Monografías de la Real Academia de Ciencias de Zaragoza. **29:** 207–213, (2006).

Symmetries, bi-Hamiltonian Structures and Harmonic Oscillators

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Abstract

The properties of the bi-Hamiltonian structures of the harmonic oscillator are studied using the geometric theory of symmetries as an approach. Two superintegrable systems related with the harmonic oscillator are also analyzed.

Keywords: *Dynamical symmetries, Non-symplectic symmetries, Bi-Hamiltonian structures, Harmonic oscillators.*

PACS: *02.30.Ik, 02.40.Yy, 45.20.Jj*

MSC (2000): *37J15, 37J35, 70H06, 70H33*

1 Introduction

It is well known that there is a close relation between integrability and the existence of alternatives structures and also that integrable systems are systems endowed with a great number of symmetries. The purpose of this lecture is to present a brief survey of some properties relating the existence of additional structures with the theory of dynamical symmetries in the particular case of the two-dimensional harmonic oscillator.

2 Non-symplectic symmetries

In differential geometric terms, the dynamics of a time-independent Hamiltonian system is determined by a vector field on the $2n$–dimensional cotangent bundle T^*Q of a n–dimensional manifold Q. Cotangent bundles are manifolds endowed, in a natural or canonical way, with a symplectic structure ω_0 that, in coordinates $\{(q_j,p_j)\,;\,j=1,2,\dots,n\}$, is given by

$$\omega_0 = dq_j \wedge dp_j\,, \quad \omega_0 = -\,d\theta_0\,, \quad \theta_0 = p_j\,dq_j$$

(we write all the indices as subscripts and we use the summation convention on the repeated index). Given a differentiable function $F = F(q,p)$, the vector field X_F defined as the solution of the equation

$$i(X_F)\,\omega_0 = dF$$

is called the Hamiltonian vector field of the function F. There are two important properties:

(i) The Hamiltonian vector field of a given function is well defined without ambiguities. This uniqueness is a consequence of the symplectic character of the two–form ω_0.

(ii) Suppose that we are given a Hamiltonian $H = H(q,p)$. Then the dynamics is given by the Hamiltonian vector field Γ_H of the Hamiltonian function. That is, $i(\Gamma_H)\,\omega_0 = dH$.

At this point we recall that a (infinitesimal) dynamical symmetry of a Hamiltonian system (T^*Q, ω_0, H) is a vector field Y such that it satisfies $[Y, \Gamma_H] = 0$. On the other hand it is known that, in some very particular cases, the Hamiltonian systems can admit dynamical but non-symplectic symmetries (for a classification of the symmetries in geometric terms see [1] and [2]). In this case we have the following property.

Proposition 1 *Suppose there is a vector field Y that is a dynamical symmetry of Γ_H but does not preserve the symplectic two-form*

$$\mathcal{L}_Y\,\omega_0 = \omega_Y \,\neq\, 0.$$

Then (i) the dynamical vector field Γ_H is bi-Hamiltonian, and (ii) the function $Y(H)$ is the new Hamiltonian, and therefore it is a constant of motion.

Proof: For a proof of this proposition see [3]–[8], and references therein. A similar property is studied in [9, 10] for the case of Poisson manifolds. A sketch of the proof of this statement is as follows: The vector field Y does not preserve ω_0 and, as it is a non-canonical transformation, it determines a new 2-form $\omega_Y = \mathcal{L}_Y \omega_0$ ($\mathcal{L}_Y$ denotes de Lie derivative with respect to Y). As Y is a symmetry, $[Y, \Gamma_H] = 0$, then $\mathcal{L}_Y \circ i_{\Gamma_H} = i_{\Gamma_H} \circ \mathcal{L}_Y$, and, consequently,

$$i_{\Gamma_H}\,\omega_Y = i_{\Gamma_H}\mathcal{L}_Y\omega_0 = \mathcal{L}_Y i_{\Gamma_H}\omega_0 = \mathcal{L}_Y(dH) = d(YH)\ .$$

