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On a general type of p-adic parabolic equations

Un tipo general de ecuaciones parabólicas p-ádicas

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Abstract. In this paper we study the existence and uniqueness of the Cauchy problem for a general type of p-adic parabolic pseudo-differential operators constructed using the Taibleson operator. The results presented here constitute an extension of some results obtained by Zúñiga-Galindo and the author [13].

Key words and phrases. Parabolic equations, Markov processes, p-adic numbers, ultrametric diffusion.

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Resumen. En este artículo se estudia la existencia y unicidad de soluciones del problema de Cauchy asociado a un tipo general de ecuación parabólica p-ádica, construida usando el operador de Taibleson. Los resultados presentados aquí constituyen una extensión de algunos de los resultados obtenidos por Zúñiga-Galindo y el autor en [13].

Palabras y frases clave. Ecuaciones parabólicas, procesos de Markov, números p-ádicos, difusión ultramétrica.

1. Introduction

In recent years p-adic analysis has received a lot of attention due to its applications in mathematical physics, see e.g. [1], [2], [3], [4], [6], [7], [10], [12], [15] and the references therein. In particular, stochastic models involving Markov processes have appeared in several physical models describing complex systems such as proteins and macromolecules.

In [13] Zúñiga-Galindo and the author studied the following Cauchy problem:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} + a\left(D_T^{\alpha} u\right)(x,t) = f(x,t), \qquad x \in \mathbb{Q}_p^n, \quad t \in (0,T], \\ u(x,0) = \varphi(x), \end{cases} \tag{1}$$

where $a > 0$, $\alpha > 0$ and D_T^{α} is the Taibleson operator of order α defined as

$$\left(D_T^{\alpha} u\right)(x) = \mathcal{F}^{-1}_{\xi \to x}\left(||\xi||_p^{\alpha} \mathcal{F}_{x \to \xi} u\right), \tag{2}$$

where $||\xi||_p = \max\{|\xi_1|_p, \dots, |\xi_n|_p\}$.

The existence and uniqueness of a solution for (1) was established when the initial datum φ belongs to a class of increasing functions (see [13, Thm 1]). Also, there it is shown that the fundamental solution is the transition density of a Markov process with space state $\mathbb{Q}_p^n$ (see [13, Thm. 2]). These results continue Kochubei's work on p-adic parabolic equations [9], [10, Sec. 4].

In this paper we considers the following initial value problem:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} + a_0(x,t)\left(D_T^{\alpha} u\right)(x,t) + \displaystyle\sum_{k=1}^{n} a_k(x,t)(D_T^{\alpha_k} u)(x,t) + \\ \qquad + b(x,t)u(x,t) = f(x,t), \quad x \in \mathbb{Q}_p^n, \quad t \in (0,T], \\ u(x,0) = \varphi(x). \end{cases} \tag{3}$$

here $\alpha > 1$, $0 < \alpha_1 < \ldots < \alpha_n < \alpha$, the coefficients $a_0(x,t)$, $a_1(x,t), \dots,$ $a_n(x,t)$, $b(x,t)$, are real functions and D_T^{β} is the Taibleson operator of order β.

Denote by $\mathfrak{M}_\lambda$ $(\lambda \geq 0)$ the class of complex-valued locally constant functions $\varphi(x)$ on $\mathbb{Q}_p^n$, satisfying

$$|\varphi(x)| \leq C\left(1 + ||x||_p^{\lambda}\right).$$

We solve (3) in the class $\mathfrak{M}_\lambda$ for a suitable λ (see Thm. 2 ahead) following the ideas introduced by Kochubei in [9](see also [10, Sec. 4], [8]).

In the case $n = 1$, our main result, (see Thm. 2), agrees with Kochubei's results (see [9, Thm. 1], [10]).

A different generalization of the p-adic parabolic equations and its Markov processes was given recently by Zúñiga-Galindo in [16].

