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Lipschitz geometry of surface germs in $\mathbb{R}^4$: metric knots

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Abstract
A link at the origin of an isolated singularity of a two-dimensional semialgebraic surface in $\mathbb{R}^4$ is a topological knot (or link) in S^3. We study the connection between the ambient Lipschitz geometry of semialgebraic surface germs in $\mathbb{R}^4$ and knot theory. Namely, for any knot K, we construct a surface X_K in $\mathbb{R}^4$ such that: the link at the origin of X_K is a trivial knot; the germs X_K are outer bi-Lipschitz equivalent for all K; two germs X_K and $X_{K'}$ are ambient semialgebraic bi-Lipschitz equivalent only if the knots K and K' are isotopic. We show that the Jones polynomial can be used to recognize ambient bi-Lipschitz non-equivalent surface germs in $\mathbb{R}^4$, even when they are topologically trivial and outer bi-Lipschitz equivalent.

Keywords Lipschitz geometry · Surface singularities · Knots · Jones polynomials

Mathematics Subject Classification 51F30 · 14P10 · 03C64 · 57K10

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1 Introduction

We study the difference between the outer and ambient bi-Lipschitz equivalence of semialgebraic surface germs at the origin in $\mathbb{R}^4$. Two surface germs are outer bi-Lipschitz equivalent if they are bi-Lipschitz equivalent as abstract metric spaces with the outer metric $d(x, y) = \|x - y\|$. Ambient bi-Lipschitz equivalence means that there exists a germ of a bi-Lipschitz, orientation preserving, homeomorphism of the ambient space mapping one of them to the other one. Note that in Singularity Theory the homeomorphism is not required to be orientation preserving. We add this condition to be consistent with the isotopy equivalence relation in Knot Theory. Also, to avoid confusion between the Singularity Theory and Knot Theory notions of the link, we always write "the link at the origin" speaking of the link of a surface germ.

If a surface germ in $\mathbb{R}^4$ with a connected link at the origin has an isolated singularity then its link is a knot in S^3. The results of [3] show that ambient equivalence is different from outer equivalence even when there are no topological obstructions. This phenomenon is called "metric knots." We consider the following question: How different are these equivalence relations? In the previous paper [3] we show that, for any given ambient topological type of a surface germ, one can find infinitely many equivalence classes with respect to ambient bi-Lipschitz equivalence. In this paper we start by showing that the question becomes nontrivial even when "there is no topology," i.e., for the germs with unknotted links at the origin. Universality Theorem (Theorem 3.1 below) implies that the ambient bi-Lipschitz classification in this case "contains all of Knot Theory."

More precisely, for any knot K, there exists a germ of a surface X_K in $\mathbb{R}^4$ such that:

1. The link at the origin of X_K is a trivial knot;
2. The germs X_K are outer bi-Lipschitz equivalent for all K;
3. Two germs X_K and $X_{K'}$ are ambient semialgebraic bi-Lipschitz equivalent only if the knots K and K' are isotopic.

In other words, although the links at the origin of all surface germs X_K are trivial knots, the map $K \mapsto X_K$ from the set of all isotopy classes of knots in S^3 to the set of ambient bi-Lipschitz equivalence classes of surface germs in $\mathbb{R}^4$ is injective.

The second theorem (Theorem 3.5 below) states that, for each germ X_K in Universality Theorem, there are infinitely many semialgebraic surfaces $X_{K,i}$ satisfying Universality Theorem, such that $X_{K,i}$ and $X_{K,j}$ are semialgebraic ambient bi-Lipschitz equivalent only if $i = j$.

The proofs are based on the following results of Sampaio [12] and Valette [13], (see also [1] and [2]).

Theorem 1.1 ([12, Theorem 3.2]) *If $(X, 0)$ and $(Y, 0)$ are ambient semialgebraic bi-Lipschitz equivalent semialgebraic germs, then their tangent cones $C_0(X)$ and $C_0(Y)$ are ambient semialgebraic bi-Lipschitz equivalent.*

Theorem 1.2 ([13, Corollary 0.2]) *If two semialgebraic germs $(X, 0)$ and $(Y, 0)$ are semialgebraic bi-Lipschitz homeomorphic, then there is a semialgebraic bi-Lipschitz homeomorphism $h\colon (X, 0) \to (Y, 0)$ preserving the distance to the origin.*

In Section 3 we define (β_1, β_2)-bridges and the saddle move, closely related to the broken bridge construction in [3]. A *one-bridge surface germ* is a surface germ containing a single (β_1, β_2)-bridge and metrically conical outside it. The saddle move relates the metric problem of ambient semialgebraic bi-Lipschitz equivalence of two one-bridge surface germs in $\mathbb{R}^4$ with the topological problem of isotopy of two knots in S^3 corresponding to the links at the origin of the surfaces obtained from these one-bridge surface germs by the saddle moves (see Definition 3.3). That is why topological knot invariants, such as the Jones polynomial, yield metric knot invariants, which can be used to recognize ambient semialgebraic bi-Lipschitz non-equivalence of surface germs.

Although one-bridge surface germs are the simplest examples of not Lipschitz normally embedded surfaces, they have rather non-trivial ambient Lipschitz geometry. Another version of Universality Theorem (Theorem 3.13 below) states that, for any two knots K and L, one can construct a one-bridge surface germ X_{KL} such that:

1. The link at the origin of X_{KL} is isotopic to L;
2. For any knots K and L, all surface germs X_{KL} are outer bi-Lipschitz equivalent;
3. Surface germs X_{K_1L} and X_{K_2L} are ambient semialgebraic bi-Lipschitz equivalent only if the knots K_1 and K_2 are isotopic.

In Section 4 we consider the Jones polynomial of the link at the origin $L = L_{S(X)}$ of a surface germ $S(X)$ obtained from a one-bridge surface germ X by the saddle move (see Definition 3.3). Since the isotopy class of L is an ambient semialgebraic Lipschitz invariant, its Jones polynomial becomes an ambient Lipschitz invariant of X. If $X = X'_{K,i}$ is a "twisted" surface constructed in [3] (see also Theorem 3.5) and K is a trivial knot, then L is a torus link. Its Jones polynomial is computed completely (see Corollary 4.2 and Remark 4.3) and determines the number i of twists. This shows that Jones polynomial can be used to prove ambient bi-Lipschitz non-equivalence of metric knots.

If we do not suppose the surface germ to be a one-bridge surface germ, we obtain a stronger version of Universality Theorem (Theorem 3.14 below). It states that, for any two knots K and L, and any two rational numbers $\alpha > 1$ and $\beta > 1$, one can construct a surface germ $X^{\alpha\beta}_{KL}$ such that:

1. The link at the origin of $X^{\alpha\beta}_{KL}$ is isotopic to L;
2. For a fixed knot K, the tangent link of $X^{\alpha\beta}_{KL}$ (i.e., the intersection of the tangent cone with the unit sphere) is isotopic to K;
3. All surface germs $X^{\alpha\beta}_{KL}$ are outer bi-Lipschitz equivalent for fixed α and β.

All sets, functions and maps in this paper are assumed to be real semialgebraic. We use semialgebraic bi-Lipschitz equivalence, because we refer to the theorem of Valette [13]. Our results are also true for subanalytic bi-Lipschitz equivalence of subanalytic surface germs, and we expect them to remain true in any polynomially bounded o-minimal structure over $\mathbb{R}$. The Universality Theorem 3.1 was announced without a proof in the expository paper [4].

2 Definitions and notations

We consider germs at the origin of semialgebraic surfaces (two-dimensional semialgebraic sets) in $\mathbb{R}^4$.

Definition 2.1 A surface X can be considered as a metric space, equipped with either the outer metric $d(x, y) = \|x - y\|$ or the inner metric $d_i(x, y)$ defined as the minimal length of a path in X connecting x and y. A germ X is *Lipschitz normally embedded* if its inner and outer metrics are equivalent.

Definition 2.2 Two germs of semialgebraic sets $(X, 0)$ and $(Y, 0)$ are *outer bi-Lipschitz equivalent* if there exists a homeomorphism $H\colon (X, 0) \to (Y, 0)$ bi-Lipschitz with respect to the outer metric. The germs are *semialgebraic outer bi-Lipschitz equivalent* if the map H can be chosen to be semialgebraic. The germs are *ambient bi-Lipschitz equivalent* if there exists an orientation preserving bi-Lipschitz homeomorphism $\widetilde{H}\colon (\mathbb{R}^4, 0) \to (\mathbb{R}^4, 0)$, such that $\widetilde{H}(X) = Y$. The germs are *semialgebraic ambient bi-Lipschitz equivalent* if the map $\widetilde{H}$ can be chosen to be semialgebraic.

Definition 2.3 The *link at the origin* L_X of a germ X is the equivalence class of the sets $X \cap S^3_{0,\varepsilon}$ for small positive ε with respect to the ambient bi-Lipschitz equivalence. The *tangent link* of X is the link at the origin of the tangent cone of X.

