

The Kepler problem on 3D spaces of variable and constant curvature from quantum algebras

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Abstract

A quantum $sl(2, \mathbb{R})$ coalgebra (with deformation parameter z) is shown to underly the construction of superintegrable Kepler potentials on 3D spaces of variable and constant curvature, that include the classical spherical, hyperbolic and (anti-)de Sitter spaces as well as their non-constant curvature analogues. In this context, the non-deformed limit $z \rightarrow 0$ is identified with the flat contraction leading to the proper Euclidean and Minkowskian spaces/potentials. The corresponding Hamiltonians admit three constants of the motion coming from the coalgebra structure. Furthermore, maximal superintegrability of the Kepler potential on the spaces of constant curvature is explicitly shown by finding an additional constant of the motion coming from an additional symmetry that cannot be deduced from the quantum algebra. In this way, the Laplace–Runge–Lenz vector for such spaces is deduced and its algebraic properties are analysed.

1 Introduction

From the very beginning of our period as Ph.D. students at the University of Valladolid it has been always a pleasure for us to meet J.F. Cariñena, *Pepín*. Since then we know that he is very interested in all the facets and approaches to the Kepler problem (see, for instance, [1]), and in this workshop-tribute for his sixty-years-*youth* we would like to dedicate this contribution on the Kepler potential to *Pepín*, with our best wishes for the future.

The scheme of the paper is as follows. In the next section we show how to construct (classical, *i.e.* commutative) curved spaces by making use of quantum algebras. In particular, we consider the non-standard quantum deformation of $sl(2, \mathbb{R})$ expressed as a deformed Poisson coalgebra and through the associated coproduct we obtain superintegrable geodesic motions on 3D spaces of variable and constant curvature. We are able to identify the resulting spaces with the classical spherical, Euclidean, hyperbolic, (anti-)de Sitter and Minkowskian spaces and with their analogues of non-constant curvature. In

section 3 we add a Kepler potential to the former free Hamiltonian by keeping its superintegrability. Moreover, for the spaces of constant curvature we present a relationship between quantum algebra symmetry and Lie algebra symmetry which, in turn, leads to additional constants of the motion. In this way, the well-known maximal superintegrability of the Kepler potential on spaces of constant curvature is obtained, and the explicit form of the Laplace–Runge–Lenz vector is given for the six spaces.

2 Geodesic motion on 3D curved spaces

Let us consider the non-standard quantum deformation of $sl(2, \mathbb{R})$ written as a Poisson coalgebra (that we denote $sl_z(2)$) where z is a *real* deformation parameter ($q = e^z$). The deformed Poisson brackets, coproduct Δ and Casimir \mathcal{C} of $sl_z(2)$ are given by [2]:

$$\{J_3, J_+\} = 2J_+ \cosh zJ_-, \quad \{J_3, J_-\} = -2 \frac{\sinh zJ_-}{z}, \quad \{J_-, J_+\} = 4J_3, \quad (1)$$

$$\Delta(J_-) = J_- \otimes 1 + 1 \otimes J_-, \quad (2)$$

$$\Delta(J_l) = J_l \otimes e^{zJ_-} + e^{-zJ_-} \otimes J_l, \quad l = +, 3,$$

$$\mathcal{C} = \frac{\sinh zJ_-}{z} J_+ - J_3^2. \quad (3)$$

A one-particle symplectic realization of (1) with $\mathcal{C}^{(1)} = 0$ reads

$$J_-^{(1)} = q_1^2, \quad J_+^{(1)} = \frac{\sinh zq_1^2}{zq_1^2} p_1^2, \quad J_3^{(1)} = \frac{\sinh zq_1^2}{zq_1^2} q_1 p_1. \quad (4)$$

All these expressions reduce to the $sl(2, \mathbb{R})$ coalgebra under the limit $z \rightarrow 0$, that is, the Poisson brackets and Casimir are non-deformed, the coproduct is primitive, $\Delta(X) = X \otimes 1 + 1 \otimes X$, and the symplectic realization is $J_-^{(1)} = q_1^2$, $J_+^{(1)} = p_1^2$ and $J_3^{(1)} = q_1 p_1$.

