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A characterization of space-filling curves

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Abstract. A famous theorem discovered in 1936 by H.Steinhaus on a sufficient condition for obtaining the coordinate functions of a curve filling the unit square is revised in the present paper. Here we point out that the converse of the above theorem fails in the Lebesgue curve. A characterization of the space-filling curves by means of a filling condition is proposed. A constructive characterization of this filling condition, in terms of the Borel measures, is also settled.

Una caracterización de curvas que llenan el espacio

Resumen. En este artículo revisamos un famoso teorema, descubierto por H. Steinhaus en 1936, en el que se da una condición suficiente que permite obtener las funciones coordenadas de una curva que llena el cuadrado unidad. Ponemos de manifiesto que el recíproco de este teorema no se cumple para la curva de Lebesgue. Aquí proponemos un teorema de caracterización de curvas que llenan el espacio, basado en una condición de llenado. Asimismo, damos una caracterización constructiva de esta condición de llenado por medio de medidas de Borel.

1. Introduction and notation

In 1890, G. Peano [9] demonstrated that the interval $I = [0, 1]$ could be mapped surjectively and continuously onto the square $Q = [0, 1]^2$. Immediately, furthers examples of such curves by D. Hilbert (1891) [3], E. H. Moore (1900) [7], H. Lebesgue (1904) [5], [6] and others followed. In spite of each curve was greatly superior in simplicity and ingenuity to the previous, a method for generating them remained unsettled. H. Steinhaus in 1936 [11] solved the problem by means of a surpresively result: *if two continuous non-constant functions on I are stochastically independent with respect to Lebesgue measure, then they are the coordinate functions of a space-filling curve* (see [10, pp. 2] and the original paper of Steinhaus [11]).

The attainment of space-filling curves by means of stochastically independent functions, begun by Steinhaus, was soon forgotten and, apparently, Garsia [1] and others (see [4]) arrived to the same conclusions about forty years later. Following the way of the stochastic independence (in brief, $s.i.$), it is neccesary to remark the work of Holbrook in [4]. Nevertheless, as we shall prove below, the $s.i.$ is a too much strong condition for giving a characterization theorem on space-filling curves, which is exactly the objective of our paper. For this reason we introduce here (Definition 1) a *filling condition* (in brief $f.c.$), which will be appropriate to characterize the space-filling curves.

The $f.c.$ is a concept, implicitly handled in [8], that was given to characterize a class of curves that contains to the family of the space-filling, namely the $\alpha - dense\ curves$ in parallelepipeds H of $\mathbb{R}^n$. These

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curves have the property of densifying H, $i.e.$ for any point of H there is a point of the curve at distance less than or equal that $\alpha \geq 0$.

To avoid that the $f.c.$ to be considered as a trivial characterization of space-filling curves, a characterization theorem on that condition , in terms of Borel measures, will be also settled. Moreover, this result will point the way to the construction of the coordinate function of a space-filling curve.

In order to facilitate the reading of the text , recall some definitions, contained in [10], concerning to the concepts of space-filling curves, stochastically independent functions and others.

Γ will denote the *Cantor set*, J_n the n-dimensional *Jordan content* of a Jordan measurable subset of $\mathbb{R}^n$ and Λ_n the n-dimensional *Lebesgue measure* of a Lebesgue measurable subset of $\mathbb{R}^n$.

A continuous function $f : I \to \mathbb{R}^n$ with $n \geq 2$, is called a *space-filling curve* if $J_n(f(I)) > 0$.

Let $\varphi_1, \ldots, \varphi_n : I \to \mathbb{R}$ be measurable functions. Then, $\varphi_1, \ldots, \varphi_n$ are called *stochastically independent* with respect to the Lebesgue measure (in brief r.L.m.) if, for any measurable sets $A_1, \ldots, A_n$ of $\mathbb{R}$,

$$\Lambda_1 \left[\varphi_1^{-1}(A_1) \cap ... \cap \varphi_n^{-1}(A_n)\right] = \Lambda_1 \left[\varphi_1^{-1}(A_1)\right] \times \ldots \times \Lambda_1 \left[\varphi_n^{-1}(A_n)\right].$$

A surjective function $f : I \to Q$ is said to be *measure-preserving* if, for any measurable set A of Q,

$$\Lambda_1(f^{-1}(A)) = \Lambda_2(A).$$

2. The quasi-stochastic independence as a filling condition.

The purpose of this section is to prove that the stochastic independence is sufficient but it is not a necessary condition to define space-filling curves. A characterization of these curves will be given by means of the following simple concept .

