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Some topics concerning the theory of singular dynamical systems

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Abstract

Some subjects related to the geometric theory of singular dynamical systems are reviewed in this paper. In particular, the following two matters are considered: the theory of canonical transformations for presymplectic Hamiltonian systems, and the Lagrangian and Hamiltonian constraint algorithms and the time-evolution operator.

Key words: *Presymplectic manifolds, singular systems, canonical transformations, Lagrangian formalism, Hamiltonian formalism.*

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1 Introduction

The aim of this paper is to carry out a brief review of several topics concerning the theory of autonomous singular dynamical systems, from a geometrical perspective. In particular, our interest will be focused on two subjects, namely: the theory of canonical transformations for singular systems, and the problem of the compatibility of the dynamical equations of Lagrangian and Hamiltonian singular systems; more precisely, the analysis of the Lagrangian and Hamiltonian constraint algorithms and their relation.

The article is based on the developments made on the references [18] and [7], for the theory of canonical transformations, and [1], [12], [13], [32], [10] and [23] for aspects related with the constraint algorithms. Thus, we will refer to these articles for more details on all these results.

All manifolds are real and C^∞. All maps are C^∞. Sum over crossed repeated indices is understood. Throughout this paper $i(X)\omega$ denotes the contraction between the vector field X and the differential form ω, and $\mathrm{L}(X)\omega$ the Lie derivative of ω with respect to the vector field X.

2 Canonical transformations

2.1 Presymplectic Locally Hamiltonian Systems

Definition 1 *A* presymplectic locally Hamiltonian system *(p.l.h.s.) is a triad* (M, ω, α), *where* (M, ω) *is a presymplectic manifold, and* $\alpha \in Z^1(M)$ *(i.e., it is a closed* 1*-form), which is called a* Hamiltonian form.

A p.l.h.s has associated the following equation

$$i(X)\omega = \alpha \quad , \quad X \in \mathfrak{X}(M)$$

If X exists, it is called a *presymplectic locally Hamiltonian vector field* associated to α. Nevertheless, in the best cases, this equation has consistent solutions only in a submanifold $\jmath_C\colon C \hookrightarrow M$, where there exist $X \in \mathfrak{X}(M)$, tangent to C, such that

$$[i(X)\omega - \alpha]|_C = 0 \tag{1}$$

Furthermore, the solution X is not unique, in general, and this non-uniqueness is known as *gauge freedom*. In general $(C, \omega_C = \jmath_C^*\omega)$ is a presymplectic manifold which is called the *final constraint submanifold* (f.c.s.).

The following theorem gives the local structure of p.l.h.s. (see [7]):

Theorem 1 *Let* (M, ω, α) *be a p.l.h.s., and* $\jmath_C\colon C \hookrightarrow M$ *the f.c.s. Then:*

1. There are a symplectic manifold (P, Ω) *and a coisotropic embedding* $\imath_C : C \hookrightarrow P$ *such that* $\omega_C = \imath_C^*\Omega$.

2. For every $X \in \mathfrak{X}(M)$*, tangent to* C*, solution to (1), there exists a family of vector fields* $\mathfrak{X}(P, C) \subset \mathfrak{X}(P)$ *such that, for every* $X_\xi \in \mathfrak{X}(P, C)$*, (a)* X_ξ *are tangent to* C*, (b)* $X_\xi|_C = X|_C$*, and (c)* X_ξ *are solutions to the equations* $i(X_\xi)\Omega = \alpha_P + \xi$*, where* $\alpha_P \in Z^1(P)$ *satisfies* $\imath_C^*\alpha_P = \jmath_C^*\alpha$*, and* $\xi \in Z^1(P)$ *is any* first-class constraint form *(i.e.,* $\imath_C^*\xi = 0$*, and the Hamiltonian vector field associated with* ξ*,* $X_\xi \in \mathfrak{X}(P)$*, is tangent to* C*).*

3. The coisotropic embedding $\imath_C$ *and the family* $\mathfrak{X}(P, C)$ *are unique, up to an equivalence relation of local symplectomorphisms reducing to the identity on* C.

