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Uniform rotundity in every direction of Orlicz function spaces equipped with the *p*-Amemiya norm

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Abstract Some conditions which guarantee that the Orlicz function spaces equipped with the p-Amemiya norm ($1 < p < \infty$) and generated by N-functions are uniformly rotund in every direction are given. Obtained result broaden the knowledge about this notion in Orlicz function spaces with the p-Amemiya norm ($1 \leq p \leq \infty$).

Keywords Orlicz spaces · P-Amemiya norm · Uniform rotundity in every direction

Mathematics Subject Classification 46E30 · 46B20

1 Introduction and preliminaries

Let us denote by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}_+$ the sets of natural, real and nonnegative real numbers, respectively. For a Banach space X, by $S(X)$ and $B(X)$ we will denote the unit sphere and the unit ball of X, respectively.

A Banach space $X = (X, \|.\|)$ is called uniformly rotund in every direction (see [16,42]) if for each $\varepsilon > 0$ and nonzero $z \in X$ there exists $\delta(z, \varepsilon) > 0$ such that if x and y belong to $S(X)$, $\|x - y\| \geq \varepsilon$ and $x - y = \alpha z$ for some $\alpha \in \mathbb{R}$, then $\left\| \frac{x+y}{2} \right\| \leq 1 - \delta(z, \varepsilon)$. We will write then $X \in (URED)$ for short. It is known (see [16]) that the URED property is equivalent to the following one: For each nonzero z in X there is a positive number $\delta(z)$ such that if $x \in B(X)$ and $\|x + z\| \leq 1$, then $\left\| x + \frac{z}{2} \right\| \leq 1 - \delta(z)$. Equivalently, one can say that $X \in (URED)$ if and only if $x_n, z \in X$, $\|x_n\| \to 1$, $\|x_n + z\| \to 1$ and $\|2x_n + z\| \to 2$ as $n \to \infty$ imply $z = 0$.

Another characterization of the uniform rotundity in every direction is also possible (see [16]), namely, $X \in (URED)$ if and only if the following condition holds: if there are sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ in X such that $\|x_n\| \leq 1$ and $\|y_n\| \leq 1$ for every $n \in \mathbb{N}$,

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$x_n - y_n \to z$ and $\|x_n + y_n\| \to 2$ as $n \to \infty$, then $z = 0$. Note also that in case of Köthe spaces, the following characterization of the URED property is known (see [32], Prop. 3.3): The Köthe space E is uniformly rotund in every direction if and only if for any $\varepsilon \in (0, 2]$ and $z \in E_+ \setminus \{0\}$ (E_+ is the positive cone of E) there exists $\delta(\varepsilon, z) \in (0, 1)$ such that $\left\| \frac{x+y}{2} \right\|_E \le 1 - \delta(\varepsilon, z)$ for any $x, y \in B(E)$ with $x - y = \lambda z$ for some $\lambda > 0$ and $\|x - y\|_E \ge \varepsilon$.

The notion of uniform rotundity in every direction was first used by Garkavi (see [18–20]) to characterize normed linear spaces for which every bounded subset has at most one Chebyshev center. Next, Zizler, Day, James and Swaminathan continued investigation of this property [16,46] and the revealed results and applications encouraged other mathematicians to consider this property in some particular Banach spaces such as Orlicz spaces, Musielak–Orlicz spaces, Musielak–Orlicz spaces of Bochner type, Orlicz–Sobolev spaces or Calderón–Lozanovskiĭ spaces (see [2,4,5,8,27–29,32,41,43,44]).

A map $\Phi : \mathbb{R} \to [0, \infty]$ is said to be an Orlicz function if $\Phi(0) = 0$, Φ is not identically equal to zero (i.e. $\lim_{u\to\infty} \Phi(u) = \infty$), Φ is even and convex on the interval $(-b(\Phi), b(\Phi))$ and Φ is left-continuous at $b(\Phi)$, i.e. $\lim_{u\to b(\Phi)^-} \Phi(u) = \Phi(b(\Phi))$. Let us notice that every Orlicz function Φ is continuous on the interval $(-b(\Phi), b(\Phi))$. Recall also that an Orlicz function Φ is called an N-function if it vanishes only at 0, takes only finite values and the following two conditions are satisfied: $\lim_{u\to 0} \frac{\Phi(u)}{u} = 0$ and $\lim_{u\to\infty} \frac{\Phi(u)}{u} = \infty$.

Let us note that whenever some result will concern Orlicz spaces generated by N-functions only, this fact will be announced in the assumptions.

For any Orlicz function $\Phi : \mathbb{R} \to [0, \infty]$ let us define

$$a(\Phi) := \sup\{u \ge 0 : \Phi(u) = 0\},\, b(\Phi) := \sup\{u > 0 : \Phi(u) < \infty\}.$$

Notice that $a(\Phi) = 0$ means that Φ vanishes only at zero while $b(\Phi) = \infty$ means that Φ takes only finite values.

An Orlicz function $\Phi : \mathbb{R} \to [0, \infty)$ is called convex if

$$\Phi\left(\frac{u+v}{2}\right) \le \frac{\Phi(u) + \Phi(v)}{2} \tag{1.1}$$

for all $u, v \in \mathbb{R}$. If the inequality in (1.1) is sharp for all $u \ne v$, then Φ is called strictly convex (on $\mathbb{R}$). An Orlicz function $\Phi : \mathbb{R} \to [0, \infty)$ is said to be uniformly convex at infinity (see also [1,34]) if for each $a \in (0, 1)$ there exists $\delta_a \in (0, 1)$ such that

$$\Phi\left(\frac{u+au}{2}\right) \le \frac{1}{2}\left(1 - \delta_a\right)\left(\Phi(u) + \Phi(au)\right) \tag{1.2}$$

for all $u \ge u_0$, where $u_0 > 0$. If condition (1.2) holds with $u_0 = 0$, we say that Φ is uniformly convex on $\mathbb{R}_+$ (so on the whole $\mathbb{R}$ by the fact that Φ is even).

For any Orlicz function Φ, we define its complementary (in the sense of Young) function Ψ by the formula

$$\Psi(v) = \sup_{u\ge 0}\{u|v| - \Phi(u)\}.$$

We say that an Orlicz function Φ satisfies condition $\Delta_2(\mathbb{R}_+)$ [resp. $\Delta_2(\infty)$] if there exists $K > 0$ such that for all $u \ge 0$ [resp. if there exist $K > 0$ and $u_0 > 0$ with $\Phi(u_0) < \infty$ such that for any $u \ge u_0$] inequality $\Phi(2u) \le K\Phi(u)$ holds. In this case, we will write $\Phi \in \Delta_2(\mathbb{R}_+)$ [resp. $\Phi \in \Delta_2(\infty)$].

Throughout the paper we will assume that (Ω, Σ, μ) is a measure space with a σ-finite non-atomic and complete measure μ and $L^0(\mu)$ is the space of all μ-equivalence classes of real and Σ-measurable functions defined on Ω. Let us note that Theorem 4 will be formulated for the Orlicz spaces equipped with the p-Amemiya norm, $(1 \le p < \infty)$, built over a finite non-atomic measure space. We shall say that an Orlicz function Φ satisfies the $\Delta_2(\mu)$-condition if $\Phi \in \Delta_2(\infty)$ when the measure space is non-atomic finite and $\Phi \in \Delta_2(\mathbb{R}_+)$ when the measure space is non-atomic infinite. Let us define the characteristic function χ_A of a subset A from Ω as

$$\chi_A = \begin{cases} 1, & \text{for } t \in A, \\ 0, & \text{otherwise.} \end{cases}$$

For a given Orlicz function Φ we define on $L^0(\mu)$ a convex semimodular (see [3,30,34–36,39]) by

$$I_\Phi(x) = \int_\Omega \Phi(x(t))d\mu.$$

The Orlicz space L^Φ generated by an Orlicz function Φ is a linear space of measurable functions defined by the formula (see [38]):

$$L^\Phi = \{x \in L^0(\mu) : I_\Phi(\lambda x) < \infty \text{ for some } \lambda > 0\}.$$

The Orlicz space is usually equipped with the Luxemburg norm (see [34])

$$\|x\|_\Phi = \inf\left\{\lambda > 0 : I_\Phi\left(\frac{x}{\lambda}\right) \le 1\right\}$$

or with the equivalent one (see [37,38])

$$\|x\|_o = \sup\left\{\int_\Omega |x(t)y(t)|d\mu : y \in L^\Psi, I_\Psi(y) \le 1\right\},$$

which is called the Orlicz norm, where Ψ is the complementary function to Φ.