Definition 1 (Filling condition) *We shall say that n measurable functions $\varphi_1, \ldots, \varphi_n : I \to \mathbb{R}$ are quasi-stochastically independent (in brief q.s.i.) with respect to the Lebesgue measure, if for any open sets $A_1, \ldots, A_n$ of $\mathbb{R}$ the condition*

$$\Lambda_1 \left[\varphi_1^{-1}(A_1)\right] \times \ldots \times \Lambda_1 \left[\varphi_n^{-1}(A_n)\right] > 0$$

implies

$$\Lambda_1 \left[\varphi_1^{-1}(A_1) \cap \ldots \cap \varphi_n^{-1}(A_n)\right] > 0.$$

Our next result is immediate.

Proposition 1 *Let $\varphi_1, \ldots, \varphi_n : I \to \mathbb{R}$ be nonconstant continuous functions such that the curve $f = (\varphi_1, \ldots, \varphi_n) : I \to \mathbb{R}^n$ fills the parallelepiped $\sqcap_{i=1}^n \varphi_i(I)$. Thus $\varphi_1, \ldots, \varphi_n$ are quasi-stochastically independent (r.L.m).*

As a generalization of the classical result of Steinhaus (see [10, pp. 109] or [4, Proposition 1]) we expose the following theorem.

Theorem 1 *Let $\varphi_1, \ldots, \varphi_n : I \to \mathbb{R}$ be quasi-stochastically independent functions (r.L.m.). Assume also that they are continuous but not constant. Thus the curve defined by $f = (\varphi_1, \ldots, \varphi_n) : I \to \mathbb{R}^n$ fills the parallelepiped $\sqcap_{i=1}^n \varphi_i(I)$.*

PROOF. Let $x = (x_i)_{i=1}^n$ be a point of $\sqcap_{i=1}^n \varphi_i(I)$, so there exist $(t_i)_{i=1}^n$ in I such that $x_i = \varphi_i(t_i)$ for each $i = 1, \ldots, n$. Given $\varepsilon > 0$, consider the open sets

$$A_i = \left(x_i - \frac{\varepsilon}{\sqrt{n}}, x_i + \frac{\varepsilon}{\sqrt{n}}\right) \quad \text{for } i = 1 \ldots, n \tag{1}$$

By continuinity, $\varphi_i^{-1}(A_i)$ is open in I and contains t_i, therefore the condition

$$\Lambda_1\left[\varphi_1^{-1}(A_1)\right]\times\ldots\times\Lambda_1\left[\varphi_n^{-1}(A_n)\right]>0$$

holds. Since $\varphi_1,\ldots,\varphi_n$ are q-s.i., one has

$$\Lambda_1\left[\varphi_1^{-1}(A_1)\cap\ldots\cap\varphi_n^{-1}(A_n)\right]>0.$$

Thus there exists $t\in I$ such that $\varphi_i(t)\in A_i$. From (1)

$$|\varphi_i(t)-x_i|<\frac{\varepsilon}{\sqrt{n}}\quad\text{for } i=1,\ldots,n,$$

so the euclidean norm $\|f(t)-x\|<\varepsilon$. This proves that $f(I)$ is dense in $\prod_{i=1}^n\varphi_i(I)$, but $f(I)$ is a compact set, therefore $f(I)=\prod_{i=1}^n\varphi_i(I)$ and the result follows. ■

To expose our next results,we need recall some elementary properties on the Cantor set and the Lebesgue curve.

The binary representation of a number $a\in I$ will be denoted by $0_2,a_1,a_2a_3\ldots$ where $a_i\in\{0,1\}$. Analogously in the ternary basis, $a\in I$ is written as $0_3,a_1a_2a_3\ldots$ with $a_i\in\{0,1,2\}$.

The Cantor set, or the set of the excluded middle thirds, can be represented by all numbers of $[0,1]$ such that, in the ternary basis, can be written only using the digits 0 and 2, $i.e.$

$$\Gamma=\{0_3,(2t_1)(2t_2)(2t_3)\ldots:\ t_j=0\text{ or }1\}.$$

A continuous mapping f can be defined from Γ onto the unit square Q by means of

$$f(0_3,(2t_1)(2t_2)(2t_3)\ldots)=(0_2,t_1t_3t_5\ldots;0_2,t_2t_4t_6\ldots).$$

H. Lebesgue extended this mapping continuously into I by linear interpolation, obtaining a continuous function f_l defined on the complement Γ^c as

$$f_l(t)=\frac{1}{b_n-a_n}\left[(b_n-t)\cdot f(a_n)+(t-a_n)\cdot f(b_n)\right],$$

(a_n,b_n) being the interval that is removed in the construction of Γ at the n-th step and $a_n\le t\le b_n$. Then, the *Lebesgue curve* (also *Lebesgue function*), denoted by L, is defined by

$$L(t)=\begin{cases} f(t), & \text{if } t\in\Gamma\\ f_l(t), & \text{if } t\in\Gamma^c.\end{cases}$$

L is a continuous and surjective function onto the square Q, so a space-filling curve (is also differentiable almost everywhere; for details see [10, Theorem 5.4.2]).

