\mathcal{Z}^\flat)) \tag{4.21}$$

where in the second expression, $\pi^!_{\mathrm{FMod}}\mathcal{Z}^\flat \cong \mathcal{Z}^\flat \underset{(\mathcal{Y}/H)_{\mathrm{dR}}}{\times} \mathcal{Y}_{\mathrm{dR}}$ is acted on by $H^\flat$ through the canonical homomorphism $H^\flat \to H_{\mathrm{dR}}$ on the $\mathcal{Y}_{\mathrm{dR}}$ factor.

Remark 4.16 Recall the $(\mathfrak{k}, H)$-Lie algebroid structure on $\pi^!_{\mathrm{LieAlgd}}(\mathcal{M})$, where $(\mathfrak{k}, H)$ is any classical action pair and $\mathcal{M}$ is a Lie algebroid on the quotient Y/H (see Sect. 4.2.3). If $H^\flat = H/\exp(\mathfrak{k})$ as in §4.3.2, then the $(H, H^\flat)$-formal moduli problem $\pi^!_{\mathrm{FMod}}(\mathcal{Z}^\flat)$ is precisely the one associated to $\pi^!_{\mathrm{LieAlgd}}(\mathcal{M})$ under the functor (4.9).

4.4.3. Let $(H^0, (H^0)^\flat) \to (H, H^\flat)$ be a morphism of geometric action pairs. We say that $(H^0, (H^0)^\flat)$ is a *normal subpair* of $(H, H^\flat)$ if there is a morphism $(H, H^\flat) \to (H_0, (H_0)^\flat)$ of geometric action pairs whose kernel identifies with $(H^0, (H^0)^\flat)$. In particular, the $(H, H^\flat)$-action on itself extends to $(H^0, (H^0)^\flat)$.

Given a normal subpair $(H^0, (H^0)^\flat)$ of $(H, H^\flat)$, we recover $(H_0, (H_0)^\flat)$ by the isomorphisms:

$$H_0 \xrightarrow{\sim} H/H^0, \quad H_0^\flat \xrightarrow{\sim} \mathbf{Q}^{(H^0,(H^0)^\flat)}(H^\flat).$$

Let $\mathcal{Y}^\flat \in \mathrm{FMod}_{/S}^{(H,H^\flat)}(\mathcal{Y})$. Then the prestack $\mathbf{Q}^{(H^0,(H^0)^\flat)}(\mathcal{Y}^\flat)$ is naturally an object of $\mathrm{FMod}_{/S}^{(H_0,H_0^\flat)}(\mathcal{Y}/H^0)$, and we have a second isomorphism theorem:

Proposition 4.17 *There is a natural isomorphism:*

$$\mathbf{Q}^{(H_0,H_0^\flat)} \circ \mathbf{Q}^{(H^0,(H^0)^\flat)}(\mathcal{Y}^\flat) \xrightarrow{\sim} \mathbf{Q}^{(H,H^\flat)}(\mathcal{Y}^\flat).$$

Proof Both sides are the quotient of $(\mathcal{Y}, \mathcal{Y}^\flat)$ by $(H, H^\flat)$ in the ∞-category $\mathrm{FMod}_{/S}$. □

4.4.4. Suppose we have a quasi-twisting $\widehat{\mathcal{Y}}^\flat \in \mathrm{QTw}_{/S}(\mathcal{Y}/\mathcal{Y}^\flat)$, such that $(\mathcal{Y}, \widehat{\mathcal{Y}}^\flat)$ is also an $(H, H^\flat)$-formal moduli problem, and the morphism $\widehat{\mathcal{Y}}^\flat \to \mathcal{Y}^\flat$ preserves this structure. We call quasi-twistings with these additional data $(H, H^\flat)$-*quasi-twistings* (based at $\mathcal{Y}^\flat$) and denote the category of them by $\mathrm{QTw}^{(H,H^\flat)}_{/S}(\mathcal{Y}/\mathcal{Y}^\flat)$. The quotient $\mathbf{Q}^{(H,H^\flat)}(\widehat{\mathcal{Y}}^\flat)$ inherits the structure of a quasi-twisting on $\mathcal{Y}/H$ based at $\mathbf{Q}^{(H,H^\flat)}(\mathcal{Y}^\flat)$. Indeed,

(a) applying $\mathbf{Q}^{(H,H^\flat)}$ to the action groupoid $\widehat{\mathcal{Y}}^\flat \times \mathrm{B}\,\widehat{\mathbb{G}}_m^\bullet$, we obtain a $\mathrm{B}\,\widehat{\mathbb{G}}_m$-action on $\mathbf{Q}^{(H,H^\flat)}(\widehat{\mathcal{Y}}^\flat)$, which gives rise to a $\widehat{\mathbb{G}}_m$-gerbe structure;
(b) the section $\mathcal{Y}/H \to \mathbf{Q}^{(H,H^\flat)}(\widehat{\mathcal{Y}}^\flat)$ is given by the composition:

$$\mathcal{Y}/H \to \widehat{\mathcal{Y}}^\flat/H \to \mathbf{Q}^{(H,H^\flat)}(\widehat{\mathcal{Y}}^\flat).$$

Therefore, we may view $\mathbf{Q}^{(H,H^\flat)}$ as a functor:

$$\mathbf{Q}^{(H,H^\flat)} : \mathrm{QTw}^{(H,H^\flat)}_{/\mathcal{Y}^\flat}(Y/S) \to \mathrm{QTw}_{/\mathbf{Q}^{(H,H^\flat)}(\mathcal{Y}^\flat)}((Y/H)/S).$$

4.5 Comparison of $\mathbf{Q}^{(\mathfrak{k},H)}_{\mathbf{inj}}$ and $\mathbf{Q}^{(H,H^\flat)}$

4.5.1. Suppose $(\mathfrak{k}, H)$ and $(H, H^\flat)$ are as in §4.3.2, and let Y be a scheme locally of finite type over S equipped with an H-action. We shall show that the two quotient functors constructed above are compatible.

Proposition 4.18 *The following diagram is commutative:*

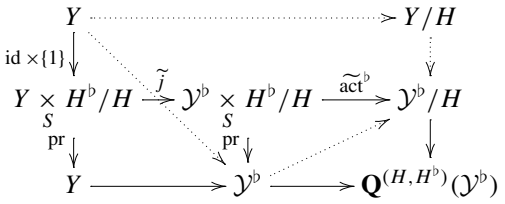
$$\begin{array}{ccc} \mathrm{LieAlgd}^{(\mathfrak{k},H)}_{\mathrm{inj}/S}(Y) & \xrightarrow{(4.9)} & \mathrm{FMod}^{(H,H^\flat)}_{/S}(Y) \\ \downarrow{\scriptstyle \mathbf{Q}^{(\mathfrak{k},H)}_{\mathrm{inj}}} & & \downarrow{\scriptstyle \mathbf{Q}^{(H,H^\flat)}} \\ \mathrm{LieAlgd}_{/S}(Y/H) & \xrightarrow{(3.11)} & \mathrm{FMod}_{/S}(Y/H). \end{array}$$

Proof Suppose $(\mathcal{L}, \eta) \in \mathrm{LieAlgd}^{(\mathfrak{k},H)}_{\mathrm{inj}/S}(Y)$, i.e., $\mathcal{L}$ is a $(\mathfrak{k}, H)$-Lie algebroid over Y such that the map $\eta : \mathfrak{k} \otimes \mathcal{O}_Y \to \mathcal{L}$ is injective. Let $\mathcal{Y}^\flat$ be the corresponding formal moduli problem under Y, equipped with the $H^\flat$-action defined by the functor (4.9). Thus $\mathbf{Q}^{(H,H^\flat)}(\mathcal{Y}^\flat)$ satisfies the universal property (4.21) for $\mathcal{Z}^\flat \in \mathrm{FMod}_{/S}(\mathcal{Y}/H)$.

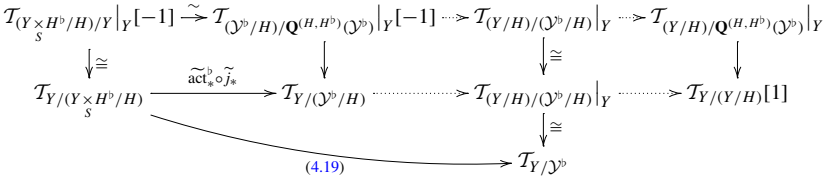
On the other hand, $\mathbf{Q}^{(\mathfrak{k},H)}_{\mathrm{inj}}(\mathcal{L})$ satisfies the universal property (4.7). Since the essential image of (3.11) consists of objects $\mathcal{Z}^\flat \in \mathrm{FMod}_{/S}(Y/H)$ such that $\mathbb{T}_{(Y/H)/\mathcal{Z}^\flat}$ belongs to $\Upsilon_{Y/H}(\mathrm{QCoh}(Y/H)^\heartsuit)$, it suffices to show that $\mathbf{Q}^{(H,H^\flat)}(\mathcal{Y}^\flat)$ has this property. The result thus follows from the lemma below and the fact that $Y \to Y/H$ is faithfully flat. □

Lemma 4.19 *Suppose $(Y, \mathcal{Y}^\flat)$ is the $(H, H^\flat)$-formal moduli problem corresponding to the $(\mathfrak{k}, H)$-Lie algebroid $(\mathcal{L}, \eta)$ under the functor (4.9). Then there is a canonical isomorphism between $\mathcal{T}_{(Y/H)/\mathbf{Q}^{(H,H^\flat)}(\mathcal{Y}^\flat)}\big|_Y$ and* $\mathrm{Cofib}(\eta)$.

Proof We will use the expression of $\mathbf{Q}^{(H,H^\flat)}(\mathcal{Y}^\flat)$ as quotient of the Hecke groupoid $\mathcal{Y}^\flat \underset{S}{\overset{H}{\times}} H^\flat/H$ (see (4.20)). Consider the following commutative diagram, which extends the commutative diagram (4.18):



where the two lower squares, as well as the dotted quadrilateral, are Cartesian. From this diagram, we obtain the following commutative diagram of objects in $\mathrm{QCoh}(Y)$:



Furthermore, the two horizontal dotted triangles are exact. Note that the composition (4.19) identifies with η, so the upper horizontal triangle allows us to identify $\mathcal{T}_{(Y/H)/\mathbf{Q}^{(H,H^\flat)}(\mathcal{Y}^\flat)}\big|_Y$ with $\mathrm{Cofib}(\eta)$. □

4.6 Example: inert quasi-twistings

4.6.1. We now specialize to Lie algebroids arising from abelian Lie algebras. They give rise to what we call "inert quasi-twistings." In the geometric Langlands theory, they arise naturally as degeneration of (non-inert) quasi-twistings as the quantum parameter κ tends to ∞. (The details of this application will appear in Sect. 6).

4.6.2. Recall that over any $\mathcal{Y} \in \mathrm{PStk}_{\text{laft-def}\,/S}$, there is a functor

$$\mathrm{triv} : \mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{Lie}(\mathrm{IndCoh}(\mathcal{Y}))$$

that associates to an ind-coherent sheaf $\mathcal{F}$ the abelian Lie algebra on $\mathcal{F}$. (The notation $\mathrm{Lie}(\mathrm{IndCoh}(\mathcal{Y}))$ means Lie algebra objects in the symmetric monoidal category $\mathrm{IndCoh}(\mathcal{Y})$). More precisely, triv is the right inverse to the forgetful functor. Because the latter is conservative and preserves limits, triv also preserves limits.

4.6.3. We also have a pair of adjunction:

$$\mathrm{diag}_{\mathcal{Y}} : \mathrm{Lie}(\mathrm{IndCoh}(\mathcal{Y})) \rightleftarrows \mathrm{FMod}(\mathcal{Y}) : \text{ker-anch}$$

where $\mathrm{diag}_{\mathcal{Y}}$ preserves fiber products.[12] It follows that the composition $\mathrm{diag}_{\mathcal{Y}} \circ \mathrm{triv}$ preserves fiber products. We call $\mathcal{Y}^\flat := \mathrm{diag}_{\mathcal{Y}} \circ \mathrm{triv}(\mathcal{F})$ the *inert* formal moduli problem on $\mathcal{F}$.

Remark 4.20 Let Y be a scheme (not necessarily locally of finite type) over S. The classical analogue of the above construction associates to an $\mathcal{O}_Y$-module $\mathcal{F}$ the Lie algebroid on $\mathcal{F}$ with *zero* Lie bracket and anchor map. If $Y \to S$ is locally of finite type, then the image of $\mathcal{F}$ under (3.11) agrees with $\mathrm{diag}_{\mathcal{Y}} \circ \mathrm{triv}(\Upsilon_{Y/S}(\mathcal{F}))$.

4.6.4. For the remainder of this section, we suppose $Y \to S$ is *smooth*. Then the identification $\Upsilon_{Y/S} : \mathrm{QCoh}(Y) \xrightarrow{\sim} \mathrm{IndCoh}(Y)$ allows us to view the universal enveloping algebra[13] of an object $\mathcal{Y}^\flat \in \mathrm{FMod}_{/S}(Y)$ as an algebra in $\mathrm{QCoh}(Y)$. If $\mathcal{Y}^\flat = \mathrm{diag}_Y \circ \mathrm{triv}(\Upsilon_Y(\mathcal{F}))$, then it is given by $\mathrm{Sym}_{\mathcal{O}_Y}(\mathcal{F})$.

4.6.5. Suppose $\mathcal{F} \in \mathrm{QCoh}(Y)^{\leq 0}$. Let $\mathbb{V}(\mathcal{F}) := \underline{\mathrm{Spec}}_Y \mathrm{Sym}_{\mathcal{O}_Y}(\mathcal{F})$. It is a prestack over Y fibered in vector DG schemes. We have an equivalence of DG categories:

$$\mathrm{IndCoh}(\mathcal{Y}^\flat) \xrightarrow{\sim} \mathrm{QCoh}(\mathbb{V}(\mathcal{F})), \tag{4.22}$$

where **oblv** : $\mathrm{IndCoh}(\mathcal{Y}^\flat) \to \mathrm{IndCoh}(Y)$ passes to the pushforward functor on QCoh (see [18, IV.4 §4.1.3, IV.2 (7.12), and IV.3 Proposition 5.1.2]).

4.6.6. Suppose, furthermore, that we have a quasi-twisting $\widehat{\mathcal{Y}}^\flat \in \mathrm{QTw}_{/S}(Y/\mathcal{Y}^\flat)$ that arises from a triangle $\mathcal{O}_Y \to \widehat{\mathcal{F}} \to \mathcal{F}$ in $\mathrm{QCoh}(Y)^{\leq 0}$ under the composition $\mathrm{diag}_Y \circ \mathrm{triv} \circ \Upsilon_{Y/S}$. We call $\widehat{\mathcal{Y}}^\flat$ the *inert quasi-twisting* on the triangle $\mathcal{O}_Y \to \widehat{\mathcal{F}} \to \mathcal{F}$.

4.6.7. Since $\underline{\mathrm{Spec}}_Y \mathrm{Sym}_{\mathcal{O}_Y}(\mathcal{O}_Y)$ is identified with $Y \times \mathbb{A}^1$, the map $\mathcal{O}_Y \to \widehat{\mathcal{F}}$ gives rise to a morphism of DG schemes:

$$\underline{\mathrm{Spec}}_Y \mathrm{Sym}_{\mathcal{O}_Y}(\widehat{\mathcal{F}}) \to Y \times \mathbb{A}^1. \tag{4.23}$$

We let $\mathbb{V}(\widehat{\mathcal{F}})_{\lambda=1}$ be the fiber of (4.23) at $\{1\} \hookrightarrow \mathbb{A}^1$. Note that the analogously defined fiber $\mathbb{V}(\widehat{\mathcal{F}})_{\lambda=0}$ identifies with $\mathbb{V}(\mathcal{F})$. There is a canonical equivalence of DG categories:

$$\widehat{\mathcal{Y}}^\flat\text{-Mod} \xrightarrow{\sim} \mathrm{QCoh}(\mathbb{V}(\widehat{\mathcal{F}})_{\lambda=1}). \tag{4.24}$$

Remark 4.21 From our point of view, the DG category $\mathrm{QCoh}(\mathrm{LocSys}_G)$ is realized by modules over some quasi-twisting on Bun_G. The DG stack LocSys_G only appears *a posteriori* through (4.24).

[12] One sees this by identifying $\mathrm{Lie}(\mathrm{IndCoh}(\mathcal{Y}))$ with $\mathrm{FMod}(\mathcal{Y})_{/\mathcal{Y}}$, where $\mathcal{Y}$ is regarded as a formal moduli problem under itself by the identity map. Under this identification, $\mathrm{diag}_{\mathcal{Y}}$ becomes the tautological forgetful functor; see [18, IV.4].

[13] This is defined as a monad on $\mathrm{IndCoh}(Y)$ in [18, IV.4.4].

