Collect. Math. (2018) 69:359–375
https://doi.org/10.1007/s13348-017-0209-3

The strong Lefschetz property of monomial complete intersections in two variables

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Received: 2 June 2017 / Accepted: 29 October 2017 / Published online: 9 November 2017

Abstract In this paper we classify the monomial complete intersections, in two variables, and of positive characteristic, which has the strong Lefschetz property. Together with known results, this gives a complete classification of the monomial complete intersections with the strong Lefschetz property.

1 Background

A graded algebra $A = \bigoplus_{i \geq 0} A_i$ is said to have the *strong Lefschetz property* (SLP) if there is a linear form such that multiplication by any power of this linear form has maximal rank in every degree. Let A be a monomial complete intersection, that is $A = K[x_1, \ldots, x_n]/(x_1^{d_1}, \ldots, x_n^{d_n})$, where K is a field and $d_1, \ldots, d_n$ some positive integers. In characteristic zero, A always has the SLP, which was first proved by Stanley in [12]. When the characteristic is positive, the algebra does not always have the SLP. A first result is that A has the SLP when $p > \sum(d_i - 1)$, where p is the characteristic. This was proved in the case $n = 2$ by Lindsey in [8], and later in the general case by Cook II in [5].

A classification of all monomial complete intersections in three or more variables with the SLP is provided in [9]. Notice that the problem is trivial when $n = 1$, so the remaining case is $n = 2$, which will be treated in this paper. The sufficient conditions in [9] hold also in two variables, but it turns out that there is an additional class of algebras $K[x, y]/(x^a, y^b)$ with the SLP. This is indicated by Cook II in [5], where the two special cases, when $a = b$, and when the characteristic is two, is studied. Cook II solves these cases, under the assumption that the residue field K is infinite.

The main result of this paper is Theorem 3.2, which is a classification of the algebras $K[x, y]/(x^a, y^b)$ with the SLP, where K is a field of characteristic $p \geq 3$. The classification is given in terms of the base p digits of the integers a and b. Together with the mentioned

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earlier results, this gives a complete classification of the monomial complete intersections with the SLP, see Theorem 3.4.

The technique used both in [5] and in this paper, is the theory of the syzygy gap function, introduced by Monsky in [10]. The syzygy gap function deals with the degrees of the relations on x^a, y^b and $(x+y)^c$. This can then be connected to the SLP using results of Brenner and Kaid in [1] and [2]. In [1], [2], and [10] the residue field is required to be algebraically closed. We will see in Sect. 4 that this assumption can be dropped. We will also give a new proof of the connection to the SLP, when working with monomial complete intersections.

2 The strong Lefschetz property

Let $A = \bigoplus_{i \geq 0} A_i$ be a graded algebra. A linear map $A_i \to A_j$ is said to have *maximal rank* if it is injective or surjective. Each homogeneous element $f \in A_d$ induces a family of linear maps $A_i \to A_{i+d}$ by $a \mapsto f \cdot a$. Let such maps be denoted by $\cdot f : A_i \to A_{i+d}$. For short, we say that multiplication by f has *maximal rank in every degree*, if all the maps induced by f have maximal rank.

Definition 2.1 A graded algebra A is said to have the *strong Lefschetz property* (SLP) if there exists an $\ell \in A_1$ such that the maps $\cdot \ell^m : A_i \to A_{i+m}$ have maximal rank for all $i \geq 0$ and all $m \geq 1$. In this case, ℓ is said to be a *strong Lefschetz element*.

We say that A has the *weak Lefschetz property* (WLP) if there exists an $\ell \in A_1$ such that the maps $\cdot \ell : A_i \to A_{i+1}$, have maximal rank for all $i \geq 0$. In this case, ℓ is said to be a *weak Lefschetz element*.

Let now K be a field, and $A = K[x_1, \dots, x_n]/I$, where I is a monomial ideal. In [9, Proposition 4.3] it is proved that A has the WLP if and only if $x_1 + \cdots + x_n$ is a weak Lefschetz element. The corresponding is also true for the strong Lefschetz property.

Theorem 2.2 *Let $R = K[x_1, \dots, x_n]$, where K is a field, and let $I \subset R$ be a monomial ideal. Then R/I has the SLP (WLP) if and only if $x_1 + \ldots + x_n$ is a strong (weak) Lefschetz element.*

Proof Suppose that $\sum_{i \in \Lambda} c_i x_i$, for some $\Lambda \subseteq \{1, \dots, n\}$ and $0 \neq c_i \in K$, is a strong Lefschetz element of $A = R/I$. The monomial ideal I is left unchanged under a change of variables $c_i x_i \mapsto x_i$. This shows that $\ell = \sum_{i \in \Lambda} x_i$ also is a strong Lefschetz element. If $\Lambda = \{1, \dots, n\}$ we are done. Assume that $\Lambda \subset \{1, \dots, n\}$, and $j \notin \Lambda$. The next step is to prove that $x_j + \ell$, is also a strong Lefschetz element. For this purpose we introduce a new element a in an extension field of the type $K' = K(a) \supset K$. We will prove that $ax_j + \ell$ is a strong Lefschetz element in $A' = A \otimes_K K'$. Let, for each i, B_i be the vector space basis for A_i that consists of monic monomials. This is also a basis for A'_i, as a vector space over K'. Let M be the matrix of the multiplication map

$$\cdot (ax_j + \ell)^m : A'_i \to A'_{i+m},$$

w. r. t. the bases B_i and B_{i+m}. The entries of M are polynomials in a. Let M_0 be the matrix we obtain by substituting $a = 0$ in M. If M does not have maximal rank, neither does M_0. But M_0 is the matrix of the map $\cdot \ell^m : A_i \to A_{i+m}$, which has maximal rank. This shows that M has maximal rank, and $ax_j + \ell$ is a strong Lefschetz element of A'. But then, since a is a non-zero element of the field K', so is $x_j + \ell$. The coefficients of $x_j + \ell$ are in K, so it is also a strong Lefschetz element of A. It follows that $x_1 + \cdots + x_n$ is a strong Lefschetz element of A. □

The *Hilbert function* of a graded algebra $A = \bigoplus_{i\geq 0} A_i$ with residue field K is a function $\mathrm{HF}_A : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ defined by $\mathrm{HF}_A(i) = \mathrm{vdim}_K A_i$, i. e. the vector space dimension of A_i over K. The *Hilbert series* of A, denoted HS_A, is the generating function of the sequence $\mathrm{HF}(i)$, that is $\mathrm{HS}_A(t) = \sum_{i\geq 0} \mathrm{HF}(i)t^i$.

Let now A be a monomial complete intersection, $A = K[x_1, \dots, x_n]/(x_1^{d_1}, \dots, x_n^{d_n})$, for some positive integers $d_1, \dots, d_n$. Let $t = \sum_{i=1}^n (d_i - 1)$. This is the highest possible degree of a monomial in A, and hence $\mathrm{HF}_A(i) = 0$ when $i > t$. It can also be seen that the Hilbert function is symmetric about $t/2$, and that $\mathrm{HF}_A(i) \leq \mathrm{HF}_A(i + d)$ when $i \leq (t - d)/2$. For a multiplication map to have maximal rank in every degree in A, it shall then be injective up to some degree i, and surjective for larger i. It can be proved that the injectiveness in this case implies the surjectiveness.

Proposition 2.3 *Let $A = K[x_1, \dots, x_n]/(x_1^{d_1}, \dots, x_n^{d_n})$ and $t = \sum_{i=1}^n (d_i - 1)$, and let $f \in A$ be a form of degree d. The maps $\cdot f : A_i \to A_{i+d}$ all have maximal rank if and only if the maps with $i \leq (t - d)/2$ are injective.*

Proof See e. g. [9, Proposition 2.6]. □

In other words, multiplication by a form f has maximal rank in every degree if all homogeneous zero divisors of f are of degree greater than $(t - d)/2$. Another interesting fact is that if we consider forms of the type ℓ^d, and $t - d$ is even, then multiplication by ℓ^{d+1} has maximal rank in every degree if multiplication by ℓ^d does. This result will be important for the classification of algebras with the SLP when $n = 2$.