Therefore, the 2-form ω_Y is admissible for the dynamical vector field Γ_H, i.e. $\mathcal{L}_{\Gamma_H}\omega_Y = 0$, which is weakly bi-Hamiltonian with respect to the original symplectic 2-form ω_0 and the new structure ω_Y. Of course the particular form of ω_Y depends on Y and, in some cases, it can be just a constant multiple of ω_0 (trivial bi-Hamiltonian system). In some other cases ω_Y may be a degenerate 2-form with a nontrivial kernel. In any case, the vector field Γ_H is a dynamical system solution of the following two equations

$$i(\Gamma_H)\,\omega_0 = dH \quad \text{and} \quad i(\Gamma_H)\,\omega_Y = d[Y(H)]\,.$$

Therefore the function $H_Y = Y(H)$, that must be a constant of motion, can be considered as a new Hamiltonian for Γ_H.

In the next two sections we will consider the particular case of the harmonic oscillator. We will analyze the existence of some bi-Hamiltonian structures and we will prove that they can be considered as associated to non-symplectic symmetries. For these properties, and some other related results, see [4]–[8] and [11] and references therein.

3 The harmonic oscillator

The Hamiltonian of the two-dimensional harmonic oscillator

$$H = \frac{1}{2}\,(p_x^2 + p_y^2) + \frac{1}{2}\,w^2(x^2 + y^2)\,,$$

can be rewriten as follows

$$H = \frac{1}{2}\left(K_x\,K_x^* + K_y\,K_y^*\right)\,,$$

where K_x and K_y are the two complex fuctions given by

$$K_x = p_x + \mathrm{i}\;wx\,, \qquad K_y = p_y + \mathrm{i}\;wy\,.$$

Let us denote by Y_x and Y_y the Hamiltonian vector fields of the functions K_x and K_y

$$i(Y_x)\,\omega_0 = dK_x\,, \qquad i(Y_y)\,\omega_0 = dK_y\,,$$

with coordinate expressions

$$Y_x = \frac{\partial}{\partial x} - \mathrm{i}\,w\,\frac{\partial}{\partial p_x}\,, \qquad Y_y = \frac{\partial}{\partial y} - \mathrm{i}\,w\,\frac{\partial}{\partial p_y}\,,$$

and by Z the following vector field

$$Z = K_y^*\,Y_x\,.$$

Then Z is neither locally-Hamiltonian with respect to ω_0

$$\mathcal{L}_Z\omega_0 = d(K_y^*\,i(Y_x)\,\omega_0) = dK_y^* \wedge dK_x \neq 0$$

nor a infinitesimal symmetry of the Hamiltonian

$$\mathcal{L}_Z\,H = K_y^*\,i(Y_x)\,dH = -\mathrm{i}\,w\,K_x K_y^* \neq 0\,.$$

Concerning the Lie bracket of Z with the dynamical vector field Γ_H, it is given by

$$[Z, \Gamma_H] = K_y^*\,[Y_x, \Gamma_H] - \Gamma_H(K_y^*)\,Y_x$$

but as

$$[Y_x, \Gamma_H] = -\mathrm{i}\;w\,Y_x \qquad \text{and} \qquad \Gamma_H(K_y^*) = -\mathrm{i}\;wK_y^*\,,$$

we arrive to

$$[Z, \Gamma_H] = 0\,.$$

Hence, Z is a dynamical but non-symplectic (non-canonical) symmetry of Γ_H.

