SPANISH JOURNAL OF STATISTICS
Vol. **4** No. 1 2022, Pages 73–79
doi:https://doi.org/10.37830/SJS.2022.1.05

REGULAR ARTICLE

On moments and entropy of the gamma-Gompertz distribution

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Received: June 22, 2022. Accepted: November 17, 2022.

Abstract: The three-parameter gamma-Gompertz family of distributions was introduced recently in the literature. We verify that the analytical expressions provided for the ordinary moments and Shannon entropy are not correct and hence cannot be used for computing such quantities. We derive two closed-form expressions for the mean and a closed-form expression for the Shannon entropy in terms of the Whittaker function.

Keywords: Whittaker function, moments, entropy

MSC: 60E05, 60E10

1 Gamma-Gompertz distribution

The gamma-Gompertz ("GGo" for short) family of distributions was defined by Shama et al. (2022), and its cumulative distribution function (CDF) and probability density function (PDF) are given, respectively, by (Shama et al., 2022, Definition 1)

$$G(x) = \frac{\gamma\left(\theta, \lambda(\mathrm{e}^{\alpha x} - 1)/\alpha\right)}{\Gamma(\theta)}, \quad x > 0,$$

$$g(x) = \frac{\lambda}{\Gamma(\theta)} \exp\left\{\alpha x - \frac{\lambda}{\alpha}\left(\mathrm{e}^{\alpha x} - 1\right)\right\}\left[\frac{\lambda}{\alpha}(\mathrm{e}^{\alpha x} - 1)\right]^{\theta - 1}, \quad x > 0,$$

where $\lambda > 0$, $\theta > 0$, $\alpha > 0$, $\Gamma(\cdot)$ is the complete gamma function, and $\gamma(a, z) = \int_0^z t^{a-1}\,\mathrm{e}^{-t}\,dt$ is the lower incomplete gamma function for $a > 0$ and $z > 0$. We can also express the PDF of the GGo distribution in the form

$$g(x) = \frac{\lambda^{\theta}\,\mathrm{e}^{\lambda/\alpha}}{\alpha^{\theta-1}\,\Gamma(\theta)} \exp\left(\alpha\,\theta\,x - \frac{\lambda}{\alpha}\,\mathrm{e}^{\alpha x}\right)(1 - \mathrm{e}^{-\alpha x})^{\theta - 1}, \quad x > 0. \tag{1}$$

 Published by the Spanish National Statistical Institute

The GGo distribution reduces to the Gompertz distribution (see, for example,c Garg et al. (1970)) when $\theta = 1$.

It is worth stressing that the failure rate (FR) function plays a substantial role in the lifetime data analysis, mainly in survival and reliability studies. Indeed, the mathematical characterization of a lifetime distribution for a certain life phenomena can be made on the basis of its failure rate pattern. In particular, many real-life data, particularly in reliability engineering, exhibit bathtub-shaped FR, which contains the three main regions: early FR region followed by constant FR region and, then, the wear-out region when the FR growths significantly. However, the assumption that the FR increases rapidly with time is not always true. In particular, Bartley (2003) provides an example from electric power industry where some high voltage transformers that survive before the mean life tend to have extremely long lives and the FR is eventually constant. Many distributions have bathtub-shaped FR, but the vast majority are V-shaped, and so these distributions may not fit appropriately bathtub-shaped data with a flat region. However, this region may be very important in real applications and, hence, the correct modeling of a flat region becomes very important. From Shama et al. (2022, Eq. (10)), we have that the FR function of the GGo distribution has the form

$$h(x) = \frac{\lambda \exp\{\alpha x - \lambda(\mathrm{e}^{\alpha x} - 1)/\alpha\}}{\Gamma(\theta, \lambda(\mathrm{e}^{\alpha x - 1} - 1)/\alpha)} \left[\frac{\lambda}{\alpha}(\mathrm{e}^{\alpha x - 1} - 1))\right]^{\theta - 1}, \quad x > 0,$$

where $\Gamma(\cdot, \cdot)$ is the upper incomplete gamma function defined by

$$\Gamma(a, z) = \int_z^\infty t^{a-1}\mathrm{e}^{-t}\, dt, \quad a \in \mathbb{C}, \quad z \in \mathbb{C}.$$

A closer look at Figure 2 in Shama et al. (2022, p. 694) reveals that the FR function of the GGo distribution can present bathtub-shaped FR with a flat region, and so this distribution can be useful in practice to fit real data with a long flat region.