(P, C, Ω) *is the* coisotropic canonical system *associated to* (M, ω, α).

2.2 Canonical Transformations for p.l.h.s.

Let $\mathfrak{X}_{lh}(C)$ be the set of locally Hamiltonian vector fields in (C, ω_C); that is, $\mathfrak{X}_{lh}(C) = \{X_C \in \mathfrak{X}(C) \mid \mathrm{L}(X_C)\omega_C = 0\}$.

Definition 2 *Let* $(M_i, \omega_i, \alpha_i)$ *(*$i = 1, 2$*) be a p.l.h.s., with f.c.s.* $\jmath_{C_i}\colon C_i \to M_i$*, and* $\omega_{C_i} = \jmath_{C_i}^*\omega_i$*, such that* $\dim M_1 = \dim M_2$, $\dim C_1 = \dim C_2$*, and* $\operatorname{rank}\omega_{C_1} = \operatorname{rank}\omega_{C_2}$.

A canonical transformation *between these systems is a pair* (Φ,ϕ)*, with* $\Phi\in\mathrm{Diff}(M_1,M_2)$ *and* $\phi\in\mathrm{Diff}(C_1,C_2)$*, such that:*

1. $\Phi\circ\jmath_{C_1}=\jmath_{C_2}\circ\phi$*; that is, we have the commutative diagram*

$$
\begin{tikzcd}
M_1 \arrow[r, "\Phi"] & M_2 \\
C_1 \arrow[u, "\jmath_{C_1}"'] \arrow[r, "\phi"] & C_2 \arrow[u, "\jmath_{C_2}"]
\end{tikzcd}
$$

2. $\phi_*(\mathfrak{X}_{lh}(C_1))\subset\mathfrak{X}_{lh}(C_2)$*.*

The generalization of Lee Hwa Chung's theorem to presymplectic manifolds allows us to prove that (see [18]):

Proposition 1 *Condition 2 is equivalent to saying that there exists* $c\in\mathbb{R}$ *such that* $\jmath^*_{C_1}(\Phi^*\omega_2-c\omega_1)=\phi^*\omega_{C_2}-c\omega_{C_1}=0$*.*

c is called the *valence* of the canonical transformation. So, *univalent canonical transformations* are the *presymplectomorfisms* between C_1 and C_2.

Let (P_i,C_i,Ω_i) be the coisotropic canonical systems associated with the p.l.h.s (M_i,ω_i,α_i), $i=1,2$. The *class of* ϕ is defined by $\{\phi\}=\{\Psi\in\mathrm{Diff}(P_1,P_2)\mid\Psi\circ\imath_{C_1}=\imath_{C_2}\circ\phi\}$. So we have the diagram

$$
\begin{tikzcd}
P_1 \arrow[rrr, "\Psi"] & & & P_2 \\
& M_1 \arrow[r, "\Phi"'] & M_2 & \\
& C_1 \arrow[ul, "\imath_{C_1}"] \arrow[u, "\jmath_{C_1}"'] \arrow[r, "\phi"] & C_2 \arrow[u, "\jmath_{C_2}"] \arrow[ur, "\imath_{C_2}"'] &
\end{tikzcd}
$$

And therefore we have (see [7]):

Theorem 2 *There exists* $\psi\in\{\phi\}$ *which is a symplectomorphism between the symplectic manifolds* (P_1,Ω_1) *and* (P_2,Ω_2)*.*

As a particular situation, we can analyze the canonical transformations in a p.l.h.s. Thus, let (M,ω,α) be a p.l.h.s., with f.c.s. $\jmath_C\colon C\hookrightarrow M$. Consider the involutive distribution $\ker\,\omega_C$ in C, and assume that the quotient space $\hat{C}=C/\ker\,\omega_C$ is a manifold with natural projection $\hat{\pi}\colon C\to\hat{C}$ (it is called the *reduced phase space* associated to the p.l.h.s.). Then the form ω_C is $\hat{\pi}$-projectable; hence there exists $\hat{\Omega}\in\mathit{\Omega}^2(\hat{C})$ such that $\omega_C=\hat{\pi}^*\hat{\Omega}$. Furthermore, $(\hat{C},\hat{\Omega})$ is a symplectic manifold, and we have the following result (see [7]):

