The Bohnenblust–Hille inequality combined with an inequality of Helson

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Abstract

We give a variant of the Bohenblust-Hille inequality which, for certain families of polynomials, leads to constants with polynomial growth in the degree.

1 Introduction

Hardy and Littlewood showed in [8] that there exists a constant $K > 0$ such that for every $f \in H^1$ we have

$$\Big(\int_{\mathbb{D}} |f(z)|^2 dm(z) \Big)^{1/2} \leq K \int_{\mathbb{T}} |f(w)| d\sigma(w) \,,$$

where dm and $d\sigma$ denote respectively the normalised Lebesgue measures on the complex unit disk $\mathbb{D}$ and the torus (or unit circle) $\mathbb{T}$. Equivalently, this means that the Hardy space $H_1(\mathbb{T})$ is contained in the Bergman space $B_2(\mathbb{D})$. Shapiro [13, p. 117-118] showed that the inequality holds with $K = \pi$ and Mateljević [11] (see also [12, 14]) showed that actually the constant could be taken $K = 1$. A simple reformulation of the Bergman norm then gives that if $\sum_{n=0}^{\infty} a_n z^n$ is the Fourier series expansion of $f \in H^1(\mathbb{D})$ we have

$$\Big(\sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} \Big)^{1/2} \leq \int_{\mathbb{T}} |f(w)| d\sigma(w) \,.$$

A few years later Helson in [10] generalised this inequality to functions in N variables. For $n \in \mathbb{N}$ denote by $d(n)$ the number of divisors and by $p^{\alpha} = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$

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the prime decomposition of n. Then we have that for every $f \in H^1(\mathbb{T}^N)$ with Fourier series expansion $\sum_{\alpha\in\mathbb{N}_0^N} c_\alpha z^\alpha$

$$\Big(\sum_{\alpha\in\mathbb{N}_0^N} \frac{|c_\alpha|^2}{d(p^\alpha)}\Big)^{1/2} \le \int_{\mathbb{T}^N} |f(w)| d\sigma(w). \tag{1}$$

Given a multiindex α, we write $\alpha+1 = (\alpha_1+1)\cdots(\alpha_k+1)$. Note that, with this notation, we have $d(p^\alpha) = \alpha+1$.

On the other hand, by the Bohnenblust-Hille inequality [4] as presented in [5] there is a constant $C>0$ such that for every m-homogeneous polynomial in N variables $P(z) = \sum_{|\alpha|=m} c_\alpha z^\alpha$ with $z\in\mathbb{C}^N$ we have

$$\Big(\sum_{|\alpha|=m} |c_\alpha|^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}} \le C^m \sup_{z\in\mathbb{D}^N} |P(z)|. \tag{2}$$

The proof of this inequality given in [5] consists basically of two steps: first to decompose the sum in (2) as the product of certain mixed sums and second to bound each one of these sums by a term including $\|P\|$, the supremum of $|P|$ in $\mathbb{D}^N$. For this second step usually the following result of Bayart [1] is used: for every m-homogeneous polynomial in N variables we have

$$\Big(\sum_{|\alpha|=m} |c_\alpha|^2\Big)^{1/2} \le 2^{m/2} \int_{\mathbb{T}^N} \Big|\sum_{|\alpha|=m} c_\alpha w^\alpha\Big| d\sigma(w). \tag{3}$$

Very recently, it was proved in [2, Corollary 5.3] that for every $\varepsilon>0$ there exists $\kappa>0$ such that we can take $\kappa(1+\varepsilon)^m$ as the constant in (2). Our aim in this note is get a variant of (2) by using (1) instead of (3). With this variant, we see that for polynomials P each of whose monomials involve a uniformly bounded number of variables, the obtained constants have polynomial growth in m.

2 Main result and some remarks

The following is our main result.

