Monografías de la Real Academia de Ciencias de Zaragoza. **28:** 59–72, (2006).

Equatorial and halo orbits around an oblate magnetic planet

M. Iñarrea[1], V. Lanchares[1], J. Palacián[2],
A. I. Pascual[1], J. P. Salas[1] and P. Yanguas[2]

[1] Universidad de La Rioja, 26004 Logroño, Spain

[2] Universidad Pública de Navarra, 31006 Pamplona, Spain

Abstract

We calculate equatorial and halo orbits around an oblate magnetic planet. It is known that circular equatorial and halo orbits exist for a dust grain orbiting a spherical magnetic planet. However, the frequency of the orbit is constrained by the charge-mass ratio of the particle. If the oblateness of the planet is taken into account this constraint is modified or, in some cases, it disappears. The conditions of stability of the orbits are also modified.

Keywords: planetary magnetospheres, Størmer problem, halo orbits, equilibria, stability.

1 Introduction

The classical model of a particle subject to a magnetic dipole field was introduced by Størmer [10, 11]. It has been taken as a starting point for the study of charged particles orbiting a planet with magnetosphere [3, 2].

One of these models is the generalised Størmer problem considered in [1, 6, 7], which describes the dynamics of a dust particle of mass m and charge q orbiting a rotating magnetic planet of mass M. In this model, the magnetic field of the planet is supposed to be a perfect magnetic dipole of strength μ aligned along the north-south poles of the planet. Moreover, the planet's magnetosphere is taken as a rigid conducting plasma which rotates with the same angular velocity $\mathbf{\Omega} = (0, 0, \Omega)$ as the planet, in such a way that the charge q is subject to a corotational electric field. In this way, using cylindrical coordinates and momenta $(\rho, z, \phi, P_\rho, P_z, P_\phi)$ and assuming that the gravitational interaction is purely Keplerian, the generalised Størmer problem can be modelled by the following dimensionless two-degrees-of-freedom Hamiltonian

$$\mathcal{H}_S = \frac{1}{2}\left(P_\rho^2 + P_z^2 + \frac{P_\phi^2}{\rho^2}\right) - \frac{1}{r} - \delta\,\frac{P_\phi}{r^3} + \frac{\delta^2}{2}\,\frac{\rho^2}{r^6} + \delta\,\beta\,\frac{\rho^2}{r^3}, \tag{1}$$

where lengths and time are expressed, respectively, in units of the planetary radius R and the Keplerian frequency $w_K = \sqrt{M/R^3}$ (Gaussian units). The variable $r = \sqrt{\rho^2 + z^2}$ stands for the distance of the charged particle to the center of mass of the planet. Cylindrical variables are natural to formulate the problem, as the system is invariant under rotations around the z–axis. Furthermore, Hamiltonian (1) depends on two external parameters δ and β which indicate, respectively, the ratio between the magnetic and the Keplerian interaction (i.e., the charge-mass ratio q/m of the particle) and the ratio between the electrostatic and the Keplerian interactions. On the other hand, the system depends on the two external parameters P_ϕ and $\mathcal{H}_S = \mathcal{E}$ (the energy).

For this model, Howard *et al.* [1, 5, 4] proved the existence and stability of orbits around the planet lying on the equatorial plane (equatorial orbits), and orbits that do not intersect the equatorial plane, i.e. the so-called halo orbits. Even more, they provided a comprehensive view of what kind of particles and what frequencies are expected for a given position away from the planet. In particular, they speculate with the possibility of finding this kind of orbits around Saturn. The electromagnetic ambient in this giant planet can be fairly modelled by the corresponding terms in (1). However, the gravitational interaction, appearing as a pure Keplerian term in (1), should be improved because Saturn is not a perfect spherical planet, but it presents a pronounced oblateness. Hence, the purpose of this paper is to study how the oblateness of a planet affects the equatorial and halo orbits.

A classical model for the oblateness of a planet is given by means of the so-called J_2 term [9]. In this way, Hamiltonian (1) becomes

$$\mathcal{H} = \frac{1}{2}\left(P_\rho^2 + P_z^2 + \frac{P_\phi^2}{\rho^2}\right) - \frac{1}{r} - \delta\,\frac{P_\phi}{r^3} + \frac{\delta^2}{2}\,\frac{\rho^2}{r^6} + \delta\,\beta\,\frac{\rho^2}{r^3} + 3J_2\frac{z^2}{2r^5} - \frac{J_2}{2r^3}, \tag{2}$$

where J_2 is a positive dimensionless parameter for an oblate planet. For example, in the case of Saturn, $J_2 = 0.09796$.