Thus, if we denote by Ω the complex 2-form defined as

$$\Omega = dK_x {\wedge} dK_y^* = \Omega_1 + \mathrm{i}\,\Omega_2\,,$$

where the two real 2-forms, $\Omega_1 = \mathrm{Re}(\Omega)$ and $\Omega_2 = \mathrm{Im}(\Omega)$, take the form

$$\Omega_1 = dx{\wedge}dy + dp_x{\wedge}dp_y\,, \qquad \Omega_2 = dx{\wedge}dp_y + dy{\wedge}dp_x\,.$$

then we have the following bi-Hamiltonian structure

$$i(\Gamma_H)\Omega_1 = -\,w\,dI_3\,, \quad i(\Gamma_H)\Omega_2 = w\,dI_4\,,$$

where the functions $I_3 = \mathrm{Im}(K_x K_y^*)$ and $I_4 = \mathrm{Re}(K_x K_y^*)$ are two constants of the motion given by

$$I_3 = xp_y - yp_x\,, \quad I_4 = p_x p_y + w^2\,xy\,.$$

4 Two superintegrable potencials

Fris, Mandrosov et al [12] studied the Euclidean $n = 2$ systems which admit separability in two different coordinate systems, and obtained four families V_r, $r = a, b, c, d$, of superintegrable potentials with constants of motion linear or quadratic in the momenta. In fact, if we call superseparable to a system that admits Hamilton-Jacobi separation of variables (Schrödinger in the quantum case) in more than one coordinate system, then quadratic superintegrability (superintegrability with linear or quadratic constants of motion) can be considered as a property arising from superseparability. The two first families, V_a and V_b, were directly related with the Harmonic oscillator

$$\begin{aligned} V_a &= (\frac{1}{2})\,w^2(x^2+y^2) + \frac{k_2}{x^2} + \frac{k_3}{y^2} \\ V_b &= (\frac{1}{2})\,w^2(4x^2+y^2) + k_2 x + \frac{k_3}{y^2} \end{aligned}$$

and can be considered as the more general deformations of the 1:1 and 2:1 oscillators (k_2, k_3, representing the intensity of the deformation) preserving quadratic superintegrability (the three-dimensional generalizations of these potentials have been studied in [13]).

4.1 Potential V_a

The potential V_a, is superintegrable with three quadratic constants of motion, I_r^a, $r = 1, 2, 3$. Since $V_{xy} = 0$, the two first constants of motion are $I_1^a = H_x^a$, and $I_2^a = H_y^a$. Concerning I_3^a, it takes the following form

$$I_3^a = (\frac{1}{2})(xp_y - yp_x)^2 + k_2(\frac{y}{x})^2 + k_3(\frac{x}{y})^2\,.$$

The function I_3^a arises from a symplectic symmetry. This symmetry is geometrically represented by the Hamiltonian vector field X_3^a of the function I_3^a

$$i(X_3^a)\,\omega_0 = dI_3^a\,, \quad X_3^a(H_a) = 0\,.$$

We can write the vector field X_3^a as follows

$$X_3^a = Y_a + Y_a'$$

with Y_a and Y_a' given by

$$\begin{aligned} Y_a &= \frac{\partial I_3^a}{\partial p_x}\frac{\partial}{\partial x} - \frac{\partial I_3^a}{\partial x}\frac{\partial}{\partial p_x} \\ Y_a' &= \frac{\partial I_3^a}{\partial p_y}\frac{\partial}{\partial y} - \frac{\partial I_3^a}{\partial y}\frac{\partial}{\partial p_y} \end{aligned}$$

It can be proved that

$$[Y_a, \Gamma_H] = 0\,, \qquad [Y_a', \Gamma_H] = 0\,.$$

So we have the following proposition

Proposition 2 *The symplectic symmetry X_3^a can be decomposed as a sum of two different "dynamical but non-symplectic symmetries" in such a way that the following properties are satisfied*

$$(i)\ \mathcal{L}_{Y_a}\omega_0 = \omega_a \neq 0\,, \quad (ii)\ Y_a(H_a) = H_Y^a = I_4^a\,, \quad (iii)\ \Gamma_H(I_4^a) = 0\,.$$

The vector filed Y_a, that turns out to be the (x, p_x)–dependent part of X_3^a, is given by

$$Y_a = (y^2 p_x - xyp_y)\,\frac{\partial}{\partial x} + \Gamma_H(y^2 p_x - xyp_y)\,\frac{\partial}{\partial p_x}$$

