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On variations of the shape Hessian and sufficient conditions for the stability of critical shapes

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Abstract. To study the nature of critical shapes in shape optimisation, some continuity properties of second order shape derivatives are needed. Since different non-equivalent topologies are involved, the Taylor-Young formula does not allow us to deduce that a critical shape is a local strict minimum of the shaping function even if the Hessian at that point is definite positive. The main result of this work provides an upper bound for the variations of the second derivative of some general shape functional of elliptic type along those paths. As an application, we state a theorem about stability of critical shapes. This estimation is then applied to some examples to solve the stability of critical shapes in some explicit examples.

Sobre las variaciones del Hessiano de la forma y condiciones suficientes para la estabilidad de formas críticas

Resumen. Para el estudio de la naturaleza de formas críticas en optimización de formas se requieren algunas propiedades de continuidad sobre las derivadas de segundo orden de las formas. Dado que la fórmula de Taylor-Young involucra a diferentes topologías que no son equivalentes, dicha fórmula no permite deducir cuando una forma crítica es un mínimo local estricto de la función forma pese a que su Hessiano sea definido positivo en ese punto. El resultado principal de este trabajo ofrece una cota superior para las variaciones de la segunda derivada de un cierto funcional de tipo elíptico a lo largo de esas curvas. Como aplicación se da un teorema sobre la estabilidad de formas críticas. Finalmente, se aplica esa estimación a algunos ejemplos para analizar la estabilidad de formas óptimas en algunos ejemplos explícitos.

1. Introduction

Motivations. The general setting of this work is shape optimisation: in a family $\mathcal{O}$ of subsets of $\mathbb{R}^d$, a real-valued function E must be minimized. This function E will be called the shaping function. Following Hadamard's approach, we will use a differential calculus developed by several authors (see [14], [16]). To find an extremum in the class of domains on which a derivative is defined, the corresponding Euler equation must be solved. Its possible solutions are called critical shapes.

In this work, we are concerned with the stability of an arbitrary critical shape denoted by Ω_0: when has the shaping function E a local strict minimum at Ω_0? Classically in optimisation, this question is adressed through the use the second derivative of E at Ω_0. Several authors ([11], [2] and [9]) have considered the use of the second derivative to study optimality of shapes in the field of shape optimisation. The first difficulty of this task is to define its second derivative. In order to guaranty existence of the second derivative, the

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class $\mathcal{O}$ of admissible domains must be restricted to the open subsets of $\mathbb{R}^d$ with a $\mathcal{C}^2$-boundary. Therefore, we deal in this work with domains of at least this regularity.

In the theory of differentiable optimisation, we only need to know the sign of the shape Hessian at Ω_0 to fully answer this question. But, in shape optimisation, specific difficulties arise due to some norm-incompatibility: for the second derivative, the coercivity-norm $\|.\|_w$ is often weaker than the differentiability-norm $\|.\|_s$. Let us refer to [8] for a concrete example of such a situation known as the magnetic shaping problem. In that example, J. Descloux pointed out that the coercivity holds only in the $H^{1/2}$ norm while differentiability holds in $\mathcal{C}^2$ topology. The Taylor-Young formula writes (here 0 denotes the critical shape and h a perturbation):

$$E(h) = E(0) + \underbrace{DE(0).h}_{=0} + \underbrace{\frac{1}{2}D^2E(0).(h,h)}_{\geq C\|h\|^2_{H^{1/2}}} + o(\|h\|_{\mathcal{C}^2}).$$

Since the quantity $o(\|h\|_{\mathcal{C}^2})$ is not smaller than $C\|h\|^2_{H^{1/2}}$, such a basic argument does not insure that the critical point is a local strict minimum of E (see the exemple given in [5]). This paper provides a method to study this problem.

The key for proving the stability of critical shapes in [5] was a precise estimate of the variations of the second derivative around a critical shape. This estimate is interesting by itself. The main result of this paper is Theorem 1 an extension of the results stated in [5]. The conditions required on the operator and, more important, on the shaping function itself are much weaker. In addition, we state the results in any space dimension and not only in dimension two. This generalisation increases deeply the difficulty of the proofs and requires new ideas. Furthermore, similar inequalities are obtained for purely geometrical functionals such as the volume and the perimeter. We use Theorem 1 to prove the stability of critical shapes in Section 4.

The leading idea to study the stability is to reduce the problem to an one-dimensional situation. Any arbitrary shape Ω_1 close to a critical shape Ω_0 may be written as $\Omega_1 = \Theta(\Omega_0)$ where Θ is a diffeomorphism of $\mathbb{R}^d$. The flow $\Phi_{\Theta,t}$ of an adequate autonomous vector field $\boldsymbol{X}_\Theta$ (i.e. the solution of $\partial_t\phi + \boldsymbol{X}_\Theta(\phi) = 0$ with the initial condition $\phi(0,x) = x$) defines a regular path $\big(\Omega_t := \Phi_{\Theta,t}(\Omega_0)\big)_{t\in(0,1)}$ in $\mathcal{O}$; this path connects the critical shape Ω_0 to $\Omega_1 = \Theta(\Omega_0)$. To study the stability of critical shapes, we write the Taylor formula with integral rest for the shaping function along this path. This requires to consider the function $e_\Theta : [0,1] \to \mathbb{R}, t \mapsto E(\Omega_t)$ and to compute its second derivative e''_Θ. The main result of this article is (3) a precise upper bound of the variations $|e''_\Theta(t) - e''_\Theta(0)|$ in terms of both the norm of differentiability and the norm of coercivity. This bound allows us to deduce that $e''_\Theta(t) > 0$ from the hypothesis of weak coercivity that states $e''_\Theta(0) > 0$. Then, stability is easily deduced from the Taylor formula.

The organisation of the paper follows this idea. The second section deals briefly with the construction of $\boldsymbol{X}_\Theta$ and with the properties of his flow stated in Propositions 2, 3 and 4. Then, we recall classical results of shape differentiability to justify the existence of the first and second derivatives of e_Θ. The third section concerns the proof of Theorem 1. In the last section of this paper, we apply the described method to some specific stability studies.

Statement of the main result. Let us make precise the class of shape functionals we will consider throughout this paper. We first introduce some notations. Let $d \geq 2$ denote the dimension. Let $L = -\operatorname{div}\big(A\nabla.\big)$ be an strictly and uniformly elliptic operator with $A = A(x)$ is a $\mathcal{C}^2$ $d \times d$-matrix with real coefficients. Let $\mathcal{O}$ be the class of admissible domains that are open bounded subsets of $\mathbb{R}^d$ of class $\mathcal{C}^{2,\alpha}$. Possible constraints (we will later consider prescribed volume constraints) are included in the definition of $\mathcal{O}$. We denote by $\mathcal{V}$ the vector space of admissible deformation fields. Let j and f be functions in $\mathcal{C}^{0,\alpha}(\mathbb{R}^d,\mathbb{R})$ and $\mathcal{C}^3(\mathbb{R}\times\mathbb{R}^d,\mathbb{R})$ respectively. For each $\Omega \in \mathcal{O}$, we define the state-function u_Ω as the solution in $\mathcal{C}^2(\overline{\Omega})$ of

$$\begin{cases} Lu = j \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1}$$

and we define a shape functional E on $\mathcal{O}$ by

$$E(\Omega) = \int_\Omega f(u_\Omega, \nabla u_\Omega). \tag{2}$$

Let Ω_0 denote a smooth shape. We now state the central result of this paper.

Theorem 1 *There exists a real $\eta_0 \in (0,1)$ and a modulus of continuity $\omega : (0,\eta_0) \to (0,+\infty)$, which depends only of Ω_0, L, f and j, such that for all $\eta \in (0,\eta_0)$ and all $\Theta \in \mathcal{C}^{2,\alpha}(\mathbb{R}^d,\mathbb{R}^d)$ with*

$$\|\Theta - Id_{\mathbb{R}^d}\|_{2,\alpha} \leq \eta,$$

there exists a vector field $\boldsymbol{X}_\Theta$ the flow $\Phi_{\Theta,t}$ of which defines a path $(\Omega_t = \Phi_{\Theta,t}(\Omega_0))_{t\in[0,1]}$ between Ω_0 and $\Theta(\Omega_0)$ such that, for all t in $[0,1]$, the following estimate holds,

$$\left| e''_\Theta(t) - e''_\Theta(0) \right| \leq \omega(\eta) \|\langle \boldsymbol{X}_\Theta, \boldsymbol{n}_{\partial\Omega_0}\rangle\|^2_{H^{1/2}(\partial\Omega_0)}, \tag{3}$$

where $\boldsymbol{n}_{\partial\Omega_0}$ denotes the exterior normal unitary field to $\partial\Omega_0$. Moreover, if the diffeomorphism Θ preserves the volume of Ω_0, then X can be chosen divergence-free.

The functionals of volume $\mathfrak{V}$ and perimeter $\mathfrak{P}$ are defined for all $\Omega \in \mathcal{O}$ as $\mathfrak{V}(\Omega) = \mathcal{L}^d(\Omega)$ and $\mathfrak{P}(\Omega) = \mathcal{H}^{d-1}(\partial\Omega)$. Here, $\mathcal{L}^d$ denotes the Lebesgue measure on $\mathbb{R}^d$ and $\mathcal{H}^{d-1}$ the $d-1$ dimensional Haussdorf measure. The same approach leads to similar statements where the weak norm turns to be either the $L^2(\partial\Omega_0)$-norm in the case of $\mathfrak{V}$ (see (45)), either the $H^1(\partial\Omega_0)$-norm in the case of $\mathfrak{P}$ (see (46)). The main application of Theorem 1 is the next result.

Theorem 2 (unconstrained problems) *Assume that $\Omega_0 \in \mathcal{O}$ is a critical shape minimizing E and assume that the Hessian $D^2E(\Omega_0)$ is coercive in the weak norm $\|.\|_{H^{1/2}(\partial\Omega_0)}$ then Ω_0 is stable in the following meaning: there exists an open ball around Ω_0 in the $\mathcal{C}^{2,\alpha}$ topology on which E has a local strict minimum at Ω_0.*

PROOF. By assumption, $D^2E(\Omega_0$ is coercive in $H^{1/2}(\partial\Omega_0)$ and there exists $C_{coer} > 0$ such that

$$e''_\Theta(0) = D^2E(\Omega_0; \boldsymbol{X}_\Theta, \boldsymbol{X}_\Theta) > C_{coer}\|\langle \boldsymbol{X}_\Theta, \boldsymbol{n}_{\partial\Omega_0}\rangle\|^2_{H^{1/2}(\partial\Omega_0)}. \tag{4}$$

Fix $\eta > 0$, and choose any Ω in $B(\Omega_0)$. Consider the path given by Theorem 1. The Taylor formula for e_Θ writtes along the path

$$e_\Theta(1) = e_\Theta(0) + \int_0^1 (1-t)e''_\Theta(t)dt,$$

and we write $e''_\Theta(t)$ as $e''_\Theta(t) = e''_\Theta(0) + e''_\Theta(0) - e''_\Theta(t)$. Therefore, we deduce from (4) and Theorem 1 that if η is chosen small enough, then $e''_\Theta(t) > 0$ on $[0,1]$ and we have $E(\Omega_0) < E(\Omega)$. ■

If we are concerned now with a constrained problem say C, the natural assumption is that the Hessian of the lagrangian is coercive on the kernel of the shape gradient of the constraints. The proof of Theorem 2 is based on the fact that $e''(0) = D^2E(\Omega_0; \boldsymbol{X}_\Theta, \boldsymbol{X}_\Theta)$ is non negative. In the constrainted case, $e''_\Theta(0)$ remains non negative only if the vector field $\boldsymbol{X}_\Theta$ satisfies $DC(\Omega_0, \boldsymbol{X}_\Theta) = 0$. This must be checked for each constraint (see section for examples). The important specific case $C = \mathfrak{V}$, *i.e* the volume of admissible domains is prescribed, can be treated by Theorem 1. The proof is the same than the proof of Theorem 2 by using the divergence-free vector so that the condition $DA(\Omega_0, \boldsymbol{X}_\Theta) = 0$ is automatically satisfied.

Theorem 3 (volume-constrained problems) *Let $v > 0$ be a given real. Assume that Ω_0 is a critical shape for the following optimisation problem*

(C) find $\Omega \in \mathcal{O}$ under the constraint $\mathfrak{V}(\Omega) = v$ that minimises E

and assume that the Hessian of the Lagrangian $L_\Lambda = E + \Lambda\mathfrak{V}$ is coercive is in the weak norm $\|.\|_{H^{1/2}(\partial\Omega_0)}$ then Ω_0 is stable in the following meaning: there exists an open ball B around Ω_0 in the $\mathcal{C}^{2,\alpha}$ topology such that, for all domain $\Omega \in B$ with $\mathfrak{V}(\Omega) = \mathfrak{V}(\Omega_0)$ and $\Omega \neq \Omega_0$, we have $E(\Omega_0) < E(\Omega)$.

2. Some particular paths within the domains

This section is devoted to construct a path in the set $\mathcal{O}$ of admissible domains between a given regular shape Ω_0 and another shape $\Theta(\Omega_0)$ close to the first one. The construction requires some technical computations. Since we are only interested in perturbations of the shape Ω_0, the behaviour of Θ far away from Ω_0 has no importance in this work. Therefore, we restrict ourselves to diffeomorphisms behaving like the identity on the complement of a ball including strictly Ω.

2.1. Construction of the transport fields

Let Ω_0 be an admissible domain with a $\mathcal{C}^{3,\alpha}$ boundary. The domain Ω_0 can be a critical shape. This last assumption is not necessary since the shaping function does not appear in this geometrical construction.

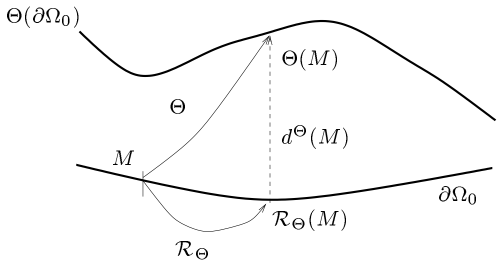
Remark 1 Since we will use normal deformations, we have to loose one rank of regularity. By definition of the shaping function E, the perturbed domains must be in $\mathcal{O}$: that is with a $\mathcal{C}^{2,\alpha}$ boundary. Hence, we assume that Ω_0 is $\mathcal{C}^{3,\alpha}$.

Let $\boldsymbol{n}$ be the outer unitary normal field to $\partial\Omega_0$. In order to define the transport field, we need to recall some basic facts of differential geometry. The application $T_{\partial\Omega_0}$ defined as:

$$T_{\partial\Omega_0}: \begin{array}{ll} \partial\Omega_0 \times \mathbb{R} & \longrightarrow \mathbb{R}^d, \\ (M,h) & \mapsto M + h\boldsymbol{n}(M). \end{array}$$

is well-known to be a local diffeomorphism on a open tubular neighbourhood of $\partial\Omega_0$ we will denote $U_{\partial\Omega_0}$. To perform the geometrical constructions, we need an extension $\breve{\boldsymbol{n}}$ of the normal $\boldsymbol{n}$. For all x in a fixed open neighbourhood $V_{\partial\Omega_0} \subset\subset U_{\partial\Omega_0}$, there exists an unique couple $(M,h) \in \partial\Omega_0 \times \mathbb{R}$ such that $x = T_{\partial\Omega_0}(M,h)\ \big(= M + h\boldsymbol{n}(M)\big)$. We define the extension $\breve{\boldsymbol{n}}$ on $V_{\partial\Omega_0}$ by $\breve{\boldsymbol{n}}(x) = \boldsymbol{n}(M)$. On $\mathbb{R}^d \setminus U_{\partial\Omega_0}$, we set $\breve{\boldsymbol{n}} = 0$ and use a cut-off technic to obtain a globally defined and smooth $\breve{\boldsymbol{n}}$.

Let us fix Θ a $\mathcal{C}^{2,\alpha}$ global diffeomorphism of $\mathbb{R}^d$ closed enough to the identity $I_{\mathbb{R}^d}$ to have $\Theta(\partial\Omega_0) \subset V_{\partial\Omega_0}$. The action of Θ on the boundary $\partial\Omega_0$ reduces to its normal component modulo a sliding term $\mathcal{R}_\Theta$ as shown in the following figure.



