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On the structure of the ultradistributions of Beurling type

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Abstract. Let Ω be a nonempty open set of the k-dimensional euclidean space $\mathbb{R}^k$. In this paper, we give a structure theorem on the ultradistributions of Beurling type in Ω. Also, other structure results on certain ultradistributions are obtained, in terms of complex Borel measures in Ω.

Sobre la estructura de las ultradistribuciones de tipo Beurling

Resumen. Sea Ω un abierto no vacío del espacio euclídeo k-dimensional $\mathbb{R}^k$. En este artículo se obtiene un teorema de estructura de las ultradistribuciones de Beurling definidas en Ω. También se obtienen otros resultados de estructura de ciertas ultradistribuciones, en términos de medidas complejas definidas en Ω.

1 Introduction and notation

Throughout this paper all linear spaces are assumed to be defined over the field $\mathbb{C}$ of complex numbers. We write $\mathbb{N}$ for the set of positive integers and by $\mathbb{N}_0$ we mean the set of nonnegative integers.

If E is a locally convex space, E' will be its topological dual and $\langle \cdot\, , \cdot \rangle$ will denote the standard duality between E and E'.

Given a Banach space X, $B(X)$ denotes its closed unit ball and X^* is the Banach space conjugate of X.

Given a positive integer k, if $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_k)$ is a multiindex of order k, i.e., an element of $\mathbb{N}_0^k$, we put $|\alpha|$ for its length, that is, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_k$, and $\alpha! := \alpha_1!\ \alpha_2!\ \ldots \alpha_k!$.

Given a complex function f defined in the points $x = (x_1, x_2, \ldots, x_k)$ of an open subset O of the k-dimensional euclidean space $\mathbb{R}^k$, and being infinitely differentiable, we write

$$D^\alpha f(x) := \frac{\partial^{|\alpha|} f(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_k^{\alpha_k}}, \qquad x \in O, \quad \alpha \in \mathbb{N}_0^k.$$

We consider a sequence $M_0, M_1, \ldots, M_n, \ldots$ of positive numbers satisfying the following conditions:

1. $M_0 = 1$.
2. Logarithmic convexity:
$$M_n^2 \leq M_{n-1} M_{n+1}, \qquad n \in \mathbb{N}.$$
3. Non-quasi-analyticity:
$$\sum_{n=1}^{\infty} \frac{M_{n-1}}{M_n} < \infty.$$

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Let us take a nonempty open set Ω in $\mathbb{R}^k$. A complex function f, defined and infinitely differentiable in Ω, is said to be *ultradifferentiable of class* (M_n) whenever, given $h > 0$ and a compact subset K of Ω, there is $C > 0$ such that

$$|D^\alpha f(x)| \leq C\, h^{|\alpha|} M_{|\alpha|}, \qquad x \in K, \quad \alpha \in \mathbb{N}_0^k.$$

We put $\mathcal{E}^{(M_n)}(\Omega)$ to denote the linear space over $\mathbb{C}$ formed by all the ultradifferentiable functions of class (M_n) defined in Ω, with the ordinary topology, [1]. The topological dual of $\mathcal{E}^{(M_n)}(\Omega)$ will be represented by $\mathcal{E}^{(M_n)'}(\Omega)$. By $\mathcal{D}^{(M_n)}(\Omega)$ we denote the linear subspace of $\mathcal{E}^{(M_n)}(\Omega)$ formed by those functions which have compact support.

Given $h > 0$, by $\mathcal{E}_0^{(M_n),h}(\Omega)$ we represent the linear space over $\mathbb{C}$ of the complex functions f, defined and infinitely differentiable in Ω, such that they vanish at infinity and also does each of its derivatives of any order, i.e., given $\epsilon > 0$ and $\beta \in \mathbb{N}^k$, there is a compact subset K of Ω such that

$$|D^\beta f(x)| < \epsilon, \qquad x \in \Omega \setminus K,$$

satisfying also that there is $C > 0$, depending only on f, for which

$$|D^\alpha f(x)| \leq C\, h^{|\alpha|} M_{|\alpha|}, \qquad x \in \Omega, \quad \alpha \in \mathbb{N}_0^k.$$

We set

$$|f|_h := \sup_{\alpha \in \mathbb{N}_0^k} \sup_{x \in \Omega} \frac{|D^\alpha f(x)|}{h^{|\alpha|} M_{|\alpha|}}$$

and assume that $\mathcal{E}_0^{(M_n),h}(\Omega)$ is endowed with the norm $|\cdot|_h$. We then put

$$\mathcal{E}_0^{(M_n)}(\Omega) := \bigcap_{m=1}^{\infty} \mathcal{E}_0^{(M_n),1/m}(\Omega).$$

We consider $\mathcal{E}_0^{(M_n)}(\Omega)$ as the projective limit of the sequence $(\mathcal{E}_0^{(M_n),1/m}(\Omega))$ of Banach spaces. We assume the topological dual $\mathcal{E}_0^{(M_n)\prime}(\Omega)$ of $\mathcal{E}_0^{(M_n)}(\Omega)$ endowed with the strong topology.

We take now a fundamental sequence of compact subsets of Ω:

$$K_1 \subset K_2 \subset \cdots \subset K_m \subset \cdots$$

If K is an arbitrary compact subset of Ω, we use $\mathcal{D}^{(M_n)}(K)$ to denote the subspace of $\mathcal{E}^{(M_n)}(\Omega)$ formed by those functions which have their support in K. We then have that

$$\mathcal{D}^{(M_n)}(\Omega) = \bigcup_{m=1}^{\infty} \mathcal{D}^{(M_n)}(K_m).$$

We consider $\mathcal{D}^{(M_n)}(\Omega)$ as the inductive limit of the sequence $(\mathcal{D}^{(M_n)}(K_m))$ of Fréchet spaces. The elements of the topological dual $\mathcal{D}^{(M_n)\prime}(\Omega)$ of $\mathcal{D}^{(M_n)}(\Omega)$ are called *ultradistributions of Beurling type* in Ω. We assume that $\mathcal{D}^{(M_n)\prime}(\Omega)$ has its strong topology.

We write $C_0(\Omega)$ for the linear space over $\mathbb{C}$ of the complex functions f, defined and continuous in Ω, which vanish at infinity. We put

$$\|f\|_\infty := \sup_{x \in \Omega} |f(x)|,$$

and assume that $C_0(\Omega)$ has the norm $\|\cdot\|_\infty$.

By $\mathcal{K}(\Omega)$ we mean the linear space over $\mathbb{C}$ of the complex functions defined in Ω which are continuous and have compact support. If K is any compact subset of Ω, $\mathcal{K}(K)$ is the subspace of $C_0(\Omega)$ formed by the functions with support contained in K. We consider $\mathcal{K}(\Omega)$ as the inductive limit of the sequence $(\mathcal{K}(K_m))$

of Banach spaces. A Radon measure in Ω is an element of the topological dual $\mathcal{K}'(\Omega)$ of $\mathcal{K}(\Omega)$. Given a Radon measure u in Ω and a compact subset K of Ω, we put $\|u\|(K)$ for the norm of the restriction of u to the Banach space $\mathcal{K}(K)$.