4.6.8. We now discuss how quotient interacts with inert quasi-twistings. Denote by pt the S-scheme S itself. Suppose $(\mathfrak{k}, H)$ is a classical action pair with *zero* map $\mathfrak{k} \to \mathfrak{h}$. Then we have

$$H^\flat := H/\exp(\mathfrak{k}) \xrightarrow{\sim} H \ltimes (\mathrm{pt}/\exp(\mathfrak{k})),$$

where the formation of the semidirect product is formed by the H-action on $\mathrm{pt}/\exp(\mathfrak{k})$. Note that the normal subpair $(\mathrm{pt}, \mathrm{pt}/\exp(\mathfrak{k}))$ of $(H, H^\flat)$ has quotient (H, H), since

$$\mathbf{Q}^{(\mathrm{pt},\mathrm{pt}/\exp(\mathfrak{k}))}(H^\flat) \xrightarrow{\sim} \mathrm{B}_{H^\flat}(H^\flat \times (\mathrm{pt}/\exp(\mathfrak{k}))^\bullet) \xrightarrow{\sim} H;$$

see Sect. 4.4.3.

4.6.9. We now assume that $\mathfrak{k}$ is also abelian. Suppose the smooth scheme Y admits an H-action, and $\mathcal{Y}^\flat$ is the inert formal moduli problem on some H-equivariant sheaf $\mathcal{F} \in \mathrm{QCoh}(Y)^\heartsuit$.

Suppose we have an H-equivariant map $\eta : \mathfrak{k} \otimes \mathcal{O}_Y \to \mathcal{F}$, giving rise to an $H^\flat$-action on $\mathcal{Y}^\flat$ (see Sect. 4.3.4). Let $\mathcal{Q} := \mathrm{Cofib}(\eta)$; it is an H-equivariant complex of $\mathcal{O}_Y$-modules, hence descends to an object $\mathcal{Q}^{\mathrm{desc}} \in \mathrm{QCoh}(Y/H)$.

Proposition 4.22 *The quotient $\mathbf{Q}^{(H,H^\flat)}(\mathcal{Y}^\flat)$ identifies with the inert formal moduli problem on $\mathcal{Q}^{\mathrm{desc}} \in \mathrm{QCoh}(Y/H)$.*

Proof By Proposition 4.17, we have

$$\mathbf{Q}^{(H,H^\flat)}(\mathcal{Y}^\flat) \xrightarrow{\sim} \mathbf{Q}^{(H,H)} \circ \mathbf{Q}^{(\mathrm{pt},\mathrm{pt}/\exp(\mathfrak{k}))}(\mathcal{Y}^\flat) \xrightarrow{\sim} \mathbf{Q}^{(\mathrm{pt},\mathrm{pt}/\exp(\mathfrak{k}))}(\mathcal{Y}^\flat)/H.$$

Note that descent of $\mathcal{O}_Y$-modules corresponds to quotient by H on the inert formal moduli problem. Hence we only need to identify $\mathbf{Q}^{(\mathrm{pt},\mathrm{pt}/\exp(\mathfrak{k}))}(\mathcal{Y}^\flat)$ as the inert formal moduli problem on $\mathcal{Q}$.

Consider the Čech nerve of $\mathcal{F} \to \mathcal{Q}$ in $\mathrm{QCoh}(Y)$, which identifies with the groupoid $\mathcal{F} \oplus (\mathfrak{k} \otimes \mathcal{O}_Y)^{\oplus \bullet}$. Since the composition $\mathrm{diag}_Y \circ \mathrm{triv}$ preserves fiber products, we see that

$$\mathrm{diag}_Y \circ \mathrm{triv}(\mathcal{F} \oplus (\mathfrak{k} \otimes \mathcal{O}_Y)^{\oplus \bullet}) \xrightarrow{\sim} \mathcal{Y}^\flat \times (\mathrm{pt}/\exp(\mathfrak{k}))^\bullet$$

identifies with the Čech nerve of the map $\mathcal{Y}^\flat \to \mathrm{diag}_Y \circ \mathrm{triv}(\mathcal{Q})$. The result follows since this is also the Čech nerve of $\mathcal{Y}^\flat \to \mathbf{Q}^{(\mathrm{pt},\mathrm{pt}/\exp(\mathfrak{k}))}(\mathcal{Y}^\flat)$. □

Remark 4.23 When Y is any scheme over S (*not* necessarily locally of finite type) but η is injective, we also have an identification of $\mathbf{Q}_{\mathrm{inj}}^{(\mathfrak{k},H)}(\mathcal{F})$ with the Lie algebroid on $\mathcal{Q}^{\mathrm{desc}}$ with zero Lie bracket and anchor map. This follows immediately from the definition of $\mathbf{Q}_{\mathrm{inj}}^{(\mathfrak{k},H)}(\mathcal{F})$.

Geometrically, the datum of η gives rise to a map $\phi : \mathbb{V}(\mathcal{F}) \to Y \underset{S}{\times} \mathfrak{k}^*$, and $\mathbb{V}(\mathcal{Q})$ identifies with its fiber at $\{0\} \hookrightarrow \mathfrak{k}^*$. Hence we have isomorphisms of DG stacks:

$$\mathbb{V}(\mathcal{Q}^{\mathrm{desc}}) \xrightarrow{\sim} \mathbb{V}(\mathcal{Q})/H \xrightarrow{\sim} \phi^{-1}(0)/H. \tag{4.25}$$

4.6.10. Suppose we have an exact sequence of H-equivariant $\mathcal{O}_Y$-modules:

$$0 \to \mathcal{O}_Y \to \widehat{\mathcal{F}} \to \mathcal{F} \to 0.$$

Let $\widehat{\mathcal{Y}}^\flat \in \mathrm{QTw}_{/\mathcal{Y}^\flat}(Y/S)$ be the corresponding inert quasi-twisting. Assume that η lifts to an H-equivariant map $\widehat{\eta} : \mathfrak{k} \otimes \mathcal{O}_Y \to \widehat{\mathcal{F}}$. Then Proposition 4.22 shows that the quotient quasi-twisting arises from a triangle in $\mathrm{QCoh}(Y/H)$:

$$\mathcal{O}_{Y/H} \to \widehat{\mathcal{Q}}^{\mathrm{desc}} \to \mathcal{Q}^{\mathrm{desc}}$$

where $\widehat{\mathcal{Q}}^{\mathrm{desc}}$ is the descent of $\widehat{\mathcal{Q}} := \mathrm{Cofib}(\widehat{\eta})$ to Y/H.

In particular, we have isomorphisms of DG stacks:

$$\mathbb{V}(\widehat{\mathcal{Q}}^{\mathrm{desc}})_{\lambda=1} \xrightarrow{\sim} \mathbb{V}(\widehat{\mathcal{Q}})_{\lambda=1}/H \xrightarrow{\sim} \widehat{\phi}_{\lambda=1}^{-1}(0)/H \tag{4.26}$$

where $\widehat{\phi}_{\lambda=1}$ is the composition

$$\mathbb{V}(\widehat{\mathcal{F}})_{\lambda=1} \hookrightarrow \mathbb{V}(\widehat{\mathcal{F}}) \xrightarrow{\mathbb{V}(\widehat{\eta})} Y \underset{S}{\times} \mathfrak{k}^*.$$

Remark 4.24 In light of (4.25) and (4.26), one may think of $\mathbf{Q}^{(H,H^\flat)}$ on inert quasi-twistings as an analogue of symplectic reduction where ϕ and $\widehat{\phi}_{\lambda=1}$ play the role of the moment map.

The universal quasi-twisting

5 Construction of $\mathcal{T}_G^{(\kappa,E)}$

Let S be an affine scheme smooth over k. To an S-point $(\mathfrak{g}^\kappa, E)$ of Par_G, we shall functorially attach a quasi-twisting $\mathcal{T}_G^{(\kappa,E)}$ over $S \times \mathrm{Bun}_G$ (relative to S).

We proceed by first constructing a Lie-$*$ algebra $\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)}$ over $S \times X$, then twisting its pullback to $S \times \mathrm{Bun}_{G,\infty x} \times X$ by the tautological G-bundle $\tilde{\mathcal{P}}_G$. Via taking sections over $\overset{\circ}{D}_x$, we produce a classical quasi-twisting $\tilde{\mathcal{T}}_G^{(\kappa,E)}$ over $S \times \mathrm{Bun}_{G,\infty x}$. Then we show that $\tilde{\mathcal{T}}_G^{(\kappa,E)}$ admits an action by the pair $(\mathfrak{g}^\kappa(\mathcal{O}_x), \mathcal{L}_x^+ G)$, so we may form the quotient $\mathcal{T}_G^{(\kappa,E)} := \mathbf{Q}^{(\mathfrak{g}^\kappa(\mathcal{O}_x), \mathcal{L}_x^+ G)}(\tilde{\mathcal{T}}_G^{(\kappa,E)})$. This last step requires both quotient functors constructed in Sect. 4 and their compatibility.

We then verify that for a simple group G and $\mathfrak{g}^\kappa$ arising from the bilinear form $\kappa = \lambda \cdot \mathrm{Kil}$, the quasi-twisting $\mathcal{T}_G^{(\kappa,0)}$ identifies with the twisting given by λ-power of the determinant line bundle $\mathcal{L}_{G,\mathrm{det}}$ over Bun_G.

5.1 Recollection on Lie-$*$ algebras

5.1.1. Let $\mathcal{X} \to S$ be a smooth curve relative to S with connected fibers.[14] The diagonal morphism $\Delta : \mathcal{X} \to \mathcal{X} \underset{S}{\times} \mathcal{X}$ is a closed immersion. Denote by $\mathcal{D}_{\mathcal{X}/S}$-Mod^r the category of $\mathcal{O}_\mathcal{X}$-modules equipped with a right action of the relative differential operators $\mathcal{D}_{\mathcal{X}/S}$.

5.1.2. A *Lie-$*$ algebra* on $\mathcal{X}$ (relative to S) is an object $\mathcal{B} \in \mathcal{D}_{\mathcal{X}/S}$-Mod^r, equipped with a $\mathcal{D}_{\mathcal{X}\underset{S}{\times}\mathcal{X}/S}$-linear morphism[15] $[-,-] : \mathcal{B}^{\boxtimes 2} \to \Delta_!(\mathcal{B})$ such that the following properties are satisfied:

(a) (*anti-symmetry*) for all sections a, b of $\mathcal{B}$, there holds

$$\tilde{\sigma}_{12}([a \boxtimes b]) = -[b \boxtimes a],$$

where $\tilde{\sigma}_{12}$ is the transposition morphism over $\mathcal{X} \underset{S}{\times} \mathcal{X}$ given by:

$$\sigma_{12}^{-1}\Delta_!(\mathcal{B}) \to \Delta_!(\mathcal{B}); \quad \text{where} \quad \sigma_{12}(x, y) = (y, x).$$

(b) (*Jacobi identity*) for all sections a, b, and c of $\mathcal{B}$, there holds

$$[[a \boxtimes b] \boxtimes c] + \tilde{\sigma}_{123}([[b \boxtimes c] \boxtimes a]) + \tilde{\sigma}_{123}^2([[c \boxtimes a] \boxtimes b]) = 0,$$

where $\tilde{\sigma}_{123}$ denotes the morphism over $\mathcal{X} \underset{S}{\times} \mathcal{X} \underset{S}{\times} \mathcal{X}$ given by:

$$\sigma_{123}^{-1}(\Delta_{x=y=z})_!(\mathcal{B}) \to (\Delta_{x=y=z})_!(\mathcal{B}); \quad \text{where} \quad \sigma_{123}(x, y, z) = (y, z, x).$$

Denote by $\mathbf{Lie}^*(\mathcal{X}/S)$ the category of Lie-$*$ algebras on $\mathcal{X}$ relative to S. Clearly, for any morphism $S' \to S$ with $\mathcal{X}' := \mathcal{X} \underset{S}{\times} S'$, we have a functor $\mathbf{Lie}^*(\mathcal{X}/S) \to \mathbf{Lie}^*(\mathcal{X}'/S')$ acting as pulling back a $\mathcal{D}_{\mathcal{X}/S}$-module, and equipping it with the induced Lie-$*$ algebra structure.

5.1.3. Lie-$*$ algebras are étale local objects. More precisely, let $\acute{\mathbf{E}}\mathbf{t}_{/\mathcal{X}}$ be the small étale site of $\mathcal{X}$. Given $\mathcal{B} \in \mathbf{Lie}^*(\mathcal{U}/S)$ where $\mathcal{U} \in \acute{\mathbf{E}}\mathbf{t}_{/\mathcal{X}}$ and a morphism $\tilde{\mathcal{U}} \to \mathcal{U}$, we may associate an object $\mathcal{B}|_{\tilde{\mathcal{U}}} \in \mathbf{Lie}^*(\tilde{\mathcal{U}}/S)$. This procedure defines a functor in groupoids:

$$\acute{\mathbf{E}}\mathbf{t}_{/\mathcal{X}}^{\mathrm{op}} \to \mathrm{Gpd}, \quad \mathcal{U} \rightsquigarrow \mathbf{Lie}^*(\mathcal{U}/S). \tag{5.1}$$

[14] For our applications, we will take $\mathcal{X} := S \times X$.

[15] We use $\boxtimes$ to denote tensoring over $\mathcal{O}_S$.

The étale local nature of Lie-$*$ algebras refers to the fact that (5.1) satisfies descent.

5.1.4. Let $\mathcal{G}$ be a presheaf of group schemes on $\mathbf{\acute{E}t}_{/\mathcal{X}}$, and $\mathcal{B} \in \mathbf{Lie}^*(\mathcal{X}/S)$. A *$\mathcal{G}$-action* on $\mathcal{L}$ consists of the following data:

- for each $\mathcal{U} \in \mathbf{\acute{E}t}_{/\mathcal{X}}$, an action of $\mathcal{G}_{\mathcal{U}}$ as endomorphisms of $\mathcal{B}\big|_{\mathcal{U}} \in \mathbf{Lie}^*(\mathcal{U}/S)$;

furthermore, this action is required to be functorial in $\mathcal{U}$.

Suppose $\mathcal{P}$ is an étale $\mathcal{G}$-torsor over $\mathcal{X}$, and $\mathcal{B} \in \mathbf{Lie}^*(\mathcal{X}/S)$ admits a $\mathcal{G}$-action. Then we can form the *$\mathcal{P}$-twisted Lie-$*$ algebra* $\mathcal{B}_{\mathcal{P}} \in \mathbf{Lie}^*(\mathcal{X}/S)$ using the descent property of (5.1).

5.2 De Rham cohomology over the disc

5.2.1. Let $x \in X$ be a closed point. Write $\mathcal{X} := S \times X$ and $\underline{x} : S \to \mathcal{X}$ for the S-point determined by x. Let $D_{\underline{x}}$ be the completion of $\mathcal{X}$ at $\underline{x}$ and $\mathring{D}_{\underline{x}}$ be its open subscheme $D_{\underline{x}} - \{\underline{x}\}$. As S is assumed affine, we have $D_{\underline{x}} \xrightarrow{\sim} \mathrm{Spec}(\mathcal{O}_S \widehat{\otimes} \mathcal{O}_x)$ and $\mathring{D}_{\underline{x}} \xrightarrow{\sim} \mathrm{Spec}(\mathcal{O}_S \widehat{\otimes} \mathcal{K}_x)$, where $\mathcal{O}_x$ denotes the completed local ring at x, and $\mathcal{K}_x$ the localization of $\mathcal{O}_x$ at its uniformizer.

5.2.2. Following [3, Sects. 2.1.13, 2.1.16], there is a right-exact functor $\Gamma_{\mathrm{dR}}(D_{\underline{x}}, -)$ carrying $\mathcal{D}_{\mathcal{X}/S}$-modules to topological $\mathcal{O}_S$-modules. (It is the functor of *zero*th de Rham cohomology, denoted by $\hat{h}_x$ in *op.cit.*) Let $\Gamma_{\mathrm{dR}}(\mathring{D}_{\underline{x}}, -)$ denote the functor $\Gamma_{\mathrm{dR}}(D_{\underline{x}}, j_* j^* -)$ where $j : \mathcal{X} - \{\underline{x}\} \hookrightarrow \mathcal{X}$ is the open immersion. According to [3, Lemma 2.1.14], the functors $\Gamma_{\mathrm{dR}}(D_{\underline{x}}, -)$, $\Gamma_{\mathrm{dR}}(\mathring{D}_{\underline{x}}, -)$ carry *coherent* $\mathcal{D}_{\mathcal{X}/S}$-modules to Tate $\mathcal{O}_S$-modules.