2. Preliminary results

Let $\mathbb{Q}_p$ be the field of the p-adic numbers. For $x \in \mathbb{Q}_p$, let $v(x)$ denote the valuation of x normalized by the condition $v(p) = 1$, and $|x|_p = p^{-v(x)}$ the normalized absolute value. We extend the p-adic norm to $\mathbb{Q}_p^n$ as follows:

$$||x||_p := \max\{|x_1|_p, \dots, |x_n|_p\}, \qquad \text{for } x = (x_1, \dots, x_n) \in \mathbb{Q}_p^n.$$

Let $S\left(\mathbb{Q}_p^n\right)$ denote the $\mathbb{C}$-vector space of Schwartz-Bruhat functions over $\mathbb{Q}_p^n$. Its dual space $S'\left(\mathbb{Q}_p^n\right)$ is the space of distribution over $\mathbb{Q}_p^n$.

If $\varphi(x) \in S\left(\mathbb{Q}_p^n\right)$, we define its exponent of local constancy as the smallest integer $l \geq 0$ with the property that for any $x \in \mathbb{Q}_p^n$

$$\varphi\left(x+x'\right)=\varphi(x), \quad \text{ if } ||x'||_p \leq p^{-l}.$$

For x, y in $\mathbb{Q}_p^n$ we put $x \cdot y=\sum_{i=1}^n x_i y_i$.

Let Ψ denote an additive character of $\mathbb{Q}_p$, trivial on $\mathbb{Z}_p$ but no on $p^{-1}\mathbb{Z}_p$. For $\varphi \in S\left(\mathbb{Q}_p^n\right)$, we define its Fourier transform by

$$(\mathcal{F}\varphi)(\xi)=\int\limits_{\mathbb{Q}_p^n} \Psi(-x \cdot \xi) \varphi(\xi)\, d^n x,$$

where $d^n x$ denotes the Haar measure of $\mathbb{Q}_p^n$ normalized in such a way that $\mathbb{Z}_p^n$ has measure 1.

2.1. The taibleson operator

We set

$$\Gamma_p^{(n)}(\alpha):=\frac{1-p^{\alpha-n}}{1-p^{-\alpha}}, \quad \alpha \neq 0.$$

This function is called the *p-adic Gamma function*. The function

$$k_\alpha(x)=\frac{||x||_p^{\alpha-n}}{\Gamma_p^{(n)}(\alpha)}, \quad \alpha \in \mathbb{R} \setminus \{0, n\}, \quad x \in \mathbb{Q}_p^n,$$

is called *the multi-dimensional Riesz kernel*. It determines a distribution on $S\left(\mathbb{Q}_p^n\right)$ as follows. If $\alpha \neq 0$, n, and $\varphi \in S\left(\mathbb{Q}_p^n\right)$,

$$\begin{aligned}\langle k_\alpha(x), \varphi(x)\rangle= & \frac{1-p^{-n}}{1-p^{\alpha-n}} \varphi(0)+\frac{1-p^{-\alpha}}{1-p^{\alpha-n}} \int_{||x||_p>1} ||x||_p^{\alpha-n} \varphi(x)\, d^n x \\ & +\frac{1-p^{-\alpha}}{1-p^{\alpha-n}} \int_{||x||_p \leq 1} ||x||_p^{\alpha-n}(\varphi(x)-\varphi(0))\, d^n x .\end{aligned}$$

Thus $k_\alpha \in S'\left(\mathbb{Q}_p^n\right)$, for $\mathbb{R} \backslash \{0, n\}$. In the case $\alpha=0$, by passing to the limit, we obtain

$$\langle k_0(x), \varphi(x)\rangle:=\lim_{\alpha \rightarrow 0}\langle k_\alpha(x), \varphi(x)\rangle=\varphi(0),$$

i.e., $k_0(x)=\delta(x)$, the Dirac delta function, and therefore $k_\alpha \in S'(\mathbb{Q}_p^n)$, for $\mathbb{R} \backslash \{n\}$.

It follows that, for $\alpha>0$,

$$\langle k_{-\alpha}(x), \varphi(x)\rangle=\frac{1-p^\alpha}{1-p^{-\alpha-n}} \int\limits_{\mathbb{Q}_p^n} ||x||_p^{-\alpha-n}(\varphi(x)-\varphi(0))\, d^n x. \tag{4}$$

Lemma 1. [14, Chap. III, Theorem 4.5] *As elements of* $S'(\mathbb{Q}_p^n)$, $(\mathcal{F}k_\alpha)(x)$ *equals* $||x||_p^{-\alpha}$, $\alpha \neq n$.