Starting from (4), the coproduct (2) determines the corresponding two-particle realization and this allows one to deduce an N -particle realization by applying it recursively [3]. In particular, the 3-sites coproduct, $\Delta^{(3)} = (\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta$, gives rise to a three-particle symplectic realization of (1) defined on $sl_z(2) \otimes sl_z(2) \otimes sl_z(2)$; namely,

$$\begin{aligned} J_-^{(3)} &= q_1^2 + q_2^2 + q_3^2 \equiv \mathbf{q}^2, \\ J_+^{(3)} &= \frac{\sinh zq_1^2}{zq_1^2} p_1^2 e^{zq_2^2} e^{zq_3^2} + \frac{\sinh zq_2^2}{zq_2^2} p_2^2 e^{-zq_1^2} e^{zq_3^2} + \frac{\sinh zq_3^2}{zq_3^2} p_3^2 e^{-zq_1^2} e^{-zq_2^2}, \\ J_3^{(3)} &= \frac{\sinh zq_1^2}{zq_1^2} q_1 p_1 e^{zq_2^2} e^{zq_3^2} + \frac{\sinh zq_2^2}{zq_2^2} q_2 p_2 e^{-zq_1^2} e^{zq_3^2} + \frac{\sinh zq_3^2}{zq_3^2} q_3 p_3 e^{-zq_1^2} e^{-zq_2^2}. \end{aligned} \quad (5)$$

The coalgebra approach introduced in [3] provides *three* functions, coming from the two- and three-sites coproduct of the Casimir (3):

$$\mathcal{C}^{(2)} \equiv \mathcal{C}_{12} = \frac{\sinh zq_1^2}{zq_1^2} \frac{\sinh zq_2^2}{zq_2^2} (q_1 p_2 - q_2 p_1)^2 e^{-zq_1^2} e^{zq_2^2},$$

$$\begin{aligned}
\mathcal{C}_{(2)} \equiv \mathcal{C}_{23} &= \frac{\sinh zq_2^2}{zq_2^2} \frac{\sinh zq_3^2}{zq_3^2} (q_2p_3 - q_3p_2)^2 e^{-zq_2^2} e^{zq_3^2}, \\
\mathcal{C}^{(3)} \equiv \mathcal{C}_{123} &= \frac{\sinh zq_1^2}{zq_1^2} \frac{\sinh zq_2^2}{zq_2^2} (q_1p_2 - q_2p_1)^2 e^{-zq_1^2} e^{zq_2^2} e^{2zq_3^2} \\
&\quad + \frac{\sinh zq_1^2}{zq_1^2} \frac{\sinh zq_3^2}{zq_3^2} (q_1p_3 - q_3p_1)^2 e^{-zq_1^2} e^{zq_3^2} \\
&\quad + \frac{\sinh zq_2^2}{zq_2^2} \frac{\sinh zq_3^2}{zq_3^2} (q_2p_3 - q_3p_2)^2 e^{-2zq_1^2} e^{-zq_2^2} e^{zq_3^2}.
\end{aligned} \tag{6}$$

Then a large family of (minimally or weak) superintegrable Hamiltonians can be constructed through the following statement:

Proposition 1. (i) *The three-particle generators (5) fulfil the commutation rules (1) with respect to the canonical Poisson bracket*

$$\{f, g\} = \sum_{i=1}^3 \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i} \right). \tag{7}$$

(ii) *These generators Poisson commute with the three functions (6).*

(iii) *Any arbitrary function defined on (5), $\mathcal{H} = \mathcal{H}(J_-^{(3)}, J_+^{(3)}, J_3^{(3)})$ (but not on \mathcal{C}), provides a completely integrable Hamiltonian as either $\{\mathcal{C}^{(2)}, \mathcal{C}^{(3)}, \mathcal{H}\}$ or $\{\mathcal{C}_{(2)}, \mathcal{C}^{(3)}, \mathcal{H}\}$ are three functionally independent functions in involution.*

(iv) *The four functions $\{\mathcal{C}^{(2)}, \mathcal{C}_{(2)}, \mathcal{C}^{(3)}, \mathcal{H}\}$ are functionally independent.*

As a byproduct, we obtain superintegrable free Hamiltonians which determine the geodesic motion of a particle on certain 3D spaces through

$$\mathcal{H} = \frac{1}{2} J_+^{(3)} f(zJ_-^{(3)}), \tag{8}$$

where f is an arbitrary smooth function such that $\lim_{z \rightarrow 0} f(zJ_-^{(3)}) = 1$, so that $\lim_{z \rightarrow 0} \mathcal{H} = \frac{1}{2} \mathbf{p}^2$. By writing the Hamiltonian (8) as a free Lagrangian, the metric on the underlying 3D space can be deduced and its sectional curvatures turn out to be, in general, non-constant. In this way, a quantum deformation can be understood as the introduction of a variable curvature on the formerly flat Euclidean space in such a manner that the non-deformed limit $z \rightarrow 0$ can then be identified with the *flat contraction* providing the proper 3D Euclidean space. Let us illustrate these ideas by recalling two specific choices for \mathcal{H} which have recently been studied in [4, 5].