Remark 2.4 By the finiteness theorems of Mostowski, Parusinski and Valette (see [10, 11, 14]) the link at the origin is well defined. We write "the link at the origin" speaking of this notion of the link from Singularity Theory, reserving the word "link" for the notion of the link in Knot Theory. If X has an isolated singularity at the origin then each connected component of L_X is a knot in S^3.

Definition 2.5 A semialgebraic germ $(X, 0) \subset \mathbb{R}^n$ is called *outer metrically conical* if there exists a germ of a bi-Lipschitz homeomorphism $H\colon (X, 0) \to C(L_X)$, where $C(L_X)$ is a straight cone over L_X. The map H is called a *conification map*. A germ $(X, 0)$ is called *ambient metrically conical* if there exists a germ of a bi-Lipschitz homeomorphism $\widetilde{H}\colon \mathbb{R}^n \to \mathbb{R}^n$, such that $\widetilde{H}(X, 0) = C(L_X)$. The map $\widetilde{H}$ is also called a *conification map*. A germ $(X, 0)$ is called *outer (ambient) semialgebraic metrically conical* if a corresponding conification map can be chosen to be semialgebraic.

Remark 2.6 Notice that the definition makes sense for semialgebraic germs of any dimension, not only for surface germs.

Definition 2.7 An *arc* in a semialgebraic germ $(X, 0)$ is a germ of a semialgebraic embedding $\gamma\colon [0, \epsilon) \to X$ such that $\gamma(0) = 0$. Unless otherwise specified, arcs are parameterized by the distance to the origin, i.e., $\|\gamma(t)\| = t$. We identify an arc with its image in X.

Definition 2.8 Let $f \not\equiv 0$ be (a germ at the origin of) a semialgebraic function defined on an arc γ. The *order* α of f on γ (notation $\alpha = ord_\gamma f$) is the value $\alpha \in \mathbb{Q}$ such that $f(\gamma(t)) = ct^\alpha + o(t^\alpha)$ as $t \to 0$, where $c \neq 0$. If $f \equiv 0$ on γ, we set $ord_\gamma f = \infty$.

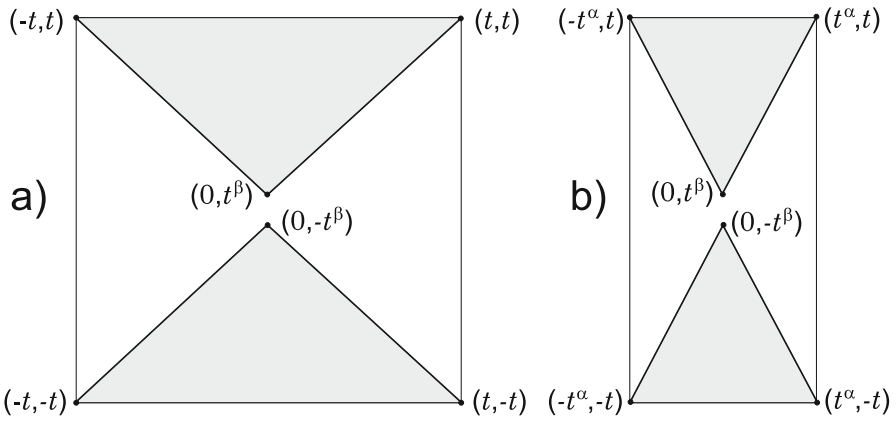


Fig. 1 **a** The set W_t. **b** The set $W_t^{\alpha\beta}$

For any two arcs γ and γ' in X one can define two orders of contact: inner and outer.

Definition 2.9 The *outer order of contact* $tord(\gamma, \gamma')$ is defined as $ord_\gamma f$, where $f(t) = \|\gamma(t) - \gamma'(t)\|$. The *inner order of contact* $itord(\gamma, \gamma')$ is defined as $ord_\gamma g$, where $g(t) = d_p(\gamma(t), \gamma'(t))$. Here d_p is a definable pancake metric (see [5]) equivalent to the inner metric. These two orders of contact are rational numbers such that $1 \leq itord(\gamma, \gamma') \leq tord(\gamma, \gamma')$.

Definition 2.10 Let $\beta > 1$ be a rational number. Consider the space $\mathbb{R}^3$ with coordinates (x, y, z). For a fixed $t \geq 0$, let $Z_t = \{|x| \leq t,\ |y| \leq t\}$ be a square in the xy-plane $\{z = t\}$ and let $Z = \bigcup_{t\geq 0} Z_t$. Let W_t^+ be the subset of Z_t bounded by the line segment $I_t^+ = \{|x| \leq t,\ y = t\}$ and the union J_t^+ of the two line segments connecting the endpoints of I_t^+ with the point $(0, t^\beta)$. Let $W_t^- = \{(x, y) : (x, -y) \in W_t^+\}$ and $J_t^- = \{(x, y) : (x, -y) \in J_t^+\}$. Let $W_t = W_t^+ \cup W_t^-$ (shaded area in Fig. 1a) and let $W = \bigcup_{t\geq 0} W_t \subset \mathbb{R}^3$. A *$\beta$-bridge* is the surface germ $B_\beta = \bigcup_{t\geq 0} J_t^+ \cup J_t^-$. Note that the tangent cone of W is the set $\{|x| \leq |y| \leq z\}$ and the tangent cone of B_β is the surface germ $\{|x| = |y| \leq z\}$.

Definition 2.11 Let $1 < \beta_1 \leq \beta_2$ be two rational numbers. For a fixed $t \geq 0$, let $Z_t = \{|x| \leq t,\ |y| \leq t,\ z = t\}$ and $Z = \bigcup_{t\geq 0} Z_t$ be as in Definition 2.10. In the xy-plane $\{z = t\}$ consider the points (see Fig. 2)

$$p_1(t) = (-t, t),\ p_2(t) = (-t^{\beta_1}, t^{\beta_2}),\ p_3(t) = (t^{\beta_1}, t^{\beta_2}),\ p_4(t) = (t, t),$$
$$p_1'(t) = (-t, -t),\ p_2'(t) = (-t^{\beta_1}, -t^{\beta_2}),\ p_3'(t) = (t^{\beta_1}, -t^{\beta_2}),\ p_4'(t) = (t, -t).$$

Let us connect the points $p_1(t),\ p_2(t),\ p_3(t),\ p_4(t)$ by three line segments, and define $\bar{J}_t^+$ as the union of these three segments. Let $U_t^+ \subset Z_t$ be the convex hull of $\bar{J}_t^+$. Let P_t^+ be the segment connecting the points $p_2(t)$ and $p_3(t)$. Similarly, let $\bar{J}_t^-$ be the union of segments connecting the points $p_1(t),\ p_2'(t),\ p_3'(t),\ p_4'(t)$, and let U_t^- be the convex hull of $\bar{J}_t^-$ and P_t^- be the segment connecting $p_2'(t)$ with $p_3'(t)$. Let $P_t = P_t^+ \cup P_t^-$

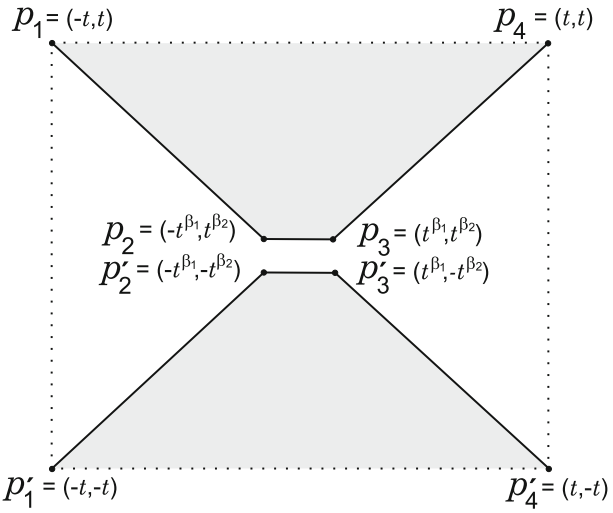


Fig. 2 The set U_t

and let $P = \bigcup_{t\geq 0} P_t \subset \mathbb{R}^3$. Let $U_t = U_t^+ \cup U_t^-$ (shaded area in Fig. 2), and let $U = \bigcup_{t\geq 0} U_t \subset \mathbb{R}^3$. A (β_1, β_2)-*bridge* is the surface germ $B_{\beta_1\beta_2} = \bigcup_{t\geq 0} \bar{J}_t$, where $\bar{J}_t = \bar{J}_t^+ \cup \bar{J}_t^-$.

Note that the set U has the same tangent cone at the origin as W, while the tangent cone at the origin of P is the positive z-axis. Note also that, for $\beta_1 = \beta_2 = \beta$, the (β, β)-bridge is outer bi-Lipschitz equivalent to the β-bridge.

