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Optimality results for dividend problems in insurance

Hansjörg Albrecher and Stefan Thonhauser

Abstract. This paper is a survey of some classical contributions and recent progress in identifying optimal dividend payment strategies in the framework of collective risk theory. In particular, available mathematical tools are discussed and some challenges are described that occur under various objective functions and model assumptions. Finally, some open research problems in this field are stated.

Resultados de optimalidad para problemas de dividendos en seguros

Resumen. Este artículo es una revista de algunos resultados clásicos y avances recientes en la identificación de estrategias óptimas de pago de dividendos en el marco de la teoría de riesgo para modelos colectivos. En particular, describimos las herramientas matemáticas disponibles y discutimos algunos de los retos que se presentan bajo diferentes funciones objetivo y supuestos del modelo. Finalmente, presentamos problemas abiertos de investigación en esta área.

1 Introduction

After the introduction of the classical collective risk model in 1903 by Lundberg [92] to describe the free surplus process of an insurance portfolio, the probability of ruin of such a portfolio was among the prime quantities of interest in this field. However, a trajectory of the surplus process that does not lead to ruin in this model will exceed every finite level, which is typically unrealistic in practice. That is why in 1957 de Finetti [39] proposed another, economically motivated, criterion to the actuarial world. Instead of focussing on the safety aspect (measured by the probability of ruin) he proposed to measure the performance of an insurance portfolio by the maximal dividend payout that can be achieved over the lifetime of the portfolio. In particular, he proposed to look for the expected discounted sum of dividend payments until the time of ruin, where the discounting is with respect to some constant discount rate $\delta > 0$. Whereas de Finetti himself solved the problem to identify the optimal such dividend strategy in a very simple discrete random walk model, since then many research groups have tried to address this optimality question under more general and more realistic model assumptions and until nowadays this turns out to be a rich and challenging field of research that needs the combination of tools from analysis, probability and stochastic control. In contrast to typical control (and consumption) problems in finance, in this insurance context a control action changes the value of the underlying, as the dividend payments are subtracted from the current surplus, so that the problems have a quite different flavor from their counterparts in mathematical finance. In this survey we would like to collect some crucial ideas and developments in this field and in particular highlight the type of mathematical techniques and challenges that occur in this field. Rather than attempting

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to provide an encyclopedic list of references we will rather focus on methodological aspects and give links to some classical and recent pertinent references. For a recent quite extensive collection of references under a slightly different focus see Avanzi [14].

The above classical criterion is in line with the so-called Gordon model [58] which uses discounted future dividend payments as an alternative to the general discounted cashflows method for valuating a company. At the same time this approach also was at debate. Miller & Modigliani [94] showed in a simple model with quite restrictive assumptions on the market, that when knowing exactly the investment strategy of the company, the knowledge about future dividends is not needed for evaluating the value of the company. However, the variety and complexity of more sophisticated stochastic models for an insurance portfolio does not fit into the simple framework of [94] (see also DeAngelo & DeAngelo [40, 41], Handley [61] and Frankfurter and Wood [45]) and it is widely believed that the discounted dividends approach (and variants of it) are still useful also from an economic point of view.

In Section 2 we will briefly describe the classical collective risk model based on a compound Poisson process and the diffusion model. These two models are the cornerstones of tractable continuous-time processes to solve stochastic control problems in this context. Although the diffusion model is not directly appealing as a model for insurance purposes, where clearly claims will cause jumps in the surplus process, we indicate an argument why it is sometimes useful in a certain approximative sense. Both the compound Poisson model and the diffusion model are Markovian and hence the dynamic programming approach can be used directly to address the problem of determining the optimal dividend strategy. In fact, it seems that all established solutions of optimal dividend problems in the literature rely in one way or the other on the dynamic programming principle. This solution procedure and its application are going to be described in Section 5 and Section 6. A potential alternative could be the *dual method*, introduced for a stochastic framework by Bismut [26], which works well for portfolio optimization problems in finance (see Kramkov & Schachermayer [76] for the general case). But due to the intervention of the control into the underlying surplus process, it seems that the resulting set of possible trajectories is too restricted to make the dual method work for insurance problems.

The dynamic programming approach leads to the so-called Hamilton-Jacobi-Bellman equation, which (depending on the underlying risk model) contains elements of a differential, partial differential or integro-differential equation. A solution to this equation is then not yet automatically the optimal solution of the optimization problem, but a good guess for it that then has to be verified in a separate step. The equation itself can be interpreted as the natural continuous limit of the dynamic programming principle from discrete optimization (see Whittle [122]) which postulates that an optimal policy for the whole time span also has to be the optimal one in each small time step. A reference for discrete-time stochastic optimization is Bertsekas & Shreve [25]. For further references on optimal dividend results in discrete-time models we refer to Avanzi [14].

The remainder of this survey is organized as follows. Section 2 introduces the type of continuous-time insurance risk models for which optimal dividend problems can typically be solved. Section 3 first discusses several possibilities of control actions on the insurance portfolio surplus process and then introduces various kinds of dividend strategies that later turn out to be optimal under certain objective functions. Furthermore references to literature that studies properties of the resulting controlled surplus process are given. Section 4 deals with various criteria to measure the value of dividend strategies and gives links to results that establish the optimality of respective strategies. In Section 5 we then discuss the dynamic programming principle and typical mathematical approaches to derive an equation for the value function of interest. Section 6 subsequently summarizes the mathematical challenges in the step of verifying whether a candidate solution is indeed the optimal solution. Finally, in Section 7 we conclude and state some open problems in the field.

2 Collective Risk Models

In the following we will always use a probability space $(\Omega, \mathcal{F}, P)$ on which all stochastic quantities are defined.

2.1 Cramér-Lundberg Model

The Cramér-Lundberg risk model (also called the classical risk model or compound Poisson model) describes the free reserve $R = (R_t)_{t\geq 0}$ in an insurance portfolio by a stochastic process of the form

$$R_t = x + c\,t - \sum_{k=1}^{N_t} Y_k. \tag{1}$$

The first ingredient is the deterministic initial capital $x \geq 0$. The premiums are assumed to be collected continuously over time with constant intensity c and the total claim amount at time t is given by a compound Poisson process $S = (S_t)_{t\geq 0}$ with $S_t = \sum_{j=1}^{N_t} Y_j$, where the number $(N_t)_{t\geq 0}$ of claim occurrences up to time $t \geq 0$ is a homogeneous Poisson process $N = (N_t)_{t\geq 0}$ with intensity $\lambda > 0$, i.e. $N_t \sim \mathrm{Poi}(\lambda t)$. The claims are a sequence of positive independent and identically distributed random variables $\{Y_i\}_{i\in\mathbb{N}}$ with distribution function F_Y. One crucial assumption in the classical risk model is the independence of N and $\{Y_i\}_{i\in\mathbb{N}}$. As a consequence of the Poisson assumption for the claim counting process N, the inter-occurrence times $\{W_i\}_{i\in\mathbb{N}}$ with $W_i = T_i - T_{i-1}$ are independent and identically exponentially distributed, $\{W_i\}_{i\in\mathbb{N}} \overset{\text{iid}}{\sim} \exp(\lambda)$.

The process R as given in (1) lies in the intersection of the class of spectrally negative Lévy processes and the class of Piecewise Deterministic Markov Processes (PDMP's, see Davis [38]). As a consequence it is itself a strong Markov process. Some of the results for R mentioned later on will have mathematically natural extensions to the class of spectrally negative Lévy processes. Whereas for actuarial applications the practical interpretation of this more general process class is somewhat limited, using the general theory sometimes leads to a quite convenient analysis (for instance in terms of scale functions), which is also applicable in the special case of the Cramér-Lundberg model.

Definition 1 *The time of ruin τ denotes the first entrance time of the reserve process R to $(-\infty, 0)$,*

$$\tau = \tau(x) = \inf\{\, t > 0 \text{ such that } R_t < 0 \mid R_0 = x \,\}.$$

The probability of ultimate ruin is defined as

$$\psi(x) = P(\tau(x) < \infty).$$

The survival probability is $U(x) = 1 - \psi(x)$.