The two following simple propositions show just how fails the converse of the Steinhaus theorem in the Lebesgue Curve.

Proposition 2 *The Lebesgue curve is not a measure-preserving function.*

PROOF. Consider the measurable set $A=\left[0,\frac{1}{2}\right)\times\left[0,\frac{1}{2}\right)$, then we claim that $L^{-1}(A)\subset\left[0,\frac{1}{9}\right)$. Indeed, let (x,y) be an element belonging to A, then in the binary basis one has

$$x=0_2,0r_2r_3\ldots\quad\text{and}\quad y=0_2,0s_2s_3\ldots$$

where $r_i,s_i\in\{0,1\}$ for any $i\ge 2$ and with some $r_i,s_i=0$ (for instance, observe that if all $r_i=1$, then $x=\frac{1}{2}\Big)$.

By denoting $t = L^{-1}(x, y)$, we have two cases.
Case 1: $t \in \Gamma$, then $t = 0_3, 00(2r_2)(2s_2)\ldots$ and consequently $t \in \left[0, \frac{1}{9}\right)$.
Case 2 : $t \in \Gamma^c$, then we have again that $t \in \left[0, \frac{1}{9}\right)$. Indeed, suppose that there is a value $t \in \Gamma^c$ with $t > \frac{1}{9}$ and $L(t) \in A$. Let (a_n, b_n) be the interval that has been removed in the construction of Γ. Thus $a_n < t < b_n$ and noticing that $\frac{1}{9} \in \Gamma$, it follows that $a_n \geq \frac{1}{9}$.
On the other hand, as $L\left(\frac{1}{3}\right) = \left(\frac{1}{2}, 1\right)$ and $L\left(\frac{2}{3}\right) = \left(\frac{1}{2}, 0\right)$ one has that $t \notin \left(\frac{1}{3}, \frac{2}{3}\right)$. Furthermore, since

$$L\left(\frac{1}{9}\right) = \left(\frac{1}{2}, \frac{1}{2}\right), \ L\left(\frac{2}{9}\right) = \left(0, \frac{1}{2}\right), \ L\left(\frac{7}{9}\right) = \left(1, \frac{1}{2}\right) \quad \text{and } L\left(\frac{8}{9}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

we conclude that $t \notin \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right)$. Therefore $t \in (a_n, b_n)$ with $a_n = \frac{m}{3^n}$, $b_n = \frac{m+1}{3^n}$ and $n \geq 3$.

Consider the following three possibilities:

i) $a_n = 0_3, 0(2r_2)(2r_3)(2r_4)\ldots$ and $b_n = 0_3, 2(2s_2)(2s_3)\ldots$ with $r_i, s_i \in \{0, 1\}$. Then $a_n \leq 0_3, 0222... = 0_3, 1$ and $b_n \geq 0_3, 2000\ldots = 0_3, 2$. Therefore $b_n - a_n \geq \frac{1}{3}$, which leads us to a contradiction since $b_n - a_n = \frac{1}{3^n}$ with $n \geq 3$.

ii) $a_n = 0_3, 02(2r_3)(2r_4)\ldots$ and $b_n = 0_3, 02(2s_3)(2s_4)\ldots$ with $r_i, s_i \in \{0, 1\}$. Then $L(a_n) = f(a_n) = (0_2, 0r_3r_5\ldots;\ 0_2, 1r_4r_6\ldots)$ and $L(b_n) = f(b_n) = (0_2, 0s_3s_5\ldots;\ 0_2, 1s_4s_6\ldots)$. Therefore, if we denote by φ and ψ the first and second component of the function L respectively, we get

$$\psi(a_n) \geq 0_2, 100\ldots = \frac{1}{2};\ \psi(b_n) \geq 0_2, 100\ldots = \frac{1}{2}.$$

Consequently, $\psi(t) \geq \frac{1}{2}$ for any $t \in (a_n, b_n)$ and then it contradits that $L(t) \in A$.

iii) $a_n = 0_3, 2(2r_2)(2r_3)\ldots$ and $b_n = 0_3, 2(2s_2)(2s_3)\ldots$ with $r_i, s_i \in \{0, 1\}$. In this case $\varphi(a_n) = 0_2, 1r_3r_5\ldots$ and $\varphi(b_n) = 0_2, 1s_3s_5\ldots$ Thus we have

$$\varphi(a_n) \geq 0_2, 100\ldots = \frac{1}{2};\ \varphi(b_n) \geq 0_2, 100\ldots = \frac{1}{2},$$

deducing that $\varphi(t) \geq \frac{1}{2}$ for any $t \in (a_n, b_n)$, which is again a contradiction.
As conclusion $L^{-1}(A) \subset \left[0, \frac{1}{9}\right)$, involving that $\Lambda_1\left[L^{-1}(A)\right] \leq \frac{1}{9}$. Since $\Lambda_2(A) = \frac{1}{4}$, we deduce that $\Lambda_1\left[L^{-1}(A)\right] \neq \Lambda_2(A)$ and consequently the Lebesgue curve is not a measure-preserving function, as claimed. ■