Definition 2.12 Let X be a semialgebraic surface germ in $\mathbb{R}^4$ with the link at the origin homeomorphic to a circle in S^3. We say that X is a *one-bridge surface germ* if

1. There exists a semialgebraic bi-Lipschitz C^1 embedding $\Theta\colon (Z, 0) \to (\mathbb{R}^4, 0)$ such that $\Theta(B_{\beta_1\beta_2}) = X \cap \Theta(Z)$.
2. The union $X \cup \Theta(Z)$ is Lipschitz normally embedded in $\mathbb{R}^4$ and ambient semialgebraic metrically conical: there exist a semialgebraic bi-Lipschitz homeomorphism $\widetilde{H}\colon (\mathbb{R}^4, 0) \to (\mathbb{R}^4, 0)$, such that $\widetilde{H}(X \cup \Theta(Z))$ is a straight cone.

Definition 2.13 Let $\alpha > 1$ and $\beta > 1$ be rational numbers. Consider the space $\mathbb{R}^3$ with coordinates (x, y, z). For a fixed $t \geq 0$, let $Z_t^\alpha = \{|x| \leq t^\alpha,\ |y| \leq t\}$ be a rectangle in the xy-plane $\{z = t\}$. Let $W_t^{\alpha+}$ be the subset of the rectangle Z_t^α bounded by the line segment $I_t^{\alpha+} = \{|x| \leq t^\alpha,\ y = t\}$ and the union $J_t^{\alpha+}$ of the line segments connecting the endpoints of $I_t^{\alpha+}$ with the point $(0, t^\beta)$. Let $W_t^{\alpha-} = \{(x, y) : (x, -y) \in W_t^{\alpha+}\}$ and $J_t^{\alpha-} = \{(x, y) : (x, -y) \in J_t^{\alpha+}\}$. Let $W_t^\alpha = W_t^{\alpha+} \cup W_t^{\alpha-}$ (shaded areas in Fig. 1b) and let $W^\alpha = \bigcup_{t\geq 0} W_t^\alpha \subset \mathbb{R}^3$. An (α, β)-*wedge* is the surface germ $E^{\alpha\beta} = \bigcup_{t\geq 0} J_t^\alpha$, where $J_t^\alpha = J_t^{\alpha+} \cup J_t^{\alpha-}$.

Note that the tangent cone at the origin of W^α is the set $\{(x, y, z) : x = 0; |y| \leq z\}$.

Remark 2.14 We define a *link diagram* in the same way as it is done in Knot Theory, choosing a generic projection of the topological link to some 2-dimensional plane in

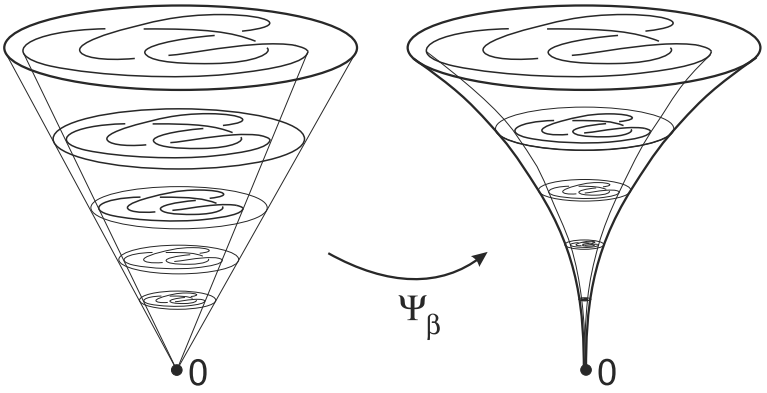


Fig. 3 Hornification of the cone over a knot

$\mathbb{R}^3$ (see [8] for details). Two diagrams are equivalent if they can be related by a finite sequence of Reidemeister moves.

The following result is a special case of the finiteness theorem of Hardt (see [6]).

Theorem 2.15 *Let X be a semialgebraic surface germ. Then, for small $t > 0$ and for any plane $\mathbb{R}^2$, such that the projections of the links $X \cap S_t$ are generic, the diagrams of the links $X \cap S_t$ are equivalent.*

Definition 2.16 Let $F_K \subset S^3$ be a smooth semialgebraic embedded surface diffeomorphic to $S^1 \times [-1, 1]$, such that the two components $\widetilde{K}$ and $\widetilde{K}'$ of the boundary ∂F_K of F_K are isotopic to the same knot K and the linking number (see [9]) of the components $\widetilde{K}$ and $\widetilde{K}'$ is zero. The surface F_K is called a *characteristic band* of the knot K. Let $\widetilde{Y}_K$ and $\widetilde{X}_K$ be the cones over F_K and ∂F_K, respectively. These cones are called *characteristic cones* of the knot K.

Definition 2.17 Let (ρ, l), where $\rho \in S^1$ and $l \in [-1, 1]$, be coordinates in F_K. Let $\xi = (\rho_0, 0)$ be an interior point of F_K. We define a *slice* $S_K = \{(\rho, l) \in F_K,\ |\rho - \rho_0| \leq \epsilon\}$.

Definition 2.18 Let $\beta > 1$ be a rational number. The standard *β-horn* in $\mathbb{R}^4$ is the set $C_\beta = \{(x, y, z, t) \in \mathbb{R}^4 \mid t \geq 0,\ x^2 + y^2 + z^2 = t^{2\beta}\}$. The standard *$\beta$-horn like neighborhood* of the positive t-axis is the set $V_\beta = \{(x, y, z, t) \in \mathbb{R}^4 \mid t \geq 0,\ x^2 + y^2 + z^2 \leq t^{2\beta}\}$.

If $\beta = 1$ then $C_1 = \{t \geq 0,\ x^2 + y^2 + z^2 = t^2\}$ is a cone and $V_1 = \{t \geq 0,\ x^2 + y^2 + z^2 \leq t^2\}$ is a conical neighborhood of the positive t-axis.

The standard *β-hornification* $\Xi_\beta \colon V_1 \to V_\beta$ is defined as $\Xi_\beta(x, y, z, t) = (xt^\beta, yt^\beta, zt^\beta, t)$.

For an arc $\gamma \subset \mathbb{R}^4$, a conical neighborhood of γ is the image $V_1(\gamma)$ of a semialgebraic bi-Lipschitz map $\Phi \colon V_1 \to \mathbb{R}^4$ such that γ is the image of the positive t-axis. A β-horn like neighborhood of γ is $V_\beta(\gamma) = \Phi(V_\beta)$, and a β-hornification to γ is the map $\Psi_\beta \colon V_1(\gamma) \to V_\beta(\gamma)$ defined as $\Psi_\beta = \Phi \circ \Xi_\beta \circ \Phi^{-1}$ (see Fig. 3). We may assume, by Valette's theorem, that Ψ_β preserves the distance to the origin. For a subset S of $V_1(\gamma)$, the set $\Psi_\beta(S)$ is called a *β-hornification* of S to γ.

3 Metric knots

Theorem 3.1 (Universality Theorem) *Let $K \subset S^3$ be a knot. Then one can associate to K a semialgebraic one-bridge surface germ $(X_K, 0)$ in $\mathbb{R}^4$ so that the following holds:*

1. *The link at the origin of each germ X_K is a trivial knot;*
2. *All germs X_K are outer bi-Lipschitz equivalent;*
3. *Two germs X_{K_1} and X_{K_2} are ambient semialgebraic bi-Lipschitz equivalent only if the knots K_1 and K_2 are isotopic.*

Proof Let $F_K \subset S^3$ be a characteristic band of the knot K, and let $\widetilde{Y}_K$ and $\widetilde{X}_K$ be the corresponding characteristic cones (see Definition 2.16). Let $S_K \subset F_K$ be a slice (see Definition 2.17). Let $\varphi_K : S_K \to Z_1$, where Z_1 is the set Z_t in Definition 2.10 with $t = 1$, be a semialgebraic bi-Lipschitz homeomorphism $(\rho, l) \mapsto ((\rho - \rho_0)/\epsilon, l)$. Let $M_K = \{t\sigma : t \geq 0,\ \sigma \in S_K\} \subset \mathbb{R}^4$ be the cone over S_K. We define a mapping $\Phi_K : M_K \to Z \subset \mathbb{R}^3$ as the corresponding mapping of the cones:

$$\Phi_K(t\sigma) = t\varphi_K(\sigma) \text{ for } \sigma \in S_K. \tag{1}$$

Note that Φ_K is a bi-Lipschitz homeomorphism. Let $W \subset \mathbb{R}^3$ be the set in Definition 2.10, and let

$$V_K = \Phi_K^{-1}(W), \quad Y_K = \left(\widetilde{Y}_K \setminus M_K\right) \cup V_K, \quad X_K = \partial Y_K. \tag{2}$$

Then X_K is a one-bridge surface germ, part of the surface germ $\widetilde{X}_K$ inside M_K being replaced by a β-bridge B_β (see Fig. 4). Let us show that X_K satisfies the conditions of Theorem 3.1.

