Revista Colombiana de Matemáticas
Volumen 42(2008)1, páginas 61-66

Nonderogatory directed windmills

Molinos de viento dirigidos no derogatorios

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Abstract. A directed graph G is nonderogatory if its adjacency matrix A is nonderogatory, i.e., the characteristic polynomial of A is equal to the minimal polynomial of A. Given integers $r \geq 2$ and $h \geq 3$, a directed windmill $M_h(r)$ is a directed graph obtained by coalescing r dicycles of length h in one vertex. In this article we solve a conjecture proposed by Gan and Koo ([3]): $M_h(r)$ is nonderogatory if and only if $r = 2$.

Key words and phrases. Nonderogatory matrix, characteristic polynomial of directed graphs, directed windmills.

2000 Mathematics Subject Classification. 05C50.

Resumen. Un grafo dirigido G es no-derogatorio si su matriz de adyacencia A es no-derogatoria, es decir el polinomio característico de A es igual al polinomio minimal de A. Dados enteros $r \geq 2$ y $h \geq 3$, el molino de viento dirigido $M_h(r)$ es un grafo dirigido que se obtiene por medio de la coalescencia de r diciclos de longitud h en un vértice. En este artículo resolvemos una conjetura propuesta por Gan y Koo ([3]) : $M_h(r)$ es no-derogatorio si, y sólo si, $r = 2$.

Palabras y frases clave. matriz no-derogatoria, polinomio característico de grafos dirigidos, molinos de viento dirigidos.

1. Introduction

A digraph (directed graph) $G = (V, E)$ is defined to be a finite set V and a set E of ordered pairs of elements of V. The sets V and E are called the set of vertices and arcs, respectively. If $(u, v) \in E$ then u and v are adjacent and (u, v) is an arc starting at vertex u and terminating at vertex v.

Let $\mathcal{M}_n(\mathbb{C})$ denote the space of square matrices of order n with entries in $\mathbb{C}$. Suppose that $\{u_1, \ldots, u_n\}$ is the set of vertices of G. The adjacency matrix of G is the matrix $A \in \mathcal{M}_n(\mathbb{C})$ whose entry a_{ij} is the number of arcs starting

[a]Financial support was received from CDCHT-ULA, Project No. C-13490505B.

at u_i and terminating at u_j. The characteristic polynomial of G is denoted by $\Phi_G(x)$ (or simply Φ_G) and it is defined as the characteristic polynomial of the adjacency matrix A of G, i.e., $\Phi_G(x) = |xI - A|$, where I is the identity matrix.

The monic polynomial of least degree which annihilates A is called the minimal polynomial of G and is denoted by $m_G(x) = m_G$; it divides every polynomial $f \in \mathbb{C}[x]$ such that $f(A) = 0$. In particular, by the Cayley-Hamilton Theorem, $m_G(x)$ divides $\Phi_G(x)$. Moreover, $\Phi_G(x)$ and $m_G(x)$ have the same roots.

A digraph G is nonderogatory if its adjacency matrix A is nonderogatory, i.e., if $\Phi_G(x) = m_G(x)$; otherwise, G is derogatory. For example, dipaths P_n, dicycles C_n, difans F_n and diwheels W_n are classes of nonderogatory digraphs. These classes of digraphs have been studied by Gan, Lam and Lim ([2],[4] and [5]). More recently ([3]), Gan and Koo considered the problem of determining when the directed windmills are nonderogatory.

Let h, r be integers such that $h \geq 3$ and $r \geq 2$. A directed windmill $M_h(r)$ is the directed graph with $r(h-1)+1$ vertices obtained from the coalescence of r dicycles of length h in one vertex (see Figure 1).

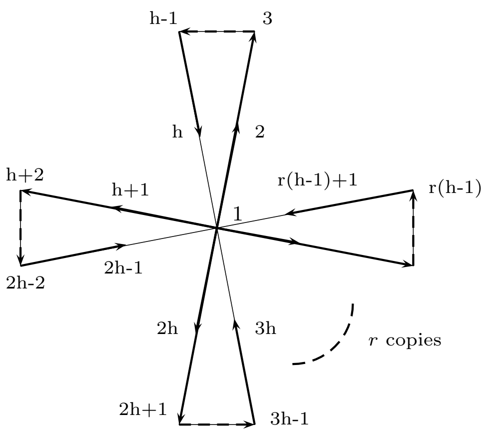

FIGURE 1. The directed windmill $M_h(r) : r$ copies of the dicycle C_h.