The so-called *net profit condition* requires to choose the premium intensity larger than the expected loss in a time interval of length 1, $c > \lambda\mu = \mathbb{E}(S_1)$ where $\mu = \mathbb{E}(Y_1)$. A result from the theory of random walks [106] shows that if $c \leq \lambda\mu = \mathbb{E}(S_1)$, ruin occurs almost surely, $\psi(x) = 1$. If $c > \lambda\mu$, then $P(\lim_{t\to\infty} R_t = \infty) = 1$. The following operator which is applied to a suitable function g (for details see [38] or [106]) is called the infinitesimal generator of the Markov process R,

$$\mathcal{L}g(x,t) = c\frac{\partial g}{\partial x}(x,t)\, I_{\{x\geq 0\}} + \frac{\partial g}{\partial t}(x,t) + I_{\{x\geq 0\}}\lambda\left(\int_0^\infty g(x-y,t)\,\mathrm{d}F_Y(y) - g(x,t)\right), \tag{2}$$

which will be needed later on.

2.2 Diffusion Approximation of the Model

We will only give a brief illustration of the ideas of diffusion approximations for risk reserve processes. Overviews and numerical comparisons of different types of approximations are given in Grandell [59], Asmussen [12] and Schmidli [110].

Let $b > 0$ and $a \in \mathbb{R}$ be two constants, then a Markov process X which, for small h, fulfills

$$\mathbb{E}\left(X_{t+h} - X_t \mid \mathcal{F}_t\right) = ha,$$
$$\mathbb{E}\left((X_{t+h} - X_t - ha)^2 \mid \mathcal{F}_t\right) = hb^2,$$

with inifinitesimal drift a and variance b^2, is of the form $X_t = at + bW_t$ for a Brownian motion W_t. Therefore X is called a Brownian motion with drift, or, loosely speaking, a diffusion process with constant drift and volatility. The basic idea behind such an approximation is to define a sequence of classical reserve processes, which converge weakly to some Brownian motion with drift. Since for that special type of process explicit results for distributions of first hitting times exist, these can be used as an approximation of the ruin probability of the classical reserve process. But in addition to their tractability, also from an optimal stochastic control point of view such approximations seem to be interesting. In [18] Bäuerle proves the convergence of values and strategies of solutions to a dividend maximization problem solved by Schäl in [107] for a PDMP risk reserve process to the value and optimal strategy of its diffusion approximation.

The following basic construction is due to Iglehart [72]. He defines a sequence of classical reserve processes $\{R^{(n)}\}_{n\in\mathbb{N}}$, where the components of the nth process are given by the initial capital $x_n > 0$, the premium intensity $c_n > 0$ and independent identically distributed claim amounts $\{Y_i^{(n)}\}_{i\in\mathbb{N}}$ with $\mathbb{E}(Y_i^{(n)}) := \mu_n > 0$ and $\mathrm{Var}(Y_i^{(n)}) := \sigma_n^2 > 0$. The claim counting process N is given by a renewal process with interclaim times $\{W_i\}_{i\in\mathbb{N}}$, see [59] or [106], for which $\mathbb{E}(W_i) = 1/\lambda > 0$ and

$$\begin{aligned} R_t^{(n)} &= x_n + c_n nt - S_{nt}^{(n)}, \qquad t \in [0,1], \\ S_{nt}^{(n)} &= \sum_{i=1}^{N_{nt}} Y_i^{(n)}. \end{aligned}$$

Note, that the distribution of the claim amounts may vary with n, whereas the claim counting process stays the same for every $R^{(n)}$. The reference reserve process R for which the approximation is valid is

$$R_t = x + ct - \sum_{i=1}^{N_t} Y_i,$$

with $\mathbb{E}(Y_i) := \mu > 0$ and $\mathrm{Var}(Y_i) := \sigma^2 > 0$. Under some technical conditions (see [72]) the sequence of classical reserve processes converges weakly to a stochastic process of the form

$$x + \Gamma + \sigma\lambda^{1/2} W,$$

where $\Gamma = (\Gamma_t)_{t\geq 0}$ with $\Gamma_t = (c - \lambda\mu)t$ and $W = (W_t)_{t\geq 0}$ is a standard Brownian motion. Later on we will sometimes refer to *general* diffusion processes, which will be solutions of stochastic differential equations of the following type

$$\mathrm{d}x_t = \mu(x_t, t)\,\mathrm{d}t + \sigma(x_t, t)\,\mathrm{d}W_t, \qquad x_0 = x.$$

The infinitesimal generator for such a process is then given by

$$\mathcal{L}g(x,t) = \frac{\partial g}{\partial t}(x,t) + \mu(x,t)\frac{\partial g}{\partial x}(x,t) + \frac{\sigma(x,t)^2}{2}\frac{\partial^2 g}{\partial x^2}(x,t). \tag{3}$$

3 Model Extensions and Possibilities of Control

The classical risk model in Section 2.1 and its diffusion approximations in Section 2.2 suggest two possibilities for an insurer to intervene in the surplus process, namely the choice of the initial capital x and the choice of the premium intensity c (respectively the drift in the diffusion approximation). In practice there will be more possibilities to influence and control the performance of the insurance portfolio. In the following we will outline some of them.

3.1 Reinsurance and Investment

In 1995, Browne [31] started to apply methods from the theory of stochastic optimal control of diffusion processes in the context of insurance. He considered a Brownian motion with drift as a model for the surplus process and included the possibility for the insurer to invest some fraction of the reserve dynamically over time into a financial asset, where the price of that financial asset is modelled by a geometric Brownian motion (cf. Black and Scholes [27]). The goal is to identify an investment strategy such that the probability of ruin of the controlled reserve process is minimized. Hipp and Plum consider the same problem for a classical compound Poisson risk reserve process in [64] and in a more general framework in [65]. In the context of the classical model, the control variable is a real-valued càdlàg process $A = (A_t)_{t\geq 0}$ adapted to the history of the aggregate claims process $S = (S_t)_{t\geq 0}$ and $W = (W_t)_{t\geq 0}$, which is the standard Brownian motion describing the price P of the financial asset with $\mathrm{d}P_t = P_t(m\,\mathrm{d}t + \sigma\,\mathrm{d}W_t)$ and $P_0 = p$. The controlled reserve process R^A (strategy A) is determined by the stochastic differential equation

$$\mathrm{d}R_t^A = (c + A_t m)\,\mathrm{d}t + A_t\sigma\,\mathrm{d}W_t - \mathrm{d}S_t, \qquad R_0^A = x.$$

The asymptotic behavior of the probability of ruin under the optimal investment strategy is e.g. considered in Hipp and Schmidli [66], Gaier and Grandits [47] and Gaier et al. [48]. Another possibility to reduce the probability of ruin in the classical model is to use reinsurance. Here the insurer passes on some of its premium income to a reinsurer, who in turn covers a certain fraction of the occurred claims. Let a function $b\colon [0,\infty) \to [0,\infty)$ with $0 < b(z) \leq z$ denote the retained amount of the insurer for a claim of size z (such that the amount $z - b(z)$ is covered by the reinsurer). This constitutes a per-risk reinsurance coverage. The premium income kept by the insurer is then $c_b(t) \leq ct$ and depends on the specification of b. The controlled process R^b is in this case given by

$$R_t^b = x + c_b(t) - \sum_{i=1}^{N_t} b(Y_i).$$

Two well-studied types of reinsurance are proportional reinsurance, where $b(z) = \gamma z$ for some $\gamma \in (0,1]$, and excess-of-loss (XL) reinsurance, where $b(z) = \min\{z, M\}$ for a retention level $M > 0$. Schmidli [108] uses modern stochastic control theory to study the optimal choice of dynamic proportional reinsurance to reduce the probability of ruin in the (otherwise) classical risk model. Here *dynamic* refers to a strategy where the proportion $\gamma = (\gamma_t)_{t\geq 0}$ is a predictable process, adapted to $\{\mathcal{F}_{t-}^{R^b}\}_{t\geq 0}$, with respect to the history of R^b, e.g. at claim time T_i the proportion has to be fixed using information only up to time T_i-. In [67] the same problem is studied for dynamic XL-reinsurance by Hipp and Vogt. Schmidli [109] uses the results from [64] and [108] to combine investment and dynamic reinsurance for the minimization of the probability of ruin. While in the diffusion setup it is sometimes possible to calculate quantities of interest explicitly (see [31]), this is not the case in models including jumps of the reserve process. The above mentioned papers dealing with the classical model give proofs of the existence of a minimal probability of ruin and the existence of an optimal strategy, but only provide ideas for their numerical evaluation.