Proposition 2 *Every canonical transformation* (Φ, ϕ) *in* (M, ω, α) *leaves the distribution* $\ker\, \omega_C$ *invariant. As a consequence, there exists a unique* $\hat{\phi} \in \mathrm{Diff}(\hat{C})$ *such that:*

1. $\hat{\phi} \circ \hat{\pi} = \hat{\pi} \circ \phi$.
2. $\hat{\phi}$ *is a symplectomorphism.*

Some applications of the geometric theory of p.l.h.s. and their canonical transformations are the following: the study of canonical transformations for regular autonomous systems (it includes a new geometrical description of these kinds of systems based on the *coisotropic embedding theorem*) [8], the analysis of the geometric structure and construction of canonical transformations for the free relativistic massive particle [8], the discussion about time scaling transformations in Hamiltonian dynamics and their application to celestial mechanics [9], and the construction of realizations of symmetry groups for singular systems and, as an example, the Poincaré realizations for the free relativistic particle [35].

3 Constraints and the Evolution Operator

3.1 Lagrangian Dynamical Systems

(See [3] for details).

Let Q be a n-dimensional differential manifold which constitutes the *configuration space* of a dynamical system. Its tangent and cotangent bundles, $\tau_Q \colon \mathrm{T}Q \to Q$ and $\pi_Q \colon \mathrm{T}^*Q \to Q$, are the (*phase spaces of velocities and momenta* of the system.

A *Lagrangian dynamical system* is a couple $(\mathrm{T}Q, \mathcal{L})$, where $\mathcal{L} \in \mathrm{C}^\infty(\mathrm{T}Q)$ is the *Lagrangian function* of the system. Using the canonical elements of $\mathrm{T}Q$, the vertical endomorphism $\mathcal{S} \in \mathfrak{T}^1_1(\mathrm{T}Q)$ and the Liouville vector field $\Delta \in \mathfrak{X}(\mathrm{T}Q)$, we can define the *Lagrangian 2-form* $\omega_\mathcal{L} = -\mathrm{d}(\mathrm{d}\mathcal{L} \circ \mathcal{S})$, and the *Lagrangian energy function* $E_\mathcal{L} = \Delta(\mathcal{L}) - \mathcal{L}$. Moreover, we define the *Legendre transformation* associated with $\mathcal{L}$, $\mathcal{FL} \colon \mathrm{T}Q \to \mathrm{T}^*Q$, as the fiber derivative of the Lagrangian.

$\mathcal{L}$ is a *singular Lagrangian* if $\omega_\mathcal{L}$ is a presymplectic form (which is assumed to have constant rank) or, what is equivalent, $\mathcal{FL}$ is no longer a local diffeomorphism. In particular, $\mathcal{L}$ is an *almost-regular Lagrangian* if: (i) $M_0 = \mathcal{FL}(\mathrm{T}Q)$ is a closed submanifold of T^*Q, (ii) $\mathcal{FL}$ is a submersion onto M_0, and (iii) for every $\mathrm{p} \in \mathrm{T}Q$, the fibres $\mathcal{FL}^{-1}(\mathcal{FL}(\mathrm{p}))$ are connected submanifolds of $\mathrm{T}Q$. Then $(\mathrm{T}Q, \mathcal{L})$ is an *almost-regular Lagrangian system.*

For almost-regular Lagrangian systems, $(\mathrm{T}Q, \Omega_\mathcal{L}, \mathrm{d}E_\mathcal{L})$ is a p.l.h.s., and we have the so-called *Lagrangian dynamical equation*

$$i(\Gamma_\mathcal{L})\omega_\mathcal{L} = \mathrm{d}E_\mathcal{L} \tag{2}$$

Variational considerations lead us to impose that, solutions $\Gamma_\mathcal{L} \in \mathfrak{X}(\mathrm{T}Q)$ to (2) must be *second-order differential equations* (SODE); that is, holonomic vector fields in $\mathrm{T}Q$.