For any $1 \le p \le \infty$ and $u \ge 0$ let us define

$$s_p(u) = \begin{cases} (1+u^p)^{\frac{1}{p}} & \text{for } 1 \le p < \infty, \\ \max\{1, u\} & \text{for } p = \infty. \end{cases} \tag{1.3}$$

In order to simplify notations, define $s_{\Phi,p}(x) = s_p \circ I_\Phi(x)$ for all $1 \le p \le \infty$ and all $x \in L^0(\mu)$. Note that the functions s_p and $s_{\Phi,p}$ are convex. Moreover, the function s_p is increasing on $\mathbb{R}_+$ for $1 \le p < \infty$, but the function s_∞ is increasing on the interval $[1, \infty)$ only.

Definition 1 Let $p \in [1, \infty]$. By the p-Amemiya norm of a function $x \in L^0$ we mean the number defined by the formula (see [11,25])

$$\|x\|_{\Phi,p} = \inf_{k>0} \frac{1}{k} s_{\Phi,p}(kx).$$

The Orlicz space equipped with the p-Amemiya norm $(L^\Phi, \|.\|_{\Phi,p})$ will be denoted by $L^{\Phi,p}$.

It is known (see [11]) that the p-Amemiya norm $\|x\|_{\Phi,p}$ $(1 \le p \le \infty)$ is equivalent to the Luxemburg norm $\|x\|_\Phi$, namely, $\|x\|_\Phi \le \|x\|_{\Phi,p} \le 2^{\frac{1}{p}}\|x\|_\Phi$ for any $x \in L^{\Phi,p}$. Recall also here the earlier result of Hudzik and Maligranda from [25], which states that the 1-Amemiya norm is equal to the Orlicz norm in general, i.e. when Φ is an arbitrary Orlicz function.

Recall that the notion of the p-Amemiya norm $(1 \le p \le \infty)$ was introduced by Reisner in 1988 in [40], where the Author defined these norms for the Calderón–Lozanovskii spaces.

Next, in 2000, Hudzik and Maligranda [25] suggested investigating a family of p-Amemiya norms ($1 \le p \le \infty$) in Orlicz spaces. After several years, the first paper constituting some basic and crutial results allowing further research and containing the complete characterization of rotundity and extreme points in Orlicz spaces equipped with the p-Amemiya norms, $1 \le p \le \infty$, was written by Cui et al. (see [11]). Since that time, an intensive development of research connected with Orlicz and Musielak–Orlicz spaces equipped with the p-Amemiya norm has taken place, many important results broaden the knowledge about the geometry of these spaces were obtained (see [6,7,9–15,17,23,24,26,33]) and some open questions were put (see [45]).

Up to the end of this section, let $p \in [1, \infty]$. Denoting by p_+ the right-side derivative of the function Φ on $[0, b(\Phi))$ and putting $p_+(b(\Phi)) = \lim_{u \to b(\Phi)^-} p_+(u)$, let us define the function $\alpha_p : L^{\Phi,p} \to [-1, \infty]$ by

$$\alpha_p(x) = \begin{cases} I_\Phi^{p-1}(x) I_\Psi(p_+(|x|)) - 1, & \text{if } 1 \le p < \infty, \\ -1, & \text{if } p = \infty \wedge I_\Phi(x) \le 1, \\ I_\Psi(p_+(|x|)), & \text{if } p = \infty \wedge I_\Phi(x) > 1, \end{cases}$$

and the functions $k_p^* : L^{\Phi,p} \to [0, \infty)$, $k_p^{**} : L^{\Phi,p} \to (0, \infty]$ by

$$\begin{aligned} k_p^*(x) &= \inf\{k \ge 0 : \alpha_p(kx) \ge 0\} \quad (\text{with } \inf \emptyset = \infty), \\ k_p^{**}(x) &= \sup\{k \ge 0 : \alpha_p(kx) \le 0\}. \end{aligned}$$

It is obvious that $k_p^*(x) \le k_p^{**}(x)$ for every $1 \le p \le \infty$ and $x \in L^{\Phi,p}$.

Denote by $K_p(x)$ the set of all $k \in (0, \infty)$ which are between $k_p^*(x)$ and $k_p^{**}(x)$, i.e., $K_p(x) = \{0 < k < \infty : k_p^*(x) \le k \le k_p^{**}(x)\}$. Note that $K_p(x) = \emptyset$ if and only if $k_p^*(x) = k_p^{**}(x) = \infty$. Moreover, the p-Amemiya norm $\|x\|_{\Phi,p}$, $x \ne 0$, is attained at every point $k \in [k_p^*(x), k_p^{**}(x))$ provided $k_p^*(x) < \infty$ and at every point $k \in [k_p^*(x), k_p^{**}(x)]$ provided $k_p^{**}(x) < \infty$ (see [11]). Recall also that an Orlicz function Φ is said to be k_p^*-finite (respectively k_p^{**}-finite) provided $k_p^*(x) < \infty$ (resp. $k_p^{**}(x) < \infty$) for every $x \in L^{\Phi,p}$. An Orlicz function Φ is said to be k_p-unique, if $0 < k_p^*(x) = k_p^{**}(x) < \infty$ for all $x \in L^{\Phi,p} \setminus \{0\}$. Evidently, Φ is k_p-unique if and only if card $K_p(x) = 1$ for every $x \in L^{\Phi,p} \setminus \{0\}$.

2 Auxiliary results

Let us first note that although the main theorem is formulated for the Orlicz spaces generated by N-functions only, we will present some auxiliary results in they original and often wider form, i.e. formulated for Orlicz spaces generated by an arbitrary Orlicz function.

Lemma 1 (See [3], Proposition 1.4) *Let Φ be a strictly convex N-function. Then:*

(1) *Φ is uniformly convex on any bounded interval.*
(2) *For any $K > 0$, $\varepsilon > 0$ and $[a, b] \subset (0, 1)$, there exists $\delta > 0$ such that*

$$\Phi(\alpha u + (1 - \alpha) v) \le (1 - \delta)[\alpha \Phi(u) + (1 - \alpha) \Phi(v))]$$

holds for all $\alpha \in [a, b]$ and $u, v \in \mathbb{R}$ satisfying $|u|, |v| \le K$ and $|u - v| \ge \varepsilon$.

Theorem 1 (Ergoroff's theorem, see [22]) *If f_n and f are measurable and almost everywhere finite in Ω, $\mu(\Omega) < \infty$ and $f_n(t) \to f(t)$ a.e. in Ω, then for every number $\varepsilon > 0$ there exists a set $A \in \Sigma$ such that $\mu(A) < \varepsilon$ and $f_n(t) \to f(t)$ uniformly in $\Omega \setminus A$.*

Theorem 2 (See [11], Theorem 6.2) *The Orlicz space $L^{\Phi,p} = (L^{\Phi}, \|.\|_{\Phi,p})$ is rotund if and only if*

1. *Φ is k_p-unique and*
2. *Φ is strictly convex on $(-b(\Phi), b(\Phi))$ and*
3. *(a) $1 \leq p < \infty$ or*
   (b) $p = \infty$ and $\Phi \in \Delta_2(\mu)$.

