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Separating Maps between Spaces of Vector-Valued Absolutely Continuous Functions

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Abstract. In this paper we give a description of separating or disjointness preserving linear bijections on spaces of vector-valued absolutely continuous functions defined on compact subsets of the real line. We obtain that they are continuous and biseparating in the finite-dimensional case. The infinite-dimensional case is also studied.

1 Introduction

V. D. Pathak obtained a characterization of linear isometries between spaces of scalar-valued absolutely continuous functions defined on compact subsets of the real line [16]. In this paper, we are interested in obtaining a complete description of maps that preserve disjointness on spaces of vector-valued absolutely continuous functions also defined on compact subsets of the real line. These maps are usually called separating or disjointness preserving.

It is well known that separating linear maps between spaces of scalar-valued continuous functions defined on compact or locally compact spaces are automatically continuous and that there exists a homeomorphism between the underlying spaces [8, 13, 14]. In a more general context, J. J. Font proved that a separating linear bijection between regular Banach function algebras which satisfy Ditkin's condition is continuous and their structure spaces are homeomorphic [7].

For spaces of vector-valued continuous functions, it is necessary to require that the inverse map be also separating to obtain a similar characterization. If a bijective map and its inverse are separating, we call it *biseparating*. There are several papers that deal with such maps on spaces of continuous functions and results about automatic continuity and topological links between the underlying spaces are obtained (see [1, 3–5, 10, 11]). Nevertheless, we do not know much about separating maps on spaces of vector-valued continuous functions. Namely, in spaces of continuous functions vanishing at infinity, just one result of automatic continuity was given by J. Araujo for spaces with finite dimension [2, Theorem 5.4].

In this paper, we study bijective and separating linear maps between spaces of absolutely continuous functions defined on compact subsets of the real line and taking

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values in arbitrary Banach spaces. We obtain a description of such maps which allows us to prove that their inverses are also separating and to deduce their automatic continuity in the finite-dimensional case. Besides we show with an example that it is not possible to obtain an analogous result when we deal with Banach spaces of infinite dimension. For this reason, we consider biseparating maps in that case.

Preliminaries and Notation

From now on, X and Y will be compact subsets of the real line and E and F will be arbitrary $\mathbb{K}$-Banach spaces, where $\mathbb{K}$ denotes the field of real or complex numbers.

If A is a subset of X, then $\operatorname{int}(A)$ denotes the interior of A in X, $\operatorname{cl}(A)$ denotes its closure and $\operatorname{bd}(A)$ its boundary. On the other hand, χ_A denotes the characteristic function on A. Finally, we define a *partition* of $A \subset X$ to be any finite family $\{x_i\}_{i=0}^n$ of points of A which satisfy $x_0 < x_1 < \cdots < x_n$.

Given a function $f\colon X \to E$, we define the *cozero set* of f as $c(f) := \{x \in X : f(x) \neq 0\}$. Also, for any $x \in X$, we denote by δ_x the functional *evaluation at the point* x, and finally, if $e \in E$, then $\hat{e}$ will be the constant function from X to E taking the value e.

Throughout this paper the word "homeomorphism" will be synonymous with "surjective homeomorphism".

Definitions and Previous Results

The space of absolutely continuous functions has usually been studied in the scalar context, that is, when the functions take real or complex values (see [12, Section 18]). In this part of the paper we study it when the functions take values in arbitrary Banach spaces.

Definition 1.1 A function $f\colon X \to E$ is said to be *absolutely continuous* on X if, given any $\varepsilon > 0$, there exists an $\delta > 0$ such that

$$\sum_{i=1}^{n} \|f(b_i) - f(a_i)\| < \varepsilon$$

for each finite family of non-overlapping open intervals $\{(a_i, b_i)\}_{i=1}^n$ whose extreme points belong to X with

$$\sum_{i=1}^{n} (b_i - a_i) < \delta.$$

Then $AC(X, E)$ will denote the space of E-valued absolutely continuous functions on X. When $E = \mathbb{K}$, we will consider $AC(X) := AC(X, \mathbb{K})$.