3.2 Dividends

Let us now define the so-called *admissible dividend strategies*. The filtration $\mathcal{F} = (\mathcal{F}_{t\geq 0})$, which we are going to use, is always the one generated by the uncontrolled processes (in the diffusion case by the driving Brownian motion, in the classical case by the compound Poisson process). The basic idea for modeling a dividend policy is to introduce a stochastic process $L = (L_t)_{t\geq 0}$ representing the cumulated dividend payments up to time t. From the interpretation as a dividend strategy, it is natural to impose the following four conditions on L:

(i) ruin does not occur due to dividend payments, i.e. $\Delta L_t \leq R_t^L$ (where R_t^L denotes the controlled risk process)

(ii) $L_0 = 0$ and the paths of L are non-decreasing,

(iii) payments have to stop after the event of ruin,

(iv) decisions have to be fixed in a predictable way.

Condition (iv) gives reason to look at càglàd processes L, which are left-continuous with existing limits from the right ($L_{t-} = L_t$). We hence call a dividend strategy $L = \{L_t\}_{t\geq 0}$ *admissible* if it is càglàd for all $t \geq 0$ and fulfills (i), (ii) and (iii) above (in particular, L_t then is previsible, i.e. $\mathcal{F}_{t-}$-measurable). The controlled process in the compound Poisson model is defined via

$$R_t^L = x + ct - \sum_{k=1}^{N_t} Y_k - L_t.$$

The càdlàg property of the reserve process and the càglàd property of the dividend process imply that $R_{t-}^L \neq R_t^L$ is always due to a claim and $R_{t+}^L \neq R_t^L$ is due to some (singular) dividend payment. This càglàd assumption is for instance used in Azcue & Muler [16] and Albrecher & Thonhauser [9] in a compound Poisson framework. Alternatively, it is also possible to consider previsible càdlàg strategies L, which preserve the càdlàg property of the risk process for the controlled process. But then —in order to allow lump sum payments at $t = 0$ (one has to take $L_{0-} = 0$) and to exclude payments at the time of ruin— the optimization criterion has to be slightly modified (we will come back to that later on, see also Schmidli [112] and Mnif & Sulem [95]). The essential difference between using a càdlàg or a càglàd control process is observing the process after or before a possible dividend payment (cf. [112]). If the uncontrolled risk process is a diffusion, the requirement that L_t is $\mathcal{F}_{t-}$-measurable is equivalent to requiring L_t to be adapted (i.e. $\mathcal{F}_t$-measurable for all $t \geq 0$). The controlled diffusion process is

$$R_t^L = x + \int_0^t \mu(R_s^L, s)\,\mathrm{d}s + \int_0^t \sigma(R_s^L, s)\,\mathrm{d}W_s - L_t.$$

This definition is in line with the one of Shreve et al. [113] and the one for general controlled Markov processes presented in Fleming & Soner [44]. Again it is also possible to use càdlàg controls instead, as is done for constant μ and σ by Asmussen & Taksar [13], see also [112].

3.2.1 Analytic Properties of L

In many situations it is useful to restrict the general class of admissible controls further. A natural subclass of admissible control strategies are the absolutely continuous ones. Such strategies L admit an adapted nonnegative density process $l = (l_s)_{s\geq 0}$ such that

$$L_t = \int_0^t l_s\,\mathrm{d}s.$$

To avoid payments after ruin, one has to additionally require $l_t = 0$ for $t \geq \tau^L$, where τ^L denotes the time of ruin of R^L. In order to exclude singularities, one usually assumes that the density process is bounded, $0 \leq l_s < l_\infty < \infty$ for all $s \geq 0$. Note that this restricted type of control, now determined by its density process, then does not include the possibility of lump sum payments (i.e. jumps of L, which would be singularities of l). In Fleming & Soner [44] a complete picture of admissible controls (or more generally *admissible control systems*) and the use of progressively measurable control processes $l = (l_s)_{s\geq 0}$ for Markov diffusion processes can be found. Moreover these authors give a construction how to move from a control represented by its —possibly unbounded— density, which a priori does not make sense, to its integrated representation L.

In some situations (examples will be given below) it can be shown that the singular parts of a strategy do not contribute to the resulting wealth, and hence the absolutely continuous controls are a sufficient choice for solving the general maximization problem. This restricted class of controls is for instance considered in Asmussen & Taksar [13], Jeanblanc-Picqué & Shiryaev [74], Schäl [107] and Gerber & Shiu [56]. In Schmidli [112] the solution of the restricted problem in the Cramér-Lundberg model is shown to converge pointwise to the general solution as $l_\infty \to \infty$.

In an insurance context the introduction of transaction costs charging the dividend payments seems to be relatively new (although there is an early discussion by Porteus [102]) and up to now mostly problems in a diffusion setup are solved, see for example Jeanblanc-Picqué & Shiryaev [74], Paulsen [98] and Cadenillas et al. [33, 34]. For the compound Poisson risk reserve process, the effect of transaction costs on the optimal control problem was recently investigated in Albrecher & Thonhauser [10]. The inclusion of transaction costs naturally leads to another restricted class of admissible strategies known as impulse controls. Let us assume that every dividend payment is charged by proportional and fixed costs such that the shareholder receives $kz - K$ from a payment of size z ($K > 0$ and $k \in (0,1)$). Then dividend strategies with an absolutely continuous component lead to an unbounded negative payoff for the shareholder and are consequently not appropriate. An *impulse control* $S = \{(\tau_i, Z_i)\}_{i\in\mathbb{N}}$ is now instead a sequence of increasing intervention times τ_i and associated control actions Z_i, which fulfills the following four conditions:

- $0 \le \tau_i \le \tau_{i+1}$ a.s. for all $i \in \mathbb{N}$,
- τ_i is a stopping time with respect to the filtration $\mathcal{F}_t = \sigma\{\, R^S_{s-} \mid s \le t \,\}$ for $t \ge 0$,
- Z_i is measurable with respect to $\mathcal{F}_{\tau_i}$,
- $P\left(\lim_{i\to\infty} \tau_i \le T\right) = 0$ for all $T \ge 0$.

The controlled process $R^S = (R^S_t)_{t\ge 0}$ based on an uncontrolled reserve R, is consequently given by

$$R^S_t = R_t - \sum_{i=1}^{\infty} I_{\{\tau_i < t\}} Z_i.$$

If the uncontrolled model has continuous sample paths, then the measurability condition on the stopping times τ_i can of course be replaced by measurability with respect to the history of the process. For further details on impulse controls for PDMPs and also on the existence of controlled processes see Davis [38]. Other standard references in this context are Bensoussan & Lions [23] and Øksendal & Sulem [96].

Remark 1 *In the literature on optimal stochastic control one often encounters the concept of* relaxed *(or* generalized*) controls, which goes back to Fleming* [43] *(for the deterministic case see Young* [123]*). The basic idea is to enlarge the set of admissible controls (which take values in a compact control space U) by defining the set of relaxed controls consisting of the set of measurable functions $m\colon [0,\infty) \to P(U)$ (where $P(U)$ denotes the set of probability measures on U). In other words, one allows for stochastic strategies and a classical control $u \in U$ then corresponds to a Dirac measure δ_u. For instance, if the drift of a controlled diffusion depends continuously on the control $u \in U$, then, applying a relaxed control m_t, one gets the process*

$$R^m_t = x + \int_0^t \int_U \mu(s,u) m_s(\mathrm{d}u)\,\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}W_s.$$

A natural question is now whether such a randomization of strategies can substantially increase the value of the objective function in the stochastic control problem (as it is for instance the case for the value of non-cooperative deterministic games in game theory). In general, in the diffusion case the so-called chattering lemma *states that for any relaxed control there exists a* simple *control approximating the relaxed one arbitrarily closely (i.e. U is dense in the set of relaxed controls, see Kushner* [78] *for an overview and Davis* [38] *for a short comment on the introduction of randomized strategies in the PDMP case). Explicit calculations for certain simple randomized dividend strategies in a compound binomial risk model are given in Tan & Yang* [118] *and Landriault* [83].

The concept of relaxed strategies can in any case often be helpful for proving the existence of an optimal control (at least in the relaxed sense), when there exists no simple *optimal strategy, maximizing or minimizing a given cost functional.*

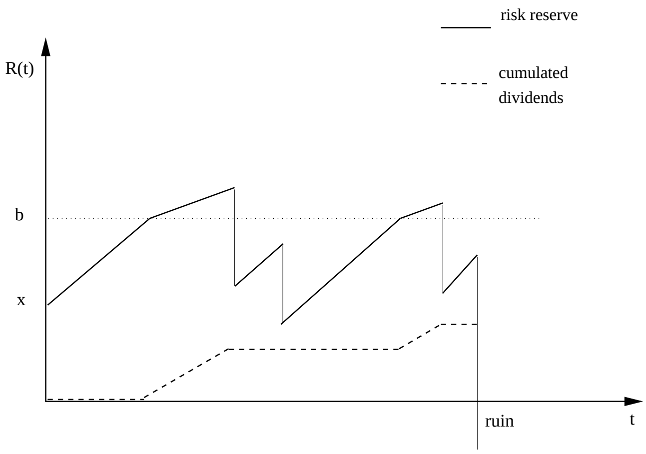

Figure 1. A sample path of the Cramér-Lundberg model under a control of *threshold* type

3.2.2 Some Particular Control Strategies

We will conclude this section by introducing some concrete well-known strategies that will turn out to be optimal in certain situations.