The new symplectic form and the new Hamiltonian, now denoted by ω_a and H_Y^a, become

$$\begin{aligned} \omega_a &= p_x p_y\, dx{\wedge}dy + 4\,(\frac{k_2 y}{x^3} + \frac{k_3 x}{y^3})\, dx{\wedge}dy + (yp_x - 2xp_y)\, dx{\wedge}dp_y \\ &\quad + (2yp_x - xp_y)\, dy{\wedge}dp_x + xy\, dp_x{\wedge}dp_y \\ H_Y^a &= p_x^2 p_y + w^2(yp_x - xp_y)xy + \frac{2k_2 y p_y}{x^3} - \frac{2k_3 x p_x}{y^3} \end{aligned}$$

and the bi–Hamiltonian structure for $H^a = T + V^a$ is given by

$$i(\Gamma_H^a)\,\omega_0 = dH^a\,, \quad \text{and} \quad i(\Gamma_H^a)\,\omega_a = dH_Y^a\,.$$

4.2 Potential V_b

The potential V_b is superintegrable with three quadratic constants of motion, I_r^b, $r = 1, 2, 3$,

$$I_1^b = H_x^b\,,$$

$$
\begin{aligned}
I_1^b &= H_y^b\,, \\
I_3^b &= (xp_y - yp_x)\,p_x + w^2x^2y \;+\; (\frac{1}{2})k_2x^2 \;-\; 2k_3(\frac{y}{x^2})\,.
\end{aligned}
$$

The function I_3^b arises from a symplectic symmetry. This symmetry is geometrically represented by the Hamiltonian vector field X_3^b of the function I_3^b

$$
i(X_3^b)\,\omega_0 = dI_3^b\,, \quad X_3^b(H_b) = 0\,.
$$

We can write the vector field X_3^b as follows

$$
X_3^b = Y_b + Y_b'
$$

with Y_b and Y_b' given by

$$
\begin{aligned}
Y_b &= \frac{\partial I_3^b}{\partial p_x}\frac{\partial}{\partial x} - \frac{\partial I_3^b}{\partial x}\frac{\partial}{\partial p_x} \\
Y_b' &= \frac{\partial I_3^b}{\partial p_y}\frac{\partial}{\partial y} - \frac{\partial I_3^b}{\partial y}\frac{\partial}{\partial p_y}
\end{aligned}
$$

It can be proved that

$$
[Y_b, \Gamma_H] = 0\,, \qquad [Y_b', \Gamma_H] = 0\,.
$$

So we have

$$
(i)\ \mathcal{L}_{Y_b}\,\omega_0 = \omega_b \neq 0\,, \quad (ii)\ Y_b(H_b) = H_Y^b = I_4^b\,, \quad (iii)\ \Gamma_H(I_4^b) = 0\,.
$$

The vector filed Y_b is given by

$$
Y_b = (y^2p_x - xyp_y)\,\frac{\partial}{\partial x} + \Gamma_H(y^2p_x - xyp_y)\,\frac{\partial}{\partial p_x}
$$

The new symplectic form and the new Hamiltonian, now denoted by ω_b and H_Y^b, become

$$
\begin{aligned}
\omega_b &= -\,2(w^2y + \frac{2k_3}{y^3})\,dx{\wedge}dy + \;2p_y\,dx{\wedge}dp_y + \;p_y\,dy{\wedge}dp_x - \;y\,dp_x{\wedge}dp_y \\
H_Y^b &= p_xp_y^2 + (\frac{2k_3}{y^2} - w^2y^2)p_x + \;(4w^2x + k_2)yp_y
\end{aligned}
$$

and the bi–Hamiltonian structure for $H^b = T + V^b$ is given by

$$
i(\Gamma_H^b)\,\omega_0 = dH^b\,, \quad \text{and} \quad i(\Gamma_H^b)\,\omega_b = dH_Y^b\,.
$$

Acknowledgments.

Support of projects BFM-2003-02532, FPA-2003-02948 and E23/1 (DGA) is acknowledged.

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