In [1, p. 76], a structure theorem for ultradistributions of Beurling type in Ω is given such that we may state in the following way: a) *If S is an element of $\mathcal{D}^{(M_n)\prime}(\Omega)$ and G is an open subset of Ω which is relatively compact, for each $\alpha \in \mathbb{N}_0^k$, we may find an element v_α in the conjugate of the Banach space $\mathcal{K}(\overline{G})$, whose norm we represent by $\|v_\alpha\|$, such that, for some $h > 0$,*

$$\sup_{\alpha\in\mathbb{N}_0^k} h^{|\alpha|} M_{|\alpha|} \|v_\alpha\| < \infty$$

and

$$S_{|G} = \sum_{\alpha\in\mathbb{N}_0^k} D^\alpha v_\alpha.$$

In this paper, we shall prove the following structure theorem on the ultradistributions of Beurling type in Ω: b) *If S is an element of $\mathcal{D}^{(M_n)\prime}(\Omega)$, there is a family $(\, u_\alpha : \alpha \in \mathbb{N}_0^k \,)$ of Radon measures in Ω so that*

$$\langle \varphi, S\rangle = \sum_{\alpha\in\mathbb{N}_0^k} \langle D^\alpha \varphi, u_\alpha\rangle, \qquad \varphi \in \mathcal{D}^{(M_n)}(\Omega),$$

where the series converges absolute and uniformly in every bounded subset of $\mathcal{D}^{(M_n)}(\Omega)$. Besides, for a given compact subset K of Ω, there is $h > 0$ such that

$$\sup_{\alpha\in\mathbb{N}_0^k} h^{|\alpha|} M_{|\alpha|} \|u_\alpha\|(K) < \infty.$$

We shall also give structure theorems for some ultradistributions of Beurling type in terms of complex Borel measures in Ω.

2 Basic constructions

Let X be a Banach space. We use $\|\cdot\|$ to denote the norm of X and also for the norm of X^*. Let $\lambda_0, \lambda_1, \ldots, \lambda_n, \ldots$ be a sequence of positive numbers. Given $r \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^k$, for each $x \in X$, we write

$$|x|_{r,\alpha} := \frac{r^{|\alpha|}\|x\|}{\lambda_{|\alpha|}}.$$

We denote by $X_{r,\alpha}$ the linear space X provided with the norm $|\cdot|_{r,\alpha}$. The Banach space conjugate of $X_{r,\alpha}$ will be $X_{r,\alpha}^*$ and we still will use $|\cdot|_{r,\alpha}$ for its norm. Clearly, if u is in X^*, then

$$|u|_{r,\alpha} = \frac{\lambda_{|\alpha|}}{r^{|\alpha|}}\|u\|.$$

We put Z_r for the linear space over $\mathbb{C}$ formed by the families $(\, x_\alpha : \alpha \in \mathbb{N}_0^k \,)$ of elements of X, for which we shall briefly write (x_α), such that

$$\|(x_\alpha)\|_r := \sup_{\alpha\in\mathbb{N}_0^k} \frac{r^{|\alpha|}\|x_\alpha\|}{\lambda_{|\alpha|}} < \infty.$$

We assume Z_r provided with the norm $\|\cdot\|_r$. It follows that $Z_r \supset Z_{r+1}$ and that the canonical injection from Z_{r+1} into Z_r is continuous.

We write Z for the Fréchet space given by the projective limit of the sequence (Z_r) of Banach spaces. We assume that Z' is endowed with its strong topology.

If (x_α) is an element of Z and β is in $\mathbb{N}_0^k$, we set

$$x_\alpha^\beta := \begin{cases} x_\beta, & \text{if } \alpha = \beta, \\ 0, & \text{if } \alpha \neq \beta. \end{cases}$$

Clearly, (x_α^β) belongs to Z and, for each $r \in \mathbb{N}$,

$$\|(x_\alpha^\beta)\|_r \leq \|(x_\alpha)\|_r.$$

If Z^β represents the subspace of Z formed by those elements (x_α) are such that $x_\alpha = 0$ when $\alpha \neq \beta$, we have that Z^β is topologically isomorphic to X, and considering Z^β as a linear subspace of Z_r, then it is isometric to $X_{r,\beta}$.

For an arbitrary element $u \in Z'$ and $r \in \mathbb{N}$, we put

$$\|u\|_{(r)} := \sup\{\, |\langle (x_\alpha), u\rangle| : (x_\alpha) \in B(Z_r) \cap Z \,\}.$$

For each $u \in Z'$ and each $\beta \in \mathbb{N}_0^k$, we identify, in a natural manner, the restriction of u to Z^β with an element u_β of X^*.

Proposition 1 *If u belongs to Z', there is $r \in \mathbb{N}$ such that*

$$\sup_{\alpha \in \mathbb{N}_0^k} r^{-|\alpha|} \lambda\, |\alpha|\, \|u_\alpha\| \leq \|u\|_{(r)} < \infty, \tag{1}$$

and

$$\langle (x_\alpha), u\rangle = \sum_{\alpha \in \mathbb{N}_0^k} \langle x_\alpha, u_\alpha\rangle, \qquad (x_\alpha) \in Z, \tag{2}$$

where the series converges absolute and uniformly in every bounded subset of Z.

PROOF. We find a positive integer r such that $\|u\|_{(r)}$ is finite. We fix β in $\mathbb{N}_0^k$. We have that

$$\begin{aligned} \|(u)\|_{(r)} &= \sup\{\, |\langle (x_\alpha), u\rangle| : (x_\alpha) \in B(Z_r) \cap Z \,\} \\ &\geq \sup\{\, |\langle (x_\alpha^\beta), u\rangle| : (x_\alpha) \in B(Z_r) \cap Z \,\} \\ &= \sup\{\, |\langle x_\beta, u_\beta\rangle| : |x_\beta|_{r,\beta} \leq 1 \,\} \\ &= |u_\beta|_{r,\beta} = \frac{\lambda_{|\beta|}}{r^{|\beta|}} \|u_\beta\| \end{aligned}$$

and from here

$$\sup_{\alpha \in \mathbb{N}_0^k} r^{-|\alpha|} \lambda_{|\alpha|}\, \|u_\alpha\| \leq \|u\|_{(r)} < \infty.$$