2.1 Spaces of non-constant curvature

The simplest Hamiltonian (8) arises by setting $f \equiv 1$: $\mathcal{H}_{\text{nc}} = \frac{1}{2} J_+^{(3)}$. This can be rewritten as the free Lagrangian

$$2\mathcal{T}_{\text{nc}} = \frac{zq_1^2}{\sinh zq_1^2} e^{-zq_2^2} e^{-zq_3^2} \dot{q}_1^2 + \frac{zq_2^2}{\sinh zq_2^2} e^{zq_1^2} e^{-zq_3^2} \dot{q}_2^2 + \frac{zq_3^2}{\sinh zq_3^2} e^{zq_1^2} e^{zq_2^2} \dot{q}_3^2, \tag{9}$$

which defines a geodesic flow on a 3D Riemannian space with a definite positive metric given by

$$ds_{\text{nc}}^2 = \frac{2zq_1^2}{\sinh zq_1^2} e^{-zq_2^2} e^{-zq_3^2} dq_1^2 + \frac{2zq_2^2}{\sinh zq_2^2} e^{zq_1^2} e^{-zq_3^2} dq_2^2 + \frac{2zq_3^2}{\sinh zq_3^2} e^{zq_1^2} e^{zq_2^2} dq_3^2. \quad (10)$$

The sectional curvatures K_{ij} in the planes 12, 13 and 23, and the scalar curvature K turn out to be

$$\begin{aligned} K_{12} &= \frac{z}{4} e^{-z\mathbf{q}^2} (1 + e^{2zq_3^2} - 2e^{2z\mathbf{q}^2}), \\ K_{13} &= \frac{z}{4} e^{-z\mathbf{q}^2} (2 - e^{2zq_3^2} + e^{2zq_2^2} e^{2zq_3^2} - 2e^{2z\mathbf{q}^2}), \\ K_{23} &= \frac{z}{4} e^{-z\mathbf{q}^2} (2 - e^{2zq_2^2} e^{2zq_3^2} - 2e^{2z\mathbf{q}^2}), \\ K &= 2(K_{12} + K_{13} + K_{23}) = -5z \sinh(z\mathbf{q}^2). \end{aligned}$$

Next we introduce new canonical coordinates (ρ, θ, ϕ) and the conjugated momenta $(p_\rho, p_\theta, p_\phi)$ (with respect to (7)) defined by [5]

$$\begin{aligned} \cosh^2(\lambda_1 \rho) &= e^{2z\mathbf{q}^2}, \\ \sinh^2(\lambda_1 \rho) \cos^2(\lambda_2 \theta) &= e^{2zq_1^2} e^{2zq_2^2} (e^{2zq_3^2} - 1), \\ \sinh^2(\lambda_1 \rho) \sin^2(\lambda_2 \theta) \cos^2 \phi &= e^{2zq_1^2} (e^{2zq_2^2} - 1), \\ \sinh^2(\lambda_1 \rho) \sin^2(\lambda_2 \theta) \sin^2 \phi &= e^{2zq_1^2} - 1, \end{aligned} \quad (11)$$

where $z = \lambda_1^2$ and $\lambda_2 \neq 0$ is an additional parameter which can be either a real or a pure imaginary number [4] and enables to deal with Riemannian and Lorentzian signatures. Thus the metric (10) is transformed into

$$ds_{\text{nc}}^2 = \frac{1}{\cosh(\lambda_1 \rho)} \left(d\rho^2 + \lambda_2^2 \frac{\sinh^2(\lambda_1 \rho)}{\lambda_1^2} \left(d\theta^2 + \frac{\sin^2(\lambda_2 \theta)}{\lambda_2^2} d\phi^2 \right) \right), \quad (12)$$

which is just the metric of the 3D Riemannian and relativistic spacetimes [6] written in geodesic polar coordinates multiplied by a global factor $e^{-zJ^{(3)}} \equiv 1/\cosh(\lambda_1 \rho)$. In the new coordinates the sectional and scalar curvatures read

$$K_{12} = K_{13} = -\frac{1}{2} \lambda_1^2 \frac{\sinh^2(\lambda_1 \rho)}{\cosh(\lambda_1 \rho)}, \quad K_{23} = \frac{1}{2} K_{12}, \quad K = -\frac{5}{2} \lambda_1^2 \frac{\sinh^2(\lambda_1 \rho)}{\cosh(\lambda_1 \rho)}.$$

Therefore, according to the pair (λ_1, λ_2) we have obtained analogues of the 3D spherical $(i, 1)$, hyperbolic $(1, 1)$, de Sitter $(1, i)$ and anti-de Sitter (i, i) spaces with *variable* radial sectional and scalar curvatures. These reduce to the flat Euclidean $(0, 1)$ and Minkowskian $(0, i)$ spaces under the limit $z \rightarrow 0$. The contraction $\lambda_2 = 0$, which is well defined in the metric (12), would lead to oscillating and expanding Newton–Hooke $(\lambda_1 = i, 1)$ spacetimes of non-constant curvature; again their limit $z \rightarrow 0$ would give the flat Galilean spacetime. Nevertheless we avoid this contraction since the metric is degenerate so that a *direct* relationship with a 3D Hamiltonian is lost.

