

Preference for Solution Methods and Mathematical Performance: A Critical Review

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Citation: Mainali, B. (2021). Preference for Solution Methods and Mathematical Performance: A Critical Review. *International Electronic Journal of Mathematics Education*, 16(3), em0651. <https://doi.org/10.29333/iejme/11089>

ARTICLE INFO

Received: 6 Jul. 2020

Accepted: 9 Jan. 2021

ABSTRACT

Preferences for solution methods have an important implication teaching and learning mathematics and students' mathematical performances. In the domain of learning mathematics, there are two modes of processing mathematical information: verbal logical and visual-pictorial. Learners who process mathematical information using verbal logical and visual -pictorial modes are respectively called verbalizers and visualizers. Based on the verbalizer-visualizer continuum, students can be placed in a continuum with regard to their preference for solution methods and correlation between the two modes of thought. They belong to one of three categories: (a) visualizers (geometric), who have a preference for the use of visual solution methods, which involve graphic representation (i.e., figures, diagrams, and pictures); (b) verbalizers (analytic), who have a preference for the use of nonvisual solution methods, which involve algebraic, numeric, and verbal representation; and (c) harmonics (mixer), who use visual and verbal methods equally. Several research studies have been conducted to examine the relationship between preferences for solution methods and mathematical performances; however, no conclusive findings were reported. Regardless of inconclusive findings, it is important for students to develop preferences for both solution methods: visual and nonvisual. The mathematical instructional strategies need to equally incorporate preferences for both solution methods, utilizing different modes of mathematical representations, in order to enhance learning mathematics.

Keywords: preference, verbal logical, visual pictorial, representation, visual and nonvisual solution methods

INTRODUCTION

Preferences in learning have an important implication in educational theory and practice (Stenberg & Grigorenko, 1997). Preference refers to an individual's preferred and habitual approach to organizing, representing, and processing information, which subsequently affects the way in which an individual perceives and responds to ideas, events, or problems (Riding & Rayner, 1998). Preference is also related to cognitive styles. Researchers have identified various types of cognitive styles, but in the domain of mathematics education, the verbalizers and visualizers continuum is the most widely accepted in the domain of mathematics education (Krutetskii, 1976).

Krutetskii (1976), a Russian scholar, laid a foundation for the verbalizers and visualizers continuum for the learning of mathematics. He identified two modes of thought or ways of processing mathematical information: verbal-logical and visual-pictorial. He contended that everyone is endowed with these two modes of thought. Verbalizers use verbal-logical modes of thought and visualizers employ visual-pictorial modes of thought while attempting to learn mathematical ideas and concepts or do mathematical tasks. Based on the verbalizer-visualizer continuum, students can be placed in a continuum with regard to their preference for thinking and correlation between the two modes of thought. They belong to one of three categories: (a) visualizers (geometric), who have a preference for the use of visual solution methods, which involve graphic representation (i.e., figures, diagrams, and pictures); (b) verbalizers (analytic), who have a preference for the use of nonvisual solution methods, which involve algebraic, numeric, and verbal representation; and (c) harmonics (mixer), who use visual and verbal methods equally.

How students process mathematical information (verbal-logical or visual-pictorial) can affect their preferences for solution methods (Galindo, 1994; Haciomeroglu et al., 2013; Krutetskii, 1976; Lowrie & Kay, 2001; Moses, 1977; Suwarsono, 1982). In-depth knowledge about what kind of solution methods students prefer to use and what difficulties they encounter when solving mathematical tasks can contribute not only to theoretical knowledge but also to the solution of the actual problems in learning mathematics (Gorgorio, 1998). Based on the Krutetskii (1976) framework, several research studies have been conducted which examined the relationships between students' preferences for solution methods and mathematical performance (Battista, 1990; Haciomeroglu, Aspinwall, & Presmeg, 2010; Krutetskii, 1976; Moses, 1977; Suwarsono, 1982). What kind of solution methods an

individual use and how the method is associated with mathematical performance have been of interest to many researchers in the mathematics education field?

Krutetskii (1963) categorized low-achieving students into different categories and investigated the factors behind their poor performance. He suggested that high-level development of analytical thinking does not determine mathematical thinking; however, low development of analytical thinking does result in an incapacity for mathematics. Krutetskii (1976) further contended that there is a correlation between the ability to visualize abstract mathematical relationships and the ability to make sense of spatial geometric concepts. However, these are not the essential components that determine students' mathematical abilities. He further stated that strengths or weaknesses of analytical or visual thinking do not determine the extent of students' mathematical giftedness; however, they determine its type. A student can be mathematically capable with different correlations between verbal-logical and visual-pictorial modes of thinking. In fact, it is the correlation between the two modes of thinking—verbal-logical and visual-pictorial—that determines each student's category (analytical, geometric, and harmonic).