Geometrically this means that

$$\mathcal{S}(\Gamma_{\mathcal{L}}) = \Delta \tag{3}$$

For singular Lagrangians this does not hold in general, and (3) must be imposed as an additional condition (SODE-*condition*). Integral curves of vector fields satisfying (2) and (3) are solutions to the Euler-Lagrange equations.

The *Lagrangian problem* consists in finding a submanifold $\jmath_{S_f} : S_f \hookrightarrow \mathrm{T}Q$, and $\Gamma_{\mathcal{L}} \in \mathfrak{X}(\mathrm{T}Q)$, tangent to S_f, such that

$$[i(\Gamma_{\mathcal{L}})\omega_{\mathcal{L}} - dE_{\mathcal{L}}]|_{S_f} = 0 \quad , \quad [\mathcal{S}(\Gamma_{\mathcal{L}}) - \Delta]|_{S_f} = 0$$

Now, if $\mathcal{FL}_0 \colon \mathrm{T}Q \to M_0$ is the restriction of $\mathcal{FL}$ to M_0, we have that $\omega_{\mathcal{L}}$ and $E_{\mathcal{L}}$ are $\mathcal{FL}_0$-projectable: there exist $\omega_0 \in \Omega^2(M_0)$, and $h_0 \in \mathrm{C}^\infty(M_0)$ such that $\omega_{\mathcal{L}} = \mathcal{FL}_0^*\omega_0$, $E_{\mathcal{L}} = \mathcal{FL}_0^* h_0$. Then, $(M_0, \omega_0, \mathrm{d}h_0)$ is a p.l.h.s. which is called the *canonical Hamiltonian system* associated with the Lagrangian system $(\mathrm{T}Q, \omega_{\mathcal{L}}, \mathrm{d}E_{\mathcal{L}})$ So we have the *Hamiltonian dynamical equation*

$$i(X_0)\omega_0 = dh_0 \quad ; \quad X_0 \in \mathfrak{X}(M_0)$$

and the *Hamiltonian problem* consists in finding a submanifold $\jmath_{M_f} : M_f \hookrightarrow M_0$ and $X_0 \in \mathfrak{X}(M_0)$ tangent to M_f such that

$$[i(X_0)\omega_0 - dh_0]|_{M_f} = 0$$

3.2 Constraint Algorithms

In order to solve the Hamiltonian problem stated for an almost-regular system different kinds of Hamiltonian constraint algorithms were developed. The first was the local-coordinate *Dirac constraint algorithm* [16], but there were also geometric algorithms: the *Presymplectic Constraint Algorithm* (PCA) of *Gotay, Nester, Hinds* [21], and others by *Marmo, Tulczyjew* et al. [30], [31], etc. All of them give a sequence of submanifolds which, in the best cases, stabilizes giving the f.c.s.: $\mathrm{T}^*Q \hookleftarrow M_0 \hookleftarrow M_1 \hookleftarrow \ldots \hookleftarrow M_f$.

For the Lagrangian problem, the first attempt was not to consider the SODE-problem (3), and then develop a Lagrangian constraint algorithm by simply applying the P.C.A. to the Lagrangian dynamical equation (2), obtaining a sequence of submanifolds $\mathrm{T}Q \hookleftarrow P_1 \hookleftarrow \ldots \hookleftarrow P_f$ [19]. The SODE problem is studied later in [20], obtaining a submanifold S_f which solves the lagrangian problem, but which is not defined by constraints and is not maximal. Later, *Kamimura* [28] and *Batlle, Gomis, Pons, Román-Roy* [1] developed local-coordinate Lagrangian constraint algorithms in which the Lagrangian dynamical equation (2) and the SODE-condition (3) were both considered at the same time. The f.c.s. S_f obtained at the end of the corresponding sequence, $\mathrm{T}^*Q \hookleftarrow S_1 \hookleftarrow \ldots \hookleftarrow S_f$, is maximal and is defined by constraints. The relation between the aforementioned sequences of submanifolds is explained in the following diagram

$$\begin{array}{ccccccccc} \mathrm{T}Q & \hookleftarrow & P_1 & \hookleftarrow & \dots & \hookleftarrow & P_f & & \\ & & & \nwarrow & & & & \nwarrow & \\ & & & & S_1 & \hookleftarrow & \dots & \hookleftarrow & S_f \end{array} \tag{4}$$

