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A Quantum Duistermaat-Heckman formula?

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Abstract. Some aspects of Duistermaat-Heckman formula in finite dimensions are reviewed. We speculate with some of its possible extensions to infinite dimensions. In particular we review the localization principle and the geometry of loop spaces following Witten and Atiyah's insight.

¿Es posible una fórmula de Duistermaat-Heckamn cuántica?

Resumen. En este trabajo se revisan algunos aspectos de la fórmula de Duistermaat–Heckman en dimensión finita. A continuación especulamos sobre sus posibles extensiones a dimensión infinita. En particular, revisaremos el principio de localización y la geometría de los espacios de lazos siguiendo las ideas de Witten y Atiyah.

1. Introduction

The evaluation of oscillatory integrals of the following form (Fourier transform of the push-forward of the measure $d\mu$ with respect to the map f)

$$I(t) = \int_M e^{itf(x)} d\mu(x) \tag{1}$$

where M is a $2n$-dimensional orientable closed manifold and $d\mu$ is the measure defined by a volume form ν on M, is both a challenging and a deep problem with multiple applications. Among the various fields where integrals like (1) appear, we can mention: Optics, Group theory, Statistical Mechanics, Quantum Theory and Quantum Field Theory, etc.

One of the most simple ways to approach the evaluation of (1) is by means of the stationary phase approximation, that consists in the computation of the first order in the perturbation expansion in t^{-1} of the previous integral. The "physical principle" on which the stationary phase approximation relies is that the relevant contributions to the integral will be those in the vecinity of critical points of the function f because of the cancellation due to destructive "interference" of contribution of terms far from them. Expanding the exponent f around its critical points, we can compute the gaussian integrals coming from the quadratic term in the expansion and obtain a reliable approximation to (1). More precisely, if $f\colon M \to \mathbb{R}$ is a Morse

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function, i.e., is a smooth function with nondegenerate critical points and $C(f)$ denotes the finite set of critical points of f, it is well-known that we arrive to the following formula for $I(t)$ [10],

$$I(t) = \frac{(2\pi)^n}{t^n} \sum_{p \in C(f)} c(p) e^{itf(p)} + O(t^{-n-1}), \tag{2}$$

where

$$c(p) = \frac{e^{i\pi \mathrm{sgn}\, f(p)/4}}{\sqrt{\det \mathrm{Hess}\ \ f(p)}},$$

is a characteristic number of the critical point p, with sgn $f(p) = 2n - 2l$ and $2l =$ ind $f(p)$ (see for instance [10] and references therein).

The heuristic infinite dimensional analoge of the previous formula (2) has been used sistematically in the computation of partition functions in Quantum Field theory and Statistical Mechanics, i.e., integrals of the form (see for instance any primer on Quantum Field Theory [18]),

$$Z_J = \int \mathcal{D}\varphi \, e^{-S(\varphi) - \int J\varphi}. \tag{3}$$

We will discuss later on the similarities and differences between (1) and the previous equation.

It was recognized long time ago that various semiclassical approximations to compute (3) gave the exact result in some particular cases. The same phenomena actually happens with the computation of integral (1). For certain functions f, Hamiltonians defining a $U(1)$ action, the stationary phase approximation is exact. The concrete realization of this fact constitutes what is called the Duistermaat-Heckman formula.

We will review in theses notes some of the fundamental facts concerning the Duistermaat-Heckman formula and we will explore some aspects of its possible extensions to infinite dimensions.

2. On the finite-dimensional Duistermaat-Heckman formula

As we were pointing before, one of the main corollaries of the results obtained by Duistermaat and Heckman [8], is the fact that in certain ocasions the stationary phase approximation is exact, i.e., the integral in the l.h.s. of equation (2) is given exactly by the finite sum over the critical points of the function in the exponent appearing on the r.h.s. of the same formula. This rather surprising fact, implies some sort of integrability on the problem under study, that in fact is hidden in the geometry of the manifold M and the function f.