We take now (x_α) in Z and we see that the family $(\,(x_\alpha^\beta) : \beta \in \mathbb{N}_0^k\,)$ is summable in Z, with sum (x_α). For this, let us choose $s, q \in \mathbb{N}$ and we have that

$$\begin{aligned} \|(x_\alpha) - \sum_{|\beta| \leq q} (x_\alpha^\beta)\|_s &= \sup_{|\alpha| > q} \frac{s^{|\alpha|} \|x_\alpha\|}{\lambda_{|\alpha|}} \\ &= \sup_{|\alpha| > q} \frac{(2s)^{|\alpha|} \|x_\alpha\|}{2^{|\alpha|} \lambda_{|\alpha|}} \\ &\leq \frac{1}{2^q} \sup_{\alpha \in \mathbb{N}_0^k} \frac{(2s)^{|\alpha|} \|x_\alpha\|}{\lambda_{|\alpha|}} \\ &= \frac{1}{2^q} \|(x_\alpha)\|_{2s}, \end{aligned}$$

from where the conclusion follows. From

$$(x_\alpha) = \sum_{\beta \in \mathbb{N}_0^k} (x_\alpha^\beta)$$

in Z, we deduce that

$$\langle (x_\alpha), u \rangle = \sum_{\beta \in \mathbb{N}_0^k} \langle (x_\alpha^\beta), u \rangle = \sum_{\beta \in \mathbb{N}_0^k} \langle x_\beta, u_\beta \rangle.$$

We now take a bounded subset B of Z. We find $b > 0$ such that $B \subset bB(Z_{2kr})$. We choose an arbitrary element (x_α) of B. We fix β in $\mathbb{N}_0^k$. Then

$$\begin{aligned}
|\langle x_\beta, u_\beta \rangle| \leq \|x_\beta\| \|u_\beta\| &= \frac{(2kr)^{|\beta|}\|x_\beta\|}{\lambda_{|\beta|}} \cdot \frac{\lambda_{|\beta|}\|u_\beta\|}{(2kr)^{|\beta|}} \\
&\leq \frac{1}{(2k)^{|\beta|}} \|(x_\alpha)\|_{2kr} \sup_{\alpha \in \mathbb{N}_0^k} \frac{\lambda_{|\alpha|}\|u_\alpha\|}{r^{|\alpha|}} \\
&\leq \frac{1}{(2k)^{|\beta|}} \|(x_\alpha)\|_{2kr} \|u\|_{(r)} \\
&\leq \frac{1}{(2k)^{|\beta|}} b\|u\|_{(r)},
\end{aligned}$$

and since

$$\sum_{\beta \in \mathbb{N}_0^k} \frac{1}{(2k)^{|\beta|}} = 2,$$

the result follows. ■

Note In the former theorem, if we take an arbitrary bounded subset M of Z', we may choose r in such a way that (1) is satisfied for each element u of M. Then, (2) converges absolute and uniformly when (x_α) runs in any bounded subset of Z, and u varies in M.

Proposition 2 *Let $(\, v_\alpha : \alpha \in \mathbb{N}_0^k \,)$ be a family of elements of X^* such that there is $h > 0$ with*

$$\sup_{\alpha \in \mathbb{N}_0^k} h^{|\alpha|} \lambda_{|\alpha|} \|v_\alpha\| < \infty.$$

Then, there is a unique element u in Z' such that $u_\alpha = v_\alpha$, $\alpha \in \mathbb{N}_0^k$.

PROOF. We find r in $\mathbb{N}$ such that $1/r < h$. Let β be fixed in $\mathbb{N}_0^k$. We take (x_α) in Z. Then

$$\begin{aligned}
|\langle x_\beta, v_\beta \rangle| \leq \|x_\beta\| \|v_\beta\| &= \frac{(2kr)^{|\beta|}\|x_\beta\|}{\lambda_{|\beta|}} \cdot \frac{\lambda_{|\beta|}\|v_\beta\|}{(2kr)^{|\beta|}} \\
&\leq \frac{1}{(2k)^{|\beta|}} \|(x_\alpha)\|_{2kr} \sup_{\alpha \in \mathbb{N}_0^k} \frac{\lambda_{|\alpha|}\|v_\alpha\|}{r^{|\alpha|}} \\
&\leq \frac{1}{(2k)^{|\beta|}} \|(x_\alpha)\|_{2kr} \sup_{\alpha \in \mathbb{N}_0^k} h^{|\alpha|} \lambda_{|\alpha|} \|v_\alpha\|
\end{aligned}$$

from we where we have that the complex function u defined in Z such that

$$u((x_\alpha)) = \sum_{\alpha \in \mathbb{N}_0^k} \langle x_\alpha, v_\alpha \rangle, \qquad (x_\alpha) \in Z,$$

which is clearly linear, is also continuous. After the former proposition we have that

$$\langle (x_\alpha), u\rangle = \sum_{\alpha\in\mathbb{N}_0^k} \langle x_\alpha, u_\alpha\rangle, \qquad (x_\alpha)\in Z.$$

We fix γ in $\mathbb{N}_0^k$. We take an arbitrary element $x_\gamma \in X$. If (x_α^γ) is the element of Z^γ such that $x_\gamma^\gamma = x_\gamma$, it follows that

$$\langle (x_\gamma), u_\gamma\rangle = \langle (x_\alpha^\gamma), u\rangle = \sum_{\alpha\in\mathbb{N}_0^k} \langle x_\alpha^\gamma, v_\alpha\rangle = \langle x_\gamma, v_\gamma\rangle,$$

and so $u_\gamma = v_\gamma$, $\gamma\in\mathbb{N}_0^k$. The uniqueness of u obtains after the linear span of $\bigcup\{\, Z^\beta : \beta\in\mathbb{N}_0^k \,\}$ being dense in Z. ■

3 On the structure of certain ultradistributions

In this section, we substitute the sequence λ_n, $n\in\mathbb{N}_0$, of the previous section by M_n, $n\in\mathbb{N}_0^k$. We take $C_0(\Omega)$ instead of X. Hence, each element of Z_r is a family $(\, f_\alpha : \alpha\in\mathbb{N}_0^k \,)$ of elements of $C_0(\Omega)$, and

$$\|(f_\alpha)\|_r = \sup_{\alpha\in\mathbb{N}_0^k}\sup_{x\in\Omega}\frac{r^{|\alpha|}|f_\alpha(x)|}{M_\alpha} < \infty.$$

We put V_r to denote the subspace of Z_r formed by the families $(\, D^\alpha f : \alpha\in\mathbb{N}_0^k \,)$ such that

$$f\in\mathcal{E}_0^{(M_n),1/r}(\Omega).$$

Let

$$\Phi_r\colon \mathcal{E}_0^{(M_n),1/r}(\Omega)\longrightarrow V_r$$

such that

$$\Phi_r(f) = (D^\alpha f), \qquad f\in\mathcal{E}_0^{(M_n),1/r}(\Omega).$$

Then Φ_r is an onto linear isometry. We set $V := \bigcap\{\, V_r : r\in\mathbb{N} \,\}$, considered as a subspace of Z. Let

$$\Phi\colon \mathcal{E}_0^{(M_n)}(\Omega)\longrightarrow V$$

such that

$$\Phi(f) = (D^\alpha f), \qquad f\in\mathcal{E}_0^{(M_n)}(\Omega).$$

Clearly, Φ is a topological isomorphism from $\mathcal{E}_0^{(M_n)}$ onto V.

