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CORRECTION

Correction To: Frobenius and homological dimensions of complexes

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The proof of Theorem 3.2 in the paper contains an error (namely in the use of Lemma 3.1 when $T = {}^eR$, which is only a faithful R-module when R is reduced). We give a new proof of this Theorem (slightly strengthened to streamline the proof) which avoids the use of Lemma 3.1.

Theorem 3.2 *Let $(R, \mathfrak{m}, k)$ be a d-dimensional Cohen–Macaulay local ring of prime characteristic p and which is F-finite. Let $e \geqslant \log_p e(R)$ be an integer, M an R-complex, and $r = \max\{1, d\}$.*

(a) *Suppose there exists an integer $t > \sup \mathrm{H}^*(M)$ such that $\mathrm{Ext}^i_R({}^eR, M) = 0$ for $t \leqslant i \leqslant t + r - 1$. Then M has finite injective dimension.*
(b) *Suppose there exists an integer $t > \sup \mathrm{H}_*(M)$ such that $\mathrm{Tor}_i^R({}^eR, M) = 0$ for $t \leqslant i \leqslant t + r - 1$. Then M has finite flat dimension.*

Proof We first note that if (a) holds in the case $\dim R = d$, then (b) also holds in the case $\dim R = d$: For, suppose the hypotheses of (b) hold for a complex M. Then by Lemma 2.5(a), $\mathrm{Ext}^i_R({}^eR, M^{\vee}) \cong \mathrm{Tor}_i^R({}^eR, M)^{\vee} = 0$ for $t \leqslant i \leqslant t + r - 1$. As $\sup \mathrm{H}^*(M^{\vee}) = \sup \mathrm{H}_*(M)$, we have by (a) that $\mathrm{id}_R M^{\vee} < \infty$. Hence, $\mathrm{fd}_R M < \infty$ by Corollary 2.6(a).

Thus, it suffices to prove (a). As in the original proof, we may assume that M is a module concentrated in degree zero and $\mathrm{Ext}^i_R({}^eR, M) = 0$ for $i = 1, \ldots, r$. We proceed by induction on d, with the case $d = 0$ being established by Proposition 2.8. Suppose $d \geqslant 1$ (so $r = d$) and we assume both (a) and (b) hold for complexes over local rings of dimension less than d.

Let $\mathfrak{p} \neq \mathfrak{m}$ be a prime ideal of R. As R is F-finite, we have $\mathrm{Ext}^i_{R_\mathfrak{p}}({}^eR_\mathfrak{p}, M_\mathfrak{p}) = 0$ for $1 \leqslant i \leqslant d$. As $d \geqslant \max\{1, \dim R_\mathfrak{p}\}$ and $e(R) \geqslant e(R_\mathfrak{p})$ (see [12]), we have $\mathrm{id}_{R_\mathfrak{p}} M_\mathfrak{p} < \infty$

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by the induction hypothesis. Hence, $\mathrm{id}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leqslant \dim R_{\mathfrak{p}} \leqslant d-1$ by [4, Proposition 4.1 and Corollary 5.3]. It follows that $\mu_i(\mathfrak{p}, M) = 0$ for all $i \geqslant d$ and all $\mathfrak{p} \neq \mathfrak{m}$.

For convenience, we let S denote the R-algebra ${}^eR$. Let J be a minimal injective resolution of M. We have by assumption that

$$\mathrm{Hom}_R(S, J^0) \xrightarrow{\phi^0} \mathrm{Hom}_R(S, J^1) \to \cdots \to \mathrm{Hom}_R(S, J^d) \xrightarrow{\phi^d} \mathrm{Hom}_R(S, J^{d+1}) \quad (3.1)$$

is exact. Let L be the injective S-envelope of $\operatorname{coker} \phi^d$ and $\psi : \mathrm{Hom}_R(S, J^{d+1}) \to L$ the induced map. Hence,

$$0 \to \mathrm{Hom}_R(S, J^0) \to \cdots \xrightarrow{\phi^d} \mathrm{Hom}_R(S, J^{d+1}) \xrightarrow{\psi} L$$

is acyclic and in fact the start of an injective S-resolution of $\mathrm{Hom}_R(S, M)$.

