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On the concept of force in Lagrangian mechanics

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Abstract

In this note we define a "Lagrangian force" in geometric terms; then, the Euler-Lagrange equation is interpreted as a Newton-like equation "mass×acceleration = force".

1 Introduction

Strictly speaking, Euler-Lagrange equations are not differential equations. They could be seen better as a formula expressing the operations to perform in order to obtain the vector field whose integral curves are the critical points of the action functional $\int L dt$ constructed with a Lagrangian L. In this sense, they are linear equations for the second derivatives of the components of these curves. This point of view is clearly stated in the traditional coordinate-dependent formulation of Lagrangian mechanics: the Euler-Lagrange equations $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$ give rise to the following linear system for the accelerations $a = (\ddot{q}_i)$

$$Ma = Q, \tag{1}$$

where $M = \left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right)$ is the Hessian of the Lagrangian with respect to the velocities and, in the autonomous case, $Q = \left(-\frac{\partial^2 L}{\partial q_i \partial \dot{q}_j}\dot{q}_j + \frac{\partial L}{\partial q_i}\right)$. This term Q could be called the "(Lagrangian) force" and (1) interpreted as a Newton-like equation "mass×acceleration = force". For natural mechanical systems $L = T - V$, the force Q includes the "true" (physical) generalized force $-\frac{\partial V}{\partial q}$ and some ficticious (non-inertial) forces $\frac{\partial T}{\partial q} - \frac{\partial^2 T}{\partial q \partial \dot{q}}\dot{q}$.

When the Lagrangian L is regular, the equations (1) can be solved for the accelerations $a = M^{-1}Q$ and the trajectories $q(t)$ are the solutions of the system of second order differential equations $\ddot{q}_i = a_i(q, \dot{q})$.

On the other hand, it is well known that either aspect of the Lagrangian formalism has its own differential geometric interpretation. As it is well established [5, 2], the geometric form of the Euler-Lagrange equations for an autonomous Lagrangian $L \in C^\infty(TM)$ is

$$i(\Gamma)d\vartheta_L = -dE_L, \tag{2}$$

where $E_L = \Delta(L) - L$ is the *energy* function. As before, this is in fact a linear equation on the manifold TM (the space of states) for the vector field Γ whose integral curves will be the dynamical trajectories. In local coordinates $(t, q_i, \dot{q}_i)$ (2) is nothing but (1). Now the point is the following: is there a direct geometric version of the Euler-Lagrange equations (2) analogous to (1) from which the geometric object representing the force Q is obtained?

The purpose of this note is to answer to this question. We will get the desired geometric equation using the affine structures related to the 1– and 2–jet bundles associated to the variational problem. In particular, we will see that the main geometric objects entering in the dynamical equation can be seen as affine morphisms between affine bundles and consequently the equation itself can be written explicitly as a linear equation.

2 Geometric structures

First of all, we will review the geometric structures appearing in the Lagrangian formulation of Classical Mechanics and explain the subsequent notation. Let M be the configuration space of a m–dimensional autonomous Lagrangian system, and $\tau_M : TM \to M$ the tangent bundle projection. The basic constructions we will make use of from now on are those of jet bundles of curves of M [4]. First order Lagrangian systems are described by a single function $L \in C^\infty(TM)$ (the "Lagrangian") encoding all the dynamical properties (kinematics, masses, active forces, etc.) of the system; as the dynamical equations are of the second order type, the spaces we need are the 1–jet bundle, which coincides with the tangent bundle TM, and the 2–jet bundle T^2M; both TM and T^2M are bundles over M, and the elements of TM are known as "velocities" or "(dynamical) states", while those of T^2M are the "accelerations". There is a natural projection $\mu : T^2M \to TM$, $j^2_x\sigma \mapsto j^1_x\sigma$, $j^k_x\sigma$ denoting the k–jet, $k = 1, 2$, of the curve $\sigma : \mathbb{R} \to M$ at the point $\sigma(0) = x$. Every curve σ can be prolonged to curves $j^k\sigma : \mathbb{R} \to T^kM$ given by $j^k\sigma(t) = j^k_{\sigma(t)}\sigma_t$, σ_t being the curve $\sigma_t(s) = \sigma(t + s)$ which starts from the point $\sigma(t)$.

