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A unified treatment for some ring-shaped potentials as a generalized 4-D isotropic oscillator

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Abstract

A generalized integrable biparametric family of 4-D isotropic oscillators is proposed which allows to treat, in a unified way, Pöschl-Teller, Hartmann and other ring-shaped systems. This approach, based in the use of two canonical extensions, helps to simplify known studies of classical and quantum aspects of those systems.

Keywords: Four dimension isotropic oscillatorsRing-shaped systemsGeneralized Hartmann potentials.

1 Introduction

This paper deals with with a 4-D integrable dynamical system defined by the parametric Hamiltonian function

$$\mathcal{H}_O = \frac{1}{2}\Big(Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2 \, + \, \omega\,(q_1^2 + q_2^2 + q_3^2 + q_4^2) + \frac{a}{q_1^2 + q_2^2} + \frac{b}{q_3^2 + q_4^2}\Big), \tag{1}$$

(where ω, a and b are parameters), and its relation with two families of 3-D integrable Hamiltonian systems $\mathcal{H} = \frac{1}{2}\|X\|^2 + V_i$ with axial symmetry, namely systems with potentials given by

$$V_1 = -\frac{\mu}{\sqrt{x_1^2 + x_2^2 + x_3^2}} + \frac{P}{x_1^2 + x_2^2} + \frac{Q\, x_3}{(x_1^2 + x_2^2)\, \sqrt{x_1^2 + x_2^2 + x_3^2}}, \tag{2}$$

dubbed as Smorodinsky-Winternitz potential (see Mardoyan 2003), and

$$V_2 = \frac{\Omega^2}{2}(x_1^2 + x_2^2 + x_3^2) + \frac{P}{2x_3^2} + \frac{Q}{2(x_1^2 + x_2^2)}, \tag{3}$$

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(where μ, Ω, P and Q are parameters). Notice that, written in spherical variables, potentials V_1 also appear in the literature under the Pöschl-Teller form

$$V_1 = -\frac{\mu}{r} + \frac{P+Q}{4\,r^2 \sin^2\frac{\phi}{2}} + \frac{P-Q}{4\,r^2\cos^2\frac{\phi}{2}}.$$

The particular case the the system (1) when $a = b$ was considered by Kibler and Négadi (1984a) when they studied the Hartman potential using KS transformation. In this sense, the proposed Hamiltonian (1) represents a generalization of theirs.

Potentials V_i belong to a larger family of integrable systems which are known to be separable from the work done by Makarov *et al.* (1967). Potentials (2) and (3) have received special attention since the pioneer work of Hartmann and collaborators because they are related with the benzene molecule, as well as other models in quantum chemistry and nuclear physics. When we take $Q = 0$ in potential V_1 we have the Hartmann (1972) model. Continuing the work done by Kibler and Négadi (1984a), the solution is given in detail in Kibler and Winternitz (1987), now in parabolic coordinates. With respect to potential V_2, the case $P = 0$ has been studied by Quesne (1988), and the study of both models is given in Kibler *et al.* (1992). The ring-shaped features come from the fact that coefficients have to be taken then within specific ranges.

Systems defined by those potentials are super-integrable, but not maximally super-integrable, having four globally defined single-valued integrals of motion. They admit two maximally super-integrable systems as limiting cases, viz, the Coulomb-Kepler system and the isotropic harmonic oscillator system in three dimensions. This relates with the fact that Schrödinger equation is separable, among others, in spherical, parabolic and spheroidal coordinates. All finite trajectories are quasi-periodical; they become truly periodical if a commensurability condition is imposed on an angular momentum component. For potential V_1 the coefficients of the interbasis expansions between three bases (spherical, parabolic and spheroidal) are studied in detail by Kibler *et al.* (1994). For the path integral approach applied to these and related systems we mention the review paper of Grosche (1992) and references therein. Recently the normalized wavefunctions and explicit expressions for their radial average values have been presented by Chen *et al.* (2002), where an updated list of references on these problem is given. Similar studies for potential V_2 were done by Kibler *et al.* (1996); see also Kibler and Winternitz (1990).