The resulting superintegrable Hamiltonian on these six curved spaces with its *three* constants of motion in the latter phase space read

$$H_{\text{nc}} = \frac{1}{2} \cosh(\lambda_1 \rho) \left(p_\rho^2 + \frac{\lambda_1^2}{\lambda_2^2 \sinh^2(\lambda_1 \rho)} \left(p_\theta^2 + \frac{\lambda_2^2}{\sin^2(\lambda_2 \theta)} p_\phi^2 \right) \right), \quad (13)$$

$$C^{(2)} = p_\phi^2, \quad C_{(2)} = \left(\cos \phi p_\theta - \lambda_2 \frac{\sin \phi p_\phi}{\tan(\lambda_2 \theta)} \right)^2, \quad C^{(3)} = p_\theta^2 + \frac{\lambda_2^2 p_\phi^2}{\sin^2(\lambda_2 \theta)}, \quad (14)$$

where $H_{\text{nc}} = 2\mathcal{H}_{\text{nc}}$, $C^{(2)} = 4\mathcal{C}^{(2)}$, $C_{(2)} = 4\lambda_2^2 \mathcal{C}_{(2)}$ and $C^{(3)} = 4\lambda_2^2 \mathcal{C}^{(3)}$.

We remark that, in general, other choices for the Hamiltonian (8) (with $f \neq 1$) give rise to more complicated spaces of non-constant curvature, for which a clear geometrical interpretation, similar to the one above developed, remains as an open problem. However a very particular choice of the function \mathcal{H} leads to spaces of constant curvature.

2.2 Spaces of constant curvature

If we now consider the function $f = e^{zJ_-^{(3)}}$ in (8), we obtain a Hamiltonian $\mathcal{H}_{\text{cc}} = \frac{1}{2} J_+^{(3)} e^{zJ_-^{(3)}}$ endowed with an additional constant of motion $\mathcal{I}^{(2)}$ [2]:

$$\mathcal{I}^{(2)} = \frac{\sinh zq_1^2}{2zq_1^2} e^{zq_1^2} p_1^2, \quad (15)$$

which does not come from the coalgebra symmetry but it is a consequence of the Stäckel system [7] defined by \mathcal{H}_{cc} . Since $\mathcal{I}^{(2)}$ is functionally independent with respect to the three previous constants of the motion (6), \mathcal{H}_{cc} is a *maximally superintegrable* Hamiltonian with free Lagrangian and associated metric given, in terms of (9) and (10), by $\mathcal{T}_{\text{cc}} = \mathcal{T}_{\text{nc}} e^{-z\mathbf{q}^2}$ and $ds_{\text{cc}}^2 = ds_{\text{nc}}^2 e^{-z\mathbf{q}^2}$. Such a metric is of Riemannian type with constant sectional and scalar curvatures: $K_{ij} = z$ and $K = 6z$.

A more familiar expression for the metric and the associated spaces can be deduced by applying the change of coordinates (11) and next introducing a new radial coordinate r as $\cos(\lambda_1 r) = 1/\cosh(\lambda_1 \rho)$ [4]. Thus we find that ds_{cc} is transformed into a metric written in terms of geodesic polar (spherical) coordinates [6]:

$$ds_{\text{cc}}^2 = dr^2 + \lambda_2^2 \frac{\sin^2(\lambda_1 r)}{\lambda_1^2} \left(d\theta^2 + \frac{\sin^2(\lambda_2 \theta)}{\lambda_2^2} d\phi^2 \right). \quad (16)$$

According to the pair (λ_1, λ_2) (we take again the simplest values: $1, 0, i$), this metric covers well known classical spaces of constant curvature $z = \lambda_1^2$: the 3D spherical $(1, 1)$, Euclidean $(0, 1)$, hyperbolic $(i, 1)$, anti-de Sitter $(1, i)$, Minkowskian $(0, i)$, de Sitter (i, i) , oscillating Newton–Hooke $(1, 0)$, expanding Newton–Hooke $(i, 0)$ and Galilean $(0, 0)$ spaces; we shall avoid the non-relativistic spacetimes with $\lambda_2 = 0$ as the metric is degenerate. Recall that r is a radial (time-like) geodesic distance, θ is either an angle in the Riemannian spaces or a rapidity in the relativistic spacetimes ($\lambda_2 = i/c$ with c being the speed of light), while ϕ is an ordinary angle for the six spaces.