In particular, the submanifolds P_i, $i = 1 \dots f$, are defined by constraints that can be expressed as *$\mathcal{FL}$-projectable* functions which give all the Hamiltonian constraints, and are related with the *first-class Hamiltonian constraints*. Futhermore, the submanifolds S_i, $i = 1 \dots f$, are defined by adding constraints that are not *$\mathcal{FL}$-projectable*, and they are related with the *second-class Hamiltonian constraints*.

The remaining question was how to describe geometrically the submanifolds S_i and their properties. First, this problem was solved in [12], [13] for S_1, i.e.; the submanifold of compatibility conditions for (2) and (3), obtaining as the main result that:

Theorem 3 *For every* SODE $\Gamma \in \mathfrak{X}(\mathrm{T}Q)$*, we have:*

$$S_1 = \{\mathrm{p} \in \mathrm{T}Q \mid [i(Z)(i(\Gamma)\omega_{\mathcal{L}} - \mathrm{d}E_{\mathcal{L}})](\mathrm{p}) = 0, \ \forall Z \in \mathcal{M}\}$$

where $\mathcal{M} = \{Z \in \mathfrak{X}(\mathrm{T}Q) \mid \mathcal{S}(Z) \in \ker \mathcal{FL}_* = \ker \omega_{\mathcal{L}} \cap \mathfrak{X}^{\mathrm{V}(\tau_Q)}(\mathrm{T}Q)\}$.

In particular, for every $Z \in \ker \omega_{\mathcal{L}} \subset \mathcal{M}$, $i(Z)(i(\Gamma)\omega_{\mathcal{L}} - \mathrm{d}E_{\mathcal{L}}) = i(Z)(\mathrm{d}E_{\mathcal{L}})$ *define the submanifold* P_1 *where (2) is compatible. They are called* dynamical constraints, *and are related to the existence of primary first-class Hamiltonian constraints.*

For every $Z \notin \ker \omega_{\mathcal{L}}$, $Z \in \mathcal{M}$, *these functions are called* SODE-constraints, *and are related to the existence of primary second-class Hamiltonian constraints.*

For S_i, $i = 2 \dots f$, the problem was studied in [32], by imposing tangency conditions for solutions to (2) and (3). The main results are the following: there are two kinds of Lagrangian constraints defining every S_i, $i = 1, \dots f$:

- *Dynamical constraints*, which define the submanifolds P_i, $i = 1, \dots f$, in the sequence (4). They are related with the solutions to the eq. (2), and all of them can be expressed as $\mathcal{FL}$-projectable functions which give all the Hamiltonian constraints.

- SODE *(non-dynamical) constraints*, coming from the SODE-condition (3). They are not *$\mathcal{FL}$-projectable.*

Hence, the relation among the sequences of submanifolds in the Lagrangian and Hamiltonian formalisms is given in the following diagram

$$\begin{tikzcd}
 & \mathrm{T}Q \arrow[ddl, "\mathcal{FL}"'] \arrow[dd, "\mathcal{FL}_0"] & P_1 \arrow[l, hook'] \arrow[dd] & & \dots \arrow[ll, hook'] & P_f \arrow[l, hook'] \arrow[dd] & \\
 & & & S_1 \arrow[ul] \arrow[dl] & \dots \arrow[l, hook'] & & S_f \arrow[ll, hook'] \arrow[ul] \arrow[dl] \\
\mathrm{T}^*Q & M_0 \arrow[l, hook'] & M_1 \arrow[l, hook'] & & \dots \arrow[ll, hook'] & M_f \arrow[l, hook'] &
\end{tikzcd}$$

Furthermore, the tangency conditions for dynamical constraints give dynamical and SODE constraints, while the tangency conditions for SODE-constraints remove gauge degrees of freedom and do not give new constraints.