Lemma 5.1 *There are canonical isomorphisms:*

$$\Gamma_{\mathrm{dR}}(D_{\underline{x}}, \omega_{\mathcal{X}/S}) \cong 0, \quad \Gamma_{\mathrm{dR}}(\mathring{D}_{\underline{x}}, \omega_{\mathcal{X}/S}) \cong \mathcal{O}_S.$$

Proof The Spencer complex defines a resolution of $\omega_{\mathcal{X}/S}$ by the complex $\mathcal{D}_{\mathcal{X}/S} \to \omega_{\mathcal{X}/S} \otimes \mathcal{D}_{\mathcal{X}/S}$. Applying $\Gamma_{\mathrm{dR}}(D_{\underline{x}}, -)$, this complex becomes $d : \mathcal{O}_S \widehat{\otimes} \mathcal{O}_x \to \mathcal{O}_S \widehat{\otimes} \omega_x$ (see [3, Sect. 2.1.13, Examples (i)]). The vanishing of $\Gamma_{\mathrm{dR}}(D_{\underline{x}}, \omega_{\mathcal{X}/S})$ thus follows. The calculation of $\Gamma_{\mathrm{dR}}(\mathring{D}_{\underline{x}}, \omega_{\mathcal{X}/S})$ follows from the canonical triangle $i_! i^!(\omega_{\mathcal{X}/S}) \to \omega_{\mathcal{X}/S} \to j_* j^* \omega_{\mathcal{X}/S}$ (for $i : S \hookrightarrow \mathcal{X}$ denoting the closed immersion $\underline{x}$) and the isomorphism $i^!(\omega_{\mathcal{X}/S}) \cong \mathcal{O}_S[-1]$. □

5.2.3. Given a Lie-$*$ algebra $\mathcal{B}$, the object $\Gamma_{\mathrm{dR}}(\mathring{D}_{\underline{x}}, \mathcal{B})$ acquires the structure of a Lie algebra in $\mathrm{QCoh}^{\mathrm{Tate}}(S)$, whose (continuous) Lie bracket is given by the composition:

$$\begin{aligned}[-,-] : \Gamma_{\mathrm{dR}}(\mathring{D}_{\underline{x}}, \mathcal{B})^{\boxtimes 2} &\xrightarrow{\sim} \Gamma_{\mathrm{dR}}(\mathring{D}_{\underline{x}} \underset{S}{\times} \mathring{D}_{\underline{x}}, \mathcal{B}^{\boxtimes 2}) \\ &\to \Gamma_{\mathrm{dR}}(\mathring{D}_{\underline{x}} \underset{S}{\times} \mathring{D}_{\underline{x}}, \Delta_!(\mathcal{B})) \xrightarrow{\sim} \Gamma_{\mathrm{dR}}(\mathring{D}_{\underline{x}}, \mathcal{B}).\end{aligned}$$

The map $\Gamma_{\mathrm{dR}}(D_{\underline{x}}, \mathcal{B}) \to \Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{x}}, \mathcal{B})$ realizes $\Gamma_{\mathrm{dR}}(D_{\underline{x}}, \mathcal{B})$ as a Lie subalgebra if $\mathcal{B}$ is $\mathcal{O}_{\mathcal{X}}$-flat.

5.3 The Kac-Moody Lie-$*$ algebra

5.3.1. Suppose now that S is equipped with a morphism $S \to \mathrm{Par}_G$, represented by $(\mathfrak{g}^\kappa, E)$ (see Sect. 2). We will construct a central extension of Lie-$*$ algebras over $\mathcal{X} := S \times X$:

$$0 \to \omega_{\mathcal{X}/S} \to \widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)} \to \mathfrak{g}_{\mathcal{D}}^{\kappa} \to 0, \tag{5.2}$$

together with $\mathcal{G}$-actions on $\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)}$ and $\mathfrak{g}_{\mathcal{D}}^{\kappa}$, where $\mathcal{G}$ is the presheaf of group schemes $\mathcal{G}_{\mathcal{U}} := \underline{\mathrm{Maps}}(\mathcal{U}, G)$ on $\acute{\mathbf{E}}\mathbf{t}_{/\mathcal{X}}$. The construction will be functorial in S.

Remark 5.2 The central extension (5.2), together with the $\mathcal{G}$-action, is called the *(generalized) Kac-Moody* central extension of Lie-$*$ algebras, and we refer to $\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)}$ as the *(generalized) Kac-Moody* Lie-$*$ algebra.

5.3.2. The Lie-$*$ algebra $\mathfrak{g}_{\mathcal{D}}^{\kappa}$ has underlying $\mathcal{D}_{\mathcal{X}/S}$-module $\mathfrak{g}^\kappa \boxtimes \mathcal{D}_{\mathcal{X}/S}$. Its Lie-$*$ algebra structure is defined using the Lie bracket (2.18) on $\mathfrak{g}^\kappa$:

$$[-,-] : (\mathfrak{g}_{\mathcal{D}}^{\kappa})^{\boxtimes 2} \to \Delta_!(\mathfrak{g}_{\mathcal{D}}^{\kappa}), \quad (\mu \otimes \mathbf{1}) \boxtimes (\mu' \otimes \mathbf{1}) \rightsquigarrow [\mu, \mu'] \otimes \mathbf{1}_{\mathcal{D}},$$

where $\mathbf{1}_{\mathcal{D}}$ is the canonical symmetric section of $\Delta_!(\mathcal{D}_{\mathcal{X}/S})$. Note that the Lie-$*$ bracket $[-,-]$ factors through the embedding $\mathfrak{g}_{\mathrm{s.s.}}^{\kappa} \boxtimes \mathcal{D}_{\mathcal{X}/S} \hookrightarrow \mathfrak{g}_{\mathcal{D}}^{\kappa}$.[16]

We construct a $\mathcal{G}$-action on $\mathfrak{g}_{\mathcal{D}}^{\kappa}$ as follows: for every $\mathcal{U} \in \acute{\mathbf{E}}\mathbf{t}_{/\mathcal{X}}$, there is an *adjoint-coadjoint* action of the group scheme $\underline{\mathrm{Maps}}(\mathcal{U}, G)$ on $\mathfrak{g}^\kappa \otimes \mathcal{O}_{\mathcal{U}}$:

$$g_{\mathcal{U}} \cdot (\xi \oplus \varphi) = \mathrm{Ad}_{g_{\mathcal{U}}}(\xi) \oplus \mathrm{Coad}_{g_{\mathcal{U}}}(\varphi). \tag{5.3}$$

where $\xi \oplus \varphi$ denotes a section of $\mathfrak{g}^\kappa \otimes \mathcal{O}_{\mathcal{U}}$, regarded as a subbundle of $(\mathfrak{g} \otimes \mathcal{O}_{\mathcal{U}}) \oplus (\mathfrak{g}^* \otimes \mathcal{O}_{\mathcal{U}})$. The action (5.3) extends to an action of $\underline{\mathrm{Maps}}(\mathcal{U}, G)$ on $\mathfrak{g}^\kappa \underset{\mathcal{O}_{\mathcal{U}}}{\otimes} \mathcal{D}_{\mathcal{U}/S}$ by Lie-$*$ algebra endomorphisms.

5.3.3. The underlying $\mathcal{D}_{\mathcal{X}/S}$-modules of (5.2) are defined by first inducing a sequence of $\mathcal{D}_{\mathcal{X}/S}$-modules from (2.20):

$$0 \to \omega_{\mathcal{X}/S} \underset{\mathcal{O}_{\mathcal{X}}}{\otimes} \mathcal{D}_{\mathcal{X}/S} \to \widehat{\mathfrak{g}}^{\kappa} \underset{\mathcal{O}_{\mathcal{X}}}{\otimes} \mathcal{D}_{\mathcal{X}/S} \to \mathfrak{g}^\kappa \boxtimes \mathcal{D}_{\mathcal{X}/S} \to 0 \tag{5.4}$$

and then taking the push-out along the action map $\omega_{\mathcal{X}/S} \underset{\mathcal{O}_{\mathcal{X}}}{\otimes} \mathcal{D}_{\mathcal{X}/S} \to \omega_{\mathcal{X}/S}$.

[16] See §2.3.2 for the notation $\mathfrak{g}_{\mathrm{s.s.}}^{\kappa}$.

In particular, the extension $\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)} \to \mathfrak{g}_{\mathcal{D}}^{\kappa}$ splits over $\mathfrak{g}_{\mathrm{s.s.}}^{\kappa} \boxtimes \mathcal{D}_{\mathcal{X}/S}$, and we have a decomposition

$$\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)} \xrightarrow{\sim} E_{\mathcal{D}} \oplus (\mathfrak{g}_{\mathrm{s.s.}}^{\kappa} \boxtimes \mathcal{D}_{\mathcal{X}/S}). \tag{5.5}$$

where $E_{\mathcal{D}}$ is the push-out of $E \underset{\mathcal{O}_{\mathcal{X}}}{\otimes} \mathcal{D}_{\mathcal{X}/S}$ along $\omega_{\mathcal{X}/S} \underset{\mathcal{O}_{\mathcal{X}}}{\otimes} \mathcal{D}_{\mathcal{X}/S} \to \omega_{\mathcal{X}/S}$.

5.3.4. The Lie-$*$ algebra structure on $\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)}$ is defined by the composition:

$$(\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)})^{\boxtimes 2} \to (\mathfrak{g}_{\mathcal{D}}^{\kappa})^{\boxtimes 2} \to \Delta_!(\omega_{\mathcal{X}/S}) \oplus \Delta_!(\mathfrak{g}_{\mathrm{s.s.}}^{\kappa} \boxtimes \mathcal{D}_{\mathcal{X}/S}) \to \Delta_!(\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)})$$

where the middle map is defined using the bilinear form (2.19) and the Lie bracket (2.18) on $\mathfrak{g}^{\kappa}$:

$$(\mu \otimes \mathbf{1}) \boxtimes (\mu' \otimes \mathbf{1}) \rightsquigarrow (\mu, \mu')\mathbf{1}'_{\omega} + [\mu, \mu'] \otimes \mathbf{1}_{\mathcal{D}};$$

the notation $\mathbf{1}'_{\omega}$ denotes the canonical anti-symmetric section of $\Delta_!(\omega_{\mathcal{X}/S})$.

5.3.5. We now construct the $\mathcal{G}$-action on $\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)}$. Let $\mathcal{U} \in \acute{\mathbf{E}}\mathbf{t}_{/\mathcal{X}}$ and $g_{\mathcal{U}}$ be a point of $\underline{\mathrm{Maps}}(\mathcal{U}, G)$. The corresponding endomorphism $g_{\mathcal{U}} : \widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)} \to \widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)}$ is defined by the sum of the following maps (using the decomposition (5.5)):

(a) identity on $E_{\mathcal{D}}$;
(b) adjoint-coadjoint action on $\mathfrak{g}_{\mathrm{s.s.}}^{\kappa} \boxtimes \mathcal{D}_{\mathcal{U}/S}$ by formula (5.3);
(c) the composition:

$$\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)}\big|_{\mathcal{U}} \to \mathfrak{g}_{\mathcal{D}}^{\kappa}\big|_{\mathcal{U}} \xrightarrow{\sim} (\mathfrak{g}^{\kappa} \boxtimes \mathcal{O}_{\mathcal{U}}) \underset{\mathcal{O}_{\mathcal{U}}}{\otimes} \mathcal{D}_{\mathcal{X}/S} \xrightarrow{\mathrm{res}(g_{\mathcal{U}})} \omega_{\mathcal{U}/S} \hookrightarrow \widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)}\big|_{\mathcal{U}} \tag{5.6}$$

where the map $\mathrm{res}(g_{\mathcal{U}})$ is defined by the formula:

$$(\xi \oplus \varphi) \otimes \mathbf{1} \rightsquigarrow \varphi(g_{\mathcal{U}}^{-1} dg_{\mathcal{U}}), \quad \xi \oplus \varphi \in \mathfrak{g}^{\kappa} \boxtimes \mathcal{O}_{\mathcal{U}}.$$

Here, $d : \mathcal{O}_{\mathcal{U}} \to \omega_{\mathcal{U}/S}$ is the exterior derivative, so $g_{\mathcal{U}}^{-1} dg_{\mathcal{U}}$ is a section of $\mathfrak{g} \boxtimes \omega_{\mathcal{U}/S}$, on which φ rightfully acts.

It is clear from the construction that $\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)} \to \mathfrak{g}_{\mathcal{D}}^{\kappa}$ is $\mathcal{G}$-equivariant.

Remark 5.3 If $\mathfrak{g}^{\kappa}$ arises from a symmetric bilinear form κ (see Sect. 2), then we have an isomorphism $\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,0)} \xrightarrow{\sim} \mathcal{B}(\mathfrak{g}, \kappa)$ where $\mathcal{B}(\mathfrak{g}, \kappa)$ is the Kac-Moody Lie-$*$ algebra at level κ in the ordinary sense (see [11]). On the other hand, the Lie-$*$ algebra $\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\infty,0)}$ is given by $\omega_{\mathcal{X}/S} \oplus \mathfrak{g}_{\mathcal{D}}^{*}$ with zero Lie-$*$ bracket (but a nontrivial $\mathcal{G}$-action).

5.3.6. Let us bring in the closed point $x \in X$, which induces a section $\underline{x} : S \to \mathcal{X}$. Applying $\Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{x}}, -)$ to the sequence (5.2) and using Lemma 5.1, we obtain a central extension of Lie algebras in $\mathrm{QCoh}^{\mathrm{Tate}}(S)$:

$$0 \to \mathcal{O}_S \to \widehat{\mathfrak{g}}^{(\kappa,E)} \to \mathfrak{g}^{\kappa}(\mathcal{K}_x) \to 0, \tag{5.7}$$

where the notation $\mathfrak{g}^{\kappa}(\mathcal{O}_x)$ (resp. $\mathfrak{g}^{\kappa}(\mathcal{K}_x)$) denotes the Tate $\mathcal{O}_S$-module $\mathfrak{g}^{\kappa}\widehat{\otimes}\mathcal{O}_x$ (resp. localization at the uniformizer of $\mathcal{O}_x$).

The Lie bracket on $\widehat{\mathfrak{g}}^{(\kappa,E)}$ is given by the composition:

$$(\widehat{\mathfrak{g}}^{(\kappa,E)})^{\boxtimes 2} \to (\mathfrak{g}^{\kappa}(\mathcal{K}_x))^{\boxtimes 2} \to \mathcal{O}_S \oplus \mathfrak{g}^{\kappa}_{\mathrm{s.s.}}(\mathcal{K}_x) \to \widehat{\mathfrak{g}}^{(\kappa,E)},$$

where the middle map is defined by

$$(\mu \otimes f) \boxtimes (\mu' \otimes f') \rightsquigarrow (\mu, \mu') \cdot \mathrm{Res}((df) f') + [\mu, \mu'] \otimes ff'.$$

Lemma 5.4 *The central extension (5.7) canonically splits over* $\mathfrak{g}^{\kappa}(\mathcal{O}_x)$.

Proof The result follows from applying $\Gamma_{\mathrm{dR}}(D_{\underline{x}}, -)$ to the sequence (5.2) and observing that $\Gamma_{\mathrm{dR}}(D_{\underline{x}}, \omega_{\mathcal{X}/S})$ vanishes (Lemma 5.1). □

Let $\mathcal{L}_x G$ (resp. $\mathcal{L}^+_x G$) denote the loop (resp. arc) group of G at x. There is an action of $\mathcal{L}_x G$ on $\widehat{\mathfrak{g}}^{(\kappa,E)}$ defined analogously to §5.3.5, with the composition (5.6) replaced by:

$$\widehat{\mathfrak{g}}^{(\kappa,E)} \to \mathfrak{g}^{\kappa}(\mathcal{K}_x) \xrightarrow{\mathrm{res}(g)} \mathcal{O}_S \hookrightarrow \widehat{\mathfrak{g}}^{(\kappa,E)}$$

where the map $\mathrm{res}(g)$ (g is a point of $\mathcal{L}_x G$) is defined by the formula:

$$(\xi \oplus \varphi) \otimes f \rightsquigarrow \mathrm{Res}(f \cdot \varphi(g^{-1} dg)).$$

Since the Lie algebra of $\mathcal{L}_x G$ identifies with $\mathfrak{g}(\mathcal{K}_x)$, this $\mathcal{L}_x G$-action induces a $\mathfrak{g}(\mathcal{K}_x)$-action on $\widehat{\mathfrak{g}}^{(\kappa,E)}$ by $\mathcal{O}_S$-linear endomorphisms.

Lemma 5.5 *The Lie bracket on* $\widehat{\mathfrak{g}}^{(\kappa,E)}$ *agrees with the composition:*

$$(\widehat{\mathfrak{g}}^{(\kappa,E)})^{\boxtimes 2} \xrightarrow{(\mathrm{pr},\mathrm{id})} \mathfrak{g}(\mathcal{K}_x) \boxtimes \widehat{\mathfrak{g}}^{(\kappa,E)} \xrightarrow{\mathrm{act}} \widehat{\mathfrak{g}}^{(\kappa,E)}.$$

Proof This is a straightforward computation. □

5.4 The classical quasi-twisting $\tilde{\mathcal{T}}_G^{(\kappa,E)}$ over $\mathrm{Bun}_{G,\infty x}$

5.4.1. Let $\mathrm{Bun}_{G,\infty x}$ denote the stack classifying pairs $(\mathcal{P}_G, \alpha)$ where $\mathcal{P}_G$ is a G-bundle on X and $\alpha : \mathcal{P}_G\big|_{D_x} \xrightarrow{\sim} \mathcal{P}^0_G$ is a trivialization over D_x. The (right) $\mathcal{L}^+_x G$-action on

$\mathrm{Bun}_{G,\infty x}$ by changing α realizes $\mathrm{Bun}_{G,\infty x}$ as a $\mathcal{L}_x^+G$-bundle over Bun_G, locally trivial in the étale topology. In particular, $\mathrm{Bun}_{G,\infty x}$ is placid; see Sect. 3.2.