In this new phase space, the Hamiltonian, $H_{\text{cc}} = 2\mathcal{H}_{\text{cc}}$, reads

$$H_{\text{cc}} = \frac{1}{2} \left(p_r^2 + \frac{\lambda_1^2}{\lambda_2^2 \sin^2(\lambda_1 r)} \left(p_\theta^2 + \frac{\lambda_2^2}{\sin^2(\lambda_2 \theta)} p_\phi^2 \right) \right), \quad (17)$$

while its *four* constants of motion are $C^{(2)}, C_{(2)}, C^{(3)}$ given in (14) and

$$I^{(2)} = \left(\lambda_2 \sin(\lambda_2 \theta) \sin \phi p_r + \frac{\lambda_1 \cos(\lambda_2 \theta) \sin \phi}{\tan(\lambda_1 r)} p_\theta + \frac{\lambda_1 \lambda_2 \cos \phi}{\tan(\lambda_1 r) \sin(\lambda_2 \theta)} p_\phi \right)^2, \quad (18)$$

where $I^{(2)} = 4\lambda_2^2 \mathcal{I}^{(2)}$. We stress that all these results can alternatively be obtained by following a Lie group approach [6] instead of a quantum algebra one. Explicitly, the Hamiltonian (17) has a Poisson–Lie algebra symmetry determined by a subset of $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ graded contractions of $so(4)$, $so_{\kappa_1, \kappa_2}(4)$, where κ_i are two *real* contraction parameters; the six generators $J_{\mu\nu}$ ($\mu, \nu = 0, 1, 2, 3; \mu < \nu$) of $so_{\kappa_1, \kappa_2}(4)$ satisfy the following Poisson–Lie brackets:

$$\begin{aligned} \{J_{12}, J_{13}\} &= \kappa_2 J_{23}, & \{J_{12}, J_{23}\} &= -J_{13}, & \{J_{13}, J_{23}\} &= J_{12}, \\ \{J_{12}, J_{01}\} &= J_{02}, & \{J_{13}, J_{01}\} &= J_{03}, & \{J_{23}, J_{02}\} &= J_{03}, \\ \{J_{12}, J_{02}\} &= -\kappa_2 J_{01}, & \{J_{13}, J_{03}\} &= -\kappa_2 J_{01}, & \{J_{23}, J_{03}\} &= -J_{02}, \\ \{J_{01}, J_{02}\} &= \kappa_1 J_{12}, & \{J_{01}, J_{03}\} &= \kappa_1 J_{13}, & \{J_{02}, J_{03}\} &= \kappa_1 \kappa_2 J_{23}, \\ \{J_{01}, J_{23}\} &= 0, & \{J_{02}, J_{13}\} &= 0, & \{J_{03}, J_{12}\} &= 0. \end{aligned} \quad (19)$$

The parameters κ_i are related to the λ_i through $\kappa_1 \equiv z = \lambda_1^2$ and $\kappa_2 \equiv \lambda_2^2$. Consequently, the above six spaces of constant curvature has a deformed coalgebra symmetry, $(sl_z(2) \otimes sl_z(2) \otimes sl_z(2))_{\lambda_2}$, and also a Poisson–Lie algebra symmetry $so_{\kappa_1, \kappa_2}(4)$; the latter comprises $so(4)$ for the spherical, $iso(3)$ for the Euclidean, $so(3, 1)$ for the hyperbolic, $so(2, 2)$ for the anti-de Sitter, $iso(2, 1)$ for the Minkowskian, and $so(3, 1)$ for the de Sitter space.

In terms of the geodesic polar phase space and the parameters λ_i the symplectic realization of the generators $J_{\mu\nu}$ is given by [6]:

$$\begin{aligned} J_{01} &= \cos(\lambda_2 \theta) p_r - \frac{\lambda_1 \sin(\lambda_2 \theta)}{\lambda_2 \tan(\lambda_1 r)} p_\theta, & J_{23} &= p_\phi, \\ J_{02} &= \lambda_2 \sin(\lambda_2 \theta) \cos \phi p_r + \frac{\lambda_1 \cos(\lambda_2 \theta) \cos \phi}{\tan(\lambda_1 r)} p_\theta - \frac{\lambda_1 \lambda_2 \sin \phi}{\tan(\lambda_1 r) \sin(\lambda_2 \theta)} p_\phi, \\ J_{03} &= \lambda_2 \sin(\lambda_2 \theta) \sin \phi p_r + \frac{\lambda_1 \cos(\lambda_2 \theta) \sin \phi}{\tan(\lambda_1 r)} p_\theta + \frac{\lambda_1 \lambda_2 \cos \phi}{\tan(\lambda_1 r) \sin(\lambda_2 \theta)} p_\phi, \\ J_{12} &= \cos \phi p_\theta - \frac{\lambda_2 \sin \phi}{\tan(\lambda_2 \theta)} p_\phi, & J_{13} &= \sin \phi p_\theta + \frac{\lambda_2 \cos \phi}{\tan(\lambda_2 \theta)} p_\phi. \end{aligned} \quad (20)$$

Hence the four constants of the motion (14) and (18) as well as the free Hamiltonian (17) are related to the generators $J_{\mu\nu}$ through

$$\begin{aligned} C^{(2)} &= J_{23}^2, & C_{(2)} &= J_{12}^2, & C^{(3)} &= J_{12}^2 + J_{13}^2 + \lambda_2^2 J_{23}^2, & I^{(2)} &= J_{03}^2, \\ 2\lambda_2^2 H_{cc} &= \lambda_2^2 J_{01}^2 + J_{02}^2 + J_{03}^2 + \lambda_1^2 (J_{12}^2 + J_{13}^2 + \lambda_2^2 J_{23}^2), \end{aligned}$$

so that H_{cc} is just the quadratic Casimir of $so_{\kappa_1, \kappa_2}(4)$ associated to the Killing–Cartan form.

3 Kepler potentials

The results of proposition 1 allows one to construct many types of superintegrable potentials on 3D curved spaces through specific choices of the Hamiltonian function $\mathcal{H} = \mathcal{H}(J_-^{(3)}, J_+^{(3)}, J_3^{(3)})$ which could be momenta-dependent potentials, central ones, etc. (see [8] for the 2D case). In order to introduce a Kepler potential we consider the free Hamiltonian (8) as the kinetic energy and add a term $\mathcal{U}(zJ_-^{(3)})$ which is a smooth function such that $\lim_{z \rightarrow 0} \mathcal{U}(zJ_-^{(3)}) = -\gamma/\sqrt{\mathbf{q}^2}$ (γ is a real constant). Thus a family of Kepler potentials is defined by

$$\mathcal{H} = \frac{1}{2}J_+^{(3)} f(zJ_-^{(3)}) + \mathcal{U}(zJ_-^{(3)}), \quad (21)$$

which can be interpreted either as deformations of the Kepler potential on the flat Euclidean space ($\mathcal{H} \rightarrow \frac{1}{2}\mathbf{p}^2 - \gamma/\sqrt{\mathbf{q}^2}$ when $z \rightarrow 0$), or as Kepler-type potentials on 3D curved spaces. All the Hamiltonians contained within (21) are *superintegrable* sharing the same set of *three* constants of the motion (6).

We propose the following functions as the Hamiltonians containing a Kepler potential on the aforementioned spaces of variable ($\mathcal{H}_{\text{nc}}^{\text{SK}}$) and constant curvature ($\mathcal{H}_{\text{cc}}^{\text{MSK}}$):

$$\begin{aligned} \mathcal{H}_{\text{nc}}^{\text{SK}} &= \frac{1}{2}J_+ - \gamma \sqrt{\frac{2z}{e^{2zJ_-} - 1}} e^{2zJ_-}, \\ \mathcal{H}_{\text{cc}}^{\text{MSK}} &= \frac{1}{2}J_+ e^{zJ_-} - \gamma \sqrt{\frac{2z}{e^{2zJ_-} - 1}}. \end{aligned} \quad (22)$$

By firstly introducing in both Hamiltonians the symplectic realization (5) and secondly the new coordinates (ρ, θ, ϕ) (11) in $\mathcal{H}_{\text{nc}}^{\text{SK}}$ and (r, θ, ϕ) in $\mathcal{H}_{\text{cc}}^{\text{MSK}}$ we find that these read

$$\begin{aligned} H_{\text{nc}}^{\text{SK}} &= \frac{1}{2} \cosh(\lambda_1 \rho) \left(p_\rho^2 + \frac{\lambda_1^2}{\lambda_2^2 \sinh^2(\lambda_1 \rho)} \left(p_\theta^2 + \frac{\lambda_2^2 p_\phi^2}{\sin^2(\lambda_2 \theta)} \right) - \frac{2\lambda_1 k}{\tanh(\lambda_1 \rho)} \right), \\ H_{\text{cc}}^{\text{MSK}} &= \frac{1}{2} \left(p_r^2 + \frac{\lambda_1^2}{\lambda_2^2 \sin^2(\lambda_1 r)} \left(p_\theta^2 + \frac{\lambda_2^2 p_\phi^2}{\sin^2(\lambda_2 \theta)} \right) \right) - \frac{\lambda_1 k}{\tan(\lambda_1 r)}, \end{aligned} \quad (23)$$