Thus, let us assume that (M, ω) is a closed symplectic manifold and $d\mu_\omega$ is the Liouville measure defined by the symplectic volume form $\nu_\omega = \omega^n/n!$. Let G be a compact Lie group acting on M by symplectic diffeomorphisms and J the momentum map of such action. Let T denote a maximal compact torus on G and ξ an infinitesimal generator of T, i.e., and element of its Lie algebra $\mathfrak{t}$. The Hamiltonian function corresponding to the Killing vector field on M defined by ξ is given by $J_\xi = \langle \xi, J \rangle$, that for generic T and ξ, will be a Morse function on M. Then the stationary phase approximation formula (2) becomes exact and gives the celebrated Duistermaat-Heckman formula,

$$\int_M d\mu_\omega(x) \, e^{it\langle \xi, J(x) \rangle} d\mu_\omega(x) = \frac{(2\pi)^n}{t^n} \sum_{p \in C(J_\xi)} c(p) e^{it\langle \xi, J(p) \rangle}, \tag{4}$$

with

$$c(p) = \frac{i^n}{\prod_{k=1}^n \lambda_k},$$

where λ_k are the infinitesimal characters of the action of T on T_pM, $p \in C(J_\xi)$. In other words, because p is a critical point of J_ξ, it is a critical point for the Hamiltonian vector field X_ξ defined by J_ξ and it is a fixed point for the action of T on M. Then the action of T on M induces an action of T on T_pM, hence a representation of the Lie algebra $\mathfrak{t}$ on the linear space T_pM. Besides we can choose a metric such that the action is orthogonal. The linearization $X'_\xi(p)$ of the vector field X_ξ at the point p can be identified with the Hessian of J_ξ at p and the eigenvalues of such matrix are precisely i times the numbers λ_k, which coincide also with the weights of the linear representation of $\mathfrak{t}$.

There are several presentation of the previous result putting the emphasis on the different ideas involved on it. We shall mention the original proof by Duistermaat-Heckman [8], the approach by Atiyah-Bott [3] in terms of equivariant cohomology, Berline *et al* [5], Witten [21]. For the sake of completness we will sketch here the most direct proof available in finite dimensions [11].

The proof we sketch here is based in the following result from equivariant cohomology.

Theorem 1 *Let X_J be the Hamiltonian vector field corresponding to an $U(1)$ hamiltonian action on M and $D = d + i_{X_J}$ the equivariant derivative. Then $D^2 = 0$ on the space of invariant forms on M. Moreover if μ is an invariant nondegenerate form which in addition is D-closed, then μ is d-exact in $M^o = M - C(J)$.*

PROOF. Let θ be a 1-form on M^o such that

$$\mathcal{L}_{X_J} = 0, \qquad i_{X_J}\theta = 1. \tag{5}$$

Such 1-form can be constructed explicitely as follows. Let g be a S^1-invariant metric on M, then define

$$\theta_x(v) = \frac{\langle X_J(x), v\rangle}{||X_J(x)||^2}.$$

The 1-form θ is well-defined because $X_J(x) \neq 0$ for all $x \in M^o$. A routine computation shows that θ verifies (5). Let μ now be a form such that

$$\mathcal{L}_{X_J}\mu = 0$$

and D-closed, i.e.,

$$d\mu = -i_{X_J}\mu.$$

Let ν be the form

$$\nu = \theta \wedge (1 + d\theta)^{-1} \wedge \mu,$$

where

$$(1 + d\theta)^{-1} = 1 - d\theta + d\theta \wedge d\theta - \cdots ,$$

thus,

$$d\nu = d\theta \wedge (1 + d\theta)^{-1} \wedge \mu - \theta \wedge (1 + d\theta)^{-1} \wedge d\mu,$$

hence

$$i_{X_J} d\nu = d\theta \wedge (1 + d\theta)^{-1} \wedge i_{X_J}\mu + (1 + d\theta)^{-1} \wedge i_{X_J}\mu = i_{X_J}\mu,$$

where we have used that $d\mu = -i_{X_J}\mu$ and $i_{X_J} d\theta = 0$. ■

As an immediate application of the previous result we obtain that the invariant form $\mu = \exp it(J - \omega)$ is d-exact on M^o. Effectively,

$$D\mu = ite^{it(J-\omega)}D(J - \omega) = ite^{it(J-\omega)}(dJ - i_{X_J}\omega) = 0.$$