Krutetskii's study (1976) also revealed a correlation between the verbalizers and success in learning algebra and, similarly, between the geometric type and success in learning geometry. However, Krutetskii further contended that the classification of verbalizers and visualizers should not be regarded as a classification of thinking according to the subject relationships (school subject—algebra and geometry). In fact, the analytic cast of minds can be shown in geometry and geometric type can be shown in algebra.

Spatial ability is defined as the ability to generate, retain, retrieve, and transform well-structured visual images (Lohman, 1996). It also refers to skill in representing, transforming, generating, and recalling symbolic, non-linguistic information (Linn & Petersen, 1985). Krutetskii contended that spatial ability does not determine students' geometric performance; he documented many cases in which students who showed good spatial ability were poor in geometry performance. Moreover, he contended that a well-developed spatial ability does not imply that students will use it while attempting mathematical tasks. For example, students may be able to solve a problem by visual methods; however, they may not prefer to solve it using visual methods. Several research studies have been conducted to examine the relationships between the preferences for solution methods and spatial ability; however, they revealed that there was little or no correlation between preferences and spatial ability (Haciomeroglu, Chicken, & Dixon, 2013; Hagarty & Kozhevnikov, 1999; Kozhevnikov, Hagarty, & Mayer, 2002; Lean & Clements, 1981; Moses, 1977; Suwarsono, 1982). Presmeg (1985) also pointed out the same issues: spatial tests may be solved by using analytic solution methods, or students with good spatial ability may not prefer to use visual solution methods. Quoting the work of Wattanawha and Clements, Clements (1984) reported that mathematically gifted students had a strong preference for analytic methods (nonvisual solution methods) on space visualization tests.

VERBAL-VISUAL CONTINUUM AND SOLUTION METHOD

Krutetskii (1976) conducted a comprehensive study on gifted students' cast of mind in connection with mathematical abilities. He identified two modes of processing mathematical information: verbal-logical and visual-pictorial, stating that everybody is endowed with these two components of thinking. In the context of mathematics, students attempt to solve mathematical tasks or learn mathematics with the aid of formulae, logical reasoning, and so forth, without using the visual images in the verbal-logical mode of thought, whereas they process mathematical information based on visual images in the visual-pictorial mode of thought. He further suggested that verbalizers employ the verbal-logical component while visualizers use the visual-pictorial component. Thus, students can be placed in a continuum with regard to their preference for solution methods for solving mathematical problems. Students belong to one of the three categories: (a) analytic (a preference for manipulating words and sentences), (b) geometric (a preference for manipulating images), and (c) harmonic (a preference for using both analytic and geometric methods equally).

Krutetskii (1976) contended that every person has two components of thinking. He also identified two propositions: (1) the two components, the ability to visualize abstract mathematical relationship and the ability to use spatial geometry concepts, are not necessary components in the structure of mathematical ability; and (2) the presence or absence of these two components does not determine the extent of mathematical giftedness, but the components do determine its type. He contended that "A pupil can be mathematically capable with a different correlation between visual-pictorial and the verbal-logical components, but the given correlation determines what type the pupil belongs to" (p. 315).

According to Krutetskii (1976), the level and quality of schoolchildren's mathematics achievements are determined by the level of development of each thinking component and by the interrelation between these two thinking components. Based on the correlation between verbal-logical and visual-pictorial components, different structures of mathematical abilities and casts of mind are formed for successful mathematical performance. In fact, the *levels* of mathematical abilities are largely determined by a verbal-logical component, while the *types* of mathematical giftedness are determined largely by a visual-pictorial component. Moreover, in the case of the visual-pictorial component, it is not only the ability to use the component but the preference for its use that determines the type of mathematical giftedness of an individual. Krutetskii observed, from his analysis of children's thinking processes while they were attempting mathematical problems, that mathematically weak students always had a very weak verbal-logical component, whereas mathematically capable students always had a very strong verbal-logical component. He claimed that the visual-pictorial component merely affects the nature of a student's mathematical ability but not its level, because Krutetskii found some students in his study were very capable in mathematics but had very weak visual-pictorial components. Thus, he associated the preference for solution methods with the visual-pictorial component, while mathematical ability would be associated with the verbal-logical component.

