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Lie Algebroid exterior algebra in gauge field theories

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Abstract

The relation between Lie algebroid and the BV formalism is revisited. The equivalence between the Lie algebroid structure and it exterior algebra is the basis for a set of examples leading to a Lie algebroid structure in a infinite dimensional setting.

1 Introduction

During the last fifty years Lie groups and Lie algebras played a relevant role in the development of physical theories [2]-[5]. However the generalization of such concepts, Lie groupoids and Lie algebroids, have only been incorporated in the physics literature during the very recent years. Here we will see that the concept of Lie algebroid is a natural structure in the BRST formulation of gauge theories (for details on the BRST construction see *e.g* [6], [7]). Specifically we will use the fact that the Lie algebroid structure is in a one to one relation with an exterior differential algebra structure (which can be understood in term of *homological vector fields* on a supermanifold see e.g. [8]). In particular the BRST operator for the Yang-Mills kind theories is a special case of a Lie algebroid on an infinite dimensional setting.

2 Lie algebroids

Let us first recall the definition and some properties of Lie algebroids. The concept of Lie algebroid, which was introduced by Pradines [9], not only generalizes the concept of Lie algebra but also that of tangent bundle of a manifold B. We recall that such tangent bundle, $\tau : TB \to B$ is a vector bundle in which the set of its sections, the vector fields, $\Gamma(\tau) = \mathfrak{X}(B)$, is endowed with a Lie algebra structure. Moreover, the sections of the bundle act as derivations on the associative and commutative algebra of functions in the base manifold B. Both properties, together with a compatibility condition, are the essential ingredients of a Lie algebroid structure: given a function $\varphi \in C^\infty(B)$ and two sections $X, Y \in \Gamma(\tau)$, the following relation holds:

$$[X, \varphi\, Y] = \varphi\, [X, Y] + (X\varphi)\, Y \ .$$

Two other properties which will be also generalized to the case of Lie algebroids are that there exists a (regular) Poisson structure on the dual bundle, for example, the cotangent bundle T^*B, and that there is a graded exterior differential operator which is a derivation of degree one in the set of forms, $d : \Omega^r(B) \to \Omega^{r+1}(B)$, such that $d^2 = 0$. Here $\Omega^r(B)$ denotes $\Omega^r(B) = \Gamma(T^*B \wedge \overset{r}{\cdots} \wedge T^*B)$.

Definition 2.1. *A Lie algebroid with base B is a vector bundle $\tau_E : E \to B$, together with a Lie algebra structure in the space of its sections given by a Lie product $[\cdot,\cdot]_E$, and a vector bundle map over the identity in the base, called anchor, $\rho : E \to TB$, inducing a map between the corresponding spaces of sections, to be denoted with the same name and symbol, such that: .- For any pair of sections for τ_E, X, Y, and each differentiable function φ defined in B,*

$$[X, \varphi Y]_E = \varphi [X, Y]_E + (\rho(X)\varphi)Y \ .$$

Where ρ is a Lie algebra homomorphism. Let $\{x^i \mid i = 1, \ldots, n\}$ be local coordinates in a chart on an open set $U \subset B$, and let $\{e_\alpha \mid \alpha = 1, \ldots, r\}$ be a basis of local sections of the bundle $U_E = \tau_E^{-1}(U) \to B$. Each local section V_U is written $V = y^\alpha e_\alpha$. The local coordinates of $p \in U_E$ are $p = (x^i, y^\alpha)$.