The top term of the form μ is obtained from the expansion,

$$e^{it(J-\omega)} = e^{itJ}e^{-it\omega} = e^{itJ}\left(1 - it\omega + \cdots + \frac{i^n t^n}{n!}\omega^n\right),$$

if dim$M = 2n$. Thus using the previous result we conclude that there exists an $(2n-1)$-form ν_{2n-1} on M^o such that

$$e^{itJ}\frac{1}{n!}\omega^n = d\nu_{2n-1}.$$

We shall proceed now to compute the integral (4). Denoting by $B_\epsilon = \cup_{p\in J_\xi} B_\epsilon(p)$ the union of a family of small balls of radius ϵ around the critical points of J, we have,

$$\int_M d\mu_\omega(x) e^{itJ_\xi(x)} = \int_{M-B_\epsilon} d\mu_\omega e^{itJ} + \int_{B_\epsilon} d\mu_\omega e^{itJ}. \tag{6}$$

The integrand of the first term in the r.h.s. of previous equation is exact because of the discussion of the paragraph before. The second term bounded is above by the quantity $C(\epsilon/t)^{2n}$ for a given constant C. The computation of the first term leads thus to

$$\int_{M-B_\epsilon} d\mu_\omega\, e^{itJ} = \int_{M-B_\epsilon} d\nu_{2n-1} = \int_{\partial(B_\epsilon)} \nu_{2n-1}.$$

Then, the integral in (6) becomes,

$$\int_M d\mu_\omega(x)\, e^{itJ_\xi(x)} = \sum_{p\in\partial(J_\xi)} \left[\int_{B_\epsilon(p)} d\mu_\omega\, e^{itJ_\xi} + \int_{\partial B_\epsilon(p)} \nu_{2n-1}\right]. \tag{7}$$

We can take the balls $B_\epsilon(p)$ small enough to be contained in a single chart and such that we can apply Morse' Lemma to the function J_ξ around p. Thus there will exist local coordinates centered at p such that

$$J_\xi(x) = J_\xi(p) + \frac{1}{2}\sum_{i=1}^{2n} \lambda_i(p)(x_j^2 + p_j^2),$$

and then (6) becomes,

$$\begin{aligned}\int_M d\mu_\omega(x) e^{itJ_\xi(x)} &= \sum_{p\in\partial(J_\xi)} \left[e^{itJ_\xi(p)} \int_{B_\epsilon(p)} e^{it\sum \lambda_i(p)(x_i^2+p_i^2)/2}\, d^n x d^n p + \int_{\partial B_\epsilon(p)} \nu_{2n-1}\right] \\ &= \sum_p e^{iJ_\xi(p)} \int_{\mathbb{R}^{2n}} e^{it\sum \lambda_i(p)(x_i^2+p_i^2)/2}\, d^n x d^n p.\end{aligned}$$

Now, a routine computation gives the desired formula. ■

3. Localization and equivariant cohomology

In the proof of DH formula we have introduced a new and crucial ingredient: the operator D, or equivariant differential. In fact, the computation before is a consequence of the analysis of equivariant cohomology groups. We will not enter a detailed discussion of these aspects here, and we refer the reader to the paper by Atiyah and Bott [3]. However we will describe the basic ingredients of the approach. The equivariant cohomology ring $H_G^*(M)$ of the G-space M "computes" the cohomology of the quotient space M/G for G a group acting on the manifold M. The equivariant cohomology ring $H_G^*(M)$ is defined as the ordinary cohomology ring of the homotopy quotient $M_G = M \times E_G/G$ where E_G denotes the universal principal G-bundle. A de Rham model for such cohomology ring is constructed using equivariant forms [16]. For $G = S^1$ they are given by invariant elements on $\Omega^*(M)[u]$, i.e., polynomials in the variable u with coefficients smooth forms on M. Such complex is equipped with the equivariant differential $D = d + ui_X$

where X is the vector field generating the S^1 action on M. The cohomology of such complex is isomorphic with $H^*_{S^1}(M)$. In [21] E. Witten discuss in detail equivariant integration, i.e., the extension of ordinary integration of forms to equivariant forms. It also amounts to construct an adequate completion for the space of equivariant forms. One possible way to proceed is to use the Bargmann-Fock quantization space [12]. We will define the integration of the polynomial part with respect to the measure $e^{-|u|^2/2}du \wedge d\bar{u}$. Such integral (conveniently normalized) is called the equivariant integral in [21]). We will keep using the same symbol for the integral of equivariant form. It is clear that