3.3 The Time Evolution Operator K

As a final remark, the complete relation between Hamiltonian and Lagrangian constraints is given by the so-called *time evolution operator* K, which relates Lagrangian and Hamiltonian constraints, and also solutions to the Lagrangian and Hamiltonian problems. (It is also known by the name of the *relative Hamiltonian vector field* [33]). It was introduced and studied for the first time by *J. Gomis* et al. [1], developing some previous ideas of *K. Kamimura* [28]. They give the local-coordinate definition of this operator, whose expression in coordinates is

$$K = v^A \left(\frac{\partial}{\partial q^A} \circ \mathcal{FL} \right) + \frac{\partial \mathcal{L}}{\partial q^A} \left(\frac{\partial}{\partial p^A} \circ \mathcal{FL} \right)$$

The intrinsic definition and the geometric study of its properties was carried out independently in [10] and [23]. In the first work, K was defined using the *Skinner-Rusk unified formalism* in $\mathrm{T}Q \oplus \mathrm{T}^*Q$. In the second article, the concept of *section along a map* plays the crucial role, and so we have:

Definition 3 *Let* $(\mathrm{T}Q, \omega_{\mathcal{L}}, E_{\mathcal{L}})$ *be a Lagrangian system, and* $\Omega \in \Omega^2(\mathrm{T}^*Q)$ *the canonical form. The time-evolution operator* K *associated with* $(\mathrm{T}Q, \omega_{\mathcal{L}}, \mathrm{E}_{\mathcal{L}})$ *is a map* $K \colon \mathrm{T}Q \longrightarrow \mathrm{T}\mathrm{T}^*Q$ *verifying the following conditions:*

1. *(Structural condition): K is vector field along $\mathcal{FL}$,* $\quad \pi_{\mathrm{T}^*Q} \circ K = \mathcal{FL}$.
2. *(Dynamical condition):* $\quad \mathcal{FL}^*[i(K)(\Omega \circ \mathcal{FL})] = \mathrm{d}E_{\mathcal{L}}$.
3. *(SODE condition):* $\quad \mathrm{T}\tau_Q \circ K = \mathrm{Id}_{\mathrm{T}Q}$.

$$\begin{tikzcd}
\mathrm{T}Q & \mathrm{T}\mathrm{T}^*Q \arrow[l, "\mathrm{T}\tau_Q"'] \arrow[d, "\pi_{\mathrm{T}^*Q}"] \\
\mathrm{T}Q \arrow[u, leftarrow, "\mathrm{Id}_{\mathrm{T}Q}"] \arrow[ur, "K"] \arrow[r, "\mathcal{FL}"] & \mathrm{T}^*Q
\end{tikzcd}$$

Then, the relation between Lagrangian and Hamiltonian constraints is established as follows (see [1], [10], [23]):

Proposition 3 *If* $\xi \in \mathrm{C}^\infty(\mathrm{T}^*Q)$ *is a ith-generation Hamiltonian constraint, then* $\mathrm{L}(K)\xi$ *is a* $(i+1)$*th-generation Lagrangian constraint.*

In particular, if ξ *is a* first-class constraint *(resp. a* second-class constraint*) for* M_f*, then* $\mathrm{L}(K)\xi$ *is a dynamical constraint (resp. a SODE constraint).*

The time-evolution operator has also been used for studying different kinds of problems concerning singular systems. For instance, the operator K has been extended for analyzing higher-order singular dynamical systems [2], [11], [26], [27]. It is also used for treating constrained systems in general (*linearly singular systems*) [24]. It has been defined and its properties studied for non-autonomous dynamical systems [6]. Finally, K has been applied for analyzing gauge symmetries and other structures for singular systems [22], [25]. Furthermore, sections along maps in general are analyzed and used in different kinds of physical and geometrical problems in [4], [14], [15].