Proposition 2.4 *Let $A = K[x_1, \dots, x_n]/(x_1^{d_1}, \dots, x_n^{d_n})$ and $t = \sum_{i=1}^n (d_i - 1)$. Let $\ell \in A$ be a linear form, and d a positive integer such that $t - d$ is even. If the maps $\cdot\ell^d : A_i \to A_{i+d}$ have maximal rank for all $i \geq 0$, so does the maps $\cdot\ell^{d+1} : A_i \to A_{i+d+1}$.*

Proof Assume that $\cdot\ell^d : A_i \to A_{i+d}$ have maximal rank for all $i \geq 0$. By Proposition 2.3 all zero divisors of ℓ^d are of degree at least $(t - d)/2$. Suppose that there is a homogeneous element f such that $\ell^{d+1} f = 0$. By Proposition 2.3, we are done if we can prove that $\deg(f) > (t - (d + 1))/2 = (t - d)/2 - 1/2$. Since $t - d$ is even, the right hand side is not an integer, and it is enough to prove $\deg(f) > (t - d)/2 - 1$. Consider first the case when $\ell^d f = 0$. That is, f is a zero divisor of ℓ^d, and it follows that $\deg(f) > (t - d)/2$. Consider instead the case when $\ell^d f \neq 0$. We know that $\ell^{d+1} f = 0$, that is ℓf is a homogeneous zero divisor of ℓ^d. Then $\deg(\ell f) > (t - d)/2$, and $\deg(f) > (t - d)/2 - 1$, which finishes the proof. □

Proposition 2.5 *The algebra $A = K[x, y]/(x^a, y^b)$ has the SLP if and only if the maps*

$$\cdot(x + y)^{a+b-2c} : A_i \to A_{i+a+b-2c}$$

have maximal rank for all $i \geq 0$ and $1 \leq c < \min(a, b)$.

Proof The "only if"-part follows from Theorem 2.2.

The numbers t and d in Proposition 2.4 are here $t = a + b - 2$, and $d = a + b - 2c$. We see that $t - d = 2c - 2$ is even, so if multiplication by $(x + y)^{a+b-2c}$ has maximal rank in every degree, so does multiplication by $(x + y)^{a+b-2c+1}$. If $c \leq 0$ then $A_{i+a+b-2c} = \{0\}$, and obviously any map $A_i \to A_{i+a+b-2c}$ is surjective. This is why we only need to consider $c \geq 1$. Without loss of generality, we can assume that $a = \min(a, b)$. To complete the proof we need to show that multiplication by $(x+y)^{a+b-2c}$ has maximal rank in every degree when

$c \geq a$. Suppose there is a non-zero homogeneous $f \in A$ such that $(x+y)^{a+b-2c} f = 0$. By Proposition 2.3 multiplication by $(x+y)^{a+b-2c}$ has maximal rank in every degree if we can prove that

$$\deg(f) > \frac{a+b-2-(a+b-2c)}{2} = c-1.$$

Let F be a homogeneous element in $K[x,y]$ whose image in A is f. Then

$$(x+y)^{a+b-2c} F = gx^a + hy^b, \quad \text{for some } g, h \in K[x,y].$$

We can not have $h = 0$, because that would imply that F is divisible by x^a, and $f = 0$ in A. Hence $h \neq 0$ and $\deg((x+y)^{a+b-2c} F) \geq b$, which is equivalent to $\deg(F) \geq 2c - a$. If $c \geq a$ this implies $\deg(f) = \deg(F) \geq c$, and we are done. □

3 Classifying the monomial complete intersections with the strong Lefschetz property

A classification of the monomial complete intersections with the SLP, in three or more variables, is given in [9, Theorem 3.8]. Here we give a slightly reformulated version of the theorem, to make the notation similar to that used later in the case of two variables. We will prove that the formulation here is equivalent to that in [9].

Theorem 3.1 *Let $A = K[x_1, \dots, x_n]/(x_1^{d_1}, \dots, x_n^{d_n})$ where $n \geq 3$, $d_i \geq 2$ for all i, and K is a field of characteristic $p > 0$. Let $t = \sum_{i=1}^n (d_i - 1)$ and let $d_1 = \max(d_1, \dots, d_n)$. Write $d_1 = N_1 p + r_1$ with $0 \leq r_1 < p$. Then A has the SLP if and only if one of the following two conditions hold*

1. *$t < p$,*
2. *$d_1 \geq p$, $d_i < p$ for $i = 2, \dots, n$ and $\sum_{i=2}^n (d_i - 1) \leq \min(r_1, p - r_1)$.*

Proof The difference, compared to [9, Theorem 3.8], is that in [9] the bound for r_1 is $0 < r_1 \leq p$, and the second condition is

$$d_1 > p, \ d_i \leq p \text{ for } i = 2, \dots, n \text{ and } \sum_{i=2}^n (d_i - 1) \leq \min(r_1, p - r_1).$$

It is easy to see that both definitions of r_1 gives the same value $\min(r_1, p - r_1)$. When $d_1 = p$ condition 2 of [9, Theorem 3.8] is not satisfied. Neither is condition 2 in Theorem 3.1, because $\min(r_1, p - r_1) = 0$, and $\sum_{i=2}^n (d_i - 1) \geq n - 1 \geq 2$. When $d_i = p$, for some $i > 1$, condition 2 in Theorem 3.1 is not satisfied. Neither is 2 in [9, Theorem 3.8], because then $\sum_{i=2}^n (d_i - 1) \geq p$, and $\min(r_1, p - r_1) < p$ in general. This shows that both formulations agree. □

The two conditions in Theorem 3.1 above can be generalized to the case $n = 2$. Next we will prove that in two variables, and characteristic $p > 2$, the algebra A has the SLP in these two cases, but also in an additional one.

Theorem 3.2 *Let $A = K[x,y]/(x^a, y^b)$, where $a, b \geq 2$ and K is a field of characteristic $p > 2$. Write a and b in base p, that is $a = a_k p^k + \cdots + a_1 p + a_0$ and $b = b_\ell p^\ell + \cdots + b_1 p + b_0$, where $0 \leq a_i, b_i < p$, and $a_k, b_\ell \neq 0$. We may assume that $\ell \geq k$. The classification of the algebras with the SLP is divided into three cases.*

1. *When $a, b < p$, A has the SLP if and only if $a + b \leq p + 1$.*
2. *When $a < p$ and $b \geq p$, A has the SLP if and only if $a \leq \min(b_0, p - b_0) + 1$.*
3. *When $a, b \geq p$, A has the SLP if and only if the following three conditions are satisfied.*
   (a) $a_0 = \frac{p\pm 1}{2}$, *and* $b_0 = \frac{p\pm 1}{2}$,
   (b) $a_i = b_i = \frac{p-1}{2}$ *for* $i = 1, 2, \ldots, k-1$,
   (c) $a_k + b_k \leq p - 1$, *and* $b_k \geq a_k$ *when* $\ell > k$.

Notice that there are no restrictions on b_i for $i > k$, in the case $\ell > k$. The theorem will be proved later in this section.

In [5, Theorem 4.9] Cook II proves the special case $a = b$ of Theorem 3.2. Cook II also proves the characteristic two case.

Theorem 3.3 ([5, Corollary 4.8]) *Let $A = K[x, y]/(x^a, y^b)$, where $2 \leq a \leq b$ and K is a field of characteristic two. A has the SLP if and only if one of the two following conditions hold.*

1. *$a = 2$ and b is odd,*
2. *$a = 3$ and $b \equiv 2 \mod 4$.*

Theorems 3.1, 3.2, and 3.3 can now be combined into a complete classification of the monomial complete intersections with the SLP.