where $H_{\text{nc}}^{\text{SK}} = 2\mathcal{H}_{\text{nc}}^{\text{SK}}$, $H_{\text{cc}}^{\text{MSK}} = 2\mathcal{H}_{\text{cc}}^{\text{MSK}}$ and $k = 2\sqrt{2}\gamma$. Hence $H_{\text{cc}}^{\text{MSK}}$ contains the proper Kepler potential, either $-k/\tan r$, $-k/r$ or $-k/\tanh r$, on six spaces of constant curvature [1, 6, 9, 10, 11, 12, 13, 14, 15, 16, 17], while $H_{\text{nc}}^{\text{SK}}$ provides a generalization to their variable curvature counterpart.

The constants of the motion $C^{(2)}$ and $C^{(3)}$, which ensure complete integrability, together with the Hamiltonian allows us to write three equations, each of them depending on a canonical pair:

$$\begin{aligned} C^{(2)}(\phi, p_\phi) &= p_\phi^2, \quad C^{(3)}(\theta, p_\theta) = p_\theta^2 + \frac{\lambda_2^2}{\sin^2(\lambda_2 \theta)} C^{(2)}, \\ H_{\text{nc}}^{\text{SK}}(\rho, p_\rho) &= \frac{1}{2} \cosh(\lambda_1 \rho) \left(p_\rho^2 + \frac{\lambda_1^2}{\lambda_2^2 \sinh^2(\lambda_1 \rho)} C^{(3)} - \frac{2\lambda_1 k}{\tanh(\lambda_1 \rho)} \right), \\ H_{\text{cc}}^{\text{MSK}}(r, p_r) &= \frac{1}{2} p_r^2 + \frac{\lambda_1^2}{2\lambda_2^2 \sin^2(\lambda_1 r)} C^{(3)} - \frac{\lambda_1 k}{\tan(\lambda_1 r)}. \end{aligned}$$

Therefore both Hamiltonians are separable and reduced to a 1D radial system. Their integration would lead to the solutions of such Kepler potentials; for $H_{\text{nc}}^{\text{SK}}$ one would find very cumbersome elliptic functions.

The constant of the motion $I^{(2)}$ (18) (coming from the Stäckel system associated to free motion) is lost for the Hamiltonian $H_{\text{cc}}^{\text{MSK}}$ defined on the spaces of constant curvature. Nevertheless, *maximal superintegrability* for $H_{\text{cc}}^{\text{MSK}}$ is preserved since there is an additional constant of the motion, a component of the Laplace–Runge–Lenz vector, which does not come from the coalgebra approach (as $I^{(2)}$) but this is provided by the Poisson–Lie symmetry; this is a consequence of the particular potential expression we have considered. This property is summed up as follows [6].

Proposition 2. *Let the following three functions written in terms of the generators $J_{\mu\nu}$ (20) of $so_{\kappa_1, \kappa_2}(4)$:*

$$\begin{aligned} L_1 &= -J_{02}J_{12} - J_{03}J_{13} + k \lambda_2^2 \cos(\lambda_2\theta), \\ L_2 &= J_{01}J_{12} - J_{03}J_{23} + k \lambda_2 \sin(\lambda_2\theta) \cos \phi, \\ L_3 &= J_{01}J_{13} + J_{02}J_{23} + k \lambda_2 \sin(\lambda_2\theta) \sin \phi. \end{aligned} \tag{24}$$

(i) *The three L_i Poisson commute with $H_{\text{cc}}^{\text{MSK}}$ and these are the components of the Laplace–Runge–Lenz vector on the 3D Riemannian (λ_2 real) and relativistic (λ_2 imaginary) spaces of constant curvature.*

(ii) *Each set $\{\mathcal{C}^{(2)} = J_{23}^2, L_1, H_{\text{cc}}^{\text{MSK}}\}$, $\{\mathcal{C}^{(3)} - \mathcal{C}_{(2)} - \lambda_2^2 \mathcal{C}^{(2)} = J_{13}^2, L_2, H_{\text{cc}}^{\text{MSK}}\}$ and $\{\mathcal{C}_{(2)} = J_{12}^2, L_3, H_{\text{cc}}^{\text{MSK}}\}$ is formed by three functionally independent functions in involution.*