Definition 1.2 Given $f \in AC(X, E)$, we define the *variation* of f on X as

$$V(f; X) := \sup\Big\{ \sum_{i=1}^{n} \|f(x_i) - f(x_{i-1})\| : \{x_i\}_{i=0}^n \text{ is a partition of } X,\ n \in \mathbb{N} \Big\}.$$

Throughout the paper we will consider $AC(X,E)$ endowed with the norm $\|\cdot\|_{AC}$ defined by $\|f\|_{AC} := \|f\|_\infty + V(f;X)$ for each $f \in AC(X,E)$, where $\|\cdot\|_\infty$ denotes the supremum norm.

The next lemmas, whose proofs are straightforward, show some properties of the space of absolutely continuous functions which are the key tools to prove some further results. In particular, Lemma 1.5 proves for $AC(X)$ the existence of a partition of the unity (see [9, Lemma 1]).

Lemma 1.3 *$(AC(X,E), \|\cdot\|_{AC})$ is a Banach space.*

Lemma 1.4 *Let $f \in AC(X)$ and $g \in AC(X,E)$. Then $f \cdot g \in AC(X,E)$.*

Lemma 1.5 *Let $\{V_i\}_{i=1}^n$ be an open covering of X. Then there exist $\{f_i\}_{i=1}^n \subset AC(X)$ with $0 \le f_i \le 1$ and $c(f_i) \subset V_i$ for $i = 1, \dots, n$ such that $\sum_{i=1}^n f_i = 1$.*

2 Separating Maps

Definition 2.1 A map $T\colon AC(X,E) \to AC(Y,F)$ is said to be *separating* if it is linear and $c(Tf) \cap c(Tg) = \varnothing$ whenever $f, g \in AC(X,E)$ satisfy $c(f) \cap c(g) = \varnothing$. Equivalently, a linear map $T\colon AC(X,E) \to AC(Y,F)$ is *separating* if $\|Tf(y)\|\|Tg(y)\| = 0$ for all $y \in Y$ whenever $f, g \in AC(X,E)$ satisfy $\|f(x)\|\|g(x)\| = 0$ for all $x \in X$. Also, T is said to be *biseparating* if it is bijective and both T and T^{-1} are separating.

From now on we will assume that $T\colon AC(X,E) \to AC(Y,F)$ is a separating and bijective map unless otherwise stated.

Definition 2.2 For each $y \in Y$, we define the map $\delta_y \circ T\colon AC(X,E) \to F$ as $(\delta_y \circ T)(f) := Tf(y)$ for each $f \in AC(X,E)$. Therefore, the *support set* associated with $\delta_y \circ T$ is defined by $\operatorname{supp}(\delta_y \circ T) := \{x \in X : \forall U$ open neighborhood of x, $\exists f \in AC(X,E)$ with $c(f) \subseteq U$ and $Tf(y) \neq 0\}$.

For more details about the next three lemmas see the references [9, 13].

Lemma 2.3 *The set $\operatorname{supp}(\delta_y \circ T)$ is a singleton for every $y \in Y$.*

Definition 2.4 The previous lemma allows us to define a map $h : Y \to X$ such that $h(y)$ is the only point that belongs to $\operatorname{supp}(\delta_y \circ T)$, for each $y \in Y$. We call h the *support map* of T.

Lemma 2.5 *Given $f \in AC(X,E)$ such that $f \equiv 0$ on an open subset U of X, then $Tf \equiv 0$ on $h^{-1}(U)$.*

Lemma 2.6 *The support map h is continuous and onto.*

Proposition 2.7 *Let $f \in AC(X,E)$ such that $f(h(y)) = 0$. Then the following statements hold:*

(i) *If $\operatorname{bd}(h^{-1}(h(y))) = \varnothing$, then $Tf \equiv 0$ on $h^{-1}(h(y))$.*
(ii) *If $\operatorname{bd}(h^{-1}(h(y))) \neq \varnothing$, then $Tf \equiv 0$ on $\operatorname{bd}(h^{-1}(h(y)))$.*

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Proof Fix $y_0 \in Y$ and suppose that $f(h(y_0)) = 0$.

(i) If we assume that $\mathrm{bd}(h^{-1}(h(y_0))) = \varnothing$, we deduce that $h^{-1}(h(y_0))$ is an open and closed set (see [6, p. 24]), and by continuity of h, so is $h(y_0)$. Then applying Lemma 2.5, $Tf \equiv 0$ on $h^{-1}(h(y_0))$.

