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VORONOVSKAYA TYPE ASYMPTOTIC FORMULA FOR LUPAŞ-DURRMEYER OPERATORS[1]

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ABSTRACT. In the present paper, we study some direct results in simultaneous approximation for linear combinations of Lupaş-Beta type operators.

1. Introduction

The Bernstein-Durrmeyer $M_n, n \in N_0$ (the set of non-negative integers), were introduced by Durrmeyer [2] and independently by Lupaş [5]. For a function $f \in L^1[0,1]$ they are defined by

$$(M_n f)(x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)f(t)dt,\ x \in [0,1],$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k(1-x)^{n-k},\ 0 \le k \le n.$$

Later, starting with this integral modification of Bernstein polynomials, Heilmann [3] first defined modified Lupaş operators (see also Heilmann and Müller [4] as well as Sinha et al. [7]). More recently the present author [1] studied another modification of Lupaş operators. Now we consider Beta operator as a weight function on $C[0,\infty)$ namely,

$$(L_n f)(x) = \sum_{k=0}^{\infty} v_{n,k}(x)\int_0^{\infty} b_{n,k}(t)f(t)dt,\ x \in [0,\infty), \tag{1.1}$$

where $v_{n,k}(x) = \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}$ and

$$b_{n,k}(t) = \frac{1}{B(k+1,n)}\frac{t^k}{(1+t)^{n+k+1}} = n\binom{n+k}{k}\frac{t^k}{(1+t)^{n+k+1}} = nv_{n+1,k}(x),$$

$B(.,.)$ denoting the Beta function. Therefore

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$$(L_n f)(x) := n\sum_{k=0}^{\infty} v_{n,k}(x)\int_0^{\infty} v_{n+1,k}(t)f(t)dt,\ x\in[0,\infty)\,.$$

Let us remark that many years ago W. Meyer-König and K. Zeller [6] have introduced, in order to approximate functions g from $C[0,1]$, the so-called Bernstein-power series $M_n g$ defined as

$$(M_n g)(z) = \begin{cases} \sum\limits_{k=0}^{\infty} m_{n,k}(z) g\left(\frac{k}{n+k-1}\right), & z\in[0,1) \\ g(1), & z=1, \end{cases}$$

with $m_{n,k}(z) = \left(\begin{array}{c} n+k-1 \\ k \end{array}\right) z^k(1-z)^n$. Because $v_{n,k}\left(\frac{y}{1-y}\right) = m_{n,k}(y)$, we see that

$$(L_n f)\left(\frac{y}{1-y}\right) = n\sum_{k=0}^{\infty} m_{n,k}(y)\int_0^1 m_{n,k}(T) f\left(\frac{T}{1-T}\right)dT, \quad y\in[0,1]\,.$$

The main object of this paper is to establish a Voronovskaya type asymptotic formula and an error estimate for the linear combination of the operators (1.1).

2. **Auxiliary Results**

In this section, we shall give certain definition and lemmas which will be used in the sequel.

For every $n \in N$ and $n > (r+1)$ we have

$$\begin{cases} \sum\limits_{k=0}^{\infty} v_{n,k}(x) = 1, & \int_0^{\infty} b_{n,k}(t)dt = 1 \\ \frac{k}{n} v_{n,k}(x) = x v_{n+1,k-1}(x), & \int_0^{\infty} t b_{n-r,k+r}(t)dt = \frac{k+r+1}{n-r-1}. \end{cases} \tag{2.1}$$

Lemma 2.1. *Let $m, r \in N_0$ (the set of non-negative integers), we define*

$$\mu_{r,n,m}(x) = \sum_{k=0}^{\infty} v_{n+r,k}(x)\int_0^{\infty} b_{n-r,k+r}(t)(t-x)^m dt$$

then

$$\mu_{r,n,0}(x) = 1,\ \ \mu_{r,n,1(x)} = \frac{1+r+x(1+2r)}{(n-r-1)}, \tag{2.2}$$

$$\mu_{r,n,2}(x) = \frac{2(2r^2+4r+n+1)x^2 + 2(2r^2+5r+2+n)x + (r^2+3r+2)}{(n-r-1)(n-r-2)}, \tag{2.3}$$

and there holds the recurrence relation:

$$(n-m-r-1)\mu_{r,n,m+1}(x) = \phi^2(x)\left[\mu'_{r,n,m}(x) + 2m\mu_{r,n,m-1}(x)\right] + \left[(m+r+1)(1+2x) - x\right]\mu_{r,n,m}(x), \tag{2.4}$$

where $\phi(x) = \sqrt{x(1+x)}$. *Consequently, for each* $x \in [0,\infty)$

$$\mu_{r,n,m}(x) = O\left(n^{-[(m+1)/2]}\right). \tag{2.5}$$

Proof. We can easily obtain (2.2) and (2.3) by using the definition of $\mu_{r,n,m}(x)$. For the proof of (2.4), we proceed as follows. First

$$\phi^2(x)\mu'_{r,n,m}(x) = \sum_{k=0}^{\infty} \phi^2(x)v'_{n+r,k}(x)\int_0^\infty b_{n-r,k+r}(t)(t-x)^m dt - \phi^2(x)m\mu_{r,n,m-1}(x).$$