There exists a remarkable structure given by the "total time derivative" operators $\mathbf{T}^0 = \mathrm{id}_{TM}$ and $\mathbf{T} : T^2M \to TTM$ defined by the rule

$$\mathbf{T} \circ j^2\sigma = (j^1\sigma)_* \circ \frac{d}{dt},$$

where $\frac{d}{dt} \in \mathfrak{X}(\mathbb{R})$ is the unique vector field in $\mathbb{R}$, which is endowed with a volume form dt (the time measure), such that $i(\frac{d}{dt})dt = 1$. These operators $\mathbf{T}^0$ and $\mathbf{T}$ are in fact vector fields along the projections τ_M and μ, respectively, and are (μ, τ_M)–related, i.e., $\tau_{M*} \circ \mathbf{T} = \mathbf{T}^0 \circ \mu$. As it is well known [1], each of these operators induces two derivations along the corresponding maps: the contraction $i_\mathbf{T}$ and the Lie derivative $d_\mathbf{T}$, and same for $\mathbf{T}^0$. In particular, given a function $f \in C^\infty(M)$ (resp., $f \in C^\infty(TM)$), its total time derivative is a function $d_{\mathbf{T}^0} f \in C^\infty(TM)$ (resp., $d_\mathbf{T} f \in C^\infty(T^2M)$).

Let us have a look at these constructions in local coordinates. Let (q^i) be a local system of coordinates in M; let (q^i, v^i) and (q^i, v^i, a^i) be the corresponding fibered coordinates in TM and T^2M. If the curve σ locally is expressed as $\sigma(t) = (\sigma^i(t))$, $\sigma(0) = x$, the k–jets are $j^k_x\sigma = (\sigma^i(t), \frac{d\sigma^i}{dt}, \ldots, \frac{d^k\sigma^i}{dt^k})$. On the other hand, the total time derivative operators are

$$\mathbf{T}^0(q, v) = v^i \frac{\partial}{\partial q^i},$$
$$\mathbf{T}(q, v, a) = v^i \frac{\partial}{\partial q^i} + a^i \frac{\partial}{\partial v^i},$$

while the total time derivative of a function $f \in C^\infty(M)$ (resp., $f \in C^\infty(TM)$) is $d_{\mathbf{T}^0} f(q, v) = \frac{\partial f}{\partial q^i} v^i$ (resp., $d_{\mathbf{T}} f(q, v, a) = \frac{\partial f}{\partial v^i} a^i + \frac{\partial f}{\partial q^i} v^i$).

For our treatment of Lagrangian dynamics the affine structure of the fibre bundle μ is of great importance: it is an affine bundle over the vector bundle $\ker \tau_{M*}$ (also denoted as $V(\tau_M)$) of the τ_M–vertical tangent vectors of TM. The τ_M-vertical vector which corresponds to a couple of accelerations $w, w' \in \mu^{-1}(z)$ is the vector $\mathbf{T}(w)-\mathbf{T}(w') \in V_z(\tau_M)$; it will be denoted by $w - w'$. By means of the affine structure and the vertical lift $\xi^{\mathrm{v}} : TM \times_M TM \to V(\tau_M)$ [5, 2] it is posible to associate to every function $f \in C^\infty(TM)$ three important affine morphisms between affine bundles: its total time derivative, its Hessian and its Euler-Lagrange 1-form.

1. *The total time derivative.* Using the general fact that a function of the total space A of a fibre bundle $\pi : A \to M$ is equivalent to a fibre bundle morphism between π and the trivial vector bundle $M \times \mathbb{R} \to M$, the total time derivative of a function $f \in C^\infty(TM)$ can be represented as a fibre bundle morphism

$$\begin{tikzcd} T^2M \arrow[rr, "d_{\mathbf{T}} f"] \arrow[dr, "\mu"'] & & TM \times \mathbb{R} \arrow[dl, "\mathrm{pr}_1"] \\ & TM & \end{tikzcd}$$