$\forall M \in \partial\Omega_0$, we define $\mathcal{R}_\Theta(M)$ and $d^\Theta(M)$ as
$\big(\mathcal{R}_\Theta(M), d^\Theta(M)\big) = T^{-1}_{\partial\Omega_0}\big(\Theta(M)\big)$.
We set

$$d^\Theta(M) := \langle \Theta(M) - \mathcal{R}_\Theta(M), \boldsymbol{n}(\mathcal{R}_\Theta(M))\rangle,$$
$$\Theta(M) = \mathcal{R}_\Theta(M) + d^\Theta(M)\boldsymbol{n}(\mathcal{R}_\Theta(M)).$$

Moreover, if $||\Theta - Id_{\mathbb{R}^d}||_{2,\alpha}$ is small enough, there exists a constant C depending only of Ω_0 (via the $\mathcal{C}^{3,\alpha}$ norm of its boundary, see [5] for the proof) such that

$$||d^\Theta||_{2,\alpha} \leq C||\Theta - Id_{\mathbb{R}^d}||_{2,\alpha}. \tag{5}$$

A first vector field. We extend d^Θ as a constant on the orbits of $\breve{\boldsymbol{n}}$ in $V_{\partial\Omega_0}$, as 0 outside of $U_{\partial\Omega_0}$, use a cut off to define a global extension we still denote d^Θ. We define a vector field as

$$\boldsymbol{X}^1_\Theta = d^\Theta \breve{\boldsymbol{n}} \text{ on } \mathbb{R}^d. \tag{6}$$

Since $\boldsymbol{X}^1_\Theta$ has a constant modulus on its orbits in $V_{\partial\Omega_0}$, the flow $\Phi^1_{\Theta,t}$ of $\boldsymbol{X}^1_\Theta$ maps any point M of the boundary $\partial\Omega$ onto the point $M\ +\ t\,d^\Theta(M)\ \boldsymbol{n}(M)$. In particular, at $t = 1$ the boundary $\partial\Omega_0$ is sent exactly on the boundary of $\partial\Theta(\Omega_0)$.

Remark 2 The vector field $\boldsymbol{X}^1_\Theta$ does not preserve the volume of Ω_0 even if Θ is an volume-preserving diffeomorphism. Moreover, $\boldsymbol{X}^1_\Theta$ does not even belong, in general, to the hyper-plane $\mathcal{H} = kerD\mathfrak{V}$ of $\mathcal{V}$. The gradient $D\mathfrak{V}$ is a continuous linear form on $\mathcal{V}$ defined by

$$D\mathfrak{V}(\Omega_0) : \boldsymbol{V} \longmapsto \int_{\Omega_0} \mathrm{div}\big(\boldsymbol{V}\big) = \int_{\partial\Omega_0} \langle \boldsymbol{V}, \boldsymbol{n}\rangle.$$

But, for a general Θ, one has (in the notations of the speed-method, see [16]) :

$$D\mathfrak{V}(\Omega_0; \boldsymbol{X}^1_\Theta) = \int_{\partial\Omega_0} \langle \boldsymbol{X}^1_\Theta, \boldsymbol{n}\rangle = \int_{\partial\Omega_0} d^\Theta \neq 0 \;\;\Rightarrow\;\; \boldsymbol{X}^1_\Theta \notin \mathcal{H}.$$

The Lagrange conditions of order two for the constraint $A = Constant$ provides coercivity for the Hessian only on $\mathcal{H}$. Therefore, the sign of $e''_\Theta(0) = D^2E(\boldsymbol{X}^1_\Theta, \boldsymbol{X}^1_\Theta)$ is unknown. Our strategy explained in Section 1**.** fails if the condition $\boldsymbol{V} \in \mathcal{H}$ is not satisfied.

We now construct a divergence-free vector field $\boldsymbol{X}^2_\Theta$. Our construction imposes the lost of an additional derivative even if we work with arbitrary transverse deformations and not only with normal ones (see [4] where I also dicuss how to avoid this loss if $d = 2$). Therefore, from now on, Ω_0 is assumed with a $\mathcal{C}^{4,\alpha}$ boundary.

Volume-preserving deformations, a second vector field $\boldsymbol{X}^2_\Theta$.

Proposition 1 *Let Ω_0 be an open subset of $\mathbb{R}^d$ with a $\mathcal{C}^{4,\alpha}$ boundary $\partial\Omega_0$. If $\Theta \in \mathcal{C}^{2,\alpha}(\mathbb{R}^d, \mathbb{R}^d)$, with $\|\Theta - Id\|_{2,\alpha}$ small enough, preserves the volume of Ω_0, then there exists a vector field $\boldsymbol{X}^2_\Theta \in \mathcal{C}^{2,\alpha}(\mathbb{R}^d, \mathbb{R}^d)$ such that :*

- *in the tubular neighbourhood $V_{\partial\Omega_0}$, the field $\boldsymbol{X}^2_\Theta$ writes $\boldsymbol{X}^2_\Theta = m\breve{\boldsymbol{n}}$ with $m \in \mathcal{C}^{2,\alpha}(V_{\partial\Omega_0}, \mathbb{R})$ and is divergence-free,*
- *the flow $\Phi^2_{\Theta,t}$ of $\boldsymbol{X}^2_\Theta$ preserves the volume of Ω_0 for all $t \in [0,1]$. Moreover, it maps $\partial\Omega_0$ onto $\Theta(\partial\Omega_0)$ at the time $t = 1$.*

PROOF. The unknown vector field $\boldsymbol{X}^2_\Theta$ is searched in $V_{\partial\Omega_0}$ as

$$\boldsymbol{X}^2_\Theta(M) = m(M)\breve{\boldsymbol{n}}(M) \text{ and } \mathrm{div}\big(\boldsymbol{X}^2_\Theta\big) = 0$$

where m is a real-valued function to-be-determined and $\breve{\boldsymbol{n}}$ is the formerly defined extension of the normal field. In a second time, we will discuss how to extend that field to the whole space. In the following lines, T stands for $T_{\partial\Omega_0}$ for the sake of readability.

We first characterise the functions m such that $\mathrm{div}\big(m\breve{\boldsymbol{n}}\big) = 0$ in $V_{\partial\Omega_0}$. This requires to compute the divergence of the extended field $\breve{\boldsymbol{n}}$. We introduce an atlas for $\partial\Omega_0$ with the maps (ψ_i) mapping the coordinates $(s,h) \in \mathbb{R}^{d-1} \times \mathbb{R}$ to $\mathbb{R}^d$. In these notations, $\partial\Omega_0$ is the set of points that can be written as $\psi_i(s,0)$. The partial derivative with respect to h of the determinant of DT is :

$$\partial_h \det\big(DT\big)\big(\psi_i(s,h)\big) = \mathrm{Tr}\big(\partial_h DT\big(\psi_i(s,h)\big)(DT)^{-1}\big(\psi_i(s,h)\big)\big) \det\big(DT\big)\big(\psi_i(s,h)\big). \tag{7}$$

From the definition of $\breve{\boldsymbol{n}}$, one has

$$\breve{\boldsymbol{n}}\big[\psi_i(s,0) + h\boldsymbol{n}\big(\psi_i(s,0)\big)\big] = \boldsymbol{n}\big(\psi_i(s,0)\big),$$

that we differentiate to get

$$D\breve{\boldsymbol{n}}\big[\psi_i(s,0) + h\boldsymbol{n}\big(\psi_i(s,0)\big)\big] = \big[D_s\boldsymbol{n}(\psi_i(s,0)\big), 0\big].(DT)^{-1}\big(\psi_i(s,h)\big).$$

Since one has

$$DT\big(\psi_i(s,h)\big) = [D_s\psi_i\big(\psi_i(s,0)\big) + h\boldsymbol{n}\big(\psi_i(s,0)\big), \boldsymbol{n}\big(\psi_i(s,0)\big)].$$

where the first coefficient is a $d \times (d-1)$ matrix, we get $\partial_h DT\big(\psi_i(s,h)\big) = [D_s\breve{\boldsymbol{n}}, 0]$ that we substitute in (7) to obtain the expression of $D\breve{\boldsymbol{n}}$. We take the trace of $D\breve{\boldsymbol{n}}$ and obtain

$$\operatorname{div}\big(\breve{\boldsymbol{n}}\big) = \frac{\partial_h \det\big(DT\big)(M,h)}{\det\big(DT\big)(M,h)}. \tag{8}$$

The divergence-free condition writes $\langle \nabla m, \breve{\boldsymbol{n}}\rangle + m \operatorname{div}\big(\breve{\boldsymbol{n}}\big) = 0$. Therefore, m solves the equation $\partial_h m + m \operatorname{div}\big(\breve{\boldsymbol{n}}\big) = 0$ that provides, after integration,

$$m(M,h) = \frac{f(M)}{\det\big(DT\big)(M,h)}, \tag{9}$$

where f is a real-valued function defined on $\partial\Omega$ and to-be-determined and M stands for $\psi_i(s,0)$ for simplicity. We use a shooting method to choose the correct f in order to map the boundary $\partial\Omega_0$ onto $\Theta(\Omega_0)$. From Hadamard's representation, $\Theta(\partial\Omega)$ is parametrised by $M + d^\theta(M)\boldsymbol{n}(M)$ with M in $\partial\Omega_0$. We work on the normal lines and solve the family of Cauchy problems

$$\begin{cases} \forall t \in [0,1], \;\; \partial_t h(M,t) + m(M, h(M,t)) = 0, \\ \forall M \in \partial\Omega, \;\; h(M,0) = 0, \end{cases} \tag{10}$$

with the additional admissibility condition $h(M,1) = d^\theta(M)\boldsymbol{n}(M)$. We get

$$\left\{ \begin{array}{rcl} f(M) & = & G\Big(M, d^\Theta(M)\Big), \\ \boldsymbol{X}^2_\Theta(M, h(M,t)) & = & -\dfrac{G\Big(M, d^\Theta(M)\Big)}{\det\big(DT\big)(M, h(M,t))}\, \breve{\boldsymbol{n}}(M, h(M,t)), \end{array} \right. \tag{11}$$

where $G(M,h)$ be the anti-derivative of $h \longmapsto \det\big(DT\big)(M,h)$ normalised in order to vanish on the boundary $\partial\Omega_0$. Moreover, we can compute $h(M,t)$ of $\partial\Omega(t)$ from $\partial\Omega_0$ by solving the equation

$$G(M, h(M,t)) + f(M)t = 0. \tag{12}$$

For $M \in \partial\Omega_0$, we define M_t as $\big(M, h(M,t)\big) = \Phi^2_{\Theta,t}(M)$.

Let us consider any regular extension of the field $\boldsymbol{X}^2_\Theta$ outside $V_{\partial\Omega_0}$. Lemma 4 (see further in Section 2.3.) shows that the application $a_\Theta\, :\; t \mapsto \mathfrak{V}(\Omega_t)$ is twice differentiable on $[0,1]$. Its second derivative writes

$$\frac{d^2}{dt^2} a_\Theta(t) = \int_{\partial\Omega_t} \operatorname{div}\big(\boldsymbol{X}^2_\Theta\big)\langle \boldsymbol{X}^2_\Theta, \boldsymbol{n}(t)\rangle. \tag{13}$$

Since, by construction, $\partial\Omega_t = \{\Phi^2_{\Theta,t}(M),\; M \in \partial\Omega_0\} \subset V_{\partial\Omega_0}$ for $t \in [0,1]$, $\boldsymbol{X}^2_\Theta$ is divergence-free in $V_{\partial\Omega_0}$ and the second derivative a''_Θ vanishes on $[0,1]$. Therefore, the function a_Θ is affine on $[0,1]$. Since, by hypothesis, its extremal values are the same, a_Θ is constant on $[0,1]$. The flow of any smooth extension of $\boldsymbol{X}^2_\Theta$ preserves the volume of Ω_0 on $[0,1]$.■

Remark 3

1) We first give some remarks on that proof.

- Since only perturbations are considered, it seems reasonnable to use the Local Inversion Theorem. In fact, the losses of regularity make such a try very difficult.

- Moreover, we need some special estimates on the deformation field $\boldsymbol{X}^2_\Theta$ to prove Theorem 1. A method based on the Local Inversion Theorem cannot easily provide those estimates stated in Proposition 3 (see further in Section 2.2.). Their proof is based on the explicit expression on $\boldsymbol{X}^2_\Theta$.

- Obviously, if Ω is another open subset of $\mathbb{R}^d$, such a property is not true in general since $\operatorname{div}\big(\boldsymbol{X}^2_\Theta\big)$ has no particular reason to vanish uniformly on $\mathbb{R}^d$.

2) We have considered only motion along the normal lines in $V_{\partial\Omega_0}$. Therefore, the flow of the vector field we have constructed does not realise a path in diffeomorphisms between $Id_{\mathbb{R}^d}$ and Θ but only a path connecting Ω_0 to $\Theta(\Omega_0)$ via the sets $\Phi_{\Theta,t}(\Omega_0)$: in other terms, $\Phi_1(\Omega_0) = \Theta(\Omega_0)$ and $\Phi_1 \neq \Theta$ hold!

2.2. Properties of that path

The results stated in this paragraph are the key-estimates to prove Theorem 1. The section is split in three parts each concerning a given kind of quantity defined along the paths. All the estimates stated in this section are uniform with respect to Θ in the ball of center $Id_{\mathbb{R}^d}$ and radius $\eta > 0$: this an essential point. The parameter η is assumed to besmall enough so that $\Theta(\partial\Omega_0) \subset V_{\partial\Omega_0}$ holds for all Θ with $||\Theta - Id_{\mathbb{R}^d}||_{2,\alpha} \leq \eta$. We will even restrict η during this section.

Propositions 2 and 4 are rather classical results, therefore we give only the sketches of the proofs. Proposition 3 is specific to the paths we have constructed in Section 2.1. From now on, $\boldsymbol{X}_\Theta$ denotes either $\boldsymbol{X}^1_\Theta$ either $\boldsymbol{X}^2_\Theta$ and Ω_0 denotes a bounded $\mathcal{C}^{4,\alpha}$ domain.

Study of purely geometrical quantities. A preliminary result shows that $\Phi_{\Theta,t} - Id_{\mathbb{R}^d}$ is controlled by $\Theta - Id_{\mathbb{R}^d}$ in the $\mathcal{C}^{2,\alpha}$ norm. Since $\boldsymbol{X}_\Theta$ is autonomous, the following lemma is trivial in the $\mathcal{C}^0$ norm.

Lemma 1 *Let Θ be a $\mathcal{C}^{2,\alpha}$-diffeomorphism of $\mathbb{R}^d$ on itself close to the identity $Id_{\mathbb{R}^d}$. For the fields $\boldsymbol{X}_\Theta$ exhibited in the former section, there is a constant C_1 such that, for all t in $[0,1]$, we get*

$$||\Phi_{\Theta,t} - Id_{\mathbb{R}^d}||_{2,\alpha} \leq C_1 ||\Theta - Id_{\mathbb{R}^d}||_{2,\alpha}.$$

SKETCH PF THE PROOF. In $V_{\partial\Omega_0}$, the field $\boldsymbol{X}_\Theta$ is given by (6) or by (11). Therefore, the derivatives of the product of f by $1/\det\big(DT\big)$ and $\breve{\boldsymbol{n}}$ must be controlled. These derivatives write as a sum given by Leibnitz's formula. The quantities $1/\det\big(DT\big)$ and $\breve{\boldsymbol{n}}$ are geometrical and $\mathcal{C}^{3,\alpha}$ quantities defined only by Ω_0. Their $||.||_{2,\alpha}$ norms on $\overline{V_{\partial\Omega_0}}$ are therefore bounded without any dependency with respect to Θ.

On the other side, the factor f depends on Θ as shown by its expression (6) or (11). Recall that G is the anti-derivative of $h \longmapsto \det\big(DT\big)(M,h)$ vanishing on $\partial\Omega_0$. Hence, G is a continuous geometrical quantity in the compact $\overline{V_{\partial\Omega_0}}$. From Faà de Bruno's formula[1], we deduce

$$||G\Big(M, d^\Theta(M)\Big)||_{2,\alpha} \leq ||G||_{\mathcal{C}^{2,\alpha}(\overline{\mathcal{T}_{\tau'}(\partial\Omega_0)})} ||d^\Theta(M)||_{2,\alpha}. \tag{14}$$

We then use the estimation (5) which bounds d^Θ to get:

$$||\boldsymbol{X}_\Theta(\Phi_{\Theta,t})||_{2,\alpha} \leq C(||\Phi_{\Theta,t}||_{2,\alpha}) ||\boldsymbol{X}_\Theta||_{2,\alpha}.$$

Then one concludes easily. ■

We now examine how the boundary $\partial\Omega_t$ behaves for $t \in [0,1]$. We assume $\eta < 1/2C_1$ to have $||\Phi_{\Theta,t} - Id_{\mathbb{R}^d}||_{2,\alpha} < 1/2$ for all $t \in [01]$. The following proposition described the evolution of geometrical quantities defined on $\partial\Omega_t$.