Proposition 3 *The coordinate functions φ and ψ of the Lebesgue curve are not stochastically independent.*

PROOF. Let us take $A_1 = A_2 = \left[0, \frac{1}{2}\right)$, then we claim that:

i) $\left[0, \frac{1}{3}\right) \setminus \Gamma \subset \varphi^{-1}(A_1) \subset \left[0, \frac{1}{3}\right)$,

ii) $\left(\left[0, \frac{1}{9}\right) \cup \left(\frac{1}{2}, \frac{7}{9}\right)\right) \setminus \Gamma \subset \psi^{-1}(A_2) \subset \left[0, \frac{1}{9}\right) \cup \left(\frac{1}{2}, \frac{7}{9}\right)$.

Indeed, let us take $a \in \left[0, \dfrac{1}{3}\right) \cap \Gamma$. In the ternary basis, a is expressed as

$$a = 0_3, 0(2r_2)(2r_3)\dots, \text{with some } r_i = 0 \text{ for } i \geq 2,$$

hence

$$\varphi(a) = 0_2, 0r_3r_5\dots \leq\ 0_2, 011\dots = \frac{1}{2}.$$

Let $a_n, b_n \in \left[0, \dfrac{1}{3}\right) \cap \Gamma$ be the end-points of the interval (a_n, b_n) removed in the construction of Γ, then

$$b_n = 0_3, 0(2s_2)(2s_3)\dots$$

has the following easy property: *there exists* $k \geq 2$ *such that* $s_k = 1$ *and* $s_i = 0$ *for all* $i > k$. As consequence

$$\varphi(b_n) < 0_2, 011\dots = \frac{1}{2}.$$

For each $t \in \left[0, \dfrac{1}{3}\right) \setminus \Gamma$ denote by λ the number $\dfrac{t - a_n}{b_n - a_n}$, then

$$\varphi(t) = (1 - \lambda)\varphi(a_n) + \lambda\varphi(b_n) \quad \text{with } 0 < \lambda < 1.$$

Since $\varphi(a_n) \leq \dfrac{1}{2}$ and $\varphi(b_n) < \dfrac{1}{2}$, we deduce that $\varphi(t) < \dfrac{1}{2}$ and so $\left[0, \dfrac{1}{3}\right) \setminus \Gamma \subset \varphi^{-1}(A_1)$ is proved.

On the other hand, if $a \in \Gamma$ with $a \geq \dfrac{1}{3}$, one has that either $a = 0_3, 022\dots \left(\text{for } a = \dfrac{1}{3}\right)$ or $a = 0_3, 2(2r_2)(2r_3)\dots \left(\text{for } a > \dfrac{1}{3}\right)$. Therefore we get $\varphi(a) \geq \dfrac{1}{2}$.

Assume $t \in \left(\dfrac{1}{3}, 1\right] \setminus \Gamma$, then there exist $a_n, b_n \in \Gamma$ with $\dfrac{1}{3} \leq a_n < t < b_n$ and so

$$\varphi(t) = (1 - \lambda)\varphi(a_n) + \lambda\varphi(b_n) \geq \frac{1}{2}.$$

Therefore $\varphi^{-1}(A_1) \subset \left[0, \dfrac{1}{3}\right)$ and it proves i). Finally, as conclusion, since $\Lambda_1(\Gamma) = 0$, it follows that

$$\Lambda_1(\varphi^{-1}(A_1)) = \frac{1}{3}. \tag{2}$$

For proving ii), first observe that if $a \in \left(\left[0, \dfrac{1}{9}\right) \cup \left[\dfrac{2}{3}, \dfrac{7}{9}\right)\right) \cap \Gamma$ thus

$$\psi(a) \leq \frac{1}{2}. \tag{3}$$

Indeed, a number $a \in \left[0, \dfrac{1}{9}\right)$ is expressed as

$$a = 0_3, 00(2r_3)(2r_4)\dots$$

with some $r_i = 0$ for $i \geq 3$. Therefore $\psi(a) = 0_2, 0r_4r_6... \leq 0_2, 011\dots = \dfrac{1}{2}$. Analogously, if $a \in \left[\dfrac{2}{3}, \dfrac{7}{9}\right)$ then $a = 0_3, 20(2r_3)(2r_4)\dots$ with some $r_i = 0$ for $i \geq 3$ and so $\psi(a) = 0_2, 0r_4r_6\dots \leq 0_2, 011\dots = \dfrac{1}{2}$.