As we have said above this paper deals with the relation of those 3-D systems with a 4-D integrable dynamical system defined by the Hamiltonian function (1). More precisely we will focus on aspects related to classical dynamics. The paper is organized as follows. In Section 2 we establish the relation between the oscillator and the systems defined by potentials (2) and in Section 3 we do the same with potentials (3), making use of well known point transformations in 4-D and their canonical extensions; for each case there is a linear system which relates parameters P and Q of the potentials with integrals and parameters of the 4-D oscillator. In Section 4, assuming $\omega > 0$, we carry out the integration of system defined by Hamiltonian (1), which is given by means of elementary functions. With respect to quantum mechanic approach, we refer to the classic paper of Calogero (1969) (Sect. 2) which, with minor changes, can be applied to the two coupled 1-DOF systems defining our model.

2 The oscillator and the generalized Hartmann potentials

We show first the relation of the Hamiltonian system defined by (1) and the generalized Hartmann potentials defined by the potentials V_1. In order to do that we make use the transformation: $(r, \phi, \lambda, \psi) \to (q_1, q_2, q_3, q_4)$ given by

$$\begin{aligned} q_1 &= \sqrt{r}\,\sin\frac{\phi}{2}\,\cos\frac{\lambda-\psi}{2}, \quad & q_3 &= \sqrt{r}\,\cos\frac{\phi}{2}\,\sin\frac{\lambda+\psi}{2}, \\ q_2 &= \sqrt{r}\,\sin\frac{\phi}{2}\,\sin\frac{\lambda-\psi}{2}, \quad & q_4 &= \sqrt{r}\,\cos\frac{\phi}{2}\,\cos\frac{\lambda+\psi}{2}, \end{aligned} \tag{4}$$

with $(r, \phi, \lambda, \psi) \in R^+ \times (0, \pi) \times [0, 2\pi] \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and whose jacobian is $-r\sin\phi/8$. Later on we will need the inverse transformation given by

$$\begin{aligned} & r = q_1^2 + q_2^2 + q_3^2 + q_4^2, \\ & \sin\phi = \frac{2\sqrt{(q_1^2+q_2^2)(q_3^2+q_4^2)}}{q_1^2+q_2^2+q_3^2+q_4^2}, \qquad && \cos\phi = \frac{q_3^2+q_4^2-q_1^2-q_2^2}{q_1^2+q_2^2+q_3^2+q_4^2}, \\ & \sin\lambda = \frac{q_1q_3+q_2q_4}{\sqrt{(q_1^2+q_2^2)(q_3^2+q_4^2)}}, && \cos\lambda = \frac{q_1q_4-q_2q_3}{\sqrt{(q_1^2+q_2^2)(q_3^2+q_4^2)}}, \\ & \sin\psi = \frac{q_1q_3-q_2q_4}{\sqrt{(q_1^2+q_2^2)(q_3^2+q_4^2)}}, && \cos\psi = \frac{q_1q_4+q_2q_3}{\sqrt{(q_1^2+q_2^2)(q_3^2+q_4^2)}}. \end{aligned} \tag{5}$$

These variables are well known in the literature. Kibler and Négadi point out that they were used by Ikeda y Miyachi (1971), and these authors refer them to a classical physics book of Synge (1960). In Cornish (1984), we find a reference to the work of Barut *et al.* (1979), and are introduced starting from the transformation $(\zeta_A, \zeta_B) \to (x, y, z, \sigma)$

$$x + iy = 2\zeta_A\bar{\zeta}_B, \quad z = \zeta_A\bar{\zeta}_A - \zeta_B\bar{\zeta}_B, \quad \sigma = \arg\zeta_A\zeta_B,$$

where ζ_A y ζ_B are two complex variables. We find them also in Stiefel and Scheifele (1971), although no further use of them. As these variables are related to Euler angles of rotation, we propose to dube them as *Euler projective* variables.