Theorem 1 *If $(\, \mu_\alpha : \alpha\in\mathbb{N}_0^k \,)$ is a family of complex Borel measures in Ω such that there is $h>0$ with*

$$\sup_{\alpha\in\mathbb{N}_0^k} h^{|\alpha|} M_{|\alpha|}|\mu_\alpha|(\Omega) < \infty,$$

then there is an element S of $\mathcal{E}_0^{(M_n)\prime}(\Omega)$ such that

$$\langle\varphi, S\rangle = \sum_{\alpha\in\mathbb{N}_0^k}\int_\Omega D^\alpha\varphi\,\mathrm{d}\mu_\alpha, \qquad \varphi\in\mathcal{E}_0^{(M_n)}(\Omega),$$

where this series converges absolute and uniformly in every bounded subset of $\mathcal{E}_0^{(M_n)}(\Omega)$.

PROOF. For each $\alpha \in \mathbb{N}_0^k$, we consider μ_α as a linear functional on $C_0(\Omega)$ by means of the usual duality

$$\langle \varphi, \mu_\alpha \rangle = \int_\Omega \varphi \, \mathrm{d}\mu_\alpha, \qquad \varphi \in C_0(\Omega).$$

Thus, the norm of μ_α is $|\mu_\alpha|(\Omega)$. We apply propositions 1 and 2 to obtain an element u in Z' such that

$$\langle (f_\alpha), u \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \langle f_\alpha, \mu_\alpha \rangle, \qquad (f_\alpha) \in Z,$$

where this series converges absolute and uniformly in every bounded subset of Z. We restrict u to V and keep denoting by u this restriction. Then

$$\langle (D^\alpha \varphi), u \rangle \;=\; \sum_{\alpha \in \mathbb{N}_0^k} \langle D^\alpha \varphi, \mu_\alpha \rangle \;=\; \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \, d\mu_\alpha, \qquad \varphi \in \mathcal{E}_0^{(M_n)}(\Omega).$$

If

$${}^t\Phi \;:\; V' \longrightarrow \; \mathcal{E}_0^{(M_n)}(\Omega)$$

is the transpose mapping of Φ, we put $S := {}^t\Phi(u)$. Then, for each $\varphi \in \mathcal{E}_0^{(M_n)}(\Omega)$, we have

$$\langle (D^\alpha \varphi), u \rangle = \langle \Phi(\varphi), u \rangle = \langle \varphi, {}^t\Phi(u) \rangle = \langle \varphi, S \rangle,$$

and so

$$\langle \varphi, S \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \, \mathrm{d}\mu_\alpha, \qquad \varphi \in \mathcal{E}_0^{(M_n)}(\Omega).$$

The absolute and uniform convergence of this series in every bounded subset of $\mathcal{E}_0^{(M_n)}(\Omega)$ is now immediate.

Theorem 2 *If S is an element of $\mathcal{E}_0^{(M_n)\prime}(\Omega)$, there are $h > 0$ and a family $(\, \mu_\alpha : \alpha \in \mathbb{N}_0^k \,)$ of complex Borel measures in Ω such that*

$$\sup_{\alpha \in \mathbb{N}_0^k} h^{|\alpha|} M_{|\alpha|} |\mu_\alpha|(\Omega) < \infty$$

and

$$\langle \varphi, S \rangle \;=\; \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \, d\mu_\alpha, \qquad \varphi \in \mathcal{E}_0^{(M_n)}(\Omega),$$

where this series converges absolute and uniformly in every bounded subset of $\mathcal{E}_0^{(M_n)}(\Omega)$.

PROOF. If we use the symbol ψ to denote the mapping Φ considered from $\mathcal{E}_0^{(M_n)}(\Omega)$ into Z, then

$${}^t\psi \colon Z' \longrightarrow \mathcal{E}_0^{(M_n)\prime}(\Omega)$$

is onto. We take u in Z' such that ${}^t\psi(u) = S$. Applying Proposition 1 we obtain $r \in \mathbb{N}$ and the family $(\, u_\alpha : \alpha \in \mathbb{N}_0^k \,)$ of elements of $C_0(\Omega)^*$ with the properties there mentioned. Making use of Riesz's representation theorem, [2, p. 130], we obtain, for each $\alpha \in \mathbb{N}_0^k$, a complex Borel measure μ_α in Ω such that

$$\langle \varphi, u_\alpha \rangle = \int_\Omega \varphi \, \mathrm{d}\mu_\alpha, \qquad \varphi \in C_0(\Omega), \quad \|u_\alpha\| = |\mu_\alpha|(\Omega).$$

Then, if $h := r^{-1}$, we have that

$$\sup_{\alpha \in \mathbb{N}_0^k} h^{|\alpha|} M_{|\alpha|} |\mu_\alpha|(\Omega) < \infty \tag{3}$$

and

$$\langle (D^\alpha \varphi), u \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \, \mathrm{d}\mu_\alpha, \qquad \varphi \in \mathcal{E}_0^{(M_n)}(\Omega),$$

where this series converges absolute and uniformly in every bounded subset of $\mathcal{E}_0^{(M_n)}(\Omega)$. Finally,

$$\langle (D^\alpha \varphi), u \rangle = \langle \psi(\varphi), u \rangle = \langle \varphi, {}^t\psi(u) \rangle = \langle \varphi, S \rangle,$$

and the conclusion now follows. ■

Theorem 3 *If M is a bounded subset of $\mathcal{E}_0^{(M_n)\prime}(\Omega)$, there are $h > 0$ and, for each $S \in M$, a family $(\mu_{\alpha,S} : \alpha \in \mathbb{N}_0^k)$ of complex Borel measures in Ω, such that*

$$\sup_{\alpha \in \mathbb{N}_0^k, S \in M} h^{|\alpha|} M_{|\alpha|} |\mu_{\alpha,S}|(\Omega) < \infty,$$

and each S of M can be represented as

$$\langle \varphi, S \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \, \mathrm{d}\mu_{\alpha,S}, \qquad \varphi \in \mathcal{E}_0^{(M_n)}(\Omega),$$

where the series converges absolute and uniformly when S varies in M and φ belongs to any given bounded subset of $\mathcal{E}_0^{(M_n)}(\Omega)$.

PROOF. Let ψ be the mapping introduced in the proof of Theorem 2. Since ψ is a topological isomorphism from $\mathcal{E}_0^{(M_n)}(\Omega)$ into Z, there is a bounded subset P of Z' such that ${}^t\psi(P) = M$ and the restriction of ${}^t\psi$ to P is one-to-one. For each $S \in M$, we put $u(S)$ for the element of P such that ${}^t\psi(u(S)) = S$. Applying Proposition 1 and the Note, we find $r \in \mathbb{N}$ such that, if $u_{\alpha,S} := (u(S))_\alpha$, $S \in M$, $\alpha \in \mathbb{N}_0^k$, then

$$\sup_{\alpha \in \mathbb{N}_0^k, S \in M} r^{-|\alpha|} M_{|\alpha|} \|u_{\alpha,S}\| < \infty$$

and

$$\langle (f_\alpha), u(S) \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \langle f_\alpha, u_{\alpha,S} \rangle, \qquad (f_\alpha) \in Z,$$

where this series converges absolute and uniformly when S varies in M and (f_α) runs in any given bounded subset of Z.