Definition 1. The Taibleson pseudo-differential operator D_T^α, $\alpha > 0$, is defined as

$$(D_T^\alpha \varphi)(x) = \mathcal{F}^{-1}_{\xi\to x}\left(||\xi||_p^\alpha \mathcal{F}_{x\to\xi}\varphi\right), \qquad \text{for } \varphi \in S(\mathbb{Q}_p^n).$$

As a consequence of the previous Lemma and (4), we get

$$\begin{aligned}(D_T^\alpha \varphi)(x) &= (k_{-\alpha} * \varphi)(x) \\ &= \frac{1-p^\alpha}{1-p^{-\alpha-n}} \int_{\mathbb{Q}_p^n} ||y||_p^{-\alpha-n}(\varphi(x-y)-\varphi(x))\, d^n y. \end{aligned} \tag{5}$$

Let us remark that the right-hand side of (5) makes sense for a wider class of functions, for example, for locally constant functions $\varphi(x)$ satisfying

$$\int_{||x||_p \geq 1} ||x||_p^{-\alpha-n}|\varphi(x)|\, d^n x < \infty.$$

Definition 2. Denote by $\mathfrak{M}_\lambda$ ($\lambda \geq 0$) the class of complex-valued locally constant functions $\varphi(x)$ on $\mathbb{Q}_p^n$, such that

$$|\varphi(x)| \leq C\left(1 + ||x||_p^\lambda\right).$$

If a function φ depends also on a parameter t, we shall say that $\varphi \in \mathfrak{M}_\lambda$ uniformly with respect to t, if C and the corresponding exponent of local constancy do not depend on t.

2.2. The parametrized equation

As in the Euclidean case, the first step is the study of the parametrized fundamental solution $Z(x,t,y,\theta)$ of the Cauchy problem

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} + a_0(y,\theta)\,(D_T^\alpha u)(x,t) = 0, & x \in \mathbb{Q}_p^n, \quad t \in (0,T], \\ u(x,0) = \varphi(x), \end{cases} \tag{6}$$

where $y \in \mathbb{Q}_p^n$ and $\theta > 0$ are parameters. This equation was studied in the recent paper [13] by Zúñiga-Galindo and the author.

In this article we consider the following fundamental solution:

$$Z(x,t,y,\theta) = \int_{\mathbb{Q}_p^n} \Psi(x\cdot\xi) e^{-a_0(y,\theta)t||\xi||_p^\alpha}\, d^n\xi.$$

Lemma 2. *The fundamental solution of (6) $Z(x,t,y,\theta)$, has the following properties*

$$Z(x,t,y,\theta) \leq Ct\left(t^{1/\alpha+||x||_p}\right)^{-\alpha-n}, \tag{7}$$

$$\left|\frac{\partial Z}{\partial t}(x,t,y,\theta)\right| \leq C\left(t^{1/\alpha+||x||_p}\right)^{-\alpha-n}, \tag{8}$$

$$|(D_T^{\gamma}Z)(x,t,y,\theta)| \leq C\left(t^{1/\alpha+||x||_p}\right)^{-\gamma-n}, \tag{9}$$

where the constants do not depend on y, θ.

Proof. These results where established in Lemmas 3 and 8 of [13]. ☑

As an [13], we get the identities

Lemma 3.

$$\int\limits_{\mathbb{Q}_p^n} Z(x,t,y,\theta)\, d^n x = 1, \tag{10}$$

$$\frac{\partial Z}{\partial t}(x,t,y,\theta) = -a_0(y,\theta)\int\limits_{\mathbb{Q}_p^n} \psi(x\cdot\xi)||\xi||_p^{\alpha} e^{-a_0(y,\theta)t||\xi||_p^{\alpha}}\, d^n\xi, \tag{11}$$

$$\left(D_T^{\gamma}Z\right)(x,t,y,\theta) = \int\limits_{\mathbb{Q}_p^n} \psi(x\cdot\xi)||\xi||_p^{\gamma} e^{-a_0(y,\theta)t||\xi||_p^{\alpha}}\, d^n\xi, \tag{12}$$

$$\int\limits_{\mathbb{Q}_p^n} (D_T^{\gamma}Z)(x,t,y,\theta)\, d^n x = 0. \tag{13}$$