$$\int D\nu = 0.$$

Hence, if we change an equivariant form in an exact equivariant term of the form $D\theta$, the integral will not change. Thus if θ is an equivariant 1-form and μ and equivariantly closed form, then,

$$\int \mu = \int \mu e^{itD\theta},$$

because $\mu(1 - e^{itD\theta}) = D(\mu \wedge \theta e^{itD\theta})$. Then, the r.h.s. of previous equation can be integrated to get,

$$\int \mu = \frac{1}{2\pi} \int du \wedge d\bar{u} \, \exp\left(td\theta + t\langle X, \theta\rangle - |u|^2/2\right).$$

Thus, performing the gaussian integral on u we get,

$$\int \mu = \frac{1}{\sqrt{2\pi}} \int_M \mu \exp\left(td\theta - \frac{t^2}{2}|\langle X, \theta\rangle|^2\right).$$

Then, the previous integral is determined by the last factor in the exponent. In the set where $\langle X, \theta\rangle$ does not vanishes, then the quadratic exponential vanishes as $\exp(-Ct^2)$ for some positive constant C, and in the limit $t \to \infty$, it vanishes. Then, taking the limit $t \to \infty$, we get

$$\int \mu = \sum_i Z_i$$

where i labels the connected component of the zero set of $\langle X, \theta\rangle$, and Z_i is the value of $t \to \infty$ limit of the previous integral in an arbitrary small tubular neighborhood of this set. Thus, the integral of an equivariant closed form μ is localized in the zero set of the field generating the action.

4. A "quantum" Duistermaat-Heckman formula

As indicated in the introduction, the purpose of this paper is to contribute to the mounting evidence towards the existence of a quantum version of Duistermaat-Heckman formula. What is the meaning of "quantum" in this context? Duistermaat-Heckman formula (4) is "classical" in the sense that it computes the classical partition function Z_{cl} of a system defined by the Hamiltonian J on the (finite dimensional) phase space (M, ω),

$$Z_{cl} = \int_{M^{2n}} d\mu_\omega e^{-\beta J}. \tag{8}$$

The quantum analogous of this situation is obtained when we compute the partition function Z_J of a quantum system whose classical counterpart is given by a system on the phase space M and J is the generator of a $U(1)$ symmetry group, i.e., a current. Hence, we are interested in an integral of the form,

$$Z_J = \int_{\mathcal{L}(M)} \mathcal{D}\mu(\gamma) \, e^{-\beta \int J dt}. \tag{9}$$

In the previous formula, the integral is taken over the space of loops on M and the measure $\mathcal{D}\mu$ defining the quantum system is not well defined. In fact, the partition function of a quantum theory with Hamiltonian H is given by an expression of the form,

$$Z = \int \mathcal{D}q\mathcal{D}p\, e^{-i/\hbar \int pdq - Hdt} \cong \int \mathcal{D}q\, e^{-i/\hbar \int Ldt}. \tag{10}$$

The measure $\mathcal{D}q$ can be defined precisely for Lagrangians of the form

$$L = \frac{1}{2} g_{ij}\dot{q}^i\dot{q}^j - V(q),$$

then the Feynman-Kac formula,

$$\mathrm{Tr}\, e^{\beta H} = \int_{\mathcal{L}(Q)} \mathcal{D}q\, e^{-\int_0^\beta \frac{1}{2} g_{ij}\dot{q}^i\dot{q}^j d\tau} e^{-\int_0^\beta V d\tau},$$

holds, for

$$e^{-\int_0^\beta \frac{1}{2} g_{ij}\dot{q}^i\dot{q}^j d\tau}\mathcal{D}q = d\mu_W$$

the Wiener measure with covariance d^2/dt^2 on the Riemannian manifold (Q, g).

