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Partial unconditionality of weakly null sequences

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Abstract. We survey a combinatorial framework for studying subsequences of a given sequence in a Banach space, with particular emphasis on weakly-null sequences. We base our presentation on the crucial notion of *barrier* introduced long time ago by Nash-Williams. In fact, one of the purposes of this survey is to isolate the importance of studying mappings defined on barriers as a crucial step towards solving a given problem that involves sequences in Banach spaces. We focus our study on various forms of "partial unconditionality" present in arbitrary weakly-null sequences in Banach spaces. We give a general notion of partial unconditionality that covers most of the known cases such as, for example, Elton's near unconditionality, convex unconditionality, and Schreier unconditionality, but we also add some new cases.

Incondicionalidad parcial de sucesiones débilmente nulas.

Resumen. Presentamos un marco combinatorio para estudiar subsucesiones de una sucesión dada en un espacio de Banach, con particular énfasis sobre las sucesiones que son débilmente nulas. Nos centramos principalmente en varias formas de incondicionalidad, y para ello introducimos una noción abstracta de incondicionalidad parcial, que cubre la mayoría de los tipos de incondicionalidad parcial conocidos.

También desarrollamos un marco combinatorio apropiado para el estudio de las subsucesiones, que trata sobre familias de conjuntos finitos de números naturales. En dicho marco la noción de *barrera* introducida por Nash-Williams es principal.

Contents.

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1. Introduction

The purpose of this paper is to survey with complete proofs a general framework for studying the structure of subsequences of a given infinite sequence in a real Banach space X using combinatorics of finite sets of integers. We focus mainly on weakly-null sequences. The main combinatorial tool is the notion of barrier so let us introduce this notion. Let FIN denote the family of finite subsets of $\mathbb{N}$. A family $\mathcal{B} \subseteq \mathrm{FIN}$ is a *barrier* on an infinite set $M \subseteq \mathbb{N}$ if no two distinct elements of $\mathcal{B}$ are comparable under the inclusion and if every infinite subset of M has an initial part on $\mathcal{B}$. Observe that the two properties that define a barrier $\mathcal{B}$ allows to define a canonical mapping $\iota_{\mathcal{B}}$ that assigns to every infinite subset N of M the unique initial part $\iota_{\mathcal{B}}(N)$ of N that belongs to $\mathcal{B}$. Note that for positive integer k, the family $\mathbb{N}^{[k]}$ of subsets of $\mathbb{N}$ of cardinality k is an example of a barrier on $\mathbb{N}$. By Ramsey's original theorem all these barriers $\mathbb{N}^{[k]}$ have something one can call the *Ramsey property*, i.e., the property that for every finite coloring of $\mathcal{B}$ there is an infinite subset M of $\mathbb{N}$ such that the *restriction*

$$\mathcal{B} \upharpoonright M = \{s \in \mathcal{B} : s \subseteq M\}$$

is monochromatic. Indeed this is one of the key properties of *every* barrier.

Let us now only indicate on how this notion may help us to understand weakly-null, or even an arbitrary, sequences in Banach spaces. Subsequences of such sequences are usually indexed by members of the set $\mathbb{N}^{[\infty]}$ of all infinite subsets of $\mathbb{N}$, so that our problem becomes a problem about Borel maps of the form $F : \mathbb{N}^{[\infty]} \to X$ for some metric space X. In fact, most of the maps F will have a countable range in X. We are going to show (see Subsection 3.5.) that in this case there is a barrier $\mathcal{B}$ and a mapping $f : \mathcal{B} \to X$ such that $F = f \circ \iota_{\mathcal{B}}$.

Recall that a sequence (x_n) in a Banach space X is called C-unconditional iff for every finite sequence of scalars (a_n) and every finite set s it happens that $\|\sum_{n\in s} a_n x_n\|_X \le C\|\sum_n a_n x_n\|_X$. So, if no subsequence of (x_n) is C-unconditional, then this means that for every infinite set M there is a finite sequence of scalars $(a_n)_{n\in s}$, supported in $s \subseteq M$ and a subset t of s such that

$$\|\sum_{n\in t} a_n x_n\|_X > C\|\sum_{n\in s} a_n x_n\|_X.$$

By continuity of the norm, we may assume that the scalars are rational numbers, so we can naturally define a mapping with countable range that assigns to each infinite set M the corresponding couple $(s, (a_n)_{n\in s})$. By the previous fact, we have a corresponding mapping defined on a barrier.