Following the work of Krutetskii, Moses (1977) placed students in a continuum with regard to their preference for solution methods for solving mathematical problems. Students belong to one of the three categories: (a) analytic (a preference for manipulating words and sentences), (b) geometric (a preference for manipulating images), and (c) harmonic (a preference for using both analytic and geometric methods equally). The analytic type operates mathematical concepts and ideas easily with abstract schemes without a need for visual supports for visualizing objects or patterns in problem solving, even when a given mathematical task demands visual schemes. These students always attempt to process mathematical information via a verbal-logical approach. However, the geometric type attempts mathematical tasks with the aid of graphic representations. According to Krutetskii, the geometric type students feel a need to interpret visually an expression of an abstract mathematical relationship, and they always try to use graphic representations even when the problem can be done easily using nonvisual solution methods. Students who belong to the visualizer type process the mathematical information with the help of a visual-pictorial component. The third type is called harmonic. The majority of capable students in Krutetskii's research study belonged to the harmonic group. Students who belong to this group are successful at implementing both visual and nonvisual solution methods while solving mathematical problems. Spatial concepts are well developed in harmonic types. Krutetskii further classified the harmonic into two subtypes: abstract-harmonic and pictorial-harmonic. Both subtypes can depict mathematical relationships equally well by visual pictorial means; however, the abstract-harmonic subtype feels no need to do so and does not strive to use visual images, whereas the pictorial-harmonic subtype does feel a need and often relies on graphic schemes while attempting mathematical tasks.

Following Krutetskii's (1976) work, Suwarsono (1982) also classified students into three groups based on the preference for solution methods. He, however, slightly modified the name of the groups. Suwarsono divided students into verbalizers, visualizers, and mixers. He also called the visual method and nonvisual method of processing mathematical information what Krutetskii called the verbal-logical method (mental method) and visual-pictorial method (visual method). In fact, there are no fundamental differences between Krutetskii and Suwarsono's classification. The analytic and verbalizers are the same. Similarly, geometric and visualizers as well as harmonic and mixers are also the same. Suwarsono called visual and nonvisual methods of solving mathematical tasks what Krutetskii called visual and verbal methods. However, for the purpose of this study, the researcher uses verbal solution method (*verbalizer*), visual solution method (*visualizer*), and *harmonic* method (use both verbal and visual solution methods).

Kozhevnikov, Hegarty, and Mayer (2002) suggested that the verbalizer-visualizer continuum needs to be revised to include two groups of visualizers. They stated that visualizers are not one homogenous group with respect to their spatial abilities. Some of them have a low spatial ability and some of them have a high spatial ability. They called these groups of students as iconic type (low spatial ability) and spatial type (high spatial ability). Kozhevnikov, Kosslyn, and Shepard (2005) even objected to the verbalizer-visualizer dichotomy. They suggested three types of groups: verbalizers, object visualizers, and spatial visualizers. Object visualizers are more accurate and faster in generating static objects, whereas spatial visualizers are good at manipulating dynamic images. Regardless of types of different categories of students, they should be able to incorporate different modes of representation while doing mathematical tasks.

MODE OF REPRESENTATION AND TRANSLATION

Representation is an important component in teaching and learning mathematics. Kaput (1987b) stated that "representation and symbolization are the heart of the content of mathematics and are simultaneously at the heart of cognitions associated with mathematical activity" (p. 22). The role of representation in mathematics is supported by the National Council of Teachers of Mathematics (NCTM, 2000), which includes representation as one of the process standards. In fact, representation acts as a tool for manipulation, communication, and conceptual understanding of mathematical ideas (Zazkis & Liljedahl, 2004). Researchers contend that representation plays an important role and its use is fundamental in teaching and learning mathematics (Arcavi, 2003; Goldin, 1987; Janvier, 1987; Kaput, 1987a; Roubicek, 2006; Zazkis & Liljedahl, 2004).

A representation is a sign or combination of signs, characters, objects, diagrams, or graphs, and it can be an actual physical product or mental process (Goldin, 2001). In fact, it may be a combination of something expressed on paper, existing in the form of physical objects, and a constructed arrangement of ideas in one's mind (Janvier, 1987). Researchers suggested various types of representational systems (Goldin, 2001; Janvier, 1987; Lesh, Post, & Behr, 1987). Gleason and Hallett (1992) proposed the rule of three which consists of three types of representation: (a) symbolic, (b) graphic, and (c) numeric. The rule of three is modified to become the rule of four, which includes four types of representation: (a) graphic, (b) numeric, (c) algebraic, and (d) verbal. The rule of four is one of the most widely used and commonly accepted classifications of representation in mathematics education.