Gan and Koo showed that $M_3(r)$ is nonderogatory if and only if $r = 2$. Moreover, they conjectured that for every $h \geq 3$

$$M_h(r) \text{ is nonderogatory } \Leftrightarrow r = 2\,.$$

In this paper we show that this conjecture is true.

2. Nonderogatory directed windmills

Recall that a linear directed graph is a digraph in which each indegree and each outdegree is equal to 1 (i.e. it consists of cycles). The coefficient theorem for digraphs ([1, Theorem 1.2]) relates the coefficients of the characteristic polynomial with the structure of the digraph.

Theorem 2.1. *Let*

$$\Phi_G(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$$

be the characteristic polynomial of the digraph G. Then for each $i = 1, \ldots, n$

$$a_i = \sum_{L \in \mathcal{L}_i} (-1)^{p(L)} ,$$

where $\mathcal{L}_i$ is the set of all linear directed subgraphs L of G with exactly i vertices; $p(L)$ denotes the number of components of L.

Lemma 2.2. *The characteristic polynomial of $M_h(r)$ is*

$$\Phi_{M_h(r)} = x^{r(h-1)+1} - rx^{r(h-1)+1-h} = x^{r(h-1)+1-h}\left[x^h - r\right] .$$

Proof. This is an immediate consequence of Theorem 2.1. ☑

Let G be a directed graph and $A = (a_{ij})$ its adjacency matrix. By a walk of length k in G we mean a sequence of vertices $v_0 v_1 \cdots v_k$ in which each (v_{i-1}, v_i) is an arc of G. It is well known that the number of walks of length k between two vertices v_i and v_j of G is $a_{ij}^{(k)}$, the entry ij of the power matrix A^k ([1, Theorem 1.9]).

Theorem 2.3. *$M_h(r)$ is nonderogatory if and only if $r = 2$.*

Proof. The characteristic polynomial of $M_h(2)$ is

$$\Phi_{M_h(2)} = x^{h-1}\left(x^h - 2\right) .$$

Let $f(x) = x^{h-2}\left(x^h - 2\right)$ and $A = (a_{ij})$ the adjacency matrix of $M_h(2)$. From the structure of $M_h(2)$ it can be easily seen that $a_{h+1,h}^{(2h-2)} = 1$ and $a_{h+1,h}^{(h-2)} = 0$. Consequently $f(A) \neq 0$, which implies that $\Phi_{M_h(2)} = m_{M_h(2)}$ and $M_h(2)$ is nonderogatory.

We next show that if $r \geq 3$ then $M_h(r)$ is derogatory. For $i = 1, \ldots, h-1$, we denote by e_i the canonical row vector of $\mathbb{C}^{h-1}$ and f_i the canonical column vector of $\mathbb{C}^{h-1}$. Labeling the vertices of $M_h(r)$ as shown in Figure 1, the adjacency matrix A of $M_h(r)$ has the form

$$A = \begin{pmatrix} 0 & e_1 & e_1 & \cdots & e_1 \\ f_{h-1} & X & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & & \ddots & & \vdots \\ f_{h-1} & \mathbf{0} & \cdots & X & \mathbf{0} \\ f_{h-1} & \mathbf{0} & \mathbf{0} & \cdots & X \end{pmatrix}$$

where $\mathbf{0} \in \mathcal{M}_{h-1}(\mathbb{C})$ is the zero matrix and $X = (x_{ij}) \in \mathcal{M}_{h-1}(\mathbb{C})$ is the matrix such that $x_{i,i+1} = 1$ for $i = 1, \ldots, h-2$, and the rest of the entries of X are zero. Set $Y_1 = X$, $Z_1 = \mathbf{0}$ and for $j = 2, \ldots, h-1$ define recursively

$$Y_j = f_{h+1-j} e_1 + Y_{j-1} X \tag{1}$$

and

$$Z_j = f_{h+1-j}\, e_1 + Z_{j-1} X\,. \tag{2}$$

We next show that for every $j = 1, \ldots, h-1$

$$A^j = \begin{pmatrix} 0 & e_j & e_j & \cdots & e_j \\ f_{h-j} & Y_j & Z_j & \cdots & Z_j \\ \vdots & & \ddots & & \vdots \\ f_{h-j} & Z_j & \cdots & Y_j & Z_j \\ f_{h-j} & Z_j & Z_j & \cdots & Y_j \end{pmatrix}. \tag{3}$$