Our primary focus will be on the study of "partial unconditionality" present in arbitrary weakly-null sequences of Banach spaces. Our method would reduce the partial unconditionality problem to the understanding of mappings of the form

$$\varphi : \mathcal{B} \to \mathrm{FIN} \times c_0,$$

where $\mathcal{B} \subseteq \mathrm{FIN}$ is a barrier, and c_0 is the Banach space of sequences of real numbers converging to zero. We present several combinatorial results concerning these mappings, starting with simpler ones that would deal with mappings of the form

$$\varphi : \mathcal{B} \to \mathrm{FIN}.$$

One of the main results here is that every mapping $\varphi : \mathcal{B} \to c_0$ has a restriction which is, up to perturbation, something that we call a *L-mapping*, which loosely speaking says that φ has a kind of Lipschitz property with the additional requirement that the support $\mathrm{supp}\varphi(s)$ of $\varphi(s)$ is included in s for every $s \in \mathcal{B}$. To every such L-mapping one can associate a natural weakly-null sequence that we call *L-sequence*. Our approach shows that if for some notion of partial unconditionality $\mathfrak{F}$ there is a weakly-null sequence with no $\mathfrak{F}$-unconditional subsequence, then there must be a L-sequence with no $\mathfrak{F}$-unconditional subsequence. This gives the desired reduction of studying of the unconditionality problem $\mathfrak{F}$ to study of mappings of the form $\varphi : \mathcal{B} \to \mathrm{FIN} \times c_0$. One of the advantage of this reduction is that one often can manage saying something about these mapping and this is primarily based on the fact that barriers $\mathcal{B}$ have already rich theory on which we can rely.

With the goal to cover a large number of cases of partial unconditionality we introduce an abstract notion of unconditionality. It is motivated by a similar though slightly less general notion appearing in [10] pp. 4. Let $\mathfrak{F}$ be a set of pairs $(t, (a_n))$, where t is a finite set of integers and (a_n) a finite sequence of real numbers. Let $w : \mathfrak{F} \to \mathbb{R}^+$ be an arbitrary mapping that we call *weight assignment*. We say that a sequence (x_n) in a Banach space X is $(\mathfrak{F}_w, C)$-unconditional iff for every couple $(t, (a_n))$ in $\mathfrak{F}$ we have that

$$\|\sum_{n\in t} a_n x_n\|_X \le Cw(t, (a_n))\|\sum_n a_n x_n\|_X.$$

Let

$$\boldsymbol{C}(\mathfrak{F}_w, (x_n)) = \inf\{C \,:\, (x_n) \text{ has a } (\mathfrak{F}_w, C)\text{-unconditional subsequence}\}.$$

We illustrate this definition with two old examples and one new:
(a) (*Bessaga-Pełczyński unconditionality*) Let

$$\mathfrak{F} = \{(t, (a_n)) \,:\, t \text{ is an initial part of the support of } (a_n)\},$$

$w \equiv 1$. Then a sequence (x_n) is $(\mathfrak{F}_w, C)$-unconditional iff (x_n) is a C-basic sequence. A classical result of C. Bessaga and A. Pełczyński [7] states that for every $\varepsilon > 0$ every semi-normalized weakly-null sequence has a $1 + \varepsilon$-basic subsequence, i.e. $\boldsymbol{C}(\mathfrak{F}, (x_n)) = 1$.

(b) (*Elton unconditionality*) For $0 < \delta < 1$, let

$$\mathfrak{F}^\delta = \{(t, (a_n)) \,:\, \|(a_n)\|_\infty \le 1, \text{ and } |a_n| \ge \delta \text{ for every } n \text{ such that } a_n \ne 0\},$$

$w \equiv 1$. Then a sequence (x_n) is $(\mathfrak{F}^\delta_w, C)$-unconditional for some C iff it is δ-nearly-unconditional in the sense of Elton [12]. A well-known result of Elton [12] says that every semi-normalized weakly-null sequence contains a δ-nearly-unconditional subsequence. Moreover Dilworth, Kalton and Kutzarova [8] has shown that

$$\boldsymbol{C}(\mathfrak{F}^\delta_w, (x_n)_n) \le K \log_2(1/\delta)$$

for every semi-normalized weakly-null sequence.