3. Uniqueness of the solution

In this section we assume that the coefficients $a_k(x,t)$, $k = 0,1,\dots,n$ are non-negative bounded continuous functions, and that $b(x,t)$ is a continuous bounded function. Let $0 \leq \gamma < \alpha_1$ (if $a_1(x,t) = \cdots = a_n(x,t) = 0$, we shall assume that $0 \leq \gamma < \alpha$). The proof of the following Theorem is a simple variation of the one given by Kochubei in [10, Thm 4.5] for the case $n = 1$.

Theorem 1. [10, Thm. 4.5] *If $u(x,t)$ is a solution of* (3) *with $f(x,t) = 0$, and such that $u \in \mathfrak{M}_{\gamma}$ uniformly with respect to t, and $u(x,0) = 0$, then $u(x,t) = 0$ for any $x \in \mathbb{Q}_p^n$ and $t \in (0,T]$.*

4. Heat potentials

We now consider the heat potential

$$u(x,t,\tau) := \int_{\tau}^{t} \int_{\mathbb{Q}_p^n} Z(x-y,t-\theta,y,\theta) f(y,\theta)\, d^n y\, d\theta,$$

where $\tau < t$, $f(x,t)$ is uniformly locally constant in $x \in \mathbb{Q}_p^n$, continuous in $(x,t) \in \mathbb{Q}_p^n \times (0,T]$, and

$$|f(x,t)| \leq C t^{-\rho} \left(1 + ||x||_p^{\lambda}\right),$$

for some $0 \leq \rho < 1$, and $0 \leq \lambda < \alpha$.

Next we calculate the derivative with respect to t and the action of the Taibleson operator on this potentials. This can be achieved using the techniques presented in [10, Sec. 4.5]. We formally summarize these facts for future reference as follows

Lemma 4. *With the above notations,*

i) $$\frac{\partial u}{\partial t}(x,t,\tau) = f(x,t) + \int_{\tau}^{t} \int_{\mathbb{Q}_p^n} \frac{\partial Z}{\partial t}(x-y,t-\theta,y,\theta)(f(y,\theta)-f(x,\theta))\, d^n y\, d\theta$$
$$+ \int_{\tau}^{t} f(x,\theta) \int_{\mathbb{Q}_p^n} \frac{\partial Z}{\partial t}(x-y,t-\theta,y,\theta)\, d^n y\, d\theta.$$

ii) *If* $\lambda < \gamma < \alpha$, *then*

$$(D_T^{\gamma} u)(x,t,\tau) = \int_{\tau}^{t} \int_{\mathbb{Q}_p^n} Z_{\gamma}(x-y,t-\theta,y,\theta) f(y,\theta)\, d^n y\, d\theta, \quad \lambda < \gamma < \alpha.$$

iii) $$(D_T^{\alpha} u)(x,t,\tau) = \int_{\tau}^{t} \int_{\mathbb{Q}_p^n} Z_{\alpha}(x-y,t-\theta,y,\theta)(f(y,\theta)-f(x,\theta))\, d^n y\, d\theta$$
$$+ \int_{\tau}^{t} f(x,\theta) \int_{\mathbb{Q}_p^n} \left(Z_{\alpha}(x-y,t-\theta,y,\theta) - Z_{\alpha}(x-y,t-\theta,x,\theta)\right) d^n y\, d\theta.$$

5. The Cauchy problem

In this section we construct a fundamental solution for the following Cauchy problem

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} + a_0(x,t)\,(D_T^{\alpha} u)\,(x,t) + \displaystyle\sum_{k=1}^{n} a_k(x,t)(D_T^{\alpha_k} u)(x,t) + \\ \qquad + b(x,t)u(x,t) = f(x,t), \quad x \in \mathbb{Q}_p^n, \quad t \in (0,T], \\ \qquad u(x,0) = \varphi(x). \end{cases} \tag{14}$$