The local expressions for the Lie product and the anchor map are (summation on repeated indices is understood):

$$[e_\alpha, e_\beta]_E = C_{\alpha\beta}{}^\gamma \, e_\gamma \, , \qquad\qquad \rho(e_\alpha) = \rho^i{}_\alpha \, \frac{\partial}{\partial x^i} \, , \tag{2.1}$$

where $\alpha, \beta, \gamma = 1, \ldots, r$ and $i = 1, \ldots, n$. The functions $C_{\alpha\beta}{}^\gamma \in C^\infty(U)$ and $\rho^i{}_\alpha \in C^\infty(U)$ are called structure functions of the Lie algebroid. The conditions for ρ to be a Lie algebra homomorphism are

$$\sum_{\mathrm{cycl}(\alpha,\beta,\gamma)} \left(\rho^i{}_\alpha \, \frac{\partial C_{\beta\gamma}{}^\mu}{\partial x^i} + C_{\alpha\nu}{}^\mu \, C_{\beta\gamma}{}^\nu \right) = 0 \ , \tag{2.2}$$

and the compatibility conditions between ρ and $[\cdot,\cdot]$ are

$$\rho^j{}_\alpha \, \frac{\partial \rho^i{}_\beta}{\partial x^j} - \rho^j{}_\beta \, \frac{\partial \rho^i{}_\alpha}{\partial x^j} = \rho^i{}_\gamma \, C_{\alpha\beta}{}^\gamma \ . \tag{2.3}$$

These equations are called structure equations.

Some examples of the Lie algebroid structure are:

Example 1. *A Lie algebra: consider a finite dimensional real Lie algebra $\mathfrak{g}$ as a vector bundle over a single point. The sections are the elements of the algebra $\mathfrak{g}$, the Lie product is that of $\mathfrak{g}$ and the anchor map is identically zero. In this case the expressions* (2.1) *take the following form:*

$$[e_\alpha, e_\beta]_E = c_{\alpha\beta}{}^\gamma \, e_\gamma \, , \quad c_{\alpha\beta}{}^\gamma \in \mathbb{R} \, , \tag{2.4}$$

$$\rho\,(e_\alpha) = 0 \, , \ \ \alpha, \beta = 1, \ldots, r \ . \tag{2.5}$$

where the structure functions $C_{\alpha\beta}{}^{\gamma}$ are the structure constants $c_{\alpha\beta}{}^{\gamma}$ of $\mathfrak{g}$ and $a^i{}_{\alpha} \equiv 0$.

Example 2. *The tangent bundle $\tau_B : TB \to B$, with anchor map $\rho = \mathbb{I}_{TB}$, the identity map in TB, and the conmutator of vector fields as Lie bracket $[\cdot,\cdot]$. In terms of the usual coordinates in TB, $(q^i, \dot{q}^i)$, the structure functions are:*

$$C_{ij}{}^{k} = 0\,, \qquad a^i{}_j = \delta^i_j\ . \tag{2.6}$$

Note however that in a set of arbitrary coordinates in TB, some of the structure constant $C_{ij}{}^{k}$ may be different from zero, and the anchor map may take another form (see e.g. [10]).

Example 3. *A Lie algebroid determined by an action of a Lie algebra $\mathfrak{g}$ on a manifold M* (see e.g. [10]), i.e. by a Lie algebra morphism $\gamma : \mathfrak{g} \to \mathfrak{X}(M)$. In this case, any section v of the trivial bundle $\pi : M \times \mathfrak{g} \to M$ can be seen as a $\mathfrak{g}$-valued function, $\widetilde{v}(x)$ by means of $v(x) = (x, \widetilde{v}(x))$, and an element of the Lie algebra $\mathfrak{g}$ can be regarded as a constant section.

Then, there exists a uniquely defined Lie algebroid structure on $M \times \mathfrak{g}$ such that the bracket reduces to that of the Lie algebra for constant sections and the image of such a section under the anchor map ρ is the corresponding fundamental vector field. The bracket is given by

$$[v, w]_{M\times\mathfrak{g}}(x) = (x, [\widetilde{v}(x), \widetilde{w}(x)] + (\gamma(\widetilde{v}(x))\,\widetilde{w})\,(x) - (\gamma(\widetilde{w}(x))\,\widetilde{v})\,(x))\ ,$$

and the anchor $\rho : M \times \mathfrak{g} \to TM$ by

$$\rho(x, v) = \gamma(\widetilde{v}(x))\ ,$$

where γ is the linear map associating each element in $\mathfrak{g}$ with the corresponding fundamental vector field, which is a Lie algebra homomorphism.