Corollary 1 (See [11], Corollary 4.7) *Every Orlicz function Φ that is strictly convex on $(-b(\Phi), b(\Phi))$ is k_p-unique for all $1 < p \leq \infty$. Moreover, if $b(\Phi) = \infty$ and Φ does not have an asymptote at ∞, then Φ is k_p-unique for all $1 \leq p \leq \infty$.*

Lemma 2 (See [15], Lemma 2.7) *Let $1 \leq p \leq \infty$. Then $I_\Phi(x) \leq \|x\|_{\Phi,p}$ for all $x \in L^{\Phi,p}$ with $\|x\|_{\Phi,p} \leq 1$.*

Lemma 3 (See [15], Lemma 4.2) *Let $1 \leq p \leq \infty$ and let Φ be an Orlicz function. In the case $p = 1$ assume, additionally, that Φ is k_1^*-finite. If the sequence $(\|x_n\|_{\Phi,p})_{n=1}^{\infty}$ is bounded, $k_n \in K_p(x_n)$ and $k_n \to \infty$, then $x_n \xrightarrow{\mu} 0$ on Ω.*

Lemma 4 (See [44], Lemma 6 or [3], Lemma 2.35) *Let Φ be an N-function. Then, for a given $\sigma \in \left(0, \frac{1}{2}\right)$, $\varepsilon > 0$ and $\gamma > 0$, there exists $\delta > 0$ such that for any $u > 0$ and any $\lambda \in \left[\sigma, \frac{1}{2}\right]$ satisfying*

$$\lambda\Phi((1+\varepsilon)u) + (1-\lambda)\Phi(u) \leq (1+\delta)[\Phi(\lambda(1+\varepsilon)u + (1-\lambda)u)],$$

there exists $\tau \in [(1+\lambda\varepsilon)u, (1+\varepsilon)u]$ such that

$$p_+(\tau) \leq (1+\gamma)p_+\left(\frac{1+\sigma\varepsilon}{1+2\sigma\varepsilon}\tau\right).$$

Recall that in 1984 Kamińska (see [27]) gave the characterization of the uniform rotundity in every direction for Musielak–Orlicz spaces endowed with the Luxemburg norm over a non-atomic measure space, which was next generalized to the Musielak–Orlicz spaces of Bochner type (see [28]). The theorem presented below can be easily deduced from the result contained in [27]. Let us also note that another proof of the characterization of the URED property for Orlicz spaces equipped with the Luxemburg norm was given by Hudzik in [8].

Theorem 3 (See [27] or [31], Chapter 12) *Let (Ω, Σ, μ) be a non-atomic complete and $\sigma-$finite measure space and $\Phi : \mathbb{R} \to \mathbb{R}_+$ be an even, continuous, convex and vanishing only at zero function. Then the Orlicz space $(L^{\Phi}, \|.\|_{\Phi})$ equipped with the Luxemburg norm is uniformly rotund in every direction if and only if Φ is strictly convex and Φ satisfies the $\Delta_2(\mu)$-condition.*

3 Results

Denote $\ell_2^p = (\mathbb{R}^2, \|.\|_p)$, where the norm $\|.\|_p$ is defined as $\|x\|_p = (|x_1|^p + |x_2|^p)^{1/p}$ for any $x = (x_1, x_2) \in \mathbb{R}^2$ and any $p \in [1, \infty)$, and $\|x\|_\infty = \max\{|x_1|, |x_2|\}$, whenever $p = \infty$. Then, the p-Amemiya norm $(1 \leq p < \infty)$ in the Orlicz space $L^{\Phi,p}$ can be expressed by the help of the norm $\|.\|_p$ in the following way:

$$\|x\|_{\Phi,p} = \inf_{k>0} \|(1, I_\Phi(kx))\|_p.$$

For $u \in (L^{\Phi}, \|.\|_{\Phi,p})$ and $v \in (L^{\Psi}, \|.\|_{\Psi,q})$, where $p, q \in [1, \infty]$ are such that $\frac{1}{p}+\frac{1}{q} = 1$, let $\langle v, u\rangle := \int_{\Omega} u(t)v(t)dt$.

Let us start from presenting some lemmas. The first one is formulated for the Orlicz spaces generated by N-functions and endowed with the p-Amemiya norm but only for $p \in (1, \infty)$ (for the case $p = 1$, we refer to [44] or [3]).

Lemma 5 *Let Φ be an N-function, Ψ be its complementary in the sense of Young function and let $p \in (1, \infty)$ and $q \in (1, \infty)$ be such numbers that $\frac{1}{p} + \frac{1}{q} = 1$. If $x_n, y_n \in B(L^{\Phi,p})$ and $v_n \in B(L^{\Psi,q})$ satisfy $\langle v_n, x_n + y_n\rangle \to 2$ as $n \to \infty$, then for any $\Omega_n \in \Sigma$, $k_n \in K_p(x_n)$ and $h_n \in K_p(y_n)$, we have that $\lim_{n\to\infty} \int_{\Omega}[k_n x_n(t) - h_n y_n(t)]v_n(t)dt = \lim_{n\to\infty} \left[\|k_n x_n\|_{\Phi,p} - \|h_n y_n\|_{\Phi,p}\right] = \lim_{n\to\infty} \left[\left(1 + I_{\Phi}^{p}(k_n x_n)\right)^{\frac{1}{p}} - \left(1 + I_{\Phi}^{p}(h_n y_n)\right)^{\frac{1}{p}}\right]$ and*

$$\lim_{n\to\infty} \int_{\Omega_n} [k_n x_n(t) - h_n y_n(t)]v_n(t)dt \le \lim_{n\to\infty} |I_{\Phi}(k_n x_n \chi_{\Omega_n}) - I_{\Phi}(h_n y_n \chi_{\Omega_n})| \tag{3.1}$$

hold provided that the limits exist and $\{\max\{k_n, h_n\}\}_n$ is bounded.

Proof Let $p \in (1, \infty)$. By the assumption, we get that $\langle v_n, x_n\rangle \to 1$ and $\langle v_n, y_n\rangle \to 1$ as $n \to \infty$, so by the Hölder inequality

$$\begin{aligned} 1 \leftarrow \langle v_n, x_n\rangle &= \int_{\Omega} v_n(t)x_n(t)dt \le \|v_n\|_{\Psi,q} \cdot \|x_n\|_{\Phi,p} \le \|x_n\|_{\Phi,p} \\ &= \frac{1}{k_n}(1 + I_{\Phi}^{p}(k_n x_n))^{\frac{1}{p}} \le 1, \end{aligned}$$

whence

$$\left(1 + I_{\Phi}^{p}(k_n x_n)\right)^{\frac{1}{p}} - \int_{\Omega} k_n x_n(t)v_n(t)dt \to 0 \tag{3.2}$$

as $n \to \infty$. In a similar way we can prove that

$$\left(1 + I_{\Phi}^{p}(h_n y_n)\right)^{\frac{1}{p}} - \int_{\Omega} h_n y_n(t)v_n(t)dt \to 0 \tag{3.3}$$

as $n \to \infty$. Formulas (3.2) and (3.3) yield

$$\lim_{n\to\infty} \int_{\Omega} [k_n x_n(t) - h_n y_n(t)]v_n(t)dt = \lim_{n\to\infty} \left[\left(1 + I_{\Phi}^{p}(k_n x_n)\right)^{\frac{1}{p}} - \left(1 + I_{\Phi}^{p}(h_n y_n)\right)^{\frac{1}{p}}\right],$$

whence

$$\begin{aligned} \lim_{n\to\infty} \int_{\Omega} [k_n x_n(t) - h_n y_n(t)]v_n(t)dt &= \lim_{n\to\infty} \left[\|k_n x_n\|_{\Phi,p} - \|h_n y_n\|_{\Phi,p}\right] \\ &\le \lim_{n\to\infty} \left|\|(1, I_{\Phi}(k_n x_n))\|_p - \|(1, I_{\Phi}(h_n y_n))\|_p\right| \\ &\le \lim_{n\to\infty} \|(0, I_{\Phi}(k_n x_n) - I_{\Phi}(h_n y_n))\|_p \\ &= \lim_{n\to\infty} |I_{\Phi}(k_n x_n) - I_{\Phi}(h_n y_n)|, \end{aligned}$$

so for any $\Omega_n \in \Sigma$, we obtain

$$\lim_{n\to\infty} \int_{\Omega_n} [k_n x_n(t) - h_n y_n(t)] v_n(t) dt \le \lim_{n\to\infty} |I_\Phi(k_n x_n \chi_{\Omega_n}) - I_\Phi(h_n y_n \chi_{\Omega_n})|$$

and the proof is finished. □

Let us now present the more general result rather then this presented in [44], Lemma 4 or in [3], Lemma 2.26.