The canonical extension associated to the transformation (4) is readely obtained as a Mathieu transformation, satisfying $\sum Q_i dq_i = R dr + \Phi d\phi + \Lambda d\lambda + \Psi d\psi$. The relations among the momenta are given by

$$
\begin{aligned}
R &= \frac{1}{2\sum q_i^2}(q_1Q_1+q_2Q_2+q_3Q_3+q_4Q_4),\\
\Phi &= \frac{(q_1Q_1+q_2Q_2)(q_3^2+q_4^2)-(q_3Q_3+q_4Q_4)(q_1^2+q_2^2)}{2\sqrt{(q_1^2+q_2^2)(q_3^2+q_4^2)}},\\
\Lambda &= \frac{1}{2}(-q_2Q_1+q_1Q_2+q_4Q_3-q_3Q_4),\\
\Psi &= \frac{1}{2}(q_2Q_1-q_1Q_2+q_4Q_3-q_3Q_4),
\end{aligned}
\tag{6}
$$

The Hamiltonian (1) in the new variables may be written as

$$
\mathcal{H} = 4r\left[\frac{\omega}{8}+\frac{1}{2}\left(R^2+\frac{\Phi^2}{r^2}+\frac{\Lambda^2}{r^2\,\sin^2\phi}\right)+\frac{\Psi^2-2\,\Lambda\Psi\,\cos\phi}{2\,r^2\,\sin^2\phi}+\frac{c+d\cos\phi}{2r^2\sin^2\phi}\right] \tag{7}
$$

where

$$
c=\frac{a+b}{2},\qquad d=\frac{a-b}{2}.
$$

Note that λ and ψ are cyclic variables, with Λ and Ψ as first integrals. In other words the differential systems is

$$
\frac{dr}{d\tau}=\frac{\partial\mathcal{H}_O}{\partial R},\qquad \frac{d\phi}{d\tau}=\frac{\partial\mathcal{H}_O}{\partial\Phi},\qquad \frac{dR}{d\tau}=-\frac{\partial\mathcal{H}_O}{\partial r},\qquad \frac{d\Phi}{d\tau}=-\frac{\partial\mathcal{H}_O}{\partial\phi}
$$

and two quadratures $\lambda=\int(\partial\mathcal{H}_O/\partial\Lambda)\,d\tau$ and $\psi=\int(\partial\mathcal{H}_O/\partial\Psi)\,d\tau$.

Using Poincaré notation and introducing a change of independent variable $\tau\to s$ given by $d\tau=4r\,ds$, the Hamiltonian takes the form

$$
\begin{aligned}
\mathcal{K}_O &= \frac{1}{4r}(\mathcal{H}_O-h_O)\\
&= \frac{\omega}{8}+\frac{1}{2}\left(R^2+\frac{\Phi^2}{r^2}+\frac{\Lambda^2}{r^2\,\sin^2\phi}\right)+\frac{\Psi^2-2\,\Lambda\Psi\,\cos\phi}{2\,r^2\,\sin^2\phi}+\frac{c+d\cos\phi}{2r^2\sin^2\phi}-\frac{h_O}{4r},
\end{aligned}
\tag{8}
$$

where h_O is a fix value of the Hamiltonian $\mathcal{H}_O$ for chosen initial conditions, and the flow is defined now on the manifold $\mathcal{K}_O=0$. We prefer to use a slightly different form; we consider the Hamiltonian

$$
\tilde{\mathcal{K}}_O=\frac{1}{2}\left(R^2+\frac{\Phi^2}{r^2}+\frac{\Lambda^2}{r^2\,\sin^2\phi}\right)-\frac{h_O}{4r}+\frac{(\Psi^2+c)/2}{r^2\,\sin^2\phi}+\frac{(d/2-\Lambda\Psi)\,\cos\phi}{r^2\,\sin^2\phi} \tag{9}
$$

in the manifold $\tilde{\mathcal{K}}_O=-\dfrac{\omega}{8}$. Denoting

$$
\mathcal{H}_K=\frac{1}{2}\left(R^2+\frac{\Phi^2}{r^2}+\frac{\Lambda^2}{r^2\,\sin^2\phi}\right)-\frac{h_O}{4r}
$$