Example 4. *The gauge algebroid: Let $P(B, G, \pi)$ be a principal bundle over a manifold B with structural group G, then the bundle*

$$E \equiv TP/G \to P/G\,,$$

is endowed with a Lie algebroid structure induced by the action of the structural group G on the tangent bundle $\sigma : TP \to P$, the Lie product is defined on sections, elements of $\Gamma(B, E) \equiv \Gamma(B, TP)^G$, identified with sections of TP invariant under G, and anchor map $\rho : TP/G \to TB$. This Lie algebroid structure is related with the Atiyah sequence (see e.g. [11])

$$0 \longrightarrow \mathrm{ad}P \longrightarrow TP/G \xrightarrow{T\pi} TB \longrightarrow 0$$

3 Exterior differential algebra of a Lie algebroid.

Given a Lie algebroid, $(E, \rho, [\cdot,\cdot]_E)$, the sections of $\tau_E : E \to M$ will play the rôle of vector fields, and will be called E-vector fields, and the sections of the dual bundle $\pi_E : E^* \to B$ that of 1-forms, and will be called E-1-forms. Similarly, we can consider sections for the projection from $E^* \wedge \cdots \wedge E^*$ onto B, which allows us to construct the exterior algebra $\bigwedge^\bullet E^*$ of the dual of E. The sections of $\bigwedge^\bullet E^*$ are called E-forms. The set of them, $\Gamma(\bigwedge^\bullet E^*)$, is a $C^\infty(B)$-module. An E-(k)-form is a E-form such that $\theta \in \Gamma(\bigwedge^k E^*)$.

The exterior differential giving rise to de Rham cohomology can be generalized to this more general framework defining a differential operator d_E, which maps in a linear way each E-(k)-form into a E-$(k+1)$-form

$$d_E : \Gamma(\wedge^k E^*) \to \Gamma(\wedge^{k+1} E^*)\,,$$

as follows:

$$\begin{aligned} d_E\theta(V_1,\ldots,V_{k+1}) &= \sum_i (-1)^{i+1}\rho(V_i)\theta(V_1,\ldots,\widehat{V}_i,\ldots,V_{k+1}) + \\ +\sum_{i<j}(-1)^{i+j} & \qquad \theta([V_i,V_j]_E, V_1,\ldots,\widehat{V}_i,\ldots,\widehat{V}_j,\ldots V_{k+1})\ , \end{aligned}$$

for $V_1,\ldots,V_{k+1} \in \Gamma(\tau_E)$, where $\widehat{V}_i$ denotes, as usual, that the element V_i is omitted.

The Lie algebroid axioms imply the following properties:

1. If $f \in C^\infty(B)$, then $\langle d_E f, V\rangle = \rho(V)f$.
2. $d_E^2 = 0$.
3. d_E is a super-derivation of degree 1, i.e. if θ is homogeneous of degree $|\theta|$, then

$$d_E(\theta \wedge \zeta) = d_E\theta \wedge \zeta + (-1)^{|\theta|}\theta \wedge d_E\zeta\ .$$

Moreover, the exterior differential d_E is fully characterized by these properties.

Observe that an exterior derivation d_E satisfying $d_E^2 = 0$ on $\Gamma(\bigwedge^\bullet E^*)$ is equivalent to the Lie algebroid structure on E, because both ρ and $[\cdot,\cdot]$ can be recovered from the expressions

$$\rho(V)f := d_E f(V)\,, \quad \theta([V,W]) := \rho(V)\theta(W) - \rho(W)\theta(V) - d_E\theta(V,W)\ ,$$

for $V, W \in \Gamma(\tau_E)$, $f \in C^\infty(B)$, $\theta \in \bigwedge^1(E)$.