Lemma 6 *Let Φ be a strictly convex N-function, let $p \in [1, \infty)$, $x_n, y_n \in B(L^{\Phi,p})$, $\|x_n + y_n\|_{\Phi,p} \to 2$ as $n \to \infty$, $k_n \in K_p(x_n)$ and $h_n \in K_p(y_n)$. Then $b = \sup_n \max\{k_n, h_n\} < \infty$ implies $k_n x_n - h_n y_n \to 0$ in measure.*

Proof Recall first that an Orlicz function Φ which vanishes only at zero and takes only finite values has the following property:

$$\Phi(|u| - |v|) \le |\Phi(2u) - \Phi(2v)| \text{ for all } u, v \in \mathbb{R}. \tag{3.4}$$

Notice that for any $p \in [1, \infty)$, the function $f(u) := s_p(|u|) - 1$, $u \in \mathbb{R}$, is an Orlicz function, so by property (3.4) applied to the f, i.e. $f(\frac{1}{2}|u| - \frac{1}{2}|v|) \le |f(u) - f(v)|$ for all $u, v \in \mathbb{R}$, as well as by the convexity of the function $s_p(.)$, we get

$$\begin{aligned}
&0 \leftarrow 2 - \|x_n + y_n\|_{\Phi,p} \ge \|x_n\|_{\Phi,p} + \|y_n\|_{\Phi,p} - \|x_n + y_n\|_{\Phi,p} \\
&\ge \tfrac{1}{k_n} s_{\Phi,p}(k_n x_n) + \tfrac{1}{h_n} s_{\Phi,p}(h_n y_n) - \tfrac{1}{\frac{k_n h_n}{k_n+h_n}} s_{\Phi,p}\left(\tfrac{k_n h_n}{k_n+h_n}(x_n + y_n)\right) \\
&= \tfrac{k_n+h_n}{k_n h_n}\left\{\tfrac{h_n s_p(I_\Phi(k_n x_n))}{k_n+h_n} + \tfrac{k_n s_p(I_\Phi(h_n y_n))}{k_n+h_n} - s_p\left(I_\Phi\left(\tfrac{k_n h_n}{k_n+h_n}(x_n + y_n)\right)\right)\right\} \\
&\ge \tfrac{k_n+h_n}{k_n h_n}\left\{s_p\left(\tfrac{h_n I_\Phi(k_n x_n)}{k_n+h_n} + \tfrac{k_n I_\Phi(h_n y_n)}{k_n+h_n}\right) - s_p\left(I_\Phi\left(\tfrac{k_n h_n}{k_n+h_n}(x_n + y_n)\right)\right)\right\} \\
&= \tfrac{k_n+h_n}{k_n h_n}\left\{s_p\left(\tfrac{h_n I_\Phi(k_n x_n)}{k_n+h_n} + \tfrac{k_n I_\Phi(h_n y_n)}{k_n+h_n}\right) - 1 - \left[s_p\left(I_\Phi\left(\tfrac{k_n h_n}{k_n+h_n}(x_n + y_n)\right)\right) - 1\right]\right\} \\
&\ge \tfrac{k_n+h_n}{k_n h_n}\left\{s_p\left[\tfrac{1}{2}\left(\tfrac{h_n I_\Phi(k_n x_n)}{k_n+h_n} + \tfrac{k_n I_\Phi(h_n y_n)}{k_n+h_n} - I_\Phi\left(\tfrac{k_n h_n}{k_n+h_n}(x_n + y_n)\right)\right)\right] - 1\right\} \ge 0,
\end{aligned}$$

so

$$s_p\left[\frac{1}{2}\left(\frac{h_n I_\Phi(k_n x_n)}{k_n + h_n} + \frac{k_n I_\Phi(h_n y_n)}{k_n + h_n} - I_\Phi\left(\frac{k_n h_n}{k_n + h_n}(x_n + y_n)\right)\right)\right] \to 1$$

as $n \to \infty$. Since $s_p(u_n) \to 1$ if and only if $u_n \to 0$ as $n \to \infty$, we get that

$$\frac{h_n I_\Phi(k_n x_n)}{k_n + h_n} + \frac{k_n I_\Phi(h_n y_n)}{k_n + h_n} - I_\Phi\left(\frac{k_n h_n}{k_n + h_n}(x_n + y_n)\right) \to 0 \tag{3.5}$$

as $n \to \infty$.

If the sequence $(k_n x_n - h_n y_n)_{n=1}^\infty$ does not converge to zero in measure, then without loss of generality, we may assume that $\mu(E_n) > \varepsilon$ for any $n \in \mathbb{N}$, where $E_n = \{t \in \Omega : |k_n x_n(t) - h_n y_n(t)| \ge \sigma\}$ and σ, ε are fixed and positive numbers.

Since the Orlicz space $L^{\Phi,p}$ is a symmetric space, the norm $\|\chi_F\|_{\Phi,p}$ does not depend on the set F but only on its measure, so let us denote $k = \frac{1}{\|\chi_F\|_{\Phi,p}}$, where $F \in \Sigma$ and $\mu(F) = \frac{\varepsilon}{4}$. Define the sets

$$A_n = \{t \in \Omega : |x_n(t)| > k\}, B_n = \{t \in \Omega : |y_n(t)| > k\}.$$

Then $1 \geq \|x_n\|_{\Phi,p} \geq \|x_n \chi_{A_n}\|_{\Phi,p} > k\|\chi_{A_n}\|_{\Phi,p}$, whence $\|\chi_{A_n}\|_{\Phi,p} < \frac{1}{k}$, which implies $\mu(A_n) < \frac{\varepsilon}{4}$. In a similar way we get that $\mu(B_n) < \frac{\varepsilon}{4}$. By Lemma 1, there exists $\delta > 0$ such that

$$\Phi(\alpha u + (1-\alpha)v) \leq (1-\delta)[\alpha\Phi(u) + (1-\alpha)\Phi(v)]$$

holds for all $\alpha \in \left[\frac{1}{1+b}, \frac{b}{1+b}\right] \in (0,1)$ and any $u, v \in \mathbb{R}$ with $|u| \leq bk$, $|v| \leq bk$ and $|u-v| \geq \sigma$. Since $\frac{k_n}{k_n+h_n}, \frac{h_n}{k_n+h_n} \in \left[\frac{1}{1+b}, \frac{b}{1+b}\right]$ for all $t \in F_n := E_n \backslash \{A_n \cup B_n\}$, we obtain that

$$\Phi\left(\frac{k_n h_n\,(x_n(t) + y_n(t))}{k_n + h_n}\right) \leq (1-\delta)\left[\frac{h_n \Phi(k_n x_n(t))}{k_n + h_n} + \frac{k_n \Phi(h_n y_n(t))}{k_n + h_n}\right]. \tag{3.6}$$