Circular periodic trajectories around the z–axis correspond to equilibria (ρ_0, z_0) in the rotating meridian plane (ρ, z), i.e. they appear as the equilibrium points of the system

$$\dot\rho = \frac{\partial\mathcal{H}}{\partial P_\rho}, \quad \dot z = \frac{\partial\mathcal{H}}{\partial P_z}, \quad \dot P_\rho = -\frac{\partial\mathcal{H}}{\partial\rho}, \quad \dot P_z = \frac{\partial\mathcal{H}}{\partial z},$$

or equivalently, they are the critical points of the effective potential U_{eff} in (2),

$$U_{\text{eff}} = -\frac{1}{r} - \delta\,\frac{P_\phi}{r^3} + \frac{\delta^2}{2}\,\frac{\rho^2}{r^6} + \delta\,\beta\,\frac{\rho^2}{r^3} + 3J_2\frac{z^2}{2r^5} - \frac{J_2}{2r^3}. \tag{3}$$

Following the procedure of Howard *et al.* [1, 5, 4], instead of P_ϕ, we introduce in (3) the particle frequency ω,

$$\omega = \dot\phi = \frac{\partial U_{\text{eff}}}{\partial P_\phi} = \frac{P_\phi}{\rho^2} - \frac{\delta}{\rho^3}.$$

This allows us to handle a function which determines when the sense of the rotation of the particle is the same as the planet's one or it is the opposite. Finally, in order to simplify the calculations, we move to spherical variables (r, θ),

$$\rho = r \sin\theta, \quad z = r\cos\theta.$$

Now, the effective potential reads as

$$U_{\text{eff}} = -\frac{1}{r} + \delta\beta\frac{\sin^2\theta}{r} + \omega^2\frac{r^2\sin^2\theta}{2} + 3J_2\frac{\cos^2\theta}{2r^3} - \frac{J_2}{2r^3}, \tag{4}$$

and after this last change of variables, the critical points are found as the solutions of the system of equations

$$\begin{aligned} \frac{\partial U_{\text{eff}}}{\partial r} &= \frac{-3\,J_2}{r^4} + \frac{1}{r^2} + \left(\frac{9\,J_2}{2\,r^4} - \frac{\beta\,\delta}{r^2} + r\,\omega^2\right)\sin^2\theta = 0, \\ \frac{\partial U_{\text{eff}}}{\partial \theta} &= \frac{(-3\,J_2 + 2\,\beta\,\delta\,r^2 + r^5\,\omega^2)\,\sin 2\theta}{2r^3} = 0. \end{aligned} \tag{5}$$

The above system gives rise to the equivalent nonlinear system

$$\begin{aligned} &-6\,J_2 + 2\,r^2 + (9\,J_2 - 2\,\beta\,\delta\,r^2 + 2\,r^5\,\omega^2)\,\sin^2\theta = 0, \\ &(-3\,J_2 + 2\,\beta\,\delta\,r^2 + r^5\,\omega^2)\,\sin 2\theta\, = 0. \end{aligned} \tag{6}$$

The roots of (6) can be divided into two classes depending on the value of θ. On the one hand if $\theta = \pi/2$ we get the equatorial orbits and when $\theta \neq \pi/2$ we obtain the halo orbits.

The paper is structured as follows. In Section 2 the existence of equatorial orbits is discussed, whereas in Section 3, halo orbits are considered. Section 4 deals with the stability of the orbits computed in Sections 2 and 3. The concluding remarks appear in Section 5.

2 Existence of equatorial orbits

As it was said above, equatorial orbits appear when $\theta = \pi/2$. In this case the second equation of (6) vanishes, and r must satisfy the polynomial equation

$$3\,J_2 + 2\,r^2 - 2\,\beta\,\delta\,r^2 + 2\,\delta\,\omega\,r^2 - 2\,\omega^2\,r^5 = 0. \tag{7}$$

By means of the Descartes rule of signs we infer that equation (7) has a unique positive real root, taking into account that J_2 is positive as we are considering an oblate planet. This means that it does not matter the values of ω and δ, as there always exists an equatorial orbit associated with them.

This situation does not occur for the case $J_2 = 0$ (see the details in [1]), where the existence of equatorial orbits is constrained to a certain region of the plane (δ, ω) whose boundary is given by the curve

$$1 - \beta\,\delta + \delta\,\omega = 0. \tag{8}$$

Indeed, for $J_2 = 0$, (7) becomes

$$2\,r^2\,(1 - \beta\,\delta + \delta\,\omega - \omega^2\,r^3) = 0.$$

There are two real roots for the above equation: on the one hand the double root $r = 0$ and, on the other hand, the nontrivial root

$$r_e = \left(\frac{1 - \beta\,\delta + \delta\,\omega}{\omega^2}\right)^{1/3}.$$

As r is the radius of the orbit, it is positive and then $1 - \beta\,\delta + \delta\,\omega > 0$. This gives rise to the existence condition for equatorial orbits in the case $J_2 = 0$.

However, if $J_2 \neq 0$ the two roots 0 and r_e will change in such a way that the double root splits into two real roots only when the single root r_e becomes negative. The two roots splitting from $r = 0$ are given by the series

$$r_{1,2} = a_{1,2}\,J_2^{1/2} + b_{1,2}\,J_2 + c_{1,2}\,J_2^{3/2} + d_{1,2}\,J_2^2 + \ldots.$$

By substitution of the above expression in equation (7), we obtain

$$a_{1,2} = \pm\sqrt{\frac{3}{2(-1 + \beta\,\delta - \delta\,\omega)}},$$

whereas the rest of coefficients are not of interest in the discussion. Thus, it is clear that we get a positive real root, say r_1, in the case

$$1 - \beta\,\delta + \delta\,\omega < 0,$$

that is, outside of the existence domain for equatorial orbits in the case $J_2 = 0$.