Proposition 2 *There exists a constant $C > 0$ such that for all t $\in [01]$:*

[1]The Faà de Bruno's formula gives the expressions of derivatives obtained by iteration of the chain rule.

1. $\|D\Phi_{\Theta,t} - I_d\|_{L^\infty} + \|D^2\Phi_{\Theta,t}\|_{L^\infty} \leq \|\Phi_{\Theta,t} - Id_{\mathbb{R}^d}\|_{2,\alpha}$.
2. $\|D\Phi_{\Theta,t}^{-1} - I_d\|_{L^\infty} + \|D[D\Phi_{\Theta,t}^{-1}]\|_{L^\infty} \leq C\|\Phi_{\Theta,t} - Id_{\mathbb{R}^d}\|_{2,\alpha}$.
3. *Let* $J(t)$ *be the Jacobian* $\det\big(D\Phi_{\Theta,t}\|{}^t(D\Phi_{\Theta,t})^{-1}n_0\|\big)$. *Then, we have*
$$\|J(t) - 1\|_{\mathcal{C}^1(\partial\Omega_0)} \leq C\|\Phi_{\Theta,t} - Id_{\mathbb{R}^d}\|_{2,\alpha}. \tag{15}$$
4. *If* $\boldsymbol{n}_t$ *denotes the unitary outer normal vector field to* $\Phi_{\Theta,t}(\partial\Omega_0)$*, then*
$$\|\boldsymbol{n}_t \circ \Phi_{\Theta,t} - \boldsymbol{n}_0\|_{C^1(\partial\Omega_0)} \leq C\|\Phi_{\Theta,t} - Id_{\mathbb{R}^d}\|_{2,\alpha}. \tag{16}$$
5. *Moreover, if* Θ *has the* $\mathcal{C}^3$*-regularity and satisfies* $\|\Theta - Id_{\mathbb{R}^d}\|_3 \leq 1/2$*, then the following estimate makes (16) more precise*
$$\|\boldsymbol{n}_t \circ \Phi_{\Theta,t} - \boldsymbol{n}_0\|_{\mathcal{C}^2(\partial\Omega_0)} \leq C\|\Phi_{\Theta,t} - Id_{\mathbb{R}^d}\|_3. \tag{17}$$

SKETCH OF THE PROOF. Points 1 and 2 are direct applications of the definition of Hölder norms and of the inversion formula using power series since Hölder spaces are Banach spaces.

The third point concerns the Jacobian determinant J involved in the change of variables for boundary integrals when an integral defined on $\partial\Omega_t$ is changed into an integral defined on $\partial\Omega_0$. Since the determinant is an $\mathcal{C}^\infty$ function on $M_n(\mathbb{R})$, we just have to bound the distance between I_d and $\|{}^t(D\Phi_{\Theta,t})^{-1}n_0\|D\Phi_{\Theta,t}$ to deduce the L^∞ part. This is a direct application of points 1 and 2. In order to show the $\mathcal{C}^1$ estimate on J, this determinant must be differentiated. The use of Faà de Bruno's formula allows to separate the contribution of the norm in order to bound the derivative.

Points 4 and 5 deal with the outer normal field to $\partial\Omega_t$. The signed distance $d_{\partial\Omega_0}$ to $\partial\Omega_0$ gives a global definition of $\boldsymbol{n}$. If Ψ is a diffeomorphism, then $d_{\partial\Omega_0} \circ \Psi^{-1}$ defines the domain $\Psi(\Omega_0)$ as its 0 level set. In particular, for all $M \in \partial\Omega_0$ and all $t \in [0, 1]$,
$$\Big[\boldsymbol{n}_t \circ \Phi_{\Theta,t} - \boldsymbol{n}\Big](M) = \frac{\nabla(d_{\partial\Omega_0} \circ \Phi_{\Theta,t}^{-1})}{\|\nabla(d_{\partial\Omega_0} \circ \Phi_{\Theta,t}^{-1})\|}\big(\Phi_{\Theta,t}(M)\big) - \frac{\nabla d_{\partial\Omega_0}}{\|\nabla d_{\partial\Omega_0}\|}(M)$$
holds. The main interest of this formulation in this work is to put the whole dependency in the variable t in $\Phi_{\Theta,t}$. We then differentiate this expression to obtain the wanted upper bounds as a consequence of the Faà de Bruno's formula. ■

Properties of $\boldsymbol{X}_\Theta$. To prove Proposition 3, we need the following lemma about estimates of a product. The main arguments to prove this lemma are simply harmonic extensions of traces and the continuity of the trace operator from $H^1(\Omega)$ into $H^{1/2}(\partial\Omega)$.

Lemma 2 *Let* Ω *be in* $\mathcal{O}$*. There is a constant* $c(\Omega)$ *depending only on* Ω *such that:*

- *if* $f \in H^{1/2}(\partial\Omega)$ *and if* $g \in \mathcal{C}^1(\partial\Omega)$*, then*
$$\|fg\|_{H^{1/2}(\partial\Omega)} \;\leq\; c(\Omega)\; \|f\|_{H^{1/2}(\partial\Omega)}\; \|g\|_{\mathcal{C}^1(\partial\Omega)}.$$
- *if* $f \in H^1(\partial\Omega)$ *and if* $g \in \mathcal{C}^1(\partial\Omega)$*, then*
$$\|fg\|_{H^1(\partial\Omega)} \;\leq\; c(\Omega)\; \|f\|_{H^1(\partial\Omega)}\; \|g\|_{\mathcal{C}^1(\partial\Omega)}.$$

We now establish very important estimates about the variations of the vector fields in some norms on $\partial\Omega_t$. The estimations stated are not true for any vector field and are essential to prove Theorem 1. This is the reason why we have to restrict ourselves to the vector fields $\boldsymbol{X}_\Theta^1$ and $\boldsymbol{X}_\Theta$ we have constructed in Section 2.1.

Proposition 3 *There is a constant C depending only of Ω_0 such that*

$$\forall t \in [0,1], \begin{cases} \left\|m \circ \Phi_{\Theta,t} - m\right\|_{L^2(\partial\Omega_0)} & \leq \; C\|m\|_{L^2(\partial\Omega_0)}\|\Theta - Id\|_{2,\alpha}, \\ \left\|m \circ \Phi_{\Theta,t} - m\right\|_{H^{1/2}(\partial\Omega_0)} & \leq \; C\|m\|_{H^{1/2}(\partial\Omega_0)}\|\Theta - Id\|_{2,\alpha}, \\ \left\|m \circ \Phi_{\Theta,t} - m\right\|_{H^1(\partial\Omega_0)} & \leq \; C\|m\|_{H^1(\partial\Omega_0)}\|\Theta - Id\|_{2,\alpha} \end{cases} \tag{18}$$

where m is defined by $m = \langle \boldsymbol{X}_\Theta, \breve{\boldsymbol{n}} \rangle$.

PROOF. The vector fields $\boldsymbol{X}_\Theta$ writes $\boldsymbol{X}_\Theta = m\breve{\boldsymbol{n}}$ in $V_{\partial\Omega_0}$ where the moving boundaries $\partial\Omega_t$ are confined for $t \in [0,1]$. The proof relays on the explicitation of $m \circ \Phi_{\Theta,t} - m$.

By definition, the quantity $m^1 = \langle \boldsymbol{X}^1_\Theta, \breve{\boldsymbol{n}} \rangle$ is constant on the normal lines and does not depend of h. Hence, equation (18) is trivial for the vector field $\boldsymbol{X}^1_\Theta$ since the left hand size vanishes. The consistent case is the case of $\boldsymbol{X}^2_\Theta$ we consider now.

By construction (see (9)), for all $t \in [0,1]$ and all $M \in \partial\Omega_0$, we have :

$$m(M_t) - m(M) \; = \; m(M)A(M,t) \text{ where } A(M,t) := \left[\frac{\det\big(DT\big)(M)}{\det\big(DT\big)(M_t)} - 1\right].$$

If there exists $C > 0$ such that

$$\forall t \in [0,1], \;\; \|A(.,t)\|_{\mathcal{C}^1(\partial\Omega_0)} \leq C\|\Theta - Id\|_{2,\alpha}, \tag{19}$$

then the estimations of Lemma 2 give

$$\forall t \in [0,1], \begin{cases} \left\|m \circ \Phi_{\Theta,t} - m\right\|_{L^2(\partial\Omega_0)} & \leq \; \|m\|_{L^2(\partial\Omega_0)}\|A\|_{L^\infty}, \\ \left\|m \circ \Phi_{\Theta,t} - m\right\|_{H^{1/2}(\partial\Omega_0)} & \leq \; c(\Omega_0)\|m\|_{H^{1/2}(\partial\Omega_0)}\|A\|_{\mathcal{C}^1(\partial\Omega_0)}, \\ \left\|m \circ \Phi_{\Theta,t} - m\right\|_{H^1(\partial\Omega_0)} & \leq \; c'(\Omega_0)\|m\|_{H^1(\partial\Omega_0)}\|A\|_{\mathcal{C}^1(\partial\Omega_0)}; \end{cases}$$

that is to say exactly (18). Let us prove (19). On the neighbourhood $V_{\partial\Omega_0}$, T is a diffeomorphism and its Jacobian never vanishes. Therefore, by compactness of $\overline{V_{\partial\Omega_0}}$, $|\det\big(DT\big)(x)| \geq H_m$ holds for some $H_m > 0$. Since $h \mapsto \det\big(DT\big)(M,h)$ is $\mathcal{C}^{2,\alpha}$ on $\overline{V_{\partial\Omega_0}}$, the same argument of compactness gives

$$\begin{aligned} \big|\det\big(DT\big)(M_t) - \det\big(DT\big)(M)\big| &\leq \sup_K |\partial_h \det\big(DT\big)| \; \|h(M,t)\|_{L^\infty(\partial\Omega_0)} \\ &\leq C\|\Theta - Id_{\mathbb{R}^d}\|_{2,\alpha}. \end{aligned}$$

To get the $\mathcal{C}^1$ semi-norm estimate, we differentiate A with respect to the tangent variables and obtain

$$D_sA(M,t) = \frac{D_s\det\big(DT\big)(M_t)\det\big(DT\big)(M) - D_s\det\big(DT\big)(M)\det\big(DT\big)(M_t)}{[\det\big(DT\big)(M_t)]^2}.$$

Since $[\det\big(DT\big)(M_t)]^2 \leq H_m^2$, we should dominate $D_s\det\big(DT\big)(M_t) - D_s\det\big(DT\big)(M)$ in order to bound $|D_sA(M,t) - D_sA(M,0)|$. This derivative of $\det\big(DT\big)$ is a geometrical $\mathcal{C}^{1,\alpha}$ quantity since $T \in \mathcal{C}^{3,\alpha}$. We conclude that

$$\big|D_s\det\big(DT\big)(M_t) - D_s\det\big(DT\big)(M)\big| \leq C\big|h(M,t)\big| \leq C\|\Theta - Id_{\mathbb{R}^d}\|_{2,\alpha}.$$

The estimation (19) is now proved. ■

Study of the state-function. To justify the differentiability of e_Θ (defined as the restriction of the shaping function E to the path), the behaviour along the path of the state-function u_{Ω_t} must be examined. We use the inverse transport by $\Psi_{\Theta,t} = (\Phi_{\Theta,t})^{-1}$ to work on the initial domain Ω_0. To work on a fixed domain implies to deal with perturbed coefficients. The transported solution $u_{\Theta,t} = u_{\Omega_t} \circ \Phi_{\Theta,t}$ does not solve a Dirichlet problem for L but for a perturbed operator $L(t)$. A classical computation shows that $L(\Theta, t)$ writes

$$\begin{aligned} L(\Theta,t)u &= \underbrace{\Big[\sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}\partial_i\Psi^{\alpha}_{\Theta,t}\partial_j\Psi^{\beta}_{\Theta,t}\Big]}\,\partial^2_{\alpha,\beta}u + \underbrace{\Big[\sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}\partial^2_{i,j}\Psi^{\alpha}_{\Theta,t}\Big]}\,\partial_\alpha u, \\ &= \qquad a_{\alpha,\beta}(\Theta,t) \qquad\quad \partial^2_{\alpha,\beta}u \;+\; \qquad b_\beta(\Theta,t) \qquad \partial_\alpha u. \end{aligned} \tag{20}$$

The following proposition gives an upper bound of variations of the transported state-function.

Proposition 4 *There exists a modulus of continuity ω such that, for all diffeomorphism Θ with $\|\Theta - Id_{\mathbb{R}^d}\|_{2,\alpha}$ small enough, one has*

$$\sup_{t\in[0,1]} \|u_{\Theta,t} - u_0\|_{2,\overline{\Omega_0}} \le \omega\big(\|\Theta - Id_{\mathbb{R}^d}\|_{2,\alpha}\big). \tag{21}$$

SKETCH OF THE PROOF. Since such a result is classical in the field of shape optimisation, we only recall the main steps of the proof. First, by matricial considerations, the next lemma is proved (see [4] for details).

Lemma 3 *Let λ_0 be the smallest eigenvalue of A. There exists $\eta_c > 0$ such that $\forall\eta \in (0,\eta_c)$, for all admissible Θ and all $t \in [0,1]$, the smallest eigenvalue of the main part of $L(t)$ that is ${}^tD\Psi_{\Theta,t}AD\Psi_{\Theta,t} = \big(a_{\alpha,\beta}(t)\big)_{1\le\alpha,\beta\le d}$ is bigger than $\lambda_0/2$.*

Then, the classical Schauder *a priori* estimates are used to get a uniform upper bound for the transported state-function $u_{\Theta,t}$ in the Hölder space $\mathcal{C}^{2,\alpha}(\overline{\Omega_0})$. Then define the function ω on $(0,\eta)$ as

$$\omega(\delta) = \sup_{\|\Theta - Id_{\mathbb{R}^d}\|_{2,\alpha}\le\delta;\ t\in[0,1]} \|u_{\Theta,t} - u_0\|_{\mathcal{C}^2(\overline{\Omega_0})}$$

Seeking a contradiction, we assume $\omega(0^+) \neq 0$. Then, there exist a real $a > 0$ and two sequences $\Theta_n,\ t_n$ with

$$\|\Theta_n - Id_{\mathbb{R}^d}\|_{2,\alpha} \le \frac{1}{n} \text{ and } \|u_{\Theta_n,t_n} - u_0\|_{\mathcal{C}^2(\overline{\Omega_0})} \ge a > 0 \tag{22}$$

By the compactness of the injection of $\mathcal{C}^{2,\alpha}(\overline{\Omega_0})$ into $\mathcal{C}^2(\overline{\Omega_0})$, the sequence (u_{Θ_n,t_n}) bounded in $\mathcal{C}^{2,\alpha}(\overline{\Omega_0})$ is compact in $\mathcal{C}^2(\overline{\Omega_0})$. By extraction, a subsequence (u_{Θ_p,t_p}) converges in $\mathcal{C}^2(\overline{\Omega_0})$ to a limit u_{lim} with $t_p \to t_{lim}$. Then, we notice that

$$L(\Theta_p,t_p)u_{\Theta_p,t_p} - Lu_{lim} = \big[L(\Theta_p,t_p) - L\big]u_{\Theta_p,t_p} - L\big[u_{\Theta_p,t_p} - u_{lim}\big].$$

Passing to the limit when $p \to +\infty$, we see that u_{lim} solves the Dirichlet problem

$$\begin{cases} Lu = k \text{ in } \Omega_0, \\ u = 0 \text{ on } \partial\Omega_0. \end{cases}$$

This problem has an unique solution hence $u_{lim} = u_0$ holds. This last point enters in contradiction with (22). We have shown that $\omega(0^+) = 0$. ■

2.3. Differentiability of the restricted shaping functional

Differentiability of the state-function. This differentiability is a classical result (see for example [16, Propositions 2.82, 2.83 and 3.1]). We recall the conclusions of [16]. The derivative $\partial_t u$ exists and solves

$$\begin{cases} Lw = 0 \text{ in } \Omega_t, \\ w + \langle \nabla u(t), \boldsymbol{X}_\Theta \rangle = 0 \text{ on } \partial\Omega_t. \end{cases} \tag{23}$$

The elliptic regularity is needed to justify the existence of the second derivative $\partial^2_{tt} u$ that solves

$$\begin{cases} Lw = 0 \text{ in } \Omega_t, \\ w + 2\langle \boldsymbol{X}_\Theta \circ \Phi_{\Theta,t}, \nabla \partial_t u_{\Theta,t} \rangle + \langle \partial_t \boldsymbol{X}_\Theta \circ \Phi_{\Theta,t}, \nabla u_{\Theta,t} \rangle \\ \qquad\qquad + \langle \boldsymbol{X}_\Theta, D^2 u_{\Theta,t} \cdot \boldsymbol{X}_\Theta \circ \Phi_{\Theta,t} \rangle = 0 \text{ on } \partial\Omega_0. \end{cases} \tag{24}$$

Proposition 4 allows to conclude that for all $\beta \in (0,\alpha)$

$$u(t) \in \mathcal{C}^0\big([0,1], \mathcal{C}^{2,\beta}(\overline{\Omega_t})\big) \cap \mathcal{C}^1\big([0,1], \mathcal{C}^{1,\beta}(\overline{\Omega_t})\big) \cap \mathcal{C}^2\big([0,1], \mathcal{C}^{0,\beta}(\overline{\Omega_t})\big).$$

The Hadamard derivation formula.