We shall assume that $\alpha > 1$ and that $0 < \alpha_1 < \ldots < \alpha_n < \alpha$, and that the coefficients $a_0(x,t)$, $a_1(x,t)$, $\ldots, a_n(x,t)$, $b(x,t)$ belong (with respect to $x \in \mathbb{Q}_p^n$) to the class $\mathfrak{M}_0$ uniformly with respect to $t \in [0,T]$, and satisfy the Hölder condition in t, with an exponent $\nu \in (0,1]$, uniformly with respect to $x \in \mathbb{Q}_p^n$. We also assume the uniform parabolicity condition $a_0(x,t) \geq \mu > 0$, and that $\alpha_{n+1} = \alpha(1-\nu) > \alpha_n$.

As in [10, Sec. 4.5] we look for a fundamental solution of (14) of the form

$$\Gamma(x,t,\xi,\tau) = Z(x-\xi, t-\tau, \xi, \tau) + \int_{\tau}^{t} \int_{\mathbb{Q}_p^n} Z(x-\eta, t-\theta, \eta, \theta)\Phi(\eta, \theta, \xi, \tau)\, d^n\eta\, d\theta.$$

Thus we formally require that

$$\frac{\partial \Gamma}{\partial t}(x,t,\xi,\tau) + a_0(x,t\,(D_T^{\alpha}\Gamma)\,(x,t,\xi,\tau) + \\ + \sum_{k=1}^{n} a_k(x,t)\,(D_T^{\alpha_k}\Gamma)\,(x,t,\xi,\tau) + b(x,t)\Gamma(x,t,\xi,\tau) = 0.$$

By using formally the formulas given in the Lemma (4), we can see that $\Phi(x,t,\xi,\tau)$ is a solution of the integral equation

$$\Phi(x,t,\xi,\tau) = R(x,t,\xi,\tau) + \int_{\tau}^{t} \int_{\mathbb{Q}_p^n} R(x,t,\eta,\theta)\Phi(\eta,\theta,\xi,\tau)\, d^n\eta\, d\theta, \tag{15}$$

where

$$R(x,t,\xi,\tau) = (a_0(\xi,\tau) - a_0(x,t))Z_{\alpha}(x-\xi, t-\tau, \xi, \tau) \\ - \sum_{k=1}^{n} a_k(x,t) Z_{\alpha_k}(x-\xi, t-\tau, \xi, \tau) - b(x,t)Z(x-\xi, t-\tau, \xi, \tau).$$

In order to solve the integral equation (15) we use the method of successive approximations (see e.g. [5], [11]). We set

$$R_1(x,t,\xi,\tau) := R(x,t,\xi,\tau),$$

and

$$R_{m+1}(x,t,\xi,\tau) := \int_{\tau}^{t}\int_{\mathbb{Q}_p^n} R(x,t,\eta,\theta)R_m(\eta,\theta,\xi,\tau)\, d^n\eta\, d\theta, \quad m \in \mathbb{N} \smallsetminus \{0\}.$$

We claim that

$$\Phi(x,t,\xi,\tau) = \sum_{m=1}^{\infty} R_m(x,t,\xi,\tau)$$

is a solution of (15). In order to prove the convergence of the series we need the followings two Lemmas, whose proof is a simple variation of those given by Kochubei in [10, Sec. 4.5] for the case $n = 1$.

Lemma 5. [10, Eq 4.64] *With the above notation,*

$$|R(x,t,\xi,\tau)| \le C \sum_{k=1}^{n+1} \left((t-\tau)^{1/\alpha} + ||x-\xi||_p\right)^{-\alpha_k - n},$$

where C is a positive constant.