Proceeding as in the proof of the previous theorem, we find complex Borel measures $\mu_{\alpha,S}$, $\alpha \in \mathbb{N}_0^k$, $S \in M$, in Ω, so that

$$\langle f_\alpha, u_{\alpha,S} \rangle = \int_\Omega f_\alpha \, \mathrm{d}\mu_{\alpha,S}, \qquad \|u_{\alpha,S}\| = |\mu_{\alpha,S}|(\Omega).$$

If we set now $h := r^{-1}$, it follows that

$$\sup_{\alpha \in \mathbb{N}_0^k, S \in M} h^{|\alpha|} M_{|\alpha|} |\mu_{\alpha,S}|(\Omega) < \infty,$$

and, for each $S \in M$,

$$\langle (D^\alpha \varphi), u(S) \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \, \mathrm{d}\mu_{\alpha,S}, \qquad \varphi \in \mathcal{E}_0^{(M_n)}(\Omega),$$

where the series converges absolute and uniformly when S varies in M and φ runs in any given bounded subset of $\mathcal{E}_0^{(M_n)}(\Omega)$. Finally,

$$\langle (D^\alpha \varphi), u(S) \rangle = \langle \psi(\varphi), u(S) \rangle = \langle \varphi, {}^t \psi(u(S)) \rangle = \langle \varphi, S \rangle$$

and the conclusion follows. ■

Theorem 4 *Let M be a subset of $\mathcal{E}_0^{(M_n)\prime}(\Omega)$. If there exist $h > 0$ and, for each S in M, a family $(\,\mu_{\alpha,S} : \alpha \in \mathbb{N}_0^k\,)$ of complex Borel measures in Ω such that*

$$\sup_{\alpha\in\mathbb{N}_0^k, S\in M} h^{|\alpha|} M_{|\alpha|} |\mu_{\alpha,S}|(\Omega) < \infty,$$

and, for each $S \in M$,

$$\langle \varphi, S\rangle = \sum_{\alpha\in\mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \,\mathrm{d}\mu_{\alpha,S}, \qquad \varphi \in \mathcal{E}_0^{(M_n)}(\Omega),$$

then M is bounded.

PROOF. We fix φ in $\mathcal{E}_0^{(M_n)}(\Omega)$ and find a positive integer r such that $r^{-1} < h$. For each S in M we have

$$\begin{aligned}
|\langle \varphi, S\rangle| &\le \sum_{\alpha\in\mathbb{N}_0^k} \int_\Omega |D^\alpha\varphi| \,\mathrm{d}|\mu_{\alpha,S}| \\
&\le \sum_{\alpha\in\mathbb{N}_0^k} (2kr)^{-|\alpha|} M_{|\alpha|} \int_\Omega \frac{(2kr)^{|\alpha|}|D^\alpha\varphi|}{M_{|\alpha|}} \,\mathrm{d}|\mu_{\alpha,S}| \\
&\le \sum_{\alpha\in\mathbb{N}_0^k} r^{-|\alpha|} M_{|\alpha|} |\mu_{\alpha,S}|(\Omega) \frac{1}{(2k)^{|\alpha|}} |\varphi|_{\frac{1}{2kr}} \\
&\le |\varphi|_{\frac{1}{2kr}} \sup_{\beta\in\mathbb{N}_0^k} r^{-|\beta|} M_{|\beta|} \mu_{\alpha,S}|(\Omega) \sum_{\alpha\in\mathbb{N}_0^k} \frac{1}{(2k)^{|\alpha|}} \\
&= 2|\varphi|_{\frac{1}{2kr}} \sup_{\alpha\in\mathbb{N}_0^k} h^{|\alpha|} M_{|\alpha|} |\mu_{\alpha,S}|(\Omega)
\end{aligned}$$

and hence

$$\sup_{S\in M} |\langle \varphi, S\rangle| \le 2|\varphi|_{\frac{1}{2kr}} \sup_{\alpha\in\mathbb{N}_0^k} h^{|\alpha|} M_{|\alpha|} |\mu_{\alpha,S}|(\Omega),$$

from where we deduce that M is bounded in $\mathcal{E}_0^{(M_n)\prime}(\Omega)$ for the weak topology, and the result follows. ∎

In the following, in order to obtain Theorem 5, we shall give the details of a previous construction. Let S be an element of $\mathcal{E}_0^{(M_n)\prime}(\Omega)$ with support F, where by support of S we mean the support of the restriction of S to $\mathcal{D}^{(M_n)}(\Omega)$. Let A be an open subset of Ω such that $F \subset A$ and the distance from F to $\mathbb{R}^k \setminus A$ is $\delta > 0$. We find an open cover of $\mathbb{R}^k$ by means of open balls B_n, $n \in \mathbb{N}$, with radius $\delta/8$ such that, if C_n is the open ball of $\mathbb{R}^k$ with the center as B_n and radius $\delta/4$, $\{\,C_n : n \in \mathbb{N}\,\}$ is a locally finite covering of $\mathbb{R}^k$. For $x \in \mathbb{R}^k$, we write $d(x, F)$ to denote the distance from x to F and put

$$G := \{\, x \in \mathbb{R}^k : d(x, F) \le \delta/4 \,\}.$$

Let $\{\, f_n : n \in \mathbb{N}\,\}$ be a partition of unity formed by continuous functions, subordinated to the cover $\{\, B_n : n \in \mathbb{N}\,\}$. Let η be an element of $\mathcal{E}^{(M_n)}(\mathbb{R}^k)$, which takes non-negative values and whose support is contained in the closed ball B of $\mathbb{R}^k$, with center the origin and with radius $\delta/8$, such that

$$\int_{\mathbb{R}^k} \eta(x)\,\mathrm{d}x = 1.$$

We denote by $m(B)$ the Lebesgue measure of B and write

$$g_n(y) := (f_n * \eta)(y) = \int_{\mathbb{R}^k} f_n(x)\eta(y - x)\,\mathrm{d}x, \qquad y \in \mathbb{R}^k.$$

It follows that $\{ g_n : n \in \mathbb{N} \}$ is a partition of unity in $\mathbb{R}^k$, subordinated to the open cover $\{ C_n : n \in \mathbb{N} \}$ of $\mathbb{R}^k$, formed by elements of $\mathcal{D}^{(M_n)}(\mathbb{R}^k)$. We set

$$g := \sum \{ g_n : C_n \cap G \neq \emptyset \}.$$

It follows that g has value one in every point of a neighborhood of G and its support is contained in A. We now apply Theorem 2 to S and obtain $h > 0$ and the family $(\mu_\alpha : \alpha \in \mathbb{N}_0^k)$ of complex Borel measures in Ω with the properties there mentioned. We take $0 < s < \frac{h}{4k}$ and put

$$|||\eta||| := \sup_{\alpha \in \mathbb{N}_0^k} \sup_{x \in \mathbb{R}^k} \frac{|D^\alpha \eta(x)|}{s^{|\alpha|} M_{|\alpha|}}.$$