Threshold strategies As an example for an absolutely continuous control fix a *threshold* level $b > 0$ and choose a Markovian density process $l_s = l(x) = a\ I_{\{x>b\}}$ with $a > 0$. The cumulated dividend payments process is then given by

$$L_t = \int_0^{t\wedge\tau^L} a\, I_{\{R_{s-}\geq b\}}\,\mathrm{d}s.$$

Such a strategy pays out dividends continuously at a rate a whenever the current reserve is above level b (cf. Figure 1).

The articles by Gerber & Shiu [56], Frostig [46] and Lin & Pavlova [86] deal with such a strategy in the classical model and Gerber & Shiu [55] in the diffusion model. Kyprianou and Loeffen [79] discuss the existence of spectrally negative Lévy processes controlled by a threshold strategy.

An extension of the threshold strategy is to fix multiple thresholds b_i and associated intensities a_i. Kerekhesha [75], Zhou [124], Albrecher & Hartinger [4] and Lin & Sendova [87] study properties of the resulting risk reserve process in the classical model, see also Badescu et al. [17].

Barrier strategies For a fixed barrier height $b \geq 0$, the cumulated dividend payments are described by

$$L_t = (x-b)I_{\{x>b\}} + \int_0^{t\wedge\tau^L} c\, I_{\{R^L_{t-}=b\}}\,\mathrm{d}t.$$

Such a strategy pays out all the reserve above b immediately at $t = 0+$ (representing a singular component in the strategy) and subsequently all incoming premiums that lead to a surplus above b are immediately distributed as dividends. For $t > 0$ the controlled risk process is hence reflected at b and there are obvious connections to concepts of first hitting times of the process at b from below and the local time of the process at b (cf. Figure 2).

This intuitively natural strategy for profit participation in the risk process was first proposed by de Finetti [39] in 1957 and he showed that a certain barrier strategy maximizes expected discounted dividend payments if the underlying risk reserve process is modelled as a simple random walk. For

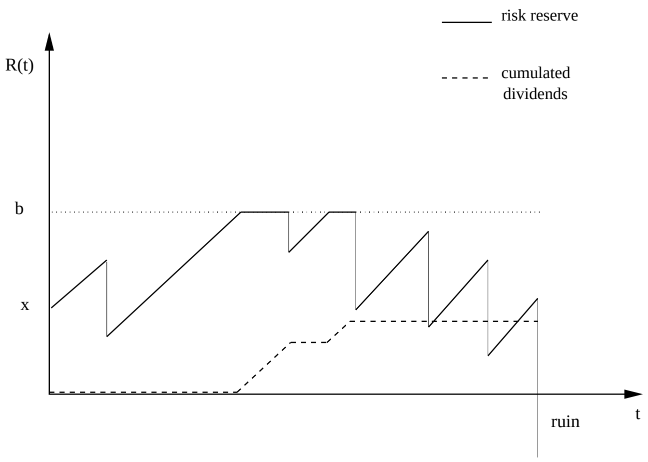

Figure 2. A sample path of the Cramér-Lundberg model under a control of *barrier* type

further situations in which barrier strategies turn out to be optimal we refer to Section 4. There are many papers in the literature that deal with specific properties of the risk reserve process resulting from a barrier strategy. For instance, Paulsen & Gjessing [100] investigate the effect of barrier strategies on risk processes in an economic environment. Irbäck [73] studies asymptotic results for high horizontal barriers. Gerber & Shiu [54] calculate the moments of the expected dividends for an underlying diffusion process. Leung et al. [84] deal with finite horizon problems in the presence of a horizontal barrier and a geometric Brownian motion. Cai et al. [35] study an Ornstein-Uhlenbeck model including credit and debit interest. Lin et al. [88] discuss properties of the classical risk reserve process controlled by a barrier strategy by means of the so-called expected discounted penalty function. For the more general spectrally negative Lévy processes, Avram et al. [15], Renaud & Zhou [104] and Kyprianou & Palmowski [81] use scale functions for calculating functionals of the expected discounted dividends under a barrier strategy. In the compound Poisson model, Højgaard [68] determines optimal premium payment schemes such that expected discounted dividend payments under a barrier strategy are maximized.

Time-dependent barriers were studied in Gerber [52], Siegl & Tichy [114] and Albrecher et al. [5] for the linear case and in Alegre et al. [11] and Albrecher & Kainhofer [7] for the non-linear case (see also Garrido [49] for the diffusion model). In [51], it was shown that barrier dividend payments constitute a complete family of Pareto-optimal dividends.

Band strategies When studying the classical reserve process, Gerber [50] showed that for general optimality one needs another type of strategy called *band* strategy. Such a strategy is characterized by three sets $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ which partition the state space of the reserve process. Each set is associated with a certain dividend payment action for the current reserve x as follows: if the current surplus $x \in \mathcal{A}$, then every incoming premium is paid out; if $x \in \mathcal{B}$, then a lump sum is paid out moving the current reserve to the closest point in $\mathcal{A}$ that is smaller than x; if $x \in \mathcal{C}$ then no dividend is paid. It is possible that several disjoint intervals belong to $\mathcal{B}$ and $\mathcal{C}$ and create a band structure for (R_t, t) over $\mathbb{R}^+ \times \mathbb{R}^+$. For further discussions on these type of strategies see also Bühlmann [32], where also other general thoughts about dividend policies can be found. In Figure 3 a sample path of the risk process with a band strategy given by $\mathcal{A} = \{b_0, b_1\}$, $\mathcal{B} = (b_0, a] \cup (b_1, \infty)$ and $\mathcal{C} = (a, b_1)$ is illustrated.

A simple type of impulse strategy Fix two levels b_1 and b_2 with $0 \leq b_1 < b_2$ and use the following rules for dividend payments: if the surplus is above or equal b_2, then pay out the amount $b_2 - b_1$ immediately; if the surplus is below b_2, do nothing until the reserve reaches the level b_2 again. Let $\theta_{b_2}^n$

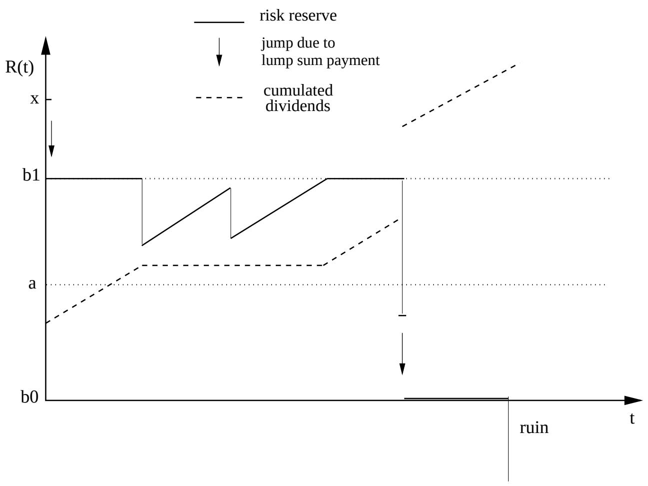

Figure 3. A sample path of the Cramér-Lundberg model under a control of *band* type

denote the nth time that the process hits b_2 from below. Then the payoff of such a dividend strategy is given by

$$L_t = (x - b_1) I_{\{x \geq b_2\}} + \sum_{n=1}^{\infty} (b_2 - b_1) I_{\{\theta_{b_2}^n < t < \tau^S\}},$$

when starting with initial capital $x \geq 0$ (cf. Figure 4).

Such a dividend strategy naturally appears for diffusion risk reserve processes and transaction costs for dividend payments (cf. Jeanblanc-Piqué & Shiryaev [74] for a simple diffusion model with constant drift and volatility, Cadenillas et al. [34] for a mean-reverting diffusion process, Paulsen [98] for general diffusion processes; Cadenillas et al. [33] also take proportional reinsurance into account).