We must then be very careful when comparing expression (9) with a quantum path integral like (10) because the measure $d\mu$ appearing in (9) is a Liouville measure whereas the measure appearing in the computation of the quantum partition function is a Wiener measure obtained from a Riemannian metric. The relation among both, the Wiener measure and the symplectic measure, is given by a generalized Radon-Nikodyn derivative, that in the particular case of dealing with the Riemann measure and the symplectic measure defined respectively by a Riemannian metric g and a symplectic form ω, coincides with the Pfaffian of ω with respect to g, i.e., the square root of the linear operator obtained lowering indexes with g and raising them with ω.

Moreover passing from formula (8) to formula (9) amounts to quantize the classical theory. Thus, a quantum Duistermaat-Heckman formula is a situation where the stationary phase approximation for the quantum partition function Z_J would be exact.

There is abondant evidence that such situations occur, but of course there is not a general theory characterizing for which quantum systems, i.e., for what measures $\mathcal{D}\mu$ and for what symmetry groups, such statement is true. M. Atiyah pointed out that the quantum Duistermaat-Heckman formula is true for Witten's supersymmetric Quantum Mechanics, the proof of such statement is given by a fresh interpretation of Atiyah-Singer index theorem for Dirac's operator [4]. E. Witten showed also that such quantum Duistermaat-Heckman formula can be used in the computation of the partition function of Yang-Mills in 2D [21]. Other such quantum Duistermaat-Heckman formulas have been used in different contexts (see for instance the quantization of coadjoint orbits by Alekseev *et al* [1], [13]). In a series of papers and communications Niemi [17], Tirkkonen [19], Blau [7], etc., have extended the equivariant localization principle to path integrals and showed some of its possible implications.

5. Duistermaat-Heckman formula and Dirac index theorem

We will review here the masterly exposition of M. Atiyah in [4] where following a suggestion by Witten, it is shown that an adequate interpretation of the Duistermaat-Heckman in the infinite dimensional loop space is equivalent to the index theorem for the Dirac operator (see also [6]).

To be more precise, let us consider a $2n$ dimensional Riemannian manifold Q and the loop space $\mathcal{L}(Q) = \{\, \gamma \colon [0,1] \to Q \mid \gamma(0) = \gamma(1) \,\}$. We shall define the 2-form Ω on $\mathcal{L}(Q)$ by the formula

$$\Omega_\gamma(\delta\gamma_1, \delta\gamma_2) = \int_0^1 \left\langle \frac{D\delta\gamma_1}{dt}, \delta\gamma_2 \right\rangle_{\gamma(t)} dt,$$

where $\gamma \in \mathcal{L}(Q)$ and $\delta\gamma_a \in T_\gamma\mathcal{L}(Q) = \Gamma(\gamma^*TQ)$, $a = 1, 2$, are two tangent vectors. The 2-form Ω is presymplectic with characteristic distribution given by Jacobi fields. As it was discussed in the previous section, the "Riemann measure" and the "Liouville measure" defined respectively by the the induced metric $\langle .,. \rangle$ on $\mathcal{L}(Q)$ and the 2-form Ω will differ by the Pfaffian of Ω which is given by the square root of the determinant of the operator D/dt.

The group S^1 acts naturally on $\mathcal{L}(Q)$ by rotating the loops. The infinitesimal generator of such action is given by the vector field $\Gamma(\gamma) = \dot{\gamma}$. The vector field Γ is Hamiltonian with Hamiltonian $E(\gamma) = \int_0^1 ||\dot{\gamma}(t)||^2 dt$. The integral we would like to compute is then,

$$\int_{\mathcal{L}(Q)} d\mu_\Omega(\gamma)\, e^{-\beta E(\gamma)} = \int_{\mathcal{L}(Q)} d\mu_W(\gamma) \sqrt{\det \frac{D}{dt}(\gamma)} = \int_{\mathcal{L}(Q)} \mathcal{D}\gamma \sqrt{\det \frac{D}{dt}(\gamma)} e^{-\beta E(\gamma)}.$$