Lemma 4 *Let $\boldsymbol{V}$ be a vector field in $\mathcal{C}^k(\mathbb{R}^d, \mathbb{R}^d)$ with $k \geq 1$. Let Ω_0 be a $\mathcal{C}^k$ open subset of $\mathbb{R}^d$ and let Ω_t denote its image by the flow of $\boldsymbol{V}$. If*

$$f \in \mathcal{C}^1((0,T), \mathcal{C}^0(\overline{\Omega_t})) \cap \mathcal{C}^0((0,T), \mathcal{C}^1(\overline{\Omega_t}))$$

then the function $e : [0,1] \to \mathbb{R}, t \mapsto E(\Omega_t)$ is differentiable with derivative

$$e'(t) = \int_{\Omega_t} \Big[\partial_t f(t,x) + div\big(f(t,x)\boldsymbol{V}(x)\big) \Big] dx. \tag{25}$$

Expression of some derivatives.

1. *The volume.* To compute the derivatives of $a_\Theta : [0,1] \to \mathbb{R}, t \mapsto \mathfrak{V}(\Omega_t)$, we set $f = 1$ and obtain

$$a'_\Theta(t) = \int_{\partial\Omega_t} \langle \boldsymbol{X}_\Theta, \boldsymbol{n}(t) \rangle, \tag{26}$$

$$a''_\Theta(t) = \int_{\partial\Omega_t} \operatorname{div}\big(\boldsymbol{X}_\Theta\big) \langle \boldsymbol{X}_\Theta, \boldsymbol{n}(t) \rangle. \tag{27}$$

2. *The perimeter.* We introduce $p_\Theta : [0,1] \to \mathbb{R}, t \mapsto \mathfrak{P}(\partial\Omega_t)$ and consider the extension of the unitary normal field $\boldsymbol{n}$ to $\partial\Omega_t$ defined by

$$\breve{\boldsymbol{n}}(t) = \frac{\nabla d_{\partial\Omega_0} \circ \Phi_t^{-1}}{\|\nabla d_{\partial\Omega_0} \circ \Phi_t^{-1}\|}. \tag{28}$$

We set $f(t,x) = \operatorname{div}\big(\breve{\boldsymbol{n}}(t)\big)$ to get

$$p'_\Theta(t) = \int_{\partial\Omega_t} \operatorname{div}\big(\breve{\boldsymbol{n}}(t)\big) \langle \boldsymbol{X}_\Theta, \boldsymbol{n}(t) \rangle, \tag{29}$$

$$p''_\Theta(t) = \int_{\partial\Omega_t} \big[\operatorname{div}\big(\partial_t \breve{\boldsymbol{n}}(t)\big) + \operatorname{div}\big(\operatorname{div}\big(\breve{\boldsymbol{n}}(t)\big)\boldsymbol{X}_\Theta\big)\big] \langle \boldsymbol{X}_\Theta, \breve{\boldsymbol{n}}(t) \rangle. \tag{30}$$

3. *The shaping functional E.* The state-functions u_{Ω_t} are only defined on the moving domains Ω_t as solution of (1). This fact introduce some technical difficulties to use Lemma 4. To overcome these difficulties , we define some linear and continuous extension operators P_t in order to deal with functions defined in the whole space. We refer to [12, lemma 6-37] for the construction of the extension operator P_0 outside the initial domain Ω_0. For $t > 0$, we define the extension operator P_t outside the domain Ω_t by $P_t = \Phi_{\Theta,t} \circ P_0 \circ \Phi_{\Theta,t}^{-1})$.

   To compute the second derivative of e_Θ with Lemma 4, we must regularise the extended solution $\tilde{u}(t) = P_t\big(u(t)\big)$ by convolution with some mollifiers ρ_ϵ. We introduce a perturbed shaping function $E_\epsilon(\Omega)$ defined by

$$E_\epsilon(\Omega) = \int_\Omega f(u_\epsilon(t), \nabla u_\epsilon(t)),$$

   where $u_\epsilon(t) = \tilde{u}(t) * \rho_\epsilon$. Then, Lemma 4 justifies the following calculus

$$\begin{aligned} e'_{\Theta,\epsilon}(t) = \int_{\Omega_t} & D_s f(u_\epsilon(t), \nabla u_\epsilon(t))\, \partial_t u_\epsilon(t) + \mathrm{div}\big(f(x, u_\epsilon(t), \nabla u_\epsilon(t)) \boldsymbol{X}_\Theta\big) \\ & + \langle D_v f(x, u_\epsilon(t), \nabla u_\epsilon(t)), \nabla \partial_t u_\epsilon(t)\rangle, \end{aligned}$$

   and then

$$\begin{aligned} e''_{\Theta,\epsilon}(t) = \int_{\Omega_t} & 2\,\mathrm{div}\big(\big[D_s f(u_\epsilon(t), \nabla u_\epsilon(t))\partial_t u_\epsilon(t) + \langle D_v f(x, u_\epsilon(t), \nabla u_\epsilon(t)), \nabla\partial_t u_\epsilon(t)\rangle\big] \boldsymbol{X}_\Theta\big) \\ & + D_s f(u_\epsilon(t), \nabla u_\epsilon(t))\, \partial^2_{tt} u_\epsilon(t) + D^2_{s,s} f(u_\epsilon(t), \nabla u_\epsilon(t))(\partial_t u_\epsilon(t))^2 \\ & + 2\langle D^2_{s,v} f(u_\epsilon(t), \nabla u_\epsilon(t)), \nabla\partial_t u_\epsilon(t)\rangle \partial_t u_\epsilon(t) \\ & + \langle D_v f(u_\epsilon(t), \nabla u_\epsilon(t)), \nabla \partial^2_{tt} u_\epsilon(t)\rangle \\ & + \langle D^2_{v,v} f(u_\epsilon(t), \nabla u_\epsilon(t))\, \nabla\partial_t u_\epsilon(t), \nabla\partial_t u_\epsilon(t)\rangle + \mathrm{div}\big(\mathrm{div}\big(f\boldsymbol{X}_\Theta\big)\boldsymbol{X}_\Theta\big) \end{aligned}$$

   We need the regularisation to give a sense to $\nabla\partial^2_{tt} u(t)$. Without this regularisation, a third order derivative of $u_{\Theta,t}$ would have appeared and $u_{\Theta,t}$ is only $\mathcal{C}^{2,\alpha}$ by Proposition 4. We apply Stocke's formula to make this third derivative disappear. Then, we pass to the limit when $\epsilon \to 0$ to get

$$e'_\Theta(t) = \int_{\Omega_t} D_s f(u(t), \nabla u(t))\, \partial_t u(t) + \langle D_v f(x, u(t), \nabla u(t)), \nabla\partial_t u(t)\rangle + \mathrm{div}\big(f\boldsymbol{X}_\Theta\big), \tag{31}$$

$$\begin{aligned} e''_\Theta(t) = \int_{\partial\Omega_t} & \langle D_v f(u(t), \nabla u(t)), \boldsymbol{n}(t)\rangle \partial^2_{tt} u(t) + \mathrm{div}\big(f(u(t), \nabla u(t))\boldsymbol{X}_\Theta\big)\langle \boldsymbol{X}_\Theta, \boldsymbol{n}(t)\rangle \\ & + 2\Big[D_s f(u(t), \nabla u(t))\, \partial_t u(t) + \langle D_v f(u(t), \nabla u(t)), \nabla\partial_t u(t)\rangle\Big]\langle \boldsymbol{X}_\Theta, \boldsymbol{n}(t)\rangle \\ + \int_{\Omega_t} & \big[D_s f(u(t), \nabla u(t)) - \mathrm{div}\big(D_v f(u(t), \nabla u(t))\big)\big]\, \partial^2_{tt} u(t) \\ & + D^2_{s,s} f(u(t), \nabla u(t))(\partial_t u(t))^2 + \langle D^2_{v,v} f(u(t), \nabla u(t))\, \nabla\partial_t u(t), \nabla\partial_t u(t)\rangle \\ & + 2\langle D^2_{s,v} f(u(t), \nabla u(t)), \nabla\partial_t u(t)\rangle \partial_t u(t). \end{aligned} \tag{32}$$

Remark 4 This writing of e''_Θ is not canonical. As well-known by the structure theorem of shape derivatives (see for example [6], [3], [15]), the shape Hessian at Ω_t is a distribution supported on the boundary $\partial\Omega_t$. The presence of integrals over Ω_t is caused by the use of Hadamard's Lemma. Nevertheless, (32) is convenient to prove Theorem 1.

3. Proof of Theorem 1

Preliminaries. In this section, Θ denotes a fixed admissible deformation in an adequate neighborhood of the identity. Therefore, we will omit in this proof the dependency with respect to Θ in order to simplify the notations that we fix now. Let C denote any constant depending only of the operator L, of $\|j\|_{0,\alpha}$ and of Ω_0. We set $\eta := \|\Theta - Id_{\mathbb{R}^d}\|_{2,\alpha}$.

Let $\boldsymbol{V}$ be either $\boldsymbol{X}^1_\Theta$ either $\boldsymbol{X}^2_\Theta$. Let $(\Omega_t)_{t\in[0,1]}$ be the path realized by the flow Φ_t of $\boldsymbol{V}$: it connects Ω_0 to $\Theta(\Omega_0)$. The transport of differential operators from Ω_t onto Ω_0 is a key to many of the following estimates. We introduce simplified notations. The use of the accent ˜ denotes the composition with the flow Φ_t of $\boldsymbol{V}$. For example, we define: $m := m_\Theta$, $\tilde{m} := m \circ \Phi_t$, $\tilde{\boldsymbol{V}} := \boldsymbol{V} \circ \Phi_t$, , $\tilde{n}(t) := n(t) \circ \Phi_t$, $L(\Theta, t) = L(t)$, $\tilde{u}_t = u_{\Theta,t} (:= u(t) \circ \Phi_t)$, $\ldots$. Let $\mathfrak{D}_t$ denotes the matrix ${}^tD\Phi_t^{-1}$ so that a gradient writes $\nabla u_t \circ \Phi_t = \mathfrak{D}_t \nabla \tilde{u}_t$ once transported.

The $\mathcal{C}^3$ non-linearity f induces no real difficulties: the following lemma easily deduced from the Mean Value Theorem is sufficient to deal with it.

Lemma 5 *Let F be one of the functions $f, \partial_s f, \partial_v f, \partial^2_{ss} f, \partial^2_{sv} f$ or $\partial^2_{vv} f$, we have*

$$\begin{cases} \|F(\tilde{u}_t, \mathfrak{D}_t \nabla \tilde{u}_t)\|_{L^\infty(\overline{\Omega_0})} \leq C, \\ \|F(\tilde{u}_t, \mathfrak{D}_t \nabla \tilde{u}_t) - F(u_0, \nabla u_0)\|^2_{L^\infty(\overline{\Omega_0})} \leq C\omega(\eta). \end{cases}$$

The Hessian given by (32) is a sum of different kinds of terms defined by the number and type of the derivations applied to the state-function. In particular, the upper bounds of their variations are obtained in terms of various weak norms. All the needed technics will be described in this section but illustrated on only one term each time to avoid repetitive proofs.

We start with the terms written in (32) as an integral on the boundary $\partial\Omega_t$ we call boundary terms in opposition to the terms written as an integral over the domain Ω_t we call internal term. One has to distinguish two cases depending if the quantity $\nabla \partial_t u$ on $\partial\Omega_t$ appears or not. From (23), this quantity is the harmonic extension on Ω_t of $-\langle \nabla u(t), \boldsymbol{V}\rangle$. In order to deal with the conormal derivative, the *Dirichlet-to-Neumann* operator corresponding to both the elliptic operator L and the domain Ω_t is to be used.

Let us introduce some notations around the L and $L(t)$ harmonic extensions of Dirichlet boundary data and *Dirichlet-to-Neumann* operators. We first define the operator R_t of L-harmonic extension on Ω_t and R^t_0 of $L(t)$-harmonic extension on Ω_0

$$\begin{array}{llll} R_t & : H^{1/2}(\partial\Omega_t) & \longrightarrow H^1(\Omega_t), \\ & \quad u & \mapsto R_t(u) \end{array} \qquad \begin{array}{llll} R^t_0 & : H^{1/2}(\partial\Omega_0) & \longrightarrow H^1(\Omega_0), \\ & \quad u & \mapsto R_t(u) \end{array} \tag{33}$$

where $R_t(u)$ (resp. $R^t_0(u)$) solves

$$\begin{cases} Lv = 0 \text{ in } \Omega_t, \\ v = u \text{ on } \partial\Omega_t; \end{cases} \quad resp. \begin{cases} L(t)v = 0 \text{ in } \Omega_0, \\ v = u \text{ on } \partial\Omega_0. \end{cases}$$

Recall that $L(t)$ is defined in (20). By abuse, we will simply writes R_0 instead of R^0_0. Then, the Dirichlet-to-Neumann operator C_t (also called interior capacity operator) associated to $\mathcal{L}$ on Ω is defined as

$$\begin{array}{lll} C_t & : H^{1/2}(\partial\Omega_t) & \to H^{-1/2}(\partial\Omega_t), \\ & \quad u & \longmapsto C_t(u) = \langle \boldsymbol{n}_L, \nabla R_t(u)\rangle, \end{array} \tag{34}$$

where the conormal vector is $\boldsymbol{n}_L = A\boldsymbol{n}$. The operator C_t has the following fundamental property:

$$\langle u, C_t u\rangle_{H^{1/2}(\partial\Omega_t)\times H^{-1/2}(\partial\Omega_t)} = \int_{\Omega_t} |\nabla R_t(u)|^2. \tag{35}$$

We now start the proof of Theorem 1.