Theorem 3.4 *Let $A = K[x_1, \ldots, x_n]/(x_1^{d_1}, \ldots, x_n^{d_n})$, where all $d_i \geq 2$ and K is a field of characteristic $p > 0$. Write each d_i in base p as $d_i = c_{ik_i} p^{k_i} + \cdots + c_{i1} p + c_{i0}$, with $c_{ik_i} \neq 0$. The algebra A has the SLP if and only if one of the following conditions hold.*

1. *$n = 1$,*
2. *$n = 2$, $p = 2$, and one of the following holds, for $d_1 \leq d_2$*

- *$d_1 = 2$ and $c_{20} = 1$,*
- *$d_1 = 3$, $c_{21} = 1$, and $c_{20} = 0$,*

3. *$n = 2$, $p > 2$ and all the following conditions are satisfied, for $k_1 \leq k_2$*

- $c_{10} = \frac{p\pm 1}{2}$, $c_{20} = \frac{p\pm 1}{2}$,
- $c_{1j} = c_{2j} = \frac{p-1}{2}$, *for* $j = 1, \ldots, k_1 - 1$,
- $c_{1k_1} + c_{2k_1} < p$, *and* $c_{2k_1} \geq c_{1k_1}$ *if* $k_1 < k_2$,

4. *$n \geq 2$, and $\sum_{i=1}^{n}(d_i - 1) < p$,*
5. *$n \geq 2$, and there is a j such that $d_j \geq p$, $d_i < p$ for all $i \neq j$, and $\sum_{i \neq j}(d_i - 1) \leq \min(c_{j0}, p - c_{j0})$.*

Proof The case $n = 1$ is trivial. Condition 3 is condition 3 of Theorem 3.2, and Condition 4 is Theorem 3.2 with $b = d_2$ written in base 2. The conditions 4 and 5 are the conditions 1 and 2 from Theorems 3.1 and 3.2 combined. Notice that 4 and 5 are not satisfied when $p = 2$. □

Both proofs of [5, Corollary 4.8] and [5, Theorem 4.9] use Theorem 3.5 below. This will also be the key to the proof of Theorem 3.2.

Theorem 3.5 *Let K be a field of characteristic $p > 0$. The algebra $K[x, y]/(x^a, y^b)$ has the SLP if and only if*

$$|a - up^i| + |b - vp^i| + |a + b - 2c - wp^i| \geq p^i$$

for all integers $i \geq 0$, $1 \leq c < \min(d_1, d_2)$, and u, v, w such that $u + v + w$ is odd.

Theorem 3.5 is proved in Sect. 4.

We will now prove that Theorem 3.5 can be reformulated as the following proposition.

Proposition 3.6 *Let $A = K[x, y]/(x^a, y^b)$, where K is a field of characteristic $p > 0$. For each integer $i \geq 1$ we can write $a = m_i p^i + r_i$, and $b = n_i p^i + s_i$, where $0 \leq r_i, s_i < p^i$. The algebra A has the SLP if and only if the following conditions hold for all i.*

1. *If $m_i > 0$, then $r_i \geq s_i - 1$,*
2. *If $n_i > 0$, then $s_i \geq r_i - 1$,*
3. *If $m_i > 0$ and $n_i > 0$, then $r_i + s_i \geq p^i - 1$,*
4. *$r_i + s_i \leq p^i + 1$.*

Proof We shall prove that the conditions above is equivalent to that in Theorem 3.5. Let us investigate for which a and b it can happen that

$$|a - up^i| + |b - vp^i| + |a + b - 2c - wp^i| < p^i.$$

Write $a = m_i p^i + r_i$ and $b = n_i p^i + s_i$, as in the proposition. Notice that

$$|a - up^i| = \begin{cases} r_i & \text{when } u = m_i \\ p^i - r_i & \text{when } u = m_i + 1. \end{cases}$$

For all other values of u we get $|a - up^i| \geq p^i$, and then of course $|a - up^i| + |b - vp^i| + |a + b - 2c - wp^i| \geq p^i$. Therefore we only need to consider $u = m_i$ and $u = m_i + 1$. The corresponding is also true for $|b - vp^i|$. This gives us four cases to examine.

I. $u = m_i$ and $v = n_i$

Here

$$|a - up^i| + |b - vp^i| = r_i + s_i.$$

To obtain $|a - up^i| + |b - vp^i| + |a + b - 2c - wp^i| \leq p^i - 1$ it is necessary that $r_i + s_i \leq p^i - 1$.

Suppose first that $r_i + s_i = p^i - 1$. Since $u + v + w = m_i + n_i + w$ is supposed to be odd, we must have $w = m_i + n_i - 2d + 1$, for some integer d. Then

$$\begin{aligned} a + b - 2c - wp^i &= n_i p^i + r_i + m_i p^i + s_i - 2c - (m_i + n_i - 2d + 1)p^i \\ &= r_i + s_i - 2c + (2d - 1)p^i = 2dp^i - 2c - 1, \end{aligned}$$

which is an odd number, and thus $|a + b - 2c - wp^i| \geq 1$. We get

$$|a - up^i| + |b - vp^i| + |a + b - 2c - wp^i| \geq p^i - 1 + 1 = p^i,$$

and we can conclude that $|a - up^i| + |b - vp^i| + |a + b - 2c - wp^i| \geq p^i$ for all w and c, when $r_i + s_i = p^i - 1$. Now suppose that $r_i + s_i \leq p^i - 2$. We want to find out what the smallest possible value of $|a + b - 2c - wp^i|$ is. For this purpose we choose the largest w such that $u + v + w$ is odd, and $a + b - wp^i > 0$. After that we choose the value for c that makes $|a + b - 2c - wp^i|$ as small as possible. Since $r_i + s_i \leq p^i - 2$, the largest w with the required properties is $w = m_i + n_i - 1$. Then

$$a + b - wp^i = p^i + r_i + s_i.$$

If $m_i = 0$, then $\min(a, b) = \min(r_i, b) \leq r_i$ and $c \leq r_i - 1$. Then

$$\begin{aligned} a + b - 2c - wp^i \geq p^i + r_i + s_i - 2(r_i - 1) = p^i - r_i + s_i + 2, \quad \text{and} \\ |a - up^i| + |b - vp^i| + |a + b - 2c - wp^i| \geq r_i + s_i + p^i - r_i + s_i + 2 > p^i. \end{aligned}$$

In a similar way we see that $|a-up^i|+|b-vp^i|+|a+b-2c-wp^i| > p^i$ if $n_i = 0$. Suppose now that $m_i > 0$ and $n_i > 0$. Then we choose $c = [(p^i + r_i + s_i)/2]$, where $[\ldots]$ denotes the integer part. This gives

$$\begin{aligned}
&a+b-2c-wp^i = 0 \text{ or } 1, \quad \text{and}\\
&|a-up^i|+|b-vp^i|+|a+b-2c-wp^i| \le r_i+s_i+1 \le p^i-1.
\end{aligned}$$

The conclusion, in this case, is that $|a-up^i|+|b-vp^i|+|a+b-2c-wp^i| < p^i$, exactly when $m_i, n_i > 0$ and $r_i + s_i \le p^i - 2$. This corresponds to condition 3 in the proposition.

II. $u = m_i$ and $v = n_i + 1$

Here

$$|a-up^i|+|b-vp^i| = r_i + p^i - s_i.$$

To obtain $|a-up^i|+|b-vp^i|+|a+b-2c-wp^i| \le p^i-1$ it is necessary that $r_i + p^i - s_i \le p^i - 1$, that is $r_i \le s_i - 1$. Let us first consider the case when $r_i = s_i - 1$. Since $u+v+w$ is supposed to be odd we must have $w = n_i + m_i - 2d$, for some integer d. This gives

$$a+b-wp^i = r_i+s_i+2dp^i = 2r_i+1+2dp^i,$$

which is odd. Then $|a+b-2c-wp^i| \ge 1$, and

$$|a-up^i|+|b-vp^i|+|a+b-2c-wp^i| \ge r_i+p^i-s_i+1 = p^i.$$

Suppose instead that $r_i \le s_i - 2$. We use that same idea as in case 1, and choose first w, and then c, such that $|a+b-2c-wp^i|$ has the smallest possible value. The best option for w is $w = n_i + m_i$. This gives

$$a+b-wp^i = r_i+s_i.$$

If $m_i = 0$, then $\min(a,b) = \min(r_i,b) = r_i$, thus $c = r_i - 1$ is the largest allowed value of c. Then

$$\begin{aligned}
&a+b-2c-wp^i = r_i+s_i-2(r_i-1) = s_i-r_i+2, \quad \text{and}\\
&|a-up^i|+|b-vp^i|+|a+b-2c-wp^i| = r_i+p^i-s_i+s_i-r_i+2 = p^i+2.
\end{aligned}$$

If $m_i > 0$ on the other hand, we are allowed tho choose $c = s_i - 1$. Then we get

$$a+b-2c-wp^i = r_i-s_i+2$$

instead. Note that this is a non-positive number. This gives

$$|a-up^i|+|b-vp^i|+|a+b-2c-wp^i| = r_i+p^i-s_i+s_i-r_i-2 = p^i-2.$$

The conclusion, in this case, is that $|a-up^i|+|b-vp^i|+|a+b-2c-wp^i| < p^i$, exactly when $m_i > 0$ and $r_i \le s_i - 2$. This corresponds to condition 1 in the proposition.