Now, using relations

$$\phi^2(x)v'_{n,k}(x) = (k-nx)v_{n,k}(x) \text{ and } \phi^2(t)b'_{n,k}(t) = \left[k-(n+1)t\right]b_{n,k}(t),$$

we obtain

$$\begin{aligned}
&\phi^2(x)\left[\mu'_{r,n,m}(x) + m\mu_{r,n,m-1}(x)\right] \\
=\ & \sum_{k=0}^{\infty}[k-(n+r)x]v_{n+r,k}(x)\int_0^\infty b_{n-r,k+r}(t)(t-x)^m dt \\
=\ & \sum_{k=0}^{\infty} v_{n+r,k}(x)\int_0^\infty [(k+r)-(n+1-r)t]b_{n-r,k+r}(t)(t-x)^m dt \\
& + \left[x - r(1+2x)\right]\mu_{r,n,m}(x) + (n+1-r)\mu_{r,n,m+1}(x) \\
=\ & \sum_{k=0}^{\infty} v_{n+r,k}(x)\int_0^\infty t(1+t)b'_{n-r,k+r}(t)(t-x)^m dt + \left[x-r(1+2x)\right]\mu_{r,n,m}(x) \\
& + (n+1-r)\mu_{r,n,m+1}(x) \\
=\ & \sum_{k=0}^{\infty} v_{n+r,k}(x)\int_0^\infty \left[(2x+1)(t-x) + (t-x)^2 + (x+1)x\right]b'_{n-r,k+r}(t)(t-x)^m dt \\
& + \left[x-r(1+2x)\right]\mu_{r,n,m}(x) + (n+1-r)\mu_{r,n,m+1}(x) \\
=\ & -(2x+1)(m+1)\mu_{r,n,m}(x) - (m+2)\mu_{r,n,m+1}(x) - m\phi^2(x)\mu_{r,n,m-1}(x) \\
& + \left[x-r(1+2x)\right]\mu_{r,n,m}(x) + (n+1-r)\mu_{r,n,m+1}(x).
\end{aligned}$$

This leads to (2.4). The proof of (2.5) easily follow from (2.2) and (2.4). □

Lemma 2.2. *If f is differentiable r times $(r = 1, 2, ...)$ on $[0,\infty)$, then we get*

$$\left(L_n^{(r)} f\right)(x) = \beta(n,r)\sum_{k=0}^{\infty} v_{n+r,k}(x)\int_0^\infty b_{n-r,k+r}(t)f(t)dt,\ x\in[0,\infty) \tag{2.6}$$

where

$$\beta(n,r) = \prod_{l=0}^{r-1}\frac{n+l}{n-(l+1)} = \frac{(n+r-1)!(n-r-1)!}{((n-1)!)^2}.$$

Proof. By using the Leibniz theorem, we obtain

$$\left(L_n^{(r)}f\right)(x) = \sum_{i=0}^{r}\sum_{k=i}^{\infty}\binom{r}{i}\frac{(n+k+r-i-1)!}{(k-1)!\,(n-1)!}\frac{(-1)^{r-i}x^{k-i}}{(1+x)^{n+k+r-i}}\int_0^\infty b_{n,k}(t)f(t)dt$$
$$= \frac{(n+r-1)!}{(n-1)!}\sum_{k=0}^{\infty}v_{n+r,k}(x)\int_0^\infty\sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}b_{n,k+i}(t)f(t)dt,$$

using again Leibniz theorem, we get

$$b_{n-r,k+r}^{(r)}(t) = \frac{(n-1)!}{(n-r-1)!}\sum_{i=0}^{r}(-1)^{i}\binom{r}{i}b_{n,k+i}(t).$$

Thus

$$\left(L_n^{(r)}f\right)(x) = \frac{(n+r-1)!(n-r-1)!}{((n-1)!)^2}\sum_{k=0}^{\infty}v_{n+r,k}(x)\int_0^\infty(-1)^r b_{n-r,k+r}^{(r)}(t)f(t)dt.$$