For risk models with jumps and an impulse strategy of the above type, the literature is still scarce. For the case of spectrally negative Lévy risk processes see Loeffen [90]. Albrecher & Thonhauser [10] study the Cramér-Lundberg model with both proportional and fixed transaction costs and also discuss the role of these simple impulse strategies.

Another somewhat intuitive payout scheme for profit participation is to pay a certain proportion of the premium income whenever it represents new gains (i.e. whenever the risk process is in a running maximum). Although there are no criteria known under which such a payment strategy is optimal, it leads to surprisingly simple identities between the survival probability with and without those payments and has another natural interpretation in terms of tax payments on profits of the insurance business (cf. Albrecher & Hipp [6] and Albrecher et al. [1, 8]).

4 Value Functions

Let us now consider in more detail ways to measure the value of a certain dividend strategy L. Let $\delta > 0$ denote a constant discount factor (this can be interpreted as reflecting the preference of shareholders to receive dividend payments earlier rather than later during the lifetime of the reserve process, see e.g. Borch [30]). The index x in the notation $\mathbb{E}_x$ will indicate in the following that the initial capital is x, i.e. $P(R_0^L = x) = 1$.

The classical performance measure for a certain dividend strategy L (in this context going back to

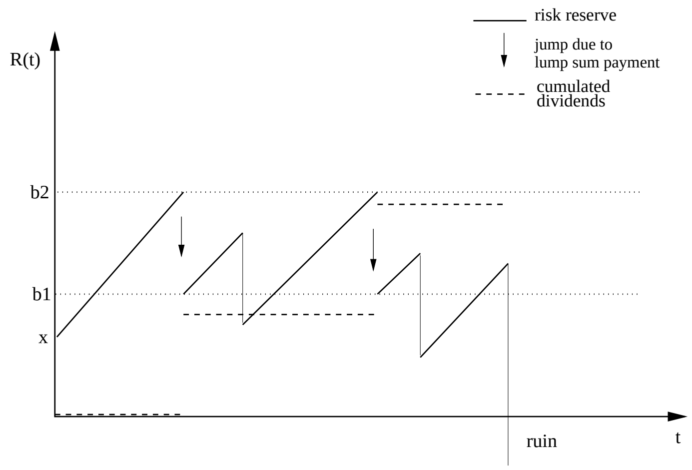

Figure 4. A sample path of the Cramér-Lundberg model under an impulse control

de Finetti [39]), is the expected value of discounted future dividend payments

$$V_L(x) = \mathbb{E}_x\left(\int_0^{\tau^L} \mathrm{e}^{-\delta t}\,\mathrm{d}L_t\right). \tag{4}$$

If instead of càglàd processes L_t one defines càdlàg processes to be admissible (cf. Section 3.2), then (4) has to be modified to

$$V^L(x) = \mathbb{E}_x\left(\int_{0-}^{\tau^L-} \mathrm{e}^{-\delta t}\,\mathrm{d}L_t\right).$$

The associated optimization problem then consists of finding

$$V(x) = \sup_{L\in\Pi} V_L(x) \tag{5}$$

and an optimal admissible strategy L^* such that $V(x) = V_{L^*}(x)$ holds. The set of admissible controls denoted by Π will vary depending on the generality one aims at. For obtaining explicit solutions and simple decision rules, one may want to focus on barrier or threshold strategies; for solving the problem in a general form one will want to deal with general càglàd cumulated dividend processes as specified in the previous section.

The general problem for the classical Cramér-Lundberg risk reserve process was first solved by Gerber in [50] via a limit of an associated discrete problem and later on by means of stochastic control theory by Azcue & Muler [16], who also included a general reinsurance strategy as a second control possibility. See also Schmidli [112] and Mnif & Sulem [95] who allow for additional dynamic XL-reinsurance and Albrecher & Thonhauser [9] for a reserve process under a force of interest. For all these cases in general a band strategy turns out to be optimal among all admissible strategies.

For the particular case of exponentially distributed claim amounts, the band strategy collapses to a barrier strategy (this was proven by Gerber [50] in 1969 as a by-product of the general characterization). In Albrecher & Thonhauser [9] it is shown that the optimality of barrier strategies in the classical model with exponential claims still holds if there is a constant force of interest. Recently, Loeffen [89] showed that barrier strategies maximize the expected discounted dividend payments until ruin also for general spectrally negative Lévy risk processes with completely monotone jump density (and Kyprianou et al. [80] relaxed this condition on the jump densities to log-convexity). This for instance establishes the optimality of barrier strategies in the Cramér-Lundberg model with Pareto claim sizes. However, despite this collection of sufficient conditions for the optimality of barrier strategies, explicit necessary conditions on the model parameters are still not available up to now.

In the general diffusion setup the optimal dividend problem (5) was completely solved by Shreve et al. [113] and a barrier strategy was identified to be optimal. The special case of constant drift and diffusion coefficient was then solved again by slighty different means in Jeanblanc-Piqué & Shiryaev [74] and Asmussen & Taksar [13] (Radner & Shepp [103] study the situation where the drift and volatility can also be controlled within a discrete set of possible values). In addition to the dividend control, Højgaard & Taksar [69, 70] also considered the possibility of proportional reinsurance and optimal investment. For an overview on this and variants of these problems for diffusion processes see Taksar [117].

If one wants to maximize (4) over the set of absolutely continuous controls with a bounded intensity, then a threshold strategy turns out to be optimal in a diffusion risk model (cf. Asmussen & Taksar [13]) as well as in the compound Poisson risk model with exponentially distributed jumps and $a < c$ (cf. Gerber & Shiu [56]).

Motivated by optimal consumption problems from mathematical finance (see e.g. Merton [93]), Hubalek & Schachermayer [71] propose a value function measuring the expected discounted utility of a dividend stream and discuss the related optimization problem for a diffusion risk reserve process. They show that under so-called Inada conditions on the utility function $u\colon [0,\infty) \to [0,\infty)$ (namely $u'(0) = \infty$ and $u'(\infty) = 0$), the optimal strategy has to be absolutely continuous. The value of a strategy L is then defined by

$$V_L(x) = \mathbb{E}_x\left(\int_0^{\tau^L} \mathrm{e}^{-\delta s} u(l_s)\,\mathrm{d}s\right). \tag{6}$$

Although the measurement of a utility of a density may seem strange at a first glance, this can be motivated by interpreting the problem as a limit of a discrete model, where the cumulated utility of the payments from each time step is considered (cf. Borch [30]). Another utility-based approach is due to Grandits et al. [60], who propose to measure a strategy by its expected (in their case exponential) utility of the cumulated discounted dividend payments,

$$V_L(x) = \mathbb{E}_x\left(u\left(\int_0^{\tau^L} \mathrm{e}^{-\delta t}\,\mathrm{d}L_t\right)\right). \tag{7}$$

For a diffusion model a certain time-dependent barrier strategy turns out to be optimal. However, the concrete form of this barrier is difficult to obtain, as it is given through a defining integral equation.

When including transaction costs, the inclusion of a utility per payment seems to be natural (e.g. $u(z) = (kz - K)^\gamma/\gamma$ with $\gamma \in (0,1]$). Then the value of an admissible impulse strategy $S = \{(\tau_i, Z_i)\}_{i\in\mathbb{N}}$ is measured by

$$V_S(x) = \mathbb{E}_x\left(\sum_{i=1}^{\infty} \mathrm{e}^{-\delta\tau_i} u(Z_i) I_{\{\tau_i < \tau^S\}}\right). \tag{8}$$

The corresponding optimization problem is considered in Paulsen [98] for a general diffusion process, in Jeanblanc-Picqué & Shiryaev [74] for the constant drift and volatility case and in Cadenillas et al. [34] for a mean-reverting diffusion. Albrecher & Thonhauser [10] characterize the value function according to (8) for the classical model. In a similar way as for general càglàd controls, Loeffen [90] proves that a simple impulse strategy, as introduced in Section 3.2, is optimal for spectrally negative Lévy risk processes when there are fixed transaction costs with each dividend payment, $\gamma = 1$ and the density of the jump distribution is log-convex.