Theorem 2.1. *Let $\Lambda \subseteq \{\alpha\in\mathbb{N}_0^N : |\alpha| = m\}$ be an indexing set. Then for every family $(c_\alpha)_{\alpha\in\Lambda}$ we have*

$$\Big(\sum_{\alpha\in\Lambda}\Big(\frac{|c_\alpha|}{\sqrt{\alpha+1}}\Big)^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}} \le m^{\frac{m-1}{2m}}\Big(1-\frac{1}{m-1}\Big)^{m-1} \sup_{z\in\mathbb{D}^N}\Big|\sum_{\alpha\in\Lambda} c_\alpha z^\alpha\Big|.$$

We give several remarks before we present the proof.

Remark 2.2.

1. It is easy to see that $\sqrt{\alpha+1} \leq \sqrt{2}^m$. Hence the preceding inequality includes the hypercontractive version of the Bohnenblust-Hille inequality from (2) as a special case.

2. Thanks to the term $\sqrt{\alpha+1}$, the constants in the previous inequality grow much more slowly than the constants in (2). Actually, we have

$$m^{\frac{m-1}{2m}}\Big(1-\frac{1}{m-1}\Big)^{m-1} = \frac{\sqrt{m}}{e}\,\big(1+o(m)\big).$$

3. Let vars(α) denote the numbers of different variables involved in the monomial z^α. In other words, vars$(\alpha) = \operatorname{card}\{j : \alpha_j \neq 0\}$. Given M we consider the set

$$\Lambda_{N,M} = \big\{\alpha \in \mathbb{N}_0^N : |\alpha| = m \text{ and } \operatorname{vars}(\alpha) \leq M\big\},$$

(note that if $M \geq N$, then $\Lambda_{N,M} = \Lambda_{N,N}$). An application of Lagrange multipliers gives that for any $\alpha \in \Lambda_{N,M}$ we have for every N and M

$$\alpha+1 = (\alpha_1+1)\cdots(\alpha_k+1)\cdots \leq \Big(\frac{m}{M}+1\Big)^M.$$

Combining this with Theorem 2.1 we obtain for every m, N, M

$$\begin{aligned}\Big(\sum_{\alpha\in\Lambda_{N,M}} |c_\alpha|^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}} &\leq \Big(\frac{m}{M}+1\Big)^{M/2}\Big(\sum_{\alpha\in\Lambda_{N,M}}\Big(\frac{|c_\alpha|}{\sqrt{\alpha+1}}\Big)^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}}\\ &\leq \Big(\frac{m}{M}+1\Big)^{M/2} m^{\frac{m-1}{2m}}\Big(1-\frac{1}{m-1}\Big)^{m-1}\sup_{z\in\mathbb{D}^N}\Big|\sum_{\alpha\in\Lambda_{N,M}} c_\alpha z^\alpha\Big|,\end{aligned}$$

hence

$$\Big(\sum_{\alpha\in\Lambda_{N,M}} |c_\alpha|^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}} \leq 2^{\frac{M}{2}}\, m^{\frac{M+1}{2}} \sup_{z\in\mathbb{D}^N}\Big|\sum_{\alpha\in\Lambda_{N,M}} c_\alpha z^\alpha\Big|. \tag{4}$$

This means that for polynomials whose monomials have a uniformly bounded number M of different variables, we get a Bohnenblust-Hille type inequality with a constant of polynomial growth in m. We remark that the dimension N plays no role in this inequality, the only important point here is the number of different variables in each monomial. As a consequence, an analogue of (4) holds for m-homogeneous polynomials on c_0: Let $P : c_0 \to \mathbb{C}$ be an m-homogeneous polynomial and

$$\Lambda_M = \{\alpha \in \mathbb{N}_0^{(\mathbb{N})} : |\alpha| = m \text{ and } \operatorname{vars}(\alpha) \leq M\}.$$

Then for every M and m

$$\Big(\sum_{\alpha\in\Lambda_M} |c_\alpha(P)|^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}} \leq 2^{\frac{M}{2}}\, m^{\frac{M+1}{2}}\, \|P\|,$$

where the $c_\alpha(P)$ are the coefficients of P and $\|P\|$ is the supremum of $|P|$ on the unit ball of c_0.

4. In [6, Theorem 5.3] a very general version of the Bohnenblust-Hille inequality is given, involving operators with values on a Banach lattice. A straightforward combination of the proof of Theorem 2.1 (see the final section) and the arguments presented in [6, Theorem 5.3] easily gives a version of Theorem 2.1 in that setting.