(c) ($\mathfrak{F}$-*unconditionality*) In subsection 4.3. we present the following generalization. Let

$$\mathfrak{F} = \{(t, (a_n)) \,:\, t \text{ is a subset of the support of } (a_n)\},$$

and let us consider the following weight assignment on this family

$$w((t, (a_n))) = \max\{1, \log_2(\frac{\|(a_n)\|_\infty}{\min_{n\in t}|a_n|})\}.$$

Then

$$\boldsymbol{C}(\mathfrak{F}_w, (x_n)) \le 8$$

for every semi-normalized weakly-null sequence (x_n).

We shall show that the problem of estimating the constants $\boldsymbol{C}(\mathfrak{F}_w,(x_n))$ is closely related to the structure theory of mappings defined on barriers (see Subsection 4.1.). Moreover, for semi-normalized weakly-null sequences (x_n) these constants $\boldsymbol{C}(\mathfrak{F}_w,(x_n))$ are always dominated by the corresponding constant $\boldsymbol{C}(\mathfrak{F}_w,(y_n))$ of a *L-sequence*, that is, a semi-normalized weakly null sequence (y_n) defined by a *Lipschitz* mapping on a barrier $\varphi:\mathcal{B}\to c_{00}$, i.e. such that

$$\varphi(s)\restriction t=\varphi(u)\restriction t,\ \text{for every } s,u\in\mathcal{B} \text{ such that } t \text{ is an initial part of } s \text{ and } u.$$

As explained above, properties of weakly null sequence translate to combinatorial and topological properties of families $\mathcal{F}\subseteq$ FIN. It turns out that quite analogous theory can be developed for bounded weakly-Cauchy sequences in Banach spaces. To capture the bounded weakly-Cauchy sequences all one needs to do is to replace FIN by the family FIN_2 of finite block-sets of doubletons from $\mathbb{N}$. Then the result that characterizes when a given sequence contains a weakly null subsequence in terms of subfamilies of FIN translates into a similar characterization for bounded weakly-Cauchy sequences in terms of subfamilies of FIN_2. We shall exemplify this with a proof of the well-known Rosenthal ℓ_1-theorem.

The paper is organized as follows. In Section 2, we present the basic combinatorial and topological notions that apply to families of finite sets and that will be useful in the rest of the paper. In particular we introduce the notion of barrier, and prove that it has the Ramsey property. In Section 3 we develop further the results of Section 2 and give some application. In particular, in Theorem 2, we present a dichotomy for families of finite sets, and prove that this dichotomy is closely related to a problem concerning a particular sort of weakly-null sequences (Proposition 3). In the following subsection 3.2. we show that a variation of Theorem 2 leads us to the famous Rosenthal's ℓ_1-theorem. In subsection 3.3. we deal with "matching properties" of members of a given barrier that will be used later in our study of weakly-null sequences. In Subsection 3.4. we introduce the main technical notions of L-mappings and U-mappings defined on barriers. We finish this section by giving some consequences to maps defined on infinite-dimensional combinatorial cubes. In the fourth Section we introduce our notion of partial unconditionality and we use some of the combinatorial results from Section 3 and give proofs of several partial unconditionality results, some of them new and some old such as, for example, near and convex unconditionality, Schreier unconditionality, and the Maurey-Rosenthal unconditionality.

We finish the introduction by saying that this paper is largely a very selective survey article inspired by a vast variety of papers on this subject found in the literature and in particular by the paper of S. J. Dilworth, E. Odell, Th. Schlumprecht and A. Zsak [10]. It also can be considered as a natural continuation of our previous paper [21]. We should note however that all uncredited results except for few trivial observations due to the authors are either part of the folklore of the subject or are to be found in the papers listed in the reference list.