the differential system defined by (9) is given by

$$
\begin{aligned}
\frac{dr}{ds} &= \frac{\partial\tilde{\mathcal{K}}_O}{\partial R} = R,\\
\frac{d\phi}{ds} &= \frac{\partial\tilde{\mathcal{K}}_O}{\partial \Phi} = \frac{\Phi}{r^2},\\
\frac{dR}{ds} &= -\frac{\partial\tilde{\mathcal{K}}_O}{\partial r} = -\frac{\partial\mathcal{H}_K}{\partial r} + 2\frac{(\Psi^2+c)/2}{r^3\,\sin^2\phi} + 2\frac{(d/2-\Lambda\Psi)\,\cos\phi}{r^3\,\sin^2\phi}\\
\frac{d\Phi}{ds} &= -\frac{\partial\tilde{\mathcal{K}}_O}{\partial \phi} = -\frac{\partial\mathcal{H}_K}{\partial \phi} + \frac{(\Psi^2+c)/2}{r^2}\frac{\partial}{\partial\phi}\Big(\frac{1}{\sin^2\phi}\Big) + \frac{(d/2-\Lambda\Psi)}{r^2}\frac{\partial}{\partial\phi}\Big(\frac{\cos\phi}{\sin^2\phi}\Big)
\end{aligned}
\tag{10}
$$

and two quadratures

$$
\lambda = \int \frac{\partial\tilde{\mathcal{K}}_O}{\partial\Lambda}\,ds = \int \Big(\frac{\Lambda}{r^2\,\sin^2\phi} - \frac{\Psi\,\cos\phi}{r^2\,\sin^2\phi}\Big)\,ds, \tag{11}
$$

$$
\psi = \int \frac{\partial\tilde{\mathcal{K}}_O}{\partial\Psi}\,ds = \int \frac{\Psi-\Lambda\,\cos\phi}{r^2\,\sin^2\phi}\,ds, \tag{12}
$$

If we consider now the differential system defined by the Hamiltonian with potential V_1, Eq. (2) in spherical variables (r,ϕ,λ),

$$
x_1 = r\sin\phi\,\cos\lambda, \qquad x_2 = r\sin\phi\,\sin\lambda, \qquad x_3 = r\cos\phi, \tag{13}
$$

and their momenta (R,Φ,Λ), we check that those equations coincide with equations (10) and (11), when we restrict to the manifold $\Psi = 0$ and we take the following values for the coefficients

$$
h_O = 4\mu, \qquad c = 2P, \qquad d = 2Q,
$$

and we identify the variable s with the physical time t.

Notice that we may also choose $\Psi \neq 0$. In that case the values will be

$$
h_O = 4\mu, \qquad c = 2P - \Psi^2, \qquad d = 2(Q + \Lambda\Psi),
$$

and for λ, instead of (11), we take

$$
\lambda = \int \frac{\Lambda}{r^2\,\sin^2\phi}\,ds,
$$

with $r(s)$ and $\phi(s)$ given by the solution of the system (10). It is an open question if there is any advantage in proceeding this way.

Thus, we have shown that the dynamics of the oscillator defined by Hamiltonian (1) corresponds to the family of the generalized Hartmann potentials. If we assume $\Psi = 0$, the particular case of the Hartmann model is obtained when $d = 0$, *i. e.* when we take for the oscillator the following values

$$
\mathcal{H}_O = 4\mu \qquad \omega = -8\tilde{\mathcal{K}}_O \qquad a = b = P.
$$

3 Relation with generalized 3-D isotropic potentials

There is still another family of potentials related to our system (1). Let us consider now the transformation (used by Kibler and Negali, 1984)

$$q_1 = r\cos\alpha\cos\beta, \quad q_2 = r\cos\alpha\sin\beta, \quad q_3 = r\sin\alpha\cos\gamma, \quad q_4 = r\sin\alpha\sin\gamma \tag{14}$$

with Jacobian: $-r^3\sin 2\alpha/2$, in other words $\alpha \in (0,\pi/2)\bigcup(\pi/2,\pi)$. The associated canonical extension $(q_1,q_2,q_3,q_4,Q_1,Q_2,Q_3,Q_4) \to (r,\alpha,\beta,\gamma,R,A^*,B^*,C^*)$ reads