Lemma 6. [10, Lemma 4.6] *Let*

$$\begin{aligned} J(x,\xi,t,\tau) = \int_{\tau}^{t} &(t-\mu)^{-\rho/\alpha}(\mu-\tau)^{-\sigma/\alpha} \\ &\left(\int_{\mathbb{Q}_p^n} \left((t-\mu)^{1/\alpha} + ||x-\eta||_p\right)^{-n-b_1}\right. \\ &\left.\left((\mu-\tau)^{1/\alpha} + ||\eta-\xi||_p\right)^{-n-b_2} d^n\eta\right) d\mu, \end{aligned}$$

where $0 \le \tau < t$, $x, \xi \in \mathbb{Q}_p^n$, $b_1, b_2 > 0$, $\rho + b_1 < \alpha$, $\sigma + b_2 < \alpha$. Then

$$\begin{aligned} J(x,\xi,t,\tau) \le C &\left((t-\tau)^{\kappa} B\left(1-\frac{\rho}{\alpha}, 1-\frac{\sigma+b_2}{\alpha}\right)\left((t-\tau)^{1/\alpha} + ||x-\xi||_p\right)^{-n-b_1}\right) \\ + C &\left((t-\tau)^{\varrho} B\left(1-\frac{\rho+b_1}{\alpha}, 1-\frac{\sigma}{\alpha}\right)\left((t-\tau)^{1/\alpha} + ||x-\xi||_p\right)^{-n-b_2}\right), \end{aligned}$$

where $\kappa = -\frac{(\rho+\sigma+b_2-\alpha)}{\alpha}$, $\varrho = -\frac{(\rho+\sigma+b_1-\alpha)}{\alpha}$, C is a positive constant depends only on b_1, b_2 and $B(z_1, z_2)$ is the Archimedean Beta function.

Lemma 7. *With the above notation,*

$$|R_m(x,t,\xi,\tau)| \leq CM^m (t-\tau)^{(m-1)\upsilon/\alpha} \frac{(\Gamma(\upsilon/\alpha))^m}{\Gamma(m\upsilon/\alpha)} \sum_{k=1}^{n+1} \big((t-\tau)^{1/\alpha} + ||x-\xi||_p\big)^{-\alpha_k - n},$$

where C is a positive constant.

Proof. We use induction on m. The case $m=1$ is clear. We assume the case m as induction hypothesis, then by Lemmas (5), (6) and (7) we have

$$\begin{aligned}
|R_{m+1}(x,t,\xi,\tau)| &\leq \int_\tau^t \int_{\mathbb{Q}_p^n} |R(x,t,\eta,\theta)| \cdot |R_m(\eta,\theta,\xi,\tau)| \, d^n\eta \, d\theta \\
&= CM^m \frac{(\Gamma(\upsilon/\alpha))^m}{\Gamma(m\upsilon/\alpha)} \sum_{k,l=1}^{n+1} \int_\tau^t (\theta-\tau)^{(m-1)\upsilon/\alpha} \\
&\quad \int_{\mathbb{Q}_p^n} \big((t-\theta)^{1/\alpha} + ||x-\eta||_p\big)^{-\alpha_k - n} \\
&\quad \big((\theta-\tau)^{1/\alpha} + ||\eta-\xi||_p\big)^{-\alpha_l - n} \, d^n\eta \, d\theta.
\end{aligned}$$

Thus it is sufficient to bound the integral

$$\begin{aligned}
I_{k,l}(x,\xi,t,\tau) = &\int_\tau^t (\theta-\tau)^{(m-1)\upsilon/\alpha} \times \\
&\int_{\mathbb{Q}_p^n} \big((t-\theta)^{1/\alpha} + ||x-\eta||_p\big)^{-\alpha_k - n} \\
&\big((\theta-\tau)^{1/\alpha} + ||\eta-\xi||_p\big)^{-\alpha_l - n} \, d^n\eta \, d\theta.
\end{aligned}$$

By using Lemma (6),

$$\begin{aligned}
I_{k,l}(x,\xi,t,\tau) \leq\ & CB\left(\frac{\alpha-\alpha_k}{\alpha}, \frac{m\upsilon+\alpha-\upsilon}{\alpha}\right)(t-\tau)^{-(\upsilon - m\upsilon + \alpha_k - \alpha)/\alpha} \\
& \big((t-\tau)^{1/\alpha} + ||x-\xi||_p\big)^{-\alpha_l - n} \\
& + CB\left(1, \frac{m\upsilon+\alpha-\upsilon-\alpha_l}{\alpha}\right)(t-\tau)^{-(\upsilon - m\upsilon + \alpha_l - \alpha)/\alpha} \\
& \big((t-\tau)^{1/\alpha} + ||x-\xi||_p\big)^{-\alpha_k - n}.
\end{aligned}$$