By the convexity of Φ and condition (3.5), we get that

$$\begin{aligned}
0 \leftarrow\ & \frac{h_n I_\Phi(k_n x_n)}{k_n + h_n} + \frac{k_n I_\Phi(h_n y_n)}{k_n + h_n} - I_\Phi\left(\frac{k_n h_n}{k_n + h_n}(x_n + y_n)\right) \\
\geq\ & \frac{h_n I_\Phi(k_n x_n \chi_{F_n})}{k_n + h_n} + \frac{k_n I_\Phi(h_n y_n \chi_{F_n})}{k_n + h_n} - I_\Phi\left(\frac{k_n h_n}{k_n + h_n}(x_n + y_n)\chi_{F_n}\right) \\
\geq\ & \frac{h_n}{k_n + h_n} I_\Phi(k_n x_n \chi_{F_n}) + \frac{k_n}{k_n + h_n} I_\Phi(h_n y_n \chi_{F_n}) \\
& - (1-\delta)\left[\frac{h_n}{k_n + h_n} I_\Phi(k_n x_n \chi_{F_n}) + \frac{k_n}{k_n + h_n} I_\Phi(h_n y_n \chi_{F_n})\right] \quad \text{by (3.6)} \\
=\ & \delta\left[\frac{h_n}{k_n + h_n} I_\Phi(k_n x_n \chi_{F_n}) + \frac{k_n}{k_n + h_n} I_\Phi(h_n y_n \chi_{F_n})\right] \\
\geq\ & \delta\left[\frac{1}{2b} I_\Phi(k_n x_n \chi_{F_n}) + \frac{1}{2b} I_\Phi(h_n y_n \chi_{F_n})\right] \\
=\ & \delta\left(\frac{1}{2b} I_\Phi(k_n x_n \chi_{F_n}) + \frac{1}{2b} I_\Phi(-h_n y_n \chi_{F_n})\right) \\
=\ & \frac{\delta}{b}\left(\frac{I_\Phi(k_n x_n \chi_{F_n}) + I_\Phi(-h_n y_n \chi_{F_n})}{2}\right) \\
\geq\ & \frac{\delta}{b} I_\Phi\left(\frac{(k_n x_n - h_n y_n)\chi_{F_n}}{2}\right) \geq \frac{\delta}{b} \int\limits_{E_n \backslash \{A_n \cup B_n\}} \Phi\left(\frac{\sigma}{2}\right) dt \\
\geq\ & \frac{\delta}{b} \Phi\left(\frac{\sigma}{2}\right) \frac{\varepsilon}{2},
\end{aligned}$$

where $\sigma, \varepsilon > 0$ were fixed, a contradiction. □

Lemma 7 *Let $(X, \|.\|)$ be a Banach space. If $x_n, y_n \in B(X)$ for any $n \in \mathbb{N}$ and $\left\|\frac{x_n+y_n}{2}\right\| \to 1$, then $\|\alpha x_n + (1-\alpha)y_n\| \to 1$ for any $\alpha \in (0,1)$ as $n \to \infty$.*

Proof Assume that $x_n, y_n \in B(X)$ for any $n \in \mathbb{N}$ and $\left\|\frac{x_n+y_n}{2}\right\| \to 1$, but there exists $\alpha \in (0,1)$ such that $\|\alpha x_n + (1-\alpha)y_n\| \nrightarrow 1$ as $n \to \infty$. Then, we can assume that there exists $\varepsilon \in (0,1)$ such that $\|\alpha x_n + (1-\alpha)y_n\| \leq 1-\varepsilon$. Denote $z_{n,\alpha} = \alpha x_n + (1-\alpha)y_n$. Then, we can find:
case a) either $\beta \in (0,1)$ such that $\frac{x_n+y_n}{2} = \beta z_{n,\alpha} + (1-\beta)y_n$ or case b) $\widetilde{\beta} \in (0,1)$ such that $\frac{x_n+y_n}{2} = \widetilde{\beta} x_n + (1-\widetilde{\beta})z_{n,\alpha}$, where $z_{n,\alpha}$ is generated either by some $\alpha \in \left(\frac{1}{2}, 1\right)$ or some $\alpha \in \left(0, \frac{1}{2}\right)$, respectively. Since the proof for case b) is similar, it is omitted. Consequently

$$1 \leftarrow \left\| \frac{x_n + y_n}{2} \right\| \leq \beta \|z_{n,\alpha}\| + (1-\beta)\|y_n\| \leq \beta(1-\varepsilon) + 1 - \beta = 1 - \beta\varepsilon,$$

a contradiction. □

Let Φ be an N-function, Ψ be its complementary (in the sense of Young) function, let p_+ be the right derivative of Φ and q_+ be the right-inverse function of p_+, that is,

$$q_+(s) = \sup\{t : p_+(t) \leq s\} = \inf\{t : p_+(t) > s\}.$$

Now, we will present the sufficient conditions for the uniform rotundity in every direction in Orlicz spaces equipped with the p-Amemiya norm, where $p \in [1, \infty)$. Although Theorem 4 is formulated for Orlicz spaces endowed with the p-Amemiya norm, $p \in [1, \infty)$, generated by N-functions and built over a non-atomic finite measure space, we will present its proof only in case of the Orlicz spaces equipped with the p-Amemiya norm, $p \in (1, \infty)$, because in case of the Orlicz spaces endowed with the Orlicz norm, the proof can be found in [44] or [3].

Theorem 4 *Let (Ω, Σ, μ) be a non-atomic finite and complete measure space and let $p \in [1, \infty)$. If*

(a) Φ is a strictly convex N-function,
(b) for any $u_0 > 0$, $\varepsilon > 0$ and $\widetilde{\varepsilon} > 0$ there exist $\gamma > 0$ and $A > 0$ such that for any $u \geq u_0$, if $p_+((1+\varepsilon)u) \leq (1+\gamma)p_+(u)$, then $p_+(u) \leq Ap_+(\widetilde{\varepsilon}u)$,

then the Orlicz space $(L^{\Phi}, \|.\|_{\Phi,p})$ equipped with the p-Amemiya norm and generated by an N-function Φ is uniformly rotund in every direction.

Proof In the whole proof we will assume that $p \in (1, \infty)$.

Assume that $L^{\Phi,p}$ satisfies conditions a) and b) but $L^{\Phi,p}$ is not uniformly rotund in every direction. Then (see [16]), there exist $x_n \in B(L^{\Phi,p})$, $z \in L^{\Phi,p}$ such that $x_n + z \in B(L^{\Phi,p})$ for any $n \in \mathbb{N}$, $\|x_n\|_{\Phi,p} \to 1$, $\|x_n + z\|_{\Phi,p} \to 1$ and $\|2x_n + z\|_{\Phi,p} \to 2$ as $n \to \infty$ but $z \neq 0$. Since Φ is a strictly convex N-function, by Corollary 1, Φ is k_p-unique. Let $K_p(x_n) = \{k_n\}$. Without loss of generality we can assume that the sequence $(x_n)_{n=1}^{\infty}$ does not converge to 0 in measure. Indeed, if $x_n \xrightarrow{\mu} 0$, then we replace x_n and z by $x_n' = x_n + \frac{z}{4}$ and $z' = \frac{z}{2}$, respectively. By virtue of Lemma 7 and by the definitions of x_n' and z', we get that $\|x_n'\|_{\Phi,p} \to 1$, $\|x_n' + z'\|_{\Phi,p} = \|x_n + \frac{3}{4}z\|_{\Phi,p} \to 1$ and $\|2x_n' + z'\|_{\Phi,p} = \|2x_n + z\|_{\Phi,p} \to 2$ as $n \to \infty$ but $z' \neq 0$. Obviously $x_n' \overset{\mu}{\nrightarrow} 0$. By Lemma 3, the sequence $(k_n)_{n=1}^{\infty}$ is bounded. Passing to a subsequence, if necessary, we may assume that $k_n \to k$ $(k \geq 1)$ and $1 \leq k_n \leq 2k$ for all $n \in \mathbb{N}$. Moreover, we can assume that $2k\|z\|_{\Phi,p} \leq 1$ and that $(x_n + z)_{n=1}^{\infty}$ does not converge to zero in measure (otherwise, instead of z we take βz for some $\beta > 0$).