3 The proof

Let us fix some notation before we prove our main result. We are going to use the following indexing sets

$$\mathcal{M}(m,N) = \{\boldsymbol{i} = (i_1,\dots,i_m) \colon 1 \le i_j \le N,\ j = 1,\dots,m\}$$
$$\mathcal{J}(m,N) = \{\boldsymbol{i} \in \mathcal{M}(m,N) \in \colon 1 \le i_1 \le \cdots \le i_m \le N\}.$$

In $\mathcal{M}(m,N)$ we define an equivalence relation by $\boldsymbol{i} \sim \boldsymbol{j}$ if there is a permutation σ of $\{1,\dots,N\}$ such that $j_k = i_{\sigma(k)}$ for every k. With this, if $\left(a_{i_1,\dots,i_m}\right)$ are symmetric then we have

$$\sum_{\boldsymbol{i}\in\mathcal{M}(m,N)} a_{\boldsymbol{i}} = \sum_{\boldsymbol{i}\in\mathcal{J}(m,N)} \sum_{\boldsymbol{j}\in[\boldsymbol{i}]} a_{\boldsymbol{j}} = \sum_{\boldsymbol{i}\in\mathcal{J}(m,N)} \operatorname{card}[\boldsymbol{i}]\, a_{\boldsymbol{i}}.$$

Also, given $\boldsymbol{i} \in \mathcal{M}(m-1,N)$ and $j \in \{1,\dots,N\}$, for $1 \le k \le m-1$ we define $(\boldsymbol{i},_k j) = (i_1,\dots,i_{k-1}, j, i_k,\dots,i_{m-1}) \in \mathcal{M}(m,N)$ (that is, we put j in the k-th possition, shifting the rest to the right).
There is a one-to-one correspondance between $\mathcal{J}(m,N)$ and $\{\alpha \in \mathbb{N}_0^N \colon |\alpha| = m\}$ defined as follows. For each $\boldsymbol{i}$ we define $\alpha = (\alpha_1,\dots,\alpha_N)$ by $\alpha_r = \operatorname{card}\{j \colon i_j = r\}$ (i.e. α_r counts how many times r comes in $\boldsymbol{i}$); on the other hand, given α we define $\boldsymbol{i} = (1,\overset{\alpha_1}{\dots},1,\dots,N,\overset{\alpha_N}{\dots},N) \in \mathcal{J}(m,N)$.
Each m-homogeneous polynomial on N variables has a unique symmetric m-linear form $L \colon \mathbb{C}^N \times \cdots \times \mathbb{C}^N \to \mathbb{C}$ such that $P(z) = L(z,\dots,z)$ for every z. If (c_α) are the coefficients of the polynomial and $a_{i_1,\dots,i_m} = L(e_{i_1},\dots,e_{i_m})$ is the matrix of L we have $c_\alpha = \operatorname{card}[\boldsymbol{i}]\, a_{\boldsymbol{i}}$, where α and $\boldsymbol{i}$ are related to each other.
Finally, if α and $\boldsymbol{i}$ are related and $p_1 < p_2 < \cdots$ denotes the sequence of prime numbers, we will write $p^\alpha = p_1^{\alpha_1}\cdots p_N^{\alpha_N} = p_{i_1}\cdots p_{i_m} = p_{\boldsymbol{i}}$.

Proof of Theorem 2.1. We follow essentially the guidelines of the proof of the Bohnenblust-Hille inequality as presented in [5]. First of all let us assume that $c_\alpha = 0$ for every $\alpha \notin \Lambda$; then we have