$$\begin{aligned}
R &= \frac{1}{\sum q_i^2}(q_1Q_1+q_2Q_2+q_3Q_3+q_4Q_4),\\
A &= \frac{(q_3Q_3+q_4Q_4)(q_1^2+q_2^2)-(q_1Q_1+q_2Q_2)(q_3^2+q_4^2)}{\sqrt{(q_1^2+q_2^2)(q_3^2+q_4^2)}},\\
B &= -q_2Q_1+q_1Q_2,\\
C &= -q_4Q_3+q_3Q_4,
\end{aligned} \tag{15}$$

The inverse transformation, needed for the construction of the explicit transformation with the old variables takes the form

$$\begin{aligned}
&r = q_1^2+q_2^2+q_3^2+q_4^2, \\
&\sin\alpha = \sqrt{\frac{q_1^2+q_2^2}{q_1^2+q_2^2+q_3^2+q_4^2}}, \qquad \cos\alpha = \sqrt{\frac{q_3^2+q_4^2}{q_1^2+q_2^2+q_3^2+q_4^2}}\\
&\cos\beta = \frac{q_1}{\sqrt{q_1^2+q_2^2}}, \qquad \sin\beta = \frac{q_2}{\sqrt{q_1^2+q_2^2}},\\
&\cos\gamma = \frac{q_3}{\sqrt{q_3^2+q_4^2}}, \qquad \sin\gamma = \frac{q_4}{\sqrt{q_3^2+q_4^2}}.
\end{aligned} \tag{16}$$

The Hamiltonian (1) in these variables is given by

$$\mathcal{H}_{OP} = \frac{1}{2}\Big(R^2+\frac{A^{*2}}{r^2}+\frac{C^{*2}}{r^2\sin^2\alpha}\Big)+\frac{\omega}{2}r^2+\frac{B^{*2}+a}{2r^2\cos^2\alpha}+\frac{b}{2r^2\sin^2\alpha}. \tag{17}$$

Note that β and γ are cyclic, thus B^* and C^* are first integrals. In other words the differential systems is

$$\frac{dr}{d\tau} = \frac{\partial\mathcal{H}_O}{\partial R}, \qquad \frac{d\alpha}{d\tau} = \frac{\partial\mathcal{H}_O}{\partial A^*}, \qquad \frac{dR}{d\tau} = -\frac{\partial\mathcal{H}_O}{\partial r}, \qquad \frac{dA^*}{d\tau} = -\frac{\partial\mathcal{H}_O}{\partial\alpha}$$

and two quadratures $\beta = \int(\partial\mathcal{H}_O/\partial B^*)\,d\tau$ and $\gamma = \int(\partial\mathcal{H}_O/\partial C^*)\,d\tau$.

If we restrict to the subsystem defined by $(r,\alpha,\gamma,R,A^*,C^*)$, and we consider the transformation defined by

$$x_1 = r\sin\alpha\cos\gamma, \qquad x_2 = r\sin\alpha\sin\gamma, \qquad x_3 = r\cos\alpha,$$

the system defined corresponds to the one given by family of potentials V_2, choosing the constants as follows

$$\omega = \Omega^2, \qquad a = P - B^{*2} \qquad \text{y} \qquad b = Q,$$

and we identify τ with the physical time t. Notice that the 3-D isotropic oscillator is obtained either choosing $B^* = a = b = 0$ or if $b = 0$ and $a = -B^*$. Kibler and Winternitz (1990) studied the case $P = 0$, and a similar analysis for the general case may be found in Kibler *et al.* (1996).

4 The biparametric oscillator and its integration

Having already shown the relation of both families of ring-shaped systems with the oscillator, we focus now the integration of our oscillator. The Hamiltonian function (1) defines an integrable system in $\Delta = \mathbb{R}^4 - \{(0,0)\times\mathbb{R}^2\} \bigcup \{\mathbb{R}^2\times(0,0)\}$.