Let $y_n = x_n + z$ and $K_p(y_n) = \{h_n\}$. We can also assume that $h_n \to h$ as $n \to \infty$ $(h \geq 1)$. Then, by Lemma 6, $k_n x_n - h_n y_n \to 0$ in measure and we conclude that $k \neq h$; otherwise we would have that $k_n x_n - h_n y_n = k_n x_n - h_n x_n - h_n z = (k_n - h_n)x_n - h_n z \xrightarrow{\mu} 0$, so if $k = h$, then $z \xrightarrow{\mu} 0$, a contradiction. In what follows, we will consider only the case when $k > h$ (the other case is similar). Without loss of generality, passing to a subsequence if necessary, we may assume that $k_n > h_n$ for any $n \in \mathbb{N}$ and $k_n x_n - h_n y_n \to 0$ μ-a.e. on Ω. Set $\lambda_n = \frac{h_n}{k_n + h_n}$. Then $\lambda_n \to \frac{h}{k+h}$ as $n \to \infty$, so $\sigma \leq \lambda_n \leq \frac{1}{2}$ for some $\sigma > 0$.

Since $z \neq 0$, there is $c > 0$ such that $\mu(E) = d > 0$, where $E = E_c := \{t \in \Omega : |z(t)| > c\}$. Let $\varepsilon > 0$ be arbitrary. By the assumption b) there exist $A > 0$ and $\gamma \in (0, 1)$ such that whenever $\tau \geq \varepsilon$ and $p_+(\tau) \leq (1+\gamma)p_+\left(\frac{1+\sigma\varepsilon}{1+2\sigma\varepsilon}\tau\right)$, then

$$p_+\left(\frac{1+\sigma\varepsilon}{1+2\sigma\varepsilon}\tau\right) \le Ap_+\left(\frac{\varepsilon}{2k}\cdot\frac{1+\sigma\varepsilon}{1+2\sigma\varepsilon}\tau\right) \le Ap_+\left(\frac{\varepsilon}{2k}\tau\right). \tag{3.7}$$

For such σ, ε, γ, by Lemma 4, it follows that there exists $\delta > 0$ such that for any $u > 0$ and $\lambda \in \left[\sigma, \frac{1}{2}\right]$ the inequality

$$\lambda\Phi((1+\varepsilon)u) + (1-\lambda)\Phi(u) \le (1+\delta)[\Phi(\lambda(1+\varepsilon)u + (1-\lambda)u)]$$

implies the existence of $\tau \in [u + \lambda\varepsilon u, u + \varepsilon u]$ satisfying the inequality

$$p_+(\tau) \le (1+\gamma)p_+\left(\frac{1+\sigma\varepsilon}{1+2\sigma\varepsilon}\tau\right).$$

Since $\|2kz\|_{\Phi,p} \le 1$, by Lemma 2, we get that $I_\Phi(2kz) \le 1$ and $\langle |z|, p_+(|z|)\rangle = I_\Phi(z) + I_\Psi(p_+(|z|)) \le 2$. Therefore, by absolute continuity of the integral, there exists $\widetilde{\alpha} \in \left(0, \frac{d}{2}\right)$ such that if $B \subset \Omega$ and $\mu(B) \le \widetilde{\alpha}$, then

$$\int_B |z(t)|p_+(|z(t)|)dt \le \frac{\varepsilon^2}{kA} \text{ and } I_\Phi(2kz\chi_B) < \varepsilon. \tag{3.8}$$

Since $k_n x_n - h_n y_n \to 0$ μ-a.e. on Ω, by Theorem 1, passing to subsequence if necessary, we can find $F \in \Sigma$ such that $\mu(\Omega\backslash F) < \widetilde{\alpha}$ and $k_n x_n - h_n y_n \to 0$ uniformly on F, and $|z| \le a_1$ on F for some $a_1 > 0$. Notice that $\mu(E \cap F) > \frac{d}{2}$. Indeed,

$$d = \mu(E) = \mu(E\cap F) + \mu(E\backslash F) \le \mu(E\cap F) + \mu(\Omega\backslash F) < \mu(E\cap F) + \frac{d}{2}.$$

Hence $\mu(E\cap F) > \frac{d}{2}$ and we deduce that

$$I_\Phi\left(\frac{h}{k-h}z\chi_F\right) \ge I_\Phi\left(\frac{h}{k-h}z\chi_{F\cap E}\right) \ge \frac{d}{2}\Phi\left(\frac{hc}{k-h}\right). \tag{3.9}$$

For every $n \in \mathbb{N}$, let us divide Ω into the following sets:

$$\begin{aligned}
A_n &= \{t \in \Omega\backslash F : x_n(t)y_n(t) < 0\},\\
I_n &= \{t \in \Omega\backslash\{F\cup A_n\} :\ \max\{|k_n x_n(t)|, |h_n y_n(t)|\} < \varepsilon\},\\
J_n &= \{t \in \Omega\backslash\{F\cup A_n\cup I_n\} : |k_n x_n(t) - h_n y_n(t)| \le \varepsilon\max\{|k_n x_n(t)|, |h_n y_n(t)|\}\},\\
H_n &= \left\{t \in \Omega\backslash\{F\cup A_n\cup I_n\cup J_n\} : (1+\delta)\Phi\left(\frac{k_n h_n}{k_n+h_n}(x_n(t)+y_n(t))\right)\right.\\
&\qquad \left. < \frac{h_n\Phi(k_n x_n(t))}{k_n+h_n} + \frac{k_n\Phi(h_n y_n(t))}{k_n+h_n}\right\},\\
Q_n &= \{t \in \Omega\backslash\{F\cup A_n\cup I_n\cup J_n\cup H_n\} : |z(t)| < \varepsilon|x_n(t)| \text{ or } |x_n(t)| < |y_n(t)|\},\\
T_n &= \Omega\backslash\{F\cup A_n\cup I_n\cup J_n\cup H_n\cup Q_n\}.
\end{aligned}$$

Let us pick $v_n \in B(L^{\Psi,q})$ ($q \in (1,\infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$) such that $v_n(t)[x_n(t) + y_n(t)] \ge 0$ for all $t \in \Omega$ and $\langle v_n, x_n + y_n\rangle \to 2$ as $n \to \infty$. Then, by the linearity of the integral and by the assumptions about x_n and $y_n = x_n + z$, we obtain that $\langle v_n, x_n\rangle \to 1$ and $\langle v_n, y_n\rangle \to 1$ as $n \to \infty$ and, consequently,

$$k - h = \lim_{n\to\infty}(k_n - h_n) = \lim_{n\to\infty}\int_\Omega v_n(t)[k_n x_n(t) - h_n y_n(t)]dt.$$

Now, we will find the upper estimates of the integrals $\int_C |[k_n x_n(t) - h_n y_n(t)]v_n(t)|dt$, where C denotes one of the following sets: F, A_n, I_n, J_n, H_n, Q_n and T_n, respectively.