It is worth to note that when the parameter δ and the variable ω lie on the boundary of the existence of r_e, the unperturbed polynomial equation has a quintuple root at $r = 0$, and the corresponding positive perturbed root can be directly calculated and it is given by

$$r = \sqrt[5]{\frac{6\,J_2\,\delta^2}{(2\,\beta\,\delta - 1)^2}}.$$

In Figure 1 it is depicted the plane (δ, ω), where the red color indicates those equatorial orbits existing for $J_2 = 0$ and the blue one indicates those equatorial orbits appearing due to the effect of the oblateness of the planet.

Nevertheless, most of the orbits arising from the effect of $J_2 \neq 0$ cannot be taken into account, because $r < 1$ and they would be inside the planet. Thus, the effective boundary for the existence of equatorial orbits is given by

$$2 - 2\,\beta\,\delta + 3\,J_2 + 2\,\delta\,\omega - 2\,\omega^2 = 0, \tag{9}$$

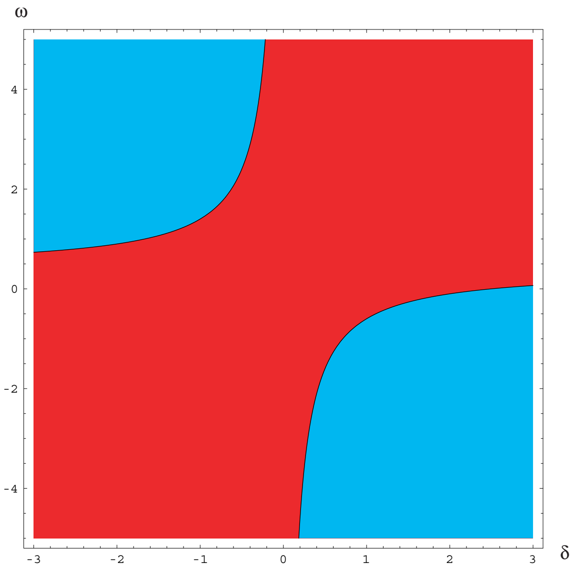

Figure 1.— Equatorial orbits in the plane (δ, ω) for $\beta = 0.4$ and $J_2 = 0.09796$. Red color stands for equatorial orbits existing for $J_2 = 0$. Blue color stands for those orbits arising for $J_2 \neq 0$.

obtained from equation (7) when $r = 1$. In Figure 2 the effective boundary is depicted for the parameters' values of Saturn ($\beta = 0.4$ and $J_2 = 0.09796$). It can be seen that the set of new equatorial orbits is not too large and it is limited to a narrow region between the two curves (8) and (9) which intersect at the points

$$\left(\frac{2}{2\beta + \sqrt{6J_2}}, -\frac{\sqrt{6J_2}}{2}\right), \qquad \left(\frac{2}{2\beta - \sqrt{6J_2}}, \frac{\sqrt{6J_2}}{2}\right).$$

It is worth noting that the set of equatorial orbits of constant r is a family of hyperbolas in the (δ, ω) plane. The asymptotic straight lines of this family correspond to the radius of the synchronous orbit (r_s) obtained by setting $\omega = \beta$ (the frequency of the synchronous orbit), that is, r_s is the positive solution of the equation

$$2\beta^2 r^5 - 2r^2 - 3J_2 = 0.$$

We stress that now r_s is not given by Kepler's third law $\beta^2 r^3 = 1$, but by a modified one. We can express it asymptotically in terms of J_2 as

$$\beta^2 r^3 = 1 + \frac{3}{2}\beta^{4/3} J_2 - \frac{3}{2}\beta^{8/3} J_2^2 + O(J_2^3).$$

The most remarkable fact is that the synchronous orbit plays the role of a separatrix in the sense that if $r > r_s$ there is a close interval of non-allowed charge-mass ratio values, but all the frequencies are possible. On the other hand, if $r < r_s$ the situation reverses: all charge-mass ratios are allowed but there is a close interval of excluded frequencies (see Figure 3).

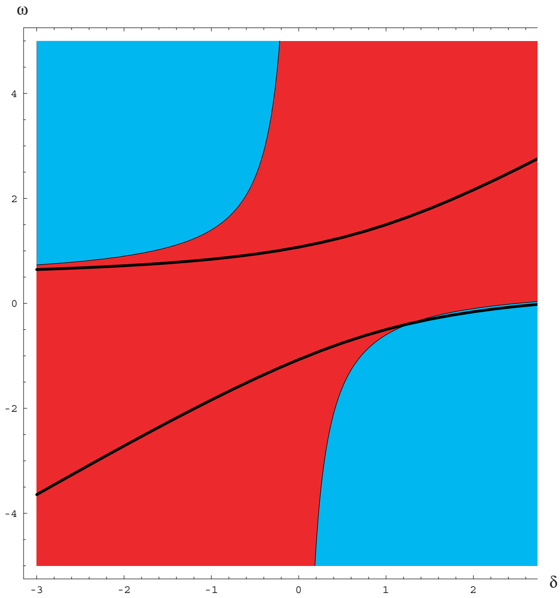

Figure 2.— Effective region for equatorial orbits, between black solid lines corresponding to $r = 1$.