1.1. Preliminaries

Let us now explain some of the notation and well-known facts that will be useful for us. We use the boldface notation for sequences of objects. For example, letters $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\dots$ are reserved for sequence of vectors from c_0, while $\boldsymbol{x},\boldsymbol{y},\dots$ are reserved for an infinite sequence of vectors in an arbitrary Banach space. We use $\boldsymbol{e}=(e_n)$ to denote the standard Hamel basis of $c_{00}:e_n(k)=1$ if $k=n$ and 0, otherwise. By default, unless otherwise stated, every infinite sequence $\boldsymbol{x}$ of elements of some Banach space will be indexed in $\mathbb{N}$. In general, an infinite sequence $\boldsymbol{x}=(x_n)_{n\in M}$ may be indexed not only by $\mathbb{N}$ but by in an arbitrary infinite subset M of $\mathbb{N}$. We interpret a sequence $\boldsymbol{x}=(x_n)_{n\in M}$ in a Banach space X as a mapping from M into X, so for $N\subseteq M$ we denote by $\boldsymbol{x}\restriction N$ the subsequence $(x_n)_{n\in N}$ of $\boldsymbol{x}$. By default every sequence $\boldsymbol{x}$ will be indexed in $\mathbb{N}$. Given a sequence $\boldsymbol{x}=(x_n)_{n\in M}$ of elements of some Banach space X and a sequence $\boldsymbol{a}=(a_n)_{n\in\mathbb{N}}\in c_0$, we define the product

$$\boldsymbol{a}\cdot\boldsymbol{x}=\sum_{n\in M}a_nx_n$$

whenever the series converges in X.

A sequence $\boldsymbol{x} = (x_n)$ is *semi-normalized* if

$$0 < \inf_n \|x_n\| \leq \sup_n \|x_n\| \leq 1.$$

Recall that a sequence $\boldsymbol{x}$ in a given a Banach space X is called *weakly-null* if the numerical sequence $x^*(\boldsymbol{x}) = (x^*(x_n))_n$ belongs to c_0 for every $x^* \in X^*$. Recall also that a subset K of c_0 is called *weakly-compact* iff K iff it is compact with respect to the weak-topology, and that in this case is the pointwise topology of $c_0 \subseteq \mathbb{R}^{\mathbb{N}}$. We call a subset $W \subseteq c_0$ *weakly-pre-compact*, iff its weak-closure is a subset of c_0. This is equivalent to say that every sequence in W has a convergent subsequence whose limit belongs to c_0.

Recall that FIN denotes the family of all finite sets of $\mathbb{N}$. The topology on FIN is the one induced from the Cantor cube $2^{\mathbb{N}}$ via the identification of subsets of $\mathbb{N}$ with their characteristics function. Observe that this topology coincides with the one induced by c_0 with the same identification of finite sets and corresponding characteristic functions. Thus, we say that a family $\mathcal{F} \subseteq \mathrm{FIN}$ is *compact* if it is a compact space under the induced topology. We say that $\mathcal{F} \subseteq \mathrm{FIN}$ is *pre-compact* if its topological closure $\overline{\mathcal{F}}^{\mathrm{top}}$ taken in the Cantor cube $2^{\mathbb{N}}$ consists only of finite subsets of $\mathbb{N}$. For example, given an infinite set M of integers, the family $M^{[\leq k]}$ of subsets of M whose cardinality is at most k is a compact family for every fixed integer k, while the family $M^{[<\infty]}$ of all finite subsets of M is not pre-compact, since for example M is an accumulation point.

A simple, but useful observation is that if $\boldsymbol{x}$ is a given weakly-null sequence, then the natural mapping from B_{X^*} to c_0 defined by

$$x^* \mapsto x^*(\boldsymbol{x}) = \sum_{n \in M} x^*(x_n) e_n.$$

is continuous, provided we equip B_{X^*} with the weak*-topology and c_0 with its weak topology. It follows that its range $K(\boldsymbol{x})$ is a weakly-compact subset of c_0.

Given $\boldsymbol{a} \in c_0$, and $\varepsilon > 0$, we define the ε-support of $\boldsymbol{a}$ as

$$\operatorname{supp}_\varepsilon \boldsymbol{a} = \{n \in \operatorname{supp} \boldsymbol{a} \,:\, |\boldsymbol{a}(n)| \geq \varepsilon\}.$$

Note that $\operatorname{supp}_\varepsilon \boldsymbol{a} \in \mathrm{FIN}$ for every $\boldsymbol{a}$. For a set $W \subseteq c_0$, we define

$$\operatorname{supp}_\varepsilon(W) = \{\operatorname{supp}_\varepsilon \boldsymbol{a} \,:\, \boldsymbol{a} \in W\}.$$