5.4.2. The Beauville-Laszlo theorem shows that $\mathrm{Bun}_{G,\infty x}$ also classifies pairs $(\mathcal{P}_{G,\Sigma}, \alpha)$, where $\mathcal{P}_{G,\Sigma}$ is a G-bundle on $\Sigma := X - \{x\}$ and $\alpha : \mathcal{P}_{G,\Sigma}\big|_{\overset{\circ}{D}_x} \xrightarrow{\sim} \mathcal{P}_G^0$ is a trivialization over $\overset{\circ}{D}_x$. This alternative description shows that the $\mathcal{L}_x^+G$-action on $\mathrm{Bun}_{G,\infty x}$ extends to an $\mathcal{L}_x G$-action.

5.4.3. Fix an S-point $(\mathfrak{g}^\kappa, E)$ of Par_G. We apply the construction of Sect. 5.3 to the relative curve

$$\tilde{\mathcal{X}} := S \times \mathrm{Bun}_{G,\infty x} \times X \quad \text{over} \quad \tilde{\mathcal{S}} := S \times \mathrm{Bun}_{G,\infty x},$$

and obtain a central extension in $\mathbf{Lie}^*(\tilde{\mathcal{X}}/\tilde{\mathcal{S}})$:

$$0 \to \omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}} \to \widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)} \to \mathfrak{g}_{\mathcal{D}}^{\kappa} \to 0. \tag{5.8}$$

In other words, (5.8) is the image of Kac-Moody extension (5.2) under the base change functor $- \boxtimes \mathcal{O}_{\mathrm{Bun}_{G,\infty x}} : \mathbf{Lie}^*(\mathcal{X}/S) \to \mathbf{Lie}^*(\tilde{\mathcal{X}}/\tilde{\mathcal{S}})$.

Let $\underline{\tilde{x}} : \tilde{\mathcal{S}} \hookrightarrow \tilde{\mathcal{X}}$ (resp. $\underline{x} : S \hookrightarrow \mathcal{X}$) denote the section given by $x \in X$. Let $\tilde{\mathcal{P}}_G$ be the tautological G-bundle over $\tilde{\mathcal{X}}$ equipped with the trivialization α over $D_{\underline{\tilde{x}}}$. Since $\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)}$ and $\mathfrak{g}_{\mathcal{D}}^{\kappa}$ are equipped with $\mathcal{G}$-actions, we can form the $\tilde{\mathcal{P}}_G$-twist of (5.8):

$$0 \to \omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}} \to (\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)})_{\tilde{\mathcal{P}}_G} \to (\mathfrak{g}_{\mathcal{D}}^{\kappa})_{\tilde{\mathcal{P}}_G} \to 0. \tag{5.9}$$

Remark 5.6 (a) Since $\mathfrak{g}_{\mathcal{D}}^{\kappa}$ is the $\mathcal{D}_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}}$-module induced from $\mathfrak{g}^\kappa \boxtimes \mathcal{O}_{\mathrm{Bun}_{G,\infty x} \times X}$ and the $\mathcal{G}$-action comes from one on $\mathfrak{g}^\kappa \boxtimes \mathcal{O}_{\mathrm{Bun}_{G,\infty x} \times X}$, we see that $(\mathfrak{g}_{\mathcal{D}}^{\kappa})_{\tilde{\mathcal{P}}_G}$ is the $\mathcal{D}_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}}$-module induced from $\mathfrak{g}^{\kappa}_{\tilde{\mathcal{P}}_G}$.

(b) the datum of α gives an isomorphism between (5.8) and (5.9) when restricted to $D_{\tilde{x}}$.

5.4.4. We apply the functors $\Gamma_{\mathrm{dR}}(\Sigma, -)$ and $\Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, -)$ to (5.9). Using the two observations above, we obtain a morphism between two triangles in $\mathrm{QCoh}^{\mathrm{Tate}}(\tilde{\mathcal{S}})$:

$$\begin{array}{ccccc} \Gamma_{\mathrm{dR}}(\Sigma, \omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}}) & \longrightarrow & \Gamma_{\mathrm{dR}}(\Sigma, (\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)})_{\tilde{\mathcal{P}}_G}) & \longrightarrow & \Gamma(\Sigma, \mathfrak{g}^{\kappa}_{\tilde{\mathcal{P}}_G}) \\ \downarrow & & \downarrow & \overset{\widehat{\gamma}}{\swarrow} & \downarrow \gamma \\ \Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}}) & \longrightarrow & \Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)}) & \longrightarrow & \mathfrak{g}^\kappa(\mathcal{K}_x)\widehat{\boxtimes}\mathcal{O}_{\mathrm{Bun}_{G,\infty x}} \end{array} \tag{5.10}$$

where $\mathfrak{g}^\kappa(\mathcal{K}_x)$ is (as before) an object of $\mathrm{QCoh}^{\mathrm{Tate}}(S)$.

Since $\omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}}$ has top de Rham cohomology (along $\tilde{\mathcal{X}} \to \tilde{\mathcal{S}}$) isomorphic to $\mathcal{O}_{\tilde{\mathcal{S}}}$, one may conclude that the first vertical map in (5.10) vanishes by comparing the canonical

triangles associated to open immersions $\Sigma \subset X$ and $\mathring{D}_{\underline{\tilde{x}}} \subset D_{\underline{\tilde{x}}}$.[17] Hence we obtain a splitting $\widehat{\gamma}$ as depicted. Note that γ (hence $\widehat{\gamma}$) is injective, so we may define two Tate $\mathcal{O}_{\tilde{\mathcal{S}}}$-modules by cokernels without running into DG issues:

$$\widehat{\mathcal{L}}^{(\kappa,E)} := \mathrm{Coker}(\widehat{\gamma}), \quad \mathcal{L}^{\kappa} := \mathrm{Coker}(\gamma).$$

Since $\Gamma_{\mathrm{dR}}(\mathring{D}_{\underline{\tilde{x}}}, \omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}})$ is canonically isomorphic to $\mathcal{O}_{\tilde{\mathcal{S}}}$ (Lemma 5.1), we arrive at an exact sequence of Tate $\mathcal{O}_{\tilde{\mathcal{S}}}$-modules:

$$0 \to \mathcal{O}_{\tilde{\mathcal{S}}} \to \widehat{\mathcal{L}}^{(\kappa,E)} \to \mathcal{L}^{\kappa} \to 0. \tag{5.11}$$

Notation 5.7 In what follows, we will show that (5.11) has the structure of a classical quasi-twisting (on Tate modules) over $\tilde{\mathcal{S}}$ (relative to S; see Sect. 3.2.5), to be denote by $\tilde{\mathcal{T}}_G^{(\kappa,E)}$.

5.4.5. We (temporarily) use the notation $\widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D},\mathcal{X}}$ to denote the Kac-Moody Lie-$*$ algebra over $\mathcal{X}$, constructed using the recipe in Sect. 5.3 for the relative curve $\mathcal{X} \to S$.

The isomorphism $\widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}} \xrightarrow{\sim} \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D},\mathcal{X}} \boxtimes \mathcal{O}_{\mathrm{Bun}_{G,\infty x}}$ gives rise to an isomorphism in $\mathrm{QCoh}^{\mathrm{Tate}}(\tilde{\mathcal{S}})$:

$$\Gamma_{\mathrm{dR}}(\mathring{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}}) \xrightarrow{\sim} \Gamma_{\mathrm{dR}}(\mathring{D}_{\underline{x}}, \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D},\mathcal{X}})\widehat{\boxtimes}\mathcal{O}_{\mathrm{Bun}_{G,\infty x}} \cong \widehat{\mathfrak{g}}^{(\kappa,E)}\widehat{\boxtimes}\mathcal{O}_{\mathrm{Bun}_{G,\infty x}} \tag{5.12}$$

Observe that the $G(\mathcal{K}_x)$-action on $\mathrm{Bun}_{G,\infty x}$ gives rise to a $\mathfrak{g}(\mathcal{K}_x)$-action[18] on $\mathcal{O}_{\mathrm{Bun}_{G,\infty x}}$ by derivations. Hence, the Lie (algebroid) bracket on $\Gamma_{\mathrm{dR}}(\mathring{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}})$ can be defined using the $\mathcal{O}_S$-linear Lie bracket on $\widehat{\mathfrak{g}}^{(\kappa,E)}$ (see Sect. 5.3.6):

$$[\mu \boxtimes f, \mu' \boxtimes f'] := [\mu, \mu'] + \overline{\mu}(f') \cdot \mu' - \overline{\mu}'(f) \cdot \mu.$$

where $\overline{\mu}$ denotes the image of $\mu \in \widehat{\mathfrak{g}}^{(\kappa,E)}$ along $\widehat{\mathfrak{g}}^{(\kappa,E)} \to \mathfrak{g}^{\kappa}(\mathcal{K}_x) \to \mathfrak{g}(\mathcal{K}_x)\widehat{\boxtimes}\mathcal{O}_S$, which acts on $\mathcal{O}_{\tilde{\mathcal{S}}}$ by $\mathcal{O}_S$-linear derivations. The anchor map $\widehat{\sigma}$ of $\Gamma_{\mathrm{dR}}(\mathring{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}})$ is defined by the composition:

$$\Gamma_{\mathrm{dR}}(\mathring{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}}) \xrightarrow{(5.12)} \widehat{\mathfrak{g}}^{(\kappa,E)}\widehat{\boxtimes}\mathcal{O}_{\mathrm{Bun}_{G,\infty x}} \to \mathfrak{g}(\mathcal{K}_x)\widehat{\boxtimes}\mathcal{O}_{\tilde{\mathcal{S}}} \to \mathcal{T}_{\tilde{\mathcal{S}}/S}. \tag{5.13}$$

We have thus equipped $\Gamma_{\mathrm{dR}}(\mathring{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}})$ with the structure of a Lie algebroid. The following lemma, whose proof is deferred to Sect. 5.4.6, extends this Lie algebroid structure to its quotient $\widehat{\mathcal{L}}^{(\kappa,E)}$:

[17] This vanishing is also reflected in the classical fact that the sum of residues of a meromorphic form is zero.

[18] Unlike the Tate $\mathcal{O}_S$-module $\mathfrak{g}^{\kappa}(\mathcal{K}_x)$, the notation $\mathfrak{g}(\mathcal{K}_x)$ is reserved for the Tate vector space $\mathfrak{g} \otimes \mathcal{K}_x$ (similar for the notation $\mathfrak{g}(\mathcal{O}_x)$).

Lemma 5.8 *The morphism $\widehat{\gamma}$ realizes $\Gamma(\Sigma, \mathfrak{g}^{\kappa}_{\tilde{\mathcal{P}}_G})$ as an ideal of $\Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}})$.*

In an analogous way, we turn $\mathfrak{g}^{\kappa}(\mathcal{K}_x)\widehat{\boxtimes}\mathcal{O}_{\mathrm{Bun}_{G,\infty x}}$ into an object of $\mathrm{LieAlgd}(\tilde{\mathcal{S}}/S)$, and the map $\Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}}) \to \mathfrak{g}^{\kappa}(\mathcal{K}_x)\widehat{\boxtimes}\mathcal{O}_{\mathrm{Bun}_{G,\infty x}}$ in (5.10) is a morphism of such. Lemma 5.8 shows that γ also realizes $\Gamma(\Sigma, \mathfrak{g}^{\kappa}_{\tilde{\mathcal{P}}_G})$ as an ideal of $\mathfrak{g}^{\kappa}(\mathcal{K}_x)\widehat{\boxtimes}\mathcal{O}_{\mathrm{Bun}_{G,\infty x}}$. Hence the cokernels (5.11) is a central extension of Lie algebroids.

5.4.6 Proof of Lemma 5.8

We first give an alternative description of the Lie bracket on $\Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}})$. Indeed, from the identification in (5.12) and the $\mathfrak{g}(\mathcal{K}_x)$-action on $\widehat{\mathfrak{g}}^{(\kappa,E)}$ (see Sect. 5.3.6), we obtain an action of $\mathfrak{g}(\mathcal{K}_x)\widehat{\boxtimes}\mathcal{O}_{\tilde{\mathcal{S}}}$ on $\Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}})$ by $\mathcal{O}_S$-linear derivations. It follows from Lemma 5.5 that the Lie bracket on $\Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}})$ agrees with the composition:

$$\Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}})^{\boxtimes 2} \xrightarrow{(\mathrm{pr},\mathrm{id})} (\mathfrak{g}(\mathcal{K}_x)\widehat{\boxtimes}\mathcal{O}_{\tilde{\mathcal{S}}}) \boxtimes \Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}}) \xrightarrow{\mathrm{act}} \Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}}), \tag{5.14}$$

where pr denotes the composition of the first two maps in (5.13).

Therefore, it suffices to show that the Tate $\mathcal{O}_{\tilde{\mathcal{S}}}$-submodule:

$$\Gamma_{\mathrm{dR}}(\Sigma, (\widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}})_{\tilde{\mathcal{P}}_G}) \hookrightarrow \Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}}) \tag{5.15}$$

is invariant under the aforementioned $\mathfrak{g}(\mathcal{K}_x)\widehat{\boxtimes}\mathcal{O}_{\tilde{\mathcal{S}}}$-action. Note that by construction, this action arises from the $S \times \mathcal{L}_x G$-equivariance structure on $\Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}})$. The following claim is immediate:

Claim 5.9 There is also an $S \times \mathcal{L}_x G$-equivariance structure on $\Gamma_{\mathrm{dR}}(\Sigma, (\widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}})_{\tilde{\mathcal{P}}_G})$, defined at every T-point $(s, \mathcal{P}_{G,\Sigma}, \alpha, g)$ of $S \times \mathrm{Bun}_{G,\infty x} \times \mathcal{L}_x G$ (for $T \in \mathbf{Sch}^{\mathrm{aff}}_{/k}$) by:

(a) first identifying the fiber of $\Gamma_{\mathrm{dR}}(\Sigma, (\widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}})_{\tilde{\mathcal{P}}_G})$ at both of the T-points

$$(s, \mathcal{P}_{G,\Sigma}, \alpha), \text{ and } (s, \mathcal{P}_{G,\Sigma}, g \cdot \alpha), \quad g \in \mathrm{Maps}(T, \mathcal{L}_x G),$$

with $\Gamma_{\mathrm{dR}}(\Sigma, (\widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}})_{\mathcal{P}_{G,\Sigma}})$;[19]

(b) relating the above two fibers via the identity map on $\Gamma_{\mathrm{dR}}(\Sigma, (\widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}})_{\mathcal{P}_{G,\Sigma}})$. □

[19] We are slightly abusing the notation $(\widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}})_{\mathcal{P}_{G,\Sigma}}$, since this is now the Kac-Moody extension associated to the parameter $T \xrightarrow{s} S \xrightarrow{(\mathfrak{g}^{\kappa},E)} \mathrm{Par}_G$, twisted by $\mathcal{P}_{G,\Sigma}$ on the open curve $T \times \Sigma$.

So we have reduced the problem to showing that (5.15) preserves the $S \times \mathcal{L}_x G$-equivariance structure. In other words, the following diagram in $\mathrm{QCoh}^{\mathrm{Tate}}(T)$ needs to commute:

$$\begin{array}{ccccc}
\Gamma_{\mathrm{dR}}(\Sigma, (\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)})_{\mathcal{P}_{G,\Sigma}}) & \xrightarrow{\sim} & \Gamma_{\mathrm{dR}}(\Sigma, (\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)})_{\mathcal{P}_G})\big|_{(s,\mathcal{P}_{G,\Sigma},\alpha)} & \xrightarrow{(5.15)} & \Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)}) \\
\downarrow \mathrm{id} & & & & \downarrow g\cdot \\
\Gamma_{\mathrm{dR}}(\Sigma, (\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)})_{\mathcal{P}_{G,\Sigma}}) & \xrightarrow{\sim} & \Gamma_{\mathrm{dR}}(\Sigma, (\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)})_{\mathcal{P}_G})\big|_{(s,\mathcal{P}_{G,\Sigma},g\cdot\alpha)} & \xrightarrow{(5.15)} & \Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)}).
\end{array} \tag{5.16}$$

Here, the two horizontal compositions express the procedure of

(a) first restricting a flat section of $(\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)})_{\mathcal{P}_{G,\Sigma}}$ to $\overset{\circ}{D}_{\underline{\tilde{x}}} \hookrightarrow T \times \Sigma$;
(b) then using the trivialization α (respectively, $g \cdot \alpha$) to identify it with a section of $\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)}$.