Shama et al. (2022) have derived various distributional properties of the GGo distribution, and have provided an extensive Monte Carlo simulation study to assess the effectiveness of some classical estimation approaches to estimate the GGo distribution parameters. They have also considered a re-parametrized log-GGo distribution and, based on this re-parametrized distribution, a log-GGo regression model was introduced. However, we note that closed-form expressions of some mathematical properties provided by these authors do not appear correct, and so cannot be recommended to users.

2 Moments and entropy

It is well-known that some important statistical measures as, for example, variance, skewness and kurtosis can be obtained in terms of moments. Thus, it is quite important to have a valid expression for the moments in order to compute such quantities. Unfortunately, the analytical expression for the rth ordinary moment of the GGo distribution provided by Shama et al. (2022) does not appear correct. Shama et al. (2022, Theorem 5) have derived a closed-form expression for the moments of the GGo distribution, which is given by

$$\mu_r' = \lambda \left(\frac{\lambda}{\alpha}\right)^{\theta-1} \mathrm{e}^{\lambda/\alpha} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{i+j+r+1}\,(\theta+j-i)^{-r-1}\,\lambda^j}{\Gamma(i+1)\,\Gamma(\theta-i)\,\Gamma(j+1)\,\alpha^{j+r+1}}\Gamma(r+1). \tag{2}$$

From (2), note that the quantity $(\theta + j - i)^{-r-1}$ may be undefined, and so is impossible to compute the moments from the above expression. For example, let $\theta = 1$, $j = 1$ and $i = 2$. In this case, it follows that

$$(\theta + j - i)^{-r-1} = \frac{1}{(1+1-2)^{r+1}} = \frac{1}{0^{r+1}},$$

which is obviously undefined for all r. In addition, a closer look at the proof of Theorem 5 in Shama et al. (2022, p. 695) reveals that closed-form expression (2) comes from an integral which is not convergent; that is, after a change of variable, Shama et al. (2022, p. 695) have provided an analytical expression for the following integral

$$\int_0^\infty x^r \, \mathrm{e}^{\alpha(\theta+j-i)x} \, dx.$$

However, if there exists at least a pair (i, j) such that $j - i > 0$, then it is evident that the above integral diverges, since $\alpha > 0$ and $\theta > 0$. In short, the above integral diverges for infinite pairs (i, j), and so the moments from expression (2) do not exist.

The entropy of a random variable is a measure of variation of the uncertainty. Entropy has been used in various situations in science and engineering, and numerous measures of entropy have been studied and compared in the literature. Let N be a random variable with PDF v. The Shannon entropy of N is defined by $\mathbb{E}[-\log(v(N))]$. Shama et al. (2022, Theorem 7) have derived a closed-form expression for the Shannon entropy of the GGo distribution, which is given by

$$H(g) = -\tau - \ln(\lambda) + \theta + \ln(\Gamma(\theta)) + (1-\theta)\Psi(\theta), \tag{3}$$

where $\Psi(\cdot)$ denotes the digamma function, and

$$\tau = \left(\frac{\lambda}{\alpha}\right)^{\theta} \mathrm{e}^{\lambda/\alpha} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{i+j} (\lambda/\alpha)^j \, (\theta + j - i)^{-2}}{\Gamma(i+1)\, \Gamma(\theta - i)\, \Gamma(j+1)\, \alpha^{j+r+1}}.$$

From (3) (i.e., specifically from τ), the quantity $(\theta + j - i)^{-2}$ may be undefined, and so is not possible to compute the Shannon entropy from the analytical expression $H(g)$. For example, let $\theta = 1$, $j = 1$ and $i = 2$. Hence, it follows that $(\theta + j - i)^{-2} = (1 + 1 - 2)^{-2} = 0^{-2}$, which is obviously undefined. Therefore, the closed-form expression for the Shannon entropy obtained in Shama et al. (2022, Theorem 7) is not a valid analytical expression and, consequently, cannot be recommended to users.