A. Boundary terms without derivatives of the state-function. They are treated by transport. They give L^2-estimates. We transform the integral over $\partial\Omega_t$ into an integral over $\partial\Omega_0$ to obtain an difference written as

$$\int_{\partial\Omega_0}\Big[\tilde{m}^2\prod_{i=1}^{N}a_i(t)-m^2\prod_{i=1}^{N}a_i(0)\Big].$$

From the estimates related either to the geometrical quantities of the boundary $\partial\Omega_t$ (see Proposition 2), either to the state-function itself (see Proposition 4), the a_i, where the quantity m does not appear, satisfy

$$\forall t\in[0,1],\quad\left\{\begin{array}{rcl}\|a_i(t,.)\|_{L^\infty(\partial\Omega_0)} & \leq & C\|a_i(0,.)\|_{L^\infty(\partial\Omega_0)},\\ \|a_i(t,.)-a_i(0,.)\|_{L^\infty(\partial\Omega_0)} & \leq & \omega(\eta)\|a_i(0,.)\|_{L^\infty(\partial\Omega_0)}.\end{array}\right. \tag{36}$$

Proposition 3 gives then

$$\Big|\int_{\partial\Omega_0}\tilde{m}^2\prod_{i=1}^{N}a_i(t)-\int_{\partial\Omega_0}m^2\prod_{i=1}^{N}a_i(0)\Big|\leq C\|a_0(0)\|^2_{L^2(\partial\Omega_0)}\omega(\eta).$$

For example, consider the term

$$\int_{\partial\Omega_t}f\,\mathrm{div}\big(\boldsymbol{V}\big)\langle\boldsymbol{V},\boldsymbol{n}(t)\rangle.$$

One sets:

$$\begin{array}{rclrcl}a_1(t) & := & f(\tilde{u}_t,\mathfrak{D}_t\nabla\tilde{u}_t), & a_2(t) & := & \langle\tilde{\boldsymbol{n}}(t),\breve{\boldsymbol{n}}\rangle,\\ a_3(t) & := & J(t), & a_4(t) & := & \mathrm{div}\big(\breve{\boldsymbol{n}}\big).\end{array}$$

For $i=1$, one gets the needed estimates from Lemma 5. The properties of the transport shown in the Section 2.2 give for $i=2,3$ and for all $t\in[0,1]$,

$$\|a_i(t)\|_{L^\infty(\partial\Omega_0)}\leq C\text{ and }\|a_i(t)-a_i(0)\|_{L^\infty(\partial\Omega_0)}\leq C\eta.$$

The inequality on $\mathrm{div}\big(\breve{\boldsymbol{n}}\big)$ id deduced from its explicit expression (9).

B. Boundary terms with $\nabla\partial_t u(t)$. The main idea is to decompose the vector which appears in a scalar product with $\nabla\partial_t u$ into its tangent and conormal components. This leads to separate the contributions of the tangential gradient of $\partial_t u$ and of the conormal derivative of $\partial_t u$. After integration by part on $\partial\Omega_t$, the part with the tangential gradient leads to a situation without any derivatives of the state function treated in the former paragraph. The part involving the conormal derivative is transformed into an integral on Ω_t via the Green formula that requires the classical elliptic *a priori* estimates in Sobolev spaces. We will carry the computations on the term

$$\int_{\partial\Omega_t}\langle D_v f(u,\nabla u),\nabla\partial_t u\rangle\langle\boldsymbol{V},\boldsymbol{n}(t)\rangle.$$

The first step in the study is to decompose $D_v f(u,\nabla u)$ in its tangential component $[D_v f(u,\nabla u)]_{\boldsymbol{\tau}}$ and its conormal one $[D_v f(u,\nabla u)]_{\boldsymbol{n}_L}$. We get

$$[D_v f(u,\nabla u)]_{\boldsymbol{n}_L}=\frac{\langle D_v f(u,\nabla u),\boldsymbol{n}\rangle}{\langle\boldsymbol{n},\boldsymbol{n}_L\rangle}\text{ and }[D_v f(u,\nabla u)]_{\boldsymbol{\tau}}=D_v f(u,\nabla u)-[D_v f(u,\nabla u)]_{\boldsymbol{n}_L}\boldsymbol{n}_L,$$

and then study separately

$$\int_{\partial\Omega_t}\langle\nabla_{\boldsymbol{\tau}}\partial_t u,[D_v f(u,\nabla u)]_{\boldsymbol{\tau}}\rangle\langle\boldsymbol{V},\boldsymbol{n}(t)\rangle,\text{ and }\int_{\partial\Omega_t}[D_v f(u,\nabla u)]_{\boldsymbol{n}_L}\langle\boldsymbol{n}_L(t),\nabla\partial_t u\rangle\langle\boldsymbol{V},\boldsymbol{n}(t)\rangle.$$

The following lemma is easily deduced from Leibnitz Formula and Propositions 2 and 4.

Lemma 6 *Both the function* $b_1 = [D_v f(u,\nabla u)]_{\boldsymbol{n}_L}$ *and the vector field* $b_2 = [D_v f(u,\nabla u)]_{\boldsymbol{\tau}}$ *are* $\mathcal{C}^1$. *Moreover, there exists a constant* C *such that*

$$\forall t \in [0,1], \forall i \in \{1,2\} \quad \begin{cases} \|b_i(t,.)\|_{\mathcal{C}^1(\partial\Omega_0)} & \leq \quad C, \\ \|b_i(t,.) - b_i(0,.)\|_{\mathcal{C}^1(\partial\Omega_0)} & \leq \quad \omega(\eta)\|b_i(0,.)\|_{\mathcal{C}^1(\partial\Omega_0)}. \end{cases}$$

The tangential part: We first perform an integration by part on $\partial\Omega_t$ to suppress the tangential gradient[2]. After the expansion

$$\nabla_{\boldsymbol{\tau}}\partial_t u = \nabla_{\boldsymbol{\tau}}(-m\langle\nabla u, \breve{\boldsymbol{n}}\rangle) = -\langle\nabla u, \breve{\boldsymbol{n}}\rangle\nabla_{\boldsymbol{\tau}} m - m\nabla_{\boldsymbol{\tau}}\langle\nabla u, \breve{\boldsymbol{n}}\rangle,$$

we get since $m\nabla_{\boldsymbol{\tau}} m = 1/2\nabla_{\boldsymbol{\tau}} m^2$:

$$\int_{\partial\Omega_t}\langle\nabla_{\boldsymbol{\tau}}\partial_t u, [D_v f(u,\nabla u)]_{\boldsymbol{\tau}}\rangle\langle\boldsymbol{V}, \boldsymbol{n}(t)\rangle = -\int_{\partial\Omega_t} m^2\langle\nabla_{\boldsymbol{\tau}}\langle\nabla u, \breve{\boldsymbol{n}}\rangle, [D_v f(u,\nabla u)]_{\tau}\rangle\langle\breve{\boldsymbol{n}}, \boldsymbol{n}(t)\rangle$$
$$+\frac{1}{2}\int_{\partial\Omega_t} m^2 \operatorname{div}_{\boldsymbol{\tau}}\left((\langle\breve{\boldsymbol{n}}, \boldsymbol{n}(t)\rangle\langle\nabla u, \breve{\boldsymbol{n}}\rangle)[D_v f(u,\nabla u)]_{\tau}\right).$$

We are a the former situation and obtain estimates in terms of $\|m\|^2_{L^2(\partial\Omega_0)}$.
The conormal part. From the expression of $\partial_t u$, we set

$$B_{\boldsymbol{n}_L}(t) = -\int_{\partial\Omega_t} C_t\big(-m\langle\nabla u, \breve{\boldsymbol{n}}\rangle\big)m[D_v f(u,\nabla u)]_{\boldsymbol{n}_L}\langle\breve{\boldsymbol{n}}, \boldsymbol{n}(t)\rangle.$$

We apply Green's formula to get:

$$B_{\boldsymbol{n}_L}(t) = \int_{\Omega_t} {}^t\Big[\nabla R_t\big(m\langle\nabla u, \breve{\boldsymbol{n}}\rangle\big)\Big] A \Big[\nabla R_t\big(m[D_v f(u,\nabla u)]_{\boldsymbol{n}_L}\langle\breve{\boldsymbol{n}}, \boldsymbol{n}(t)\rangle\big)\Big]. \tag{37}$$

As in [5], we are interested in the dependency of those quantities with respect to m. The H^1-norm of the extensions (that is the $H^{1/2}$-norm of the traces on the boundary $\partial\Omega_t$) appears. Following the idea to use *a priori* estimates, we first study the boundary condition set in $H^{1/2}(\partial\Omega_0)$ and we define:

$$z_1(t) = m\,[D_v f(u,\nabla u)]_{\boldsymbol{n}_L}\langle\breve{\boldsymbol{n}}, \boldsymbol{n}(t)\rangle \;\Rightarrow\; \tilde{z}_1(t) = \tilde{m}\,[D_v f(\tilde{u}_t, \nabla\tilde{u}_t)]_{\boldsymbol{n}_L}\,\langle\tilde{\boldsymbol{n}}(t), \breve{\boldsymbol{n}}\rangle,$$
$$z_2(t) = m\,\langle\nabla u, \breve{\boldsymbol{n}}\rangle \;\Rightarrow\; \tilde{z}_2(t) = \tilde{m}\,\langle\mathfrak{D}_t\nabla\tilde{u}_t, \breve{\boldsymbol{n}}\rangle.$$

Our analysis starts with the following lemma the proof of which is postponed.

Lemma 7 *There exists a constant* C *such that for all* $t \in [0,1]$ *and all* $i \in \{1,2\}$

$$\begin{aligned} \|R_0^t(\tilde{z}_i(t))\|_{H^1(\Omega_0)} &\leq C\|m\|_{H^{1/2}(\partial\Omega_0)}, \\ \|R_0(z_i(0)) - R_0^t(\tilde{z}_i(t))\|_{H^1(\Omega_0)} &\leq C\omega(\eta)\|m\|_{H^{1/2}(\partial\Omega_0)}. \end{aligned}$$

The second step is to transport (37) on Ω_0 :

$$B_{2,2,\boldsymbol{n}_L}(t) = \int_{\Omega_0} {}^t\Big[\mathfrak{D}_t\nabla R_0^t(\tilde{z}_2)\Big]\,\Big[A\circ\Phi_t\Big]\,\Big[\mathfrak{D}_t\nabla R_0^t(\tilde{z}_1)\Big]J(t).$$

Note that we have used that $R_t[z_i(t)\circ\Phi_t] = R_0^t[\tilde{z}_i]$. We set $X_t^i = \mathfrak{D}_t\nabla R_0^t(\tilde{z}_i)$ for $i = 1,2$ and $A_t = J(t)A\circ\Phi_t$. We get the punctual[3] estimations

$$\begin{aligned}\Big|{}^tX_t^2 A_t X_t^1 - {}^tX_0^2 A_0 X_0^1\Big| &\leq \Big|{}^tX_t^2[A_t - A_0]X_t^1\Big| + \Big|{}^t(X_t^2 - X_0^2)A_0X_t^1\Big| + \Big|{}^tX_0^2A_0(X_t^1 - X_0^1)\Big|, \\ &\leq \|A_t - A_0\|_\infty\|X_t^1\|\|X_t^2\| + \|A_0\|_\infty\big[\|X_t^1 - X_0^1\|\|X_t^2\| + \|X_0^1\|\|X_t^2 - X_0^2\|\big].\end{aligned}$$

[2] Note that both this operator and the tangential divergence are relative to $\partial\Omega_t$ even if we drop this dependency in the notations for the sake of readability.

[3] This is possible since we deal, in fact, with continuous functions even if only Lebesgue or Sobolev norms count.

From the expressions of coefficients of $L(t)$ given (20) and the regularity of A, one deduces the existence of a constant C such that $\|A_t - A_0\|_\infty \le C\eta$ and $\|A_t\|_\infty \le C$ for all $t \in [0,1]$. We obtain

$$\begin{aligned}\left|B_{2,2,\boldsymbol{n}_L}(t) - B_{2,2,\boldsymbol{n}_L}(0)\right| \le C\eta \int_{\Omega_0} \|X_t\|\|Y_t\| \\ + C\int_{\Omega_0} \|X_t - X_0\|\|X_t\| + C\int_{\Omega_0} \|Y_t - Y_0\|\|Y_t\|\end{aligned}$$

We conclude with Lemma 7 that

$$\left|B_{2,2,\boldsymbol{n}_L}(t) - B_{2,2,\boldsymbol{n}_L}(0)\right| \le \omega(\eta)\|m\|^2_{H^{1/2}(\partial\Omega_0)}.$$

PROOF OF LEMMA 7. We first gives upper bounds in terms of $H^{1/2}(\partial\Omega_0)$-norm of the normal deformation m of the traces on the boundary.

Lemma 8 *There exists a constant C such one has for all Θ , all t in $[0,1]$ and $i := 1,2$*

$$\begin{aligned}\|\tilde z_i(t)\|_{H^{1/2}(\partial\Omega_0)} &\le C\|m\|_{H^{1/2}(\partial\Omega_0)},\\ \|z_i(0) - \tilde z_i(t)\|_{H^{1/2}(\partial\Omega_0)} &\le C\omega(\eta)\|m\|_{H^{1/2}(\partial\Omega_0)}.\end{aligned}$$

PROOF OF LEMMA 8. First, remark that the z_i write $z = m\ f(t)$ with $f \in \mathcal{C}^1(\partial\Omega_t)$ as shown in Proposition 4 and in Lemma 6. Let $\tilde f(t)$ denotes $f(t)\circ\Phi_t \in \mathcal{C}^1(\partial\Omega_0)$, then one has $\forall t \in [0,1]$:

$$\begin{cases}\|\tilde f\|_{\mathcal{C}^1(\partial\Omega_0)} \le C,\\ \|\tilde f(t) - f(0)\|_{\mathcal{C}^1(\partial\Omega_0)} \le C\omega(\eta).\end{cases}$$

The product Lemma 2 provides the estimate

$$\|\tilde z\|_{H^{1/2}(\partial\Omega_0)} = \|\tilde m\tilde f\|_{H^{1/2}(\partial\Omega_0)} \le \|\tilde m\|_{H^{1/2}(\partial\Omega_0)}\|\tilde f\|_{\mathcal{C}^1(\partial\Omega_0)} \le C\|\tilde m\|_{H^{1/2}(\partial\Omega_0)}. \tag{38}$$

To an upper bound of $\|\tilde z(t) - z(0)\|_{H^{1/2}(\partial\Omega_0)}$, we use the triangular decomposition

$$\tilde z_i(t) - z_i(0) \;=\; \tilde m(t)\tilde f(t) - m(0)f(0) \;=\; \big(\tilde m(t) - m(0)\big)\tilde f(t) \;+\; m(0)\big(\tilde f(t) - f(0)\big).$$

Then, we applied Proposition 3 and Lemma 2 to get

$$\begin{aligned}\|\tilde z_i(t) - z_i(0)\|_{H^{1/2}(\partial\Omega_0)} &\le \|\tilde m(t) - m(0)\|_{H^{1/2}(\partial\Omega_0)}\ \|\tilde f(t)\|_{\mathcal{C}^1(\partial\Omega_0)}\\ &+ \|m\|_{H^{1/2}(\partial\Omega_0)}\ \|\tilde f(t) - f(0)\|_{\mathcal{C}^1(\partial\Omega_0)} \le C\eta\|m\|_{H^{1/2}(\partial\Omega_0)}.\end{aligned}$$

This concludes the proof of Lemma 8. ■

From those estimates on the traces, one deduces the wanted estimations on the extension through the classical *a priori* estimate

$$\|u\|_{H^1} \le C(\Omega, L)\Big[\|k\|_{H^{-1}} + \|g\|_{H^{1/2}}\Big],$$

that holds for the solution of

$$\begin{cases}L(u) = k \text{ in } \Omega,\\ u = g \text{ on } \partial\Omega,\end{cases}$$

We apply this estimate to (38) and get

$$\|R_0^t(\tilde z(t))\|_{H^1(\Omega_0)} \le \|\tilde{z(t)}\|_{H^{1/2}(\partial\Omega_0)} \le C\|m\|_{H^{1/2}(\partial\Omega_0)}. \tag{39}$$

Here, the constant C depends only of Ω_0 and of the extremal eigenvalues of the order two part of $L(t)$. From Lemma 3, we know that this constant C can be chosen independently from both Θ and t. If $t = 0$, it means

$$\|R_0(z(0))\|_{H^1(\Omega_0)} \leq \|z(0)\|_{H^{1/2}(\partial\Omega_0)} \leq C\|m\|_{H^{1/2}(\partial\Omega_0)}.$$

We turn ourselves to the study of the difference between the extensions. The starting point is the equation

$$L(t)\Big(R_0^t(\tilde{z}(t))\Big) = 0 \text{ in } \Omega_0.$$

It can also be written as

$$L\Big(R_0^t(\tilde{z}(t)) - R_0(z(0))\Big) = \big[L - L(t)\big]\Big(R_0^t(\tilde{z}(t))\Big) \text{ in } \Omega_0.$$

In our particular case, we get $R_0^t(\tilde{z}(t)) - R_0(z(0)) = \tilde{z}(t) - z(0)$ on $\partial\Omega_0$ and hence

$$\Big\|R_0^t(\tilde{z}(t)) - R_0(z)\Big\|_{H^1(\Omega_0)} \leq C\Big[\|\tilde{z}(t) - z\|_{H^{1/2}(\partial\Omega_0)} + \Big\|\big[L(t) - L\big]\big(R_0^t(\tilde{z}(t))\big)\Big\|_{H^{-1}(\Omega_0)}\Big].$$