$$\begin{aligned}
\Big(\sum_{\alpha\in\Lambda} \Big(\frac{|c_\alpha|}{\sqrt{\alpha+1}}\Big)^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}} &= \Big(\sum_{\boldsymbol{i}\in\mathcal{J}(m,N)} \Big|\operatorname{card}[\boldsymbol{i}]\frac{a_{\boldsymbol{i}}}{\sqrt{d(p_{\boldsymbol{i}})}}\Big|^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}} \\
&= \Big(\sum_{\boldsymbol{i}\in\mathcal{M}(m,N)} \frac{1}{\operatorname{card}[\boldsymbol{i}]}\Big|\operatorname{card}[\boldsymbol{i}]\frac{a_{\boldsymbol{i}}}{\sqrt{d(p_{\boldsymbol{i}})}}\Big|^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}} \\
&= \Big(\sum_{\boldsymbol{i}\in\mathcal{M}(m,N)} \Big|\operatorname{card}[\boldsymbol{i}]^{1-\frac{m+1}{2m}}\frac{a_{\boldsymbol{i}}}{\sqrt{d(p_{\boldsymbol{i}})}}\Big|^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}}.
\end{aligned}$$

We now use an inequality due to Blei [3, Lemma 5.3] (see also [5, Lemma 1]): for any family of complex numbers $\big(b_{\boldsymbol{i}}\big)_{\boldsymbol{i}\in\mathscr{M}(m,N)}$ we have

$$\sum_{\boldsymbol{i}\in\mathscr{M}(m,N)} |b_{\boldsymbol{i}}|^{\frac{2m}{m+1}} \leq \prod_{k=1}^{m}\Big(\sum_{j=1}^{N}\Big(\sum_{\boldsymbol{i}\in\mathscr{M}(m-1,N)} |b_{(\boldsymbol{i},_k j)}|^2\Big)^{1/2}\Big)^{\frac{2}{m-1}}. \tag{5}$$

Using this and the fact that $\operatorname{card}[(\boldsymbol{i},_k j)] \leq m\operatorname{card}[\boldsymbol{i}]$ we get

$$\begin{aligned}
&\Big(\sum_{\alpha\in\Lambda}\Big(\frac{|c_\alpha|}{\sqrt{\alpha+1}}\Big)^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}} \\
&\qquad\leq \prod_{k=1}^{m}\Big(\sum_{j=1}^{N}\Big(\sum_{\boldsymbol{i}\in\mathscr{M}(m-1,N)}\Big|\operatorname{card}[(\boldsymbol{i},_k j)]^{\frac{m-1}{2m}}\frac{a_{(\boldsymbol{i},_k j)}}{\sqrt{d(p_{(\boldsymbol{i},_k j)})}}\Big|^2\Big)^{1/2}\Big)^{\frac{1}{m}} \\
&\qquad\leq \prod_{k=1}^{m}\Big(\sum_{j=1}^{N}\Big(\sum_{\boldsymbol{i}\in\mathscr{M}(m-1,N)}\Big|\operatorname{card}[\boldsymbol{i}]^{\frac{m-1}{2m}} m^{\frac{m-1}{2m}}\frac{a_{(\boldsymbol{i},_k j)}}{\sqrt{d(p_{(\boldsymbol{i},_k j)})}}\Big|^2\Big)^{1/2}\Big)^{\frac{1}{m}} \\
&\qquad= m^{\frac{m-1}{2m}}\prod_{k=1}^{m}\Big(\sum_{j=1}^{N}\Big(\sum_{\boldsymbol{i}\in\mathscr{M}(m-1,N)}\operatorname{card}[\boldsymbol{i}]\Big|\frac{a_{(\boldsymbol{i},_k j)}}{\sqrt{d(p_{(\boldsymbol{i},_k j)})}}\Big|^2\Big)^{1/2}\Big)^{\frac{1}{m}}.
\end{aligned}$$

We now bound each one of the sums in the product. We use the fact that the coefficients $a_{\boldsymbol{j}}$ are symmetric. Also, if q divides $p_{i_1}\cdots p_{i_m} = p_{\boldsymbol{i}}$, then it also divides $p_{i_1}\cdots p_{i_m}p_j = p_{(\boldsymbol{i},_k j)}$; hence $d(p_{\boldsymbol{i}}) \leq d(p_{(\boldsymbol{i},_k j)})$ for every $\boldsymbol{i}$ and every j. This altogether gives