Note that if $W \subseteq c_0$ is weakly-pre-compact then $\operatorname{supp}_\varepsilon(W)$ is a pre-compact subset of FIN. To see this consider a sequence $(s_n)_n$ in $\operatorname{supp}_\varepsilon(W)$ with limit $A \subseteq \mathbb{N}$. We need to show that the set A is finite. Pick for each n an element $\boldsymbol{a}_n \in W$ such that $\operatorname{supp}_\varepsilon \boldsymbol{a}_n = s_n$. As W is weakly-pre-compact, we can find a convergent subsequence of $(\boldsymbol{a}_n)$ with limit $\boldsymbol{a} \in c_0$. It is easy to see that then $A \subseteq \operatorname{supp}_\varepsilon \boldsymbol{a}$, so A is finite.

Recall that two basic sequences $\boldsymbol{x} = (x_n)_{n \in N}$ and $\boldsymbol{y} = (y_n)_{n \in N}$ of Banach spaces X and Y respectively are called C-equivalent ($C \geq 1$) if for every sequence $(a_n)_{n \in N}$ of scalars we have that

$$\frac{1}{C} \| \sum_{n \in N} a_n x_n \|_X \leq \| \sum_{n \in M} a_{\pi(n)} x_{\pi(n)} \|_X \leq C \| \sum_{n \in N} a_n x_n \|_X.$$

where $\pi : N \to M$ is the unique order-preserving onto mapping.

2. Families of finite sets of integers

That families of finite sets of integers are relevant to any study of sequences in Banach spaces is a well understood fact, and it is therefore not surprising that the study of such families is a predominant theme in the literature of this subject. As we shall see later the study of *mappings* defined on certain families of finite sets of integers seems to be a theme that is much more to the point and that is not always explicit in papers of this area. One of the purpose of this survey is to give a more complete exposition of this theme.

Given $X, Y \subseteq \mathbb{N}$ we write

(1) $X < Y$ iff $\max X < \min Y$. We will use the convention $\emptyset < X$ and $X < \emptyset$ for every X.

(2) $X \sqsubseteq Y$ iff $X \subseteq Y$ and $X < Y \setminus X$.

A sequence (s_n) of finite sets of integers is called a *block sequence* iff $s_n < s_m$ for every $n < m$, and it is called a Δ-*sequence* iff there is some finite set s such that $s \sqsubseteq s_n$ $(n \in \mathbb{N})$ and $(s_n \setminus s)$ is a block sequence. The set s is called the *root* of (s_n). Note that $s_n \to_n s$ iff every subsequence of (s_n) has a further Δ-subsequence with root s. It follows that the topological closure $\overline{\mathcal{F}}$ of a pre-compact family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ is included in its downwards closure

$$\overline{\mathcal{F}}^{\subseteq} = \{s \subseteq t \,:\, t \in \mathcal{F}\}$$

with respect to the inclusion relation and also included in its downwards closure

$$\overline{\mathcal{F}}^{\sqsubseteq} = \{s \sqsubseteq t \,:\, t \in \mathcal{F}\}$$

with respect to the relation $\sqsubseteq$. We say that a family $\mathcal{F} \subseteq$ FIN is $\subseteq$-*hereditary* if $\mathcal{F} = \overline{\mathcal{F}}^{\subseteq}$ and $\sqsubseteq$-*hereditary* if $\mathcal{F} = \overline{\mathcal{F}}^{\sqsubseteq}$. The $\subseteq$-hereditary families will simply be called *hereditary families*. We shall consider the following two restrictions of a given family $\mathcal{F}$ of subsets of $\mathbb{N}$ to a finite or infinite subset X of $\mathbb{N}$

$$\begin{aligned}\mathcal{F} \upharpoonright X =& \{s \in \mathcal{F} \,:\, s \subseteq X\},\\ \mathcal{F}[X] =& \{s \cap X \,:\, s \in \mathcal{F}\}.\end{aligned}$$

The first family $\mathcal{F} \upharpoonright M$ is called the *restriction* of $\mathcal{F}$ on M, while the second one is called the *trace* of $\mathcal{F}$ on M.