III. $u = m_i + 1$ and $v = n_i$

In the same way as above, we see that this corresponds to condition 2.

IV. $u = m_i + 1$ and $v = n_i + 1$

Here

$$|a-up^i|+|b-vp^i| = 2p^i - r_i - s_i,$$

so for this to be smaller than p^i we must have $2p^i - r_i - s_i \le p^i - 1$, which is $r_i + s_i \ge p^i + 1$. Consider first the case when $r_i + s_i = p^i + 1$. Then we must choose $w = m_i + n_i - 2d + 1$, for some integer d. Then

$$a + b - wp^i = r_i + s_i + (2d - 1)p^i = 2dp^i + 1,$$

and $|a + b - 2c - wp^i| \ge 1$. Then we get

$$|a - up^i| + |b - vp^i| + |a + b - 2c - wp^i| \ge 2p^i - r_i - s_i + 1 = p^i.$$

Suppose now that $r_i + s_i \ge p^i + 2$. We choose $w = m_i + n_i + 1$ and $c = [(r_i + s_i - p^i)/2]$, because this gives

$$\begin{aligned} &a + b - 2c - wp^i = r_i + s_i - p^i - 2c = 0 \text{ or } 1, \quad \text{and} \\ &|a - up^i| + |b - vp^i| + |a + b - 2c - wp^i| \le 2p^i - r_i - s_i + 1 \le p^i - 1. \end{aligned}$$

This shows that $|a - up^i| + |b - vp^i| + |a + b - 2c - wp^i| < p^i$ when $r_i + s_i \ge p^i + 2$, which is condition 4. □

Proposition 3.6 will be used later in this section to prove Proposition 3.7, which says something about the structure of an algebra that does not have the SLP. Now we shall use Proposition 3.6, with $p > 2$, to prove Theorem 3.2.

Proof of Theorem 3.2 Let $A = K[x, y]/(x^a, y^b)$, and suppose throughout this proof that the characteristic of K is greater than 2. Write a and b in base p as $a = a_k p^k + \cdots + a_1 p + a_0$ and $b = b_\ell p^\ell + \cdots + b_1 p + b_0$, where $0 \le a_i, b_i < p$. We assume that $\ell \le k$. With the notation $a = m_i p^i + r_i$ from Proposition 3.6 we have $r_i = a_{i-1}p^{i-1} + \cdots + a_1 p + a_0$, and $m_i = a_k p^{k-i} + a_{k-1}p^{k-i-1} + \cdots + a_i$, and similar for b.

If $a, b < p$ then $n_i = m_i = 0$ in Proposition 3.6, for all i, and the conditions 1, 2 and 3 are trivially satisfied. Since $a + b < 2p$ condition 4 is satisfied for $i > 1$. The only restriction we get comes from condition 4 when $i = 1$, and states that A has the SLP if and only if $a + b \le p + 1$.

If $a < p$ and $b \ge p$ we get $b_0 \ge a_0 - 1$ and $a_0 + b_0 \le p + 1$ from the conditions 2 and 4 with $i = 1$. These two inequalities can be written as $a_0 \le \min(b_0, p - b_0) + 1$. In condition 1 and 3 there is nothing to check, and for $i > 1$ all conditions are satisfied. We get that A has the SLP if and only if $a_0 \le \min(b_0, p - b_0) + 1$.

Assume now that $a, b \ge p$. The idea now is to translate the four conditions of Proposition 3.6 into the base p digits of a and b.

Let us first look at $i = 1$ in Proposition 3.6. We know that $m_1, n_1 > 0$, so 1 and 2 gives $a_0 - 1 \le b_0 \le a_0 + 1$. The conditions 3 and 4 gives $p - 1 \le a_0 + b_0 \le p + 1$. Both these inequality are satisfied exactly when $a_0 = \frac{p\pm1}{2}$ and $b_0 = \frac{p\pm1}{2}$. This is condition (a) in Theorem 3.2. Suppose that this is the case, and move on to $i = 2$. If $k \ge 2$ then m_2 and n_2 are positive. The conditions 1 and 2 gives

$$a_1 p + a_0 - 1 \le b_1 p + b_0 \le a_1 p + a_0 + 1,$$

which implies $a_1 = b_1$. For 3 and 4 to be satisfied

$$p^2 - 1 \le (a_1 + b_1)p + (a_0 + b_0) \le p^2 + 1$$

is required. This is true if and only if $a_1 + b_1 = p - 1$. Hence we get $a_1 = b_1 = \frac{p-1}{2}$. We suppose that this is true and continue with $i = 3, \ldots k$. In the same way as above we get $a_2 = \cdots = a_{k-1} = b_2 = \cdots = b_{k-1} = \frac{p-1}{2}$. This is condition (b) in Theorem 3.2.

Suppose that the conditions for $i = 1, 2 \ldots, k$ are satisfied, and move on to $i = k+1$. Now $m_{k+1} = 0$, so in condition 1 and 3 there is nothing to check. If $\ell > k$ then $n_{k+1} > 0$. In this case condition 2 says

$$b_k p^k + \cdots + b_1 p + b_0 \geq a_k p^k + \cdots + a_1 p + a_0 - 1,$$

which holds if and only if $b_k \geq a_k$. Condition 4 says

$$(a_k + b_k) p^k + \cdots + (a_1 + b_1) p + (a_0 + b_0) \leq p^{k+1} + 1,$$

which holds if and only if $a_k + b_k \leq p - 1$. This proves (c).

We must also show that there are no further restrictions on b_j for $j > k$, when such b_j exist. Suppose that the four conditions of Proposition 3.6 are satisfied for $i = 1, 2, \ldots, k+1$. We continue by looking at $i = k+2$. The conditions 1 and 3 are satisfied, since $m_i = 0$. Notice also that $r_{k+2} = r_{k+1} = a$, and $s_{k+2} \geq s_{k+1}$. This means that if condition 2 is satisfied for $i = k+1$, so it is for $i = k+2$. Condition 4 requires

$$b_{k+1} p^{k+1} + (a_k + b_k) p^k + \cdots + (a_1 + b_1) p + (a_0 + b_0) \leq p^{k+2} + 1.$$

But this is no restriction on b_{k+1}, other than $b_{k+1} < p$. The same reasoning works for larger i. □

The proof in [9] of when an algebra in three or more variables does not have the SLP, is carried out by finding a monomial zero-divisor of $(x_1 + \cdots + x_n)^m$, for some m. We will now see that this can also be done in two variables. This gives an alternative proof of the "only if"-part of Theorem 3.2.

Proposition 3.7 *Let $A = K[x_1, \ldots, x_n]/(x_1^{d_1}, \ldots, x_n^{d_n}) = \bigoplus_{i \geq 0} A_i$ be an algebra of characteristic $p > 0$ which does not possess the SLP. Let ℓ be a linear form in A. Then there are integers d and m such that $\mathrm{HF}_A(d) \leq \mathrm{HF}_A(d+m)$, and the kernel of the multiplication map $\cdot\ell^m : A_d \to A_{d+m}$ contains a non-zero monomial.*

Proof For the case $n \geq 3$, see [9].