3 Statistical properties

In the following, we provide explicit closed-form expressions of some statistical properties of the GGo distribution.

3.1 Moment generation function

From Gradshteyn and Ryzhik (2015, p. 340), we have that

$$\int_0^\infty (1 - \mathrm{e}^{-z})^{\nu-1} \exp\left(-\mu z - \beta \mathrm{e}^{z}\right) dz = \Gamma(\nu)\, \beta^{\frac{\mu-1}{2}}\, \mathrm{e}^{-\beta/2}\, W\left(\frac{1-\mu-2\nu}{2}, \frac{-\mu}{2}; \beta\right), \tag{4}$$

where $\mu \in \mathbb{C}$, $\beta \in \mathbb{C}$ such that the real part of β is positive, $\nu \in \mathbb{C}$ such that the real part of ν is positive, and $W(a, b; v)$ denotes the Whittaker function with $a \in \mathbb{C}$, $b \in \mathbb{C}$ and $v \in \mathbb{C}$ (Whittaker, 1903). We have the following proposition.

Proposition 1. *The moment generation function of the GGo distribution is given by*

$$M(t) = \mathrm{e}^{\lambda/2\alpha} \left(\frac{\lambda}{\alpha}\right)^{\theta/2-1/2-t/2\alpha} W\left(\frac{1}{2} + \frac{t}{2\alpha} - \frac{\theta}{2}, \frac{\theta}{2} + \frac{t}{2\alpha}; \frac{\lambda}{\alpha}\right). \tag{5}$$

Proof. We have that $M(t) = \int_0^\infty \mathrm{e}^{tx} g(x) dx$, for $t \in \mathbb{R}$. Hence, from the PDF in (1), we have that

$$M(t) = \frac{\lambda^\theta \, \mathrm{e}^{\lambda/\alpha}}{\alpha^{\theta-1}\,\Gamma(\theta)} \int_0^\infty \mathrm{e}^{tx} \exp\left\{\alpha\,\theta\,x - \frac{\lambda}{\alpha}\,\mathrm{e}^{\alpha\,x}\right\} (1-\mathrm{e}^{-\alpha\,x})^{\theta-1}\,dx.$$

Let $z = \alpha x$, and so

$$M(t) = \frac{\lambda^\theta \, \mathrm{e}^{\lambda/\alpha}}{\alpha^{\theta}\,\Gamma(\theta)} \int_0^\infty \exp\left\{(t/\alpha+\theta)\,z - \frac{\lambda}{\alpha}\,\mathrm{e}^{z}\right\} (1-\mathrm{e}^{-z})^{\theta-1}\,dz.$$

From (4), the result follows. □

Corollary 1. *If $\theta = 1$, the moment generating function of the GGo distribution reduces to moment generating function of the Gompertz distribution given by*

$$M(t) = \mathrm{e}^{\lambda/\alpha} \left(\frac{\lambda}{\alpha}\right)^{-t/\alpha} \Gamma\left(1 + \frac{t}{\alpha}, \frac{\lambda}{\alpha}\right), \quad t \in \mathbb{R}.$$

Proof. From (5) and when $\theta = 1$, we have that

$$M(t) = \mathrm{e}^{\lambda/2\alpha} \left(\frac{\lambda}{\alpha}\right)^{-t/2\alpha} W\left(\frac{t}{2\alpha}, \frac{1}{2} + \frac{t}{2\alpha}; \frac{\lambda}{\alpha}\right).$$

From Olver et al. (2010, p. 177), we have that $\Gamma(\xi+1; v) = \mathrm{e}^{-v/2}\, v^{\xi/2}\, W(\xi/2, (\xi+1)/2; v)$, where $\xi > 0$ and $v > 0$. The result follows by considering $\xi = t/\alpha$ and $v = \lambda/\alpha$ in $\Gamma(\xi+1; v)$. □

Proposition 2. *The characteristic function of the GGo distribution is given by $\varphi(s) = M(is)$, where $s \in \mathbb{R}$, and $i = \sqrt{-1}$ is the imaginary unit.*

3.2 Moments

We have the following proposition.