On integrating r times by parts, we get the required result. □

3. Voronovskaya Asymptotic Formula

Theorem 3.1. *Let f integrable in $[0,\infty)$, admits its $(r+1)-th$ and $(r+2)-th$ derivatives, which are bounded at a fixed point $x\in[0,\infty)$ and $f^{(r)}(t)=O(t^\alpha)$ as $t\to\infty$ for some $\alpha>0$, then*

$$\lim_{n\to\infty} n\left[\frac{1}{\beta(n,r)}\left(L_n^{(r)}f\right)(x)-f^{(r)}(x)\right] = \{1+r+x(1+2r)\}f^{(r+1)}(x)+2\phi^2(x)f^{(r+2)}(x),$$

where

$$\beta(n,r)=\prod_{l=0}^{r-1}\frac{n+l}{n-(l+1)}=\frac{(n+r-1)!(n-r-1)!}{((n-1)!)^2},\quad \phi(x):=\sqrt{x(1+x)}$$

Proof. Using Taylor's formula, we have

$$f^{(r)}(t)-f^{(r)}(x)=(t-x)f^{(r+1)}(x)+\frac{(t-x)^2}{2}f^{(r+2)}(x)+\frac{(t-x)^2}{2}\zeta(t,x), \quad (3.1)$$

where

$$\zeta(t,x) = \frac{f^{(r)}(t)-f^{(r)}(x)-(t-x)f^{(r+1)}(x)-\frac{(t-x)^2}{2}f^{(r+2)}(x)}{\frac{(t-x)^2}{2}} \quad if \quad x\neq t$$
$$=0 \quad if \quad x=t.$$

Now, for arbitrary $\varepsilon>0$, $A>0$ there exists a $\delta>0$ such that

$$\left|\zeta(t,x)\right|\le\varepsilon \quad for \quad \left|t-x\right|\le\delta,\ x\le A. \quad (3.2)$$

Using the value of (2.6) in (3.1), we get

$$\begin{aligned}\frac{1}{\beta(n,r)}\left(L_n^{(r)}f\right)(x)-f^{(r)}(x)&=\sum_{k=0}^{\infty}v_{n+r,k}(x)\int_0^{\infty}b_{n-r,k+r}(t)\,f^{(r)}(t)dt-f^{(r)}(x)\\&=\sum_{k=0}^{\infty}v_{n+r,k}(x)\int_0^{\infty}b_{n-r,k+r}(t)\,[f^{(r)}(t)-f^{(r)}(x)]dt\\&=\mu_{r,n,1}f^{(r+1)}(x)+\frac{\mu_{r,n,2}}{2}f^{(r+2)}(x)+R_{n,r}(x)\end{aligned}$$

where

$$R_{n,r}(x)=\frac{1}{2}\sum_{k=0}^{\infty}v_{n+r,k}(x)\int_0^{\infty}b_{n-r,k+r}(t)(t-x)^2\xi(t,x)dt$$

In order to completely prove the theorem, it is sufficient to show that

$$nR_{n,r}(x)\to 0\quad as\quad n\to\infty.$$

Now

$$nR_{n,r}(x)=Q_{n,r,1}(x)+Q_{n,r,2}(x)$$

where

$$Q_{n,r,1}(x)=\frac{n}{2}\sum_{k=0}^{\infty}v_{n+r,k}(x)\int_{\left|t-x\right|\leq\delta}b_{n-r,k+r}(t)(t-x)^2\xi(t,x)dt$$

and

$$Q_{n,r,2}(x)=\frac{n}{2}\sum_{k=0}^{\infty}v_{n+r,k}(x)\int_{\left|t-x\right|>\delta}b_{n-r,k+r}(t)(t-x)^2\xi(t,x)dt$$

Using (3.2) and (2.3).

$$\begin{aligned}\left|Q_{n,r,1}(x)\right|&<\frac{n\varepsilon}{2}\sum_{k=0}^{\infty}v_{n+r,k}(x)\int_{\left|t-x\right|\leq\delta}v_{n-r,k+r}(t)(t-x)^2dt\\&\leq 2\varepsilon\phi^2(x)\quad as\ \ n\to\infty\end{aligned}\tag{3.3}$$