4 Discussion and outlook

Some of the previous problems have been studied in the sphere of first-order classical field theories, specially their *multisymplectic formalism* [5]. So, a geometric constraint algorithm has recently been completed for Lagrangian and Hamiltonian singular field theories [29], and the definition and properties of the operator K have been carried out for field theories [17], [34].

Other potentially interesting topics could be the generalization of some of the above results; such as: to study the local structure of pre-multisymplectic Hamiltonian field theories (previous generalization of the *coisotropic embedding theorem* for premultisymplectic manifolds); the study of canonical transformations for Hamiltonian field theories (*multisymplectomorphisms* and *pre-multisymplectomorphisms*), and the application of the operator K to analyze the relation between Lagrangian and Hamiltonian constraints of singular field theories (which could require prior development of the non-covariant formulation, i.e., *space-time splitting*, of field theories).

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References

[1] C. Batlle, J. Gomis, J.M. Pons, N. Román-Roy: "Equivalence between the Lagrangian and Hamiltonian formalism for constrained systems". *J. Math. Phys.* **27** (1986) 2953-2962.

[2] C. Batlle, J. Gomis, J.M. Pons, N. Román-Roy: "Lagrangian and Hamiltonian constraints for second order singular Lagrangians". *J. Phys. (A): Math. Gen.* **21**(12) (1988) 2693-2703.

[3] J.F. Cariñena: "Theory of singular Lagrangians". *Fortschr. Phys.* **38** (1990) 641-679.

[4] J.F. Cariñena: "Section along maps in geometry and physics", *Rend. Sem. Mat. Univ. Pol. Torino* **54**(3) (1996) 245-256.

[5] J.F. Cariñena, M. Crampin, A. Ibort: "On the multisymplectic formalism for first-order field theories". *Diff. Geom. Appl.* **1** (1991) 345-374.

[6] J.F. Cariñena, J. Fernández-Núñez, E. Martínez: "Time-dependent K-operator for singular Lagrangians", (unpublished) (1995).

[7] J.F. Cariñena, J. Gomis, L.A. Ibort, N. Román-Roy: "Canonical transformations theory for presymplectic systems". *J. Math. Phys.* **26**(8) (1985) 1961-1969.

[8] J.F. Cariñena, J. Gomis, L.A. Ibort, N. Román-Roy: "Applications of the canonical transformations theory for presymplectic systems". *Nuovo Cim. (B)* **98**(2) (1987) 172-196.

[9] J.F. Cariñena, L.A. Ibort, E. Lacomba: "Time scaling as an infinitesimal canonical transformation". *Celest. Mech.* **42**(1–4) (1987/88) 201–213.

[10] J.F. Cariñena, C. López: "The time evolution operator for singular Lagrangians". *Lett. Math. Phys.***14** (1987) 203-210.

[11] J.F. Cariñena, C. López: "The time evolution operator for higher-order singular Lagrangians". *J. Mod. Phys.* **7** (1992) 2447-2468.

[12] J.F. Cariñena, C López, N. Román-Roy: "Geometric study on the connection of Lagrangian and Hamiltonian constraints". *J. Geom. Phys.* **IV**(3) (1987) 315-334.

[13] J.F. Cariñena, C López, N. Román-Roy: "Origin of Lagrangian constraints and their relation with the Hamiltonian formalism". *J. Math. Phys.* **29**(5) (1988) 1143-1149.

[14] J.F. Cariñena, E. Martínez, W. Sarlet: "Derivations of differential forms along the tangent bundle projection". *Diff. Geom. and Appl.* **2**(1) (1992) 17-43.

[15] J.F. Cariñena, E. Martínez, W. Sarlet: "Derivations of differential forms along the tangent bundle projection II". *Diff. Geom. and Appl.* **3**(1) (1993) 1-29.

[16] P.A.M. Dirac: *Lectures on Quantum Mechanics*. Yeshiva Univ., New York, 1964.