Representation is an important component for two modes of mathematical information processing: verbal logical and visual-pictorial. Based on the continuum of two modes of thought, learners can be placed into three groups: verbalizer, visualizer and harmonic. The preferences for solution methods for both verbalizers and visualizers are connected with the utilization of representations. Verbalizers preferred to use nonvisual solution methods, whereas visualizers tend to utilize visual solution methods. In nonvisual solution method, students use mathematical formulae, rules, axioms, and postulates, while attempting mathematical tasks. In visual solution methods, A solution method, students utilize given diagrams and figures, or draw diagrams and figures, or visualize diagrams and figures in their mind while attempting mathematical tasks. The diagrams and figures play a dominant role while attempting mathematical tasks. The fact is that visualizers have preference for using graphic representations while verbalizers have preference for employing algebraic, numeric, and verbal representations. Thus, it is apparent that representation has fundamental role in preferences for solutions methods.

In learning mathematics, different modes of representations are equally used and have important roles in preferences for solution methods and understanding mathematical concepts. For example, students constantly change the employing one mode of representation to another while solving mathematics problems because they have different preferences for solution methods. The fact is that using different types of representation often illuminates different aspects of complex mathematical ideas or relationships (NCTM, 2000). Thus, it is important to develop skills in students where they can translate one mode of representation to another based on the nature and situation of mathematics tasks. Lesh, Post and Behr (1987) state that translations (dis)abilities are significant factors influencing and problem-solving performance, and that fortifying and remediating these abilities facilitates the acquisition and use of elementary mathematical ideas. Therefore, translation among the representations and transferring within them is an important process (Lesh, Post, & Behr, 1987) for effective teaching and learning of mathematics.

Janvier (1987) states that translation ability refers to psychological involvement going from one mode of representation to another mode of representation. Most researchers agree that translation is an important process of successful use of representation (Dufour, Bednarz, & Belanger, 1987; Janvier, 1987; Lesh, Post, & Behr, 1987a, 1987b) because translation of one mode of representation to another will provide flexibility in solving mathematics problems. Thus, one of the important goals of teaching mathematics is to teach students to translate one mode of representation to another without falling into contradictions (Hitt, 1998). In fact, the instructional strategies should include translation of all modes of representation because each representation has its own characteristics and poses different challenges for students doing mathematics problems (Gegatsis & Shiakalli, 2004). While students solve mathematical problems, primarily they utilize visual and nonvisual solution methods depending on their preferences for solution methods. If students have a specific preference for solution methods, then they might need to translate a mathematical task from one mode to the other mode of representation.

Consider the following verbal descriptive representation of a mathematics problem:

From a ship on the sea at night, the captain can see three lighthouses and can measure the angles between them. If the captain knows the positions of the light houses from a map, can the captain determine the position of the ship? (NCTM, 2000, p. 69).

If a student has preference for nonvisual solution methods, then he/she simply attempt to solve the problem without translating the problem into a different mode of representation. However, if a student has preference for visual solution method, the he/she tend to translate the problem into graphic mode of representation. In graphic representation the ship and the light house become points in the plane and to solve the problem students do not need to know about the ship and the lighthouse. However, even for nonvisualizers it is important to understand of translating the problem into graphic representation. Similarly, if a problem is given using graphic mode of representation (for example the above problem is presented with a picture), then it is important to understand for visualizers to translate the problem into a verbal mode of representation. The fact is that students need to learn to translate one mode of representation to another, which provides them flexibility in solving mathematical tasks as well as they understanding ideas and concepts in an effective way. Students being able to translate problems into different modes of representation certainly provide flexibility in solving mathematics task; however, at the meantime students can chose their preferences for solution methods in order to solve the given problem. For example, whether an algebraic or graphic representation is used; students might need verbal descriptive representation in order to present mathematical ideas or concepts in a clear and coherent way. Despite some differences, most researchers contend that being able to translate between different modes of representation will result in a more in-depth understanding of mathematics (De Jong & van Joolingen, 1998; Lesh et al., 1987).

PREFERENCES FOR SOLUTION METHODS

In the domain of mathematics education, students prefer to use two modes of processing mathematical information (modes of thought): verbal-logical and visual-pictorial (Krutetskii, 1976). The preferred mode of processing refers to how students prefer to process information, not whether they possess particular skills or abilities (Haciomeroglu, Chicken, & Dixon, 2013). Based on the modes of thought, students prefer to solve mathematics problems using different solution methods. Students who prefer to use visual solution methods are called visualizers, whereas students who use nonvisual solution methods are called nonvisualizers. Harmonic students employ both visual and nonvisual solution methods. Diagrams and figures play a dominant role in a visual solution method. In a nonvisual solution method, students use mathematical formulae, rules, axioms, postulates, numbers, equations etc., while attempting mathematical tasks. The extent which students use a visual or nonvisual solution method is also called *visuality*. Thus, *visuality* refers to the extent to which the students use visual solution methods to solve given mathematical problems (Mainali, 2019).