We now recall that if $\epsilon, \delta > 0$, then $B(x+\epsilon, y+\delta) \leq B(x,y)$, thus

$$B\left(\frac{\alpha-\alpha_k}{\alpha}, \frac{m\lambda+\alpha-\lambda}{\alpha}\right) \leq B\left(\frac{\lambda}{\alpha}, \frac{m\lambda}{\alpha}\right),$$
$$B\left(1, \frac{m\lambda+\alpha-\lambda-\alpha_l}{\alpha}\right) \leq B\left(\frac{\lambda}{\alpha}, \frac{m\lambda}{\alpha}\right),$$

and

$$(t-\tau)^{-(\upsilon-m\upsilon+\alpha_k-\alpha)\alpha} \leq C'(t-\tau)^{(m+1-1)\upsilon\alpha}.$$

Therefore,

$$\begin{aligned}|R_{m+1}(x,t,\xi,\tau)| \leq C M^{m+1}(t-\tau)^{m\upsilon/\alpha} &\frac{(\Gamma(\upsilon/\alpha))^{m+1}}{\Gamma((m+1)\upsilon/\alpha)} \\ &\sum_{k=1}^{n+1}\left((t-\tau)^{1/\alpha}+||x-\xi||_p\right)^{-\alpha_k-n}.\end{aligned}$$

By using Stirling's formula we verify the absolute convergence of

$$\Phi(x,t,\xi,\tau) = \sum_{m=1}^{\infty} R_m(x,t,\xi,\tau),$$

and also that

$$|\Phi(x,t,\xi,\tau)| \leq C\sum_{k=1}^{n+1}\left((t-\tau)^{1/\alpha}+||x-\xi||_p\right)^{-\alpha_k-n} \tag{16}$$

☑

We now come to the main result. This result is an n-dimensional version of Theorem 4.6, p. 156 in [10]. Here we assume that $0 \leq \lambda < \alpha_1$; if all the coefficients $a_1(x,t), \dots, a_n(x,t)$ vanish identically, then we may assume $0 \leq \lambda < \alpha$.

Theorem 2. *The Cauchy problem*

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} + a_0(x,t)\left(D_T^{\alpha}u\right)(x,t) + \displaystyle\sum_{k=1}^{n} a_k(x,t)\left(D_T^{\alpha_k}u\right)(x,t) \\ \qquad\qquad + b(x,t)u(x,t) = f(x,t), \quad x \in \mathbb{Q}_p^n, \quad t \in (0,T], \\ u(x,0) = \varphi(x), \end{cases} \tag{17}$$

has a solution

$$u(x,t)=\int_0^t\int_{\mathbb{Q}_p^n}\Gamma(x,t,\xi,\tau)f(\xi,\tau)\,d^n\xi\,d\tau+\int_{\mathbb{Q}_p^n}\Gamma(x,t,\xi,0)\varphi(\xi)\,d^n\xi, \tag{18}$$

which is continuous on $\mathbb{Q}_p^n\times[0,T]$*, continuously differentiable in* t*, and belonging to* $\mathfrak{M}_\lambda$ *uniformly with respect to* t*. The fundamental solution* $\Gamma(x,t,\xi,\tau)$*,* $x,\xi\in\mathbb{Q}_p^n$*,* $0\le\tau<t\le T$*, is then of the form*

$$\Gamma(x,t,\xi,\tau)=Z(x-\xi,t-\tau,\xi,\tau)+W(x,t,\xi,\tau), \tag{19}$$

and finally

$$\begin{aligned}|W(x,t,\xi,\tau)|\le C\Big\{&(t-\tau)^{n+\lambda}\left[(t-\tau)^{1/\alpha}+||x-\xi||_p\right]^{-\alpha-n}\\&+(t-\tau)\sum_{k=1}^{n+1}\left[(t-\tau)^{1/\alpha}+||x-\xi||_p\right]^{-\alpha_k-n}\Big\}.\end{aligned} \tag{20}$$