On the other hand, if $a \in \left(\left[\frac{1}{9}, \frac{2}{3}\right) \bigcup \left[\frac{7}{9}, 1\right)\right) \cap \Gamma$ thus

$$\psi(a) \geq \frac{1}{2}. \tag{4}$$

Indeed, whether $a \in \left[\frac{1}{9}, \frac{2}{3}\right)$ its expression is given by $a = 0_3, 02(2r_3)(2r_4)\ldots$ and then $\psi(a) = 0_2, 1r_4 \ldots \geq 0_2, 1 = \frac{1}{2}$. If $a \in \left[\frac{7}{9}, 1\right)$, thus $a = 0_3, 2(2r_2)(2r_3)(2r_4)\ldots$. Now, noticing that $\frac{7}{9} = 0_3, 2022\ldots$ we have that, either $r_2 = 1 \left(a > \frac{7}{9}\right)$ or $r_2 = 0$ with $r_i = 1$ for all $i \geq 3 \left(a = \frac{7}{9}\right)$. Therefore, in both cases

$$\psi(a) = 0_2, r_2 r_4 \ldots \geq 0_2, 1 = \frac{1}{2}.$$

From (3) and (4) we deduce that if $a \in \Gamma$, the condition

$$\psi(a) < \frac{1}{2} \tag{5}$$

implies, necessary, that a is a number belonging to

$$\left[0, \frac{1}{9}\right) \bigcup \left[\frac{2}{3}, \frac{7}{9}\right).$$

Let us take $t \in \left(\left[0, \frac{1}{9}\right) \cup \left(\frac{1}{2}, \frac{7}{9}\right)\right) \setminus \Gamma$, then there exist $a_n, b_n \in \Gamma$ with $a_n < t < b_n$ and then we have either $[a_n, b_n] \subset \left[0, \frac{1}{9}\right) \bigcup \left(\frac{2}{3}, \frac{7}{9}\right)$ or $[a_n, b_n] = \left[\frac{1}{3}, \frac{2}{3}\right]$.

In the first case, as we have already showed, $\psi(a_n) \leq \frac{1}{2}$ and $\psi(b_n) < \frac{1}{2}$, so $\psi(t) < \frac{1}{2}$. Whether $[a_n, b_n] = \left[\frac{1}{3}, \frac{2}{3}\right]$, since $\frac{1}{2} < t < \frac{2}{3}$, one has

$$\begin{aligned} \psi(t) &= (1-\lambda)\psi(a_n) + \lambda\psi(b_n) = (1-\lambda)\psi(\frac{1}{3}) + \lambda\psi(\frac{2}{3}) = 1 - \lambda \\ &= 1 - \frac{t - a_n}{b_n - a_n} = \frac{\frac{2}{3} - t}{\frac{2}{3} - \frac{1}{3}} < \frac{1}{2} \end{aligned}$$

and this shows the first part of ii).

Finally, if $\psi(x) < \frac{1}{2}$ and $x = a \in \Gamma$, from (5) we deduce that

$$x \in \left[0, \frac{1}{9}\right) \bigcup \left[\frac{2}{3}, \frac{7}{9}\right) \subset \left[0, \frac{1}{9}\right) \bigcup \left(\frac{1}{2}, \frac{7}{9}\right).$$

On the other hand, if $\psi(x) < \frac{1}{2}$ and $x = t \notin \Gamma$, we obtain the same conclusion. Indeed, by supposing that

$$x \notin \left[0, \frac{1}{9}\right) \bigcup \left[\frac{2}{3}, \frac{7}{9}\right)$$

and by applying again (4) to the end-points a_n and b_n of the removed interval (a_n, b_n), with $a_n < t < b_n$, we are led to $\psi(t) \geq \frac{1}{2}$, which is a contradiction.

The above involves that for any x with $\psi(x) < \frac{1}{2}$, one follows that

$$x \in \left[0, \frac{1}{9}\right) \bigcup \left[\frac{2}{3}, \frac{7}{9}\right) \subset \left[0, \frac{1}{9}\right) \bigcup \left(\frac{1}{2}, \frac{7}{9}\right)$$

and therefore ii) is proved.

Now, i) and ii) imply

$$\Lambda_1(\psi^{-1}(A_2)) = \frac{7}{18}$$

and from (2) one has

$$\Lambda_1(\varphi^{-1}(A_1)).\Lambda_1(\psi^{-1}(A_2)) = \frac{7}{54}.$$

On the other hand, since $\varphi^{-1}(A_1) \cap \psi^{-1}(A_2) \subset \left[0, \frac{1}{9}\right)$, one deduces

$$\Lambda_1\left(\varphi^{-1}(A_1) \cap \psi^{-1}(A_2)\right) \leq \frac{1}{9} < \frac{7}{54} = \Lambda_1(\varphi^{-1}(A_1)).\Lambda_1(\psi^{-1}(A_2)),$$

concluding that φ and ψ are not stochastically independent. ■

From this the following is clear.