Why students solving mathematical problems prefer one solution method over another when multiple solution methods are possible could be an important field of investigation in mathematics education. In this regard, Krutetskii (1976) laid a foundation for the distinction between preferences and abilities in relation to doing mathematics tasks. He contended that ability and preference are not the same thing. For example, students might have the *ability* to solve a problem with visual methods, but they might not *prefer* to solve it by visual methods; rather, they might prefer to solve it by a verbal method. Similarly, students might have the *ability* to solve a problem by a verbal method, which does not necessarily imply that they *prefer* to solve it by the verbal method. Thus, as far as the verbal-logical and visual-pictorial frameworks are concerned, students demonstrate different preferences for solution methods while doing mathematical tasks.

Verbal and visual are the predominantly used solution methods in the domain of mathematics. Even though researchers may have used different terms to represent these solution methods, most of the researchers, if not all, share common concepts about verbal and visual solution methods. Many researchers have investigated preferences for verbal and visual solution methods while attempting mathematical tasks (Haciomeroglu, 2012; Lean & Clements, 1981; Lowrie & Kay, 2001; Moses, 1977; Presmeg, 1986b). Various distinctions are made between verbal and visual solution methods. Presmeg (1986b) stated that:

A visual solution method is one which involves visual imagery, with or without a diagram, as an essential part of the solution method; even if algebraic methods are also employed while verbal solution method involves no visual imagery (p. 42).

Based on the use of visual imagery, Presmeg (1986b), Suwarsono (1982), and Moses (1977) defined and explained mathematical visuality—the extent to which a person prefers to use visual imagery when attempting mathematical problems. Moses stated that *degree of visuality* refers to the extent to which the subject uses visual solution processes to solve the given mathematical problems. In fact, the visual approach involves the act of visualization, which consists of any mental constructions and/or transformation of objects or processes (Suwarsono, 1982). In general, visualizers primarily rely on graphs, pictures, or symbols. In contrast, verbalizers attempt to solve problems by relying on rules, formulas, and algorithms (Moses, 1977).

It is worthwhile to mention that visual solution methods may also use some verbal and mathematical symbols, verbal statements, and mathematical statements. The fact is that diagrams, pictures, or similar constructions need to be labeled or they require verbal description in order to communicate about the constructions. Verbal and mathematical symbols are merely the shorthand for ordinary language and mathematical language (Skemp, 1987). However, the role of diagrams and figures is significantly important, and without using diagrams and figures it is not possible to solve problems using visual solution methods, regardless of whether the answer is correct or incorrect. In summary, in visual solution methods students use given diagrams and figures, or draw diagrams and figures, or visualize diagrams and figures in their head while attempting mathematical tasks. The diagrams and figures play a dominant role in visual solution methods to find the answer while attempting mathematical tasks.

A verbal solution method is one that involves analytic reasoning while attempting mathematical tasks. Analytic reasoning implies the use of mathematical formulae, algebra, arithmetic, rules, postulates, axioms, conjectures, and so forth while solving mathematical tasks. With this method, students do not use diagrams and figures. Suwarsono (1982) stated that in verbal solution methods, the reasoning is conducted purely on the basis of the processing or manipulation of verbal and mathematical statements and these manipulations are performed using the rules of language and mathematics. Zazkis, Dubnisky, and Dautermann (1996) stated that verbal solution methods involve an act of any mental manipulation of objects with or without the aid of symbols. It is apparent that primarily there are two types of preferences for solution methods: visual and nonvisual.

PREFERENCES AND MATHEMATICAL PERFORMANCE

Preferences for solution methods and mathematical performance have been of great interest to researchers for several decades (Battista, 1990; Fennema & Sherman, 1978; Haciomeroglu et al., 2013; Lean & Clements, 1981; Moses, 1977; Samuels, 2010; Suwarsono, 1982). Students can choose different solution methods when a mathematical task can be solved in multiple ways by employing either a visual-pictorial or a verbal-logical mode of thought. For example, a study conducted for a nationally representative sample in the UK and the USA identified that males preferred to use visual solution methods but females preferred to use verbal solution methods (Lohman & Larkin, 2009; Strand, Deary, & Smith, 2006). In contrast, Calvin, Farnandes, Smith, Visscher, and Deary (2010) revealed that the association between preferences and mathematics, there was no significant difference in employing solution methods based on gender. These are just two examples of the findings of the research studies that are not consistent with each other in this area. In this section, different research studies will be described that have been performed in the arena of preferences for solution methods and mathematical performance.