We also note that the curves do not cross, that is to say, two orbits of different radius cannot share the same frequency and the same charge-mass ratio particles. This is not the case when $J_2 = 0$ where all the orbits share the pair of values $\omega = 0$ and $\delta = 1/\beta$.

Finally, we remark that the trajectories discussed through this section correspond to the circular equatorial orbits found in [6, 7, 8] using normalisation and reduction techniques.

3 Existence of halo orbits

Halo orbits appear as the solutions of system (6) when $\theta \neq \pi/2$. In this case, the term in r in the second equation of (6) must be zero, and then the following equation is satisfied

$$3J_2 + 2\delta(\omega - \beta)r^2 + \omega^2 r^5 = 0. \tag{10}$$

We note that the presence of J_2 prevents from an easy expression of r. Furthermore, from the Descartes rule of signs, it follows that if equation (10) has positive real roots, then it has two different real roots. This is what happens when

$$128\,\delta^5\,(\beta - \omega)^5 - 3125\,{J_2}^3\,\omega^4 > 0.$$

Now we have to discuss if the two roots, when they exist, correspond to a halo orbit. To begin with, we note that we can eliminate J_2 from system (6) to obtain

$$r^3\,\omega^2\,(5\,\sin^2\theta - 2) - 2\,[1 + 2\,\delta\,(\beta - \omega)(\sin^2\theta - 1)] = 0.$$

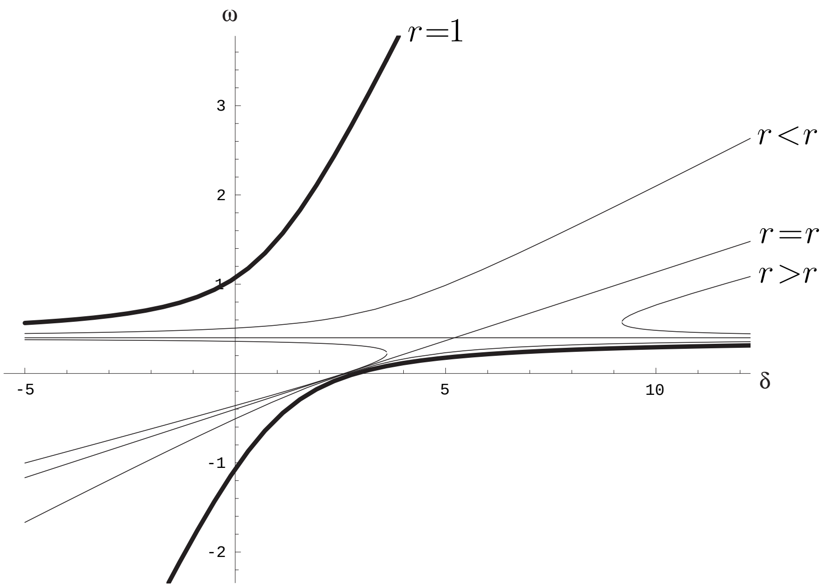

Figure 3.— Curves of constant radius for equatorial orbits.

This is a linear equation in r^3 and in $\sin^2\theta$. This means that each value of r is related with a unique value of $\sin^2\theta$ and vice versa. Thus, for a given r, i.e., a root of (10), we can obtain two different halo orbits which are symmetric with respect to the equatorial plane if

$$\sin^2\theta = 1 - \frac{3\omega^2 r^3 - 2}{5\omega^2 r^3 + \delta(\omega-\beta)} \leq 1.$$

This condition can be obtained in terms of δ, β, ω and θ by elimination of r in system (6). In this way, we obtain the following equation in $\sin^2\theta$

$$32\left[-1+3\delta(\beta-\omega)\sin^2\theta\right]^3\left[1-2\delta(\beta-\omega)(1-\sin^2\theta)\right]^2 - 27J_2^3\omega^4(5\sin^2\theta-2)^5 = 0. \quad (11)$$

Now it is easy to find the limit curves for an admissible value of $\sin^2\theta$, by substitution of $\sin^2\theta = 1$ and $\sin^2\theta = 0$ in (11). Thus, we find

$$\sin^2\theta = 1 \longrightarrow 32[-1+3\delta(\beta-\omega)]^3 - 6561J_2^3\omega^4 = 0, \quad (12)$$

$$\sin^2\theta = 0 \longrightarrow 864J_2^3\omega^4 - 32[1-2\delta(\beta-\omega)]^2 = 0. \quad (13)$$

These two curves lie in the area of the plane (δ,ω) where the polynomial equation (10) has two positive real roots. Even more, all the lines are tangent at the points (δ_t,ω_t) given by

$$\omega_t^4 = \frac{4}{243J_2^3}, \qquad \delta_t = \frac{5}{6(\beta-\omega_t)},$$

as it is depicted in Figure 4.