However, the following diagram is tautologically commutative:

$$\begin{array}{ccc}
\Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, (\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)})_{\mathcal{P}_{G,\Sigma}}) & \xrightarrow{\alpha_*} & \Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)}) \\
\downarrow \mathrm{id} & & \downarrow g\cdot \\
\Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, (\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)})_{\mathcal{P}_{G,\Sigma}}) & \xrightarrow{(g\cdot\alpha)_*} & \Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)}),
\end{array}$$

so we obtain the commutativity of (5.16). □ (Lemma 5.8)

5.5 Descent to Bun_G

5.5.1. We continue to fix the S-point $(\mathfrak{g}^\kappa, E)$ of Par_G. The goal of this section is to "descend" the classical quasi-twisting $\tilde{\mathcal{T}}_G^{(\kappa,E)}$ to Bun_G. Recall the action of $H := S \times \mathcal{L}_x^+ G$ on $\tilde{\mathcal{S}} = S \times \mathrm{Bun}_{G,\infty x}$, whose quotient is given by $\tilde{\mathcal{S}}/H \xrightarrow{\sim} S \times \mathrm{Bun}_G$. Let $\mathfrak{k} := \mathfrak{g}^\kappa(\mathcal{O}_x)$. Then $(\mathfrak{k}, H)$ forms a classical action pair (see Sect. 4.1).

5.5.2. We now equip (5.11) with the structure of a $(\mathfrak{k}, H)$-action. Indeed, applying the functor $\Gamma(D_{\underline{\tilde{x}}}, -)$ to (5.9) and using $\Gamma_{\mathrm{dR}}(D_{\underline{\tilde{x}}}, \omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}}) = 0$ (Lemma 5.1), we obtain a commutative diagram:

$$\begin{array}{ccccc}
 & & \Gamma_{\mathrm{dR}}(D_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)}) & \xrightarrow{\sim} & \Gamma(D_{\underline{\tilde{x}}}, \mathfrak{g}^\kappa \boxtimes \mathcal{O}_{\mathrm{Bun}_{G,\infty x} \times X}) \\
 & & \downarrow & \overset{\widehat{\eta}}{\swarrow} & \downarrow \eta \\
\Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}}) & \to & \Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)}) & \longrightarrow & \mathfrak{g}^\kappa(\mathcal{K}_x) \widehat{\boxtimes} \mathcal{O}_{\mathrm{Bun}_{G,\infty x}}
\end{array} \tag{5.17}$$

where the splitting $\widehat{\eta}$ exists for obvious reasons. Since $\Gamma(D_{\underline{\tilde{x}}}, \mathfrak{g}^\kappa \boxtimes \mathcal{O}_{\mathrm{Bun}_{G,\infty x} \times X})$ is canonically isomorphic to $\mathfrak{k} \widehat{\otimes} \mathcal{O}_{\tilde{\mathcal{S}}}$, we obtain the $(\mathfrak{k}, H)$-action datum on $\widehat{\mathcal{L}}^{(\kappa,E)}$ via the

composition:

$$\mathfrak{k}\widehat{\otimes}\mathcal{O}_{\tilde{\mathcal{S}}} \xrightarrow{\widehat{\eta}} \Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)}) \to \widehat{\mathcal{L}}^{(\kappa,E)},$$

which we again denote by $\widehat{\eta}$.

Remark 5.10 Ideally, we would like to directly define $\mathcal{T}^{(\kappa,E)}$ as the quotient $\mathbf{Q}^{(\mathfrak{k},H)}(\tilde{\mathcal{T}}^{(\kappa,E)})$. However, we run into problems because $\tilde{\mathcal{S}}$ is not locally of finite type (so we cannot use $\mathbf{Q}^{(H,H^\flat)}$ (4.20)), and $\widehat{\eta}$ is not injective (so we cannot use $\mathbf{Q}_{\mathrm{inj}}^{(\mathfrak{k},H)}$ (4.6)). In what follows, we circumvent this technical problem using a combination of the two functors.

5.5.3. For each integer $n \geq 0$, let $\mathrm{Bun}_{G,nx}$ denote the stack classifying pairs $(\mathcal{P}_G, \alpha_n)$ where $\mathcal{P}_G$ is a G-bundle on X and $\alpha_n : \mathcal{P}_G\big|_{\mathrm{Spec}(\mathcal{O}_x^{(n)})} \xrightarrow{\sim} \mathcal{P}_G^0$ is a trivialization over the nth infinitesimal neighborhood $\mathrm{Spec}(\mathcal{O}_x^{(n)})$ of x. Then $\mathrm{Bun}_{G,nx}$ is an $\mathcal{L}_{nx}G$-torsor over Bun_G, where $\mathcal{L}_{nx}G$ classifies maps from $\mathrm{Spec}(\mathcal{O}_x^{(n)})$ to G.

Remark 5.11 In particular, $\mathcal{L}_{nx}G$ is a group scheme of finite type.

Set $H_n := S \times \mathcal{L}_{nx}G$, and we have an exact sequence of group schemes over S:

$$1 \to H^n \to H \to H_n \to 1.$$

Define $\mathfrak{k}^n := \mathfrak{k} \otimes \mathfrak{m}_x^n$, and $\mathfrak{k}_n := \mathfrak{k}/\mathfrak{k}^n \cong \mathfrak{k} \otimes \mathcal{O}_x^{(n)}$. Then the above sequence extends to an exact sequence of action pairs (see Sect. 4.1.2):

$$1 \to (\mathfrak{k}^n, H^n) \to (H, \mathfrak{k}) \to (H_n, \mathfrak{k}_n) \to 1. \tag{5.18}$$

5.5.4. We briefly review the Harder-Narasimhan truncation of Bun_G. For this, we need to fix a Borel $B \hookrightarrow G$, whose quotient torus is denoted by T. There are canonical maps

$$\begin{array}{ccccc} & & \mathrm{Bun}_B & & \\ & \overset{\mathfrak{p}}{\swarrow} & & \overset{\mathfrak{q}}{\searrow} & \\ \mathrm{Bun}_G & & & & \mathrm{Bun}_T\,. \end{array}$$

Let Λ_G denote the coweight lattice of G, and $\Lambda_G^+, \Lambda_G^{\mathrm{pos}} \subset \Lambda_G$ denote the submonoid of dominant coweights, respectively the submonoid generated by positive simple coroots. Denote by $\Lambda_G^{+,\mathbb{Q}}$ and $\Lambda_G^{\mathrm{pos},\mathbb{Q}}$ the corresponding rational cones.

There is a partial ordering on $\Lambda_G^{\mathbb{Q}}$, given by:

$$\lambda_1 \underset{G}{\leq} \lambda_2 \iff \lambda_2 - \lambda_1 \in \Lambda_G^{\mathrm{pos},\mathbb{Q}}.$$

Given $\lambda \in \Lambda_G^{\mathbb{Q}}$, define Bun_B^λ as the pre-image of λ under the composition:

$$\mathrm{Bun}_B \xrightarrow{\mathfrak{q}} \mathrm{Bun}_T \xrightarrow{\deg} \Lambda_T^{\mathbb{Q}} \cong \Lambda_G^{\mathbb{Q}}.$$

For each $\theta \in \Lambda_G^{+,\mathbb{Q}}$, define $\mathrm{Bun}_G^{(\leq\theta)}$ as the substack of Bun_G classifying G-bundles $\mathcal{P}_G$ with the following property:

- for each B-bundle $\mathcal{P}_B \in \mathrm{Bun}_B^{\lambda}$ with $\mathfrak{p}(\mathcal{P}_B) \cong \mathcal{P}_G$, we have $\lambda \underset{G}{\leq} \theta$.

The following result is proved in [8]:

Lemma 5.12 *$\mathrm{Bun}_G^{(\leq\theta)}$ is an open, quasi-compact substack of Bun_G.* □

Remark 5.13 The definition of $\mathrm{Bun}_G^{(\leq\theta)}$ in [8] refers to all standard parabolics P of G, rather than just the Borel. However, the two definitions are equivalent; see the discussion in §7.3.3 in *loc.cit*.

5.5.5. For each integer $n \geq 0$ (as well as $n = \infty$), we let $\mathrm{Bun}_{G,nx}^{(\leq\theta)}$ denote the preimage of $\mathrm{Bun}_G^{(\leq\theta)}$ under the canonical map $\mathrm{Bun}_{G,nx} \to \mathrm{Bun}_G$. We denote the universal G-bundle over $\mathrm{Bun}_G^{(\leq\theta)} \times X$ by $\mathcal{P}_G$, and that over $\mathrm{Bun}_{G,\infty x}^{(\leq\theta)} \times X$ by $\tilde{\mathcal{P}}_G$; their pullbacks to $S \times \mathrm{Bun}_G^{(\leq\theta)} \times X$ and $S \times \mathrm{Bun}_{G,\infty x}^{(\leq\theta)} \times X$ are denoted by the same characters.

5.5.6. The key technical assertion we need is:

Proposition 5.14 *For each $\theta \in \Lambda_G^{+,\mathbb{Q}}$, there exists an integer $N(\theta)$ such that whenever $n \geq N(\theta)$, we have*

$$(\mathfrak{g}^\kappa(\mathfrak{m}_x^n)\widehat{\boxtimes}\mathcal{O}_{\mathrm{Bun}_{G,\infty x}^{(\leq\theta)}}) \cap \Gamma(\Sigma, \mathfrak{g}^\kappa_{\tilde{\mathcal{P}}_G}) = 0$$

as submodules of $\mathfrak{g}^\kappa(\mathcal{K}_x)\widehat{\boxtimes}\mathcal{O}_{\mathrm{Bun}_{G,\infty x}^{(\leq\theta)}}$ (via η and γ).

Proof Fix $\theta \in \Lambda_G^{+,\mathbb{Q}}$. For each integer $n \geq 0$, we have an isomorphism:

$$(\mathfrak{g}^\kappa(\mathfrak{m}_x^n)\widehat{\boxtimes}\mathcal{O}_{\mathrm{Bun}_{G,\infty x}^{(\leq\theta)}}) \cap \Gamma(\Sigma, \mathfrak{g}^\kappa_{\tilde{\mathcal{P}}_G}) \xrightarrow{\sim} \mathrm{R}^0(\mathrm{pr}_{\infty x})_* \mathfrak{g}^\kappa_{\tilde{\mathcal{P}}_G}(-nx),$$

where $\mathrm{pr}_{\infty x}$ is the projection map in the following Cartesian diagram:

$$\begin{array}{ccc} S \times \mathrm{Bun}_{G,\infty x}^{(\leq\theta)} \times X & \to & S \times \mathrm{Bun}_G^{(\leq\theta)} \times X \\ \downarrow{\scriptstyle \mathrm{pr}_{\infty x}} & & \downarrow{\scriptstyle \mathrm{pr}} \\ S \times \mathrm{Bun}_{G,\infty x}^{(\leq\theta)} & \longrightarrow & S \times \mathrm{Bun}_G^{(\leq\theta)}. \end{array}$$

Since $\tilde{\mathcal{P}}_G$ is the pullback of the universal G-bundle $\mathcal{P}_G$ over $S \times \mathrm{Bun}_G^{(\leq\theta)} \times X$, it suffices to show that $\mathrm{R}^0(\mathrm{pr})_* \mathfrak{g}^\kappa_{\mathcal{P}_G}(-nx)$ vanishes for sufficiently large n (relative to θ). (Identification of $\mathrm{R}^0(\mathrm{pr}_{\infty x})_* \mathfrak{g}^\kappa_{\mathcal{P}_G}(-nx)$ with the pullback of $\mathrm{R}^0(\mathrm{pr})_* \mathfrak{g}^\kappa_{\mathcal{P}_G}(-nx)$ follows from flatness of the projection $S \times \mathrm{Bun}_{G,\infty x}^{(\leq\theta)} \to S \times \mathrm{Bun}_G^{(\leq\theta)}$). We verify this in a more abstract setting:

Claim 5.15 Let T be a finite type k-scheme. Suppose E is a vector bundle on $T \times X$. Write $\mathrm{pr} : T \times X \to T$ for the projection map. Then there exists some n such that $\mathrm{R}^0(\mathrm{pr})_* E(-nx) = 0$.

Indeed, let $t_0 \in T$ be a k-point. Since $\mathrm{H}^0(X, E|_{t_0}(-n_0 x)) = 0$ for some n_0, the coherent sheaf $\mathrm{R}^0(\mathrm{pr})_* E(-n_0 x)$ vanishes in an open neighborhood $\mathring{T}$ of t_0 (cohomology and base change). Let $T_1 \hookrightarrow T$ be a closed subscheme whose complement is $\mathring{T}$. If T_1 is nonempty, pick a k-point $t_1 \in T_1$. The same argument shows that $\mathrm{R}^0(\mathrm{pr})_* E(-n_1 x)$ vanishes in an open neighborhood of t_1 for some $n_1 \geq n_0$. We find the desired n by iterating this process, which must terminate after finitely many steps since T is Noetherian. □

It follows from Proposition 5.14 that the $(\mathfrak{k}, H)$-algebroid $\mathcal{L}^\kappa$ (hence also $\mathcal{L}^{(\kappa, E)}$) is an object of $\mathrm{LieAlgd}_{\mathrm{inj}}^{(\mathfrak{k}^n, H^n)}(S \times \mathrm{Bun}_G^{(\leq\theta)} / S)$ whenever $n \geq N(\theta)$.

5.5.7. For each $\theta \in \Lambda_G^{+,\mathbb{Q}}$, denote by $\tilde{\mathcal{T}}_G^{(\leq\theta)}$ the restriction of the classical quasi-twisting $\tilde{\mathcal{T}}_G^{(\kappa,E)}$ to $S \times \mathrm{Bun}_{G,\infty x}^{(\leq\theta)}$.[20] Given $n \geq N(\theta)$, we can define a quasi-twisting over $S \times \mathrm{Bun}_G^{(\leq\theta)}$ by the formula:

$$\mathcal{T}_{G,n}^{(\leq\theta)} := \mathbf{Q}^{(H_n, H_n^\flat)} \circ \mathbf{Q}_{\mathrm{inj}}^{(\mathfrak{k}^n, H^n)}(\tilde{\mathcal{T}}_G^{(\leq\theta)}), \tag{5.19}$$

where $H_n^\flat$ denotes the quotient $H_n / \exp(\mathfrak{k}_n)$ (see Sect. 4.3.2).

Remark 5.16 Note that $\mathbf{Q}_{\mathrm{inj}}^{(\mathfrak{k}^n, H^n)}(\mathcal{T}_G^{(\leq\theta)})$ is well-defined as a classical quasi-twisting over $S \times \mathrm{Bun}_{G,nx}^{(\leq\theta)}$, equipped with a $(\mathfrak{k}_n, H_n)$-action. Since the stack $S \times \mathrm{Bun}_{G,nx}^{(\leq\theta)}$ is locally of finite type, any classical quasi-twisting gives rise to a quasi-twisting, and the $(\mathfrak{k}_n, H_n)$-action induces an $(H_n, H_n^\flat)$-action (see Sect. 4.3.4). Hence the formula (5.19) makes sense.

5.5.8. Suppose $n_1 \geq n_2 \geq N(\theta)$. We would like to construct a canonical isomorphism of quasi-twistings

$$\mathcal{T}_{G,n_1}^{(\leq\theta)} \xrightarrow{\sim} \mathcal{T}_{G,n_2}^{(\leq\theta)}. \tag{5.20}$$

Indeed, let $(\mathfrak{k}', H')$ be the kernel of the map $(\mathfrak{k}_{n_1}, H_{n_1}) \to (\mathfrak{k}_{n_2}, H_{n_2})$. In particular, H' is of finite type. Furthermore, we have an exact sequence of classical action pairs:

$$1 \to (\mathfrak{k}^{n_1}, H^{n_1}) \to (\mathfrak{k}^{n_2}, H^{n_2}) \to (\mathfrak{k}', H') \to 1.$$

Hence, there are isomorphisms:

$$\begin{aligned} \mathcal{T}_{G,n_1}^{(\leq\theta)} &\xrightarrow{\sim} \mathbf{Q}^{(H_{n_2}, H_{n_2}^\flat)} \circ \mathbf{Q}^{(H', (H')^\flat)} \circ \mathbf{Q}_{\mathrm{inj}}^{(\mathfrak{k}^{n_1}, H^{n_1})}(\tilde{\mathcal{T}}_G^{(\leq\theta)}) \\ &\xrightarrow{\sim} \mathbf{Q}^{(H_{n_2}, H_{n_2}^\flat)} \circ \mathbf{Q}_{\mathrm{inj}}^{(\mathfrak{k}', H')} \circ \mathbf{Q}_{\mathrm{inj}}^{(\mathfrak{k}^{n_1}, H^{n_1})}(\tilde{\mathcal{T}}_G^{(\leq\theta)}) \xrightarrow{\sim} \mathcal{T}_{G,n_2}^{(\leq\theta)}, \end{aligned}$$

[20] We temporarily suppress the notational dependence on the parameter $(\mathfrak{g}^\kappa, E)$.

using Propositions 4.17, 4.18, and 4.9. In light of the isomorphism (5.20), we may let $\mathcal{T}_G^{(\leq\theta)}$ denote the quasi-twisting $\mathcal{T}_{G,n}^{(\leq\theta)}$ over $S \times \mathrm{Bun}_G^{(\leq\theta)}$ for any $n \geq N(\theta)$.

