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A recursion formula for expected negative and positive powers of the central Wishart distribution

Heinz Neudecker

Amsterdam School of Economics

Abstract

We use Haff's fundamental identity to express the expectation of S^p in lower-order terms, where S follows the central Wishart distribution.

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1 The recursion formula (positive powers)

Let $S \sim W_m(\Omega, n)$. Hence S follows the Central Wishart distribution with scale matrix $\Omega > 0$ and n degrees of freedom. We use Haff's Fundamental Identity (FI):

$$\mathcal{E}\, F_1 \Omega^{-1} F_2 = 2\, \mathcal{E} F_1 \nabla F_2 + 2\, (\mathcal{E}\, F_2' \nabla F_1')' + (n - m - 1)\, \mathcal{E}\, F_1\, S^{-1}\, F_2,$$

with $F_1 = F_1(S)$ a differentiable matrix function of S and $n > m + 1$. Further ∇ is a matrix of differential operators with typical element

$$d_{ij} = \frac{1}{2}\,(1 + \delta_{ij})\,\frac{\partial}{\partial\, s_{ij}} \quad (i, j = 1, \ldots, m),$$

where δ_{ij} is a Kronecker delta: $\delta_{ii} = 1$ and $\delta_{ij} = 0$ $(i \neq j)$.

Address for correspondence: Oosterstraat, 13. 1741 GH Schagen. The Netherlands.
E-mail: H.Neudecker@uva.nl
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All matrices are square of dimension m. A very useful property is: when $dF = P'(dS)Q$ then $2\nabla F = PQ + (\operatorname{tr} P)Q$, where dF is the differential of F. For these properties see Neudecker (2001). Further $\mathcal{E}$ is the expectation operator.

We choose $F_1 = I_m$ and $F_2 = S^p, (p > 0)$. By Haff's FI we get:

$$\mathcal{E}\,\Omega^{-1} S^p = 2\,\mathcal{E}\,\nabla S^p + (n - m - 1)\,\mathcal{E}\,S^{p-1},$$

because $\nabla I = 0$. For definition and computation of $\nabla(\cdot)$ see also Neudecker (2000).

Clearly

$$d\,S^p = \sum_{k=0}^{p-1} S^k (dS) S^{p-k-1}$$

where $S^0 = I_m$.

Hence

$$2\,\nabla S^p = (m + p) S^{p-1} + \sum_{k=1}^{p-1} \left(\operatorname{tr} S^k\right) S^{p-k-1}.$$

Then

$$\Omega^{-1}\,\mathcal{E}\,S^p = (n + p - 1)\,\mathcal{E}\,S^{p-1} + \mathcal{E}\sum_{k=1}^{p-1} \left(\operatorname{tr} S^k\right) S^{p-k-1}$$

or equivalently

$$\mathcal{E}\,S^p = (n + p - 1)\Omega\,\mathcal{E}\,S^{p-1} + \Omega\,\mathcal{E}\sum_{k=1}^{p-1} \left(\operatorname{tr} S^k\right) S^{p-k-1}.$$

2 Discussion and application (positive powers)

The second recursion formula is not really suited for finding $\mathcal{E}\,S^p$, because this requires knowledge of $\mathcal{E}\,(\operatorname{tr} S^k)\,S^{p-k-1}$ for $1 \le k \le p - 1$. This expression has to be found by other methods. See Neudecker (2001). The formula is, however, suited for establishing Loewner orderings. Relevant is that $\Omega\,\mathcal{E}\,S^p = \mathcal{E}\,S^p\,\Omega > 0$ (positive definite). This follows from the first recursion formula, which yields the identity $\Omega^{-1}\,\mathcal{E}\,S^p = \mathcal{E}\,S^p\,\Omega^{-1}$ and further the positive definiteness of the two expressions. Pre- and postmultiplication of the identity by matrix Ω yields: $\Omega\,\mathcal{E}\,S^p = \mathcal{E}\,S^p\,\Omega$. As Ω and $\mathcal{E}\,S^p$ are two commuting positive definite matrices, they are simultaneously diagonizable by an orthogonal similarity transformation. Hence $T'\,\Omega\,T = \Lambda$ and $T'\,\mathcal{E}\,S^p\,T = M$, say, and consequently

$\Omega\,\mathcal{E}\,S^p = T\Lambda T' T M T' = T\Lambda M T' > 0$ (positive definite), as both Λ and M are diagonal positive definite. See e.g. Hadley (1972, ex. 7-14).

We further conclude that $\Omega^{-1}\,\mathcal{E}\,S^p > (n+p-1)\mathcal{E}\,S^{p-1}$ or equivalently $(\mathcal{E}\,S)^{-1}\,\mathcal{E}\,S^p > n^{-1}(n+p-1)\mathcal{E}\,S^{p-1}$ as $\mathcal{E}\,S = n\,\Omega$ $(p \geq 2)$. The first inequality follows from the second recursion formula. Repeated substitution finally yields the inequality

$$\mathcal{E}\,S^p > (\mathcal{E}\,S)^p$$

3 The recursion formula (negative powers)

With $F_1 = I_m$ and $F_2 = S^{-p}$, we get by Haff's FI

$$\Omega^{-1}\,\mathcal{E}\,S^{-p} = 2\,\mathcal{E}\,\nabla S^{-p} + (n-m-1)\,\mathcal{E}\,S^{-(p+1)} \tag{1}$$