4.1 First integrals in involution

There is a large literature on the issue of integrability and superintegrability which we do not consider necessary to treat here. We wish only to mention that Liouville-Arnold contidions for integrability are satisfied for our system. Indeed, we chek immediately that the functions

$$I_1 = -Q_1 q_2 + q_1 Q_2, \qquad I_3 = \frac{1}{2}\Big(Q_1^2 + Q_2^2 + \omega(q_1^2 + q_2^2) + \frac{\omega_1}{q_1^2 + q_2^2}\Big), \tag{18}$$

$$I_2 = -Q_3 q_4 + q_3 Q_4, \qquad I_4 = \frac{1}{2}\Big(Q_3^2 + Q_4^2 + \omega(q_3^2 + q_4^2) + \frac{\omega_2}{q_3^2 + q_4^2}\Big), \tag{19}$$

are invariants which are in involution. Moreover, in order to see that they are independent, the dimension where rank of the Jacobian defined by those functions is not four is of dimension 2. In other words, there is an open domain in the cotangent space where the rank defined by these invariants is maximal equal to four. Finally, as

$$\mathcal{H} = I_3 + I_4,$$

we may take as the basic set of four first integrals the functions

$$\begin{aligned} \mathcal{H} = \frac{1}{2}\Big(&Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2 \\ &+ \omega\,(q_1^2 + q_2^2 + q_3^2 + q_4^2) + \frac{\omega_1}{q_1^2 + q_2^2} + \frac{\omega_2}{q_3^2 + q_4^2}\Big), \end{aligned} \tag{20}$$

$$I_1 = -Q_1 q_2 + q_1 Q_2, \tag{21}$$

$$I_2 = -Q_3 q_4 + q_3 Q_4, \tag{22}$$

$$I_3 = \frac{1}{2}\Big(Q_1^2 + Q_2^2 + \omega(q_1^2 + q_2^2) + \frac{\omega_1}{q_1^2 + q_2^2}\Big). \tag{23}$$

If this Hamiltonian defines a superintegrable system will not be study here. Those systems, apart from being integrable in the Liouville-Arnold sense, they possess more constants of motion than degrees of freedom. Moreover, if the number N of independent constants takes the value $N = 2n - 1$, then the system is called maximally superintegrable. Join with the three well known classic cases, more recently the existence of other less simple such as the Calogero-Moser, the Smorodinsky-Winternitz and the hyperbolic Calogero-Sutherland-Moser models have been identify as superintegrable n-dimensional systems (for more details we refer to López *et al.* 1999).

4.2 The explicit solution

Related to the previous integrals, we make use of the polar-polar transformation $(q_1, q_2, q_3, q_4) \rightarrow (\rho_1, \rho_2, \alpha_1, \alpha_2)$, considered by Kibler and Winternitz (1987), formulae (18), given by:

$$q_1 = \rho_1 \cos\alpha_1, \quad q_2 = \rho_1 \sin\alpha_1, \quad q_3 = \rho_2 \cos\alpha_2, \quad q_4 = \rho_2 \sin\alpha_2 \tag{24}$$

and its canonical extension, $(q_1, q_2, q_3, q_4, Q_1, Q_2, Q_3, Q_4) \rightarrow (P_1, P_2, A_1, A_2)$

$$P_1 = \frac{q_1 Q_1 + q_2 Q_2}{\sqrt{q_1^2 + q_2^2}}, \quad A_1 = q_1 Q_2 - q_2 Q_1, \quad P_2 = \frac{q_3 Q_3 + q_4 Q_4}{\sqrt{q_3^2 + q_4^2}}, \quad A_2 = q_3 Q_4 - q_4 Q_3$$

Then,the Hamiltonian (1) in the new variables reads

$$\mathcal{H}_O = \frac{1}{2}\Big(P_1^2 + P_2^2 + \frac{A_1^2}{\rho_1^2} + \frac{A_2^2}{\rho_2^2} + \omega(\rho_1^2 + \rho_2^2) + \frac{a}{\rho_1^2} + \frac{b}{\rho_2^2}\Big) \tag{25}$$