As it always happens with path integrals some sort of regularization will be necessary. Thus, using ζ-function regularization of operator determinants [18], we obtain immediately for the determinant of the operator D/dt that

$$\det \frac{D}{dt}(\gamma) = \det(I - T_\gamma) = \mathrm{Tr}(S^+(T_\gamma)) - \mathrm{Tr}(S^-(T_\gamma)),$$

where T_γ denotes the parallel transport along the loop γ and $S^\pm$ are the spin representations of $O(2n)$. Thus, if $D\colon S^+ \to S^-$ denotes the Dirac operator (we assume the manifold Q to be spin), then it is well known that the path integral representation of the index of D is given by

$$\begin{aligned} \mathrm{ind}\, D &= \dim\ker D - \dim\ker D^\dagger = \mathrm{Tr}(-1)^F e^{-4\pi^2\Delta} \\ &= \int_{\mathcal{L}(Q)} \mathcal{D}\gamma\, [\mathrm{Tr}S^+(T_\gamma) - \mathrm{Tr}S^-(T_\gamma)] e^{-E(\gamma)} = \int_{\mathcal{L}(Q)} d\mu_\Omega\, e^{-E(\gamma)}. \end{aligned}$$

A naive application of Duistermaat-Heckman formula will allow us to evaluate the r.h.s. of the previous equation by summing the contributions of the exponent over the fixed points of the action. The fixed points of the action of S^1 are constant loops on Q. But the set of constant loops is naturally identified with Q itself. Thus, the set of fixed points is not made of isolated points. This can be easily remedied by using the appropriate extension of Duistermaat-Heckman formula (4) [8]. If J is a function with connected critical submanifolds N_i, $C(J) = \cup_{i=1}^s N_i$, $J(N_i) = J_i$, and $\nu(N_i)$ denotes the normal bundle to N_i on M, the rank of $\nu(N_i)$ will be given by $2n - 2k_i$, the linearization of the vector field X_J on TN_i will be given by $diag(m_1(N_i), \ldots, m_{k_i}(N_i))$. Moreover the Chern class of the (complex) fibre bundle $\nu(N_i)$ will be factorized as

$$c(\nu(N_i)) = \prod_{j=1}^{k_i} (1 + \alpha_{ij}).$$

Then, the generalization of Duistermaat-Heckman formula, eq. (4), is given by,

$$\int_M e^{itJ} d\mu_\omega = \sum_i \int_{N_i} \frac{e^{itJ_i} e^\omega}{\prod_{j=1}^{k_i}(itm_j(N_i) - i\alpha_{ij})}, \tag{11}$$

where

$$\prod_{j=1}^{k_i} \frac{1}{(tm_j(N_i) - \alpha_{ij})} = \prod \left(\frac{1}{tm_j(N_i)} + \frac{\alpha_{ij}}{(tm_j(N_i))^2} + \cdots \right) = \frac{1}{t^{k_i} \prod_{j=1}^{k_i} m_j(N_i)} + \cdots .$$

In our particular case the normal space to Q in $\mathcal{L}(Q)$ is given by nonconstant maps $S^1 \to T_qQ$. Thus the normal bundle $\nu(Q)$ can be decomposed as

$$\nu(Q) = T_1 \oplus T_2 \oplus \cdots ,$$

with T_k the complexified tangent bundle TQ with S^1 acting with rotation number k. Thus the Chern class $c(T_k)$ can be factorized as

$$c(T_k) = \prod_{j=1}^{m} (k + \alpha_j)(k - \alpha_j),$$

and the denominator in DH formula (11) becomes

$$\prod_{j=1}^{m} \prod_{k=1}^{\infty} (k^2 - \alpha_j^2),$$

that using the same regularization than in the case of the Pfaffian, we obtain,

$$\int_Q e^{iE(q)} e^{\Omega(q)} \prod_{j=1}^{m} \frac{\alpha_j/2}{\sinh \alpha_j/2},$$

and we arrive to the formula

$$\operatorname{ind} D = \int_Q \prod_{j=1}^{m} \frac{\alpha_j/2}{\sinh \alpha_j/2} = \hat{A}(Q),$$

which is the index theorem for the Dirac operator.