1) The link at the origin of X_K is a trivial knot, because it bounds the closure of $F_K \setminus S_K$ homeomorphic to a disk.

2) Let K_1 and K_2 be any two knots. Let $\Psi : \widetilde{Y}_{K_1} \to \widetilde{Y}_{K_2}$ be a semialgebraic bi-Lipschitz map sending each point $(\rho, l, t) \in \widetilde{Y}_{K_1}$ to the point $(\rho, l, t) \in \widetilde{Y}_{K_2}$. By definition $\Psi(M_{K_1}) = M_{K_2}$. By the definition of the maps Φ_{K_1} and Φ_{K_2} (see (1)) we have $\Psi(Y_{K_1}) = Y_{K_2}$ and $\Psi(X_{K_1}) = X_{K_2}$.

3) Note that, for any knot K, the link of the tangent cone C_0X_K of the set X_K is the union of two knots isotopic to K, with a single common point. Thus if K_1 and K_2 are not isotopic, then the tangent cones $C_0X_{K_1}$ and $C_0X_{K_2}$ are not ambient topologically equivalent. This contradicts Sampaio's theorem [12] (see also Theorem 1.1) which implies that tangent cones of ambient Lipschitz equivalent semialgebraic sets are ambient Lipschitz equivalent. In our case, the links of the tangent cones are not even ambient topologically equivalent.

This concludes the proof of Theorem 3.1. □

Definition 3.2 A surface germ X_K obtained by the above construction is called a *band-bridge surface germ* corresponding to the knot K and a β-bridge (or a (β_1, β_2)-bridge as in the proof of Theorem 3.5 below).

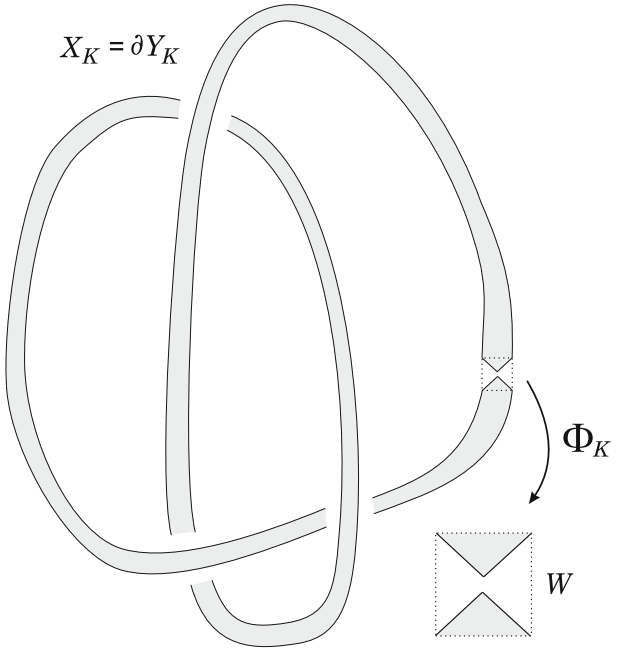


Fig. 4 The links of the sets $X_K = \partial Y_K$ and W in the proof of Theorem 3.1

Definition 3.3 Consider the (β_1, β_2)-bridge $B_{\beta_1\beta_2} = \bigcup_{t\geq 0} \bar{J}_t$ (see Definition 2.11). The set $\bar{J}_t$ has two components $\bar{J}_t^+$ and $\bar{J}_t^-$, each of them consisting of three line segments connecting the points $p_1(t)$, $p_2(t)$, $p_3(t)$, $p_4(t)$ and $p_1'(t)$, $p_2'(t)$, $p_3'(t)$, $p_4'(t)$, respectively (see Fig. 2). Let $\hat{J}_t$ be the set obtained by replacing the line segments $[p_2(t), p_3(t)]$ and $[p_2'(t), p_3'(t)]$ in $\bar{J}_t$ with the line segments $[p_2(t), p_2'(t)]$ and $[p_3(t), p_3'(t)]$ (see Fig. 5a). Let $S_{\beta_1\beta_2} = \bigcup_{t\geq 0} \hat{J}_t$. Let X be a one-bridge surface germ (see Definition 2.12). Replacing $\widetilde{B} = \Theta(B_{\beta_1\beta_2}) \subset X$ with $\widetilde{S} = \Theta(S_{\beta_1\beta_2})$, we obtain a new surface germ $S(X)$. This defines the *saddle move* operation applied to X.

Lemma 3.4 *Let X_1 and X_2 be semialgebraic ambient bi-Lipschitz equivalent one-bridge surface germs. Then the surface germs $S(X_1)$ and $S(X_2)$, obtained by the saddle move applied to X_1 and X_2, are ambient topologically equivalent, the links at the origin $L_{S(X_1)}$ and $L_{S(X_2)}$ are isotopic as topological links in S^3, and the diagrams of the links $L_{S(X_1)}$ and $L_{S(X_2)}$ are equivalent.*

Proof Let $Z \subset \mathbb{R}^3$ be as in Definitions 2.10 and 2.11. Let $\Theta_1 \colon Z \to \mathbb{R}^4$ and $\Theta_2 \colon Z \to \mathbb{R}^4$ be bi-Lipschitz embeddings such that $\widetilde{B}_1 = \Theta_1(B_{\beta_1\beta_2}) \subset X_1$ and $\widetilde{B}_2 = \Theta_2(B_{\beta_1\beta_2}) \subset X_2$. Since X_1 and X_2 are one-bridge surfaces, we can suppose that $X_1 \cup \Theta_1(Z)$ and $X_2 \cup \Theta_2(Z)$ are straight cones over their links. Let $H \colon \mathbb{R}^4 \to \mathbb{R}^4$ be a bi-Lipschitz homeomorphism isotopic to identity such that $H(X_1) = X_2$. By Valette's Theorem [13] (see also Theorem 1.2) we may suppose that H preserves the

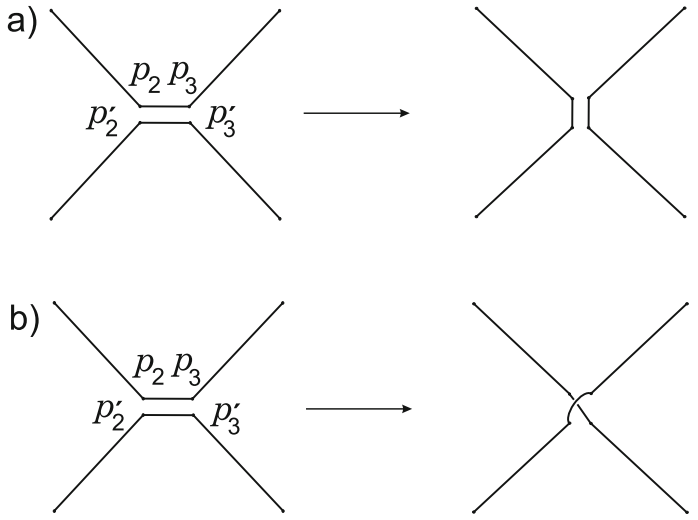


Fig. 5 **a** The saddle move. **b** The crossing move

distance to the origin, and that the maps Θ_1 and Θ_2 send each section Z_t of Z to the sphere S_t of radius t centered at the origin.

Let $\widetilde{P}_1 = \Theta_1(P)$ and $\widetilde{P}_2 = \Theta_2(P)$, where $P = \bigcup_{t\geq 0} P_t \subset B_{\beta_1\beta_2}$ (see Definition 2.11) and let $\widetilde{P}_1(t) = \Theta_1(P_t) = \widetilde{P}_1 \cap S_t$ and $\widetilde{P}_2(t) = \Theta_2(P_t) = \widetilde{P}_2 \cap S_t$. Since the tangent cone C_0P of P is the positive z-axis, the tangent cones $C_0\widetilde{P}_1$ and $C_0\widetilde{P}_2$ of $\widetilde{P}_1$ and $\widetilde{P}_2$ are rays in $\mathbb{R}^4$. For a small positive ϵ, let $N_t \subset S_t$ be a ball of radius ϵt centered at the point $C_0\widetilde{P}_2 \cap S_t$, and let $N = \bigcup_{t\geq 0} N_t$ be a conical ϵ-neighbourhood of $C_0\widetilde{P}_2$. Note that $\widetilde{P}_2 \subset N \cap X_2 \subset \widetilde{B}_2$ for small $\epsilon > 0$.