Assume $n = 2$, and let $\ell = c_1 x_1 + c_2 x_2$ for some $c_1, c_2 \in K$. Recall that $\mathrm{HF}_A(d) \leq \mathrm{HF}_A(d+m)$ when $d \leq (d_1 + d_2 - 2 - m)/2$. We shall prove that when one of the conditions in Proposition 3.6 fails, we can find a monomial of degree low enough, which is a zero divisor of some power of ℓ. Write $d_1 = m_i p^i + r_i$ and $d_2 = n_i p^i + s_i$, for some i, as in Proposition 3.6, and suppose that condition 1 fails for this i. This means that $m_i > 0$ and $r_i \leq s_i - 2$. Then $r_i < d_1$, and therefore $x_1^{r_i} \neq 0$. Recall that

$$\ell^{p^i} = (c_1 x_1 + c_2 x_2)^{p^i} = c_1^{p^i} x_1^{p^i} + c_2^{p^i} x_2^{p^i},$$

since we are in a ring of characteristic p. Also,

$$(c_1^{p^i} x_1^{p^i} + c_2^{p^i} x_2^{p^i})^{m_i + n_i} = e x_1^{m_i p^i} x_2^{n_i p^i} \quad \text{in } A, \text{ for some } e \in K$$

since all the other terms in the expansion will be of the form $c x_1^{\alpha} x_2^{\beta}$ where either $\alpha \geq d_1$ or $\beta \geq d_2$. We have

$$\ell^{(m_i+n_i)p^i} x_1^{r_i} = (c_1^{p^i} x_1^{p^i} + c_2^{p^i} x_2^{p^i})^{m_i+n_i} x_1^{r_i} = e x_1^{m_i p^i} x_2^{n_i p^i} x_1^{r_i} = e x_1^{m_i p^i + r_i} x_2^{n_i p^i} = 0.$$

In other words, $x_1^{r_i}$ is a monomial in the kernel of the multiplication map $\cdot\ell^{(m_i+n_i)p^i} : A_{r_i} \to A_{r_i + (m_i+n_i)p^i}$, and since

$$r_i \leq s_i - 2 \iff r_i \leq \frac{r_i + s_i - 2}{2} \iff r_i \leq \frac{d_1 + d_2 - 2 - (m_i + n_i)p^i}{2}$$

we have $\mathrm{HF}_A(r_i) \le \mathrm{HF}_A(r_i + (m_i + n_i)p^i)$.

If instead conditions 2 of Proposition 3.6 fails, the proof is carried out in the same way, but with $x_1^{r_i}$ replaced by $x_2^{s_i}$. Suppose now that condition 3 fails for some i. That is $m_i, n_i > 0$, and $r_i + s_i \le p^i - 2$. Then $x_1^{r_i} x_2^{s_i} \ne 0$. We have

$$\ell^{(m_i+n_i-1)p^i} = (c_1^{p^i} x_1^{p^i} + c_2^{p^i} x_2^{p^i})^{m_i+n_i-1} = e_1 x_1^{(m_i-1)p^i} x_2^{n_i p^i} + e_2 x_2^{m_i p^i} x_2^{(n_i-1)p^i}$$

for some $e_1, e_2 \in K$, and we see that $\ell^{(m_i+n_i-1)p^i} x_1^{r_i} x_2^{s_i} = 0$. Also,

$$r_i + s_i \le p^i - 2 \iff r_i + s_i \le \frac{r_i + s_i - 2 + p^i}{2} = \frac{d_1 + d_2 - 2 - (m_i + n_i - 1)p^i}{2},$$

which implies that $\mathrm{HF}_A(r_i + s_i) \le \mathrm{HF}_A(r_i + s_i + (m_i + n_i - 1)p^i)$.

At last, suppose that condition 4 of Proposition 3.6 fails. Then $r_i + s_i \ge p^i + 2$. This implies that $d_1 + d_2 - 2 = m_i p^i + r_i + n_i p^i + s_i \ge (m_i + n_i + 1)p^i$, and $\mathrm{HF}((m_i + n_i + 1)p^i) \ge 1$. But

$$\ell^{(m_i+n_i+1)p^i} = (c_1^{p^i} x_1^{p^i} + c_2^{p^i} x_2^{p^i})^{m_i+n_i+1} = 0,$$

since all terms in the expansion will be of the form $c x_1^{\alpha} x_2^{\beta}$ where either $\alpha \ge d_1$ or $\beta \ge d_2$. This shows that 1 is in the kernel of the multiplication map $\cdot \ell^{(m_i+n_i+1)p^i} : A_0 \to A_{(m_i+n_i+1)p^i}$. Since $\mathrm{HF}((m_i + n_i + 1)p^i) \ge 1 = \mathrm{HF}(0)$, this completes the proof. □

4 The syzygy gap

The main purpose of this section is to prove Theorem 3.5. If we require the residue field to be algebraically closed, the theorem follows from combining a theorem by Han [6] and results by Brenner and Kaid in [1] and [2]. Han's result is also proved in a different way by Monsky in [10]. Monsky deals with the syzygy module of three pairwise relatively prime polynomials in two variables, and the so called "syzygy gap", while Brenner and Kaid connects this to the Lefschetz properties. We will go through the results from [10], and give a new proof of the connection to the SLP in the case of monomial complete intersections. The reason to go though the results of [10] is to prove that the residue field does not need to be algebraically closed, but also to give a deeper understanding of Theorem 3.5 and the theory behind it.

4.1 Mason–Stothers' Theorem

First we need a review of Mason–Stothers' Theorem. Suppose f is a polynomial in $K[x_1, \ldots, x_n]$, where K is some field. The polynomial f can be factorized as $f = \prod_{i=1}^{s} p_i^{e_i}$, where the p_i's are distinct irreducible factors. Define $\mathrm{r}(f) = \deg(\prod_{i=1}^{s} p_i)$. Note that $\mathrm{r}(fg) \le \mathrm{r}(f) + \mathrm{r}(g)$, with equality when f and g are relatively prime. Let f'_{x_j} denote the formal derivative of f w. r. t. the variable x_j. When in a polynomial ring with just one variable, we write f' for the derivative. Mason–Stothers' theorem is usually formulated over one variable, as follows.

Theorem 4.1 *(Mason–Stothers) Let K be a field, and let f, g and h be polynomials in $K[x]$ such that*

- *f, g and h are pairwise relatively prime,*
- *f', g' and h' are not all zero,*

- $f + g + h = 0$.

Then $\max(\deg(f), \deg(g), \deg(h)) \le \mathrm{r}(fgh) - 1$.

An elementary proof can be found in [11]. There is also a version of this theorem for homogeneous polynomials in two variables. For clarity we will prove how it can be deduced from Theorem 4.1.

Theorem 4.2 *Let K be a field, and let f, g and h be homogeneous polynomials of degree d in $K[x, y]$ such that*

- *f, g and h are pairwise relatively prime,*
- *f'_x, f'_y, g'_x, g'_y, h'_x and h'_y are not all zero,*
- *$f + g + h = 0$.*

Then $d \le \mathrm{r}(fgh) - 2$.

Proof Let K' be the splitting field of f. Over this field f can be factorized as follows

$$f(x, y) = \sum_{i=0}^{d} \alpha_i x^i y^{d-i} = y^d \sum_{i=0}^{d} \alpha_i \left(\frac{x}{y}\right)^i = y^d \prod_{j=1}^{d} \left(u_j \frac{x}{y} - v_j\right) = \prod_{j=1}^{d} (u_i x - v_j y),$$

where the α_i, u_j and v_j's are elements in K'. After a possible linear change of variables, we can assume that $f(x, y) = y^m \prod_{j=1}^{d-m} (r_j x - s_j y)$, where $m \ge 1$. Let $\hat{f}(x) = f(x, 1) = \prod_{j=1}^{d-m} (r_j x - s_j)$, $\hat{g}(x) = g(x, 1)$ and $\hat{h}(x) = h(x, 1)$. Then $\mathrm{r}(\hat{f}) = \mathrm{r}(f) - 1$, while $\mathrm{r}(g) = \mathrm{r}(\hat{g})$ and $\mathrm{r}(h) = \mathrm{r}(\hat{h})$. Note also that $\deg(\hat{g}) = d$. By Theorem 4.1 it now follows that

$$\begin{aligned} d = \deg(\hat{g}) &\le \mathrm{r}(\hat{f}\hat{g}\hat{h}) - 1 \\ &= \mathrm{r}(\hat{f}) + \mathrm{r}(\hat{g}) + \mathrm{r}(\hat{h}) - 1 = \mathrm{r}(f) + \mathrm{r}(g) + \mathrm{r}(h) - 2 = \mathrm{r}(fgh) - 2, \end{aligned}$$

which we wanted to prove. □

4.2 The syzygy gap

Let now $R = K[x, y]$, where K is any field. Let f_1, f_2 and f_3 be non-zero homogeneous, pairwise relatively prime, polynomials in R, with $d_i = \deg(f_i)$, and let $I = (f_1, f_2, f_3)$. The R-module R/I has a free resolution of length 2, by Hilbert's syzygy theorem. If $\{f_1, f_2, f_3\}$ is a minimal set of generators of I, then