5.5.9. Finally, we check that the quasi-twistings $\mathcal{T}_G^{(\leq\theta)}$ glue along various Harder-Narasimhan truncations. Indeed, suppose $\theta_1, \theta_2 \in \Lambda_G^{+,\mathbb{Q}}$. Then we have isomorphisms:

$$\begin{aligned}
&\mathcal{T}_{G,n}^{(\leq\theta_1)}\big|_{S\times(\mathrm{Bun}_G^{(\leq\theta_1)}\cap\mathrm{Bun}_G^{(\leq\theta_2)})}\\
&\quad\xrightarrow{\sim}\mathbf{Q}^{(H_n,(H_n)^\flat)}\circ\mathbf{Q}_{\mathrm{inj}}^{(\mathfrak{k}^n,H^n)}(\mathcal{T}_{\infty x}\big|_{S\times(\mathrm{Bun}_{G,\infty x}^{(\leq\theta_1)}\cap\mathrm{Bun}_{G,\infty x}^{(\leq\theta_2)})})\\
&\quad\xrightarrow{\sim}\mathcal{T}_{G,n}^{(\leq\theta_2)}\big|_{S\times(\mathrm{Bun}_G^{(\leq\theta_1)}\cap\mathrm{Bun}_G^{(\leq\theta_2)})},
\end{aligned}$$

whenever $n \geq N(\theta_1), N(\theta_2)$. Therefore we obtain a quasi-twisting $\mathcal{T}_G^{(\kappa,E)}$ on $S\times\mathrm{Bun}_G$ (relative to S) whose restriction to each $S \times \mathrm{Bun}_G^{(\leq\theta)}$ agrees with $\mathcal{T}_G^{(\leq\theta)}$.

Notation 5.17 We write $\mathcal{T}_G^{(\kappa,E)} = \mathbf{Q}^{(\mathfrak{g}^\kappa(\mathcal{O}_x),\mathcal{L}_x^+G)}(\tilde{\mathcal{T}}_G^{(\kappa,E)})$, although it is tacitly understood that the construction of $\mathcal{T}_G^{(\kappa,E)}$ requires two quotient steps and gluing. In a similar way, we write:

$$\mathcal{T}_{G,n}^{(\kappa,E)} := \mathbf{Q}^{(\mathfrak{g}^\kappa(\mathfrak{m}_x^{(n)}),H^n)}(\tilde{\mathcal{T}}_G^{(\kappa,E)}), \tag{5.21}$$

for the corresponding quasi-twisting on $S \times \mathrm{Bun}_{G,nx}$. Since the construction of $\mathcal{T}_G^{(\kappa,E)}$ (resp. $\mathcal{T}_{G,n}^{(\kappa,E)}$) is functorial in S, we obtain a *universal* quasi-twisting $\mathcal{T}_G^{\mathrm{univ}}$ over $\mathrm{Par}_G \times \mathrm{Bun}_G$ (resp. $\mathcal{T}_{G,n}^{\mathrm{univ}}$ over $\mathrm{Par}_G \times \mathrm{Bun}_{G,nx}$).

Remark 5.18 The construction of $\mathcal{T}_G^{\mathrm{univ}}$ depends *a priori* on the choice of the closed point $x \in X$. To remove this dependence, one may consider a multiple point version $\mathcal{T}_{G,x^I}^{\mathrm{univ}}$ associated to any collection x^I of closed points of X. For each inclusion $x^I \subset x^J$, there is a canonical isomorphism $\mathcal{T}_{G,x^I}^{\mathrm{univ}} \xrightarrow{\sim} \mathcal{T}_{G,x^J}^{\mathrm{univ}}$ of quasi-twistings. Hence, the quasi-twisting $\mathcal{T}_{G,x}^{\mathrm{univ}}$ associated to any individual point $x \in X$ is canonically isomorphic to $\mathrm{colim}_{x^I\subset X(k)}\,\mathcal{T}_{G,x^I}^{\mathrm{univ}}$.

Remark 5.19 Note that the DG category $\mathcal{T}_G^{(\kappa,E)}$-Mod is naturally a $\mathrm{QCoh}(S)$-module. Again from the functoriality in maps $(\mathfrak{g}^\kappa, E) : S \to \mathrm{Par}_G$, we obtain a sheaf of DG categories over Par_G, denoted by $\mathcal{T}_G^{\mathrm{univ}}$-Mod.

The *naïve* version of the quantum Langlands duality claims an equivalence of sheaves of DG categories:

$$\mathcal{T}_G^{\mathrm{univ}}\text{-Mod} \xrightarrow{\sim} \mathcal{T}_{\check{G}}^{\mathrm{univ}}\text{-Mod} \tag{5.22}$$

over the common base $\mathrm{Par}_G \xrightarrow{\sim} \mathrm{Par}_{\check{G}}$ (by (2.11)). However, the hypothetical equivalence (5.22) is false whenever G is not a torus, and a renormalization procedure is required for stating the correct version of quantum Langlands duality.

5.5.10 Recovering the classical TDOs

Suppose G is simple, and we fix a k-valued parameter $(\mathfrak{g}^\kappa, 0)$ of Par_G corresponding to some bilinear form κ on $\mathfrak{g}$. Let λ and c be as in Example 2.15. Let $\mathcal{L}_{G,\det}$ denote the determinant line bundle over Bun_G. It is the inverse of the relative determinant of the vector bundle $\mathfrak{g}_{\mathcal{P}_G}$ ($\mathcal{P}_G$ being the universal G-bundle) along the map $\mathrm{Bun}_G \times X \to \mathrm{Bun}_G$ (see [25, Sect. 6.1]). Write $\tilde{\mathcal{L}}_{G,\det}$ for its pullback to $\mathrm{Bun}_{G,\infty x}$.

Proposition 5.20 *The classical quasi-twisting (5.11) at the parameter* $(\mathfrak{g}^{\mathrm{Kil}}, 0)$*:*

$$0 \to \mathcal{O}_{\mathrm{Bun}_{G,\infty x}} \to \widehat{\mathcal{L}}^{(\mathrm{Kil},0)} \to \mathcal{L}^{\mathrm{Kil}} \to 0$$

identifies with the Picard algebroid $\mathrm{Diff}^{\leq 1}(\tilde{\mathcal{L}}_{G,\det})$.

Proof Via the isomorphism $\mathrm{pr}_{\mathfrak{g}} : \mathfrak{g}^{\mathrm{Kil}} \xrightarrow{\sim} \mathfrak{g}$, the lower triangle of (5.10) identifies with:

$$0 \to \mathcal{O}_{\mathrm{Bun}_{G,\infty x}} \to \widehat{\mathfrak{g}}^{\mathrm{Tate}} \widehat{\boxtimes} \mathcal{O}_{\mathrm{Bun}_{G,\infty x}} \to \mathfrak{g}(\mathcal{K}_x) \widehat{\boxtimes} \mathcal{O}_{\mathrm{Bun}_{G,\infty x}} \to 0. \tag{5.23}$$

where $\widehat{\mathfrak{g}}^{\mathrm{Tate}}$ is the central extension of $\mathfrak{g}(\mathcal{K}_x)$ defined by the cocycle

$$(\xi \otimes f, \xi' \otimes f') \rightsquigarrow \mathrm{Kil}(\xi, \xi') \cdot \mathrm{Res}(df \cdot f').$$

Recall that (5.23) is a classical quasi-twisting, where the Lie algebroid brackets are induced from the $\mathcal{L}_x G$-action on $\mathrm{Bun}_{G,\infty x}$.

It is well known (see, e.g. [25, Sect. 7, §10]) that $\widehat{\mathfrak{g}}^{\mathrm{Tate}}$ comes from a central extension of group ind-schemes:

$$1 \to \mathbb{G}_m \to \widehat{G}^{\mathrm{Tate}} \to \mathcal{L}_x G \to 1,$$

and the $\mathcal{L}_x G$-action on $\mathrm{Bun}_{G,\infty x}$ extends to an action of $\widehat{G}^{\mathrm{Tate}}$ on $\widetilde{\mathcal{L}}_{G,\det}$. Hence $\widehat{\mathfrak{g}}^{\mathrm{Tate}}$ acts as derivations on $\widetilde{\mathcal{L}}_{G,\det}$, and we obtain a morphism $\widehat{\mathfrak{g}}^{\mathrm{Tate}} \widehat{\boxtimes} \mathcal{O}_{\mathrm{Bun}_{G,\infty x}} \to \mathrm{Diff}^{\leq 1}(\widetilde{\mathcal{L}}_{G,\det})$ of Lie algebroids. Note that the following diagram commutes:

$$\begin{array}{ccccccc}
0 & \to & \mathcal{O}_{\mathrm{Bun}_{G,\infty x}} & \to & \widehat{\mathfrak{g}}^{\mathrm{Tate}} \widehat{\boxtimes} \mathcal{O}_{\mathrm{Bun}_{G,\infty x}} & \to & \mathfrak{g}(\mathcal{K}_x) \widehat{\boxtimes} \mathcal{O}_{\mathrm{Bun}_{G,\infty x}} \to 0. \\
 & & \downarrow \wr & & \downarrow & & \downarrow \\
0 & \to & \mathcal{O}_{\mathrm{Bun}_{G,\infty x}} & \to & \mathrm{Diff}^{\leq 1}(\widetilde{\mathcal{L}}_{G,\det}) & \to & \mathcal{T}_{\mathrm{Bun}_{G,\infty x}} \to 0
\end{array}$$

Furthermore, the $\mathcal{O}_{\mathrm{Bun}_{G,\infty x}}$-submodule $\Gamma(\Sigma, \mathfrak{g}_{\widetilde{\mathcal{P}}_G})$ of $\widehat{\mathfrak{g}}^{\mathrm{Tate}} \widehat{\boxtimes} \mathcal{O}_{\mathrm{Bun}_{G,\infty x}}$ acts by zero on $\widetilde{\mathcal{L}}_{G,\det}$, so by modding out $\Gamma(\Sigma, \mathfrak{g}_{\widetilde{\mathcal{P}}_G})$, we obtain a morphism of classical quasi-twistings:

$$\begin{array}{ccccccc}
0 & \to & \mathcal{O}_{\mathrm{Bun}_{G,\infty x}} & \to & \widehat{\mathcal{L}}^{(\mathrm{Kil},0)} & \to & \mathcal{L}^{\mathrm{Kil}} \to 0. \\
 & & \downarrow \wr & & \downarrow & & \downarrow \\
0 & \to & \mathcal{O}_{\mathrm{Bun}_{G,\infty x}} & \to & \mathrm{Diff}^{\leq 1}(\widetilde{\mathcal{L}}_{G,\det}) & \to & \mathcal{T}_{\mathrm{Bun}_{G,\infty x}} \to 0
\end{array}$$

where the last terms $\mathcal{L}^{\mathrm{Kil}}$ and $\mathcal{T}_{\mathrm{Bun}_{G,\infty x}}$ are identified. As such, it is an isomorphism of classical quasi-twistings. □

It follows from Proposition 5.20 that the classical quasi-twisting at $(\mathfrak{g}^\kappa, 0)$ operates on the virtual line bundle $\tilde{\mathcal{L}}^\lambda_{G,\det}$. Since quotient by the action pair $(\mathfrak{g}(\mathcal{O}_x), \mathcal{L}^+_x G)$ agrees with strong quotient of Picard algebroids, we obtain an equivalence

$$\mathcal{T}_G^{(\kappa,\mathrm{triv})}\text{-Mod} \xrightarrow{\sim} \mathrm{Diff}(\mathcal{L}^\lambda_{G,\det})\text{-Mod}(\mathrm{Bun}_G).$$

In particular, the hypothetical equivalence (5.22) specializes to (1.2).

6 Recovering QCoh(LocSys$_G$) at $\kappa = \infty$

In this section, we show that at level ∞, the quasi-twisting $\mathcal{T}_G^{(\kappa,E)}$ constructed in Sect. 5 recovers the DG algebraic stack LocSys_G in the following sense: $\mathcal{T}_G^{(\infty,0)}$ is the *inert* quasi-twisting on some triangle $\mathcal{O}_{\mathrm{Bun}_G} \to \widehat{\mathcal{Q}}^{(\infty,0)}_{\mathrm{desc}} \to \mathcal{Q}^{(\infty,0)}_{\mathrm{desc}}$ in $\mathrm{QCoh}(\mathrm{Bun}_G)$ (see Sect. 4.6.6. for what this means). Furthermore, the corresponding stack $\mathbb{V}(\widehat{\mathcal{Q}}^{(\infty,0)}_{\mathrm{desc}})_{\lambda=1}$ over Bun_G identifies with LocSys_G, so we obtain an equivalence of DG categories $\mathcal{T}_G^{(\infty,0)}\text{-Mod} \xrightarrow{\sim} \mathrm{QCoh}(\mathrm{LocSys}_G)$.

Finally, we comment on the role of certain additional parameters E when $\mathfrak{g}^\kappa = \mathfrak{g}^\infty$.

6.1 The underlying $\mathcal{O}_{S\times \mathrm{Bun}_G}$-modules of $\mathcal{T}^{(\kappa,0)}_{G,n}$

6.1.1. We adopt the following notations from the previous section: let $\mathcal{S}_n := S \times \mathrm{Bun}_{G,nx}$, and $\mathcal{X}_n := S \times \mathrm{Bun}_{G,nx} \times X$ which is a curve over $\mathcal{S}_n$. The tautological G-bundle over $\mathcal{X}_n$ is denoted by $\mathcal{P}^{(n)}_G$. Write $\widetilde{S} := S \times \mathrm{Bun}_{G,\infty x}$ and similarly for $\widetilde{\mathcal{X}}$ and $\widetilde{\mathcal{P}}_G$.

Recall the quasi-twisting $\mathcal{T}^{(\kappa,0)}_{G,n}$ and $\mathcal{T}^{(\kappa,0)}_G = \mathcal{T}^{(\kappa,0)}_{G,0}$ which are special cases of (5.21) for the S-valued parameter $(\mathfrak{g}^\kappa, 0)$. Suppose $\mathcal{T}^{(\kappa,0)}_{G,n}$ is expressed as a map of some formal moduli problems $\widehat{\mathcal{S}}^\flat_n \to \mathcal{S}^\flat_n$ under $\mathcal{S}_n$.

6.1.2. Since $\mathcal{T}^{(\kappa,0)}_{G,n}$ is the quotient of $\tilde{\mathcal{T}}^{(\kappa,0)}_G$ by the pair $(\mathfrak{g}^\kappa(\mathfrak{m}^n_x), H^n)$, the underlying ind-coherent sheaves of $\widehat{\mathcal{S}}^\flat_n$ and $\mathcal{S}^\flat_n$ arise from a triangle in $\mathrm{QCoh}(\mathcal{S}_n)$:

$$\mathcal{O}_{\mathcal{S}_n} \to \widehat{\mathcal{Q}}^{(\kappa,0)}_{n,\mathrm{desc}} \to \mathcal{Q}^\kappa_{n,\mathrm{desc}}, \tag{6.1}$$

where $\widehat{\mathcal{Q}}^{(\kappa,0)}_{n,\mathrm{desc}}$ is the descent of the H^n-equivariant complex of $\mathcal{O}_{\widetilde{\mathcal{S}}}$-modules:

$$\widehat{\mathcal{Q}}^{(\kappa,0)}_n := \mathrm{Cofib}(\mathfrak{g}^\kappa(\mathfrak{m}^n_x) \boxtimes \mathcal{O}_{\mathrm{Bun}_{G,\infty x}} \to \widehat{\mathcal{L}}^{(\kappa,0)}),$$

and a similar description is valid for $\mathcal{Q}^\kappa_{n,\mathrm{desc}}$.

6.1.3. The Atiyah bundle construction gives rise to a triangle:

$$\omega_{\mathcal{X}_n/\mathcal{S}_n} \to \mathrm{At}(\mathcal{P}_G^{(n)})^* \to \mathfrak{g}^*_{\mathcal{P}_G^{(n)}}$$

over $\mathcal{X}_n$. Its pullback along the projection $\mathfrak{g}^\kappa_{\mathcal{P}_G^{(n)}} \to \mathfrak{g}^*_{\mathcal{P}_G^{(n)}}$ is denoted by:

$$\omega_{\mathcal{X}_n/\mathcal{S}_n} \to \mathcal{E}^\kappa(\mathcal{P}_G^{(n)}) \to \mathfrak{g}^\kappa_{\mathcal{P}_G^{(n)}}. \tag{6.2}$$

Note that there is a canonical isomorphism $\mathcal{Q}^\kappa_{n,\mathrm{desc}} \xrightarrow{\sim} \mathrm{R}\,\Gamma(X, \mathfrak{g}^\kappa_{\mathcal{P}_G^{(n)}}(-nx))[1]$.

Proposition 6.1 *The triangle (6.1) is identified with the push-out of*

$$\mathrm{R}\,\Gamma(X, \omega_{\mathcal{X}_n/\mathcal{S}_n}(-nx))[1] \to \mathrm{R}\,\Gamma(X, \mathcal{E}^\kappa(\mathcal{P}_G^{(n)})(-nx))[1] \to \mathrm{R}\,\Gamma(X, \mathfrak{g}^\kappa_{\mathcal{P}_G^{(n)}}(-nx))[1] \tag{6.3}$$

along the trace map $\mathrm{R}\,\Gamma(X, \omega_{\mathcal{X}_n/\mathcal{S}_n}(-nx))[1] \to \mathcal{O}_{\mathcal{S}_n}$.