Let $p_2(t),\ p_2'(t),\ p_3(t),\ p_3'(t)$ be the boundary points of P_t (see Definition 2.11). Let $q_2(t) = \Theta_1(p_2(t))$, $q_2'(t) = \Theta_1(p_2'(t))$, $q_3(t) = \Theta_1(p_3(t))$, $q_3'(t) = \Theta_1(p_3'(t))$ be the boundary points of $\widetilde{P}_1(t)$, and let $v_2(t) = \Theta_2(p_2(t))$, $v_2'(t) = \Theta_2(p_2'(t))$, $v_3(t) = \Theta_2(p_3(t))$, $v_3'(t) = \Theta_2(p_3'(t))$ be the boundary points of $\widetilde{P}_2(t)$. Then $\tilde{q}_2(t) = H(q_2(t))$, $\tilde{q}_2'(t) = H(q_2'(t))$, $\tilde{q}_3(t) = H(q_3(t))$, $\tilde{q}_3'(t) = H(q_3'(t))$ are the boundary points of $H(\widetilde{P}_1(t))$.

The saddle move operation applied to X_1 replaces $\widetilde{P}_1$ with $Q = \bigcup_{t\geq 0} Q_t$, where $Q_t = Q_2(t) \cup Q_3(t)$, $Q_2(t) = \Theta_1([p_2(t), p_2'(t)])$, $Q_3(t) = \Theta_1([p_3(t), p_3'(t)])$. The saddle move operation applied to X_2 replaces $\widetilde{P}_2$ with $V = \bigcup_{t\geq 0} V_t$, where $V_t = V_2(t) \cup V_3(t)$, $V_2(t) = \Theta_2([p_2(t), p_2'(t)])$, $V_3(t) = \Theta_2([p_3(t), p_3'(t)])$. Let $\widetilde{Q} = H(Q)$, $\widetilde{Q}_2 = H(Q_2)$, $\widetilde{Q}_3 = H(Q_3)$. Note that the boundary points $\tilde{q}_2(t)$, $\tilde{q}_2'(t)$, $\tilde{q}_3(t)$, $\tilde{q}_3'(t)$ of $\widetilde{Q}_t$ are the same as the boundary points of $H(\widetilde{P}_1(t))$, and the boundary points $v_2(t),\ v_2'(t), v_3(t),\ v_3'(t)$ of V_t are the same as the boundary points of $\widetilde{P}_2(t)$. In particular, all these points belong to the bridge $\widetilde{B}_2$ of X_2, and to the ϵt-ball N_t (see Fig. 6).

Note that $tord(\widetilde{Q}_2, \widetilde{Q}_3) = tord(V_2, V_3) = \beta_1$ and $tord(\widetilde{Q}_2, V_2) = tord(\widetilde{Q}_3, V_3) = \beta_2$. Consider $diam(V_2(t))$, $diam(V_3(t))$, $diam(\widetilde{Q}_2(t))$, $diam(\widetilde{Q}_3(t))$ as functions of t. Note that the order of all of these functions at the origin is β_2. Let N_2 be the family of

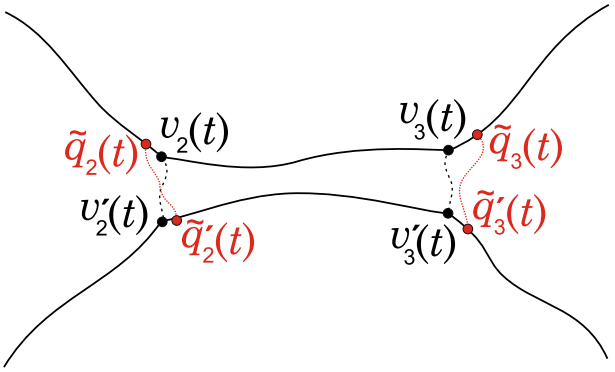


Fig. 6 The images by the map H in the proof of Lemma 3.4

balls $N_{2,t}$ on S_t centered at $v_2(t)$ with the radius $t^{\tilde{\beta}}$ for $\tilde{\beta} \in (\beta_1, \beta_2)$, and let N_3 be the family of balls $N_{3,t}$ on S_t centered at $v_3(t)$ with the radius $t^{\tilde{\beta}}$. Clearly $N_{2,t} \cap N_{3,t} = \emptyset$ and also $\widetilde{Q}_2(t) \subset N_{2,t}$, $V_2(t) \subset N_{2,t}$, $\widetilde{Q}_3(t) \subset N_{3,t}$, $V_2(t) \subset N_{3,t}$.

Since $\widetilde{Q}_2(t)$, $\widetilde{Q}_3(t)$, $V_2(t)$, $V_3(t)$ are homeomorphic to segments, there exists a homeomorphism $\bar{H}_2 \colon N_1 \to N_1$ isotopic to identity, such that :

1. $\bar{H}_2$ maps the sections $z = t$ to the sections $z = t$.
2. $\bar{H}_2$ is identity on the boundary of N_1.
3. $\bar{H}_2(H(\widetilde{Q}_2)) = V_2$.
4. The bridge $\widetilde{B}_2 = H(\widetilde{B}_1)$ is invariant under $\bar{H}_2$.

Similarly, there exists a homeomorphism $\bar{H}_3 \colon N_2 \to N_2$ isotopic to identity, such that :

1. $\bar{H}_3$ maps the sections $z = t$ to the sections $z = t$.
2. $\bar{H}_3$ is identity on the boundary of N_2.
3. $\bar{H}_3(H(\widetilde{Q}_3)) = V_3$.
4. The bridge $\widetilde{B}_2 = H(\widetilde{B}_1)$ is invariant under $\bar{H}_3$.

Then we define a homeomorphism $H' \colon R^4 \to R^4$ to be equal to H outside $H^{-1}(N_1 \cup N_2)$, to $\bar{H}_2 \circ H$ on $H^{-1}(N_1)$ and to $\bar{H}_3 \circ H$ on $H^{-1}(N_2)$, thus $H'(S(X_1)) = S(X_2)$. This proves that $S(X_1)$ and $S(X_2)$ are ambient topologically equivalent, and the links at the origin $L_{S(X_1)}$ and $L_{S(X_2)}$ are isotopic as topological links. □

Theorem 3.5 *For any knot $K \subset S^3$ and all integers $i \geq 0$, there exist semialgebraic surface germs $(X'_{K,i}, 0)$ in $\mathbb{R}^4$ such that:*

1. *The tangent cones at the origin of all $X'_{K,i}$ are topologically equivalent to the cone over two knots isotopic to K with a single common point.*
2. *All $X'_{K,i}$ are outer bi-Lipschitz equivalent.*
3. *$X'_{K,i}$ and $X'_{K,j}$ are semialgebraic ambient bi-Lipschitz equivalent only when $i = j$.*

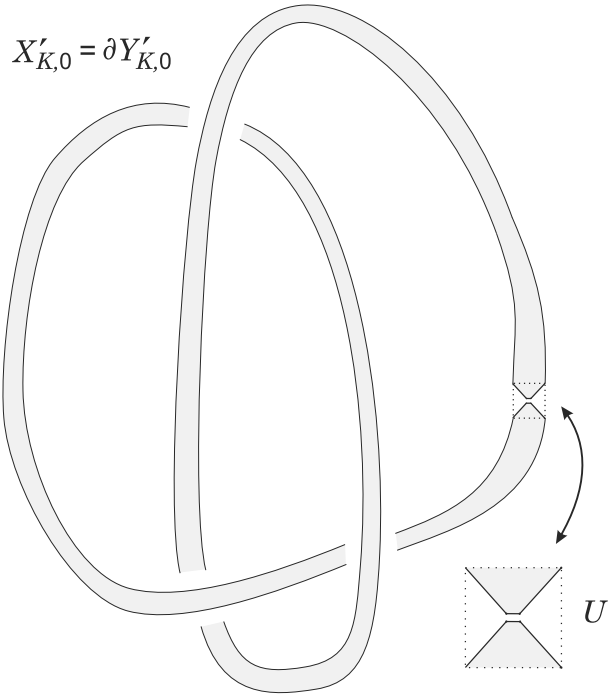


Fig. 7 The links of the surface $X'_{K,0} = \partial Y'_{K,0}$ and U in the proof of Theorem 3.5

Proof Consider a characteristic band $F_K \subset S^3$, a slice $S_K \subset F_K$, and characteristic cones $\widetilde{Y}_K$ and $\widetilde{X}_K$ (see Definitions 2.16 and 2.17). We construct a band-bridge surface germ with a (β_1, β_2)-bridge corresponding to K as follows. Let $M_K = \{t\sigma : t \geq 0,\ \sigma \in S_K\} \subset \mathbb{R}^4$ be the cone over S_K (as in Theorem 3.1). Let $\Phi_K \colon M_K \to Z \subset \mathbb{R}^3$ be the map defined in (1):

$$\Phi_K(t\sigma) = t\varphi_K(\sigma) \text{ for } \sigma \in S_K. \tag{3}$$

Note that Φ_K is a bi-Lipschitz homeomorphism. For $1 < \beta_1 \leq \beta_2$, let $U \subset \mathbb{R}^3$ be the set in Definition 2.11. We define

$$V'_{K,0} = \Phi_K^{-1}(U), \quad Y'_{K,0} = \left(\widetilde{Y}_K, \setminus M_K\right) \cup V'_{K,0} \quad X'_{K,0} = \partial Y'_{K,0}. \tag{4}$$

The set $Y'_{K,0}$ is obtained by replacing the set W (see Definition 2.10) with the set U in construction of the set X_K in the proof of Theorem 3.1. Let $X'_{K,0} = \partial Y'_{K,0}$ be its boundary (see Fig. 7). This construction replaces a β-bridge in Theorem 3.1 by a (β_1, β_2)-bridge. In particular, the one-bridge surface germ $(X'_{K,0}, 0)$ satisfies conditions of Theorem 3.1.