Corollary 1 *There are space-filling curves whose coordinate functions are not stochastically independent.*

Corollary 2 *There are Q.S.I. functions that are not stochastically independent.*

3. Characterization of the Q.S.I. condition

Though it is obvious that Theorem 1 is not a trivial result, the characterization of space-filling curves by means of the Q.S.I. condition could seem it. Therefore, to avoid that appearance, in this section we are going to prove that the Borel measures characterize the Q.S.I. condition, in such a way that the coordinate functions of a space-filling curve can be easily determined.

From the countably additivity of the Lebesgue measure and Theorem 1, the easy technical lemma follows immediately.

Lemma 1 *Let $\varphi_1, \varphi_2, \ldots, \varphi_n : I \to \mathbb{R}$ be nonconstant continuous functions. Suppose also they are Q.S.I. Then the set function μ defined by*

$$\mu\left(\prod_{i=1}^{n} B_i\right) = \Lambda_1\left[\cap_{i=1}^{n} \varphi_i^{-1}(B_i)\right],$$

is countably additive on the class C_H of all cubes $C = \prod_{i=1}^{n} B_i$ contained in the parallelepiped $H = \prod_{i=1}^{n} \varphi_i(I)$. Furthermore, $\mu(H) = 1$.

With the help of this lemma we finally have what we wanted all along, the connection between the curves filling a parallelepiped and the Borel measures defined on it.

Theorem 2 *Assume $\varphi_1, \ldots\ldots, \varphi_n : I \to \mathbb{R}$ are continuous nonconstant functions verifying the Q.S.I. condition. Then the set function $\mu(\prod_{i=1}^{n} B_i) = \Lambda_1\left[\cap_{i=1}^{n} \varphi_i^{-1}(B_i)\right]$, on the class C_H of all cubes $\prod_{i=1}^{n} B_i$ contained in $H = \prod_{i=1}^{n} \varphi_i(I)$, defines a Borel measure on H such that*

$$\mu(H) = 1 \quad \textit{and} \quad \mu(C) > 0 \tag{6}$$

for any cube C of C_H with $int(C) \neq \emptyset$.
Reciprocally, any Borel measure μ on a parallelepiped $H = \prod_{i=1}^{n} [a_i, b_i]$, $(a_i < b_i, i = 1, 2, \ldots, n)$ satisfying (6) defines n continuous nonconstant functions that are Q.S.I.

PROOF. Suposse that the functions $\varphi_1, \ldots, \varphi_n$ are Q.S.I., then, by Theorem 1, the curve $\varphi = (\varphi_1, \ldots, \varphi_n)$ fills $H = \prod_{i=1}^{n} \varphi_i(I)$. On the other hand, by the previous Lemma, μ is countably additive on the class C_H and satisfies $\mu(H) = 1$. Clearly, then, μ defines on the ring $\Re(K)$, of all finite disjoint unions of sets of C_H, a unique finite measure which is extended to an unique measure on the σ-ring $\Sigma(H)$ (see for instance [2]) that contains the Borel sets. Therefore μ defines a Borel measure on H, and from the Q.S.I condition

$$\mu(C) = \Lambda_1 \left[\cap_{i=1}^{n} \varphi_i^{-1}(B_i)\right] > 0$$

for any cube $C = \prod_{i=1}^{n} B_i$,provided that $int(C) \neq \emptyset$.
Conversely, let μ be a Borel measure on $H = \prod_{i=1}^{n} [a_i, b_i]$ verifying the condition (6) of the statement of the theorem. Let P_1 denote the partition of H into the 2^n equal disjoint subcubes $\left\{ C_p^{(1)} : 1 \leq p \leq 2^n \right\}$. Assume, for $N > 1$, that P_N is the partition of H, into the 2^{Nn}equal disjoint subcubes $\left\{ C_p^{(N)} : 1 \leq p \leq 2^{Nn} \right\}$, obtained from the partition P_{N-1} by the division of each $C_p^{(N-1)}$ into 2^n equal disjoint subcubes.
For an integer $M \geq 1$, we arrange the 2^{Mn} cubes $C_p^{(M)}$, of the partition P_M, in such a way that $C_{p+1}^{(M)}$ is adjacent to $C_p^{(M)}$ for each $1 \leq p \leq 2^{Mn}$. Assume the same has been also done for each of the 2^n subcubes produced by the construction of the partition P_{M+1}.
Since $\mu(H) = 1$, we have $\sum_{p=1}^{2^{Mn}} \mu(C_p^{(M)}) = 1$. Noticing that $\mu(C_p^{(M)}) > 0$ for all $1 \leq p \leq 2^{Mn}$, the 2^{Mn} real intervals defined as