Of course there are various possibilities to extend the definition of the value function. We now mention two more examples that may be of particular interest for insurance issues. The first one introduces some sort of *reward* for avoiding early ruin, modelled by a discounted stream of payments with density $\Lambda > 0$ until ruin and a corresponding value function

$$V_L(x) = \mathbb{E}_x\left(\int_0^{\tau} \mathrm{e}^{-\delta t}\,\mathrm{d}L_t + \int_0^{\tau} \mathrm{e}^{-\delta t}\Lambda\,\mathrm{d}t\right). \tag{9}$$

The additional parameter Λ can be used for balancing between safety and profit in the portfolio (alternatively, one can interpret the additional summand as a certain discounted penalty at ruin, cf. Gerber et al. [53]; for the special case of expected time to ruin ($\delta = 0$) see Borch [29]). For this value function, Shreve et al. [113] and Boguslavskaya [28] identify the optimality of barrier strategies in diffusion models, and in [28] also the inclusion of transaction costs is investigated. Thonhauser & Albrecher [119] establish the optimality of barrier strategies under (9) for the classical risk model with exponentially distributed claim amounts. For recent extensions to general Lévy risk models see Loeffen [91] and for an inclusion of additional investment possibilities see Wang & Zhang [121].

Another approach is to allow for capital injections from the shareholders when the surplus falls below zero to make it again positive and avoid bankruptcy. Dickson & Waters [42] and Gerber et al. [57] assumed that the deficit at ruin has to be paid by the shareholders and hence looked at choosing an optimal barrrier that minimizes the expected difference between discounted dividend payments until ruin and deficit at ruin for a compound Poisson model. Assume now more generally that these capital injections can occur at any point in time with the goal that the surplus does not become negative and denote by $Z = (Z_t)_{t\geq 0}$ the injection process. Then the controlled process is of the form

$$R_t^{L,Z} = x + ct - \sum_{n=1}^{N_t} Y_t - L_t + Z_t.$$

The value of such a control pair (L, Z) can naturally be defined by

$$V_{L,Z}(x) = \mathbb{E}_x \left(\int_0^\infty \mathrm{e}^{-\delta t} \, \mathrm{d} L_t - \theta \int_0^\infty \mathrm{e}^{-\delta t} \, \mathrm{d} Z_t \right),$$

where $\theta > 1$ is a weight for the expected discounted capital injections. The associated general maximization problem was recently solved in Kulenko & Schmidli [77] for the classical risk model, see also Avram et al. [15]. It turns out that the optimal strategy is now for arbitrary claim size distributions a barrier strategy and injections should only take place when the process is negative. Shreve et al. [113] solved the analogous problem for a general diffusion process. He & Liang [62] deal with this problem in a diffusion framework allowing general dividend strategies and including transaction costs on the reinvestments, and Paulsen [99] investigates the diffusion setup when both dividend payments and reinvestments are charged by transaction costs.

The idea of putting different constraints on the probability of ruin of the controlled reserve process is used in Paulsen [97] for a general diffusion model and Bayraktar & Young [19] for a diffusion model and a utility criterion on the value of a strategy. Hipp [63] solves such a problem in a discrete framework. When fixing a dividend strategy, Bayraktar & Young [20, 21] use an investment control possibility for minimizing the probability of ruin of the controlled diffusion reserve process.

5 The Dynamic Programming Approach

In the following sections we will describe in more detail the nature of the mathematical challenges when trying to identify optimal control strategies in an insurance environment. We will start with a discussion of the dynamic programming approach, which is at the heart of the solution of most dividend maximization problems in a Markovian environment (for a general overview see Fleming & Soner [44] and Schmidli [112]).

The dynamic programming principle has its origin in discrete-time optimization (see e.g. [122]) and basically states that one tries to behave optimally in a first time interval and then optimally from there on. In continuous time this leads to a so-called *Hamilton-Jacobi-Bellman (HJB) equation*. Typically the derivation of this equation for $V(x)$ involves several assumptions that are difficult to verify directly. Hence the usual procedure is to derive the equation heuristically and finally prove separately (in a so-called *verification step*) that its solution is indeed the required value function of the optimal control problem. This verification step can consist of two alternative procedures: either one can show that the value function indeed fulfills the

HJB equation (by justifying all steps in the derivation of the equation rigorously), or one is able to show that the obtained solution of the HJB equation actually dominates the values of all other possible strategies (usually by martingale arguments).

5.1 Non-Singular Controls —the Classical Case

Let Π be a set of admissible strategies and R be one of the risk reserve processes introduced before. The value function $V(x)$ of the maximization problem is said to fulfill the dynamic programming principle if for any stopping time γ the equation

$$V(x) = \sup_{L\in\Pi} \mathbb{E}_x \left(\int_0^{\tau^L\wedge\gamma} \mathrm{e}^{-\delta s}\,\mathrm{d}L_s + \mathrm{e}^{-\delta(\tau\wedge\gamma)} V(R^L_{\tau^L\wedge\gamma}) \right) \tag{10}$$

holds. In other words, maximizing the dividend payments in an interval $[0,\gamma)$ and from there on continuing in an optimal way is equivalent to maximizing the payments over the whole lifetime of the reserve process. Now replace γ by some small $h>0$ and suppose that a certain admissible control L admits a density process $l=(l_t)_{t\geq 0}$ which is constant for $t\in[0,h)$. Then clearly

$$V(x) \geq \mathbb{E}_x \left(l \int_0^{h\wedge\tau^L} \mathrm{e}^{-\delta t}\,\mathrm{d}t + \mathrm{e}^{-\delta(h\wedge\tau^L)} V(R^L_{h\wedge\tau^L}) \right). \tag{11}$$

Dividing (11) by h and subtracting $V(x)$ results in

$$0 \geq \frac{1}{h}\,\mathbb{E}_x \left(l \int_0^{h\wedge\tau^L} \mathrm{e}^{-\delta t}\,\mathrm{d}t + \mathrm{e}^{-\delta(h\wedge\tau^L)} V(R^L_{h\wedge\tau^L}) - V(x) \right). \tag{12}$$

We now want to take the limit $h\to 0$ and assume that V is in the domain of the generator $\mathcal{L}^l$ of the reserve controlled by the constant *dividend density* l (at this point, several other assumptions enter that make the derivation heuristic). For the compound Poisson model this generator (compare with (2)) is for instance given by

$$\mathcal{L}^l g(x) = (c-l)g'(x) - \lambda g(x) + \lambda \int_0^x g(x-y)\,\mathrm{d}F_Y(y). \tag{13}$$

One then arrives at

$$0 \geq \mathcal{L}^l V(x) - \delta V(x) + l.$$

Suppose now that in (10) the supremum is attained for a strategy L^* (again assumed to be absolutely continuous but now not necessarily constant in $[0,h)$), so that (12) holds with equality:

$$0 = \frac{1}{h}\,\mathbb{E}_x \left(\int_0^{h\wedge\tau^{L^*}} \mathrm{e}^{-\delta t} l^*_t\,\mathrm{d}t + \mathrm{e}^{-\delta(h\wedge\tau^{L^*})} V(R^{L^*}_{(h\wedge\tau^{L^*})}) - V(x) \right).$$

This indicates that

$$0 = \sup_l \left\{ \mathcal{L}^l V(x) - \delta V(x) + l \right\} \tag{14}$$

should hold. All this is under the assumption that interchanging limit and expectation, and taking the supremum is allowed. Equation (14) is called the Hamilton-Jacobi-Bellman (HJB) equation associated with the dividend maximization problem

$$V(x) = \sup_{L\in\Pi^{\mathrm{ac}}} \mathbb{E}_x \left(\int_0^{\tau^L} \mathrm{e}^{-\delta t}\, l_t\,\mathrm{d}t \right), \tag{15}$$

where Π^{ac} denotes the set of absolutely continuous admissible strategies (a solution of (15) is given in [111]).

For a diffusion risk reserve process (constant drift μ, volatility $\sigma > 0$ and generator (3)), the HJB equation corresponding to (15) is given by

$$0 = \sup_{0 \le l \le l_\infty} \left\{ (\mu - l)V'(x) + \frac{\sigma^2}{2} V''(x) - \delta V(x) + l \right\}, \qquad V(0) = 0, \tag{16}$$

(see e.g. [13]) and the one corresponding to (6) with $u(l) = l^\alpha/\alpha$ and $\alpha \in (0,1)$, by

$$0 = \sup_{0 \le l} \left\{ (\mu - l)V'(x) + \frac{\sigma^2}{2} V''(x) - \delta V(x) + u(l) \right\}, \qquad V(0) = 0, \tag{17}$$

(cf. [71]), where due to the Inada conditions in the utility framework the upper bound l_∞ does not need to be specified.

When the reserve process is given by a diffusion, $V(0) = 0$ is an obvious initial condition for the HJB equation, because when starting in 0 the driving Brownian motion immediately becomes negative with probability 1 (see Rogers & Williams [105]) and there will be no future dividend payments. In contrast, there is no obvious initial value in the compound Poisson model, because there is a positive probability for the reserve to recover from the value 0. We will see later how this fact influences the mathematical characterization of a solution of the maximization problem.