Let now $F'_{K,i}$ be the set obtained by removing the slice S_K from F_K, making i complete twists and adding S_K back (see Fig. 8a–d). Let $Y'_{K,i}$ be the set obtained from the cone over $F'_{K,i}$ by replacing the set M_K (the cone over S_K) with the set U (see

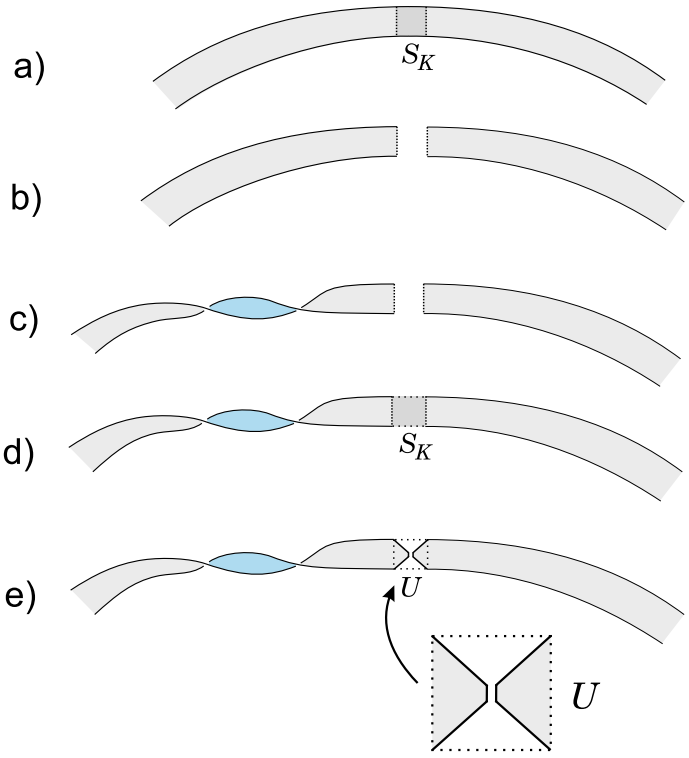


Fig. 8 Cut and twist in the proof of Theorem 3.5

Fig. 8e) and let $X'_{K,i} = \partial Y'_{K,i}$ be its boundary. The same arguments as in the proof of Theorem 3.1 show that the link of $X'_{K,i}$ is a trivial knot and the tangent cone of $X'_{K,i}$ is a cone over the union of two knots isotopic to K, pinched at one point.

We are going to prove that $X'_{K,i}$ and $X'_{K,j}$ are not semialgebraic ambient bi-Lipschitz equivalent if $i \neq j$. The result of the saddle move applied to each of these surface germs is a surface germ such that its tangent link is the union of two copies of the knot K, with the linking number of the two copies being twice the number of complete twists. Thus the links $S(X'_{K,j})$ and $S(X'_{K,i})$ are not isotopic when $i \neq j$. It follows from Lemma 3.4 that surface germs $X'_{K,i}$ and $X'_{K,j}$ are not ambient semialgebraic bi-Lipschitz equivalent when $i \neq j$.

Note that the topology of the tangent link of $X'_{K,i}$ does not depend on i. The tangent link is formed by two copies of K pinched at one point. □

Remark 3.6 Let $X'_{K,i}$ be the surface germ constructed in the proof of Theorem 3.5. Then the link at the origin of the surface germ $S(X'_{K,i})$, obtained from $X'_{K,i}$ by the saddle move, is a subset of $F'_{K,i}$ isotopic to $\partial F'_{K,i}$.

Proposition 3.7 *Let $X'_{K,i}$ be a surface germ constructed in Theorem 3.5, and let $S(X'_{K,i})$ be the surface germ obtained by a saddle move applied to $X'_{K,i}$. If K is a trivial knot, then the link at the origin of $S(X'_{K,i})$ is a torus link.*

Proof For a small $\epsilon > 0$, the boundary of the ϵ-neighbourhood of K is an unknotted two-dimensional torus $T_K \subset S^3$. One can define coordinates (ϕ, ψ) on T_K, where $\phi \in K$, $\psi \in S^1$, so that the curves $\tilde{K} = \{\phi \in K,\ \psi = 0\}$ and $\tilde{K}' = \{\phi \in K,\ \psi = \pi\}$ have the linking number zero. Then $F_K = \{\phi \in K, 0 \leq \psi \leq \pi\} \subset T_K$ is a characteristic band of the knot K (see Definition 2.16) bounded by the curves $\tilde{K}$ and $\tilde{K}'$. If $X'_{K,0}$ is the surface germ constructed in Theorem 3.5, then the link at the origin of $S(X'_{K,0})$, isotopic to the union of $\tilde{K}$ and $\tilde{K}'$, is a trivial torus link.

The surgery for constructing a surface germ $X'_{K,i}$ in Theorem 3.5 (see Fig. 8) corresponds to the choice of a coordinate system (ϕ, ψ_i) on T_K such that the band $F_{K,i} = \{\phi \in K,\ 0 \leq \psi_i \leq \pi\} \subset T_K$ is bounded by the curves $\tilde{K}_i = \{\phi \in K,\ \psi_i = 0\}$ and $\tilde{K}'_i = \{\phi \in K,\ \psi_i = \pi\}$ with the linking number $2i$. Since the link at the origin of $S(X'_{K,i})$ is isotopic to the union of $\tilde{K}_i$ and $\tilde{K}'_i$ (see Remark 3.6) it is a torus link. □

Proposition 3.8 *Let X_1 and X_2 be two one-bridge surface germs. If the germs are ambient bi-Lipschitz equivalent, then the links of the origin $L_{S(X_1)}$ and $L_{S(X_2)}$ are isotopic.*

Remark 3.9 Saddle move on the level of knot diagrams is described as follows:  is replaced by  .

Here we are going to define the crossing move, that will be useful for further calculations.

Definition 3.10 We proceed in a similar way to the definition of the saddle move. Consider the subset B of a one-bridge surface X outer bi-Lipschitz equivalent to a (β_1, β_2)-bridge $B_{\beta_1\beta_2} = \bigcup_{t\geq 0} \bar{J}_t$ (see Definition 2.11). The set $\bar{J}_t$ has two components $\bar{J}_t^+$ and $\bar{J}_t^-$, consisting of three line segments connecting the points $p_1(t)$, $p_2(t)$, $p_3(t)$, $p_4(t)$ and $p'_1(t)$, $p'_2(t)$, $p'_3(t)$, $p'_4(t)$, respectively, in the plane $\{z = t, w = 0\}$ (see Fig. 2). Let us embed this set to $\mathbb{R}^4$ with coordinates (x, y, z, w). Replacing the line segments $[p_2(t), p_3(t)]$ and $[p'_2(t), p'_3(t)]$ with the line segment $[p_2(t), p'_3(t)]$ and a circle arc in the half-space $\{w \geq 0\}$ with the ends at $p'_2(t)$ and $p_3(t)$, orthogonal to the plane $\{w = 0\}$ (see Fig. 5b), we replace the set $\bar{J}_t$ with the set $\hat{J}_t$. Let $\widehat{B}_{\beta_1\beta_2} = \bigcup_{t\geq 0} \hat{J}_t$. Note that the surface germs $B_{\beta_1\beta_2}$ and $\widehat{B}_{\beta_1\beta_2}$ have the same boundary arcs. Replacing the subset B of X with the subset $\widehat{B}$ outer bi-Lipschitz equivalent to $\widehat{B}_{\beta_1\beta_2}$, so that B and $\widehat{B}$ have the same boundary arcs, we get a new surface germ $C(X)$. This defines a *crossing move* operation applied to X.

Remark 3.11 Crossing move on the level of knot diagrams is described as follows:  is replaced by  .

Remark 3.12 One can show that, for the fixed orientation on L_X, the isotopy class of the resulting knot or link is an ambient bi-Lipschitz invariant. However, in what follows we do not need this result.