In local coordinates of E as indicated above, d_E is determined by

$$d_E x^i = \rho^i{}_\alpha\, e^\alpha\,, \qquad d_E e^\gamma = C_{\alpha\beta}{}^\gamma\, e^\alpha \wedge e^\beta\ ,$$

where $\{e^\alpha \mid \alpha = 1, \ldots, r\}$ is the dual basis of $\{e_\alpha \mid \alpha = 1, \ldots, r\}$.

The conditions $d_E^2 x^i = 0$ and $d_E^2 e^\alpha = 0$ are equivalent to the structure equations (2.2) and (2.3).

Some examples of de exterior algebra of a Lie algebroid are:

Example 5. *The Chevalley-Eilenberg operator of Lie algebra cohomology: let $E = \mathfrak{g}$ be a finite dimensional Lie algebra, and consider the bundle $\mathfrak{g} \to \{\bullet\}$ with the Lie algebroid of the example* (1)*, it is found that d_E takes the form:*

$$d_{\mathfrak{g}}\theta(v_1, \ldots, v_{k+1}) = \sum_{i<j} (-1)^{i+j} \theta([v_i, v_j], v_1, \ldots, \check{v}_i, \ldots, \check{v}_j, \ldots v_{k+1}) \ , \tag{3.1}$$

where $v_i \in \mathfrak{g}$ and $\theta \in \bigwedge^\bullet \mathfrak{g}^*$.

A local form of the differential operator, $d_{\mathfrak{g}}$, can be obtained, after the choose of a local basis of section, and is given by (see e.g. [7])*:*

$$\widehat{s} = \frac{1}{2} f_{ij}^k c^i c^j \frac{\partial}{\partial c^k} \ , \tag{3.2}$$

which is also known as the BRST form of the differential operator of Lie algebra cohomology. In this case some anticonmuting variables, the $c^1, \ldots, c^k$ are introduced, which are the so called ghost variables (in general the ghost fields), such that:

$$c^i c^j = -c^j c^i \ , \qquad i, j = 1, \ldots, (\dim \mathfrak{g}) \ ,$$

Example 6. *The de Rahm exterior operator: consider the tangent bundle with base manifold B, ($TB \to B$), with anchor map the identity in TB, and the Lie algebra bracket is the Lie algebra bracket of vector field, so the differential operator $d_E \equiv d$ has the form:*

$$d\theta(v_1, \ldots, v_{k+1}) = \sum_i (-1)^{i+1} v_i \theta(v_1, \ldots, \check{v}_i, \ldots, v_{k+1}) \ , \tag{3.3}$$

where $v_i \in \Gamma(\wedge^\bullet TB)$ and $\theta \in \Gamma(\wedge^k TB^*)$, being this the de Rahm differential operator.

The local form of this differential operator, in a local basis, is given by (see e.g. [7, 8])*:*

$$\widehat{s} = a_i^\alpha c^i \frac{\partial}{\partial x^\alpha} \ , \tag{3.4}$$

which is also known as the BRST form of the differential operator.

Example 7. *Let us consider an extension of the example* (3)*, the action Lie algebroid, to the case of a Lie group G acting on a manifold M, in this case the anchor map is given by*

$$\rho(X)(x) = Y \ , \quad \text{with} \quad X \in \mathfrak{g} \ , \tag{3.5}$$

which is a vector field on M, associated to the action induced on $\Omega^n(\mathfrak{g}, C^\infty(M))$ by the action of the group G on M, where $\Omega^n(\mathfrak{g}, C^\infty(M))$ is the set of anti-symmetric multilinear maps