Then

$$|D^\alpha \eta(x)| \leq s^{|\alpha|} M_{|\alpha|} |||\eta|||, \qquad x \in \mathbb{R}^k, \quad \alpha \in \mathbb{N}_0^k,$$

and

$$|D^\alpha g(y)| \leq \int_{\mathbb{R}^k} |D^\alpha \eta(y - x)| \, \mathrm{d}x = \int_{\mathbb{R}}^k |D^\alpha \eta(x)| \, \mathrm{d}x \leq s^{|\alpha|} M_{|\alpha|} |||\eta||| \, m(B), \qquad y \in \mathbb{R}^k.$$

We take φ in $\mathcal{E}_0^{(M_n)}(\Omega)$. We then have that $g\varphi$ belongs to $\mathcal{E}_0^{(M_n)}(\Omega)$ and so

$$\langle \varphi, S \rangle = \langle g\varphi, S \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega D^\alpha (g\varphi) \, \mathrm{d}\mu_\alpha = \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega \left(\sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha - \beta)!} D^\beta g \, D^{\alpha - \beta} \varphi \right) \mathrm{d}\mu_\alpha.$$

For each $x \in \Omega$, we have

$$\begin{aligned}
&\sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha - \beta)!} |D^\beta g(x) D^{\alpha - \beta} \varphi(x)| \\
&\qquad \leq \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha - \beta)!} s^{|\beta|} M_{|\beta|} |||\eta||| \, m(B) |\varphi|_s s^{|\alpha - \beta|} M_{|\alpha - \beta|} \\
&\qquad = s^{|\alpha|} |||\eta||| \, m(B) |\varphi|_s \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha - \beta)!} M_{|\beta|} M_{|\alpha - \beta|} \\
&\qquad \leq s^{|\alpha|} |||\eta||| \, m(B) |\varphi|_s 2^{|\alpha|} M_{|\alpha|} \\
&\qquad \leq \frac{h^{|\alpha|}}{(2k)^{|\alpha|}} |||\eta||| \, m(B) |\varphi|_s M_{|\alpha|}
\end{aligned}$$

and thus

$$\begin{aligned}
&\sum_{\alpha \in \mathbb{N}_0^k} \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha - \beta)!} \int_\Omega |D^\beta g \, D^{\alpha - \beta} \varphi| \, \mathrm{d}\mu_\alpha \\
&\qquad \leq |||\eta||| \, m(B) |\varphi|_s \sum_{\alpha \in \mathbb{N}_0^k} \frac{1}{(2k)^{|\alpha|}} h^{|\alpha|} M_{|\alpha|} |\mu_\alpha|(\Omega) \\
&\qquad \leq |||\eta||| \, m(B) |\varphi|_s \sum_{\alpha \in \mathbb{N}_0^k} \frac{1}{(2k)^{|\alpha|}} \sup_{\gamma \in \mathbb{N}_0^k} h^{|\gamma|} M_{|\gamma|} |\mu_\gamma|(\Omega) \\
&\qquad = 2 |||\eta||| \, m(B) |\varphi|_s \sup_{\alpha \in \mathbb{N}_0^k} h^{|\alpha|} M_{|\alpha|} |\mu_\alpha|(\Omega),
\end{aligned}$$

from where we deduce that we can write, putting $\gamma := \alpha - \beta$,

$$\sum_{\alpha\in\mathbb{N}_0^k}\sum_{\beta\le\alpha}\frac{\alpha!}{\beta!(\alpha-\beta)!}\int_\Omega D^\beta g\, D^{\alpha-\beta}\varphi\,\mathrm{d}\mu_\alpha = \sum_{\gamma\in\mathbb{N}_0^k}\sum_{\beta\in\mathbb{N}_0^k}\frac{(\beta+\gamma)!}{\beta!\gamma!}\int_\Omega D^\beta g\, D^\gamma\varphi\,\mathrm{d}\mu_{\beta+\gamma}.$$

Let us now take an arbitrary element θ of $C_0(\Omega)$. Then

$$\begin{aligned}
\Big|\sum_{\beta\in\mathbb{N}_0^k}&\frac{(\beta+\gamma)!}{\beta!\gamma!}\int_\Omega D^\beta g\,\theta\,\mathrm{d}\mu_{\beta+\gamma}\Big| \\
&\le \sum_{\beta\in\mathbb{N}_0^k}\frac{(\beta+\gamma)!}{\beta!\gamma!}\int_\Omega |D^\beta g\,\theta||\mathrm{d}\mu_{\beta+\gamma}| \\
&\le \sum_{\beta\in\mathbb{N}_0^k}\frac{(\beta+\gamma)!}{\beta!\gamma!}\int_\Omega s^{|\beta|}|||\eta|||\, m(B)M_{|\beta|}\|\theta\|_\infty\,\mathrm{d}|\mu_{\beta+\gamma}| \\
&= |||\eta|||\, m(B)\|\theta\|_\infty\sum_{\beta\in\mathbb{N}_0^k}\frac{(\beta+\gamma)}{\beta!\gamma!}s^{|\beta|}M_{|\beta|}|\mu_{\beta+\gamma}|(\Omega) \\
&\le |||\eta|||\, m(B)\|\theta\|_\infty\sum_{\beta\in\mathbb{N}_0^k}2^{|\beta+\gamma|}s^{|\beta|}M_{|\beta|}|\mu_{\beta+\gamma}|(\Omega).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\sum_{\beta\in\mathbb{N}_0^k}2^{|\beta+\gamma|}s^{|\beta|}M_{|\beta|}|\mu_{\beta+\gamma}|(\Omega) &= \frac{1}{M_{|\gamma|}s^{|\gamma|}}\sum_{\beta\in\mathbb{N}_0^k}(2s)^{|\beta+\gamma|}M_{|\beta|}M_{|\gamma|}|\mu_{\beta+\gamma}|(\Omega) \\
&\le \frac{1}{M_{|\gamma|}s^{|\gamma|}}\sum_{\beta\in\mathbb{N}_0^k}(2s)^{|\beta+\gamma|}M_{|\beta+\gamma|}|\mu_{\beta+\gamma}|(\Omega) \\
&\le \frac{1}{M_{|\gamma|}s^{|\gamma|}}\sum_{\beta\in\mathbb{N}_0^k}\frac{1}{(2k)^{|\beta+\gamma|}}(4ks)^{|\beta+\gamma|}M_{|\beta+\gamma|}|\mu_{\beta+\gamma}|(\Omega) \\
&\le \frac{1}{M_{|\gamma|}s^{|\gamma|}}\sum_{\beta\in\mathbb{N}_0^k}\frac{1}{(2k)^{|\beta|}}\sup_{\alpha\in\mathbb{N}_0^k}h^{|\alpha|}M_{|\alpha|}|\mu_\alpha|(\Omega) \\
&= \frac{2}{M_{|\gamma|}s^{|\gamma|}}\sup_{\alpha\in\mathbb{N}_0^k}h^{|\alpha|}M_{|\alpha|}|\mu_\alpha|(\Omega).
\end{aligned}$$