Proposition 3. *The rth ordinary moment of the GGo distribution is given by*

$$\mu_r' = \frac{d^r}{dt^r} M(t)\bigg|_{t=0}.$$

Remark 1. *proposition 3 relies on the fact that the moments of a distribution can be obtained from the moment generating function.*

Remark 2. *It is worth stressing that the computation of the ordinary moments of the GGo distribution from proposition 3 is not a trivial problem, since the analytical derivatives of the Whittaker function are not easy to obtain.*

The next propositions present the mean of the GGo distribution.

Proposition 4. *If $\theta = n \in \mathbb{N}$, the first moment (mean) of the GGo distribution reduces to*

$$\mathbb{E}[X] = \frac{\mathrm{e}^{\lambda/2\alpha}}{\alpha} \sum_{k=0}^{n} \left(\frac{\lambda}{\alpha}\right)^{(k-1)/2} W\left(\frac{-k-1}{2}, \frac{k}{2}; \frac{\lambda}{\alpha}\right). \tag{6}$$

Proof. The survival function of the GGo distribution when $\theta = n \in \mathbb{N}$ can be expressed as

$$\bar{G}(x) = \mathrm{e}^{\lambda/\alpha} \sum_{k=0}^{n} \frac{(\lambda/\alpha)^k}{k!} (1 - \mathrm{e}^{-\alpha x})^k \exp\left\{\alpha\, k\, x - \frac{\lambda}{\alpha}\, \mathrm{e}^{\alpha x}\right\}, \quad x > 0.$$

We have that $\mathbb{E}[X] = \int_0^\infty \bar{G}(x)dx$, and so

$$\mathbb{E}[X] = \mathrm{e}^{\lambda/\alpha} \sum_{k=0}^{n} \frac{(\lambda/\alpha)^k}{k!} \int_0^\infty (1 - \mathrm{e}^{-\alpha x})^k \exp\left\{\alpha\, k\, x - \frac{\lambda}{\alpha} \mathrm{e}^{\alpha x}\right\} dx.$$

Let $z = \alpha x$. We have that

$$\mathbb{E}[X] = \frac{\mathrm{e}^{\lambda/\alpha}}{\alpha} \sum_{k=0}^{n} \frac{(\lambda/\alpha)^k}{k!} \int_0^\infty (1 - \mathrm{e}^{-z})^k \exp\left\{k\, z - \frac{\lambda}{\alpha} \mathrm{e}^{z}\right\} dz.$$

From (4) with $\nu = k + 1$, $\mu = -k$ and $\beta = \lambda/\alpha$, the result follows. □

Proposition 5. *If $\theta > 0$, the first moment (mean) of the GGo distribution reduces to*

$$\begin{aligned}\mathbb{E}[X] = {} & \frac{\mathrm{e}^{\lambda/\alpha}}{\alpha\Gamma(\theta)} \sum_{m=0}^{\infty} \frac{(\lambda/\alpha)^m\, (1-\theta)_m}{m!} \left[\Psi(\theta - m) - \log(\lambda/\alpha)\right] \Gamma(\theta - m) \\ & + \frac{\mathrm{e}^{\lambda/\alpha} (\lambda/\alpha)^\theta}{\alpha\Gamma(\theta)} \sum_{m=0}^{\infty} \frac{(1-\theta)_m}{m!(\theta - m)^2}\, {}_2F_2(\theta - m, \theta - m; \theta - m + 1, \theta - m + 1; -\lambda/\alpha),\end{aligned}$$