Proof. Denote by $u_1(x,t)$ and $u_2(x,t)$ the first and the second summands in the right hand side of (18). We find that

$$\begin{aligned}u_1(x,t)=&\int_0^t\int_{\mathbb{Q}_p^n}Z(x-\xi,t-\tau,\xi,\tau)f(\xi,\tau)\,d^n\xi\,d\tau\\&+\int_0^t\int_{\mathbb{Q}_p^n}Z(x-\eta,t-\theta,\eta,\theta)F(\eta,\theta)\,d^n\eta\,d\theta,\end{aligned}$$

and

$$\begin{aligned}u_2(x,t)=&\int_{\mathbb{Q}_p^n}Z(x-\xi,t,\xi,0)\varphi(\xi)\,d^n\xi\\&+\int_0^t\int_{\mathbb{Q}_p^n}Z(x-\eta,t-\theta,\eta,\theta)G(\eta,\theta)\,d^n\eta\,d\theta,\end{aligned}$$

where

$$F(\eta,\theta)=\int_0^\theta\int_{\mathbb{Q}_p^n}\Phi(\eta,\theta,\xi,\tau)f(\xi,\tau)\,d^n\xi\,d\tau,$$

$$G(\eta,\theta)=\int_{\mathbb{Q}_p^n}\Phi(\eta,\theta,\xi,0)\varphi(\xi)\,d^n\xi.$$

Now by (16) and Proposition 2 in [13],

$$|F(\eta, \theta)| \le C, \text{ and } |G(\eta, \theta)| \le C\theta^{-\alpha_{n+1}/\alpha},$$

for all $\eta \in \mathbb{Q}_p^n$ and $\theta \in (0, T]$. In addition the functions F and G are uniformly locally constant. Indeed, by the recursive definition of the function Φ we see that if N is a local constancy exponent for all the functions a_i, b, Z_{α_i} and Z, and if $|\delta| \le q^{-N}$, then

$$\phi(x + \delta, t, \xi + \delta, \tau) = \phi(x, t, \xi, \tau),$$

whence

$$F(\eta + \delta, \theta) = F(\eta, \theta), \quad G(\eta + \delta, \theta) = G(\eta, \theta).$$

Thus the potentials in the expressions for $u_1(x,t)$ and $u_2(x,t)$ satisfy the conditions under which the differentiation formulas of the Lemmas (4) were obtained. By using these formulas one verifies after some simple transformations that $u(x,t)$ is a solution of the equation (17).

Let us show that $u(x,t) \to \varphi(x)$ as $t \to 0$. Due to (19) and (20), it is sufficient to verify that $u_2(x,t) \to \varphi(x)$ as $t \to 0$. By virtue of (10) we have

$$\begin{aligned} u_2(x,t) = & \int_{\mathbb{Q}_p^n} [Z(x-\xi, t, \xi, 0) - Z(x-\xi, t, x, 0)]\varphi(\xi)\, d^n\xi \\ & + \int_{\mathbb{Q}_p^n} Z(x-\xi, t, x, 0)[\varphi(\xi) - \varphi(x)]\, d^n\xi + \varphi(x). \end{aligned}$$

Since as functions of their third argument Z and φ are locally constant, both integrals in the previous expression are performed over the set

$$\left\{\xi \mid ||x - \xi||_p \ge p^{-N}\right\}.$$

By applying (7) we see that both integrals tend to zero as $t \to 0$. ☑

6. Markov processes

By using Theorems (1) and (2), we obtain a probabilistic interpretation for the function $\Gamma(x, t, \xi, \tau)$.

Theorem 3. *The fundamental solution $\Gamma(x, t, \xi, \tau)$ is the transition density of a bounded right-continuous strict Markov process without second kind discontinuities. If $b(x,t) = 0$, then the process does not explode.*

The proof uses the same argument given in [10, pg. 162].

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