Consequently, there is a constant $C_\gamma > 0$ such that

$$\left|\sum_{\beta\in\mathbb{N}_0^k}\frac{(\beta+\gamma)!}{\beta!\gamma!}\int_\Omega D^\beta g\,\theta\,\mathrm{d}\mu_{\beta+\gamma}\right| \le C_\gamma\|\theta\|_\infty \tag{4}$$

Setting

$$v_\gamma(\theta) := \sum_{\beta\in\mathbb{N}_0^k}\frac{(\beta+\gamma)!}{\beta!\gamma!}\int_\Omega D^\beta g\,\theta\,\mathrm{d}\mu_{\beta+\gamma}, \qquad \theta\in C_0(\Omega),$$

we have that v_γ is a complex function, clearly linear, which, after (4), is in $C_0(\Omega)^*$. Making use of Riesz's representation theorem, we obtain a complex Borel measure ν_γ in Ω such that

$$v_\gamma(\zeta) = \int_\Omega \zeta\,\mathrm{d}\nu_\gamma, \qquad \zeta\in C_0(\Omega).$$

Then, for each $\varphi \in \mathcal{E}_0^{(M_n)}(\Omega)$, we have that

$$\left| \sum_{\beta \in \mathbb{N}_0^k} \frac{(\beta+\gamma)!}{\beta!\gamma!} \int_\Omega D^\beta g \, D^\gamma \varphi \, \mathrm{d}\mu_{\beta+\gamma} \right| = \int_\Omega D^\gamma \varphi \, \mathrm{d}\nu_\gamma.$$

Clearly, the Borel measures ν_γ, $\gamma \in \mathbb{N}_0^k$, have their supports in A.

We take $0 < 2kl < s$. We choose $\gamma \in \mathbb{N}_0^k$ and $\zeta \in C_0(\Omega)$ such that $\|\zeta\|_\infty < 2$ and

$$v_\gamma(\zeta) = |\nu_\gamma|(\Omega).$$

Then

$$\begin{aligned} l^{|\gamma|} M_{|\gamma|} |\nu_\gamma|(\Omega) &= l^{|\gamma|} M_{|\gamma|} v_\gamma(\zeta) \\ &\le l^{|\gamma|} M_{|\gamma|} \left| \sum_{\beta \in \mathbb{N}_0^k} \frac{(\beta+\gamma)!}{\beta!\gamma!} \int_\Omega D^\beta g \, \zeta \, \mathrm{d}\mu_{\beta+\gamma} \right| \\ &\le l^{|\gamma|} \, M_{|\gamma|} \, m(B) \, |||\eta||| \, \|\zeta\|_\infty \sum_{\beta \in \mathbb{N}_0^k} 2^{|\beta+\gamma|} s^{|\beta|} M_{|\beta|} |\mu_{\beta+\gamma}|(\Omega) \\ &\le l^{|\gamma|} \, M_{|\gamma|} \, m(B) |||\eta||| \, \|\zeta\|_\infty \frac{2}{M_{|\gamma|} s^{|\gamma|}} \sup_{\alpha \in \mathbb{N}_0^k} h^{|\alpha|} M_{|\alpha|} |\mu_\alpha|(\Omega) \end{aligned}$$

and therefore there is a constant $C > 0$ for which

$$l^{|\gamma|} M_{|\gamma|} |\nu_\gamma|(\Omega) \le C (l/s)^{|\gamma|} \le C \frac{1}{(2k)^{|\gamma|}} \le C \sum_{\alpha \in \mathbb{N}_0^k} \frac{1}{(2k)^{|\alpha|}} = 2\, C.$$

Thus

$$\sup_{\alpha \in \mathbb{N}_0^k} l^{|\gamma|} M_{|\gamma|} |\nu_\gamma|(\Omega) < \infty.$$

Summarizing, we have obtained $l > 0$ and a family ν_α, $\alpha \in \mathbb{N}_0^k$, of complex Borel measures in Ω, with supports contained in A, such that

$$\sup_{\alpha \in \mathbb{N}_0^k} l^{|\alpha|} M_{|\alpha|} |\nu_\alpha|(\Omega) < \infty$$

We apply Theorem 1 and so obtain an element T in $\mathcal{E}_0^{(M_n)\prime}(\Omega)$ such that

$$\langle \varphi, T \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \, \mathrm{d}\nu_\alpha, \qquad \varphi \in \mathcal{E}_0^{(M_n)}(\Omega),$$

where the series converges absolute and uniformly in every bounded subset of $\mathcal{E}_0^{(M_n)}(\Omega)$. It follows that, for each $\varphi \in \mathcal{E}_0^{(M_n)}(\Omega)$,

$$\langle \varphi, T \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \, \mathrm{d}\nu_\alpha = \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega D^\alpha (g\varphi) \, \mathrm{d}\mu_\alpha = \langle g\varphi, S \rangle.$$

Since, for each $\varphi \in \mathcal{D}^{(M_n)}(\Omega)$, we have that $\langle g\varphi, S \rangle = \langle \varphi, S \rangle$, we may then write the following

Theorem 5 *Let S be an element of $\mathcal{E}_0^{(M_n)\prime}(\Omega)$ whose support is F. Let A be an open subset of Ω containing F. If the distance from F to $\mathbb{R}^k \setminus A$ is positive, there are $h > 0$ and a family $(\, \mu_\alpha : \alpha \in \mathbb{N}_0^k \,)$ of complex Borel measures in Ω such that*

$$\sup_{\alpha \in \mathbb{N}_0^k} h^{|\alpha|} M_{|\alpha|} |\mu_\alpha|(\Omega) < \infty, \quad \operatorname{supp} \mu_\alpha \subset A, \qquad \alpha \in \mathbb{N}_0^k,$$

and

$$\langle \varphi, S\rangle = \sum_{\alpha\in\mathbb{N}_0^k} \int_\Omega D^\alpha\varphi \, \mathrm{d}\mu_\alpha, \qquad \varphi \in \mathcal{D}^{(M_n)}(\Omega),$$

where this series converges absolute and uniformly in every bounded subset of $\mathcal{D}^{(M_n)}(\Omega)$.

4 Structure of the ultradistributions of Beurling type

Theorem 6 *Let* $\{ u_\alpha : \alpha \in \mathbb{N}_0^k \}$ *be a family of Radon measures in* Ω*. If, given any compact subset* K *of* Ω*, there is* $h > 0$ *such that*

$$\sup_{\alpha\in\mathbb{N}_0^k} h^{|\alpha|} M_{|\alpha|} \|u_\alpha\|(K) < \infty,$$

then there is an element S *in* $\mathcal{D}^{(M_n)\prime}(\Omega)$ *so that*

$$\langle \varphi, S\rangle = \sum_{\alpha\in\mathbb{N}_0^k} \langle D^\alpha\varphi, u_\alpha\rangle, \qquad \varphi \in \mathcal{D}^{(M_n)}(\Omega),$$

where this series converges absolute and uniformly in every bounded subset of $\mathcal{D}^{(M_n)}(\Omega)$.