The plane (δ,ω) is divided into different regions where the number of halo orbits change. In this way, there are no halo orbits between the curve limiting the existence of positive real roots of (10) and the curve (12), for $\sin^2\theta = 1$. There is a unique halo orbit

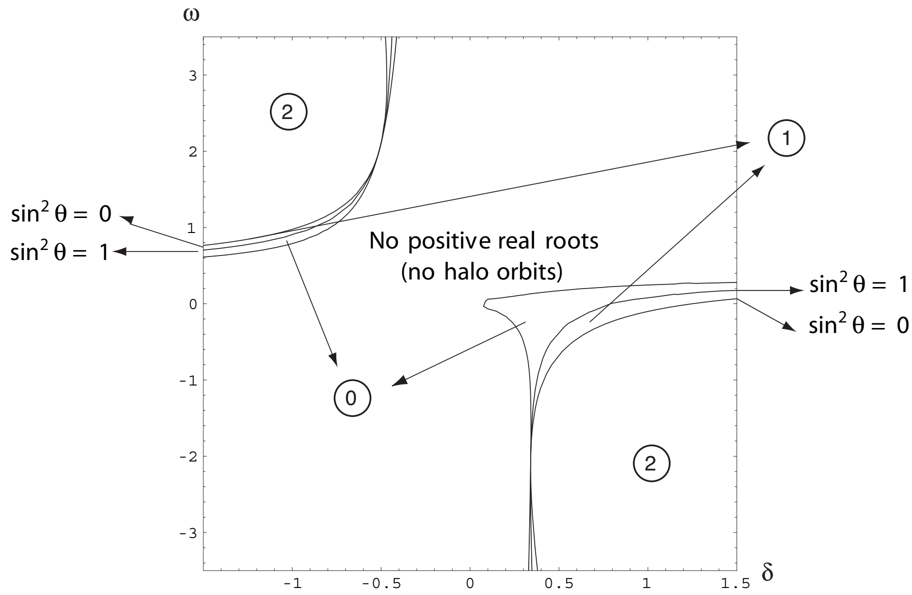

Figure 4.— Conditions for the existence of halo orbits. The inner curve gives the limit for the existence of real roots in equation (10). The next curve gives the condition for $\sin^2\theta = 1$ and the outer one gives the condition for $\sin^2\theta = 0$. Encircled, the number of halo orbits in the region.

in the region between (12) and (13) and two halo orbits below (13) for $\delta > 0$ and (13) for $\delta < 0$.

It is important to remark that the lines depicted in Figure 4 do not constitute parametric bifurcation lines. In fact, in the region where two halo orbits exist they share δ and ω, but not the radius of the orbit. As a consequence, the corresponding third component of the angular momentum P_ϕ is different for the two orbits, and they do not exist at the same time for fixed values of the external parameters. These orbits sharing δ and ω are located at the same latitude, but at different altitudes from the equator, when

$$6\delta(\beta - \omega) = 5.$$

On the other hand, there is an effective boundary for halo orbits similarly to what happens for the equatorial orbits. In order to really have halo orbits, their radii must be greater than $\sin\theta$. This effective boundary can be calculated by substituting r by $\sin\theta$ in (10) and in the first equation of (6) and then eliminating $\sin\theta$ from the two equations. As a result it is obtained an algebraic curve of degree eight in ω and seventh degree in δ. This curve is depicted in Figure 5 joint to the curve (12) and it lies on the region where two halo orbits exist. For positive charged particles only one of the two halo orbits is not admissible below the limit. For negative charged particles the situation is similar, but the occurrence of the two halo orbits now is not admissible above the limit curve.

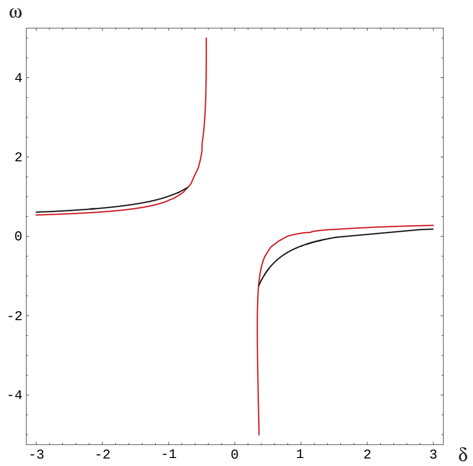

Figure 5.— Effective boundary for the existence of halo orbits (black line).

4 Stability

Stability for both equatorial and halo orbits follows from their character as critical points of the effective potential. In this way, they are stable if they are local minima of U_{eff}, that is, the Hessian matrix at an equilibrium point gives rise to a positive defined quadratic form [1]. The Hessian matrix is given by the second order partial derivatives of the effective potential U_{eff} and they are

$$\frac{\partial^2 U_{\text{eff}}}{\partial r^2} = \frac{(\delta^2 + 2\beta\delta r^3 - 6\delta\omega r^3 + 3\omega^2 r^6)\sin^2\theta + 18 J_2 r\cos^2\theta - 6J_2 r - 2r^3}{r^6},$$

$$\frac{\partial^2 U_{\text{eff}}}{\partial \theta^2} = \frac{2(\delta + \omega r^3)^2 + [2\delta^2 - 3J_2 r + \omega^2 r^6 + 2\delta(\beta+\omega)r^3]\cos 2\theta}{r^4},$$

$$\frac{\partial^2 U_{\text{eff}}}{\partial r\partial \theta} = \frac{-2\delta^2 + 9J_2 r - 2\delta(\beta - 2\omega)r^3 + 2\omega^2 r^6}{r^5}\sin\theta\cos\theta.$$

Due to the complex expressions arising, we satisfy ourselves by computing the boundaries of stability which appear in the case that the determinant of the Hessian matrix is equal to zero.