Remark 2 *From the statement of the HJB equation (14), we immediately get that a candidate solution suggests a Markov control as the optimal strategy, i.e. the density only depends on the present state x of the process (and not on the whole filtration up to a certain time t). This means $l_t = l(R_t)$ with*

$$l(x) := \arg\max_l \left\{ \mathcal{L}^l V(x) - \delta V(x) + l \right\}.$$

Further note that classical dividend maximization problems are stated as infinite-time horizon optimization problems and therefore stationary controls are natural (for an exception see [60]).

The common notation of specifying the HJB equation through the value function $V(x)$ is, due to its heuristic derivation, a bit misleading, as one still needs to check by the verification arguments whether the actual value function indeed satisfies the HJB equation.

5.2 The Singular Control Case

Let us now drop the assumption of absolute continuity of L, i.e. we deal with the case of general admissible controls, so that the density process $l = (l_s)_{s \ge 0}$ of a dividend strategy L is not necessarily bounded. Focussing on the classical model and now plugging in the generator of the controlled reserve (13) into (14) explicitly, we obtain

$$0 = \sup_{l \ge 0} \left\{ (1 - V'(x))\, l + c\, V'(x) - (\lambda + \delta) V(x) + \lambda \int_0^x V(x-y)\, \mathrm{d}F_Y(y) \right\}. \tag{18}$$

One immediately observes that in the case of $V'(x) < 1$ for some $x \ge 0$ the local maximizer $l^*(x)$ and more generally (18) is unbounded, so that both quantities do not make sense any more. On the other hand, in the case of $V'(x) > 1$ we get $l^*(x) = 0$ and

$$0 = c\, V'(x) - (\lambda + \delta)\, V(x) + \lambda \int_0^x V(x-y)\, \mathrm{d}F_Y(y).$$

Restricting to $1 - V'(x) \le 0$ for all $x \ge 0$, we hence obtain the following rewritten HJB equation

$$0 = \max \left\{ 1 - V'(x),\ \ c\, V'(x) - (\lambda + \delta) V(x) + \lambda \int_0^x V(x-y)\, \mathrm{d}F_Y(y) \right\}. \tag{19}$$

The following observations also motivate heuristically the form of equation (19). First suppose that at some point $x \geq 0$ it is optimal to pay a (possibly very small) lump sum dividend $h > 0$ and then continue with capital $x - h$ (or stop if $x = 0$), so that $V(x) = h + V(x-h)$ which for $h \to 0$ indicates $V'(x) = 1$. Secondly, if waiting and not paying dividends in some small interval around $x \geq 0$ is optimal, one obtains the second part of the right side of (19).

For diffusion risk reserve processes with constant drift $\mu > 0$ and constant volatility $\sigma > 0$ one obtains along the same lines of arguments

$$0 = \max\left\{1 - V'(x),\ \ \mu V'(x) + \frac{\sigma^2}{2} V''(x) - \delta V(x)\right\}, \qquad V(0) = 0. \tag{20}$$

For the value function (7) and exponential utility function $u(z) = (1 - \mathrm{e}^{-\gamma z})/\gamma$, the corresponding HJB equation for the singular control problem then is

$$0 = \max\{V_t(x,t) + \mu V_x(x,t) + \frac{\sigma^2}{2} V_{xx}(x,t),\ \ -V_x(x,t) + \mathrm{e}^{-\delta t}(1 - \gamma V(x,t)\}, \qquad V(0,t) = 0.$$

Here it turns out necessary to use the additional time variable $t \geq 0$ (for details see [60]).

5.3 The Impulse Control Case

Let us first consider the compound Poisson model. For the value function (8) in the impulse control framework, we first observe that at points $x \geq 0$ where it would be optimal to intervene, we should have $\mathrm{MV}(x) = V(x)$, where the operator

$$\mathrm{MV}(x) := \sup_{y \text{ admissible}} \{u(y) + V(x-y)\},$$

gives the value of the best admissible intervention at the reserve level x. On the other hand, if it would be optimal not to intervene in an open interval around the point x, then conditioning on the first claim occurrence in a small time interval $[0,h]$ and letting $h \to 0$ will result in

$$c\,V'(x) + \lambda\left(\int_0^x V(x-y)\,\mathrm{d}F_Y(y) - V(x)\right) - \delta V(x) = 0.$$

These observations heuristically motivate the so-called *quasi-variational inequalities* (QVI):

$$\begin{aligned} c\,V'(x) + \lambda\left(\int_0^x V(x-y)\,\mathrm{d}F_Y(y) - V(x)\right) - \delta V(x) &\leq 0,\\ \mathrm{MV} - V &\leq 0,\\ \left(cV'(x) + \lambda\left(\int_0^x V(x-y)\,\mathrm{d}F_Y(y) - V(x)\right) - \delta V(x)\right)(\mathrm{MV} - V) &= 0, \end{aligned}$$

or equivalently

$$\max\left\{cV'(x) + \lambda\left(\int_0^x V(x-y)\,\mathrm{d}F_Y(y) - V(x)\right) - \delta V(x),\ \ \mathrm{MV} - V\right\} = 0.$$

For a rigorous treatment cf. [10]. For a diffusion model one just needs to replace the generator accordingly and arrives at a similar equation (see [34] or [74]). The dynamic approach for stochastic impulse control problems was introduced by Bensoussan & Lions [22, 23].

6 Discussion of the HJB equation —Verification Arguments

In the previous section we saw how one can (heuristically) derive the HJB equation associated with a given stochastic optimization problem. Now we want to link a solution of this equation to the value function of the optimization problem. Crucial questions in this context are: Which types of solutions exist? Is the value function a solution? Is the solution unique?

In general there are two ways to obtain a solution for the optimization problem based on the HJB equation.

- It is possible to prove that there exists a unique solution to the HJB equation of the given dividend maximization problem. In the ideal case it is also possible to construct an explicit solution. Then a so-called *verification theorem* is needed that states that this solution dominates all other values that can be achieved by admissible strategies, and that a strategy obtained by this solution is admissible (and hence optimal). We then get that this unique solution of the HJB equation is the value function.

- It is possible to show that there exist solutions (in some sense) of the HJB equation, but uniqueness is doubtful. Then a precise characterization of the value function is needed and one has to prove that the value function indeed fulfills the HJB equation by verifying that all steps in the derivation of the HJB equation are actually justified.

Once the value function is determined one has to identify the corresponding dividend payment strategy that realizes this value function (this is often non-trivial and it may even happen that such a strategy does not exist, see e.g. Shreve et al. [113, Th. 4.3]).

Remark 3 *As an alternative to the above full characterization of the optimization problem (the "analytic way"), another quite common (Bensoussan et al.* [24] *call it "probabilistic") approach in the literature is to maximize a certain value function over a (small) restricted class of admissible strategies, say barrier type strategies or simple impulse controls (cf. Avram et al.* [15]*, Loeffen* [90, 89]*, Gerber & Shiu* [54, 56]*). Then in some cases it is possible to verify by comparison that the —within the restricted class— optimal strategy is also optimal within the bigger class of general admissible strategies.*

6.1 There is a unique solution

In some cases it is possible to calculate an explicit solution to the HJB equation (e.g. for (16) and (20), cf. [13, 113]), whereas in other cases it is only possible to prove the existence of a classical solution (e.g. for (17), cf. [71]). *Classical solution* in this context means that the solution is as regular as required by the equation (note that the crucial points in that respect are the junction points of the various parts of the equation).

In many cases an explicit solution can be obtained along the following lines of argument: One can reformulate the HJB equation (14) as

$$0 = \sup_l \left\{ \mathcal{L}V(x) - \delta V(x) + l\left(1 - V'(x)\right)\right\}, \tag{21}$$

where $\mathcal{L}$ is the generator of the uncontrolled reserve process given in Section 2 (for notational convenience we restrict ourselves here to the absolutely continuous case).