The trace is already known but we need some control on the H^{-1}-term. We use a duality method. Let ψ be a test-function in $\mathcal{C}_0^\infty(\Omega_0)$, we get

$$\int_{\Omega_0} \psi[L(t) - L]w = \int_{\Omega_0} \Big[[a_{i,j}(t) - a_{i,j}(0)]\partial^2_{i,j}w + [b_j(t) - b_j(0)]\partial_j w\Big]\psi$$

The main argument is that the coefficients of $L(t)$ and L are close in the $\mathcal{C}^1$-norm. This is deduced from Proposition 2. Since ψ vanishes on the boundary, the Green formula gives

$$\begin{aligned}\Big|\int_{\Omega_0} \psi[a_{i,j}(t) - a_{i,j}(0)]\partial^2_{i,j}w\Big| &\leq \Big|\int_{\Omega_0} \Big[\partial_j[a_{i,j}(t) - a_{i,j}(0)]\psi + [a_{i,j}(t) - a_{i,j}(0)]\partial_j\psi\Big]\partial_i w\Big|,\\ &\leq C\|a_{i,j}(t) - a_{i,j}(0)\|_{\mathcal{C}^1(\Omega_0)}\|\psi\|_{H_0^1(\Omega_0)}\|w\|_{H^1(\Omega_0)},\\ &\leq C\eta\|\psi\|_{H_0^1(\Omega_0)}\|w\|_{H^1(\Omega_0)}.\end{aligned}$$

For the order one term, we get :

$$\begin{aligned}\Big|\int_{\Omega_0} \psi[b_j(t) - b_j(0)]\partial_j w\Big| &\leq C\|b_j(t) - b_j(0)\|_{L^\infty(\Omega_0)}\|\psi\|_{H_0^1(\Omega_0)}\|w\|_{H^1(\Omega_0)},\\ &\leq C\eta\|\psi\|_{H_0^1(\Omega_0)}\|w\|_{H^1(\Omega_0)}.\end{aligned}$$

That is

$$\Big\|\big[L(t) - L\big]\big(R_0^t(\tilde{z})\big)\Big\|_{H^{-1}(\Omega_0)} \leq C\eta\big\|R_0^t(\tilde{z})\big\|_{H^1(\Omega_0)}. \tag{40}$$

Hence, we get from (39)

$$\Big\|\big[L(t) - L\big]\big(R_0^t(\tilde{z})\big)\Big\|_{H^{-1}(\Omega_0)} \leq C\eta\big\|m\big\|_{H^{1/2}(\partial\Omega_0)}. \quad \blacksquare$$

We now study the internal terms. The main difference between those terms is the presence of one or two derivations with respect to t. Recall from (23) and (24) that the trace on $\partial\Omega_t$ of $\partial_t u$ (and $\partial^2_{tt}u$) is linear (resp. bilinear) with respect to m. Therefore, different methods are required.

C) Quadratic internal terms in $\partial_t u$ or $\nabla\partial_t u$ One uses again the L-harmonic extensions and the *a priori* H^1-estimates. This apply to the following terms

$$\int_{\Omega_t} D^2_{s,s} f(u,\nabla u)(\partial_t u)^2, \int_{\Omega_t} \langle D^2_{s,v} f(u,\nabla u), \nabla\partial_t u\rangle \partial_t u, \int_{\Omega_t} \langle D^2_{v,v} f(u,\nabla u)\nabla\partial_t u, \nabla\partial_t u\rangle.$$

The extension operators defined in (33) are essential to study those terms since they allow to control $\partial_t u$ by its trace on $\partial\Omega_t$ since it solves (23). We mimic the former study of the conormal case and set :

$$z = -m\langle\nabla u, n(t)\rangle \;\Rightarrow\; \tilde{z} = -\tilde{m}\langle \mathfrak{D}_t\nabla\tilde{u}_t, \breve{\boldsymbol{n}}(t)\rangle.$$

We then establish some estimates.

Lemma 9 *There exists C such that*

$$\begin{aligned} \|\tilde{z}(t)\|_{H^{1/2}(\partial\Omega_0)} &\le C\|m\|_{H^{1/2}(\partial\Omega_0)}, &(41)\\ \|z(0)-\tilde{z}(t)\|_{H^{1/2}(\partial\Omega_0)} &\le C\omega(\eta)\|m\|_{H^{1/2}(\partial\Omega_0)}, &(42)\\ \|R_0^t(\tilde{z}(t))\|_{H^1(\Omega_0)} &\le C\|m\|_{H^{1/2}(\partial\Omega_0)}, &(43)\\ \|R_0(z(0)) - R_0^t(\tilde{z}(t))\|_{H^1(\Omega_0)} &\le C\omega(\eta)\|m\|_{H^{1/2}(\partial\Omega_0)}. &(44)\end{aligned}$$

PROOF. The $H^{1/2}$-estimations are direct applications of the product estimate stated in Lemma 2. We factorise m in the expressions of $z(0)$ and $\tilde{z}(t)$. Propositions 2 and 4 gives upper bounds for the $\mathcal{C}^1$-norm of the factorised quantities. The difference is treated as follows

$$\begin{aligned}\|z(0)-\tilde{z}(t)\|_{H^{1/2}(\partial\Omega_0)} &\le \|m-\tilde{m}(t)\|_{H^{1/2}(\partial\Omega_0)}\|\langle\mathfrak{D}_t\nabla\tilde{u}_t, \breve{\boldsymbol{n}}(t)\rangle\|_{\mathcal{C}^1(\partial\Omega_0)}\\ &\quad + \|m\|_{H^{1/2}(\partial\Omega_0)}\|\langle\nabla u, n(t)\rangle - \langle\mathfrak{D}_t\nabla\tilde{u}_t, \breve{\boldsymbol{n}}(t)\rangle\|_{\mathcal{C}^1(\partial\Omega_0)},\\ &\le C\omega(\eta)\|m\|_{H^{1/2}(\partial\Omega_0)}.\end{aligned}$$

We conclude as in the proof of Lemma 7. ∎

As example, we treat the term involving $D^2_{v,v}f$. First, we transport the term on the fixed domain

$$\int_{\Omega_t}\langle D^2_{v,v} f(u,\nabla u)\nabla\partial_t u, \nabla\partial_t u\rangle = \int_{\Omega_0}\langle\big[\mathfrak{D}_t D^2_{v,v} f(\tilde{u}_t, \mathfrak{D}_t\nabla\tilde{u}_t)\,{}^t\mathfrak{D}_t\big]\mathfrak{D}_t\nabla R_0^t(\tilde{z}(t)), \mathfrak{D}_t\nabla R_0^t(\tilde{z}(t))\rangle.$$

Then, we decompose the difference into

$$\begin{aligned}&\int_{\Omega_0}\langle\big[\mathfrak{D}_t D^2_{v,v} f(\tilde{u}_t,\, {}^t\mathfrak{D}_t\nabla\tilde{u}_t)\,{}^t\mathfrak{D}_t - D^2_{v,v} f(u_0,\nabla u_0)\big]\mathfrak{D}_t\nabla R_0^t(\tilde{z}(t)), \mathfrak{D}_t\nabla R_0^t(\tilde{z}(t))\rangle\\ &\quad + \int_{\Omega_0}\langle D^2_{v,v} f(u_0,\nabla u_0)\big[\mathfrak{D}_t\nabla R_0^t(\tilde{z}(t)) - \nabla R_0(z(0))\big], \mathfrak{D}_t\nabla R_0^t(\tilde{z}(t))\rangle\\ &\quad + \int_{\Omega_0}\langle D^2_{v,v} f(u_0,\nabla u_0)\nabla R_0(z(0)), \big[\mathfrak{D}_t\nabla R_0^t(\tilde{z}(t)) - \nabla R_0(z(0))\big]\rangle.\end{aligned}$$

From Lemma 5 and Cauchy-Schwartz inequality, we can bound the first term as

$$C\omega(\eta)\|\mathfrak{D}_t\nabla R_0^t(\tilde{z}(t))\|^2_{L^2(\Omega_0)} \le C\omega(\eta)\|R_0^t(\tilde{z}(t))\|^2_{H^1(\Omega_0)} \le C\omega(\eta)\|m\|^2_{H^{1/2}(\partial\Omega_t)}.$$

The others terms are bounded by

$$C\|D^2_{v,v} f(u_0,\nabla u_0)\|_{L^\infty(\Omega_0)}\|\mathfrak{D}_t\nabla R_0^t(\tilde{z}(t)) - \nabla R_0(z(0))\|_{L^2(\Omega_0)}\|\nabla R_0^t(\tilde{z}(t))\|_{L^2(\Omega_0)}.$$

We then apply Lemma 9 to obtain the upper bound $C\omega(\eta)\|m\|^2_{H^{1/2}(\partial\Omega_0)}$.

D) Internal terms involving $\partial^2_{tt}u$. The second derivative of the state-function $\partial^2_{tt}u$ is given as solution of (24): that is the L harmonic extension of a $\mathcal{C}^{0,\alpha}(\partial\Omega_t)$ Dirichlet data. We use the following L^1 lemma that is sufficient for our purposes. This approach is relevant as soon as appears $\partial^2_{tt}u$ without any spatial derivatives. Recall that the $\nabla\partial^2_{tt}u$, that appears by a direct application of Hadamard's derivation Lemma, is removed by the use of Green's formula (see Section 2.3.). Note also that the $p = 1$ case is the critical one for elliptic estimates in $W^{1,p}$ and that the following lemma is not true in $W^{1,1}$

Lemma 10 *Let Ω be a domain in $\mathcal{O}$, let $E = \partial_i(a_{i,j}(x)\partial_j)$ denote a strictly elliptic operator and z be a continuous function on $\partial\Omega$. Then, there exists a constant C such that the solution $R(z)$ of*

$$\begin{cases} Eu = 0 \text{ in } \Omega, \\ u = z \text{ on } \partial\Omega \end{cases}$$

satisfies the following a priori estimate

$$\|R(z)\|_{L^1(\Omega)} \leq C\|z\|_{L^1(\partial\Omega)}.$$

PROOF. We use a duality method to use known L^∞-estimations. We set $u = R(z)$ and we consider (P^*) the adjoint problem to (P) defined as

$$(P)\ \begin{cases} Eu = 0 \text{ in } \Omega, \\ u = z \text{ on } \partial\Omega, \end{cases} \qquad (P^*)\ \begin{cases} E^*\phi = \theta \text{ in } \Omega, \\ \phi = 0 \text{ on } \partial\Omega, \end{cases}$$

where E^* denotes the adjoint of E in distribution meaning. Green formula writes

$$\int_\Omega \phi E(u) = -\int_\Omega \sum_{i,j} a_{i,j}\partial_i u \partial_j \phi + \int_{\partial\Omega} \phi \sum_{i,j} a_{i,j}\partial_i u \partial_j \boldsymbol{n}.$$

We then get

$$\int_\Omega uE^*(\phi) = -\int_{\partial\Omega} z\partial_{\boldsymbol{n}_E}\phi.$$

Therefore, we need to control $\|\nabla\phi\|_{L^\infty(\partial\Omega)}$ with θ. This is done via a L^∞-estimation of the gradient (see [12]) where the constant C depends of d, $\partial\Omega$ and of the smallest and the biggest eigenvalue of the matrix $(a_{i,j})$

$$\|\nabla\phi\|_{L^\infty} \leq C\|\theta\|_{L^\infty}.$$

We obtain

$$\Big|\int_\Omega u\theta\Big| \leq \|z\|_{L^1(\partial\Omega)}\|\theta\|_{L^\infty(\Omega)}.$$

We use the duality $L^1 \times L^\infty$ to conclude. ■

Let z denote the trace of $\partial^2_{tt}u$ that is

$$\begin{aligned} z &= -\Big[2\langle \boldsymbol{V}, \nabla\partial_t u\rangle + \langle D\boldsymbol{V}\cdot\boldsymbol{V}, \nabla u\rangle + \langle \boldsymbol{V}, D^2u\boldsymbol{V}\rangle\Big], \\ &= -\Big[2\langle m\check{\boldsymbol{n}}, \nabla R_t(-m\langle\check{\boldsymbol{n}}, \nabla u\rangle)\rangle + \langle D\boldsymbol{V}\cdot\boldsymbol{V}, \nabla u\rangle + m^2\langle\check{\boldsymbol{n}}, D^2u\check{\boldsymbol{n}}\rangle\Big] \end{aligned}$$

on the moving boundary $\partial\Omega_t$. We transport z on the fixed boundary and get

$$\begin{aligned} \tilde{z} = -\Big[&\tilde{m}\langle\check{\boldsymbol{n}}, \mathfrak{D}_t\nabla R_0^t(-\tilde{m}\langle\check{\boldsymbol{n}}, \mathfrak{D}_t\nabla\tilde{u}_t)\rangle \\ &+ \langle\mathfrak{D}_t D\boldsymbol{V}(\Phi_t)\cdot\boldsymbol{V}(\Phi_t), \mathfrak{D}_t\nabla\tilde{u}_t)\rangle + \tilde{m}^2\langle\check{\boldsymbol{n}}, \mathfrak{D}_t D^2\tilde{u}_t\, {}^t\mathfrak{D}_t\check{\boldsymbol{n}}\rangle\Big]. \end{aligned}$$

This time, we are interested in the L^1-norm of the difference between the traces on $\partial\Omega_0$ from which the estimations on the extensions is easily deduced.

Lemma 11 *There exists a constant C such that for all t in* $[0,1]$,

$$\|\tilde{z}(t) - z(0)\|_{L^1(\partial\Omega_0)} \leq C\omega(\eta)\|m\|^2_{H^{1/2}(\partial\Omega_0)}.$$

PROOF. The term with ∇R_0^t requires the transverse-conormal decomposition already used in the second method for boundary integrals. This leads to the $H^{1/2}$-norm. To study the term with $D\boldsymbol{V}.\boldsymbol{V}$, we have to distinguish between $\boldsymbol{V}$ and $\boldsymbol{X}^1_\Theta$, we obtain in both cases a control in L^2-norm. Let us examine the term with D^2u. From the triangular inequality, we have

$$\begin{aligned}&\left|\tilde{m}^2\langle\breve{\boldsymbol{n}}, \mathfrak{D}_t D^2\tilde{u}_t\, {}^t\mathfrak{D}_t\breve{\boldsymbol{n}}\rangle - m^2\langle\breve{\boldsymbol{n}}, D^2u_0\breve{\boldsymbol{n}}\rangle\right| \\ &\quad\leq |\tilde{m}^2 - m^2|\ \|\langle\breve{\boldsymbol{n}}, \mathfrak{D}_t D^2\tilde{u}_t\, {}^t\mathfrak{D}_t\breve{\boldsymbol{n}}\rangle\|_{L^\infty(\partial\Omega_0)} \ +\ m^2\left\|D^2u_0 - \mathfrak{D}_t D^2\tilde{u}_t\, {}^t\mathfrak{D}_t\right\|_{L^\infty(\partial\Omega_0)}.\end{aligned}$$

From Cauchy-Schwartz inequality, we get that

$$\begin{aligned}\|\tilde{m}^2 - m^2\|_{L^1} &= \|(\tilde{m} - m)(\tilde{m} + m)\|_{L^1} \leq \|\tilde{m} - m\|_{L^2}\|\tilde{m} + m\|_{L^2}, \\ &\leq C\omega(\eta)\|m\|_{L^2}\left[\|\tilde{m}\|_{L^2} + \|m\|_{L^2}\right] \leq C\omega(\eta)\|m\|^2_{L^2}.\end{aligned}$$

Hence, we conclude that

$$\left\|\tilde{m}^2\langle\breve{\boldsymbol{n}}, \mathfrak{D}_t D^2\tilde{u}_t\, {}^t\mathfrak{D}_t\breve{\boldsymbol{n}}\rangle J(t) - m^2\langle\breve{\boldsymbol{n}}, D^2u_0\breve{\boldsymbol{n}}\rangle\right\|_{L^1(\partial\Omega_0)} \leq C\omega(\eta)\|m\|^2_{L^2(\partial\Omega_0)}. \quad \blacksquare$$