If G is a compact Lie group acting on Q, then, there is an induced action of $\mathcal{L}(G)$, the loop group of G, on $\mathcal{L}(Q)$. If we denote by ξ an element on the Lie algebra $\mathfrak{g}$, then $\hat{\xi}\colon S^1 \to \mathfrak{g}$ will denote the elements on the Lie algebra $\mathcal{L}(\mathfrak{g})$ of $\mathcal{L}(G)$. The group G is the subgroup of constant loops of $\mathcal{L}(G)$. The action of G on $\mathcal{L}(Q)$ is symplectic with momentum map $\mathcal{J}$ given by

$$\langle \mathcal{J}(\gamma), \xi \rangle = \int_0^1 \left\langle \frac{D\xi_Q}{dt}, \dot{\gamma}(t) \right\rangle_{\gamma(t)} dt, \qquad \forall \xi \in \mathfrak{g}.$$

Thus we can ask about the evaluation of the integral

$$\int_{\mathcal{L}(Q)} d\mu_\Omega(\gamma)\, e^{-\beta(E(\gamma) + \mathcal{J}_\xi(\gamma))}. \tag{12}$$

A similar discussion to the previous one shows that the previous integral (12) leads to the equivariant index theorem for the Dirac operator D (the computation of the path integral can be repeated easily following the ideas in [2]).

6. Some aspects of the geometry of loop spaces

The previous discussion seems a bit forced by the need to introduce a metric and the restriction on the dimension of the manifold Q. Everything is much more natural formulated directly on a symplectic manifold.

Let (M, ω) be as usual a symplectic manifold. The space of loops $\mathcal{L}(M)$ carries a natural symplectic structure Ω defined as follows:

$$\Omega_\gamma(\delta\gamma_1, \delta\gamma_2) = \int_0^1 \omega_{\gamma(t)}(\delta\gamma_1(t), \delta\gamma_2(t)) dt,$$

where $\delta\gamma_a \in T_\gamma\mathcal{L}(M) = \Gamma(\gamma^*(TM))$, $a = 1, 2$, are tangent vectors to $\mathcal{L}(M)$ at $\gamma \in \mathcal{L}(M)$.

In the particular case of $M = T^*Q$, we immediately see that

$$\mathcal{L}(T^*Q) = T^*\mathcal{L}(Q),$$

moreover the canonical Liouville 1-form θ_0 on T^*Q lifts to the canonical 1-form Θ on $T^*\mathcal{L}(Q)$,

$$\Theta_\gamma(\delta\gamma) = \int_0^1 (\theta_0)_{\gamma(t)}(\delta\gamma(t))dt,$$

and $d\Theta = \Omega_0$, where Ω_0 is the canonical symplectic form on $T^*\mathcal{L}(Q)$. Notice that for any closed 1-form α on $\mathcal{L}(Q)$ its graph will define a Lagrangian submanifold on $T^*\mathcal{L}(Q)$. Similar considerations will stand for arbitrary 1-forms on $\mathcal{L}(Q)$, i.e., if β is a 1-form on $\mathcal{L}(Q)$ then its graph will define a submanifold of $T^*\mathcal{L}(Q)$ whose characteristic distribution is given by $\ker\beta$. This is the construction in the previous paragraphs with Atiyah's construction for the 1-form η,

$$\eta_\gamma(\delta\gamma) = \int_0^1 \langle\dot\gamma, \delta\gamma\rangle dt.$$

The generator of the canonical vector field $\Gamma(\gamma) = \dot\gamma$, is the action functional closed 1-form $\mathcal{A}$ given by,

$$\mathcal{A}_\gamma(\delta\gamma) = \int_0^1 \omega(\dot\gamma, \delta\gamma)dt.$$

If the symplectic manifold M is Floer, i.e., $\langle\pi_2(M), \omega\rangle = 0$, then the closed 1-form $\mathcal{A}$ is exact, and $\mathcal{A} = d\mathcal{S}$, with $\mathcal{S}$ the action functional (further properties of the symplectic manifold $\mathcal{L}(M)$ in connection with Arnold's conjecture can be found in [14]).

The computation of the integral

$$\int_{\mathcal{L}(M)} d\mu_\Omega(\gamma)\, e^{-\mathcal{S}(\gamma)},$$

leads again to the index of the Dirac operator D constructed out of a Riemannian metric g on M compatible with the symplectic form ω.

Further discussion on path intgrals on the loop space of a symplectic manifold will be presented elsewhere.

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