Note that α_1 and α_2 are cyclic, thus A_1 and A_2 are first integrals. In other words, the system is made separable in two subsystems of 1-DOF, defined by the Hamiltonian functions

$$\mathcal{H}_a = \frac{1}{2}\Big(P_1^2 + \frac{A_1^2}{\rho_1^2} + \omega\rho_1^2 + \frac{a}{\rho_1^2}\Big), \qquad \mathcal{H}_b = \frac{1}{2}\Big(P_2^2 + \frac{A_2^2}{\rho_2^2} + \omega\rho_2^2 + \frac{b}{\rho_2^2}\Big), \tag{26}$$

such that

$$\mathcal{H}_O = \mathcal{H}_a + \mathcal{H}_b.$$

We integrate the differential system defined by (25) immediately, following closely the steps of Deprit (1991). Introducing

$$\tilde{Q} = 2\mathcal{H}_a - \omega\rho_1^2 - \frac{A_1^2 + a}{\rho_1^2},$$

and the quantities a_1 and b_1 by

$$a_1 + b_1 = \sqrt{2\Big(\frac{\mathcal{H}_a}{\omega} + \sqrt{\frac{A_1^2 + a}{\omega}}\Big)}, \qquad a_1 - b_1 = \sqrt{2\Big(\frac{\mathcal{H}_a}{\omega} - \sqrt{\frac{A_1^2 + a}{\omega}}\Big)},$$

then, we may write

$$\tilde{Q} = \frac{\omega}{\rho_1^2}(a_1^2 - \rho_1^2)(\rho_1^2 - b_1^2).$$

We see that the equation $\tilde{Q} = 0$ has real roots when $\mathcal{H}_a \geq \sqrt{\omega\,(A_1^2 + a)}$. The system defined by $\mathcal{H}_a$ reduces to

$$\dot{\rho}_1 = P_1 = \sqrt{\tilde{Q}}, \qquad \dot{\alpha}_1 = \frac{A_1}{\rho_1^2},$$

i.e., to two quadratures. From the first quadrature we obtain immediately

$$\rho_1(\tau) = \sqrt{a_1^2 \sin^2 \sqrt{\omega}\,\tau \;+\; b_1^2 \cos^2 \sqrt{\omega}\,\tau}$$

where $\rho_1(0) = b_1$ and the angle $\alpha_1 = \alpha_1(\tau)$, after some computations, is given by

$$\sin(\alpha_1) = a_1 \frac{\sin \sqrt{\omega}\,\tau}{\rho_1(\tau)}, \qquad \cos(\alpha_1) = b_1 \frac{\cos \sqrt{\omega}\,\tau}{\rho_1(\tau)}. \tag{27}$$

where we have chosen $\alpha_1^0 = 0$. Similar expressions are obtained for ρ_2 and α_2. With the quantities a_2 and b_2 given by

$$a_2 + b_2 = \sqrt{2\Big(\frac{\mathcal{H}_b}{\omega} + \sqrt{\frac{A_2^2 + b}{\omega}}\Big)}, \qquad a_2 - b_2 = \sqrt{2\Big(\frac{\mathcal{H}_b}{\omega} - \sqrt{\frac{A_2^2 + b}{\omega}}\Big)},$$

we have

$$\rho_2(t) = \sqrt{a_2^2 \sin^2 \sqrt{\omega}\,\tau \;+\; b_2^2 \cos^2 \sqrt{\omega}\,\tau}$$

where $\rho_2(0) = b_2$ and the angle $\alpha_2 = \alpha_2(\tau)$, after some computations, is given by

$$\sin(\alpha_2) = a_2 \frac{\sin \sqrt{\omega}\,\tau}{\rho_2(\tau)}, \qquad \cos(\alpha_2) = b_2 \frac{\cos \sqrt{\omega}\,\tau}{\rho_2(\tau)}. \tag{28}$$

where we have chosen $\alpha_2^0 = 0$. Finally replacing in Eqs. (24) we obtain the q_i variables.