4.1 Stability for equatorial orbits

For equatorial orbits it must be $\theta = \pi/2$ and then the crossed derivative $\frac{\partial^2 U_{\text{eff}}}{\partial r\partial\theta}$ vanishes. Thus, stability boundaries arise when one of the elements in the principal diagonal of the Hessian matrix is equal to zero, that is

$$\left.\frac{\partial^2 U_{\text{eff}}}{\partial r^2}\right|_{\text{equatorial}} = 0 \qquad \text{or} \qquad \left.\frac{\partial^2 U_{\text{eff}}}{\partial \theta^2}\right|_{\text{equatorial}} = 0.$$

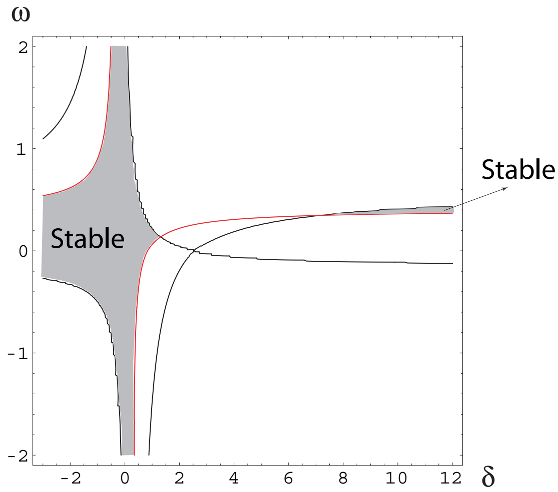

Figure 6.— Stability zones for equatorial orbits (in grey).

For the first equality we get

$$\begin{aligned}
&-64\,\delta^4\,(1-2\,\beta\,\delta+\beta^2\,\delta^2-2\,\delta\,\omega+2\,\beta\,\delta^2\,\omega-2\,\delta^2\,\omega^2)^3\\
&+864\,J_2^3\,\big(-1+5\,\beta\,\delta-10\,\beta^2\,\delta^2+10\,\beta^3\,\delta^3-5\,\beta^4\,\delta^4+\beta^5\,\delta^5\\
&\qquad+11\,\delta\,\omega-44\,\beta\,\delta^2\,\omega+66\,\beta^2\,\delta^3\,\omega-44\,\beta^3\,\delta^4\,\omega+11\,\beta^4\,\delta^5\,\omega\\
&\qquad-48\,\delta^2\,\omega^2+144\,\beta\,\delta^3\,\omega^2-144\,\beta^2\,\delta^4\,\omega^2+48\,\beta^3\,\delta^5\,\omega^2\\
&\qquad+105\,\delta^3\,\omega^3-210\,\beta\,\delta^4\,\omega^3+105\,\beta^2\,\delta^5\,\omega^3-118\,\delta^4\,\omega^4\\
&\qquad+\;118\,\beta\,\delta^5\,\omega^4+58\,\delta^5\,\omega^5\big)+729\,J_2^6\,\omega^4=0
\end{aligned}\tag{14}$$

and it determines the bifurcation of equatorial orbits into the equatorial plane. This is a saddle-center bifurcation, the tangent bifurcation mentioned in [1].

The second equality involves bifurcation of equatorial and halo orbits, as the variation of the angle θ is taken into account. It is not surprising that we get the limit curve for the existence of halo orbits given by $\sin^2\theta = 1$, that is

$$6561\,J_2^3\,\omega^4-32\,(-1+3\,\beta\,\delta-3\,\delta\,\omega)^3=0. \tag{15}$$

The corresponding bifurcation is a pitchfork one, as two symmetric halo orbits merge with an equatorial orbit.

The two curves given by (14) and (15) determines a partition between stable and unstable orbits in the plane (δ,ω), as it is depicted in Figure 6.

We note that no significant differences with the case $J_2 = 0$ are appreciated although the regions of stability are slightly modified. Indeed, if $J_2 = 0$, equations (14) and (15) recover the stability boundaries established in [1]. Nonetheless, we remark the gap for stable orbits in the range of charge-mass ratio between 1.3 and 7.2, approximately.

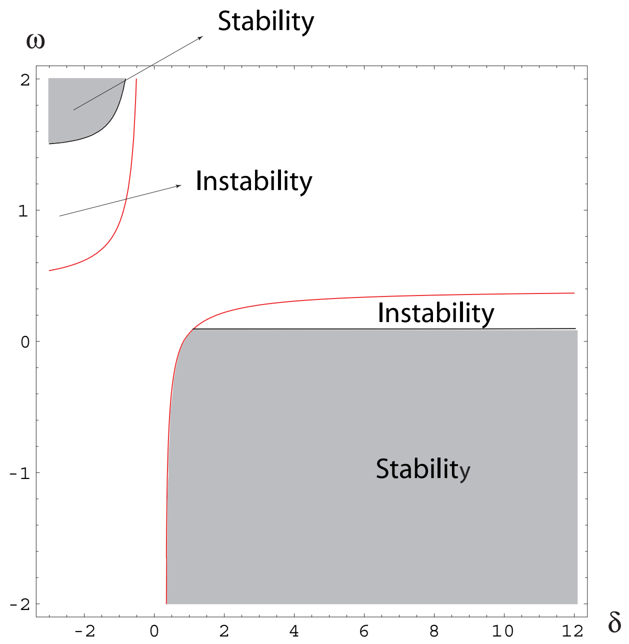

Figure 7.— Stability zones for halo orbits (in grey).