$$0 \to \ker\phi \to R^3 \xrightarrow{\phi} R \to R/I \to 0, \tag{1}$$

where ϕ is given by the matrix $\begin{pmatrix} f_1 & f_2 & f_3 \end{pmatrix}$, is an exact sequence of free modules. We have $\operatorname{rank}\ker\phi = 3 - 1 = 2$. That is, $\ker\phi = \mathrm{Syz}(f_1, f_2, f_3)$ is generated by two homogeneous elements. If $\{f_1, f_2, f_3\}$ is not a minimal set of generators of I, we have e. g. $f_3 = g_1 f_1 + g_2 f_2$, for some homogeneous polynomials g_1 and g_2. Then every relation $Af_1 + Bf_2 + Cf_3 = 0$ can be written as $(A + Cg_1)f_1 + (B + Cg_2)f_2 = 0$. Since f_1 and f_2 are relatively prime $A + Cg_1 = hf_2$, and $B + Cg_2 = -hf_1$, for some homogeneous h. It follows that $\ker\phi$ is generated by $(f_2, -f_1, 0)$ and $(g_1, g_2, -1)$. This shows that (1) is always a free resolution (but not necessarily minimal), and $\ker\phi$ is generated by two homogeneous elements of degrees, say α and β. We have a graded resolution

$$0 \to R(-\alpha) \oplus R(-\beta) \to R(-d_1) \oplus R(-d_2) \oplus R(-d_3) \to R \to R/I \to 0,$$

of R/I. Define $\Delta(f_1, f_2, f_3) = |\alpha - \beta|$. This is the *syzygy gap function* introduced in [10]. From the graded resolution we see that the Hilbert series of R/I is

$$\mathrm{HS}_{R/I}(t) = \frac{1 - t^{d_1} - t^{d_2} - t^{d_3} + t^{\alpha} + t^{\beta}}{(1-t)^2}.$$

We also know that R/I has dimension 0, thus the Hilbert series is a polynomial, say $\mathrm{HS}_{R/I}(t) = p(t)$. Then

$$(1-t)^2 p(t) = 1 - t^{d_1} - t^{d_2} - t^{d_3} + t^{\alpha} + t^{\beta}.$$

By taking the derivative of both sides, and substituting $t = 1$ we get $0 = -d_1 - d_2 - d_3 + \alpha + \beta$, that is $\alpha + \beta = d_1 + d_2 + d_3$. This is one of the so called Herzog-Kühl equations, see e. g. [4]. From this follows also the below lemma.

Lemma 4.3 *Let f_1, f_2 and f_3 be non-zero, pairwise relatively prime homogeneous polynomials in $K[x, y]$, with $d_i = \deg(f_i)$. Then $\Delta(f_1, f_2, f_3) \equiv d_1 + d_2 + d_3 \mod 2$.*

We shall also see some other properties of the function Δ.

Lemma 4.4 *Let K be a field of characteristic $p > 0$, and let f_1, f_2 and f_3 be non-zero, pairwise relatively prime homogeneous polynomials in $K[x, y]$. Then*

$$\Delta(f_1^{p^s}, f_2^{p^s}, f_3^{p^s}) = p^s \Delta(f_1, f_2, f_3),$$

for all non-negative integers s.

Proof Let $R = K[x, y]$, and $I = (f_1, f_2, f_3)$. For a fixed s, let $q = p^s$, and $I^{(q)} = (f_1^q, f_2^q, f_3^q)$. We let $\mathcal{F}$ denote the Frobenius functor on the category of R-modules, induced by the endomorphism $a \mapsto a^q$ on R. For a review of the Frobenius functor, see e. g. [3]. By [7, Corollary 2.7], $\mathcal{F}$ is an exact functor. Now, suppose $\mathrm{Syz}(f_1, f_2, f_3)$ is generated by (A_1, A_2, A_3) and (B_1, B_2, B_3), of degrees α and β. When we apply $\mathcal{F}$ to the resolution

$$0 \to R^2 \xrightarrow{\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \\ A_3 & B_3 \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} f_1 & f_2 & f_3 \end{pmatrix}} R \to R/I \to 0$$

we get an exact sequence

$$0 \to R^2 \xrightarrow{\begin{pmatrix} A_1^q & B_1^q \\ A_2^q & B_2^q \\ A_3^q & B_3^q \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} f_1^q & f_2^q & f_3^q \end{pmatrix}} R \to R/I^{(q)} \to 0.$$

This proves that $\mathrm{Syz}(f_1^q, f_2^q, f_3^q)$ is generated by (A_1^q, A_2^q, A_3^q) and (B_1^q, B_2^q, B_3^q). Then

$$\Delta(f_1^q, f_2^q, f_3^q) = |\alpha q - \beta q| = q|\alpha - \beta| = q\Delta(f_1, f_2, f_3),$$

which we wanted to prove. □

Let us now investigate what happens with $\Delta(f_1, f_2, f_3)$ when, for example, f_1 is replaced by ℓf_1, for some linear form ℓ. By Lemma 4.3, $\Delta(f_1, f_2, f_3)$ and $\Delta(\ell f_1, f_2, f_3)$ has different parity, so they can not be equal. If we have a relation $A_1 f_1 + A_2 f_2 + A_3 f_3 = 0$, we also get a relation on $\ell f_1, f_2, f_3$ by multiplying the expression by ℓ. This means that the two elements that generates $\mathrm{Syz}(\ell f_1, f_2, f_3)$ can have degrees at most $\alpha + 1$ and $\beta + 1$. On the other hand, a relation $A_1\ell f_1 + A_2 f_2 + A_3 f_3 = 0$ on $\ell f_1, f_2, f_3$ can also be considered a syzygy $(A_1\ell, A_2, A_3)$ on f_1, f_2, f_3. Hence, the two generators of $\mathrm{Syz}(\ell f_1, f_2, f_3)$ have degrees at least α and β. This shows that Δ must either increase of decrease by 1 when f_1 is replaced by ℓf_1. We summarize this in a lemma.

Lemma 4.5 *Let f_1, f_2 and f_3 be non-zero, pairwise relatively prime homogeneous polynomials in $K[x, y]$. Let ℓ be a linear form, relatively prime to f_2 and f_3. Then*

$$\Delta(\ell f_1, f_2, f_3) = \Delta(f_1, f_2, f_3) \pm 1.$$

We shall look more carefully into two special cases where Lemma 4.5 applies. Let (A_1, A_2, A_3) be the element in $\mathrm{Syz}(f_1, f_2, f_3)$ of the lowest degree α. If $\ell | A_1$ then $(\ell^{-1}A_1, A_2, A_3)$ is a syzygy of $\ell f_1, f_2, f_3$ of degree α. The other generating syzygy can have degree β or $\beta + 1$, as we saw above. But since $\Delta(\ell f_1, f_2, f_3) \neq \Delta(f_1, f_2, f_3)$ it must have degree $\beta + 1$. Hence, $\Delta(\ell f_1, f_2, f_3) = \Delta(f_1, f_2, f_3) + 1$ in this case.

It follows also from Lemma 4.5 that $\Delta(\ell^{-1} f_1, f_2, f_3) = \Delta(f_1, f_2, f_3) \pm 1$, if $\ell | f_1$. If, in addition, $\ell | A_2$, it follows from the equality $A_1 f_1 + A_2 f_2 + A_3 f_3 = 0$ that ℓ also divides A_3. Then we can divide the whole expression by ℓ, and get a syzygy $(A_1, \ell^{-1}A_2, \ell^{-1}A_3)$ on $\ell^{-1} f_1, f_2, f_3$, of degree $\alpha - 1$. We see that we must have $\Delta(\ell^{-1} f_1, f_2, f_3) = \Delta(f_1, f_2, f_3) + 1$, in this case.

This, together with Theorem 4.2, can now be used to prove the following proposition.