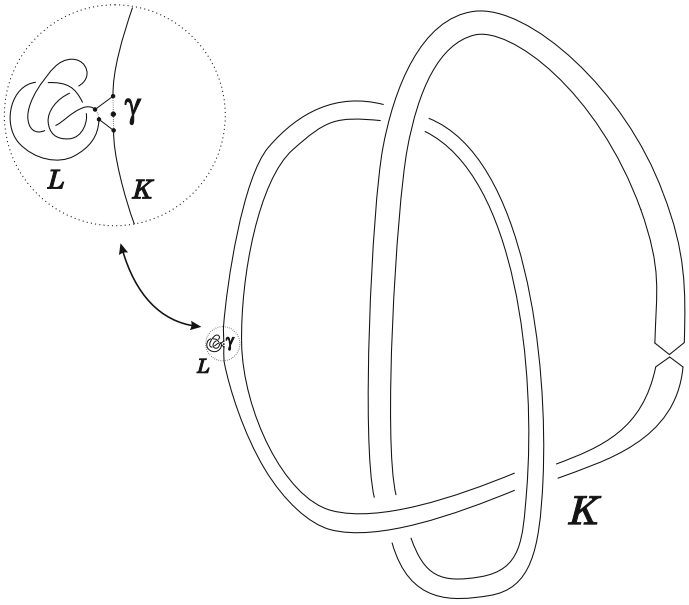


Fig. 9 Construction of X_{KL}

The next statement is a modification of the Universality Theorem.

Theorem 3.13 *For any two knots K and L, there exists a germ of a semialgebraic one-bridge surface germ X_{KL} such that:*

1. *The link of X_{KL} at the origin is isotopic to L.*
2. *For a fixed knot K all surface germs X_{KL} have isotopic tangent links. In particular, surface germs X_{K_1L} and X_{K_2L} are ambient semialgebraic bi-Lipschitz equivalent only if the knots K_1 and K_2 are isotopic.*

Proof We use the construction from the proof of Theorem 3.1. Consider a characteristic band F_K, the characteristic cones $\widetilde{Y}_K$ and $\widetilde{X}_K$ (see Definition 2.16). Consider the surface germ X_K defined in (2) for the knot K. Let $\gamma \subset X_K$ be an arc not tangent to the set V_K defined in (2) (i.e., $tord(\gamma', \gamma) = 1$ for any $\gamma' \subset V_K$). Let $V(\gamma)$ be a small conical neighbourhood of γ in $\mathbb{R}^4$, such that $V(\gamma) \cap X_K$ is a Hölder triangle. Let us embed the straight cone Z_L over L inside $V(\gamma)$ so that its image $\tilde{Z}_L$ does not intersect X_K, and its is ambient topologically equivalent to L. Let us choose two arcs γ_1 and γ_2 in $X_K \cap V(\gamma)$, and two arcs γ_1' and γ_2' in $\tilde{Z} \cap V(\gamma)$, satisfying the following conditions:

a. $tord(\gamma_1, \gamma_2) = tord(\gamma_1', \gamma_2') = 1$.

b. Replacing the union of the Hölder triangles $T(\gamma_1, \gamma_2) \subset X_K$ and $T(\gamma_1', \gamma_2') \subset \tilde{Z}$ with the union of Hölder triangles $T(\gamma_1, \gamma_1') \subset V(\gamma)$ and $T(\gamma_2, \gamma_2') \subset V(\gamma)$, as shown in Fig. 9, we obtain a semialgebraic set X_{KL} such that $X_{KL} \cap V(\gamma)$ is conical and its link is isotopic to the connected sum of K and L. Note that construction of X_{KL} is similar to the saddle move construction in Definition 3.3.

Let us check that the surface germ X_{KL} satisfies conditions of Theorem 3.13.

1. Since the link of X_K is unknotted, the connected sum is isotopic to L.

The proof of the fact that, for a fixed knot L, all surface germs X_{KL} are outer bi-Lipschitz equivalent is the same as the proof that all surface germs X_K are outer bi-Lipschitz equivalent in the proof of Theorem 3.1.

2. Since X_{KL} is a one-bridge surface germ, its tangent link is the union of two knots with a single common point. One of these two knots is isotopic to K, and the other one is isotopic to the connected sum of K and L. Since the first knot is isotopic to K, condition 2 is satisfied. □

The next result is another modification of the Universality Theorem. In contrast to the previous results, we consider surface germs with the metric structure more complicated than one-bridge.

Theorem 3.14 *For any two knots K and L, and for any two rational numbers α and β such that $1 \leq \alpha \leq \beta$, there exists a semialgebraic surface germ $X^{\alpha\beta}_{KL}$ such that:*

1. *For any knots K and L, the link at the origin of $X^{\alpha\beta}_{KL}$ is isotopic to L.*
2. *For any knots K and L, the tangent link of $X^{\alpha\beta}_{KL}$ is isotopic to K.*
3. *For fixed α and β, all surface germs $X^{\alpha\beta}_{KL}$ are outer bi-Lipschitz equivalent.*

Proof Let $F_K \subset S^3$ be the characteristic band of a knot K (see Definition 2.16). It is diffeomorphic to $S^1 \times [-1, 1]$, and its boundary has two components $\widetilde{K}$ and $\widetilde{K}'$ isotopic to K. Let (ρ, l), where $\rho \in S^1$ and $l \in [-1, 1]$, be coordinates in F_K. Let $\widetilde{Y}_K$ and $\widetilde{X}_K$ be the corresponding characteristic cones (see Definition 2.16). Then (ρ, l, t) are coordinates in $\widetilde{Y}_K$, where t is the distance to the origin. Let $\widetilde{Y}^{\alpha}_K$ be a subset of $\widetilde{Y}_K$ defined as follows:

$\widetilde{Y}^{\alpha}_K = \{(\rho, l, t) : |l| \leq t^{\alpha}\}$. The set $\widetilde{Y}^{\alpha}_K$ is called *α-contraction* of $\widetilde{Y}_K$. Notice that the tangent link of $\widetilde{Y}^{\alpha}_K$ is a knot isotopic to K.

Let $S_K = \{(\rho, l) : |\rho - \rho_0| \leq \epsilon\}$ be a slice of F_K (see Definition 2.17) for a small $\epsilon > 0$, and let M_K be the cone over S_K. Let $M^{\alpha}_K = \{(\rho, l, t) : \rho_0 - \epsilon \leq \rho \leq \rho_0 + \epsilon,\ |l| \leq t^{\alpha}\}$ be α-contraction of M_K. Replacing M^{α}_K by the (α, β)-wedge $W^{\alpha\beta}$ (see Definition 2.13 and Fig. 1b) as in the proof of Theorem 3.1, we get the set $Y^{\alpha\beta}_K$. Let $X^{\alpha\beta}_K$ be the boundary of $Y^{\alpha\beta}_K$.

Let $\gamma \subset X^{\alpha\beta}_K$ be an arc far from the set $W^{\alpha\beta}$, i.e., $tord(\gamma, \gamma') = 1$ for any arc $\gamma' \subset W^{\alpha\beta}$. Let Z_L be the straight cone over L. Let $V_\beta(\gamma) \subset \mathbb{R}^4$ be a β-horn like neighbourhood of γ. Let $Z^{\beta}_{L,\gamma} = \Psi_\beta(Z_L) \subset V^{\beta}(\gamma)$ be a β-hornification of Z_L to γ (see Definition 2.18 and Fig. 3). Let us choose two arcs γ_1 and γ_2 in $X_K \cap V_\beta(\gamma)$, and two arcs γ'_1 and γ'_2 in $Z^{\beta}_{L,\gamma} \cap V_\beta(\gamma)$ satisfying the following conditions:

a. $tord(\gamma_1, \gamma_2) = \beta,\ tord(\gamma'_1, \gamma'_2) = \beta$.

b. If we remove from X_K the Hölder triangle bounded by the arcs γ_1 and γ_2, remove from $Z^{\beta}_{L,\gamma}$ the Hölder triangle bounded by the arcs γ'_1 and γ'_2, and add to the set $X_K \cup \tilde{Z}$ the Hölder triangle obtained as the union of line segments connecting $\gamma_1(t)$ and $\gamma'_1(t)$, and the Hölder triangle obtained as the union of line segments connecting $\gamma_1(t)$ and $\gamma'_2(t)$, we obtain a semialgebraic set $X^{\alpha\beta}_{KL}$ with the link isotopic to the connected sum

of the links of X_K and $Z^{\beta}_{L,\gamma}$ (see Fig. 9). Note that construction of $X^{\alpha\beta}_{KL}$ is similar to construction of X_{KL} in the proof of Theorem 3.13 and to the saddle move construction in Definition 3.3.

Let us check that the surface germ $X^{\alpha\beta}_{KL}$ satisfies conditions of Theorem 3.14.