As in the original proof, we obtain that the map ψ is injective.

Now consider the complex J, which is a minimal injective resolution of M:

$$0 \to J^0 \xrightarrow{\partial^0} J^1 \to \cdots \to J^{d-1} \xrightarrow{\partial^{d-1}} J^d \xrightarrow{\partial^d} \cdots$$

The proof will be complete upon proving:

Claim: ∂^{d-1} is surjective.

Proof: As ψ is injective we have from (3.1) that $\phi^d = 0$, and thus ϕ^{d-1} is surjective. Let $C = \operatorname{coker} \partial^{d-1}$ and $(-)^{\vee}$ the Matlis dual functor (as defined in Corollary 2.6). Then

$$0 \to C^{\vee} \to (J^d)^{\vee} \to \cdots \to (J^0)^{\vee} \to M^{\vee} \to 0$$

is exact. Note that $(J^i)^{\vee}$ is a flat R-module for all i (e.g., Corollary 2.6(b)). As the set of associated primes of any flat R-module is contained in the set of associated primes of R, and as R is Cohen–Macaulay of dimension greater than zero, to show $C^{\vee} = 0$ it suffices to show $(C^{\vee})_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \neq \mathfrak{m}$. So fix a prime $\mathfrak{p} \neq \mathfrak{m}$. As S is a finitely generated R-module, we have $\mathrm{Tor}_i^R(S, M^{\vee}) \cong \mathrm{Ext}_R^i(S, M)^{\vee} = 0$ for $i = 1, \ldots, d$ by Lemma 2.5(b). This implies $\mathrm{Tor}_i^{R_{\mathfrak{p}}}(S_{\mathfrak{p}}, (M^{\vee})_{\mathfrak{p}}) = 0$ for $i = 1, \ldots, d$. As $R_{\mathfrak{p}}$ is an F-finite Cohen–Macaulay local ring of dimension less than d, and $p^e \geqslant e(R) \geqslant e(R_{\mathfrak{p}})$, we have that $\mathrm{fd}_{R_{\mathfrak{p}}} (M^{\vee})_{\mathfrak{p}} < \infty$ by the induction hypothesis on part (b). In particular, by [4, Corollary 5.3], $\mathrm{fd}_{R_{\mathfrak{p}}} (M^{\vee})_{\mathfrak{p}} \leqslant \dim R_{\mathfrak{p}} \leqslant d-1$ and thus $(C^{\vee})_{\mathfrak{p}}$ is a flat $R_{\mathfrak{p}}$-module. Then by either [15, Corollary 3.5] or [6, Theorem 3.1], we have

$$0 \to S_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} (C^{\vee})_{\mathfrak{p}} \to S_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} ((J^d)^{\vee})_{\mathfrak{p}} \to S_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} ((J^{d-1})^{\vee})_{\mathfrak{p}} \quad (3.3)$$

is exact. Now, since $\phi^{d-1} = \mathrm{Hom}_R(S, \partial^{d-1})$ is surjective, we have, using duality and Lemma 2.5(b), that

$$0 \to S \otimes_R (J^d)^{\vee} \to S \otimes_R (J^{d-1})^{\vee}$$

is exact. Localizing this exact sequence at $\mathfrak{p}$ and comparing with (3.3), we have $S_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} (C^{\vee})_{\mathfrak{p}} = 0$. However, tensoring with $S_{\mathfrak{p}}$ over $R_{\mathfrak{p}}$ is faithful (e.g., [13, Proposition 2.1(c)]) and hence $(C^{\vee})_{\mathfrak{p}} = 0$. Hence, $C^{\vee} = 0$, and thus $C = 0$, which completes the proof of the Claim. $\square$

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