Proposition 4.6 ([10, Theorem 8]) *Let K be a field of characteristic $p > 0$. Let f_1, f_2, and f_3 be homogeneous relatively prime polynomials in $K[x, y]$. Assume there is a linear form ℓ such that $f_1 = \ell^m h$, where $\ell \nmid h$ and $p \nmid m$. Assume also that $\Delta(f_1, f_2, f_3)$ decreases when f_1 is replaced by ℓf_1 or $\ell^{-1} f_1$. Then $\Delta(f_1, f_2, f_3) \leq \mathrm{r}(f_1 f_2 f_3) - 2$.*

Proof Let (A_1, A_2, A_3) be one of the two generators of $\mathrm{Syz}(f_1, f_2, f_3)$ of minimal degree α. We saw above that if $\ell | A_1$ then $\Delta(\ell f_1, f_2, f_3) = \Delta(f_1, f_2, f_3) + 1$. We also saw that if $\ell | A_2$ then $\Delta(\ell^{-1} f_1, f_2, f_3) = \Delta(f_1, f_2, f_3) + 1$. The same holds if $\ell | A_3$. By assumption, none of this is the case, and hence A_1, A_2 and A_3 are not divisible ℓ. Let $M = \gcd(A_1 f_1, A_2 f_2, A_3 f_3)$. Then

$$\frac{A_1 f_1}{M} + \frac{A_2 f_2}{M} + \frac{A_3 f_3}{M} = 0$$

and the three terms $A_i f_i / M$ are relatively prime. Notice that every irreducible factor of M must divide one of f_1, f_2 or f_3. Also ℓ does not divide M, since ℓ does not divide A_2, A_3, f_2 or f_3. We shall now see that the formal derivative of $A_1 f_1 / M$ w. r. t. x or y is non-zero, so that we can use Theorem 4.2. One of ℓ'_x and ℓ'_y must be non-zero, otherwise $\ell = 0$. Say that $\ell'_x = c \neq 0$. Then

$$\left(\frac{A_1 f_1}{M}\right)'_x = \left(\ell^m \frac{A_1 h}{M}\right)'_x = mc\ell^{m-1}\frac{A_1 h}{M} + \ell^m \left(\frac{A_1 h}{M}\right)'_x.$$

The two terms can not cancel each other, and the first one is non-zero, since $m \neq 0$ in K. Hence $(A_1 f_1/M)'_x \neq 0$. By Theorem 4.2

$$\deg\left(\frac{A_1 f_1}{M}\right) \leq \mathrm{r}\left(\frac{A_1 f_1 A_2 f_2 A_3 f_3}{M^3}\right) - 2. \tag{2}$$

We know that $\deg(A_1 f_1/M) = \alpha - \deg(M)$. Let $d_i = \deg(f_i)$, for $i = 1, 2, 3$, and recall that $d_1 + d_2 + d_3 = \alpha + \beta$, where β is the degree of the other generator of $\mathrm{Syz}(f_1, f_2, f_3)$. We have

$$\begin{aligned}
\mathrm{r}\left(\frac{A_1 f_1 A_2 f_2 A_3 f_3}{M^3}\right) &\le \mathrm{r}\left(\frac{f_1 f_2 f_3}{M}\right) + \deg\left(\frac{A_1 A_2 A_3}{M^2}\right) \\
&\le \mathrm{r}(f_1 f_2 f_3) + \deg(A_1) + \deg(A_2) + \deg(A_3) - 2\deg(M) \\
&= \mathrm{r}(f_1 f_2 f_3) + (\alpha - d_1) + (\alpha - d_2) + (\alpha - d_3) - 2\deg(M) \\
&= \mathrm{r}(f_1 f_2 f_3) + 3\alpha - (\alpha + \beta) - 2\deg(M) \\
&= \mathrm{r}(f_1 f_2 f_3) + 2\alpha - \beta - 2\deg(M).
\end{aligned}$$

Inserted in (2), this gives

$$\alpha - \deg(M) \le \mathrm{r}(f_1 f_2 f_3) + 2\alpha - \beta - 2\deg(M) - 2,$$

which is rewritten as

$$\beta - \alpha \le \mathrm{r}(f_1 f_2 f_3) - \deg(M) - 2.$$

We can now conclude that $\Delta(f_1, f_2, f_3) = \beta - \alpha \le \mathrm{r}(f_1 f_2 f_3) - 2$. □

4.3 Application of the syzygy gap function to monomial complete intersections

We will now specialize to the case $f_1 = x^{d_1}$, $f_2 = y^{d_2}$, and $f_3 = (x+y)^{d_3}$. This is allowed, since these polynomials are pairwise relatively prime. For an easier notation we introduce a new function $\delta : \mathbb{Z}_+^3 \to \mathbb{Z}_{\ge 0}$ defined by $\delta(d_1, d_2, d_3) = \Delta(x^{d_1}, y^{d_2}, (x+y)^{d_3})$. We will now see how the theory of the syzygy gap connects to the SLP.

Proposition 4.7 *Let $S = K[x, y]/(x^{d_1}, y^{d_2})$. The maps $\cdot(x+y)^{d_3} : S_i \to S_{i+d_3}$, with $d_3 < d_1 + d_2$, have maximal rank for all i if and only if $\delta(d_1, d_2, d_3) \le 1$.*

This result can be proved for general f_1, f_2 and f_3 using [1, Theorem 2.2] and [2, Corollary 3.2]. Below follows an easier proof for this special case.

Proof We know that the syzygy module $\mathrm{Syz}(x^{d_1}, y^{d_2}, (x+y)^{d_3})$ is generated by two homogeneous elements (A_1, A_2, A_3) and (B_1, B_2, B_3) of degrees α and β. We may assume that $\alpha \le \beta$. Provided that $A_3 \ne 0$, this can be formulated as $(x+y)^{d_3} A_3 = 0$ in S, and A_3 is a homogeneous element of lowest degree with this property. The degree of A_3 is $\alpha - d_3$. By Proposition 2.3 multiplication by $(x+y)^{d_3}$ has maximal rank in every degree if and only if

$$\alpha - d_3 > \frac{d_1 + d_2 - 2 - d_3}{2} \quad \text{or equivalently} \quad \alpha > \frac{d_1 + d_2 + d_3 - 2}{2}.$$

Recall that $\alpha + \beta = d_1 + d_2 + d_3$. This inserted in the above inequality gives, after simplification, $\alpha > \beta - 2$. Since $\alpha \le \beta$ this is exactly the property $\delta(d_1, d_2, d_3) = \beta - \alpha \le 1$.

It remains to prove that $A_3 \ne 0$. If $A_3 = 0$ we would have a relation $A_1 f_1 + A_2 f_2 = 0$. Since f_1 and f_2 are relatively prime, this gives $A_1 = c f_2$ and $A_2 = -c f_1$, for some $c \in K$. Then $\alpha = d_1 + d_2$, and since $\alpha + \beta = d_1 + d_2 + d_3$, we get $\beta = d_3$. But $\beta \ge \alpha$ and $d_3 < d_1 + d_2$ yields a contradiction. □

This result combined with Proposition 2.5 now gives the following.

Theorem 4.8 *The algebra $K[x, y]/(x^{d_1}, y^{d_2})$ has the SLP if and only if*

$$\delta(d_1, d_2, d_1 + d_2 - 2c) = 0 \ \textit{for all} \ 1 \le c < \min(d_1, d_2).$$

Proof It follows directly from Propositions 4.7 and 2.5 that $K[x, y]/(x^{d_1}, y^{d_2})$ has the SLP if and only if $\delta(d_1, d_2, d_1 + d_2 - 2c) \le 1$. By Lemma 4.3 $\delta(d_1, d_2, d_1 + d_2 - 2c)$ is even, so it must be 0 in this case. □

The problem now is to determine for which d_1, d_2, d_3 we have $\delta(d_1, d_2, d_3) = 0$. Let us define

$$L = \{(u, v, w) \in \mathbb{Z}_+^3 \mid 2\max(u, v, w) \leq u + v + w\}.$$

Also, let $L_=$ be the subset of L where equality holds, and $L_< = L \setminus L_=$.

Lemma 4.9 *Let* $(d_1, d_2, d_3) \in L_=$. *Then* $\delta(d_1, d_2, d_3) = 0$.

Proof Suppose $d_1 \leq d_2 < d_3 = d_1+d_2$. We are in the situation when $x^{d_1}, y^{d_2}, (x+y)^{d_3}$ is not a minimal generating set; there are polynomials g and h such that $(x+y)^{d_1+d_2} = gx^{d_1} + hy^{d_2}$ As we saw in the beginning of Sect. 4.2, the module $\mathrm{Syz}(x^{d_1}, y^{d_2}, (x+y)^{d_1+d_2})$ is, in this case, generated by $(g, h, -1)$ and $(y^{d_2}, -x^{d_1}, 0)$. Both these relations have degree $d_1 + d_2$, which gives $\delta(d_1, d_2, d_3) = 0$.