6.1.4. We now begin the proof of Proposition 6.1. Since both triangles in question are descent of triangles over $\tilde{\mathcal{S}}$, we ought to establish an H^n-equivariant isomorphism between the triangle:

$$\mathcal{O}_{\tilde{\mathcal{S}}} \to \widehat{\mathcal{Q}}_n^{(\kappa,0)} \to \mathcal{Q}_n^\kappa \tag{6.4}$$

and the push-out of the analogous triangle:

$$\mathrm{R}\,\Gamma(X, \omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}}(-nx))[1] \to \mathrm{R}\,\Gamma(X, \mathcal{E}^\kappa(\widetilde{\mathcal{P}}_G)(-nx))[1] \to \mathrm{R}\,\Gamma(X, \mathfrak{g}^\kappa_{\widetilde{\mathcal{P}}_G}(-nx))[1] \tag{6.5}$$

under the trace map $\mathrm{R}\,\Gamma(X, \omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}}(-nx))[1] \to \mathcal{O}_{\tilde{\mathcal{S}}}$.

6.1.5. We describe more explicitly the $\mathcal{D}_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}}$-modules underlying the extension sequence of Lie-$*$ algebras (5.9):

$$0 \to \omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}} \to (\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,0)})_{\tilde{\mathcal{P}}_G} \to (\mathfrak{g}^\kappa_{\mathcal{D}})_{\tilde{\mathcal{P}}_G} \to 0,$$

in the case where the $E = 0$. Namely, consider the $\mathcal{D}_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}}$-modules induced from the sequence (6.2) (where we use $\tilde{\mathcal{X}}$ instead of $\mathcal{X}^{(n)}$ in the Atiyah bundle construction):

$$0 \to (\omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}})_{\mathcal{D}} \to \mathcal{E}^\kappa(\widetilde{\mathcal{P}}_G)_{\mathcal{D}} \to (\mathfrak{g}^\kappa_{\mathcal{D}})_{\tilde{\mathcal{P}}_G} \to 0$$

Let $\mathcal{E}^\kappa(\widetilde{\mathcal{P}}_G)^{\mathrm{push}}_{\mathcal{D}}$ be the push-out along $\mathrm{act} : (\omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}})_{\mathcal{D}} \to \omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}}$ of the $\mathcal{D}_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}}$-module $\mathcal{E}^\kappa(\widetilde{\mathcal{P}}_G)_{\mathcal{D}}$.

Lemma 6.2 *The $\mathcal{D}_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}}$-module underlying the extension $(\widehat{\mathfrak{g}}^{(\kappa,0)}_{\mathcal{D}})_{\tilde{\mathcal{P}}_G}$ identifies with $\mathcal{E}^\kappa(\widetilde{\mathcal{P}}_G)^{\mathrm{push}}_{\mathcal{D}}$.*

Proof Recall that $(\widehat{\mathfrak{g}}^{(\kappa,0)}_{\mathcal{D}})_{\tilde{\mathcal{P}}_G}$ is the $\tilde{\mathcal{P}}_G$-twist of the trivial extension $\widehat{\mathfrak{g}}^{(\kappa,0)}_{\mathcal{D}} \xrightarrow{\sim} \omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}} \oplus \mathfrak{g}^\kappa_{\mathcal{D}}$. Consider the push-out diagram:

$$\begin{array}{ccc} (\omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}})_{\mathcal{D}} & \twoheadrightarrow & (\omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}} \oplus (\mathfrak{g}^\kappa \otimes \mathcal{O}_{\tilde{\mathcal{X}}}))_{\mathcal{D}} \\ \downarrow{\scriptstyle \mathrm{act}} & & \downarrow \\ \omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}} & \twoheadrightarrow & \omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}} \oplus \mathfrak{g}^\kappa_{\mathcal{D}}. \end{array} \tag{6.6}$$

Note that the entire diagram is acted on by the sheaf of groups $\mathcal{G}$, as described below:

(a) the $\mathcal{G}$-actions on $(\omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}})_{\mathcal{D}}$ and $\omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}}$ are trivial, and the action on $\omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}} \oplus \mathfrak{g}^\kappa_{\mathcal{D}}$ is given by §5.3.5;

(b) the $\mathcal{G}$-action on $(\omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}} \oplus (\mathfrak{g}^\kappa \otimes \mathcal{O}_{\tilde{\mathcal{X}}}))_{\mathcal{D}}$ is the $\mathcal{D}_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}}$-linear extension of the following $\mathcal{G}$-action on $\omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}} \oplus (\mathfrak{g}^\kappa \otimes \mathcal{O}_{\tilde{\mathcal{X}}})$ centralizing $\omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}}$:

$$g_{\mathcal{U}} \cdot (\xi \oplus \varphi) = \varphi(g_{\mathcal{U}}^{-1} d g_{\mathcal{U}}) + (\mathrm{Ad}_{g_{\mathcal{U}}}(\xi) \oplus \mathrm{Coad}_{g_{\mathcal{U}}}(\varphi)) \tag{6.7}$$

where $g_{\mathcal{U}} \in \mathcal{G}(\mathcal{U})$ and $\xi \oplus \varphi \in \mathfrak{g}^\kappa \otimes \mathcal{O}_{\mathcal{U}}$.

If we twist the trivial $\mathcal{O}_{\tilde{\mathcal{X}}}$-module extension equipped with the $\mathcal{G}$-action (6.7):

$$0 \to \omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}} \to \omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}} \oplus (\mathfrak{g}^\kappa \otimes \mathcal{O}_{\tilde{\mathcal{X}}}) \to \mathfrak{g}^\kappa \otimes \mathcal{O}_{\tilde{\mathcal{X}}} \to 0$$

by the G-bundle $\tilde{\mathcal{P}}_G$, we obtain precisely the Atiyah sequence (pulled back along $\mathfrak{g}^\kappa_{\tilde{\mathcal{P}}_G} \to \mathfrak{g}^*_{\tilde{\mathcal{P}}_G}$):

$$0 \to \omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}} \to \mathcal{E}^\kappa(\tilde{\mathcal{P}}_G) \to \mathfrak{g}^\kappa_{\tilde{\mathcal{P}}_G} \to 0.$$

Therefore, twisting the diagram (6.6) by $\tilde{\mathcal{P}}_G$, we obtain a push-out diagram:

$$\begin{array}{ccc} (\omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}})_{\mathcal{D}} & \twoheadrightarrow & \mathcal{E}^\kappa(\tilde{\mathcal{P}}_G)_{\mathcal{D}} \\ \downarrow{\scriptstyle \mathrm{act}} & & \downarrow \\ \omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}} & \twoheadrightarrow & (\widehat{\mathfrak{g}}^{(\kappa,0)}_{\mathcal{D}})_{\tilde{\mathcal{P}}_G}. \end{array}$$

This proves the Lemma. □

6.1.6. By construction of $\widehat{\mathcal{Q}}^{(\kappa,0)}_n$ and $\mathcal{Q}^\kappa_n$, the required isomorphism shall follow from a general claim. We first explain the set-up (which is quite involved): let $\mathcal{S}$ be a scheme, and $\mathcal{X} := X \times \mathcal{S}$ with section $\underline{x}$ given by the closed point $x \in X$. Suppose we have an exact sequence of $\mathcal{O}_{\mathcal{X}}$-modules:

$$0 \to \omega_{\mathcal{X}/\mathcal{S}} \to \mathcal{E} \to \mathcal{F} \to 0.$$

Let $\mathcal{E}_{\mathcal{D}}$ denote the induced $\mathcal{D}$-module of $\mathcal{E}$ and $\mathcal{E}_{\mathcal{D}}^{\mathrm{push}}$ its push-out along act : $(\omega_{\mathcal{X}/\mathcal{S}})_{\mathcal{D}} \to \omega_{\mathcal{X}/\mathcal{S}}$.

Then we may form a map between exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma_{\mathrm{dR}}(\Sigma, \omega_{\mathcal{X}/\mathcal{S}}) & \longrightarrow & \Gamma_{\mathrm{dR}}(\Sigma, \mathcal{E}_{\mathcal{D}}^{\mathrm{push}}) & \longrightarrow & \Gamma(\Sigma, \mathcal{F}) & \longrightarrow & 0 \\ & & \downarrow 0 & & \downarrow & \widehat{\gamma} \swarrow & \downarrow \gamma & & \\ 0 & \longrightarrow & \Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{x}}, \omega_{\mathcal{X}/\mathcal{S}}) & \longrightarrow & \Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{x}}, \mathcal{E}_{\mathcal{D}}^{\mathrm{push}}) & \longrightarrow & \Gamma(\overset{\circ}{D}_{\underline{x}}, \mathcal{F}) & \longrightarrow & 0, \end{array}$$

as well as a section $\widehat{\gamma}$ from the residue theorem. On the other hand, let $\mathcal{E}_{\mathcal{D}}^{\mathrm{push}}(\mathfrak{m}^{(n)})$ denote the $\mathcal{O}_{\mathcal{S}}$-submodule of $\Gamma_{\mathrm{dR}}(D_{\underline{x}}, \mathcal{E}_{\mathcal{D}}^{\mathrm{push}})$ annihilated by the restriction to $D_{\underline{x}}^{(n)}$; we use the notation $\mathcal{F}(\mathfrak{m}^{(n)})$ for a similar meaning. We have a triangle:

$$\mathcal{O}_{\mathcal{S}} \to \widehat{\mathcal{Q}} \to \mathcal{Q} \tag{6.8}$$

where:

(a) $\widehat{\mathcal{Q}} := \mathrm{Cofib}(\Gamma(\Sigma, \mathcal{F}) \to \Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{x}}, \mathcal{E}_{\mathcal{D}}^{\mathrm{push}})/\mathcal{E}_{\mathcal{D}}^{\mathrm{push}}(\mathfrak{m}^{(n)}))$;

(b) $\mathcal{Q} := \mathrm{Cofib}(\Gamma(\Sigma, \mathcal{F}) \to \Gamma(\overset{\circ}{D}_{\underline{x}}, \mathcal{F})/\mathcal{F}(\mathfrak{m}^{(n)}))$.

Remark 6.3 For $\mathcal{S} := \tilde{\mathcal{S}}$, $\mathcal{E} := \mathcal{E}^{\kappa}(\tilde{\mathcal{P}}_G)$, and $\mathcal{F} := \mathfrak{g}^{\kappa}_{\tilde{\mathcal{P}}_G}$, we see from the construction of (6.4) that it identifies with the triangle (6.8).

Claim 6.4 The triangle (6.8) identifies with the push-out of the canonical triangle:

$$\mathrm{R}\,\Gamma(X, \omega_{\mathcal{X}/\mathcal{S}}(-nx))[1] \to \mathrm{R}\,\Gamma(X, \mathcal{E}(-nx))[1] \to \mathrm{R}\,\Gamma(X, \mathcal{F}(-nx))[1] \tag{6.9}$$

along the trace map $\mathrm{R}\,\Gamma(X, \omega_{\mathcal{X}/\mathcal{S}}(-nx))[1] \to \mathcal{O}_{\mathcal{S}}$.

Proof Recall the identification:

$$\mathcal{Q} = \mathrm{Cofib}(\Gamma(\Sigma, \mathcal{F}) \to \Gamma(\overset{\circ}{D}_{\underline{x}}, \mathcal{F})/\mathcal{F}(\mathfrak{m}^{(n)})) \xrightarrow{\sim} \mathrm{R}\,\Gamma(X, \mathcal{F}(-nx))[1],$$

which is also valid when $\mathcal{F}$ is replaced by any $\mathcal{O}_{\mathcal{X}}$-module. It suffices to produce a morphism of triangles from (6.9) to (6.8), whose first and third terms are the trace map, respectively the above isomorphism.

Consider the diagram defining $\mathcal{E}_{\mathcal{D}}^{\mathrm{push}}$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\omega_{\mathcal{X}/\mathcal{S}})_{\mathcal{D}} & \longrightarrow & \mathcal{E}_{\mathcal{D}} & \longrightarrow & \mathcal{F}_{\mathcal{D}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \wr & & \\ 0 & \longrightarrow & \omega_{\mathcal{X}/\mathcal{S}} & \longrightarrow & \mathcal{E}_{\mathcal{D}}^{\mathrm{push}} & \longrightarrow & \mathcal{F}_{\mathcal{D}} & \longrightarrow & 0. \end{array}$$

Using the functors $\Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{x}}, -)$ and $\mathcal{M} \rightsquigarrow \mathcal{M}(\mathfrak{m}^{(n)})$, we obtain a diagram:

$$\begin{array}{ccccccccc}
0 & \to & \overset{\circ}{\omega}/\omega(\mathfrak{m}^{(n)}) & \longrightarrow & \Gamma(\overset{\circ}{D}_{\underline{x}}, \mathcal{E})/\mathcal{E}(\mathfrak{m}^{(n)}) & \longrightarrow & \Gamma(\overset{\circ}{D}_{\underline{x}}, \mathcal{F})/\mathcal{F}(\mathfrak{m}^{(n)}) & \to & 0 \\
 & & \downarrow{\scriptstyle \mathrm{res}} & & \downarrow & & \downarrow\wr & & \\
0 & \longrightarrow & \mathcal{O}_{\tilde{\mathcal{S}}} & \longrightarrow & \Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{x}}, \mathcal{E}^{\mathrm{push}}_{\mathcal{D}})/\mathcal{E}^{\mathrm{push}}_{\mathcal{D}}(\mathfrak{m}^{(n)}) & \to & \Gamma(\overset{\circ}{D}_{\underline{x}}, \mathcal{F})/\mathcal{F}(\mathfrak{m}^{(n)}) & \to & 0
\end{array}$$

where the rows are still exact sequences by the Snake lemma. We now take cofibers of the map from the triangle $\Gamma(\Sigma, \omega) \to \Gamma(\Sigma, \mathcal{E}) \to \Gamma(\Sigma, \mathcal{F})$ to the top row, and the cofibers of the map from $0 \to \Gamma(\Sigma, \mathcal{F}) \to \Gamma(\Sigma, \mathcal{F})$ to the bottom row:

$$\begin{array}{ccccc}
\mathrm{R}\,\Gamma(X, \omega_{\mathcal{X}/\mathcal{S}}(-nx))[1] & \to & \mathrm{R}\,\Gamma(X, \mathcal{E}(-nx))[1] & \to & \mathrm{R}\,\Gamma(X, \mathcal{F}(-nx))[1] \\
\downarrow & & \downarrow & & \downarrow\wr \\
\mathcal{O}_{\mathcal{S}} & \longrightarrow & \widehat{\mathcal{Q}} & \longrightarrow & \mathcal{Q}
\end{array}$$

This is a morphism between triangles. Finally, we observe that the residue morphism from $\overset{\circ}{\omega}/\omega(\mathfrak{m}^{(n)})$ passes to the trace map from $\mathrm{R}\,\Gamma(X, \omega_{\mathcal{X}/\mathcal{S}}(-nx))[1]$. □

We have now constructed an isomorphism from (6.4) to the push-out of (6.5) along the trace map $\mathrm{R}\,\Gamma(X, \omega_{\widetilde{\mathcal{X}}/\widetilde{\mathcal{S}}}(-nx))[1] \to \mathcal{O}_{\widetilde{\mathcal{S}}}$. We omit checking that this map is compatible with the H^n-equivariance structure. □ (Proposition 6.1)

Remark 6.5 Combined with Sect. 5.5.10, we have showed that the Picard algebroid $\mathrm{Diff}^{\leq 1}(\mathcal{L}_{G,\mathrm{det}})$ has as its underlying triangle of $\mathcal{O}_{\mathrm{Bun}_G}$-modules constructed explicitly by the following procedure:

(a) Consider the triangle $\mathrm{R}\,\Gamma(X, \omega_{\mathcal{X}/\mathcal{S}})[1] \to \mathrm{R}\,\Gamma(X, \mathcal{E}^{\kappa}(\mathcal{P}_G))[1] \to \mathrm{R}\,\Gamma(X, \mathfrak{g}^*_{\mathcal{P}_G})[1]$;
(b) Obtain a push-out along the trace map $\mathrm{R}\,\Gamma(X, \omega_{\mathcal{X}/\mathcal{S}})[1] \to \mathcal{O}_{\mathcal{S}}$:

$$\mathcal{O}_{\mathcal{S}} \to \mathcal{E} \to \mathrm{R}\,\Gamma(X, \mathfrak{g}^*_{\mathcal{P}_G})[1]$$

(c) The extension associated to $\mathrm{Diff}^{\leq 1}(\mathcal{L}_{G,\mathrm{det}})$ is the pullback of the above triangle along:

$$\mathcal{T}_{\mathrm{Bun}_G} \xrightarrow{\sim} \mathrm{R}\,\Gamma(X, \mathfrak{g}_{\mathcal{P}_G})[1] \xrightarrow{\mathrm{Kil}} \mathrm{R}\,\Gamma(X, \mathfrak{g}^*_{\mathcal{P}_G})[1].$$

where the Killing form Kil is regarded as a G-invariant isomorphism $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$.