(ii) In this case, we suppose that $\mathrm{bd}(h^{-1}(h(y_0))) \neq \varnothing$. We must see that $Tf(y) = 0$ for every $y \in \mathrm{bd}(h^{-1}(h(y_0)))$.

We consider the following functions in $AC(X, E)$:

- $f_A := f \cdot \chi_A$ with $A = (-\infty, h(y_0)) \cap X$,
- $f_B := f \cdot \chi_B$ with $B = (h(y_0), \infty) \cap X$,

which satisfy $f = f_A + f_B$, so $Tf = Tf_A + Tf_B$. On the other hand, taking into account the definitions of A and B, it is not hard to see that

$$[\mathrm{cl}(h^{-1}(A)) \backslash h^{-1}(A)] \cup [\mathrm{cl}(h^{-1}(B)) \backslash h^{-1}(B)] = \mathrm{bd}\big(h^{-1}(h(y_0))\big),$$

so we need to prove that $Tf(y) = 0$ for each $y \in \mathrm{cl}(h^{-1}(A)) \backslash h^{-1}(A)$ and $y \in \mathrm{cl}(h^{-1}(B)) \backslash h^{-1}(B)$.

We next prove that if $y \in \mathrm{cl}(h^{-1}(A)) \backslash h^{-1}(A)$, then $y \in h^{-1}(h(y_0))$ and $Tf(y) = 0$. Since $y \in \mathrm{cl}(h^{-1}(A))$, there exists a sequence (y_n) in $h^{-1}(A)$ converging to y. By continuity of h, we obtain that $h(y_n)$ converges to $h(y)$. Besides $\mathrm{cl}(A) \setminus A = \{h(y_0)\}$, so $h(y_n)$ converges to $h(y_0)$ and then $h(y) = h(y_0)$. In order to show that $Tf(y) = 0$, we will prove that $Tf_A(y) = 0$ and $Tf_B(y) = 0$. By Lemma 2.5, it is obvious that $Tf_B(y_n) = 0$ for each $n \in \mathbb{N}$, and by continuity of Tf_B we deduce that $Tf_B(y) = 0$. We now see that $Tf_A(y) = 0$. Suppose that $Tf_A(y) \neq 0$. Let (z_n) be a sequence in $h^{-1}(A)$ converging to y and such that $\|f_A(h(z_n))\| < 1/n^3$ for each $n \in \mathbb{N}$. Taking a subsequence if necessary, we can consider disjoint open neighborhoods U_n of $h(z_n)$ for each $n \in \mathbb{N}$, such that $\|f_A(x)\| < 1/n^3$ for all $x \in U_n$ and $V(f_A; U_n) < 1/n^3$. Also, we take compact neighborhoods K_n of $h(z_n)$ with $K_n \subset U_n$ for every $n \in \mathbb{N}$. As each K_n is a compact subset of the real line, we can consider the least compact interval $[m_n, M_n]$ in $\mathbb{R}$ such that $K_n \subset [m_n, M_n]$ for each $n \in \mathbb{N}$. Then since each $K_n \subset U_n$ and U_n is an open set, there exists $\varepsilon_n > 0$ satisfying that $(m_n - \varepsilon_n, m_n + \varepsilon_n) \subset U_n$ and $(M_n - \varepsilon_n, M_n + \varepsilon_n) \subset U_n$ for every $n \in \mathbb{N}$. Finally, we define $g_n \in AC(X)$ for each $n \in \mathbb{N}$ in the following way:

- $g_n \equiv n$ on $[m_n, M_n] \cap X$,
- $g_n \equiv 0$ on $X \backslash (m_n - \frac{\varepsilon_n}{2}, M_n + \frac{\varepsilon_n}{2}) \cap X$,
- g_n is linear on $(m_n - \frac{\varepsilon_n}{2}, m_n) \cup (M_n, M_n + \frac{\varepsilon_n}{2})$.

Each function g_n satisfies $g_n \equiv n$ on K_n, $c(g_n) \subset U_n$, $\|g_n\|_\infty = n$, and $V(g_n; X) = 2n$. Now we define the function $g_0 := \sum_{n=1}^{\infty} f_A g_n$ and we are going to see that g_0 belongs to $AC(X, E)$.