There are various ways to associate an ordinal index to a pre-compact family $\mathcal{F}$ of finite subsets of $\mathbb{N}$. For example, one may consider the *Cantor-Bendixson index* $r(\mathcal{F})$, the minimal ordinal α for which the iterated *Cantor-Bendixson derivative* $\partial^\alpha(\mathcal{F})$ is equal to $\emptyset$. Recall that $\partial\mathcal{F}$ is the set of all proper accumulation points of $\mathcal{F}$ and that

$$\partial^\alpha(\mathcal{F}) = \bigcap_{\xi<\alpha} \partial(\partial^\xi(\mathcal{F})).$$

The Cantor-Bendixson index is well defined since $\overline{\mathcal{F}}$ is countable and therefore a scattered compactum so the sequence $\partial^\xi(\mathcal{F})$ of iterated derivatives must vanish. Observe that if $\mathcal{F}$ is a nonempty compact, then necessarily $r(\mathcal{F})$ is a successor ordinal. An important feature of this ordinal index and all other considered in this paper is that for every $n \in \mathbb{N}$, the index of the family

$$\mathcal{F}_{\{n\}} = \{s \in \text{FIN} \,:\, n < s,\ \{n\} \cup s \in \mathcal{F}\}$$

is strictly smaller than the index of $\mathcal{F}$ whenever the last nonempty set of the form $\partial^\xi(\mathcal{F})$ is equal to $\{\emptyset\}$.

Let us now introduce the other basic combinatorial concepts of this section. For this we need the following piece of notation, where X and Y are subsets of $\mathbb{N}$

$${}_*X = X \setminus \{\min X\} \text{ and } X/Y = \{m \in X \,:\, \max Y < m\}.$$

The set ${}_*X$ is called the *shift* of X. Given integer $n \in \mathbb{N}$, we write X/n to denote $X/\{n\} = \{m \in X \,:\, m > n\}$. The following notions have been introduced by Nash-Williams [25].

Definition 1 *Let $\mathcal{F} \subseteq$ FIN.*

(1) $\mathcal{F}$ is called thin *if $s \not\sqsubseteq t$ for every pair s, t of distinct members of $\mathcal{F}$.*

(2) $\mathcal{F}$ is called Sperner *if $s \not\subseteq t$ for every pair $s \neq t \in \mathcal{F}$.*

(3) $\mathcal{F}$ is called Ramsey *if for every finite partition*

$$\mathcal{F} = \mathcal{F}_0 \cup \cdots \cup \mathcal{F}_k \tag{1}$$

there is an infinite set $M \subseteq \mathbb{N}$ such that at most one of the restrictions $\mathcal{F}_i \upharpoonright M$ is non-empty.
(4) $\mathcal{F}$ is called a front *on M if $\mathcal{F} \subseteq \mathcal{P}(M)$, it is thin, and for every infinite $N \subseteq M$ there is some $s \in \mathcal{F}$ such that $s \sqsubseteq N$.*
(5) $\mathcal{F}$ is called a barrier *on M if $\mathcal{F} \subseteq \mathcal{P}(M)$, it is Sperner, and for every infinite $N \subseteq M$ there is some $s \in \mathcal{F}$ such that $s \sqsubseteq N$.*

Definition 2 *Given $\mathcal{F} \subseteq \mathrm{FIN}$, let*

$$\mathcal{F}^{\sqsubseteq-\max} = \{s \in \mathcal{F} \,:\, (\forall t \in \mathcal{F})(s \sqsubseteq t \to s = t)\}$$
$$\mathcal{F}^{\sqsubseteq-\min} = \{s \in \mathcal{F} \,:\, (\forall t \in \mathcal{F})(t \sqsubseteq s \to s = t)\}.$$

It is clear that both sets are thin.

Definition 3 *Given a front $\mathcal{B}$ on M, let $\iota_{\mathcal{B}} : M^{[\infty]} \to \mathcal{B}$ be the mapping that assigns to every $N \in M^{[\infty]}$ the unique initial part $\iota_{\mathcal{B}}(N)$ of N that belongs to $\mathcal{B}$.*