$$\begin{aligned}
&\sum_{j=1}^{N}\Big(\sum_{\boldsymbol{i}\in\mathscr{M}(m-1,N)}\operatorname{card}[\boldsymbol{i}]\Big|\frac{a_{(\boldsymbol{i},_k j)}}{\sqrt{d(p_{(\boldsymbol{i},_k j)})}}\Big|^2\Big)^{1/2} \\
&\qquad= \sum_{j=1}^{N}\Big(\sum_{\boldsymbol{i}\in\mathscr{J}(m-1,N)}\operatorname{card}[\boldsymbol{i}]^2\frac{|a_{(\boldsymbol{i},_k j)}|^2}{d(p_{(\boldsymbol{i},_k j)})}\Big)^{1/2} \\
&\qquad\qquad\leq \sum_{j=1}^{N}\Big(\sum_{\boldsymbol{i}\in\mathscr{J}(m-1,N)}\frac{|\operatorname{card}[\boldsymbol{i}]\,a_{(\boldsymbol{i},_k j)}|^2}{d(p_{\boldsymbol{i}})}\Big)^{1/2}.
\end{aligned}$$

Let us note that what we have here are the coefficients of an $(m-1)$-homogeneous polynomial in N variables, we use now (1) to conclude our argument

$$\begin{aligned}
\sum_{j=1}^{N}\Big(\sum_{\boldsymbol{i}\in\mathscr{J}(m-1,N)}\frac{|\operatorname{card}[\boldsymbol{i}]\,a_{(\boldsymbol{i},_k j)}|^2}{d(p_{\boldsymbol{i}})}\Big)^{1/2}
&\le \sum_{j=1}^{N}\int_{\mathbb{T}^N}\Big|\sum_{\boldsymbol{i}\in\mathscr{J}(m-1,N)}\operatorname{card}[\boldsymbol{i}]\,a_{(\boldsymbol{i},_k j)}\,w_{i_1}\cdots w_{i_{m-1}}\Big|\,d\sigma(w)\\
&\le \int_{\mathbb{T}^N}\sum_{j=1}^{N}\Big|\sum_{\boldsymbol{i}\in\mathscr{M}(m-1,N)}a_{(\boldsymbol{i},_k j)}\,w_{i_1}\cdots w_{i_{m-1}}\Big|\,d\sigma(w)\\
&\le \sup_{z\in\mathbb{D}^N}\sum_{j=1}^{N}\Big|\sum_{\boldsymbol{i}\in\mathscr{M}(m-1,N)}a_{(\boldsymbol{i},_k j)}\,z_{i_1}\cdots z_{i_{m-1}}\Big|\\
&= \sup_{z\in\mathbb{D}^N}\sup_{y\in\mathbb{D}^N}\Big|\sum_{j=1}^{N}\sum_{\boldsymbol{i}\in\mathscr{M}(m-1,N)}a_{(\boldsymbol{i},_k j)}\,z_{i_1}\cdots z_{i_{m-1}}\,y_j\Big|\\
&\le \Big(1-\frac{1}{m-1}\Big)^{m-1}\sup_{z\in\mathbb{D}^N}\Big|\sum_{\alpha\in\Lambda}c_\alpha z^\alpha\Big|,
\end{aligned}$$

where the last inequality follows from a result of Harris [9, Theorem 1] (see also [5, (13)]). This completes the proof. □

As we have already mentioned, very recently [2, Corollary 5.3] has shown that for every $\varepsilon>0$ there exists $\kappa>0$ such that (2) holds with $\kappa(1+\varepsilon)^m$. The main idea for the proof is to replace (5) by a similar inequality in which we have mixed sums with k and $m-k$ indices (instead of 1 and $m-1$, as we have here). This allows to use instead of (3) the following inequality:

$$\Big(\sum_{|\alpha|=m}|c_\alpha|^2\Big)^{1/2}\le c_p^m\Big(\int_{\mathbb{T}^N}\Big|\sum_{|\alpha|=m}c_\alpha w^\alpha\Big|^p d\sigma(w)\Big)^{\frac{1}{p}}\ \text{for } 1\le p\le 2.$$

A good control on the constants c_p (that tend to 1 as p goes to 2) gives the improvement on the constant in (2) presented in [2]. In our setting, by dividing by $\alpha+1$, we are using (1), which already has constant 1. Hence this new approach does not improve the constants in our setting.

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