(iii) *The four functions $\{\mathcal{C}^{(2)}, \mathcal{C}_{(2)}, \mathcal{C}^{(3)}, H_{\text{cc}}^{\text{MSK}}\}$ together with any of the components L_i are functionally independent.*

To end with, we present some properties satisfied by the components L_i . The three generators $\{J_{12}, J_{13}, J_{23}\}$ span a rotation subalgebra $so(3)$ for the three Riemannian spaces and a Lorentz one $so(2, 1)$ for the three relativistic spacetimes (see (19)). According to the signature of the metric, determined by λ_2 , the following Poisson–Lie brackets show that the three components L_i are transformed either as a vector under rotations when λ_2 is real, or as a vector under Lorentz transformations when λ_2 is imaginary:

$$\begin{aligned} \{J_{12}, L_1\} &= \lambda_2^2 L_2, & \{J_{12}, L_2\} &= -L_1, & \{J_{12}, L_3\} &= 0, \\ \{J_{13}, L_1\} &= \lambda_2^2 L_3, & \{J_{13}, L_2\} &= 0, & \{J_{13}, L_3\} &= -L_1, \\ \{J_{23}, L_1\} &= 0, & \{J_{23}, L_2\} &= L_3, & \{J_{23}, L_3\} &= -L_2. \end{aligned} \tag{25}$$

The commutation rules among the components L_i are found to be

$$\{L_i, L_j\} = 2 (\lambda_1^2 C^{(3)} - \lambda_2^2 H_{\text{cc}}^{\text{MSK}}) J_{ij}, \quad i < j, \quad i, j = 1, 2, 3. \tag{26}$$

Next we scale the components as $\mathcal{P}_1 = L_1/\lambda_2$, $\mathcal{P}_2 = \lambda_2 L_2$ and $\mathcal{P}_3 = \lambda_2 L_3$, and write the

Poisson brackets for J_{ij}, \mathcal{P}_i ($i, j = 1, 2, 3$):

$$\begin{aligned}
\{J_{12}, J_{13}\} &= \lambda_2^2 J_{23}, & \{J_{12}, J_{23}\} &= -J_{13}, & \{J_{13}, J_{23}\} &= J_{12}, \\
\{J_{12}, \mathcal{P}_1\} &= \mathcal{P}_2, & \{J_{13}, \mathcal{P}_1\} &= \mathcal{P}_3, & \{J_{23}, \mathcal{P}_2\} &= \mathcal{P}_3, \\
\{J_{12}, \mathcal{P}_2\} &= -\lambda_2^2 \mathcal{P}_1, & \{J_{13}, \mathcal{P}_3\} &= -\lambda_2^2 \mathcal{P}_1, & \{J_{23}, \mathcal{P}_3\} &= -\mathcal{P}_2, \\
\{\mathcal{P}_1, \mathcal{P}_2\} &= \mu J_{12}, & \{\mathcal{P}_1, \mathcal{P}_3\} &= \mu J_{13}, & \{\mathcal{P}_2, \mathcal{P}_3\} &= \mu \lambda_2^2 J_{23}, \\
\{\mathcal{P}_1, J_{23}\} &= 0, & \{\mathcal{P}_2, J_{13}\} &= 0, & \{\mathcal{P}_3, J_{12}\} &= 0,
\end{aligned} \tag{27}$$

where $\mu = 2(\lambda_1^2 C^{(3)} - \lambda_2^2 H_{cc}^{MSK})$ is a quadratic function on the three J_{ij} through $C^{(3)}$ (note that $C^{(3)}$ does not Poisson commute with \mathcal{P}_i). Hence when comparing (27) with the Poisson–Lie algebra $so_{\kappa_1, \kappa_2}(4)$ (19) we find that the former can be seen as a cubic generalization of the latter under the identification $\kappa_2 \equiv \lambda_2^2$ and the replacement of the translations $J_{0i} \rightarrow \mathcal{P}_i$. The cubic Poisson brackets are those involving two \mathcal{P}_i and the former contraction/deformation parameter $\kappa_1 \equiv \lambda_1^2$ (the constant curvature of the space) has been replaced by the function μ . Notice that each set of three generators $J_{ij}, \mathcal{P}_i, \mathcal{P}_j$ (for i, j fixed) define a cubic Higgs algebra [10].

More details on this construction as well its generalization to arbitrary dimension will be presented elsewhere.

Acknowledgements

This work was partially supported by the Ministerio de Educación y Ciencia (Spain, Project FIS2004-07913) and by the Junta de Castilla y León (Spain, Project VA013C05).

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