E) Case of geometrical quantities. The aim of this small section is to use the former approach to recover similar estimates for the volume and perimeter (in the class of regular shapes).
Variations of $\mathfrak{V}$. If we use $\boldsymbol{X}^2_\Theta$, a_Θ is constant along the path and therefore $a''_\Theta = 0$ on $[0,1]$. If we consider the field $\boldsymbol{X}^1_\Theta$ we get

$$a''_\Theta(t) = \int_{\partial\Omega(t)} \operatorname{div}\big(\boldsymbol{X}^1_\Theta\big)\langle\boldsymbol{X}^1_\Theta, \boldsymbol{n}(t)\rangle = \int_{\partial\Omega(t)} m^2 \operatorname{div}\big(\breve{\boldsymbol{n}}\big)\langle\breve{\boldsymbol{n}}, \boldsymbol{n}(t)\rangle.$$

We treat this integral by transport and get:

$$\left|a''_\Theta(t) - a''_\Theta(0)\right| \leq C\omega(\eta)\|m\|^2_{L^2(\partial\Omega_t)}. \tag{45}$$

Variations of $\mathfrak{P}$. Recall that

$$p''_\Theta(t) = \int_{\partial\Omega(t)} \Big[\operatorname{div}\big(\partial_t\breve{\boldsymbol{n}}(t)\big) + \operatorname{div}\big(\operatorname{div}\big(\breve{\boldsymbol{n}}(t)\big)\boldsymbol{V}\big)\Big]\langle\boldsymbol{V}, \breve{\boldsymbol{n}}(t)\rangle.$$

For a fixed t, $\breve{\boldsymbol{n}}(t,x)$ defined by (28) is an extension of $\boldsymbol{n}(t)$ that is $\mathcal{C}^1$ with respect to t. After an expansion to let m appear, we perform an integration by part on $\partial\Omega_t$ and get

$$\begin{aligned}\int_{\partial\Omega_t} m^2\Big[&\operatorname{div}\big(\breve{\boldsymbol{n}}(t)\big)\langle\breve{\boldsymbol{n}}, \breve{\boldsymbol{n}}(t)\rangle\langle\breve{\boldsymbol{n}}(t), D\breve{\boldsymbol{n}}.\breve{\boldsymbol{n}}(t)\rangle - \langle\breve{\boldsymbol{n}}, \breve{\boldsymbol{n}}(t)\rangle^2 \operatorname{Tr}\big({}^tD\breve{\boldsymbol{n}}(t)D\breve{\boldsymbol{n}}(t)\big) \\ &- \operatorname{div}\big(\breve{\boldsymbol{n}}(t)\big)\langle\nabla_{\boldsymbol{\tau}}\langle\breve{\boldsymbol{n}}, \breve{\boldsymbol{n}}(t)\rangle, \breve{\boldsymbol{n}}\rangle\Big] \\ &+ m\langle\breve{\boldsymbol{n}}, \breve{\boldsymbol{n}}(t)\rangle \operatorname{div}\big(\breve{\boldsymbol{n}}(t)\big)\Big[\langle\breve{\boldsymbol{n}}, \breve{\boldsymbol{n}}(t)\rangle\langle\nabla m, \breve{\boldsymbol{n}}(t)\rangle - \langle\breve{\boldsymbol{n}}, \nabla_{\boldsymbol{\tau}}m\rangle\Big] \\ &- m\Big[\langle\partial_t\breve{\boldsymbol{n}}(t), \nabla_{\boldsymbol{\tau}}\langle\breve{\boldsymbol{n}}, \breve{\boldsymbol{n}}(t)\rangle + \langle\breve{\boldsymbol{n}}, \breve{\boldsymbol{n}}(t)\rangle\langle\breve{\boldsymbol{n}}(t), D\breve{\boldsymbol{n}}.\partial_t\breve{\boldsymbol{n}}(t)\rangle\Big] \\ &- \langle\breve{\boldsymbol{n}}, \breve{\boldsymbol{n}}(t)\rangle\langle\partial_t\breve{\boldsymbol{n}}(t), \nabla_{\boldsymbol{\tau}}m\rangle.\end{aligned}$$

We compute the derivative $\partial_t\breve{\boldsymbol{n}}(t)$

$$\partial_t\breve{\boldsymbol{n}}(t) = \frac{\partial_t\mathbf{A}(t)}{\|\mathbf{A}(t)\|} - \frac{\mathbf{A}(t)\langle\partial_t\mathbf{A}(t), \mathbf{A}(t)\rangle}{\|\mathbf{A}(t)\|^3} = \frac{\partial_t\mathbf{A}(t)}{\|\mathbf{A}(t)\|} - \langle\frac{\partial_t\mathbf{A}(t)}{\|\mathbf{A}(t)\|}, \breve{\boldsymbol{n}}(t)\rangle\breve{\boldsymbol{n}}(t),$$

where
$$\partial_t \mathbf{A}(t) = \nabla\Big[\partial_t\big(d_{\partial\Omega_0} \circ \Phi_t^{-1}\big)\Big] = \nabla\langle\big(\nabla d_{\partial\Omega_0}\big) \circ \Phi_t^{-1}, \partial_t \Phi_t^{-1}(\Phi_t^{-1})\rangle.$$

By differentiating with respect to t the relation $\Phi_t \circ \Phi_t^{-1} = Id_{\mathbb{R}^d}$, we obtain the expression of $\partial_t \Phi_t^{-1}$

$$\partial_t \Phi_t^{-1} = -\big[D\Phi_t\big]^{-1}.\boldsymbol{V} = -m\big[D\Phi_t\big]^{-1}.\breve{\boldsymbol{n}}.$$

The properties of the used deformation fields (either $\partial_h m = 0$ either $\partial_h m = -m\,\mathrm{div}\big(\breve{\boldsymbol{n}}\big)$) and Proposition 2 leads to
$$\|\langle(\nabla m) \circ \Phi_t, \boldsymbol{n}(t) \circ \Phi_t\rangle\|_{L^2(\partial\Omega_0)} \leq C\|\Theta - Id_{\mathbb{R}^d}\|_{2,\alpha}\|m\|_{H^1(\partial\Omega_0)}.$$

Plugging the expression of $\partial_t \breve{\boldsymbol{n}}$ in $p''_\Theta(t)$, we check that this second derivatives writes

$$p''_\Theta(t) = \int_{\partial\Omega_t} m^2 a_1 + m\langle \mathbf{a}_{2,1}, a_{2,2}\nabla m\rangle + m\langle \mathbf{a}_{3,1}, \nabla_{\boldsymbol{\tau}} m\rangle\langle a_{4,1}\nabla_{\boldsymbol{\tau}} m, a_{4,2}\nabla m\rangle.$$

where both the a and $\mathbf{a}$ are obtained by additions or multiplications from the geometrical quantities $\breve{\boldsymbol{n}}, D\breve{\boldsymbol{n}}$, $\breve{\boldsymbol{n}}(t), D\breve{\boldsymbol{n}}(t)$ and Φ_t (and its derivatives up to the order two). Since all quantities satisfy (36) by Proposition 2, and since such an property is stable by both addition and multiplication, we obtain by a now classical argument the existence of a constant C such that, for all $t \in [0, 1]$,

$$\big|p''_\Theta(t) - p''_\Theta(0)\big| \leq C\omega(\eta)\|m\|^2_{H^1(\partial\Omega_0)}. \tag{46}$$

4. Study of the stability of critical shapes on some examples

Our motivation in this section is to study the stability of critical shapes on some specific examples. In particular, we use Theorem 1 to achieve this aim. The first example we study here is the mimimisation of the Dirichlet energy already studied in [5] in a particular case. The second example presented in this section is a pathological case for the method developped in this paper : it presents some new difficulties, we will discuss.

We fix α in $(0, 1)$ and $k \neq 0$ a smooth radial function in $\mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R})$ with a constant sign. We define the set $\mathcal{O}$ of admissible domains Ω as the set of open bounded subsets of $\mathbb{R}^d$ $(d \geq 2)$ with a $\mathcal{C}^{2,\alpha}$ boundary. As a constraint, the measure of each admissible domain in $\mathcal{O}$ is assumed fixed. For convinience, this constant is chosen as ω_d the volume of the unit ball B_d in $\mathbb{R}^d$. For each $\Omega \in \mathcal{O}$, we define the state-function u_Ω as the solution in $H_0^1(\Omega)$ of
$$\begin{cases} -\Delta u &= k \quad \text{in } \Omega, \\ \quad u &= 0 \quad \text{on } \Omega. \end{cases} \tag{47}$$

Study of Dirichlet energy : Let us consider J_0 the Dirichlet energy defined as

$$J_0(\Omega) = \frac{1}{2}\int_\Omega |\nabla u_\Omega|^2 - \int_\Omega k u_\Omega = -\frac{1}{2}\int_\Omega |\nabla u_\Omega|^2. \tag{48}$$

We study the stability of Ω_0 which minimizes this functionnal J_0. The analysis of [8], [5] can be followed and leads to
$$DJ_0(\Omega; \boldsymbol{V}) = -\int_{\partial\Omega} |\nabla u_\Omega|^2\langle \boldsymbol{V}, \boldsymbol{n}\rangle.$$

A critical shape Ω_0 satisfies therefore the Euler-Lagrange equation: there exists a Lagrange multiplier $\Lambda \in \mathbb{R}$ such that
$$|\nabla u_{\Omega_0}|^2 = \Lambda \text{ on } \partial\Omega_0. \tag{49}$$

From symmetry invariances, the solution u_{B_d} of (47) posed on the unit ball B_d is radially symmetric and therefore B_d is a critical shape. Let us investigate its stability.

$$D^2[J_0 + \Lambda\mathfrak{V}](B_d; \boldsymbol{V}, \boldsymbol{V}) = \int_{\partial B_d} 2\Lambda\langle \boldsymbol{V}, \boldsymbol{n}\rangle C_0(\langle \boldsymbol{V}, \boldsymbol{n}\rangle) + \langle \boldsymbol{V}, \boldsymbol{n}\rangle^2 \langle \nabla|\nabla u_0|^2, \boldsymbol{n}\rangle$$

where C_0 the Dirichlet-to-Neumann operator defined in (34). The changes occur when we explicit the term $\langle \nabla|\nabla u_0|^2, \boldsymbol{n}\rangle$. We fix x_0 on ∂B_d and compute the quantity $\langle \nabla|\nabla u_0|^2, \boldsymbol{n}\rangle$ at the point x_0. The following compution is rather general and can directly be extended to general smooth sub-manifold. The sphere ∂B_d is locally given as the graph of a function $f :]-\epsilon, \epsilon[^{d-1} \to \mathbb{R}$ with $f(0) = \partial_i f(0) = 0$ where the origin 0 is the point x_0. For all $x \in]-\epsilon, \epsilon[^{d-1}$, the homogenous Dirichlet condition writes

$$u_0(x, f(x)) = 0.$$

We differentiate this expression twice with respect to the tangential variables $(x_i)_{i\in\{1,\dots,d-1\}}$ and get for $(i,j) \in \{1, \dots, d-1\}$

$$\left\{\begin{array}{lcl} 0 & = & \partial_i u_0(x, f(x)) + \partial_i f(x)\partial_n u_0(x, f(x)), \\ 0 & = & \partial^2_{i,j} u_0(x, f(x)) + \partial_j f(x)\partial^2_{i,n} u_0(x, f(x)) + \partial_i f(x)\partial^2_{j,n} u_0(x, f(x)) + \\ & & \partial_j f(x)\partial_i f(x)\partial^2_{n,n} u_0(x, f(x)) + \partial^2_{i,j} f(x)\partial_n u_0(x, f(x)). \end{array}\right.$$

At the point x_0, these equations simplify. For $(i,j) \in \{1, \dots, d-1\}$

$$\left\{\begin{array}{lcl} 0 & = & \partial_i u_0(x_0), \\ 0 & = & \partial^2_{i,j} u_0(x_0) + \partial^2_{i,j} f(x_0)\partial_n u_0(x_0). \end{array}\right.$$

In particular, this means that

$$\sum_{i=1}^{d-1} \partial^2_{i,i} u_0 + (d-1)H\partial_n u_0(x_0) = 0$$

where H stands for the mean curvature (here $H = 1$ but we keep it in the following computations since those computations are rather general). Since $\langle \nabla u_0, \boldsymbol{n}\rangle = \partial_n u_0$, we have

$$\langle \nabla|\nabla u_0|^2, \boldsymbol{n}\rangle = \partial_n \Big[\sum_{i=1}^{d-1} (\partial_i u_0)^2 + (\partial_n u_0)^2\Big] = 2\Big[\sum_{i=1}^{d-1} \partial_i u_0 \partial^2_{i,n} u_0 + \partial_n u_0 \partial^2_{n,n} u_0\Big] = 2\partial_n u_0 \partial^2_{n,n} u_0.$$

From the state equation (47), we deduce that $-\partial^2_{n,n} u_0 = k + \sum_{i=1}^{d-1} \partial^2_{i,i} u_0$ and therefore

$$\begin{aligned} \langle \nabla|\nabla u_0|^2, \boldsymbol{n}\rangle &= 2\partial_n u_0 \partial^2_{n,n} u_0 = -2\partial_n u_0 \Big[k + \sum_{i=1}^{d-1} \partial^2_{i,i} u_0\Big], \\ &= -2(d-1)(\partial_n u_0)^2 H - 2\partial_n u_0 k = -2\Big[(d-1)\Lambda H + \langle \nabla u_0, \boldsymbol{n}\rangle k\Big], \end{aligned}$$

where Λ denotes the Lagrange multiplier. Since the unit sphere ∂B_d is connected and $\langle \nabla u_0, \boldsymbol{n}\rangle$ is continuous on ∂B_d, there exists $\epsilon = \pm 1$ such that $\sqrt{\Lambda} = \epsilon\langle \nabla u_0, \boldsymbol{n}\rangle$. To determine the value of ϵ, we write that

$$\int_{B_d} k = -\int_{B_d} \Delta u_0 = -\int_{\partial B_d} \langle \nabla u_0, \boldsymbol{n}\rangle = -\mathfrak{P}(\partial B_d)\frac{\sqrt{\Lambda}}{\epsilon} = -\frac{d\omega_d\sqrt{\Lambda}}{\epsilon}.$$

(We used the classical relationship $\mathfrak{P}(\partial B_d) = d\omega_d$). Then we get

$$\epsilon = -\frac{d\omega_d\sqrt{\Lambda}}{\int_{B_d} k}.$$

The Shape Hessian writes:

$$D^2[J_0+\Lambda\mathfrak{V}](B_d;\boldsymbol{V},\boldsymbol{V}) = 2\Lambda\int_{\partial B_d}\langle\boldsymbol{V},\boldsymbol{n}\rangle C_0(\langle\boldsymbol{V},\boldsymbol{n}\rangle) + \Big[(d-1)H - \frac{d\omega_d k}{\int_{B_d} k}\Big]\langle\boldsymbol{V},\boldsymbol{n}\rangle^2. \qquad (50)$$

Since we are concerned with vector fields such that $\int_{\partial B_d}\langle\boldsymbol{V},\boldsymbol{n}\rangle = 0$, this Hessian is coercive in $H^{1/2}(\partial\Omega_0)$ (see (35)) if the term in $\langle\boldsymbol{V},\boldsymbol{n}\rangle^2$ is non negative, that is if

$$k(1) \;<\; \frac{(d-1)}{d\omega_d}\int_{B_d} k.$$

We can be more precise : coercivity still remains if the coefficient in $\langle\boldsymbol{V},\boldsymbol{n}\rangle^2$ is negative but not too much. There exists $\lambda_d > 0$ such that

$$\langle m, C_0(m)\rangle_{H^{1/2}\times H^{-1/2}} \geq \lambda_d \|m\|_{L^2} \text{ for } m \text{ with } \int_{\partial B_d} m = 0.$$

In dimension two, the constant λ_2 can be computed explicitly through the Poisson kernel see later and we get $\lambda_2 = 1$. Therefore, we have that for all $\eta\in(0,1)$,

$$\begin{aligned} D^2[J_0+\Lambda\mathfrak{V}](B_d;\boldsymbol{V},\boldsymbol{V}) \geq & 2\Lambda(1-\eta)\int_{\partial B_d}\langle\boldsymbol{V},\boldsymbol{n}\rangle C_0(\langle\boldsymbol{V},\boldsymbol{n}\rangle) \\ & + \int_{\partial B_d}\Big[\eta\lambda_d + (d-1)H - \frac{d\omega_d k}{\int_{B_d} k}\Big]\langle\boldsymbol{V},\boldsymbol{n}\rangle^2. \end{aligned}$$

Therefore, the Hessian is still coercive in $H^{1/2}(\partial B_d)$ if

$$k(1) \;<\; \frac{\lambda_d + (d-1)}{d\omega_d}\int_{B_d} k.$$

Under this condition, Theorem 3 applies and one deduces that the ball B_d is a stable minimum. If, the opposite inegality is satisfied that is if

$$k(1) \;>\; \frac{\lambda_d + (d-1)}{d\omega_d}\int_{B_d} k,$$

there exists at least one direction of deformation for which, the Hessian is negative and the critical shape does not realize a minimum of the shaping function. Of course, the case of equality remains open. In the next example, we consider that limit case in dimension two.