Moses (1977) conducted a comprehensive study with fifth-grade students ($N=131$) to measure relationships between problem-solving performance, and mathematical visuality. Her study revealed that there was no correlation between mathematical performance and preferences for solution method. Moses's study (1977) had, however, some limitations. She measured the preferences for solution methods based on students' written response only, but some students may not express their solution process in their written response. Moreover, students at the primary school level may not be able to express all or some of their thinking process on paper. Thus, the Moses study is criticized by many researchers, including Lean and Clements (1981). Students' mean score was also too low for the problem-solving inventory both in pretest and the posttest in Moses study.

In order to avoid Moses's limitation, Suwarsono (1982) conducted a study with middle school students ($N=112$) in which he developed an instrument called the Mathematical Processing Instrument (MPI) to investigate the students' degree of preference for solution methods and its effects on their mathematical performance. The MPI consists of two parts. Suwarsono designed the questionnaires to elude the limitation of Moses's study. The questionnaires contained various solution methods (visual, verbal, and other) for each problem. Students were asked to solve the word problems in the first part of MPI. In the second stage, students were required to choose the solution methods from the questionnaires. Beyond this, if students' methods were different from the ones that were listed in the questionnaires, the researcher instructed them to describe their solution methods. Thus, the researcher could understand the solution methods of those students who did not indicate their solution methods while attempting the word problems. Consistent to Moses's findings, Suwarsono (1982) also found that preferences for solution methods did not have a significant effect on mathematical performance. Students who preferred using visual solution methods in problem solving were likely to do as well as students who used verbal solution methods. In a study conducted about high school

students' preferences for solution methods and geometry performance, Mainali (2019) reported that there is no correlation between preferences for solution methods and geometry performance. Similarly, Pitta-Pantazi and Christou (2009) found that preference for solution methods was not related to mathematical performances. Their results also corroborated Moses and Suwarsono's findings. Suwarsono's instrument (MPI) have been utilized by many scholars to investigate the relationship between preferences for solution methods and mathematical performances.

Lean and Clements (1981) conducted a study with foundation year engineering college students ($N=116$) in which they used a slightly modified version of Suwarsono's Mathematical Processing Instrument (MPI) in order to investigate relationships between preference for solution methods and mathematical performance. They found that preferences had significant influence on students' mathematical performance. Their study further revealed that students who employed verbal solution methods performed significantly better than the students who employed visual solution methods. They also contended that the verbalizers developed logical reasoning ability and were able to avoid unnecessary visual information. Their finding also supports the Krutetskii (1976) thesis that spatial ability does not determine students' mathematical performance. However, their findings conflicted with those of Moses (1977, 1980) and Webb (1979), who reported that students who preferred to use visual solution methods tend to outperform those who use less visual solution methods. Recently, Manali (2019) conducted a study to examine the preferences for solution methods and mathematical performance, particularly to high school students' geometry performance. He reported that there is no correlation between preferences for solution methods and geometry performance. He further suggested that most of the students tended to utilize visual solution methods. However, students got incorrect answer if they tend to utilize one solution methods more over the other solution methods or vice versa.

Haciomeroglu, Aspinwall, and Presmeg (2009) conducted an empirical case study for calculus students to explore the relationship between preference for solution methods and calculus performance. Rather than using the verbal representation of MPI, the researchers used graphic representations to present the derivative problems. They found that students used visual as well as verbal solution methods to complete the given tasks, but students who used visual solution methods showed limited understanding and were not able to provide a complete answer, which contradicts the Lowrie and Kay (2001) findings. They also suggested that teachers need to incorporate both visual and nonvisual solution methods in their teaching strategies to support the successful mathematical performance of students. This study supported the Krutetskii (1976) thesis that regardless of the mode of representation used to present a problem, verbal-logical and visual-pictorial modes of mathematical processing were equally likely a student response.

Haciomeroglu, Aspinwall, and Presmeg (2010) further investigated the relationship between students' preference for solution methods and calculus performance. Though a graphic representation was used to present the calculus problems, students translated the problems into algebraic representation based on their preferences for solution method, according to the researchers. Similar to the findings of Haciomeroglu et al., (2009), this study also concluded that both visual and verbal solution methods are essential components for successful mathematical performance. They emphasized the need in both modes of thinking—verbal and visual—to deepen students' understanding. Additionally, they contended that students need to be able to translate one mode of representation to another for successful mathematical performance.