It is enough to see that $\|f_A g_n\|_{AC} < 4/n^2$ for each $n \in \mathbb{N}$. Notice at this point that $c(f_A g_n) \subset U_n$, so we just need to study $\|f_A g_n\|_{AC}$ on U_n for every $n \in \mathbb{N}$. It is obvious that $\|f_A g_n\|_\infty < 1/n^2$ on U_n for each $n \in \mathbb{N}$. On the other hand, if we consider $\{x_i\}_{i=0}^{m}$ any partition of U_n, we have that

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$$\sum_{i=1}^{m} \|(f_A g_n)(x_i) - (f_A g_n)(x_{i-1})\|$$
$$\leq \sum_{i=1}^{m} \|(f_A g_n)(x_i) - f_A(x_i) g_n(x_{i-1})\| + \sum_{i=1}^{m} \|f_A(x_i) g_n(x_{i-1}) - (f_A g_n)(x_{i-1})\|$$
$$\leq \|f_{A|_{U_n}}\|_\infty \sum_{i=1}^{m} |g_n(x_i) - g_n(x_{i-1})| + \|g_n\|_\infty \sum_{i=1}^{m} \|f_A(x_i) - f_A(x_{i-1})\|$$
$$\leq \|f_{A|_{U_n}}\|_\infty V(g_n; U_n) + \|g_n\|_\infty V(f_A; U_n) < \frac{3}{n^2},$$

and then $V(f_A g_n; U_n) \leq 3/n^2$ for each $n \in \mathbb{N}$.

Now let V_n be an open neighborhood of $h(z_n)$ with $V_n \subset K_n$ for every $n \in \mathbb{N}$. It is obvious that $g_0 \equiv n f_A$ on V_n, and by Lemma 2.5 we deduce that $Tg_0 \equiv nTf_A$ on $h^{-1}(V_n)$. Consequently, $Tg_0(z_n) = nTf_A(z_n)$ for each $n \in \mathbb{N}$. Taking into account that $Tf_A(y) \neq 0$ and the fact that $Tf_A(z_n)$ converges to $Tf_A(y)$, we can conclude that $\|Tg_0(z_n)\|$ converges to ∞. This behavior implies that Tg_0 is not continuous, which is absurd.

In a similar way, we can see that $y \in h^{-1}(h(y_0))$ and $Tf(y) = 0$ whenever $y \in \mathrm{cl}(h^{-1}(B)) \setminus h^{-1}(B)$. ■

3 The Finite-Dimensional Case

In this section, we study separating bijections between spaces of absolutely continuous functions that take values in finite-dimensional normed spaces. We suppose that the spaces E and F are both n-dimensional and $\{e_1, \dots, e_n\}$ is a basis of E.

Lemma 3.1 *Let $f \in AC(X, E)$ such that $f(h(y_0)) = 0$. Then there exists $y_1 \in h^{-1}(h(y_0))$ such that $\{T\widehat{e_i}(y_1) : i = 1, \dots, n\}$ is a basis of F.*

Proof By Proposition 2.7, we know that there exists $y_1 \in h^{-1}(h(y_0))$ such that $Tf(y_1) = 0$. We will prove that $\{T\widehat{e_i}(y_1) : i = 1, \dots, n\}$ is a basis of F. As E and F have the same dimension, it is enough to show that they are linearly independent.

Suppose that $T\widehat{e_1}(y_1), \dots, T\widehat{e_n}(y_1)$ are linearly dependent. Therefore, we can take $\mathrm{f} \in F$ linearly independent with them and consider the non-vanishing function $T^{-1}\widehat{\mathrm{f}}$. Now as $\{e_1, \dots, e_n\}$ is a basis of E, there exist $\alpha_1, \dots, \alpha_n \in \mathbb{K}$, not all of them equal to zero, such that $T^{-1}\widehat{\mathrm{f}}(h(y_0)) = \sum_{i=1}^n \alpha_i e_i$. For this reason, if we define the function $g := \sum_{i=1}^n \alpha_i \widehat{e_i} \in AC(X, E)$, we obtain that $(T^{-1}\widehat{\mathrm{f}} - g)(h(y_0)) = 0$, and then $(\widehat{\mathrm{f}} - Tg)(y_1) = 0$ applying Proposition 2.7 again. This implies that $\mathrm{f} = \sum_{i=1}^n \alpha_i T\widehat{e_i}(y_1)$, which is a contradiction. ■

Theorem 3.2 *h is a homeomorphism.*

Proof We know that h is a continuous, onto and closed map. We only need to prove that h is injective. Suppose that there exist two distinct points $y_0, y_1 \in Y$ such that $h(y_0) = h(y_1)$ and we will study the three possible situations.