$$\begin{aligned}
I_1^{(M)} &= \left[0, \mu(C_1^{(M)})\right), \\
I_2^{(M)} &= \left[\mu(C_1^{(M)}), \mu(C_1^{(M)}) + \mu(C_2^{(M)})\right), \\
&\vdots \\
I_{2^{Mn}}^{(M)} &= \left[\sum_{p=1}^{2^{Mn}-1} \mu(C_p^{(M)}), 1\right]
\end{aligned}$$

form a partition on $I = [0, 1]$ for any $M \geq 1$. Furthermore, in view of the above arrangement,

$$I_1^{(M)} = \cup_{j=1}^{2^n} I_j^{(M+1)}, \; I_2^{(M)} = \cup_{j=2^n+1}^{2^{n+1}} I_j^{(M+1)}, \ldots$$

and so on.
Now, we distinguish an interior point of each $C_p^{(M)}$ with $1 \leq p \leq 2^{Mn}$, say, for instance, its center $P_p^{(M)} = (x_{(p)i}^{(M)})_{i=1}^{n}$ with $1 \leq p \leq 2^{Mn}$ and $M \geq 1$. This allows us to define on I the n functions

$$\left\{ \begin{aligned}
h_1^{(M)}(t) &= x_{(p)1}^{(M)} \text{ if } t \in I_p^{(M)}, \\
h_2^{(M)}(t) &= x_{(p)2}^{(M)} \text{ if } t \in I_p^{(M)}, \\
&\vdots \\
h_n^{(M)}(t) &= x_{(p)n}^{(M)} \text{ if } t \in I_p^{(M)}, \quad 1 \leq p \leq 2^{Mn}.
\end{aligned} \right. \tag{7}$$

Now, we are going to prove that the limits

$$\lim_{M\to\infty} h_1^{(M)}, \; \lim_{M\to\infty} h_2^{(M)}, \ldots, \lim_{M\to\infty} h_n^{(M)}$$

there exist, define functions that are continuous (observe that $h_1^{(M)}, \ldots, h_n^{(M)}$ are not) and satisfy the Q.S.I. condition. Indeed, let us take an index i with $1 \leq i \leq n$ and denote by L_i the length of the interval $[a_i, b_i]$. Directly from the definition of $h_i^{(M)}$,

$$\left| h_i^{(M)}(t) - h_i^{(M+1)}(t) \right| = \frac{1}{4} L_i 2^{-M} \text{for any } t \in I. \tag{8}$$

For $N > M$

$$\left|h_i^{(N)}(t) - h_i^{(M)}(t)\right| \leq \left|h_i^{(N)}(t) - h_i^{(N-1)}(t)\right| + \left|h_i^{(N-1)}(t) - h_i^{(N-2)}(t)\right| + \dots + \left|h_i^{(M+1)}(t) - h_i^{(M)}(t)\right|.$$

Hence, given $\varepsilon > 0$, there exists a large enough M_0 such that for $M \geq M_0$,

$$\left|h_i^{(N)}(t) - h_i^{(M)}(t)\right| \leq \frac{1}{4} L_i \sum_{j=M}^{N} 2^{-j} < \varepsilon. \tag{9}$$

This proves that $\left\{h_i^{(N)}(t)\right\}_{N=1,2,\dots}$ is a Cauchy sequence for any $t \in I$. Hence, there exists the pointwise limit

$$h_i(t) = \lim_{N \to \infty} h_i^{(N)}(t). \tag{10}$$

Taking limits in inequality (9) when $N \to \infty$, one has

$$\left|h_i(t) - h_i^{(M)}(t)\right| \leq \frac{1}{4} L_i \sum_{j=M}^{\infty} 2^{-j} < \varepsilon \text{ for all } M \geq M_0 \tag{11}$$

and , certainly, then, the limit (10) is also uniform.

Now, it remains to show that the $h_i(t)$ are continuous. Indeed, given $\varepsilon > 0$, let $M > 1$ be such that $L_i 2^{-M+1} < \varepsilon$. If t_0 is a fixed point of I, there exists some p for which $t_0 \in I_p^{(M)}$. Choose a number δ so that $0 < \delta < \min\left\{\mu(C_p^{(M)}) : 1 \leq p \leq 2^{Mn}\right\}$.