Its linear part $[d_{\mathbf{T}} f] : V(\tau_M) \to TM \times \mathbb{R}$ is nothing but the vertical differential of f, i.e., the differential df restricted to the subbundle $V(\tau_M)$: in fact, for two accelerations w, w' over one and the same state $z \in TM$ we have

$$d_{\mathbf{T}} f(w) - d_{\mathbf{T}} f(w') = \langle df(z), \mathbf{T}(w) - \mathbf{T}(w')\rangle = \langle df(z), w - w'\rangle.$$

In local coordinates essentially we have $[d_{\mathbf{T}} f] \cdot (X^j \frac{\partial}{\partial v^j}) = X^j \frac{\partial f}{\partial v^j}$.

2. *The Hessian.* The concept of the fibre derivative of a morphism of affine bundles [6] is another useful geometric tool in our constructions. Given a function $f \in C^\infty(TM)$, its *fibre derivative* is a morphism $\mathcal{F}f : TM \to T^*M$ given by $\mathcal{F}f(z) = Df_{\tau_M(z)}(z)$, where f_x is the restriction of f to T_xM and the symbol D stands for the derivative of an application between linear spaces. The second fibre derivative is a linear morphism $\mathcal{F}^2 f$:

$TM \to \mathcal{S}^2(TM)$ from TM to the bilinear symmetric functions of TM given by $\mathcal{F}^2 f(z) = D^2 f_{\tau_M(z)}(z)$. By means of the vertical lift, we extend this morphism to a bundle morphism $\mathcal{H}f : TM \to \mathcal{S}^2(V(\tau_M))$ according to the rule

$$\mathcal{H}f(z) \cdot (\xi^{\mathrm{v}}(z,v), \xi^{\mathrm{v}}(z,v')) = \mathcal{F}^2 f(z) \cdot (v, v'), \quad v, v' \in T_{\tau_M(z)}M.$$

In this sense, the second fibre derivative of a function f is usually called the *Hessian map of f*. The induced linear morphism between the vector bundle $V(\tau_M)$ and its dual vector bundle $V^*(\tau_M)$ (the bundle of τ_M-semibasic 1-forms), will be denoted by $\widehat{\mathcal{H}f}$ and the following diagram holds:

$$\begin{tikzcd} V(\tau_M) \arrow[rr, "\widehat{\mathcal{H}f}"] \arrow[dr] & & V^*(\tau_M) \arrow[dl] \\ & TM & \end{tikzcd} \tag{3}$$

This is the interpretation of the second fibre derivative we need in the following, and it can be proved [6] that $\ker(\mathcal{F}f)_* = \ker \widehat{\mathcal{H}f}$, a property saying that $\mathcal{F}f$ is a (local) diffeomorphism at a point $z \in TM$ if and only if $\widehat{\mathcal{H}f}_{|V_z(\tau_M)}$ is a linear isomorphism. When this condition holds, the function f is said to be *regular* (or *hyper-regular*, for a global diffeomorphism). Consequently, the vertical bundle $V(\tau_M)$ will be endowed with a scalar product structure $\mathcal{H}f$; when, furthermore, the Hessian map is positive, the scalar product in $V(\tau_M)$ is Euclidean.

In local coordinates the expressions for the fibre derivatives are

$$\mathcal{F}f(q,v) = \frac{\partial f}{\partial v^i} dq^i,$$
$$\widehat{\mathcal{H}f} \cdot (X^i \frac{\partial}{\partial v^i}) = X^i \frac{\partial^2 f}{\partial v^i \partial v^j} dq^j,$$

The regularity condition will be $\det\left(\frac{\partial^2 f}{\partial v^i \partial v^j}\right) \neq 0$.