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Assume that $y_0, y_1 \in \mathrm{bd}(h^{-1}(h(y_0)))$. If $\{\mathrm{f}_1, \dots, \mathrm{f}_n\}$ is a basis of F, since T is an onto map, we know that there exist $g_1, \dots, g_n \in AC(X, E)$ such that $Tg_i = \widehat{\mathrm{f}_i}$ for each i. We claim that $g_1(h(y_0)), \dots, g_n(h(y_0))$ are linearly independent. Suppose that it is not true. Therefore, there exist $\alpha_1, \dots, \alpha_n \in \mathbb{K}$, not all of them equal to zero, such that $\sum_{i=1}^n \alpha_i g_i(h(y_0)) = 0$. By Proposition 2.7, we obtain that $T \sum_{i=1}^n \alpha_i g_i(y_0) = 0$, and this implies that $\sum_{i=1}^n \alpha_i \mathrm{f}_i = 0$, which is not possible. Then for each $f \in AC(X, E)$ we have that $f(h(y_0)) = \sum_{i=1}^n \beta_i g_i(h(y_0))$ for $\beta_1, \dots, \beta_n \in \mathbb{K}$ not all of them equal to zero. Applying Proposition 2.7, we obtain that $Tf(y_0) = T\sum_{i=1}^n \beta_i g_i(y_0) = \sum_{i=1}^n \beta_i \widehat{\mathrm{f}_i}(y_0) = \sum_{i=1}^n \beta_i \mathrm{f}_i$ and $Tf(y_1) = T\sum_{i=1}^n \beta_i g_i(y_1) = \sum_{i=1}^n \beta_i \widehat{\mathrm{f}_i}(y_1) = \sum_{i=1}^n \beta_i \mathrm{f}_i$, that is, $Tf(y_0) = Tf(y_1)$ for each $f \in AC(X, E)$. This behavior implies that T is not onto, in contradiction with our assumption.

Suppose now that $\mathrm{bd}(h^{-1}(h(y_0))) = \varnothing$. With similar reasoning as in the previous situation we obtain the same contradiction.

Finally, we assume that $y_0 \in \mathrm{bd}(h^{-1}(h(y_0)))$ and $y_1 \in \mathrm{int}(h^{-1}(h(y_0)))$. Let $g \in AC(Y, F)$ be a non-zero function with $c(g) \subset \mathrm{int}(h^{-1}(h(y_0)))$ and consider $T^{-1}g$. We are going to prove that there exists an open subset V of X satisfying that $V \cap \{h(y_0)\} = \varnothing$ and $T^{-1}g(x) \neq 0$ for all $x \in V$. If it is not true, $T^{-1}g$ is equal to zero on $X \setminus \{h(y_0)\}$. Besides, we know that $h(y_0)$ is not an isolated point, so we deduce that $T^{-1}g \equiv 0$ on X, which is a contradiction since g is a non-zero function. Therefore, if we consider $x_1 \in V$ and a basis $\{e_i : i = 1, \dots, n\}$ of E, we have that $T^{-1}g(x_1) = \sum_{i=1}^n \alpha_i \widehat{e_i}(x_1)$ for $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ not all of them equal to zero. Applying Proposition 2.7 and Lemma 3.1, we can deduce that $g(y_2) = \sum_{i=1}^n \alpha_i T\widehat{e_i}(y_2) \neq 0$ for some $y_2 \in h^{-1}(x_1)$, which is not possible since $c(g) \subset \mathrm{int}(h^{-1}(h(y_0)))$. ■

Corollary 3.3 *Let $f \in AC(X, E)$ such that $f(h(y)) = 0$. Then $Tf(y) = 0$.*

Proof It is an obvious application of Proposition 2.7 and Theorem 3.2. ■

Remark 3.4 For any $y \in Y$ fixed, we define the function $g := f - \widehat{f(h(y))} \in AC(X, E)$ for each $f \in AC(X, E)$. It is obvious that $g(h(y)) = 0$, and by the previous corollary, we deduce that $Tg(y) = 0$. For this reason, we obtain $Tf(y) = T\widehat{f(h(y))}(y)$ for all $f \in AC(X, E)$ and $y \in Y$. Therefore, we define the map J_y for each $y \in Y$ in the following way:

$$\begin{aligned} J_y \colon E &\longrightarrow F \\ e &\mapsto J_y(e) := T\hat{e}(y). \end{aligned}$$

Lemma 3.5 *The map J_y is linear, bijective and continuous for every $y \in Y$.*

Proof Obviously each J_y is linear. We next see that J_y is a bijective map.