Study of the Dirichlet energy in the critical case. We set $d=2$ and $k=1$. Since B_2 is a critical shape, we can restrict $\mathcal{H}$ to normal fields. We introduce $\mathcal{H}_{\boldsymbol{n}}$ the space of the normal component $m=\langle\boldsymbol{V},\boldsymbol{n}\rangle$ of admissible deformations fields that is the subspace of $\mathcal{C}^{2,\alpha}(\partial D)$ of functions the integral of which vanishes. We obtain the quadratic form $Q_0(m,m)$ corresponding to $D^2[J_0+\Lambda\mathfrak{V}](B_2,\boldsymbol{V},\boldsymbol{V})$ defined on $\mathcal{H}_{\boldsymbol{n}}$ by

$$Q_0(m,m) = \int_{\partial B_2} mC_0(m) - m^2 = \int_D |\nabla R_0 m|^2 - \int_{\partial B_2} m^2,$$

where the operator R_0 of harmonic extension is defined in (33). To study the sign of Q_0, the normal component m is written as its Fourier series. We set

$$m(\theta) = \sum_{n\in\mathbb{Z}} c_n e^{in\theta} \Rightarrow R_0 m(r,\theta) = \sum_{n\in\mathbb{Z}} c_n r^{|n|} e^{in\theta} \qquad (51)$$

by application of Poisson kernel. Note that $c_0 = 0$ because of the constraint. We get after a straightforward computation

$$Q_0(m,m) = \pi \sum_{n\in\mathbb{Z}} \big(|n|-1\big)|c_n|^2$$

A first remark is that Q_0 vanishes for the first Fouriers modes $e^{i\theta}$ and $e^{-i\theta}$. The reason of this cancellation is the invariance of the functional with respect to translations. The second point is the coercivity of Q_0 on $\mathcal{H}^1_{\boldsymbol{n}}$ the closure in $\mathcal{H}$ of the subspace generated by $\{exp(in\theta),\ |n|\geq 2\}$ where we have

$$\forall m \in \mathcal{H}^1_{\boldsymbol{n}},\ \ Q_0(m,m) \geq \frac{\pi}{2}\sum_{n\in\mathbb{Z}} |n||c_n|^2 \geq \frac{\pi}{4}\|m\|^2_{H^{1/2}(\partial B_2)}. \tag{52}$$

Remark 5 (On the norm in $H^{1/2}(\partial B_2)$) We define the $H^{1/2}(\partial B_2)$ norm through Fourier's series as

$$\|m\|^2_{H^{1/2}(\partial B_2)} = \sum_{n\in\mathbb{Z}} \sqrt{1+n^2}|c_n|^2.$$

Since, one has $|n| \leq \sqrt{1+n^2} \leq 2|n|$ for $n \neq 0$, the inequality (52) holds.

To take into account the invariance by translation, we restrict $\mathcal{O}$. The center of gravity of any admissible domain is assumed to be the origin O. This can be seen as the addition of two constraints :

$$C_1(\Omega) = \int_\Omega x_1 = 0 \text{ and } C_2(\Omega) = \int_\Omega x_2 = 0,$$

where (x_1, x_2) denotes the coordinates in the plane. This leads to the derivatives :

$$DC_1(D;\boldsymbol{V}) = \int_{\partial B_2} x_1\langle \boldsymbol{V},\boldsymbol{n}\rangle = \int_0^{2\pi} \cos\theta\langle \boldsymbol{V},\boldsymbol{n}\rangle,$$
$$DC_2(D;\boldsymbol{V}) = \int_{\partial B_2} x_2\langle \boldsymbol{V},\boldsymbol{n}\rangle = \int_0^{2\pi} \sin\theta\langle \boldsymbol{V},\boldsymbol{n}\rangle.$$

Then, the natural space to check the sign is $\mathcal{H}^1_{\boldsymbol{n}}$. The restriction of Q_0 to $\mathcal{H}^1_{\boldsymbol{n}}$ is coercive in the norm given by Theorem 1. But a major difficulty arises since the divergence-free vector field $\boldsymbol{X}^2_\Theta$ does not belong to $\mathcal{H}^1_{\boldsymbol{n}}$. Therefore, the sign of $j_0''(0) = D^2J_0(B_2;\boldsymbol{X}^2_\Theta,\boldsymbol{X}^2_\Theta)$ is *a priori* unknown.

We take advantage of the invariance under translations to solve this difficulty in that particular case. Let Φ_t be the flow of $\boldsymbol{X}^2_\Theta$. We define $\mathbf{G}_\Theta(t)$ as the position of the center of gravity of $\Omega(t) = \Phi_t(B_2)$ that is

$$\mathbf{G}_\Theta(t) = \frac{1}{\mathfrak{V}(\Omega_t)}\int_{\Omega_t} x = \frac{1}{\omega_2}\int_{B_2} \Phi_{\Theta,t}(y)\det\big(D\Phi_t(y)\big) = \frac{1}{\omega_2}\int_{B_2}\Phi_t(y), \tag{53}$$

since $\boldsymbol{X}^2_\Theta$ is divergence free, the jacobian $\det\big(D\Phi_t(y)\big)$ is identically 1. We consider now the diffeomorphism $\Psi_{\Theta,t} = \Phi_t - \mathbf{G}_\Theta(t)$ *i.e.* the flow of $\boldsymbol{X}^2_\Theta$ minus the translation of vector $\mathbf{G}_\Theta(t)$. By construction, this is a familly of diffeomomorphisms which preserve the center of gravity. This path satisfy therefore the additional constraint that the center of gravity of admissible domain should remains fixed. This path can be seen as the flow of the non-autonomous vector field $\partial_t\Psi_t = \boldsymbol{X}^2_\Theta - \partial_t\mathbf{G}_\Theta(t)$. And therefore, the vector field $(\partial_t\Psi_t)_{|t=0}$ belongs to $\mathcal{H}^1_{\boldsymbol{n}}$ and satisfies

$$\tilde{j}_0''(0) = D^2J(B_2;(\partial_t\Psi_t)_{|t=0},(\partial_t\Psi_t)_{|t=0}) \geq \frac{\pi}{2}\|\langle(\partial_t\Psi_t)_{|t=0},\boldsymbol{n}\rangle\|_{H^{1/2}(\partial B_2)}. \tag{54}$$

Since $J_0(\Omega_1) = J_0(\Omega_2)$ as soon as there exists a translation mapping Ω_1 on Ω_2,

$$\tilde{j}_0(t) = J_0(\Psi_t(B_2)) = J_0(\Phi_t(B_2) = j_0(t)$$

for all $t \in [0,1]$ and, in particular, $\tilde{j}_0''(0) = j_0''(0)$. At $t = 0$, the derivative $\partial_t \mathbf{G}_\Theta(t)$ is deduced from

$$\mathbf{G}_\Theta(t) - \mathbf{G}_\Theta(0) = \frac{1}{\omega_2}\int_{B_2} \Phi_t - I_{\mathbb{R}^2} = \frac{1}{\omega_2}\int_{B_2} t\boldsymbol{X}_\Theta^2 + o(t)$$
$$\Rightarrow (\partial_t \Psi_t)_{|t=0} = \boldsymbol{X}_\Theta^2 - \frac{1}{\omega_2}\int_{B_2} \boldsymbol{X}_\Theta^2.$$

From Stocke's formula and $\operatorname{div}\left(\boldsymbol{X}_\Theta^2\right) = 0$, we get

$$\int_{B_2} \boldsymbol{X}_\Theta^2 = \int_{\partial B_2} x\langle \boldsymbol{X}_\Theta^2, \boldsymbol{n}\rangle = \int_0^{2\pi} \sum_{n\in\mathbb{Z}} c_n e^{in\theta} \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} d\theta = \begin{pmatrix} c_{-1} + c_1 \\ -i[c_{-1} - c_1]\end{pmatrix}.$$

To compute the $H^{1/2}(\partial B_2)$ norm of $\langle(\partial_t\Psi_t)_{|t=0}, \boldsymbol{n}\rangle$, we compute its Fouriers coefficients with respect to the coefficients of $\langle \boldsymbol{X}_\Theta^2, \boldsymbol{n}\rangle$ and get

$$\langle(\partial_t\Psi_t)_{|t=0}, \boldsymbol{n}\rangle = \sum_{n\in\mathbb{Z}} \tilde{c}_n e^{in\theta} \text{ with } \begin{cases} |n| \neq 1 & \Rightarrow \tilde{c}_n = c_n, (\text{ in particular, } c_0 = 0) \\ |n| = 1 & \Rightarrow \tilde{c}_n = 0. \end{cases}$$

Since $0 < 1 - \frac{1}{\pi} < 1$, we get

$$\begin{aligned} \|\langle(\partial_t\Psi_t)_{|t=0}, \boldsymbol{n}\rangle\|^2_{H^{1/2}(\partial B_2)} &= \sum_{n\in\mathbb{Z}} \sqrt{1+n^2}|\tilde{c}_n|^2 \\ &\geq (1 - \frac{1}{\pi})\sum_{n\in\mathbb{Z}} \sqrt{1+n^2}|c_n|^2 = (1-\frac{1}{\pi})\|\langle \boldsymbol{X}_\Theta^2, \boldsymbol{n}\rangle\|^2_{H^{1/2}(\partial B_2)}. \end{aligned} \tag{55}$$

Along the path $\Omega_t = \Phi_t(B_2)$, Taylor's formula writtes

$$j_0(1) = j_0(0) + \int_0^1 (1-t)j_0''(t)dt, \tag{56}$$

and we have

$$j_0''(t) = \underbrace{j_0''(0)}_{\substack{= \tilde{j}_0''(0) \geq \frac{\pi}{2}\|\langle(\partial_t\Psi_t)_{|t=0}, \boldsymbol{n}\rangle\|^2_{H^{1/2}(\partial B_2)} \\ \geq \frac{\pi}{2}(1-\frac{1}{\pi})\|m\|^2_{H^{1/2}} \\ \text{from (54) and (55)}}} + \underbrace{\left[j_0''(0) - j_0''(t)\right]}_{\substack{\leq C\omega(\eta)\|m\|^2_{H^{1/2}(\partial B_2)} \\ \text{from Theorem 1}}},$$

Therefore, as soon as the diameter η of the ball in $\mathcal{O}$ (for the $\mathcal{C}^{2,\alpha}$ norm) around the critical shape D is small enough, $j_0''(t) > 0$ for all $t \in [0,1]$ and (56) shows that D is a local strict minimum of J_0.

Study of a functional without ∇u_0 : We still consider the plane case and $k = 1$. Then u_D is given in polar coordinates as $u_D(r,\theta) = (1-r^2)/4$. Therefore, u_D has the natural extension $(1-r^2)/4$ on the whole space. We still denote this extension by u_D. Note also that $\partial_{\boldsymbol{n}} u_d = \partial_r u_d = -1/2$ on the unit circle ∂B_2. On $\mathcal{O}$, we define

$$J_1(\Omega) = \|u_\Omega - u_D\|^2_{L^2(\Omega)} = \int_\Omega (u_\Omega - u_D)^2. \tag{57}$$

We will show that the unit disk is a stable minimum of J_1 but that this stability cannot be deduced from Theorem 1.

By definition, we have $E(\Omega) \geq 0 = E(D)$, therefore the unit disk D is a global minimum of J_1. Moreover, for all Ω in $\mathcal{O}$, the state-function u_Ω is $\mathcal{C}^{2,\alpha}$. From the maximum principle, u_Ω does not vanish in Ω. Then, $u_\Omega = u_D$ if and only if $\Omega = D$. That is D is then the global strict minimum of J_1. We now try to prove this elementary result with the presented method.

We first apply Lemma 4 to compute the successive derivatives of the restricted functional $j_1(t) = J_1\big(\Phi_{\Theta,t}(\Omega_0)\big)$. For all $t \in [0,1]$,

$$\begin{aligned} j_1'(t) &= \int_{\Omega_t} 2\partial_t u(u-u_D) + \operatorname{div}\big(|u-u_D|^2 \boldsymbol{V}\big), \\ j_1''(t) &= \int_{\partial\Omega_t} \Big[4\partial_t u(u-u_D) + \operatorname{div}\big(|u-u_D|^2\boldsymbol{V}\big)\Big]\langle \boldsymbol{V}, \boldsymbol{n}\rangle + \int_{\Omega_t} 2\partial_{tt}^2 u(u-u_D) + 2(\partial_t u)^2. \end{aligned}$$

Since the expression of the directional derivative remains valid for any deformation field in $\mathcal{V}$, D is a critical shape for J_1. Moreover, the proof of Theorem 1 shows that the integral over the boundary varies with a control in the L^2-norm and that the integral over the domain is controlled by the $H^{1/2}$-norm.

Let us study the coercivity of the Hessian. Since $u_{|t=0} = u_D$, cancellations occur in $j_1''(0)$ and we obtain only

$$j_1''(0) = 2\int_D (\partial_t u)^2 \geq 0. \tag{58}$$

Recall that $\partial_t u$ solves (23). Here the boundary condition simplifies as

$$\langle \boldsymbol{V}, \nabla u_D\rangle = m\langle \check{\boldsymbol{n}}, \nabla u_D\rangle = m\partial_r u_D = -\frac{m}{2}.$$

Let $m \in \mathcal{H}_{\boldsymbol{n}}$ given by its Fourier series (51) with $c_0 = 0$. We get with the Poisson kernel

$$\begin{aligned} Q_1(m,m) &= \frac{1}{2}\int_0^1\int_0^{2\pi} \Big(\sum_{n\in\mathbb{Z}} c_n r^{|n|} e^{in\theta}\Big)\overline{\Big(\sum_{n\in\mathbb{Z}} \overline{c_n} r^{|n|} e^{-in\theta}\Big)} dr d\theta, \\ &= \pi\int_0^1 \sum_{n\in\mathbb{Z}} |c_n|^2 r^{2|n|+1} dr = \frac{\pi}{2}\sum_{n\in\mathbb{Z}} \frac{|c_n|^2}{|n|+1}. \end{aligned}$$

Therefore, $Q_1(m,m)$ is coercive on $\mathcal{H}_{\boldsymbol{n}}$ but only for the $H^{-1/2}(\partial B_2)$ norm since

$$\frac{\pi}{4}\|m\|^2_{H^{-1/2}(\partial B_2)} \leq Q_1(0)(m,m) \leq \frac{\pi}{2}\|m\|^2_{H^{-1/2}(\partial B_2)}$$

and Theorem 1 does not allow us to conclude to the stability of D.

Remark 6 In this example, the dominant term, the one which behaves in the strongest norm, in the expression of the Hessian at the critical point vanishes. Therefore, only weaker terms remains at the critical point and provide the coercivity of the Hessian. This cancellation is specific to the critical point and the dominant term reappears for non critical shape.

On the stabilising effect of surface tension. To modelise surface tensions, one can use a penalisation of E by the perimeter. Assume the existence of a critical shape Ω_0 for the penalised shaping function $E_\varepsilon = E + \varepsilon\mathfrak{P}$ and assume that the shape Hessian of E_ε is coercive in a weak norm $\|.\|_w$.

In order to be in the situation of J_0, the coercivity needed (at least in this approach) to claim stability is a coercivity in H^1-norm shown by Theorem 1 and equation (46). But the H^1 norm can only appear through derivatives of $\mathfrak{P}$.

The physical interpretation is that the tension force are strong enough to insure a geometry of the critical boundary $\partial\Omega_0$ rather close to a sphere. When the surface forces are weaker (that is ε is smaller), we are in the situation of J_1 where three different norms (the norm of differentiability, of coercivity and of continuity of the second derivative) are involved and where further studies are necessary.

The conclusions of this paper are the following ones. Two cases are possible: either no additional weak norms appears in the study and then the weak coercivity insures stability (see [5] and example J_0 in Section 4.), either a third non equivalent topology will appear and then a new problem arises (see example J_1 in Section 4.).

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