In fact, this is clear for $j = 1$. Assume (3) holds for $1 \leq i \leq h-2$. Note that

$$e_i f_{h-1} = \mathbf{0} \text{ and } e_i X = e_{i+1}\,. \tag{4}$$

On the other hand, since $X f_j = f_{j-1}$ for every $j = 2, \ldots, h-1$ then

$$Y_i f_{h-1} = f_{h+1-i} e_1 f_{h-1} + Y_{i-1} X f_{h-1} = Y_{i-1} f_{h-2}$$

and after $i-1$ steps we deduce

$$Y_i f_{h-1} = Y_{i-1} f_{h-2} = Y_{i-2} f_{h-3} = \cdots = Y_1 f_{h-i}\,.$$

But recall that $Y_1 = X$ and so

$$Y_i f_{h-1} = f_{h-(i+1)}\,. \tag{5}$$

Similarly,

$$Z_i f_{h-1} = Z_{i-1} f_{h-2} = \cdots = Z_1 f_{h-i}\,,$$

but $Z_1 = 0$ implies

$$Z_i f_{h-1} = 0\,. \tag{6}$$

Also we know that

$$f_{h-i} e_1 + Y_i X = f_{h+1-(i+1)} e_1 + Y_{(i+1)-1} X = Y_{i+1} \tag{7}$$

and

$$f_{h-i} e_1 + Z_i X = Z_{i+1}\,. \tag{8}$$

Consequently, it follows from equations (4)-(8) that

$$A^{i+1} = A^i A = \begin{pmatrix} 0 & e_{i+1} & e_{i+1} & \cdots & e_{i+1} \\ f_{h-(i+1)} & Y_{i+1} & Z_{i+1} & \cdots & Z_{i+1} \\ \vdots & & \ddots & & \vdots \\ f_{h-(i+1)} & Z_{i+1} & \cdots & Y_{i+1} & Z_{i+1} \\ f_{h-(i+1)} & Z_{i+1} & Z_{i+1} & \cdots & Y_{i+1} \end{pmatrix},$$

hence (3) holds for every $j = 1, \ldots, h-1$.

On the other hand,

$$e_{h-1}f_{h-1} = 1,\ e_{h-1}X = \mathbf{0},$$

$$Y_{h-1}f_{h-1} = \mathbf{0} = Z_{h-1}f_{h-1},$$

and from repeated use of (1) and the fact that $X^h = \mathbf{0}$,

$$\begin{aligned} f_1e_1 + Y_{h-1}X &= f_1e_1 + (f_2e_1 + Y_{h-2}X)\,X \\ &= f_1e_1 + f_2e_2 + Y_{h-2}X^2 = \cdots \\ &= \sum_{k=1}^{h-2} f_ke_k + Y_2X^{h-2} = \sum_{k=1}^{h-2} f_ke_k + (f_{h-1}e_1 + Y_1X)\,X^{h-2} \\ &= \sum_{k=1}^{h-2} f_ke_k + f_{h-1}e_1X^{h-2} + X^h = \sum_{k=1}^{h-1} f_ke_k = I\,. \end{aligned}$$

Similarly, using (2) it can be shown that $f_1e_1 + Z_{h-1}X = I$. It follows from these relations and (3) that

$$A^h = A^{h-1}A = \begin{pmatrix} r & 0 & \cdots & 0 \\ 0 & I & \cdots & I \\ \vdots & \vdots & & \vdots \\ 0 & I & \cdots & I \end{pmatrix}, \tag{9}$$

where the $0's$ in the first row are the zero vectors in $\mathbb{C}^{h-1}$, the $0's$ in the first column are the zero column vectors of $\mathbb{C}^{h-1}$ and $I \in \mathcal{M}_{h-1}(\mathbb{C})$ is the identity.

Relation (9) implies that for every integer $k \geq 2$

$$A^{kh} = \begin{pmatrix} r^k & 0 & \cdots & 0 \\ 0 & r^{k-1}I & \cdots & r^{k-1}I \\ \vdots & \vdots & & \vdots \\ 0 & r^{k-1}I & \cdots & r^{k-1}I \end{pmatrix} = rA^{(k-1)h}\,. \tag{10}$$

Now consider the polynomial $g \in \mathbb{C}[x]$ defined as

$$g(x) = x^{rh-r-h}\left(x^h - r\right),$$

we will show that $g(A) = 0$. To see this, note that since $r \geq 3$ and $h \geq 3$, by the division algorithm, we can find integers $q \geq 2$ and $0 \leq s \leq h-1$ such that

$$rh - r = qh + s\,.$$

From relation (10) we deduce that

$$A^{rh-r} = A^{qh+s} = rA^{(q-1)h+s} = rA^{qh+s-h} = rA^{rh-r-h}$$

which implies $g(A) = 0$ and so $M_h(r)$ is derogatory. ☑

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(Recibido en octubre de 2007. Aceptado en febrero de 2008)

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