First, we will prove that J_y is injective. If $e \neq 0$ and $\{e_i : i = 1, \dots, n\}$ is a basis of E, then there exist $\alpha_1, \dots, \alpha_n \in \mathbb{K}$, not all of them equal to zero, such that $e = \sum_{i=1}^n \alpha_i e_i$, and this implies that $\hat{e}(h(y)) = \sum_{i=1}^n \alpha_i \widehat{e_i}(h(y))$. By Lemma 3.1 and Corollary 3.3 we deduce that $T\hat{e}(y) = \sum_{i=1}^n \alpha_i T\widehat{e_i}(y) \neq 0$, and by definition of J_y we conclude that $J_y(e) \neq 0$.

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Secondly, we will see that J_y is an onto map. Given f $\in F$, since T is surjective, there exists $g \in AC(X,E)$ such that $Tg = \widehat{\text{f}}$, in particular, $Tg(y) = $ f. We define $e := g(h(y)) \in E$. It is obvious that $(\hat{e} - g)(h(y)) = 0$, and by Corollary 3.3 we deduce that $T(\hat{e} - g)(y) = 0$. This implies that $J_y(e) =$ f.

Finally, it is trivial to see that each J_y is continuous, since it is a linear map and E is a finite-dimensional normed space. ∎

Theorem 3.6 *Let $T\colon AC(X,E) \to AC(Y,F)$ be a separating and bijective map with E and F n-dimensional normed spaces. Then there exist a homeomorphism $h\colon Y \to X$ and a map $J_y\colon E \to F$ linear, bijective and continuous for each $y \in Y$, such that*

$$Tf(y) = J_y(f(h(y)))$$

for every $f \in AC(X,E)$ and $y \in Y$. Also, T is continuous and biseparating.

Proof The representation of T follows by Remark 3.4 and from the definition of J_y above. To see that T is a continuous map we apply the closed graph theorem, so we just need to prove that T is a closed map (see [15, Theorem 7.3.2]). Therefore, it is enough to see that if we take (f_n) in $AC(X,E)$ converging to 0 and (Tf_n) converges to g, then $g \equiv 0$.

First, we are going to prove that $\delta_y \circ T$ is a continuous map for each $y \in Y$. Fix $y \in Y$. It is obvious that $\delta_y \circ T$ is linear, and, by the representation of T, we have that $\|\delta_y \circ T(f)\| \leq \|J_y\| \|f\|_\infty$ for each $f \in AC(X,E)$. From this inequality, we obtain that $\delta_y \circ T$ is continuous and consequently that $(\delta_y \circ T(f_n))$ converges to 0.

On the other hand, $\|Tf_n(y) - g(y)\| \leq \|Tf_n - g\|_\infty \leq \|Tf_n - g\|_{AC}$ for each $n \in \mathbb{N}$, and since we assume that (Tf_n) converges to g, we deduce that $(Tf_n(y))$ converges to $g(y)$ for each $y \in Y$. Combined with the above, we conclude that $g(y) = 0$ for all $y \in Y$ and this completes the proof that T is continuous.

Finally, we prove that T is a biseparating map. It is enough to see that $T^{-1}\colon AC(Y,F) \to AC(X,E)$ is separating. Suppose that T^{-1} is not separating. Then there exist $f, g \in AC(Y,F)$ with $c(f) \cap c(g) = \varnothing$ such that $c(T^{-1}f) \cap c(T^{-1}g) \neq \varnothing$. For this reason, there exists $x_0 \in X$ with $T^{-1}f(x_0) \neq 0$ and $T^{-1}g(x_0) \neq 0$. As $\{e_1, \dots, e_n\}$ is a basis of E, we can take $\alpha_1, \dots, \alpha_n \in \mathbb{K}$, not all of them equal to zero, such that $T^{-1}f(x_0) = \sum_{i=1}^n \alpha_i \widehat{e_i}(x_0)$ and $\beta_1, \dots, \beta_n \in \mathbb{K}$, not all of them equal to zero, such that $T^{-1}g(x_0) = \sum_{i=1}^n \beta_i \widehat{e_i}(x_0)$. Applying Lemma 3.1 and Corollary 3.3, we can deduce that $f(h^{-1}(x_0)) = \sum_{i=1}^n \alpha_i T\widehat{e_i}(h^{-1}(x_0)) \neq 0$ and $g(h^{-1}(x_0)) = \sum_{i=1}^n \beta_i T\widehat{e_i}(h^{-1}(x_0)) \neq 0$, which contradicts the fact that f and g have disjoint cozeros. ∎