The case when d_1 or d_2 is the largest among d_1, d_2, d_3 follows from the above after a linear change of the variables x and y. □

Lemma 4.10 *For any two points* (c_1, c_2, c_3) *and* (d_1, d_2, d_3) *in* $\mathbb{Z}_+^3$ *it holds that*

$$|\delta(c_1, c_2, c_3) - \delta(d_1, d_2, d_3)| \leq |c_1 - d_1| + |c_2 - d_2| + |c_3 - d_3|. \tag{3}$$

Moreover, for $(d_1, d_2, d_3) \in L_<$ *we can find a point* (c_1, c_2, c_3) *such that*

$$\delta(c_1, c_2, c_3) = \delta(d_1, d_2, d_3) + |c_1 - d_1| + |c_2 - d_2| + |c_3 - d_3|,$$

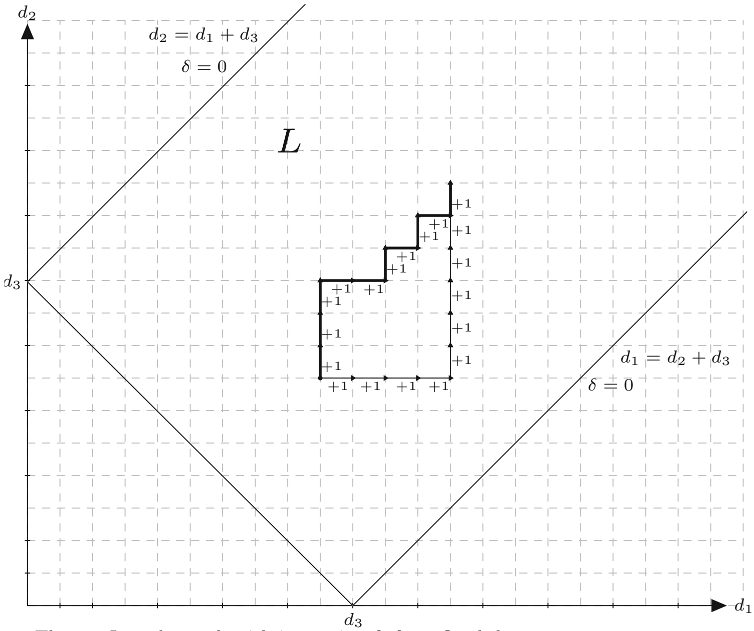
and $\delta(c_1, c_2, c_3)$ *decreases when any* c_i *is replaced by* $c_i \pm 1$.

Proof Recall from Lemma 4.5 that $\delta(d_1, d_2, d_3)$ increases or decreases by 1 when we "take a step" in $\mathbb{Z}_+^3$, that is when one d_i is replaced by $d_i \pm 1$. This proves (3).

Imagine now that we start in the point (d_1, d_2, d_3), and take a step in some direction, if it makes the value of δ increase. We continue in this way, as long as we can make the value of δ increase in each step. What we want to prove is that such a path can not be infinitely long. Let us fix a point (d_1', d_2', d_3') on our path. Any other path between (d_1, d_2, d_3) and (d_1', d_2', d_3') must give the same value of δ at (d_1', d_2', d_3'). It follows that a path where the value of δ increases in each step must be of minimal length, among all paths between these two points. Any other path of minimal length must also have the property that δ increases in each step. Hence we can replace our path by the path that first increases/decreases d_1, then d_2 and last d_3. But when d_2 and d_3 are fixed, we can only increase of decrease d_1 a finite number of times, before we hit $L_=$. The corresponding holds for d_2 and d_3. At $L_=$ the value of δ is zero, as we saw in Lemma 4.9, so δ must have decreased. This shows that there is a bound for the length of a path that starts in a given point $(d_1, d_2, d_3) \in L_<$, and increases δ in each step. Eventually we will reach a point (c_1, c_2, c_3) such that

$$\delta(c_1, c_2, c_3) = \delta(d_1, d_2, d_3) + |c_1 - d_1| + |c_2 - d_2| + |c_3 - d_3|,$$

and $\delta(c_1, c_2, c_3)$ decreases when any c_i is replaced by $c_i \pm 1$. □



The set L, and a path with increasing δ, for a fixed d_3.

Theorem 4.11 *Let the function δ be defined over $K[x, y]$ where K is a field of characteristic $p > 0$. Let d_1, d_2, d_3 be positive integers, such that $d_1 \leq d_2 \leq d_3 < d_1 + d_2$. Then $\delta(d_1, d_2, d_3) = 0$ if and only if*

$$|d_1 - up^s| + |d_2 - vp^s| + |d_3 - wp^s| \geq p^s$$

for all integers s, u, v, w such that $s \geq 0$ and $u + v + w$ is odd.

Proof Assume first that $\delta(d_1, d_2, d_3) = 0$. Let s be a non-negative integer, and u, v, w integers with odd sum. From Lemma 4.3 we know that $\delta(u, v, w)$ is odd, in particular $\delta(u, v, w) \geq 1$. Lemmas 4.4 and 4.10 now gives

$$\begin{aligned} |d_1 - up^s| + |d_2 - vp^s| + |d_3 - wp^s| &\geq |\delta(up^s, vp^s, wp^s) - \delta(d_1, d_2, d_3)| \\ &= \delta(up^s, vp^s, wp^s) = p^s \delta(u, v, w) \geq p^s. \end{aligned}$$

For the other implication, assume that

$$|d_1 - up^s| + |d_2 - vp^s| + |d_3 - wp^s| \geq p^s$$

for all $s \geq 0$ and integers u, v, w with odd sum. By Lemma 4.10 there is a point (c_1, c_2, c_3) such that

$$\delta(c_1, c_2, c_3) = \delta(d_1, d_2, d_3) + |d_1 - c_1| + |d_2 - c_2| + |d_3 - c_3|,$$

and $\delta(c_1, c_2, c_3)$ decreases by one if we replace any c_i by $c_i \pm 1$. Write $c_1 = p^s u, c_2 = p^s v$ and $c_3 = p^s w$, such that (at least) one of u, v, and w is not divisible by p. Notice that $\delta(u, v, w)$ also must decrease when u, v or w is increased or decreased by one. Otherwise we would have e. g. $\delta(u, v, w + 1) = \delta(u, v, w) + 1$, which implies $\delta(c_1, c_2, c_3 + p^s) = \delta(c_1, c_2, c_3) + p^s$. This can only hold if δ increases in each step from (c_1, c_2, c_3) to $(c_1, c_2, c_3 + p^s)$, which is not the case. Now we can use Proposition 4.6 on $\delta(u, v, w)$ with $\ell = x, y$, or $x + y$, depending on which of u, v and w are not divisible by p. Since $\mathrm{r}(x^u y^v (x+y)^w) = 3$ we get $\delta(u, v, w) \leq 1$. Since $\delta(u, v, w) - 1 = \delta(u, v, w+1) \geq 0$, we must have $\delta(u, v, w) = 1$. By Lemma 4.3 $u + v + w$ is odd, and we can use our assumption to get

$$\begin{aligned}\delta(d_1, d_2, d_3) &= \delta(c_1, c_2, c_3) - (|d_1 - c_1| + |d_2 - c_2| + |d_3 - c_3|) \\ &= p^s \delta(u, v, w) - (|d_1 - up^s| + |d_2 - vp^s| + |d_3 - wp^s|) \\ &= p^s - (|d_1 - up^s| + |d_2 - vp^s| + |d_3 - wp^s|) \leq 0.\end{aligned}$$

By definition $\delta(d_1, d_2, d_3) \geq 0$, so we can conclude $\delta(d_1, d_2, d_3) = 0$. □

Proof of Theorem 3.5 By Theorem 4.8, $K[x, y]/(x^{d_1}, y^{d_2})$ has the SLP if and only if

$$\delta(d_1, d_2, d_1 + d_2 - 2c) = 0 \text{ for all } 1 \leq c < \min(d_1, d_2).$$

With $d_3 = d_1 + d_2 - 2c$, clearly $d_1 \leq d_3$, $d_2 \leq d_3$ and $d_3 < d_1 + d_2$, so we can use Theorem 4.11. Substituting $d_3 = d_1 + d_2 - 2c$ into the inequality in Theorem 4.11, gives Theorem 3.5. □

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