where $(a)_n := a(a+1)\cdots(a+n-1)$ is the rising factorial with $a \in \mathbb{R}$, $(a)_0 := 1$ and $n \geq 1$, and ${}_2F_2(a, b; c, d; z)$ is the generalized hypergeometric function defined by

$${}_2F_2(a, b; c, d; z) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n\, (b)_n}{(c)_n\, (d)_n} \frac{z^n}{n!}, \quad z \in \mathbb{R},$$

and $\Psi(\cdot)$ is the digamma function defined by

$$\Psi(z) = \frac{d}{dz} \log(\Gamma(z)).$$

Proof. We have that $0 < \mathrm{e}^{-\alpha x} < 1$ for all $x > 0$, and so $(1 - \mathrm{e}^{-\alpha x})^{\theta - 1} = \sum_{m=0}^{\infty} \frac{(1-\theta)_m}{m!} \mathrm{e}^{-m\,\alpha x}$. Hence, we can express the PDF in (1) of the form

$$g(x) = \frac{\alpha(\lambda/\alpha)^\theta\, \mathrm{e}^{\lambda/\alpha}}{\Gamma(\theta)} \sum_{m=0}^{\infty} \frac{(1-\theta)_m}{m!} \exp\left\{(\theta - m)\alpha\, x - \frac{\lambda}{\alpha}\, \mathrm{e}^{\alpha x}\right\}, \quad x > 0.$$

Using the above PDF, it follows that

$$M(t) = \frac{\alpha(\lambda/\alpha)^\theta\, \mathrm{e}^{\lambda/\alpha}}{\Gamma(\theta)} \sum_{m=0}^{\infty} \frac{(1-\theta)_m}{m!} \int_0^\infty \exp\left\{(\theta - m + t/\alpha)\alpha\, x - \frac{\lambda}{\alpha}\, \mathrm{e}^{\alpha x}\right\} dx.$$

Let $z = \alpha x$, and so

$$M(t) = \frac{(\lambda/\alpha)^{\theta}\,\mathrm{e}^{\lambda/\alpha}}{\Gamma(\theta)} \sum_{m=0}^{\infty} \frac{(1-\theta)_m}{m!} \int_0^{\infty} \exp\left\{(\theta - m + t/\alpha)z - \frac{\lambda}{\alpha}\,\mathrm{e}^{z}\right\} dz.$$

From Gradshteyn and Ryzhik (2015, p. 340), we have that $\int_0^{\infty} \exp\left(-pz - q\mathrm{e}^{z}\right) dz = q^{p}\,\Gamma(-p, q)$, where $p \in \mathbb{C}$, and $q \in \mathbb{C}$ such that the real part of q is positive. Hence,

$$M(t) = \frac{\mathrm{e}^{\lambda/\alpha}}{\Gamma(\theta)} \sum_{m=0}^{\infty} \frac{(\lambda/\alpha)^{m}\,(1-\theta)_m}{m!} \left(\frac{\alpha}{\lambda}\right)^{t/\alpha} \Gamma\left(t/\alpha + \theta - m, \frac{\lambda}{\alpha}\right).$$

From Brychkov (2008, p. 22), we have that

$$\frac{d}{da}\Gamma(a,z) = [\psi(a) - \log(z)]\Gamma(a) + \Gamma(a,z)\,\log(z) + \frac{z^{a}}{a^{2}}\,{}_2F_2(a,a;a+1,a+1;-z).$$

Now, using the above derivative and that $\mathbb{E}[X] = (d/dt)M(t)|_{t=0}$, the result follows. □

Remark 3. *The algebraic developments considered in this section reveal that is not easy to obtain a general closed-form expression for the ordinary moments of the GGo distribution. This is still an open problem regarding the three-parameter GGo family of distributions introduced by Shama et al. (2022).*

3.3 Entropy

We have the following proposition

Proposition 6. *The Shannon entropy of the GGo distribution is given by*

$$H(g) = -\log\left[\frac{\lambda^{\theta}\,\mathrm{e}^{\lambda/\alpha}}{\alpha^{\theta-1}\,\Gamma(\theta)}\right] - \alpha\,\theta\,\mathbb{E}[X] + \frac{\lambda\,M(\alpha)}{\alpha} + (\theta-1)\sum_{n=1}^{\infty}\frac{M(-\alpha\,n)}{n}, \tag{7}$$

where $\mathbb{E}[X]$ is the mean of the GGo distribution provided in proposition 5, and $M(\cdot)$ is the moment generating function of the GGo distribution provided in proposition 1.