3. The Euler-Lagrange 1-form. We define a 1–form on the space of accelerations, known as the *Euler–Lagrange form* δf, as [1]

$$\delta f = d_{\mathbf{T}}\theta_f - \mu^* df \in \bigwedge^1(T^2 M), \tag{4}$$

where $\theta_f = df \circ S \in \bigwedge^1(TM)$ is a τ_M–semibasic form, i.e., $\theta_f \in \mathrm{Sec}(V^*(\tau_M))$, defined by means of the almost-tangent structure (or vertical endomorphism) S of the tangent bundle $T(TM)$. Note that the 1–form along τ_M equivalent to θ_f is nothing but the fibre derivative of f, and also that δf can be expressed in terms of the "energy" $E_f = \Delta(f) - f$, where $\Delta \in \mathfrak{X}(TM)$ is the vector field of dilations along the fibres of TM (the Liouville vector field) [5, 2]:

$$\delta f = i_{\mathbf{T}} d\theta_f + \mu^* dE_f.$$

As the Euler-Lagrange form δf is $\tau \circ \mu$–semibasic, it is equivalent to a 1-form along $\tau \circ \mu$, i.e., to a bundle morphism $\delta f^{\vee} : T^2M \to T^*M$; using the vertical lift again, we can extend δf, by duality, to an equivalent morphism $\widetilde{\delta f}$ from T^2M to the space of τ_M–semibasic 1-forms $V^*(\tau_M)$:

$$\begin{tikzcd} T^2M \arrow[rr, "\widetilde{\delta f}"] \arrow[dr, "\mu"'] & & V^*(\tau_M) \arrow[dl] \\ & TM & \end{tikzcd} , \tag{5}$$

the relation between $\widetilde{\delta f}$ and $\delta f^{\vee}$ being

$$\langle \widetilde{\delta f}(w), \xi^{\mathrm{v}}(z,v)\rangle = \langle \delta f^{\vee}(w), v\rangle, \quad w \in \mu^{-1}(z),\ v \in T_{\tau_M(z)}M. \tag{6}$$

It is this interpretation (5) of the Euler-Lagrange 1-form we will need later. The coordinate representation of the Euler-Lagrange form is

$$\delta f = \left[d_{\mathbf{T}} \left(\frac{\partial f}{\partial v^j} \right) - \mu^* \frac{\partial f}{\partial q^j} \right] dq^j.$$

The main property we want to point out is the following.

Proposition 1 *The Euler–Lagrange 1-form $\widetilde{\delta f}$ (5) is an affine morphism of affine bundles whose linear part is the Hessian map of f.*

Dem.- Let us consider two accelerations $w, w' \in \mu^{-1}(z)$; thus $\widetilde{\delta f}(w), \widetilde{\delta f}(w') \in V_z^*(\tau_M)$ and for a vector $v \in T_zM$ we have, using (6), $\langle \widetilde{\delta f}(w) - \widetilde{\delta f}(w'), \xi^{\mathrm{v}}(z,v)\rangle = \langle \delta f^{\vee}(w) - \delta f^{\vee}(w'), v\rangle$. Now, picking two vectors $Y \in T_wT^2M$ and $Y' \in T_{w'}T^2M$ such that $(\tau_M \circ \mu)_{*w} \cdot Y = (\tau_M \circ \mu)_{*w'} \cdot Y' = v$, a direct calculation, using the properties of all the differential forms involved, gives

$$\langle \widetilde{\delta f}(w) - \widetilde{\delta f}(w'), \xi^{\mathrm{v}}(z,v)\rangle = d\theta_f(z) \cdot (\mathbf{T}(w) - \mathbf{T}(w'), \mu_{*w} \cdot Y)\,. \tag{7}$$

On the other hand, a direct calculation, this time using the (local) flow of the vertical lift $X^{\mathrm{v}} \in \mathfrak{X}(TM)$ of a vector field $X \in \mathfrak{X}(M)$, shows that

$$\begin{aligned} d\theta_f(z) \cdot (X^{\mathrm{v}}(z), V) &= \langle \pounds_{X^{\mathrm{v}}} \theta_f(z), V\rangle \\ &= \mathcal{F}^2 f(z) \cdot (X(\tau_M(z)), \tau_{M*z} \cdot V) \\ &= \mathcal{H}f(z) \cdot (X^{\mathrm{v}}(z), \xi^{\mathrm{v}}(z, \tau_{M*z} \cdot V)); \end{aligned}$$

applying this result to the previous partial one (7) we finally arrive at

$$\langle \widetilde{\delta f}(w) - \widetilde{\delta f}(w'), \xi^{\mathrm{v}}(z,v)\rangle = \mathcal{H}f(z) \cdot (w - w', \xi^{\mathrm{v}}(z,v)), \qquad \forall v \in T_zM,$$

that is

$$\widetilde{\delta f}(w) - \widetilde{\delta f}(w') = \widehat{\mathcal{H}f} \cdot (w - w'),$$

$\forall w, w' \in T^2M$ such that $\mu(w) = \mu(w')$.