Haciomeroglu and Chicken (2011) examined the relationships among student cognitive ability, preference for solution method, and calculus performance of high school students ($N=169$). This study revealed that students' preferences for solution methods were positively correlated with calculus performance, where the problems were presented with the aid of graphic representation; however, the preferences were not associated with calculus performance, where the problems were presented with the aid of algebraic representation. Moreover, this study also found that Suwarsono's (1982) Mathematical Processing Instrument (MPI) is not an appropriate instrument to measure preferences and calculus performance. In another similar study, Haciomeroglu, Chicken, and Dixon (2013) examined high school students' ($N=150$) preference for solution methods and calculus performance by employing a graphic-calculus test. The preference for visual solution methods was significantly correlated with calculus performance, which was not consistent with Moses (1977), Lean and Clements (1981) and Suwarsono's (1982) findings. Similar to Haciomeroglu and Chicken (2011), they also argued that the MPI, which is considered an ideal test to examine students' preference for solution methods and mathematical performance, was not an appropriate test for the calculus students. Moreover, they explained that visual schemes involved in calculus tasks may not be captured by the algebraic test.

With the help of MPI, Hegarty and Kozhevnikov (1999) investigated how visual-spatial representations affect problem-solving performance of sixth graders ($N=33$). They found that preference for visual solution methods was positively correlated with mathematical performance. They made further distinctions among visual solution methods. They contended that there are, in fact, two types of visualizers: schematic types (representing the spatial relationships between objects and imagining spatial transformation), who are generally successful in mathematics problem solving, and pictorial types (constructing vivid and detailed visual images), who are less successful than schematic types. The distinction between two visual solution processes was further supported by Kozhevnikov, Hegarty and Mayer (2002). The fact is that the verbalizers and visualizers were the same on all parameters except their preferences for solution methods. Verbalizers did not have any clearly marked preference for using verbal solution methods. In contrast, visualizers showed a consistent preference for using visual solution processes. They claimed that various studies (Krutetskii, 1976; Lean & Clements, 1981; Presemeg, 1986a, 1986b) did not take the two types of visualizers into account which led them not to find the relationships between preferences for visual solution methods and mathematical performance.

Ling and Ghazali (2007) examined primary school students' preferences for solution methods ($N=5$) and pre-algebra problems. The problems were presented with the aid of verbal and graphic representation. Students equally used visual and verbal solution methods to solve the problems and noted no differences in preferences in solution method. Similarly, Sevimli and Delice (2011) investigated the relationships between calculus students' preferences for solution methods and representation preference while

solving mathematics problems using the modified version of MPI developed by Presmeg (1985). They concluded that the mode of representation used to present the problems affected students' preference for solution methods. Verbalizers and harmonic were observed to have similar preference tendencies. However, visualizers altered their preference based on the mode of representation used to present the problems. This study corroborated findings of Haciomeroglu, Chicken, and Dixon (2013) and Haciomeroglu and Chicken (2011). The greater variance in preferences of solution methods particularly for visualizers was consistent with the findings of the Kozhevnikov et al. study (2002). Moreover, this study also found that most of the verbalizers predominantly preferred verbal solution methods (algebraic representation). Similarly, Kolloffel (2012) examined the relationships between preferences and mathematical performance with college students ($N=40$). The researchers experimented with two different modes of representation (graphic and verbal) as an instructional strategy. Despite the differing teaching strategies used, no correlation was observed between preferences and mathematical performance. However, participants in the verbal instruction condition obtained significantly higher posttest scores than did students in the visual instruction condition. The findings of this study contradicted the findings of various other studies, including Moses (1977).

The related literature suggested an inconclusive finding in regards to the relationship between preferences for solution methods and mathematical performance. What we can infer from the related research studies that students equally likely to use both solution methods. Being able to utilize both solution methods equally in learning mathematics seems to be important factor in mathematical performance.

IMPLICATIONS FOR TEACHING AND LEARNING

How students process mathematical information (verbal-logical or visual-pictorial) can affect the preferences for solution methods (Galindo, 1994; Haciomeroglu et al., 2013; Krutetskii, 1976; Lowrie & Kay, 2001; Moses, 1977; Suwarsono, 1982). In-depth knowledge about what kind of solution methods students prefer to use and what difficulties they encounter when solving mathematical tasks can contribute not only to theoretical knowledge but also to the solution of the actual problems in learning mathematics (Gorgorio, 1998).