$$\omega_n : \mathfrak{g} \times \cdots \times \mathfrak{g} \to C^\infty(M) : (X_1, \ldots, X_n) \mapsto \omega_n(\cdot\,;\, X_1, \ldots, X_n) : M \to \mathbb{R}$$
$$\omega_n(\cdot\,;\, X_1, \ldots, X_n)(x) = \omega_n(x\,;\, X_1, \ldots, X_n)$$

where the action of $\rho(X)$ on $\Omega^n(\mathfrak{g}, C^\infty(M))$ is given by:

$$\rho(X)\omega_n(\cdot\,;\, X_1, \ldots, X_n)(x) = Y\omega_n(x\,;\, X_1, \ldots, X_n)\ ,$$

finally the exterior differential operator is given by:

$$\begin{aligned} d\omega_n(x; X_1, \ldots, X_{n+1}) &= \sum_i (-1)^{i+1} Y_i \omega_n(x; X_1, \ldots, \check{X}_i, \ldots, X_{n+1}) + \\ &+ \sum_{i<j} (-1)^{i+j} \omega_n(x; [X_i, X_j]_{\mathfrak{g}}, X_1, \ldots, \check{X}_i, \ldots, \check{X}_j, \ldots X_{n+1})\,. \end{aligned} \tag{3.6}$$

The BRST, or local, representation of the differential operator in this case is given by:

$$\widehat{s} = \rho(X_i, x)c^i + \frac{1}{2} C^k_{ij} c^i c^j \frac{\partial}{\partial c^k}\,. \tag{3.7}$$

Example 8. *Consider a principal bundle with structure group $G(M)$, let A denote the connection 1-form with values in the Lie algebra $\mathfrak{G}(M)$, in this setting we will considering the action on the set of local functionals $\mathcal{C}(A)$ of the gauge fields, i.e., let*

$$A(x) = A^i(x) T_i\,, \ \text{ where } \ A^i(x) = A^i_\mu(x) dx^\mu$$

where T_i is a basis of the Lie algebra $\mathfrak{G}$. The curvature two-form of the one-form is given by

$$F = dA + A \wedge A\,.$$

The gauge transformation of A generated by the action of an element $g(x) \in G(M)$ define an action on $\mathcal{C}(A)$, the infinitesimal generators of this action are given by the vector fields (for a complete analysis of the problem see e.g. [7])*:*

$$Y_i(x) = -\frac{\partial}{\partial x^\mu} \frac{\delta}{\delta A^i_\mu(x)} - C^k_{ij} A^j_\mu(x) \frac{\delta}{\delta A^k_\mu(x)}\,, \tag{3.8}$$

which generalize the action given in example (7)*. It is important at this stage to notice that these vector fields satisfies*

$$[Y_i(x), Y_j(y)] = \delta^m(x-y) C^k_{ij} Y_k(y)\,. \tag{3.9}$$

Special attention deserve the fact that this set of vector field (infinitesimal generators) are in involution, i.e., the vector field form an integrable distribution in the field space *(the space of functional of A). Now we turn our attention to the exterior operator which acts on $\Omega^n(\mathfrak{G}(M), \mathcal{C}(A))$, i.e., the set of antisymmetric multilinear mappings*

$$\omega_n : \mathfrak{G}(M) \times \cdots \times \mathfrak{G}(M) \to \mathcal{LC}(A) : (X_1, \ldots, X_n) \mapsto \omega_n[\cdot]\,(X_1, \ldots, X_n)$$
$$\omega_n[A]\,(X_1, \ldots, X_n) : M \to \mathbb{R}\,,$$

where $\mathcal{LC}(A)$ denote the local functionals of A, and $X_i(x)$ are the base elements of the $\mathfrak{G}(M)$. Continuing with the construction it is found the following general differential operator acting on $\Omega^n(\mathfrak{G}(M), \mathcal{C}(A))$:

$$d_{\mathfrak{G}}\,\omega_n[A](X_1, \ldots, X_{n+1}) = \sum_b (-1)^{b+1} \rho(X_{i_b}) \omega_n[A](X_1, \ldots, \check{X}_{i_b}, \ldots, X_{n+1}) +$$
$$+ \sum_{a<b} (-1)^{a+b} \omega_n[A]([X_i, X_j], X_1, \ldots, \check{X}_i, \ldots, \check{X}_j, \ldots X_{n+1})\,, \tag{3.10}$$