3 The equation of motion and the Lagrangian force

Now let us study the motion of a first order autonomous Lagrangian system. There are two alternative but equivalent geometrical interpretations of the Euler-Lagrange equations of the variational problem with Lagrangian L [3]: the extremals can be regarded either as the integral curves of a Second Order Differential Equation $\Gamma \in \mathfrak{X}(TM)$ solution of the equation (2) $i(\Gamma)d\theta_L = -dE_L$ or as those of a section $\gamma \in \mathrm{Sec}(\mu)$ such that $\gamma^*\delta L = 0$, that is $\delta L^\vee \circ \gamma = 0$ or

$$\widetilde{\delta L} \circ \gamma = 0; \tag{8}$$

thus, the dynamical equation is interpreted as a linear equation in affine bundles, the point of view suitable in our constructions. The relation between γ and Γ is given by [1, 3]

$$\Gamma = \mathbf{T} \circ \gamma; \tag{9}$$

in local coordinates, if $\Gamma(q,v) = v^j \frac{\partial}{\partial q^j} + \Gamma^j(q,v)\frac{\partial}{\partial v^j}$, then the equivalent section reads $\gamma(q,v) = (q^j, v^j, \Gamma^j(q,v))$.

For regular Lagrangians, i.e., Lagrangians L such that $\widehat{\mathcal{H}L}$ is a (local) isomorphism, it is well established that the solution is unique, that is, there is a unique (local) SODE $\gamma \in \mathrm{Sec}(\mu)$ solution of the linear equation (8). This is the motion obeying the Hamilton principle with L as Lagrangian, and is completely determined by L.

Using the affine structures involved, we can write the equation of motion (8) as a linear equation for the deviation $\gamma - \gamma_0 \in \mathrm{Sec}(V(\tau_M))$ of the actual motion from another one $\gamma_0 \in \mathrm{Sec}(\mu)$ taken as "the reference motion":

$$\widehat{\mathcal{H}L} \circ (\gamma - \gamma_0) = Q, \tag{10}$$

where

$$Q = -\widetilde{\delta L} \circ \gamma_0 \in \mathrm{Sec}(V^*(\tau_M)) \tag{11}$$

is the "(Lagrangian) force". Thus, the equation of motion (10) can be interpreted as a Newton-type equation. The force Q is a τ_M-semibasic 1-form representing the influences which deviate the motion γ from the reference γ_0.

Now the dynamical problem is to look for the sections $\gamma - \gamma_0 \in \mathrm{Sec}(V(\tau_M))$ satisfying the linear equation (10); we can visualize it in the following commutative diagram:

$$\begin{tikzcd} V(\tau_M) \arrow[rr, "\widehat{\mathcal{H}L}"] \arrow[dr, shift left] & & V^*(\tau_M) \arrow[dl, shift left] \\ & TM \arrow[ul, shift left, "\gamma-\gamma_0"] \arrow[ur, shift left, "Q"] & \end{tikzcd} \tag{12}$$

The equation of motion (10) immediately provides the unique solution $\gamma - \gamma_0 = \widehat{\mathcal{H}L}^{-1} Q$.

The reference motion is arbitrary and has no special physical meaning; however, if we took $\gamma_0(q, v) = (q, v; 0)$ the components of the τ_M-semibasic 1-form Q representing the Lagrangian force (11) are the ones pointed out in the introduction. It is clear that this choice has sense only locally; in general, there is no vector structure in the space of accelerations (and in the physical space itself). For natural systems $L = T - V$, the true or inertial generalized force is the basic 1-form $-\tau_M^* dV$ and consequently $-\widetilde{\delta L} \circ 0 + \tau_M^* dV = -\widetilde{\delta T} \circ 0$ is the non-inertial force.

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