Clearly, every barrier is a front but not vice-versa. For example, the family $\mathbb{N}^{[k]}$ of all k-element subsets of $\mathbb{N}$ is a barrier. The basic result of Nash-Williams [25] says that every front (and therefore every barrier) is Ramsey. Since as we will see soon there are many more barriers than those of the form $\mathbb{N}^{[k]}$ this is a far reaching generalization of the classical result of Ramsey. To see a typical application, let $\mathcal{F}$ be a front on some infinite set M and consider its partition $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$, where $\mathcal{F}_0$ is the family of all $\subseteq$-minimal elements of $\mathcal{F}$. Since $\mathcal{F}$ is Ramsey there is an infinite $N \subseteq M$ such that one of the restrictions $\mathcal{F}_i \upharpoonright M$ is empty. Note that $\mathcal{F}_1 \upharpoonright N$ must be empty. Since $\mathcal{F}_0 \upharpoonright N$ is clearly a Sperner family, it is a barrier on N. Thus we have shown that every front has a restriction that is a barrier.

Since barrier are more pleasant to work with one might wonder why introducing the notion of front at all. The reason is that inductive constructions lead more naturally to fronts rather than barriers. To get an idea about this, it is instructive to consider the following notion introduced by Pudlak and Rödl (see [26]).

Definition 4 *Given a countable ordinal α, the family $\mathcal{F}$ is called* α-uniform *on M provided that:*
(a) $\alpha = 0$ implies $\mathcal{F} = \{\emptyset\}$,
(b) $\alpha = \beta + 1$ implies that $\mathcal{F}_{\{n\}}$ is β-uniform on M/n,
(c) $\alpha > 0$ limit implies that there is an increasing sequence $\{\alpha_n\}_{n\in M}$ of ordinals converging to α such that $\mathcal{F}_{\{n\}}$ is α_n-uniform on M/n for all $n \in M$.
$\mathcal{F}$ is called uniform on M if it is α-uniform on M for some countable ordinal α.

Remark 1 *(a) If $\mathcal{F}$ is a front on M, then $\overline{\mathcal{F}} = \overline{\mathcal{F}}^{\sqsubseteq}$.*
(b) If $\mathcal{F}$ is uniform on M, then it is a front (though not necessarily a barrier) on M.
(c) If $\mathcal{F}$ is α-uniform (front, barrier) on M and $\Theta : M \to N$ is the unique order-preserving onto mapping between M and N, then $\Theta"\mathcal{F} = \{\Theta" s \,:\, s \in \mathcal{F}\}$ is α-uniform (front, barrier) on M.
(d) If $\mathcal{F}$ is α-uniform (front, barrier) on M then $\mathcal{F} \upharpoonright N$ is α-uniform (front, barrier) on N for every $N \subseteq M$.
(e) If $\mathcal{F}$ is uniform (front, barrier) on M, then for every $s \in \overline{\mathcal{F}}^{\sqsubseteq}$ the family

$$\mathcal{F}_s = \{t \,:\, s < t \text{ and } s \cup t \in \mathcal{F}\}$$

is uniform (front, barrier) on M/s.
(e) If $\mathcal{F}$ is α-uniform on M, then $\partial^\alpha(\overline{\mathcal{F}}) = \{\emptyset\}$, hence $r(\mathcal{F}) = \alpha + 1$. (Hint: use that $\partial^\beta(\mathcal{F}_{\{n\}}) = (\partial^\beta(\mathcal{F}))_{\{n\}}$ for every β and every compact family $\mathcal{F}$).

(f) An important example of a ω*-uniform barrier on* $\mathbb{N}$ *is the family* $\mathcal{S} = \{s : |s| = \min(s)+1\}$*. We call* $\mathcal{S}$ *a Schreier barrier since its downwards closure is commonly called the Schreier family. Note that unlike to the case of finite ranks there are many different* ω*-uniform families on* $\mathbb{N}$*. For example* $\{s : |s| = 2\min(s)+1\}$ *is another such family.*

The following is one of the most important results connecting arbitrary families of finite subsets of $\mathbb{N}$ with fronts and barriers. It is also the key result in the development of the topological Ramsey theory of $\mathbb{N}^{[\infty]}$ (See Subsection 3.5.).

Lemma 1 (Galvin's Lemma) *For every family* $\mathcal{F} \subseteq \mathrm{FIN}$ *there exists an infinite* $M \subseteq \mathbb{N}$ *such that the restriction* $\mathcal{F} \upharpoonright M$ *is either empty or it contains a barrier.* ■

The following result based on Galvin's lemma and Nash-Williams' extension of Ramsey's theorem explains the relationship between the concepts introduced above (see [5] for proofs and fuller discussion).