6.2 An alternative description of LocSys_G

6.2.1. Recall that LocSys_G is defined as the mapping stack $\underline{\mathrm{Maps}}(X_{\mathrm{dR}}, \mathrm{B}\,G)$; it is represented by a DG algebraic stack ([1, Sect. 10]). We give an alternative description of LocSys_G in terms of "G-bundles with connections." This description is more closely related to the quasi-twisting at level ∞.

6.2.2. Let LocSys'_G denote the prestack over Bun_G such that for every affine DG scheme S, the groupoid $\mathrm{Maps}(S, \mathrm{LocSys}'_G)$ classifies:

(a) a G-bundle $\mathcal{P}_G$ over $S \times X$;
(b) a splitting of the canonical triangle in $\mathrm{QCoh}(S \times X)$:

$$\mathfrak{g}_{\mathcal{P}_G} \to \mathrm{At}(\mathcal{P}_G) \to \mathcal{T}_{S\times X/S}. \tag{6.10}$$

Recall that for such S, the complex $\mathrm{At}(\mathcal{P}_G)$ can be described as the relative tangent complex associated to the map $S \times X \to \mathrm{B}\, G$ represented by $\mathcal{P}_G$, and the triangle (6.10) is the corresponding canonical triangle.
6.2.3. Note that a lift of $\mathcal{P}_G$ to an S-point of LocSys_G supplies the dotted arrow in the following commutative diagram:

$$\begin{array}{ccc} S \times X & \xrightarrow{\mathcal{P}_G} & S \times \mathrm{B}\, G \\ \downarrow & \nearrow & \downarrow \\ S \times X_{\mathrm{dR}} & \longrightarrow & S \end{array}$$

This arrow gives rise to a splitting of (6.10) as $\mathcal{T}_{S\times X/S\times X_{\mathrm{dR}}}$ is isomorphic to $\mathcal{T}_{S\times X/S}$. In other words, we have a morphism of stacks over Bun_G:

$$\mathrm{LocSys}_G \to \mathrm{LocSys}'_G\,. \tag{6.11}$$

Proposition 6.6 *The morphism (6.11) is an isomorphism.*

Proof Let us first introduce some auxiliary objects. For an affine open $U \subset X$, denote by $\mathrm{LocSys}_G(U)$ (resp. $\mathrm{LocSys}'_G(U)$) the prestack over Bun_G such that a lift of an S-point $\mathcal{P}_G$ of Bun_G to $\mathrm{LocSys}_G(U)$ corresponds to a flat connection of $\mathcal{P}_G|_U$ (resp. a splitting of (6.10) over $S \times U$). Denote by $\mathrm{Hitch}_G(U)$ the prestack over Bun_G classifying a G-bundle $\mathcal{P}_G$ together with a section of $\mathfrak{g}^*_{\mathcal{P}_G} \otimes \omega_X$ over U. It is known that both prestacks $\mathrm{LocSys}_G(U)$ and $\mathrm{Hitch}_G(U)$ are classical (see [1, Proposition 10.5.3]).

We claim that $\mathrm{LocSys}'_G(U)$ is also classical. Indeed, since any choice of a splitting of (6.10) over U supplies an isomorphism between $\mathrm{LocSys}'_G(U)$ and $\mathrm{Hitch}_G(U)$, it suffices to show that such a splitting exists. The extension (6.10) over U corresponds to an element of the groupoid:

$$\tau^{\leq 0}\,\mathrm{Hom}_{\mathrm{QCoh}(S\times U)}(\mathcal{T}_{S\times U/S}, \mathfrak{g}_{\mathcal{P}_G}[1]) \cong \tau^{\leq 0}\,\mathrm{Hom}_{\mathrm{QCoh}(U)}(\mathcal{T}_U, \mathfrak{g}_{\mathcal{P}_G}[1]).$$

Since $\mathfrak{g}_{\mathcal{P}_G}$ is in cohomological degree ≤ 0 and U is affine, any such element is null-homotopic.

Next, we claim that the morphism of prestacks analogous to (6.11):

$$\mathrm{LocSys}_G(U) \to \mathrm{LocSys}'_G(U)$$

is an isomorphism. Indeed, since both sides are classical, it suffices to verify the claim for classical test affine schemes S. In this case, note that lifting an S-point $\mathcal{P}_G$ of

Bun_G to $\mathrm{LocSys}'_G(U)$ amounts to supplying a connection on $\mathcal{P}_G$, whereas a lift to $\mathrm{LocSys}_G(U)$ amounts to supplying a *flat* connection on $\mathcal{P}_G$. Their equivalence follows from the fact that $\dim(X) = 1$.

Finally, we find that (6.11) is an equivalence by covering X with two affine opens U_1 and U_2, and using the Cartesian squares:

$$\begin{array}{ccc} \mathrm{LocSys}_G & \longrightarrow & \mathrm{LocSys}_G(U_1) \\ \downarrow & & \downarrow \\ \mathrm{LocSys}_G(U_2) & \to & \mathrm{LocSys}_G(U_1 \cap U_2) \end{array} \qquad \begin{array}{ccc} \mathrm{LocSys}'_G & \longrightarrow & \mathrm{LocSys}'_G(U_1) \\ \downarrow & & \downarrow \\ \mathrm{LocSys}'_G(U_2) & \to & \mathrm{LocSys}'_G(U_1 \cap U_2) \end{array}$$

These follow straightforwardly from the descent property of $\mathrm{B}\,G$, respectively QCoh. □

6.3 Identification of the fiber at ∞

6.3.1. We now specialize to the parameter $(\mathfrak{g}^\infty, 0) : \mathrm{pt} \to \mathrm{Par}_G$, where $\mathfrak{g}^\infty$ identifies with the subspace $\mathfrak{g}^* \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}^*$. The quasi-twisting $\mathcal{T}_G^{(\infty,0)}$ over Bun_G is obtained as the quotient of $\widetilde{\mathcal{T}}_G^{(\infty,0)}$ (i.e., (5.11) at parameter $(\mathfrak{g}^\infty, 0)$) by the pair $(\mathfrak{g}^\infty(\mathcal{O}_x), \mathcal{L}_x^+ G)$ along the $\mathcal{L}_x^+ G$-torsor $\mathrm{Bun}_{G,\infty x} \to \mathrm{Bun}_G$.

Proposition 6.7 *(a) $\mathcal{T}_G^{(\infty,0)}$ is the inert quasi-twisting associated to the triangle (6.1) (for $n = 0$):*

$$\mathcal{O}_{\mathrm{Bun}_G} \to \widehat{\mathcal{Q}}_{\mathrm{desc}}^{(\infty,0)} \to \mathcal{Q}_{\mathrm{desc}}^{\infty} \tag{6.12}$$

(b) there is a canonical isomorphism of DG stacks:

$$\mathbb{V}(\widehat{\mathcal{Q}}_{\mathrm{desc}}^{(\infty,0)})_{\lambda=1} \xrightarrow{\sim} \mathrm{LocSys}_G\,.$$

Combined with (4.24), we obtain an equivalence of DG categories:

$$\mathcal{T}_G^{(\infty,0)}\text{-Mod} \xrightarrow{\sim} \mathrm{QCoh}(\mathrm{LocSys}_G).$$

Proof of Proposition 6.7 It is clear from the construction that the classical quasi-twisting $\widetilde{\mathcal{T}}_G^{(\infty,0)}$ is given by the central extension of Lie algebroids (with zero Lie bracket and anchor map)

$$0 \to \mathcal{O}_{\mathrm{Bun}_{G,\infty x}} \to \widehat{\mathcal{L}}^{(\infty,0)} \to \mathcal{L}^{\infty} \to 0.$$

Since $\mathcal{T}_G^{(\infty,0)}$ arises from the quotient of $\widetilde{\mathcal{T}}_G^{(\infty,0)}$ by $(\mathfrak{g}^\infty(\mathcal{O}_x), \mathcal{L}_x^+ G)$, the paradigm of §4.6.9 applies, and $\mathcal{T}_G^{(\infty,0)}$ is the inert quasi-twisting on the triangle (6.12). For the

second statement, note that we have a push-out diagram in $\mathrm{QCoh}(\mathrm{Bun}_G)$:

$$\begin{array}{ccc} \mathrm{R}\,\Gamma(X, \mathcal{O}_{\mathrm{Bun}_G \times X})^* & \to & \mathrm{R}\,\Gamma(X, \mathrm{At}(\mathcal{P}_G) \otimes \omega_X)^* \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathrm{Bun}_G} & \longrightarrow & \widehat{\mathcal{Q}}^{(\infty,0)}_{\mathrm{desc}}, \end{array}$$

by Proposition 6.1 and Serre duality. Hence $\mathbb{V}(\widehat{\mathcal{Q}}^{(\infty,0)}_{\mathrm{desc}})_{\lambda=1}$ fits into the commutative diagram:

$$\begin{array}{ccc} \mathbb{V}(\mathrm{R}\,\Gamma(X, \mathcal{O}_{\mathrm{Bun}_G \times X})^*) & \leftarrow & \mathbb{V}(\mathrm{R}\,\Gamma(X, \mathrm{At}(\mathcal{P}_G) \otimes \omega_X)^*) \\ \{1\} \uparrow & & \uparrow \\ \mathrm{Bun}_G & \longleftarrow & \mathbb{V}(\widehat{\mathcal{Q}}^{(\infty,0)}_{\mathrm{desc}})_{\lambda=1}. \end{array}$$

For any DG scheme S mapping to Bun_G (represented by the G-bundle $\mathcal{P}_G$ over $S \times X$), a computation using the projection formula shows:

(a) $\mathrm{Maps}_{\mathrm{Bun}_G}(S, \mathbb{V}(\mathrm{R}\,\Gamma(X, \mathrm{At}(\mathcal{P}_G) \otimes \omega_X)^*)) \xrightarrow{\sim} \tau^{\leq 0}\,\mathrm{R}\,\Gamma(S \times X, \mathrm{At}(\mathcal{P}_G) \otimes \omega_X)$, and
(b) $\mathrm{Maps}_{\mathrm{Bun}_G}(S, \mathbb{V}(\mathrm{R}\,\Gamma(X, \mathcal{O}_{\mathrm{Bun}_G \times X})^*)) \xrightarrow{\sim} \tau^{\leq 0}\,\mathrm{R}\,\Gamma(S \times X, \mathcal{O}_{S \times X})$.

Hence $\mathrm{Maps}_{\mathrm{Bun}_G}(S, \mathbb{V}(\widehat{\mathcal{Q}}^{(\infty,0)}_{\mathrm{desc}})_{\lambda=1})$ is identified with the ∞-groupoid

$$\tau^{\leq 0}\,\mathrm{R}\,\Gamma(S \times X, \mathrm{At}(\mathcal{P}_G) \otimes \omega_X) \underset{\tau^{\leq 0}\,\mathrm{R}\,\Gamma(S \times X, \mathcal{O}_{S \times X})}{\times} \{1\}$$

i.e., the ∞-groupoid of splittings of the Atiyah sequence $\mathfrak{g}_{\mathcal{P}_G} \to \mathrm{At}(\mathcal{P}_G) \to \mathcal{T}_{S \times X/S}$. We obtain an isomorphism $\mathbb{V}(\widehat{\mathcal{Q}}^{(\infty,0)}_{\mathrm{desc}})_{\lambda=1} \xrightarrow{\sim} \mathrm{LocSys}'_G$ so the result follows from Proposition 6.6. □

Remark 6.8 An alternative argument (one that avoids using the results of Sect. 6.1) runs as follows: by a local computation, one identifies the universal envelope of the classical quasi-twisting (5.11) with the (topological) ring of functions over $\mathrm{LocSys}_{G,\infty x}(\Sigma)$, the stack classifying $(\mathcal{P}_G, \alpha) \in \mathrm{Bun}_{G,\infty x}$ together with a connection over $\mathcal{P}_G\big|_\Sigma$. One then shows that the closed subscheme $\mathbb{V}(\widehat{\mathcal{Q}}^{(\infty,0)})_{\lambda=1}$ identifies with $\mathrm{LocSys}_{G,\infty x}$, and (4.26) gives rise to isomorphisms:

$$\mathbb{V}(\widehat{\mathcal{Q}}^{(\infty,0)}_{\mathrm{desc}})_{\lambda=1} \xrightarrow{\sim} \mathrm{LocSys}_{G,\infty x}\,/\mathcal{L}^+_x G \xrightarrow{\sim} \mathrm{LocSys}_G\,.$$

6.3.2. We comment on the role of *integral* additional parameters at ∞, i.e., the ones arising from $Z(G)$-bundles. More precisely, let $E := \mathrm{At}(\mathcal{P}_{Z(G)})^*$ for some $Z(G)$-bundle $\mathcal{P}_{Z(G)}$. Then E is an extension of $\mathfrak{z}^*_G \otimes \mathcal{O}_X$ by ω_X, so $(\mathfrak{g}^\infty, E)$ is a well defined k-point of Par_G.

Proposition 6.9 *Let $E = \mathrm{At}(\mathcal{P}_{Z(G)})^*$ for a $Z(G)$-bundle $\mathcal{P}_{Z(G)}$. Then there is a canonical isomorphism of DG stacks:*

$$\mathbb{V}(\widehat{\mathcal{Q}}^{(\infty,E)}_{\mathrm{desc}})_{\lambda=1} \xrightarrow{\sim} \mathrm{LocSys}_G \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_G, \tag{6.13}$$

where the second map is the central shift $- \otimes \mathcal{P}_{Z(G)}$.

Proof Note that the $\mathcal{D}_{\mathrm{Bun}_{G,\infty x} \times X/\mathrm{Bun}_{G,\infty x}}$-module (5.9) at parameter $(\mathfrak{g}^\infty, E)$ is induced from the following sequence:

$$0 \to \omega_{\mathrm{Bun}_{G,\infty x} \times X/\mathrm{Bun}_{G,\infty x}} \to \mathrm{At}(\mathcal{P}_{Z(G)} \otimes \mathcal{P}_G)^* \to \mathfrak{g}^*_{\mathcal{P}_G} \to 0$$

via the functor $(-)_{\mathcal{D}}$ and pushing out (see Sect. 6.1). An argument similar to the above shows that $\mathcal{T}^{(\infty,E)}_G$ is the inert quasi-twisting associated to the triangle in $\mathrm{QCoh}(\mathrm{Bun}_G)$:

$$\mathcal{O}_{\mathrm{Bun}_G} \to \widehat{\mathcal{Q}}^{(\infty,E)}_{\mathrm{desc}} \to \mathcal{Q}^{\infty}_{\mathrm{desc}},$$

where we have a canonical isomorphism $\widehat{\mathcal{Q}}^{(\infty,E)}_{\mathrm{desc}}\big|_{\mathcal{P}_G} \xrightarrow{\sim} \widehat{\mathcal{Q}}^{(\infty,0)}_{\mathrm{desc}}\big|_{\mathcal{P}_{Z(G)} \otimes \mathcal{P}_G}$. Hence the result follows from Proposition 6.7. □

Remark 6.10 A connection on $\mathcal{P}_{Z(G)}$ gives rise to a splitting of E, hence an isomorphism $\mathbb{V}(\widehat{\mathcal{Q}}^{(\infty,E)}_{\mathrm{desc}})_{\lambda=1} \xrightarrow{\sim} \mathbb{V}(\widehat{\mathcal{Q}}^{(\infty,0)}_{\mathrm{desc}})$. Geometrically, this corresponds to a lift of the isomorphism $- \otimes \mathcal{P}_{Z(G)} : \mathrm{Bun}_G \xrightarrow{\sim} \mathrm{Bun}_G$ to LocSys_G.

Remark 6.11 Specializing the hypothetical equivalence (5.22) to the parameter $(\mathfrak{g}^{\mathrm{crit}}, 0)$, we obtain the usual, naïve statement of the geometric Langlands correspondence:

$$\mathrm{Diff}(\mathcal{L}^{-\frac{1}{2}}_{G,\det})\text{-}\mathrm{Mod}(\mathrm{Bun}_G) \xrightarrow{\sim} \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}).$$

Specializing to $(\mathfrak{g}^{\mathrm{crit}}, E)$ where $E = \mathrm{At}(\mathcal{P}_{Z(\check{G})})^*$, we obtain from (6.13) a hypothetical equivalence:

$$\mathrm{Diff}(\mathcal{L}^{-\frac{1}{2}}_{G,\det} \otimes \mathcal{M})\text{-}\mathrm{Mod}(\mathrm{Bun}_G) \xrightarrow{\sim} \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}} \underset{\mathrm{Bun}_{\check{G}}}{\times} \mathrm{Bun}_{\check{G}})$$

where $\mathcal{M}$ is the pullback to Bun_G of the line bundle on $\mathrm{Bun}_{G/[G,G]}$ corresponding to $\mathcal{P}_{Z(\check{G})}$. This equivalence can be viewed as an expected compatibility of the geometric Langlands duality with central shift. Let us reiterate that when G is not a torus, none of these equivalences are true without a renormalization process.

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