Proof. The Shannon entropy of the GGo distribution is given by $H(g) = -\mathbb{E}[\log g(X)]$, where $g(x)$ is the PDF of the GGo distribution. Note that

$$\ln(g(x)) = \ln\left[\frac{\lambda^{\theta}\,\mathrm{e}^{\lambda/\alpha}}{\alpha^{\theta-1}\,\Gamma(\theta)}\right] + \alpha\,\theta\,x - \frac{\lambda}{\alpha}\,\mathrm{e}^{\alpha\,x} + (\theta-1)\,\ln(1-\mathrm{e}^{-\alpha\,x}), \quad x > 0.$$

Using the expansion $-\ln(1-z) = \sum_{n=1}^{\infty}\frac{z^{n}}{n}$, for $|z| < 1$, and taking the expected value, the result follows. □

4 Numerical study

In what follows, we provide some numerical values for the mean and Shannon entropy of the GGo distribution. We use the proposed closed-form expression in proposition 5 to obtain numerical values for $\mathbb{E}[X]$, and the proposed closed-form expression in (7) to obtain numerical values for the entropy. Table 1 lists the values of $\mathbb{E}[X]$ and $H(g)$ for $\lambda = 0.8$, $\alpha = 1.0$, and different values of θ. In this table, $n_{\max}$ means the number of terms considered in the expansion for $H(g)$ in (7) , while the last column shows the corresponding values of the entropy by numerical integration. Note that the numerical values delivered by the expression (7) and numerical integration for the entropy are near, mainly when $n_{\max} = 1000$.

		$H(g)$				
θ	$\mathbb{E}[X]$	$n_{\max}=10$	$n_{\max}=60$	$n_{\max}=150$	$n_{\max}=1000$	numerical integration
0.6	0.454810	0.292393	0.200059	0.179110	0.159559	0.150320
0.7	0.518039	0.370634	0.320716	0.310929	0.302876	0.299965
0.8	0.578384	0.435732	0.411674	0.407606	0.404650	0.403818
0.9	0.636056	0.488960	0.480238	0.478969	0.478153	0.477971
1.0	0.691245	0.531898	0.531898	0.531898	0.531898	0.531898
1.1	0.744126	0.566114	0.570739	0.571234	0.571485	0.571521
1.2	0.794856	0.593027	0.599791	0.600410	0.600690	0.600721
1.3	0.843579	0.613869	0.621310	0.621891	0.622125	0.622147
1.4	0.890428	0.629689	0.636984	0.637469	0.637644	0.637657
1.5	0.935522	0.641361	0.648084	0.648465	0.648589	0.648594
1.6	0.978973	0.649613	0.655578	0.655864	0.655950	0.655951
1.7	1.020881	0.655048	0.660206	0.660416	0.660472	0.660472
1.8	1.061341	0.658159	0.662540	0.662690	0.662726	0.662726
1.9	1.100437	0.659356	0.663028	0.663134	0.663157	0.663157
2.0	1.138249	0.658975	0.662021	0.662096	0.662110	0.662110

Table 1: Mean and entropy.

Acknowledgments

The authors would like to thank the Editor and an anonymous reviewer for their insightful comments and suggestions. Fredy Castellares gratefully acknowledges the financial support from FAPEMIG (Belo Horizonte/MG, Brazil). Artur Lemonte gratefully acknowledges the financial support of the Brazilian agency CNPq (grant 304776/2019-0).

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