4.2 Stability for halo orbits

For the halo orbits the crossed derivatives do not vanish and the computation of the stability boundaries is not so easy. However, a standard procedure of successive steps to eliminate r and θ from the determinant of the Hessian matrix yields the desired result. We obtain an algebraic equation in δ and ω composed by two factors. The first one corresponds to the pitchfork bifurcation addressed in the previous section. The other one corresponds to a saddle-center bifurcation, and it is given by the equation

$$\begin{aligned}&32768\,\delta^{15}\,(\beta-\omega)^5\,(1+2\,\beta\,\delta-2\,\delta\,\omega)^2\,(\beta^2-4\,\beta\,\omega+\omega^2)^3+\\&J_2^3Q_1(\delta,\omega;\beta)+J_2^6Q_2(\delta,\omega;\beta)+J_2^9Q_3(\delta,\omega;\beta)-61509375\,\omega^{12}J_2^{12}=0,\end{aligned}\tag{16}$$

where $Q_1(\delta,\omega;\beta)$, $Q_2(\delta,\omega;\beta)$ and $Q_3(\delta,\omega;\beta)$ are polynomials in δ and ω whose coefficients are listed in Table 1.

The first term of equation (16) is a product of different factors. Each factor equated to zero defines a curve in the plane (δ,ω) which is a bifurcation line in the case $J_2=0$. However, only the factor $\beta^2-4\beta\omega+\omega^2$ is in the existence domain of halo orbits. Thus, the bifurcation lines for $J_2\neq 0$ will be close to these ones if J_2 is small enough, as it is the rule for real planets. In fact, it is not difficult to see that the stability boundaries are asymptotic to $\beta^2-4\beta\omega+\omega^2=0$ as δ tends to infinity. In this way the most noticeable deviations for the case $J_2=0$ take place for δ small, as it is observed in Figure 7, where the deviation from the straight line $\omega=\beta(2+\sqrt{3})$ is appreciated for negative charged dust particles when, δ is small.

Q_1	δ^{10}	δ^{11}	δ^{12}	δ^{13}	δ^{14}	δ^{15}
ω^0	$-442368\beta^{10}$	$1769472\beta^{11}$	$-1769472\beta^{12}$	0	0	0
ω^1	$3538944\beta^9$	$-19464192\beta^{10}$	$24772608\beta^{11}$	0	0	0
ω^2	$-774144\beta^8$	$66797568\beta^9$	$-141115392\beta^{10}$	0	0	0
ω^3	$-55627776\beta^7$	$-36716544\beta^8$	$448118784\beta^9$	0	0	0
ω^4	$121487360\beta^6$	$-231442432\beta^7$	$-933756928\beta^8$	$24182784\beta^9$	$-58392576\beta^{10}$	$23887872\beta^{11}$
ω^5	$47087616\beta^5$	$625272832\beta^6$	$1472823296\beta^7$	$-245071872\beta^8$	$334430208\beta^9$	$-119439360\beta^{10}$
ω^6	$-298927104\beta^4$	$-1222539264\beta^5$	$-1893486592\beta^6$	$812408832\beta^7$	$-745832448\beta^8$	$238878720\beta^9$
ω^7	$285958144\beta^3$	$1705388032\beta^4$	$2246459392\beta^5$	$-1235902464\beta^6$	$788299776\beta^7$	$-238878720\beta^8$
ω^8	$-123454464\beta^2$	$-1336324096\beta^3$	$-2346143744\beta^4$	$881418240\beta^5$	$-338411520\beta^6$	$119439360\beta^7$
ω^9	26941440β	$554913792\beta^2$	$1665630208\beta^3$	$-172744704\beta^4$	$-47775744\beta^5$	$-23887872\beta^6$
ω^{10}	-2587648	-118319104β	$-666210304\beta^2$	$-128360448\beta^3$	$90243072\beta^4$	0
ω^{11}	0	10663936	135839744β	$76750848\beta^2$	$-23887872\beta^3$	0
ω^{12}	0	0	-11161600	-13049856β	$1327104\beta^2$	0
ω^{13}	0	0	0	368640	0	0