[17] A. Echeverría-Enríquez, J. Marín-Solano, M.C.Muñoz-Lecanda, N. Román-Roy: "On the construction of $\mathcal{K}$-operators in field theories as sections along Legendre maps", *Acta Appl. Math.* **77**(1) (2003) 1-40.

[18] J. Gomis, J. Llosa, N. Román-Roy: "Lee Hwa Chung theorem for presymplectic manifolds. Canonical transformations theory for constrained systems". *J. Math. Phys.* **25**(5) (1984) 1348-1355.

[19] M.J. Gotay, J.M. Nester: "Presymplectic Lagrangian systems I: the constraint algorithm and the equivalence theorem". *Ann. Inst. H. Poincaré A* **30** (1979) 129-142.

[20] M.J. Gotay, J.M. Nester: "Presymplectic Lagrangian systems II: the second order equation problem". *Ann. Inst. H. Poincaré A* **32** (1980) 1–13.

[21] M.J. Gotay, J.M. Nester, G. Hinds: "Presymplectic manifolds and Dirac-Bergmann theory of constraints". *J.Math. Phys.* **27** (1978) 2388-2399.

[22] X. Gràcia, J.M. Pons: "Gauge generators, Dirac conjecture and degreees of freedom for constrained systems". *Ann. Phys.* **187**(2) (1988) 355–368.

[23] X. Gràcia, J.M. Pons: "On an evolution operator connecting Lagrangian and Hamiltonian formalisms". *Lett. Math. Phys.* **17** (1989) 175-180.

[24] X. Gràcia, J.M. Pons: "A generalized geometric framework for constrained systems". *Diff. Geom. Appl.* **2** (1992) 223-247.

[25] X. Gràcia, J.M. Pons: "Singular Lagrangians: some geometric structures along the Legendre map". *J. Phys. A* **34** (2001) 3047–3070.

[26] X. Gràcia, J.M. Pons, N. Román-Roy: "Higher order Lagrangian systems: geometric structures, dynamics and constraints". *J. Math. Phys.* **32** (1991) 2744-2763.

[27] X. Gràcia, J.M. Pons, N. Román-Roy: "Higher order conds. for singular Lagrangian dynamics". *J. Phys. A: Math. Gen.* **25** (1992) 1989-2004.

[28] K. Kamimura: "Singular Lagrangians and constrained Hamiltonian systems, generalized canonical formalism". *Nuovo Cim. B* **69** (1982) 33-54.

[29] M. de León, J. Marín-Solano, J.C. Marrero, M.C. Muñoz-Lecanda, N. Román-Roy: "Pre-multisymplectic constraint algorithm for field theories". *Int. J. Geom. Meth. Mod. Phys.* **2**(5) (2005) 1418

[30] G. Marmo, G. Mendella, W.M. Tulczyjew: "Constrained hamiltonian systems as implicit differential equations". *J. Phys. A: Math. Gen.* **30** (1997) 277–293.

[31] M.R. Menzio, W.M. Tulczyjew: "Infinitesimal symplectic relations and generalized Hamiltonian dynamics". *Ann. Inst. H. Poincaré A* **28** (1978) 349–367.

[32] M.C. Muñoz-Lecanda, N. Román-Roy: "Lagrangian theory for presymplectic systems", *Ann. I. H. Poincaré: Phys. Teor.* **57**(1) (1992) 27–45.

[33] F. Pugliese, A.M. Vinogradov: "On the geometry of singular Lagrangians". *J. Geom. Phys.* **35** (2000) 35-55.

[34] A.M. Rey, N. Román-Roy, M. Salgado: "Günther's formalism (k-symplectic formalism) in classical field theories: Skinner-Rusk approach and the evolution operator". *J. Math. Phys.* **46**(5) (2005) 052901, 24 pp.

[35] N. Román-Roy: "Symplectic and non-symplectic realizations of groups for singular systems. An example". sl Int. J. Theor. Phys. **30**(1)(1991) 89-95.