It follows that the optimal action with current reserve x depends on whether $1 - V'(x)$ is larger than zero or equal to zero. A first approach often is to assume that $V(x)$ is concave, in which case there will only be one switching point x_0 such that $V'(x) > 1$ for $x < x_0$ and $V'(x) \le 1$ for $x \ge x_0$. This then immediately suggests the control

$$l^*(x) = \begin{cases} 0 & x < x_0, \\ l_\infty & x \ge x_0, \end{cases} \tag{22}$$

where x_0 still has to be determined. The principle of smooth fit suggests a method to determine x_0. It states that the value function should be sufficiently regular at the *free boundary* x_0 (sometimes also called *decision boundary*, cf. Peskir & Shiryaev [101] and Kyprianou & Surya [82]), i.e.

$$V_l(x_0) = V_r(x_0),$$
$$V_l'(x_0) = V_r'(x_0) = 1,$$

where V_l and V_r denote the solutions of the HJB equation for $x < x_0$ and $x \geq x_0$ under the concavity assumption (in the diffusion case one has the additional assumption $V_l''(x_0) = V_r''(x_0)$). These conditions allow for the calculation of the individual parts of the solution and an implicit determination of the crucial point x_0. In the diffusion setup it follows by easy calculations that actually x_0 is the only value of x for which one can paste V_l and V_r such that the resulting function is twice differentiable.

The form of the problem indicated by the concavity assumption on the candidate value function is called *free boundary value problem*. In Whittle [122] there are some conditions when a value function is twice differentiable at the optimal decision boundary. When the guess on the concavity of V and the smooth-fit conditions were successful to obtain a solution of (21), then it remains to verify that this solution is indeed the value function (the verification step). The basic idea in the verification theorem is often that for an arbitrary admissible strategy L with density process $l = (l_s)_{s\geq 0}$, the process $\mathrm{e}^{-\delta(t\wedge\tau^L)}V(R^L_{t\wedge\tau^L})$, by virtue of an (appropriate) Itô-formula and the dynamic programming principle, leads to a supermartingale that then can be compared to a martingale resulting from the process $\mathrm{e}^{-\delta(t\wedge\tau^{L^*})}V(R^{L^*}_{t\wedge\tau^{L^*}})$ with strategy L^* given by (22). This then establishes $V(x) \geq V_L(x)$ for any other strategy L and $V(x) = V_{L^*}(x)$.

The mentioned martingale properties are usually established by a suitable application of the Itô formula (or its extension for jumps, respectively). In particular, one has to make sure that differentiability properties of V needed in the Itô formula actually hold (this is for instance automatically the case if the construction of the solution via the smooth-fit principle succeeds). This step can sometimes require considerable technical expertise.

Davis [38] considers the verification theorem as the mathematical motivation of the HJB equation.

6.2 There is no unique solution

For dividend maximization problems (4) in the classical risk model with arbitrary claim size distribution, some difficulties may arise. This problem was first solved by Gerber [50] via a discretization and taking the continuous-time limit. As already mentioned, he identified band strategies to be optimal in this context. Only recently Azcue & Muler [16] used the dynamic programming approach to obtain the HJB equation (19) for this problem (they also included a dynamic reinsurance possibility, see also Schmidli [111]).

The two main difficulties which arise when looking at (19) are the question of differentiability and uniqueness of a solution. The uniqueness question is really crucial, because when starting with a wrong solution to (19) the construction of an associated admissible strategy fails. There are certain parameter constellations (e.g. huge λ) such that the simple linear function $f(x) = x + c/\delta$ solves (19) but the associated strategy attaining this value (which is *pay out the initial capital* x *and subsequently all incoming premiums ignoring potential ruin*) is of course not admissible. This problem is mainly due to the missing initial value for the HJB equation. As one can in this case usually not find a solution that is sufficiently differentiable, one has to introduce other non-classical solution concepts, for instance *viscosity solutions*. For the latter, one replaces a function around a problematic point $x \geq 0$ locally by smooth functions that upperbound and lowerbound (respectively) the original function V. If V can be approximated from below (above) such that the HJB equation becomes an inequality bigger (smaller) than zero for the approximating function, one calls V a viscosity subsolution (supersolution, respectively). If both approximations are possible, V is called a viscosity solution. This extended solution concept was first introduced in Crandall & Lions [37], see also Crandall et al. [36]. For PDMPs this notion was used by Soner [115]. Schmidli [111] uses weak solutions as a further alternative solution concept. For a mixed stochastic control problem that arises in a real options situation in a diffusion framework, where one has to choose between paying dividends or investing, viscosity solutions were recently employed by Vath et al. [120].

Let us now sketch the derivation of the solution of the HJB equation

$$\max\left\{1 - V'(x),\;\; c\,V'(x) - (\delta+\lambda)V(x) + \lambda \int_0^x V(x-y)\,\mathrm{d}F_Y(y)\right\} = 0. \tag{23}$$

It is possible to obtain directly from the definition (5) of V and the definition of the general admissible strategies from Section 3.2 that V is absolutely continuous and linearly bounded. The further characterization splits into two steps.

First a so-called comparison result (if a viscosity supersolution is bigger than a viscosity subsolution in zero then this relation holds over $\mathbb{R}^+$) is needed and in a second step one shows that every viscosity supersolution is dominating the value function (this is done in a similar way as one proves a verification theorem). Because V is both a super- and subsolution, it has to be the smallest viscosity solution fulfilling a linear growth condition.

Now it only remains to determine the dividend strategy associated to the correct solution V to (23). In the above example it turns out to be a band strategy defined by the sets

- $\mathcal{A} = \{\, x \in [0,\infty) \mid c - (\delta+\lambda)V(x) + \lambda \int_0^x V(x-y)\,\mathrm{d}F_Y(y) = 0 \,\}$,
- $\mathcal{B} = \{\, x \in (0,\infty) \mid V'(x) = 1$ and $c - (\delta+\lambda)V(x) + \lambda \int_0^x V(x-y)\,\mathrm{d}F_Y(y) < 0 \,\}$,
- $\mathcal{C} = (\mathcal{A} \cup \mathcal{B})^c$.

In [111] an algorithmic procedure for obtaining V is described, in [16] and [9] explicit examples are constructed which demonstrate the necessity of an extended solution concept. A policy iteration algorithm for a related problem is constructed in [95].

7 Conclusion and Open Problems

It turned out that the *complete* solution of the seemingly simple problem of determining optimal dividend strategies for insurance risk processes requires advanced techniques from analysis, probability and stochastic control. Although the understanding of the problem in the classical risk model as well as in the diffusion model has now reached a certain state of matureness, there are still open questions.

- As discussed in Section 4, a barrier-type strategy turns out to be the optimal choice in several model situations, but even for the classical risk model there are still no explicit criteria on the model parameters available that are both necessary and sufficient for a barrier strategy to be optimal. Similarly, necessary and sufficient conditions for a threshold strategy to be optimal are still unknown. Furthermore, a rigorous numerical analysis for the determination of the optimal strategies for given parameter values needs to be developed.

- The research on dividend maximization problems under transactions costs and/or under utility criteria in the classical model is just starting to develop. For the case of transaction costs, up to now a complete characterization of the value function and some numerical ideas have been developed [10], but a formal description of a strategy that is in general optimal is not available yet.

- Optimal dividend strategies under additional constraints on the probability of ruin (see e.g. Hipp [63] for a particular case) and in general under constraints on the trajectories of the controlled process, seem to be a very hard problem for a risk reserve process with jumps.

Furthermore, there are a wealth of open problems under modified model assumptions. In this context, a particular line of potential future research is to consider the optimal dividend problem when the Poisson claim number process is replaced by a general renewal process, i.e. the Sparre Andersen risk model [116]. Li & Garrido [85] study properties of the renewal risk reserve process under a barrier strategy and Albrecher et al. [2] calculate the moments of the expected discounted dividend payments under a barrier strategy in this framework, but Albrecher & Hartinger [3] show that even in the case of $\mathrm{Erlang}(2)$ distributed interclaim times and exponentially distributed claim amounts a horizontal barrier strategy is not optimal anymore, as it can be outperformed by a strategy that depends on the time elapsed since the previous claim occurrence. It is still an open problem to identify optimal dividend strategies in this model. One can markovize the Sparre Andersen model by extending the dimension of the state space of the risk process, taking into account the time that has elapsed since the last claim occurrence. A reasonable strategy should also depend on this additional variable. But correspondingly also the dimension of the associated HJB equation will be extended which considerably increases the difficulties one is facing when analytically approaching this equation.

Finally, for risk reserve processes modelled by general spectrally negative Lévy processes, Loeffen [89] and Avram et al. [15] study the dividend optimization problem from a probabilistic point of view. It is still open to approach and solve this problem in this general setup by means of stochastic optimal control.

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Hansjörg Albrecher
Institute of Actuarial Science,
Faculty HEC,
University of Lausanne,
CH-1015 Lausanne,
Switzerland

Stefan Thonhauser
Institute of Actuarial Science,
Faculty HEC,
University of Lausanne,
CH-1015 Lausanne,
Switzerland