Q_2	δ^5	δ^6	δ^7	δ^8	δ^9	δ^{10}
ω^4	$149084928\beta^5$	$-925793280\beta^6$	$2224696320\beta^7$	$-2548039680\beta^8$	$1433272320\beta^9$	$-322486272\beta^{10}$
ω^5	$640949760\beta^4$	$-2710195200\beta^5$	$3792199680\beta^6$	$-2587852800\beta^7$	$716636160\beta^8$	0
ω^6	$271939680\beta^3$	$-1363132800\beta^4$	$6592337280\beta^5$	$-8510054400\beta^6$	$5016453120\beta^7$	$-1209323520\beta^8$
ω^7	$-684326880\beta^2$	$-1292803200\beta^3$	$1025308800\beta^4$	$-2983495680\beta^5$	$2194698240\beta^6$	$-403107840\beta^7$
ω^8	340511040β	$2854137600\beta^2$	$409363200\beta^3$	$-1811980800\beta^4$	$2225180160\beta^5$	$-868700160\beta^6$
ω^9	-64758528	-1398798720β	$-3223653120\beta^2$	$2528686080\beta^3$	$-691960320\beta^4$	$48107520\beta^5$
ω^{10}	0	273585600	1693422720β	$356659200\beta^2$	$-612956160\beta^3$	$-20321280\beta^4$
ω^{11}	0	0	-309674880	-470361600β	$90823680\beta^2$	$16865280\beta^3$
ω^{12}	0	0	0	42439680	-3939840β	$-6635520\beta^2$
ω^{13}	0	0	0	0	-207360	829440β
ω^{14}	0	0	0	0	0	-27648

Q_3	δ^0	δ^1	δ^2	δ^3	δ^4	δ^5
ω^8	-553584375	246037500β	$1109902500\beta^2$	$-1119744000\beta^3$	$93312000\beta^4$	$163296000\beta^5$
ω^9	0	2214337500	-1016955000β	$-2122848000\beta^2$	$1912896000\beta^3$	$-513216000\beta^4$
ω^{10}	0	0	-2690010000	1043928000β	$699840000\beta^2$	$-166212000\beta^3$
ω^{11}	0	0	0	922914000	-148716000β	$-96228000\beta^2$
ω^{12}	0	0	0	0	-5832000	32076000β
ω^{13}	0	0	0	0	0	-2916000

Table 1.— Coefficients of the polynomials $Q_i \equiv Q_i(\delta, \omega; \beta),\ i = 1, 2, 3.$

5 Conclusions

We have studied the occurrence of equatorial and halo periodic orbits of charged particles around a magnetic planet, where the perturbations taken into account include the gravity potential with the oblateness coefficient.

The main novelty of our approach is that if the oblateness coefficient is included there is always a circular equatorial orbit whose stability character depends on δ and ω.

Outside of the equatorial plane, for given values of δ and ω there may appear a couple of halo orbits symmetric to the equatorial plane whose stability depend upon the specific values of δ and ω.

Our analysis generalises the work by Howard *et al.* [1] and exhibits a richer dynamics due to the effect of the oblateness of the planet.

Acknowledgments

We acknowledge financial support from Ministerio de Ciencia y Tecnología (Project # BFM2002-03157), Gobierno de Navarra (Resolución 18/2005) and Ministerio de Educación y Ciencia (Project # MTM2005-08595) from Spain.

References

[1] H. R. Dullin, M. Horányi and J. E. Howard, Generalizations of the Størmer problem for dust grain orbits, *Phys. D*, **171**, 178–195, 2002.

[2] A. M. Fridman and N. N. Gor'kavyi, *Physics of Planetary Rings: Celestial Mechanics of Continuous Media*, Springer–Verlag, Berlin and New York, 1999.

[3] D. P. Hamilton, Motion of dust in a planetary magnetosphere: Orbit–averaged equations for oblateness, electromagnetic and radiation forces with application to Saturn's E ring, *Icarus*, **101**, 244–264, 1993. Erratum, *Icarus*, **103**, 161, 1993.

[4] J. E. Howard, H. R. Dullin and M. Horányi, Stability of halo orbits, *Phys. Rev. Lett.*, **84**, 3244–3247, 2000.

[5] J. E. Howard, M. Horányi and G. E. Stewart, Global dynamics of charged dust particles in planetary magnetospheres, *Phys. Rev. Lett.*, **83**, 3993–3996, 1999.

[6] M. Iñarrea, V. Lanchares, J. Palacián, A. I. Pascual, J. P. Salas and P. Yanguas, The Keplerian regime of a charged particle in planetary magnetospheres, *Phys. D*, **197**, 242–268, 2004.

[7] M. Iñarrea, V. Lanchares, J. Palacián, A. I. Pascual, J. P. Salas and P. Yanguas, Global dynamics of dust grains in magnetic planets, *Phys. Lett. A*, **338**, 247–252, 2005.

[8] M. Iñarrea, V. Lanchares, J. Palacián, A. I. Pascual, J. P. Salas and P. Yanguas, Reduction of some perturbed Keplerian problems, *Chaos Solitons & Fractals*, **27**, 527–536, 2006.

[9] C. D. Murray and S. F. Dermott, *Solar System Dynamics*, Cambridge University Press, Cambridge, 1999.

[10] C. Størmer, Sur les trajectories des corpuscules electriques, *Arch. Sci. Phys. Nat.*, **24**, 5–18; ibid 113–158; ibid 221–247, 1907.

[11] C. Størmer, *The Polar Aurora*, Clarendon Press, Oxford, 1955.