Clearly, then, for t such that $|t - t_0| < \delta$ one has that either $t \in I_p^{(M)}$ or $t \in I_{p-1}^{(M)}$ or $t \in I_{p+1}^{(M)}$. Anyway, from (11) and (7) we have

$$\begin{aligned} |h_i(t) - h_i(t_0)| &\leq \left|h_i(t) - h_i^{(M)}(t)\right| + \left|h_i^{(M)}(t) - h_i^{(M)}(t_0)\right| + \left|h_i^{(M)}(t_0) - h_i(t_0)\right| \\ &\leq L_i 2^{-M-1} + L_i 2^{-M} + L_i 2^{-M-1} \\ &= L_i 2^{-M+1} < \varepsilon. \end{aligned}$$

This shows the continuity of h_i for all $i = 1, \dots, n$.

Finally, let $\{A_i : i = 1, \dots, n\}$ be an arbitrary open sets of $\mathbb{R}$ such that the condition

$$\Lambda_1\left[h_1^{-1}(A_1)\right] \times \dots \times \Lambda_1\left[h_n^{-1}(A_n)\right] > 0$$

holds. Evidently, then, there exists a closed cube C in H with $int(C) \neq \emptyset$, such that $C \subset A = \sqcap_{i=1}^{n} A_i$. Given C, determine a cube $C_p^{(M)}$ of a certain partition P_M so that $C_p^{(M)} \subset C$. Denoting by h the function defined by $(h_1, \dots, h_n)$, we are going to prove that $I_p^{(M)}$ (the corresponding interval to the cube $C_p^{(M)}$) verifies

$$I_p^{(M)} \subset h^{-1}(C). \tag{12}$$

Indeed, let t be a point of $I_p^{(M)}$. From (7), the function $h^{(M)}$, defined as $\left(h_1^{(M)}, \dots, h_n^{(M)}\right)$, satisfies

$$h^{(M)}(t) = P_p^{(M)}. \tag{13}$$

According to the above partitions, there exists a cube $C_{p_1}^{(M+1)} \subset C_p^{(M)}$ such that $t \in I_{p_1}^{(M+1)}$ and then,

$$h^{(M+1)}(t) = \left(h_1^{(M+1)}(t), h_2^{(M+1)}(t), \dots, h_n^{(M+1)}(t)\right) = P_{p_1}^{(M+1)}.$$

In this way, we can inductively determine a sequence of cubes

$$\ldots \subset C_{p_N}^{(M+N)} \subset \ldots \subset C_{p_1}^{(M+1)} \subset C_p^{(M)} \subset C$$

and a sequence $\{h^{(M+N)}(t) : N = 1, \ldots\}$ of points of $\mathbb{R}^n$. Now, taking the limit, we have

$$\lim_{N \to \infty} h^{(M+N)}(t) = h(t) = \lim_{N \to \infty} P_{p_N}^{(M+N)} = P \in C.$$

Therefore $t \in h^{-1}(C)$ and so (12) is showed. Consequently, we have

$$\Lambda_1 \left[\cap_{i=1}^{n} h_i^{-1}(A_i)\right] = \Lambda_1 \left[h^{-1}(A)\right] \geq \Lambda_1(I_p^{(M)}). \tag{14}$$

By using (6), $\Lambda_1(I_p^{(M)}) = \mu(C_p^{(M)})$. Because of the assumption on the measure μ, $\mu(C_p^{(M)}) > 0$. Hence, from (14), $\Lambda_1 \left[\cap_{i=1}^{n} h_i^{-1}(A_i)\right] > 0$ and the proof of the theorem ends. ■

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References

[1] Garsia, A.M. (1976). Combinatorial inequalities and smoothness of functions, *Bull. Amer. Math. Soc.*, **82**, 157–170.

[2] Halmos, P. (1974). *Measure Theory*, Springer-Verlag , New York.

[3] Hilbert, D. (1891). Über die stetige Abbildung einer Linie auf ein Flächenstück, *Math. Annln.*, **38**, 459–460.

[4] Holbrook, J. A. R. (1981). Stochastic independence and space-filling curves, *Amer. Math. Monthly*, **88**, 426–432.

[5] Lebesgue, H. (1904). *Leçons sur l'Intégration et la Recherche des Fonctions Primitives*, Gauthier-Villars, Paris.

[6] Lebesgue, H. (1905). Sur les fonctions représentables analytiquement, *J. de Math.*, **6** (1), 139–216.

[7] Moore, E.H. (1900). On certain crinkly curves, *Trans. Amer. Math. Soc.*, **1**, 72–90.

[8] Mora, G. and Cherruault, Y. (1997). Characterization and Generation of α-dense Curves, *Computers Math. Applic.*, **9**, 83–91.

[9] Peano, G. (1890). Sur une courbe qui remplit toute une aire plaine, *Math. Ann.*, **36**, 157–160.

[10] Sagan, H. (1994). *Space-filling Curves*, Springer-Verlag, New York.

[11] Steinhaus, H. (1936). La courbe de Peano et les Fonctions Indépendantes , C. R. Acad. Sci. Paris **202**, 1961–1963.

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