PROOF. For each $m \in \mathbb{N}$, we identify in a natural way $\mathcal{K}(K_m)$ with $C_0(\overset{\circ}{K}_m)$. We put u_α^m for the restriction of u_α to $\mathcal{K}(K_m)$. If μ_α^m is the complex Borel measure in $\overset{\circ}{K}_m$ for which

$$\langle f, u_\alpha^m\rangle = \int_{\overset{\circ}{K}_m} f \, \mathrm{d}\mu_\alpha^m, \qquad f \in C_0(\overset{\circ}{K}_m)$$

we have that

$$\|u_\alpha\|(K_m) = |\mu_\alpha^m|(\overset{\circ}{K}_m)$$

and there is $h_m > 0$ such that

$$\sup_{\alpha\in\mathbb{N}_0^k} h^{|\alpha|} M_{|\alpha|} \|u_\alpha\|(K_m) < \infty.$$

Thus, we apply Theorem 1 and so obtain an element S_m in $\mathcal{E}_0^{(M_n)\prime}(\overset{\circ}{K}_m)$ such that

$$\langle \varphi, S_m\rangle = \sum_{\alpha\in\mathbb{N}_0^k} \int_{\overset{\circ}{K}_m} D^\alpha\varphi \, \mathrm{d}\mu_\alpha^m, \qquad \varphi \in \mathcal{E}_0^{(M_n)}(\Omega),$$

where this series converges absolute and uniformly in every bounded subset of $\mathcal{E}_0^{(M_n)}(\overset{\circ}{K}_m)$.

Given an arbitrary element φ of $\mathcal{D}^{(M_n)}(\Omega)$, we find $m \in \mathbb{N}$ such that

$$\operatorname{supp} \varphi \subset \overset{\circ}{K}_m$$

and we put

$$\langle \varphi, S\rangle := \langle \varphi, S_m\rangle.$$

It is easy to see that S belongs to $\mathcal{D}^{(M_n)\prime}(\Omega)$ and also that it fulfills the requirements of the statement. ■

Theorem 7 *If S is an element of $\mathcal{D}^{(M_n)\prime}(\Omega)$, there is a family $(\, u_\alpha : \alpha \in \mathbb{N}_0^k \,)$ of Radon measures in Ω, such that, given any compact subset K of Ω, there is $h > 0$ with*

$$\sup h^{|\alpha|} M_{|\alpha|} \|u_\alpha\|(K) < \infty,$$

and

$$\langle \varphi, S \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \langle D^\alpha \varphi, u_\alpha \rangle, \qquad \varphi \in \mathcal{D}^{(M_n)}(\Omega),$$

where the series converges absolute and uniformly in every bounded subset of $\mathcal{D}^{(M_n)}(\Omega)$.

PROOF. Let $\{\, O_m : m \in \mathbb{N} \,\}$ be a locally finite open covering of Ω such that O_m is relatively compact in Ω, $m \in \mathbb{N}$. Let $\{\, g_m : m \in \mathbb{N} \,\}$ be a partition of unity of class (M_n) subordinated to the above open cover. It follows that $g_m S$ is an element with compact support F_m contained in O_m and thus it belongs to $\mathcal{E}_0^{(M_n)\prime}(\Omega)$. The distance from F_m to $\mathbb{R}^k \setminus O_m$ is positive, hence we may apply Theorem 5 to obtain $h_m > 0$ and a family $(\, \mu_\alpha^m : \alpha \in \mathbb{N}_0^k \,)$ of complex Borel measures in Ω so that

$$\sup_{\alpha \in \mathbb{N}_0^k} h_m^{|\alpha|} M_{|\alpha|} |\mu_\alpha^m|(\Omega) < \infty, \quad \operatorname{supp} \mu_\alpha^m \subset O_m, \qquad \alpha \in \mathbb{N}_0^k,$$

and

$$\langle \varphi, g_m S \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \, \mathrm{d}\mu_\alpha^m, \qquad \varphi \in \mathcal{E}_0^{(M_n)}(\Omega),$$

where this series converges absolute and uniformly in every bounded subset of $\mathcal{E}_0^{(M_n)}(\Omega)$.

Given an arbitrary element f of $\mathcal{K}(\Omega)$, there is a finite number of subindex m such that

$$O_m \cap \operatorname{supp} f \neq \emptyset.$$

Consequently, we may define, for each $\alpha \in \mathbb{N}_0^k$,

$$u_\alpha(f) := \sum_{m \in \mathbb{N}} \int_\Omega f \, \mathrm{d}\mu_\alpha^m.$$

It follows that u_α is a linear functional on $\mathcal{K}(\Omega)$. Given an arbitrary compact subset K of Ω, there is a positive integer m_0 such that $K \cap O_m = \emptyset$, $m > m_0$. Hence, if f has its support contained in K, we have that

$$|u_\alpha(f)| \leq \sum_{m \in \mathbb{N}} \int_\Omega |f| \, \mathrm{d}|\mu_\alpha^m| \leq \sum_{m=1}^{m_0} |\mu_\alpha^m|(\Omega) \cdot \|f\|_\infty,$$

from where we deduce that u_α is a Radon measure in Ω. Besides,

$$\|u_\alpha\|(K) \leq \sum_{m=1}^{m_0} |\mu_\alpha^m|(\Omega),$$

and, if

$$h := \inf\{\, h_m : m = 1, 2, \ldots, m_0 \,\},$$

it follows that

$$\sup_{\alpha \in \mathbb{N}_0^k} h^{|\alpha|} M_{|\alpha|} \|\mu_\alpha\|(K) \leq \sum_{m=1}^{m_0} \sup_{\alpha \in \mathbb{N}_0^k} h_m^{|\alpha|} M_{|\alpha|} |\mu_\alpha^m|(\Omega) < \infty.$$

We now take φ in $\mathcal{D}^{(M_n)}(\Omega)$ with support in K. Then

$$\begin{aligned}\langle \varphi, S\rangle &= \langle \varphi \sum_{m=1}^{\infty} g_m, S\rangle = \langle \varphi \sum_{m=1}^{m_0} g_m, S\rangle \\ &= \sum_{m=1}^{m_0} \langle \varphi, g_m S\rangle = \sum_{m=1}^{m_0} \sum_{\alpha\in\mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \,\mathrm{d}\mu_\alpha^m \\ &= \sum_{\alpha\in\mathbb{N}_0^k} \sum_{m=1}^{m_0} \int_\Omega D^\alpha \varphi \,\mathrm{d}\mu_\alpha^m = \sum_{\alpha\in\mathbb{N}_0^k} \langle D^\alpha \varphi, u_\alpha\rangle. \qquad (5)\end{aligned}$$

It is now simple to show that the series in (5) converges absolute and uniformly in every bounded subset of $\mathcal{D}^{(M_n)}(\Omega)$. ■

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