Since $k_n x_n - h_n y_n \to 0$ uniformly on F, for large n, we obtain that

$$\int_F |[k_n x_n(t) - h_n y_n(t)]v_n(t)|dt < \varepsilon. \tag{3.10}$$

Notice that if $t \in A_n$, then $x_n(t)y_n(t) < 0$, i.e. $x_n(t)[x_n(t) + z(t)] < 0$, yields that $x_n(t)z(t) < 0$ and $|x_n(t)| < |z(t)|$. Therefore, for any $A \in \Sigma$ contained in the set A_n, by (3.8) we obtain that

$$\begin{aligned}\int_A |\Phi(k_n x_n(t)) - \Phi(h_n y_n(t))|dt &= \int_A |\Phi(k_n x_n(t)) - \Phi(h_n(x_n(t) + z(t)))|dt \\ &\le \int_A [\Phi(k_n z(t)) + \Phi(h_n z(t))]dt \\ &\le 2\int_A \Phi(2kz(t))dt < 2\varepsilon.\end{aligned}$$

Hence, as well as by virtue of Lemma 5, for n large enough, we get that

$$\int_{A_n} |[k_n x_n(t) - h_n y_n(t)]v_n(t)|dt \le 2\varepsilon. \tag{3.11}$$

Notice that the Hölder inequality give us that

$$\begin{aligned}&\int_{I_n} |[k_n x_n(t) - h_n y_n(t)]v_n(t)|dt \le \|(k_n x_n - h_n y_n)\chi_{I_n}\|_{\Phi,p} \cdot \|v_n\|_{\Psi,q} \\ &\le 2\varepsilon\|\chi_\Omega\|_{\Phi,p} \le \varepsilon 2^{1+\frac{1}{p}}\|\chi_\Omega\|_\Phi.\end{aligned} \tag{3.12}$$

By the definition of the set J_n and by the conditions that $1 \leftarrow \langle v_n, x_n\rangle \le 1$ and $1 \leftarrow \langle v_n, y_n\rangle \le 1$, we get that

$$\begin{aligned}\int_{J_n} |[k_n x_n(t) - h_n y_n(t)]v_n(t)|dt &\le \varepsilon \int_{J_n} (|k_n x_n(t)| + |h_n y_n(t)|)\,|v_n(t)|dt \\ &\le \varepsilon(k_n + h_n).\end{aligned} \tag{3.13}$$

For each set $H \in \Sigma$, being a subset of H_n, using the method from the proof of Lemma 6, by continuity of $s_p(.)$ and by the fact that $\frac{k_n h_n}{k_n+h_n} \ge \frac{k_n h_n}{2k_n} \ge \frac{1}{2}$, we conclude that

$$\begin{aligned}0 &\leftarrow \frac{h_n I_\Phi(k_n x_n)}{k_n + h_n} + \frac{k_n I_\Phi(h_n y_n)}{k_n + h_n} - I_\Phi\left(\frac{k_n h_n}{k_n + h_n}(x_n + y_n)\right) \\ &\ge \frac{h_n I_\Phi(k_n x_n \chi_H)}{k_n + h_n} + \frac{k_n I_\Phi(h_n y_n \chi_H)}{k_n + h_n} - I_\Phi\left(\frac{k_n h_n}{k_n + h_n}(x_n + y_n)\chi_H\right) \\ &\ge \frac{1}{2}\left(\frac{I_\Phi(k_n x_n \chi_H)}{k_n} + \frac{I_\Phi(h_n y_n \chi_H)}{h_n}\right) \\ &\quad -\frac{1}{2}\left(1 - \frac{\delta}{1+\delta}\right)\int_H \left[\frac{1}{k_n}\Phi(k_n x_n(t)) + \frac{1}{h_n}\Phi(h_n y_n(t))\right]dt \\ &= \frac{\delta}{2(1+\delta)}\int_H \left[\frac{1}{k_n}\Phi(k_n x_n(t)) + \frac{1}{h_n}\Phi(h_n y_n(t))\right]dt,\end{aligned}$$

whence, for n large enough, we obtain that

$$\int_{H_n} |(k_n x_n(t) - h_n y_n(t))v_n(t)|dt \le \varepsilon. \tag{3.14}$$

By the facts that $k_n x_n - h_n y_n \to 0$ uniformly on F and $y_n = x_n + z$, we conclude that $x_n \to \frac{h}{k-h} z$ uniformly on the set F. Hence, and by inequality (3.9), we get that

$$\frac{1}{k_n} I_\Phi(k_n x_n \chi_F) \ge I_\Phi(x_n \chi_F) \to I_\Phi\left(\frac{h}{k-h} z\chi_F\right) \ge \frac{d}{2}\Phi\left(\frac{hc}{k-h}\right). \tag{3.15}$$

By the assumption $\|x_n\|_{\Phi,p} \to 1$ as $n \to \infty$ and the fact that $k_n \in K_p(x_n)$, we obtain that $\lim_{n\to\infty} \frac{1}{k_n^p}\left(1 + I_\Phi^p(k_n x_n)\right) = 1$. Consequently, $1 + \lim_{n\to\infty} I_\Phi^p(k_n x_n) = \lim_{n\to\infty} k_n^p$, whence

$$\lim_{n\to\infty}\left(k_n^p - I_\Phi^p(k_n x_n)\right) = 1. \tag{3.16}$$

But by the superadditivity of the function $g(u) = u^p$, $p > 1$, and by (3.15), we get that

$$\begin{aligned}
\lim_{n\to\infty} I_\Phi^p(k_n x_n) &= \lim_{n\to\infty}[I_\Phi(k_n x_n \chi_{Q_n}) + I_\Phi(k_n x_n \chi_{\Omega\setminus Q_n})]^p \\
&\ge \lim_{n\to\infty}[I_\Phi^p(k_n x_n \chi_{Q_n}) + I_\Phi^p(k_n x_n \chi_{\Omega\setminus Q_n})] \\
&\ge \lim_{n\to\infty}[I_\Phi^p(k_n x_n \chi_{Q_n}) + I_\Phi^p(k_n x_n \chi_F)] \\
&\ge \lim_{n\to\infty}\left[I_\Phi^p(k_n x_n \chi_{Q_n}) + \left(k_n \frac{d}{2}\Phi\left(\frac{hc}{k-h}\right)\right)^p\right].
\end{aligned}$$

This and (3.16) yield that

$$\begin{aligned}
1 &= \lim_{n\to\infty} k_n^p\left(1 - \left(\frac{1}{k_n} I_\Phi(k_n x_n)\right)^p\right) \\
&\le \lim_{n\to\infty} k_n^p\left(1 - \frac{1}{k_n^p} I_\Phi^p(k_n x_n \chi_{Q_n}) - \left(\frac{d}{2}\Phi\left(\frac{hc}{k-h}\right)\right)^p\right) \\
&< \lim_{n\to\infty} k_n^p\left(1 - \frac{1}{k_n^p} I_\Phi^p(k_n x_n \chi_{Q_n}) - \left(\frac{d}{3}\Phi\left(\frac{hc}{k-h}\right)\right)^p\right),
\end{aligned}$$

whence

$$\lim_{n\to\infty} \frac{1}{k_n^p}\left(1 + I_\Phi^p(k_n x_n \chi_{Q_n})\right) < 1 - \left(\frac{d}{3}\Phi\left(\frac{hc}{k-h}\right)\right)^p.$$

Therefore, for n big enough,

$$\begin{aligned}
\int_{Q_n} |x_n(t) v_n(t)|dt &\le \|x_n \chi_{Q_n}\|_{\Phi,p} \le \frac{1}{k_n}\left(1 + I_\Phi^p(k_n x_n \chi_{Q_n})\right)^{\frac{1}{p}} \qquad (3.17) \\
&< \left(1 - \left(\frac{d}{3}\Phi\left(\frac{hc}{k-h}\right)\right)^p\right)^{\frac{1}{p}}.
\end{aligned}$$

Since $Q_n \subset \Omega\setminus\{F \cup A_n\}$, we have that $x_n(t)y_n(t) = x(t)(x(t) + z(t)) \ge 0$. Since $v_n(t)[x_n(t) + y_n(t)] \ge 0$, then both $x_n(t)v_n(t) \ge 0$ and $y_n(t)v_n(t) \ge 0$ (so $z(t)v_n(t) \ge 0$). Hence, if $|x_n(t)| < |y_n(t)|$, then $x_n(t)z(t) > 0$ and, consequently,

$$\begin{aligned}
v_n(t)[k_n x_n(t) - h_n y_n(t)] &= (k_n - h_n)x_n(t)v_n(t) - h_n z(t)v_n(t) \\
&< (k_n - h_n)x_n(t)v_n(t)
\end{aligned}$$

and if $|z(t)| < \varepsilon|x_n(t)|$, then

$$v_n(t)[k_nx_n(t) - h_ny_n(t)] \leq (k_n - h_n)x_n(t)v_n(t) + \varepsilon h_nx_n(t)v_n(t).$$