Some students prefer to use visual solution methods based on the visual-pictorial thought process, while others like to use nonvisual solution methods based on the verbal-logical thought process. Some research studies focused on visual solution methods, while others emphasize nonvisual solution methods. Some studies reported that there is a correlation between preferences for solution methods, while other suggested that there are no correlations. Some research studies also showed that students need to have both problem-solving skills—visual and nonvisual solution methods—for successful mathematical performance. For instance, the balance between visual and analytical reasoning ability is likely to be an important factor in mathematical performance (Battista, 1990).

To be proficient in mathematics, students are encouraged to develop preference for both solution methods: visual and nonvisual. From problem-solving methods and mathematical performance perspectives, it is essential for students to develop both solution methods because some problems are easier to solve using visual solution methods over nonvisual solution methods and vice-versa (Mainali, 2019). Thus, the development of only one-sided preferred mode of mathematical processing results in narrow mathematical development for students because they do not have an opportunity to see mathematics problems from the other perspective. Thus, instructional strategies in mathematics lesson needs to focus on students' development of balance in their knowledge and skills between visual and nonvisual solution methods (Haciomeroglu, Chicken, & Dixon 2013; Clements, 2014). In fact, students who use only (non)visual solution methods may have a limited understanding, and will not be able to provide a complete answer. In practice, mathematics instructional strategies often unaware of the importance of preference for solution methods.

The fact is that some mathematics teachers might over-emphasize nonvisual solution methods—rote memorization of mathematics rules and formulae— for success in mathematics, whereas other teachers might be over reliant on visual solution methods— utilize figures, diagrams etc.— to assist their students to learn mathematics. In doing so, teachers inhibit students' opportunity learning mathematics employing visual as well as nonvisual solution methods (Mainali, 2019). Mathematics teachers might be unaware of the fact that they are over reliant on only one instructional strategy, which might lead their students to develop preference for using only visual or nonvisual solution methods. Thus, it is suggested that instruction should be focused on incorporating both visual and nonvisual solution methods during instructional strategies in mathematics lessons because it is equally important to develop both visual and nonvisual preference for solution methods in order to be a successful learner and performer of mathematics.

Students need to learn how to construct and interpret different modes of representational systems because they are essential tools for communication and reasoning about concept and information in mathematics (Grenoo & Hall, 1997). In order to equally use visual and nonvisual solution methods, students should be able to employ different modes of representation as well as need to translate the representation. Thus, mathematics teachers need to emphasize various modes of representations in their instructional strategies in order encourage students to utilize both visual and nonvisual solution methods while solving mathematical tasks. However, mathematics teachers are unaware of using different modes of representations in appropriate in their instructional strategies. For example, Ma (1999) reported that many U.S. elementary school teachers lacked knowledge of representations in the teaching of division of fractions concept. We can find various textbooks, lesson materials, and other mathematical resources, where we can see predominantly utilization of certain mode of representation. Indeed, some mathematical areas might require one mode of representation more than over the other such as geometry might need more graphic representation than algebra. However, still balance integration of different modes of representation tend to develop both

visual and nonvisual solution methods in learning mathematics. The fact is that the developments of only one-sided preferences to utilize mode of representation result in narrow mathematical development for students because they do not have an opportunity to see mathematics problems from the other perspective. In fact, students who use only one modes of representation to solve mathematics problems might have limited understanding since they develop preference for only solution methods: visual or nonvisual. Thus, the instructional strategies need to focus in students' development of utilizing different modes of representations. The fact is that students should be able to utilize various modes of representation in order to be more proficient in mathematics since some mathematics problems, for example, can be solved in an easier way using certain modes of representation than the others (Mainali, 2019). The instructional strategies also need to focus where students would be able to get chances to translate mathematics problem from one mode of representation to other mode based on their preferences for solution methods. Mainali (2014) reported that when students utilized certain mode of representation to solve geometry problems, the majority of them had incorrect answers. However, students who employed different modes of representation, the majority of them were able to get the correct answer. Thus, based on the nature of mathematics problems, one specific mode of representation not always useful to solve problems. Thus, it is equally important to infuse different types of modes of representation in teaching learning mathematics. Relying on only one mode of representation in instructional strategies, teachers inhibit students' opportunity learning mathematics employing different modes of representations. Thus, it is suggested that instructional strategies should be focused on incorporating different modes of representation in order to students develop equally both visual and nonvisual solution methods.

Funding: No funding source is reported for this study.

Declaration of interest: No conflict of interest is declared by author.

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