4 The Infinite-Dimensional Case

The next example shows that it is not possible to obtain a similar result as in the previous case when we deal with infinite-dimensional Banach spaces. For this reason, we study biseparating maps instead of separating in this case.

Example 4.1 Let c_0 be the space of all scalar-valued sequences that converge to zero and let $T\colon AC([0,1], c_0) \to AC([0,1] \cup [2,3], c_0)$ be a bijective, separating and continuous map defined by $Tf(x) = (\lambda_1, \lambda_3, \lambda_5, \dots)$ and $Tf(2+x) = (\lambda_2, \lambda_4, \lambda_6, \dots)$, when $f(x) = (\lambda_1, \lambda_2, \lambda_3, \dots) \in c_0$ for each $x \in [0, 1]$. It is easy to see that T^{-1} is not a separating map.

Remark 4.2 Similarly to the previous example, a separating bijection from $AC([0, 1], \mathbb{R}^2)$ to $AC([0, 1] \cup [2, 3], \mathbb{R})$ can be constructed that is not biseparating. This fact allows us to conclude that Theorem 3.6 is not true if we do not suppose that E and F have the same dimension.

Remark 4.3 In this final section, $T\colon AC(X, E) \to AC(Y, F)$ will be a biseparating map and E and F will be Banach spaces of infinite dimension. Since T is biseparating, we obtain two different continuous support maps h and k asociated with T and T^{-1}, respectively.

Theorem 4.4 *h is a homeomorphism.*

Proof It is not difficult to see that h and k are inverse maps. The proof given in [9, Theorem 1(8)] for group algebras can easily be adapted to our context. ■

Corollary 4.5 *Let $f \in AC(X, E)$ such that $f(h(y)) = 0$. Then $Tf(y) = 0$.*

Proof It is clear applying Proposition 2.7 and previous theorem. ■

Remark 4.6 With the same construction as in Remark 3.4, we obtain $Tf(y) = T\widehat{f(h(y))}(y)$ for all $f \in AC(X, E)$ and $y \in Y$, and we define the map J_y for each $y \in Y$, as in the previous case.

Lemma 4.7 *J_y is linear and bijective for every $y \in Y$.*

Proof We obtain that each J_y is linear and onto in a similar way as in the finite-dimensional case. We next prove that J_y is injective. Suppose that it is not true. Thus we consider $e \in E$ with $e \neq 0$ such that $J_y(e) = 0$. We have proved that k is a homeomorphism, so there exists $x \in X$ such that $y = k(x)$, and then $J_{k(x)}(e) = 0$. Since $T\hat{e}(k(x)) = 0$, applying Corollary 4.5 to the separating map T^{-1}, we obtain that $T^{-1}(T\hat{e})(x) = 0$, which implies that $\hat{e}(x) = 0$ in contradiction with $e \neq 0$. ■

Theorem 4.8 *Let $T\colon AC(X, E) \to AC(Y, F)$ be a biseparating map with E and F infinite-dimensional Banach spaces. Then there exist a homeomorphism $h\colon Y \to X$ and a map $J_y\colon E \to F$ linear and bijective for each $y \in Y$, such that*

$$Tf(y) = J_y(f(h(y)))$$

for every $f \in AC(X, E)$ and $y \in Y$. Also, if Y has no isolated points, then T is continuous.

Proof By Remark 4.6 and the definition of J_y we deduce the representation of T. We only need to prove that T is continuous if Y has no isolated points. We will prove that $\delta_y \circ T$ is continuous for every $y \in Y$, and then applying the closed graph theorem in a similar way as in Theorem 3.6, we will deduce that T is a continuous map.