Finally we estimate $Q_{n,r,2}(x)$, using the assumption of theorem,

$$\begin{aligned}
Q_{n,r,2}(x) =& O\left(\frac{n}{2}\sum_{k=0}^{\infty} v_{n+r,k}(x)\int_{\left|t-x\right|>\delta} b_{n-r,k+r}(t)t^{\alpha}dt\right)\\
=& O\left(\frac{n}{2}\sum_{k=0}^{\infty} v_{n+r,k}(x)\int_{\left|t-x\right|>\delta} v_{n-r,k+r}(t)\Big(\sum_{i=0}^{\alpha}\binom{\alpha}{i}(t-x)^{i}x^{\alpha-i}\Big)dt\right)\\
=& O\left(\frac{n}{2}\sum_{k=0}^{\infty} v_{n+r,k}(x)\int_{\left|t-x\right|>\delta} b_{n-r,k+r}(t)\frac{(t-x)^3}{\delta^3}\right.\\
&\left.\cdot\Big(\sum_{i=0}^{\alpha}\binom{\alpha}{i}(t-x)^{i}x^{\alpha-i}\Big)dt\right)\\
=& O\left(\frac{n}{2\delta^3}\sum_{k=0}^{\infty} v_{n+r,k}(x)\int_{0}^{\infty} b_{n-r,k+r}(t)\Big(\sum_{i=0}^{\alpha}\binom{\alpha}{i}(t-x)^{i+3}x^{\alpha-i}\Big)dt\right)\\
=& O\left(\frac{1}{n}\right), \text{ in view of (2.5)}
\end{aligned}\tag{3.4}$$

Thus, from (3.3) and (3.4), we have

$$\lim_{n\to\infty}\left|nR_{n,r}(x)\right| \le 2\varepsilon\phi^2(x).$$

Since ε is arbitrary, therefore

$$\lim_{n\to\infty}(nR_{n,r}(x)) = 0.$$

This completes the proof. □

Theorem 3.2. *Let $f^{(r+1)} \in C[0,\infty)$ and $[0,\lambda] \subseteq [0,\infty)$ and let $\omega\left(f^{(r+1)};.\right)$ be the modulus of continuity of $f^{(r+1)}$ then for $r = 0, 1, 2, ...$*

$$\begin{aligned}
&\left\|\frac{1}{\beta(n,r)}\left(L_n^{(r)}f\right) - f^{(r)}\right\|_{C[0,\lambda]}\\
&\le \frac{1+r+\lambda(1+2r)}{(n-r-1)}\left\|f^{(r+1)}\right\| + C(n,r)\left(\sqrt{\eta}+\frac{\eta}{2}\right)\omega\left(f^{(r+1)};C(n,r)\right),
\end{aligned}$$

where the norm is sup-norm over $[0,\lambda]$,

$$\eta = 2(2r^2+4r+n+1)\lambda^2 + 2(2r^2+5r+2+n)\lambda + (r^2+3r+2)$$

and

$$C(n,r) = \frac{1}{(n-r-1)(n-r-2)}.$$

Proof. Applying the Taylor formula

$$f^{(r)}(t) - f^{(r)}(x) = (t-x)f^{(r+1)}(x) + \int_x^t \left\{(f^{(r+1)}(y) - f^{(r+1)}(x)\right\}dy.$$

Thus

$$\begin{aligned}\frac{1}{\beta(n,r)}\left(L_n^{(r)}f\right)(x)-f^{(r)}(x) &= \sum_{k=0}^{\infty} v_{n+r,k}(x)\int_0^{\infty} b_{n-r,k+r}(t)\big(f^{(r)}(t)-f^{(r)}(x)\big)dt\\ &= \sum_{k=0}^{\infty} v_{n+r,k}(x)\int_0^{\infty} b_{n-r,k+r}(t)\Big[(t-x)f^{(r+1)}(x)\\ &\qquad + \int_x^t \big\{(f^{(r+1)}(y)-f^{(r+1)}(x)\big\}dy\Big]dt.\end{aligned}$$

Since,

$$\left|f^{(r+1)}(y)-f^{(r+1)}(x)\right| < \left(1+\frac{|y-x|}{\delta}\right).\omega\left(f^{(r+1)};\delta\right).$$

Hence, by Schwartz's inequality

$$\left|\frac{1}{\beta(n,r)}\left(L_n^{(r)}f\right)(x)-f^{(r)}(x)\right| \le \left|\mu_{r,n,1}\right|\left|f^{(r+1)}(x)\right| + \left(\left|\sqrt{\mu_{r,n,2}}\right| + \frac{\left|\mu_{r,n,2}\right|}{2\delta}\right)\omega\left(f^{(r+1)};\delta\right).$$

Further, choosing $\delta = C(n,r)$ and using Lemma 2.1, we get the required result. □

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