Therefore, applying also (3.17) and Lemma 5, for n large enough, we obtain

$$\begin{aligned}\int_{Q_n} (k_nx_n(t) - h_ny_n(t))\, v_n(t)dt &\leq (k_n - h_n + \varepsilon h_n) \int_{Q_n} x_n(t)v_n(t)dt \\ &< (k_n - h_n + \varepsilon h_n)\left[1 - \left(\frac{d}{3}\Phi\left(\frac{hc}{k-h}\right)\right)^p\right]^{\frac{1}{p}}. \end{aligned} \tag{3.18}$$

Noticing that $t \notin Q_n \cup J_n \cup A_n$ whenever $t \in T_n$, we get that

$$\varepsilon h_n|y_n(t)| \leq \varepsilon k_n|x_n(t)| \leq k_n|x_n(t)| - h_n|y_n(t)|,$$

i.e. $\frac{k_n|x_n(t)|}{h_n|y_n(t)|} \geq 1 + \varepsilon$. By virtue of Lemma 4 and by the fact that $t \notin H_n$, i.e.

$$\frac{\lambda_n\Phi(k_nx_n(t)) + (1 - \lambda_n)\Phi(h_ny_n(t))}{\Phi(\lambda_nk_nx_n(t) + (1 - \lambda_n)h_ny_n(t))} \leq 1 + \delta,$$

there exists

$$\tau_n(t) \in [\lambda_nk_n|x_n(t)| + (1 - \lambda_n)h_n|y_n(t)|, k_n|x_n(t)|] \tag{3.19}$$

such that

$$p_+(\tau_n(t)) \leq (1 + \gamma)p_+\left(\frac{1 + \sigma\varepsilon}{1 + 2\sigma\varepsilon}\tau_n(t)\right). \tag{3.20}$$

Noticing that $t \notin I_n \cup Q_n$ implies $\tau_n(t) \geq \lambda_nk_n|x_n(t)| \geq \sigma\varepsilon$, by (3.20) and (3.7), we have

$$p_+(\tau_n(t)) \leq (1 + \gamma)p_+\left(\frac{1 + \sigma\varepsilon}{1 + 2\sigma\varepsilon}\tau_n(t)\right) \leq Ap_+\left(\frac{\varepsilon}{2k}\tau_n(t)\right). \tag{3.21}$$

From $t \notin H_n \cup Q_n$, by (3.19), (3.21) and the fact that $k_n \leq 2k$ for any $n \in \mathbb{N}$, we conclude that

$$\begin{aligned}\lambda_n\Phi(k_nx_n(t)) + (1 - \lambda_n)\Phi(h_ny_n(t)) &\leq (1 + \delta)\Phi(\lambda_nk_nx_n(t) + (1 - \lambda_n)h_ny_n(t)) \\ &\leq (1 + \delta)\Phi(\tau_n(t)) \leq (1 + \delta)\tau_n(t)p_+(\tau_n(t)) \\ &\leq A(1 + \delta)\tau_n(t)p_+\left(\frac{\varepsilon}{2k}\tau_n(t)\right) \\ &\leq A(1 + \delta)k_n|x_n(t)|p_+\left(\frac{\varepsilon}{2k}k_n|x_n(t)|\right) \\ &\leq A(1 + \delta)k_n|x_n(t)|p_+\left(\varepsilon|x_n(t)|\right) \\ &\leq \frac{1}{\varepsilon}Ak(1 + \delta)|z(t)|p_+(|z(t)|).\end{aligned}$$

This and conditions (3.8) (recall that $\mu(\Omega\backslash F) < \widetilde{\alpha} < \frac{d}{2}$) imply

$$\begin{aligned}\int_{T_n} [\lambda_n\Phi(k_nx_n(t)) + (1 - \lambda_n)\Phi(h_ny_n(t))]\, dt &\leq \frac{1}{\varepsilon}Ak(1 + \delta)\int_{\Omega\backslash F} |z(t)|p_+(|z(t)|)dt \\ &\leq (1 + \delta)\varepsilon,\end{aligned}$$

whence

$$\begin{aligned}\frac{1}{k_n + h_n}\int_{T_n} [\Phi(k_nx_n(t)) + \Phi(h_ny_n(t))]\, dt &\leq \int_{T_n} [\lambda_n\Phi(k_nx_n(t)) + (1 - \lambda_n)\Phi(h_ny_n(t))]dt \\ &\leq (1 + \delta)\varepsilon,\end{aligned}$$

i.e.

$$\int_{T_n} [\Phi(k_n x_n(t)) + \Phi(h_n y_n(t))]\,dt \le (k_n + h_n)(1+\delta)\varepsilon.$$

Hence, by Lemma 5, for all large n, we obtain that

$$\begin{aligned}\int_{T_n} |v_n(t)(k_n x_n(t) - h_n y_n(t))|dt &\le \int_{T_n} [\Phi(k_n x_n(t)) + \Phi(h_n y_n(t))]\,dt \\ &\le (k_n + h_n)(1+\delta)\varepsilon. \end{aligned} \tag{3.22}$$

Denoting $v_n(t)(k_n x_n(t) - h_n y_n(t))$ shortly as $w_n(t)$, we get that

$$\begin{aligned} k - h &= \lim_{n\to\infty}(k_n - h_n) = \lim_{n\to\infty} \int_{\Omega} v_n(t)(k_n x_n(t) - h_n y_n(t))dt \\ &= \lim_{n\to\infty}\left(\int_F w_n(t)dt + \int_{A_n} w_n(t)dt + \int_{I_n} w_n(t)dt + \int_{J_n} w_n(t)dt + \int_{H_n} w_n(t)dt \right. \\ &\quad \left. + \int_{Q_n} w_n(t)dt + \int_{T_n} w_n(t)dt \right), \end{aligned}$$

whence, as well as by inequalities (3.10), (3.11), (3.12), (3.13), (3.14), (3.18) and (3.22) and by the arbitrariness of $\varepsilon > 0$, we conclude that

$$k - h \le (k-h)\left[1 - \left(\frac{d}{3}\Phi\left(\frac{hc}{k-h}\right)\right)^p\right]^{\frac{1}{p}},$$

a contradiction, which shows that $z = 0$ a.e. in Ω.

□

Corollary 2 *If Φ is a strictly convex N-function satisfying the $\Delta_2(\infty)-$condition, then the Orlicz space $(L^{\Phi}, \|.\|_{\Phi,p})$, $p \in [1,\infty)$ equipped with the p-Amemiya norm and built over a non-atomic finite measure space is uniformly rotund in every direction.*

Proof Assume that $\Phi \in \Delta_2(\infty)$. Then for any $a > 1$ we can find $K > a$ such that $\Phi(2au) \le K\Phi(u)$ for all u large enough, whence

$$ap_+(au) \le \frac{\Phi(2au)}{u} \le K\frac{\Phi(u)}{u} \le Kp_+(u)$$

for all u large enough, so by strict convexity of N-function Φ and by Theorem 4, we conclude that Orlicz space with the p-Amemiya norm is uniformly rotund in every direction. □

At the end, let us recall that the uniform rotundity in every direction plays an important role in the fixed point theory. Namely, every uniformly rotund in every direction Banach space $(X, \|.\|)$ has normal structure (see for example [8]). Consequently, it has the weak fixed point property (we refer, for instance, to [8], [21] or [31] for the suitable definitions and results). By this, and by Theorem 4, we get the following

Corollary 3 *Let (Ω, Σ, μ) be a non-atomic finite and complete measure space and let $p \in [1, \infty)$. Then the Orlicz space $(L^{\Phi}, \|.\|_{\Phi,p})$ equipped with the p-Amemiya norm and generated by a strictly convex N-function Φ satisfying the $\Delta_2(\infty)$-condition has the weak fixed point property.*

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