Suppose that there exists $y_0 \in Y$ such that $\delta_{y_0} \circ T$ is not continuous. Then we consider a sequence (e_n) in E such that $\|e_n\| \leq 1/n^3$ and $\|T\widehat{e_n}(y_0)\| > 1$ for all $n \in \mathbb{N}$. In this way, we can find a sequence (y_n) in Y, strictly monotone and converging to y_0, such that $\|T\widehat{e_n}(y_n)\| > 1$ for each $n \in \mathbb{N}$.

We now take disjoint open neighborhoods U_n of $h(y_n)$ for each $n \in \mathbb{N}$, and define $f_n \in AC(X)$ such that $f_n(h(y_n)) = 1$, $0 \leq f_n \leq 1$ and $c(f_n) \subset U_n$ for all $n \in \mathbb{N}$. Finally, we consider the function $f := \sum_{n=1}^{\infty} f_n\widehat{e_n}$ that belongs to $AC(X, E)$.

It is obvious that $f(h(y_0)) = 0$ and, by Corollary 4.5, $Tf(y_0) = 0$. On the other hand, $(f - \widehat{e_n})(h(y_n)) = 0$ and then $Tf(y_n) = T\widehat{e_n}(y_n)$ for all $n \in \mathbb{N}$. This implies that $\|Tf(y_n)\| > 1$ for each $n \in \mathbb{N}$, and we obtain a contradiction, since Tf is continuous. ■

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References

[1] Y. A. Abramovich and A. K. Kitover, *Inverses of disjointness preserving operators*. Mem. Amer. Math. Soc. **143**(2000), no. 679.

[2] J. Araujo, *Separating maps and linear isometries between some spaces of continuous functions*. J. Math. Anal. Appl. **226**(1998), no. 1, 23–39. doi:10.1006/jmaa.1998.6031

[3] ———, *Realcompactness and spaces of vector-valued functions*. Fund. Math. **172**(2002), no. 1, 27–40. doi:10.4064/fm172-1-3

[4] ———, *Realcompactness and Banach-Stone theorems*. Bull. Belg. Math. Soc. Simon Stevin **11**(2004), no. 2, 247–258.

[5] J. Araujo and K. Jarosz, *Automatic continuity of biseparating maps*. Studia Math. **155**(2003), no. 3, 231–239. doi:10.4064/sm155-3-3

[6] R. Engelking, *General Topology*. Heldermann Verlag, Berlín, 1989.

[7] J. J. Font, *Automatic continuity of certain isomorphisms between regular Banach function algebras*. Glasgow Math. J. **39**(1997), no. 3, 333–343. doi:10.1017/S0017089500032250

[8] J. J. Font and S. Hernández, *On separating maps between locally compact spaces*. Arch. Math. (Basel) **63**(1994), no. 2, 158–165.

[9] ———, *Automatic continuity and representation of certain linear isomorphisms between group algebras*. Indag. Math. **6**(1995), no. 4, 397–409. doi:10.1016/0019-3577(96)81755-1

[10] H. L. Gau, J. S. Jeang, and N. C. Wong, *Biseparating linear maps between continuous vector-valued function spaces*. J. Aust. Math. Soc. **74**(2003), no. 1, 101–109. doi:10.1017/S1446788700003153

[11] S. Hernández, E. Beckenstein, and L. Narici, *Banach-Stone theorems and separating maps*. Manuscripta Math. **86**(1995), no. 4, 409–416. doi:10.1007/BF02568002

[12] E. Hewitt and K. Stromberg, *Real and Abstract Analysis.* Graduate Texts in Mathematics 25. Springer-Verlag, New York, 1975.

[13] K. Jarosz, *Automatic continuity of separating linear isomorphisms*. Canad. Math. Bull. **33**(1990), no. 2, 139–144.

[14] J. S. Jeang and N. C. Wong, *Weighted composition operators of $C_0(X)$'s*. J. Math. Anal. Appl. **201**(1996), no. 3, 981–993. doi:10.1006/jmaa.1996.0296

[15] R. Larsen, *Functional Analysis. An Introduction.* Pure and Applied Mathematics 15. Marcel Dekker, New York, 1973.

[16] V. D. Pathak, *Linear isometries of spaces of absolutely continuous functions*. Canad. J. Math. **34**(1982), no. 2, 298–306.

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