Finally, the local or BRST form of the differential operator, is:

$$\widehat{s} = \int d^m x \left\{ -D^k_{i\,\mu} c^i(x) \frac{\delta}{\delta A^k_\mu(x)} - \frac{1}{2} c^i(x) c^j(x) C^k_{ij} \frac{\delta}{\delta c^k(x)} \right\}. \tag{3.11}$$

where

$$\rho(X_i, x) = Y_i(x) = D^k_{l\,\mu} \frac{\delta}{\delta A^i_\mu(x)}\,, \tag{3.12}$$

with

$$D^k_{i\;\mu} = \delta^k_i \frac{\partial}{\partial x^\mu} - C^k_{ji} A^j_\mu(x)$$

the covariant derivative.

4 Final comments

The last examples show a Lie algebroid structure over an infinite dimensional manifold, in particular it is a Lie algebroid structure build on the interaction bundle [12] with basis on the space of connection of a principal bundle. The Lie algebroid structure can be constructed on a convenient setting (see e.g. [13]) this construction will be prosecuted elsewhere.

References

[1] E. Martínez, *Lagrangian mechanics on Lie algebroids*, Acta Appl. Math. **67** (2001) 295–320.

[2] E. Martínez, *Classical field theory on Lie algebroids: variational aspects*, J. Phys. A **38** 7145-7160, (2005).

[3] J. Cortés, E. Martínez, *Mechanical control systems on Lie algebroids*, IMA J. Math. Control Inform. **21** 457-492 (2004).

[4] Y. Kosmann–Schwarzbach, *Graded Poisson brackets and field theory*, In the book *Modern group theoretical methods in physics* (Paris, 1995), 189–196, Math. Phys. Stud. **18**, Kluwer Acad. Publ., Dordrecht, 1995.

[5] J.F. Cariñena and E. Martínez, *Lie Algebroid Generalization of Geometric mechanics*, In: "Lie Algebroids and related topics in differential geometry", Banach Center Publications **54**, p. 201 (2001)

[6] M. Hennaux and C. Teitelboim, *Quantization of gauge systems*, Princeton Univ. Press, (1992).

[7] J.A. de Azcárraga and J.M. Izquierdo, *Lie groups, Lie algebras, cohomology and some applications in physics*, Cambridge University Press 1995.

[8] M. Alexandrov, A. Schwarz, O. Zaboronsky and M. Kontsevich, *The geometry of the master equation and topological quantum field theory*, Int. J. Mod. Phys **A 12**, 1405–1429 (1997).

[9] J. Pradines, *Théorie de Lie pour les groupoïdes différentiables. Relations entre propiétés locales et globales*, C. R. Acad. Sci. Paris Sér. I Math. **236** (1966) 907–910.

[10] A. C. da Silva, A. Weinstein, *Lectures on geometrical models for noncommutative algebra,* **Berkeley Mathematics Lecture Notes, 10**. American Mathematical Society, Providence, RI; Berkeley Center for Pure and Applied Mathematics, Berkeley, CA, 1999.

[11] K. Mackenzie. *General theory of Lie groupoids and Lie algebroids.* London Mathematical Society Lecture Note Series, 213. Cambridge University Press, Cambridge, 2005.

[12] M. Castrillón López, L. Hernández Encinas, J Muñoz Masqué, *Gauge invariance on interaction* U(1) *bundles.* J. Phys. A **33**, 3253–3267 (2000).

[13] A. Kriegl, P. Michor. *The convenient setting of global analysis.* Mathematical Surveys and Monographs, 53. American Mathematical Society, Providence, RI, 1997.