1. Since $X^{\alpha\beta}_{K}$ has a trivial link, the connected sum is isotopic to L.
2. Since Z_L is a subset of a β-horn neighbourhood of γ, it corresponds to a single point in the tangent link. Thus the tangent link of $X^{\alpha\beta}_{KL}$ is the same as the tangent link of $X^{\alpha\beta}_{K}$, which is isotopic to K.
3. The proof of the fact that the surface germs $X^{\alpha\beta}_{KL}$ are outer bi-Lipschitz equivalent for a fixed L is the same as the proof that all surfaces X_{KL} are outer bi-Lipschitz equivalent in the proof of Theorem 3.13.

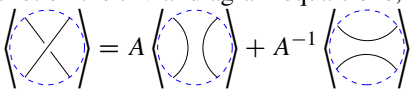
□

4 Knot invariants

In this section we make slight changes of notations. In the previous sections we used the notation L_X for the link at the origin of a surface germ X. Here we are going to use the notation K_X if the link at the origin of X is a knot, and L_X if it is a topological link with more than one component.

Let us first recall the definition of the Jones polynomial $J(L)$ of a link L via Kauffman bracket polynomial $\langle D_L\rangle$, where D_L is a link diagram of L. Kauffman bracket polynomial [8] is a polynomial in a variable A which is uniquely determined by the following properties:

(1) Kauffman bracket on the trivial diagram equals one, i.e., $\langle O\rangle = 1$

(2) Skein relation  $= A$ $+ A^{-1}$

(3) For any link diagram $D_{L'}$ we have $\langle O \cup D_{L'}\rangle = (-A^2 - A^{-2})\langle D_{L'}\rangle$

The Jones polynomial of an oriented link L can be defined as

$$J(L) = (-A^3)^{-\omega(D_L)}\langle D_L\rangle,$$

after the substitution $A = t^{-\frac{1}{4}}$. Here $\omega(D_L)$ is the writhe number of the diagram D_L, i.e., the number of positive crossings minus the number of negative crossings in D_L.

Proposition 4.1 *Let X be a one bridge surface such that the link of X at the origin is a knot K_X. Let $K_{C(X)}$ be the knot, obtained from $K(X)$ by the crossing move. Let Y be a one-bridge germ such that the link at the origin of Y is the same knot $K_Y = K_X$ as the link at the origin of X. Let $S(Y)$ be a germ obtained from Y by the saddle move. Suppose that Y is such that the link at the origin of the surface $S(Y)$ is a 2-component link $L_{S(Y)}$. If the Jones polynomial $J(K_{C(X)})$ of the knot $K_{C(X)}$ satisfies*

$$J(K_{C(X)}) \neq -t^{\frac{1}{2}}J(L_{S(Y)}) + (-1)^{\omega(D_{K'})-\omega(D_K)}\, t^{\frac{3(\omega(D_{K'})-\omega(D_K))+1}{4}} J(K),$$

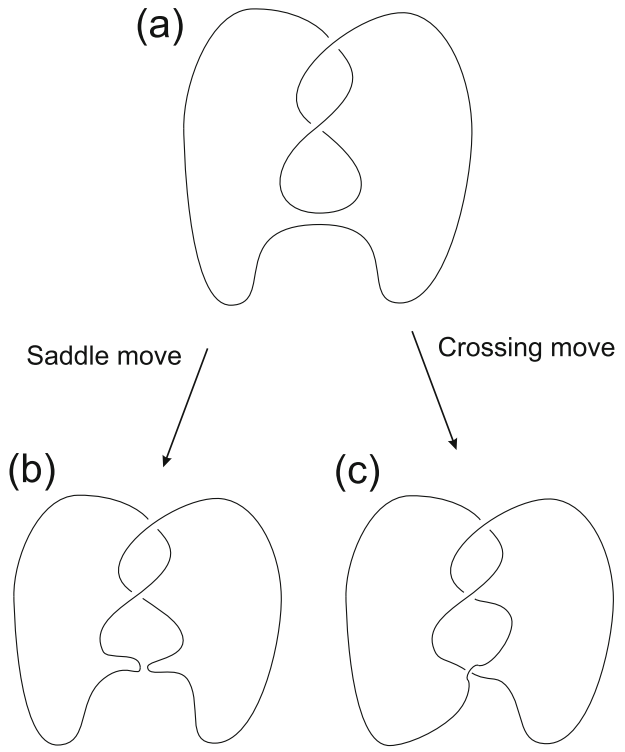


Fig. 10 The Saddle move and the Crossing move

where D_K is a diagram of a knot K determined by X, and $D_{K'}$ is a diagram (determined by the crossing move) of a knot $K_{C(X)}$ then X and Y are not semialgebraic ambient bi-Lipschitz equivalent.

Proof Let D_K be a diagram of a knot $K(X)$ determined by X and let $D_{K'}$ be a diagram of a knot $K_{C(X)}$. Let us orient D_K in an arbitrary way. We orient $D_{K'}$ so that the intersection, corresponding to the crossing move (see Fig. 10) on the diagram is positive, i.e., it looks like  . Let $S(X)$ be a germ of a surface obtained from X by a saddle move. Let D_L be the corresponding diagram of the characteristic link $L_{S(X)}$. We orient D_L such that the part, corresponding to the saddle move (see Fig. 10) looks like 

. Before the substitution $A = t^{-\frac{1}{4}}$ we have

$$\langle D_{K'}\rangle = (-A^3)^{\omega(D_{K'})} J(K_{C(X)}) \qquad \langle D_L\rangle = (-A^3)^{\omega(D_L)} J(L_{S(X)}).$$

Now it follows from the condition (2) of the Kauffman bracket that

$$(-A^3)^{\omega(D_{K'})} J(K_{C(X)}) = A(-A^3)^{\omega(D_L)} J(L_{S(X)}) + A^{-1}(-A^3)^{\omega(D_K)} J(K).$$

Using the fact that $\omega(D_{K'}) = \omega(D_L) + 1$ and after the substitution $A = t^{-\frac{1}{4}}$ we get

$$J(K_{C(X)}) = -t^{\frac{1}{2}} J(L_{S(X)}) + (-1)^{\omega(D_{K'})-\omega(D_K)} t^{\frac{3(\omega(D_{K'})-\omega(D_K))+1}{4}} J(K). \quad (5)$$

Recall that Proposition 3.8 implies that if the link $L_{S(X)}$ is not isotopic to the link $L_{S(Y)}$, then X and Y are not semi-algebraic ambient bi-Lipschitz equivalent. Hence if $J(L_{S(X)}) \neq J(L_{S(Y)})$, then X and Y are not semialgebraic ambient bi-Lipschitz equivalent. Now equality (5) yields the proof of the proposition. □

Corollary 4.2 *If K is a trivial knot and $L_{S(Y)}$ is $(2, 2m)$-torus link $L(2, 2m)$, where m is a non-negative integer, then we get the following closed formula: If the Jones polynomial $J(K_{C(X)})$ of the knot $K_{C(X)}$ satisfies*

$$J(K_{C(X)}) \neq t^m + t^{m+2}\left(\frac{1+t^{2m-1}}{1+t}\right) + (-1)^{\omega(D_{K'})-\omega(D_K)} t^{\frac{3(\omega(D_{K'})-\omega(D_K))+1}{4}}, \quad (6)$$

then X and Y are not semialgebraic ambient bi-Lipschitz equivalent.

Proof Recall that for each n the Jones polynomial of the torus knot $K(2, 2n + 1)$ equals

$$J(K(2, 2n+1)) = t^n \frac{1 - t^3 - t^{2n+2} + t^{2n+3}}{1 - t^2},$$

see e.g. [7]. The skein relation for the Jones polynomial together with the above equality yield

$$J(L(2, 2m)) = -t^{\frac{2m-1}{2}} - t^{\frac{2m+3}{2}}\left(\frac{1+t^{2m-1}}{1+t}\right). \quad (7)$$

Noting that if K is a trivial knot, then its Jones polynomial $J(K) = 1$, and applying equalities (5) and (7) we obtain the proof of the corollary. □

Remark 4.3 The above proposition has two advantages: it has a computational value, and as its immediate corollary we obtain the main result of Birbrair-Gabrielov [3, Theorem 4.1]. Let us illustrate this on the following example. Let X be such that it determines a knot diagram D_K which has no intersections, and after the crossing move the diagram $D_{K'}$ has exactly one positive intersection. It follows that $\omega(D_{K'}) - \omega(D_K) = 1$, and $J(K_{C(X)}) = 1$ since $K_{C(X)}$ is a trivial knot. Let Y be such that it determines a trivial knot diagram presented in Fig. 10a. The diagram of the link $L_{S(Y)}$ is presented in Fig. 10b. Note that it is a $(2, 2)$-torus link (Hopf link). The diagram of the knot $K_{C(Y)}$ is presented in Fig. 10c. Note that it is a trefoil knot. Noting that $m = 1$ the right hand side of equation (6) equals t^3. Hence $J(K_{C(X)}) \neq t^3$ and thus X and Y are not semialgebraic ambient bi-Lipschitz equivalent.

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