For $p = 1, 2, \ldots, n-m-2$, we clearly have

$$dS^{-p} = -\sum_{k=1}^{p} S^{-k}(dS)S^{k-(p+1)}$$

Hence

$$\begin{aligned} 2\nabla\,S^{-p} &= \sum_{k=1}^{p} S^{-(p+1)} - \sum_{k=1}^{p}\left(\operatorname{tr} S^{-k}\right)S^{k-(p+1)} \\ &= -p\,S^{-(p+1)} - \sum_{k=1}^{p}\left(\operatorname{tr} S^{-k}\right)S^{k-(p+1)} \end{aligned}$$

Insertion in (1) leads to the recursion formula

$$\Omega^{-1}\,\mathcal{E}\,S^{-p} = (n-m-p-1)\mathcal{E}\,S^{-(p+1)} - \sum_{k=1}^{p}\mathcal{E}\left(\operatorname{tr} S^{-k}\right)S^{k-(p+1)} \tag{2}$$

or equivalently

$$\mathcal{E}\,S^{-p} = (n-m-p-1)\Omega\,\mathcal{E}\,S^{-(p+1)} - \Omega\sum_{k=1}^{p}\mathcal{E}\left(\operatorname{tr} S^{-k}\right)S^{k-(p+1)} \tag{3}$$

4 Discussion and application (negative powers)

Because of the symmetry of the RHS of (2) we have

$$\Omega^{-1}\,\mathcal{E}\,S^{-p} = (\mathcal{E}\,S^{-p})\,\Omega^{-1}. \tag{4}$$

We shall prove that $\Omega^{-1}\,\mathcal{E}\,S^{-p} > 0$ (positive definite).

Proof. From the commutativity property (4) we conclude that $\mathcal{E}\,S^{-p} = T\,\Lambda_p\,T'$ say and $\Omega^{-1} = T\,M\,T'$ where T is orthogonal and M and Λ_p are diagonal positive definite. Hence

$$\Omega^{-\frac{1}{2}} = T\,M^{\frac{1}{2}}\,T'$$

and

$$\begin{aligned}\Omega^{-\frac{1}{2}}\,(\mathcal{E}\,S^{-p})\,\Omega^{-\frac{1}{2}} &= T\,M^{\frac{1}{2}}\,T'\,T\,\Lambda_p\,T'\,T\,M^{\frac{1}{2}}\,T' \\ &= T\,M^{\frac{1}{2}}\,\Lambda_p\,M^{\frac{1}{2}}\,T' = T\,M\,\Lambda_p\,T' = T\,M\,T'\,T\,\Lambda_p\,T' \\ &= \Omega^{-1}\,\mathcal{E}\,S^{-p}\end{aligned}$$

Hence

$$\Omega^{-1}\,\mathcal{E}\,S^{-p} > 0 \quad \text{as} \quad \Omega^{-\frac{1}{2}}\,(\mathcal{E}\,S^{-p})\,\Omega^{-\frac{1}{2}} > 0 \quad \square$$

Because $\mathcal{E}\,S^{-(p+1)} > 0$ and $\sum_{k=1}^{p} \mathcal{E}\left(\operatorname{tr} S^{-k}\right) S^{k-(p+1)} > 0$ we conclude from (2) that

$$\Omega^{-1}\,\mathcal{E}\,S^{-p} < (n-m-p-1)\mathcal{E}\,S^{-(p+1)} \tag{5}$$

when $p < n - m - 1$. Or equivalently

$$\left(\mathcal{E}\,S^{-1}\right)\mathcal{E}\,S^{-p} < (n-m-1)^{-1}(n-m-p-1)\mathcal{E}\,S^{-(p+1)} < \mathcal{E}\,S^{-(p+1)} \tag{6}$$

as

$$\mathcal{E}\,S^{-1} = (n-m-1)^{-1}\,\Omega^{-1}$$

and $(n-m-1)^{-1}(n-m-p-1) < 1$.

The inequality (6) ultimately leads by successive substitution to the inequality

$$\mathcal{E}\,S^{-p} > \left(\mathcal{E}\,S^{-1}\right)^{p} \quad (p \geq 2) \tag{7}$$

As $\mathcal{E} S^{-1} = n(n - m - 1)^{-1} (\mathcal{E} S)^{-1} > (\mathcal{E} S)^{-1}$, we can get the inequality

$$\mathcal{E} S^{-p} > \left(\mathcal{E} S^{-1}\right)^{p} > (\mathcal{E} S)^{-p} \tag{8}$$

It is known that

$$\mathcal{E} S^{p} > (\mathcal{E} S)^{p} \qquad p \geq 2 \tag{9}$$

See the end of section 2.

Combining (8) and (9) we finally have the inequality

$$\mathcal{E} S^{-p} > \left(\mathcal{E} S^{-1}\right)^{p} > (\mathcal{E} S)^{-p} > (\mathcal{E} S^{p})^{-1} \tag{10}$$

5 References

Hadley, G. (1972). *Linear Algebra*. Addison-Wesley Publishing Company Inc., Reading.

Neudecker, H. (2000). «A note on the matrix Haffian». *Qüestiió*, 24, 419-424.

Neudecker, H. (2001). «Some applications of the matrix Haffian in connection with differentiable matrix functions of a central Wishart variate». *Qüestiió*, 25, 187-210.