Conclusion and future work

We have established the relation of two families of ring-shaped type systems with a 4-D isotropic oscillator. This allows a unified treatment which is of interest both in quantum and classical studies. We may even consider a slightly more general potential

$$\begin{aligned}\mathcal{H}_{OP} = {} & \frac{1}{2}\Big(Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2 \;+\; \omega\,(q_1^2 + q_2^2 + q_3^2 + q_4^2)\Big) \\ & + \frac{a}{q_1^2 + q_2^2} + \frac{b}{q_3^2 + q_4^2} + \frac{c^*}{q_1^2 + q_2^2 + q_3^2 + q_4^2}.\end{aligned} \tag{29}$$

and this will be presented in a future paper now in progress by Ferrer and Lara (2007), identifying common features of them such as conditions periodic families and equilibria.

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References

[1] Barut, A. O., Schneider, C. K. and Wilson, R., *J. Math Phys.* **20**, 2244–2256 (1979).

[2] Calogero, F., Solution of a Three-Body Problem in One Dimension, *J. Math Phys.* **10**, 2191–2196 (1969).

[3] Chen, C.Y., Liu, C. L. and Sun, D. S., The normalized wavefunctions of the Hartmann potential and explicit expressions for their radial average values *Physics Letters A* **305**, 341–348 (2002).

[4] Cornish, F. H., The hydrogen atom and the four-dimensional harmonic oscillator, *J. Phys A: Math Gen* **17** 323–327 (1984).

[5] Deprit, A.: The Lissajous Transformation: I. Basics. *Celest. Mech & Dyn. Astron.* ...

[6] Ferrer, S. and Lara, M., Families of periodic orbits in the generalized Hartmann potentials, In preparation

[7] Grosche, C., Coulomb Potentials by Path Integration, *Fortschr. Phys.* **40**, 695–737 (1992).

[8] Hartmann, H., Die Bewegung eines Körpers in einen ringförmigen Potentialfeld, *Theoret. Chim. Acta* **24**, 201–206 (1972).

[9] Ikeda, M. and Miyachi, Y., On the Mathematical Structure of the Symmetry of Some Simple Dynamical Systems, *Matematica Japoniae*, 127–142 (1971).

[10] Kibler, M., Lamot, G.-H. and Winternitz, P., Classical Trajectories for Two Ring-Shaped Potentials *Int. J. Quantum Chem.* **43**, 625 (1992).

[11] Kibler, M., Mardoyan, L. G. and Pogosyan, G. S., On a Generalized Kepler-Coulomb System: Interbasis Expansions *Int. J. Quantum Chem.* **52**, 1301 (1994).

[12] Kibler, M., Mardoyan, L. G. and Pogosyan, G. S. On a Generalized Oscillator System: Interbasis Expansions, *Int. J. Quantum Chem.* (1996).

[13] Kibler, M. and Négadi, T., Motion of a particle in a ring-shaped potential: An approach via a Nonbijective Canonical transformation, *Int. J. Quantum Chemistry* **26**, 405–410 (1984a).

[14] Kibler, M. and Winternitz, P. , Dynamical invariance algebra of the Hartmann potential, J. Phys. A: Math. Gen. **20**, 4097–4108 (1987).

[15] Kibler, M. and Winternitz, P., Periodicity and Quasi-Periodicity for Super-integrable Hamiltonian Systems, *Phys. Lett. A* **147**, 338–342 (1990).

[16] López, C., Martínez, E. and Rañada, M. F., Dynamical symmetries, non-Cartan symmetries and superintegrability of the n-dimensional harmonic oscillator, *J. Phys. A: Math. Gen.* **32**, 1241–1249 (1999).

[17] Makarov, A. A., Smorodinsky, J. A., Valiev, Kh. and Winternitz, P. *Nuovo Cimento* A 52, 1061 (1967).

[18] Mardoyan, L., The generalized MIC-Kepler system, arXiv:quant-ph/0306168 v2 27 Jun (2003)

[19] Quesne, C., *J. Phys. A : Math. Gen.* 21, 3093 (1988).

[20] Stiefel, E.L. and Scheifele, G., *Linear and Regular Celestial mechanics*, Springer, Berlin (1971).

[21] Synge, J.L., *Classical Dynamics*. Handbuch der Physik **III-1**, Springer, Berlin (1960).