Theorem 1 *The following are equivalent for a family* $\mathcal{F}$ *of finite subsets of* $\mathbb{N}$*:*
(a) $\mathcal{F}$ *is Ramsey.*
(b) There is an infinite $M \subseteq \mathbb{N}$ *such that* $\mathcal{F} \upharpoonright M$ *is thin.*
(c) There is an infinite $M \subseteq \mathbb{N}$ *such that* $\mathcal{F} \upharpoonright M$ *is Sperner.*
(d) There is an infinite $M \subseteq \mathbb{N}$ *such that* $\mathcal{F} \upharpoonright M$ *is either empty or a front on* M*.*
(e) There is an infinite $M \subseteq \mathbb{N}$ *such that* $\mathcal{F} \upharpoonright M$ *is either empty or a barrier on* M*.*
(f) There is an infinite $M \subseteq \mathbb{N}$ *such that* $\mathcal{F} \upharpoonright M$ *is either empty or uniform on* M*.*
(g) There is an infinite $M \subseteq \mathbb{N}$ *such that for every infinite* $N \subseteq M$ *the restriction* $\mathcal{F} \upharpoonright N$ *cannot be split into two disjoint families that are uniform on* N*.* ■

In this kind of Ramsey theory one frequently performs diagonalisation arguments that can be formalized using the following notion.

Definition 5 *An infinite sequence* $(M_k)_{k\in\mathbb{N}}$ *of infinite subsets of* $\mathbb{N}$ *is called a* fusion sequence of subsets of $M \subseteq \mathbb{N}$ *if for all* $k \in \mathbb{N}$*:*
(a) $M_{k+1} \subseteq M_k \subseteq M$*,*
(b) $m_k < m_{k+1}$*, where* $m_k = \min M_k$*.*
The infinite set $M_\infty = \{m_k\}_{k\in\mathbb{N}}$ *is called the* fusion set (or limit) *of the sequence* $(M_k)_{k\in\mathbb{N}}$*.*

The following are simple fact to prove. We leave the details to the reader.

Proposition 1 *Let* $\mathcal{F} \subseteq \mathrm{FIN}$*.*
(a) $\mathcal{F}$ *is pre-compact iff* $\overline{\mathcal{F}}^{\sqsubseteq}$ *is pre-compact iff* $\overline{\mathcal{F}}$ *is pre-compact.*
(b) Suppose further that $\mathcal{F}$ *is either* $\subseteq$*-hereditary or* $\sqsubseteq$*-hereditary. Then* $\mathcal{F}$ *is compact iff it is pre-compact.*
(c) If $\mathcal{F}$ *is* $\subseteq$*-hereditary then for every subset* M *of* $\mathbb{N}$ *we have* $\mathcal{F}[M] = \mathcal{F} \upharpoonright M$*.*
(d) $\overline{\mathcal{F}}^{\subseteq}[M] = \overline{\mathcal{F}[M]}^{\subseteq}$*.* ■

Proposition 2 *Suppose that* $\mathcal{B} \subseteq \mathrm{FIN}$ *is a barrier on* M*. Then*
(a) $\overline{\mathcal{B}}^{\subseteq} = \overline{\mathcal{B}}^{\sqsubseteq} = \overline{\mathcal{B}}$*, and hence* $\overline{\mathcal{B}}^{\subseteq}$ *is a compact family.*
(b) For every $N \subseteq M$*,* $\overline{\mathcal{B} \upharpoonright N}^{\subseteq} = \overline{\mathcal{B}}^{\subseteq} \upharpoonright N$*.*
(c) For every $N \subseteq M$ *such that* $M \setminus N$ *is infinite we have that* $\mathcal{B}[N] = \overline{\mathcal{B} \upharpoonright N}$*, and in particular* $\mathcal{B}[N]$ *is downwards closed.*

PROOF. (a): It is clear that $\overline{\mathcal{B}}^{\subseteq} \supseteq \overline{\mathcal{B}}^{\sqsubseteq} \supseteq \overline{\mathcal{B}}$. Let us show that $\overline{\mathcal{B}}^{\subseteq} \subseteq \overline{\mathcal{B}}$: Let $t \subsetneq u \in \mathcal{B}$. For $N \subseteq M$, let $s_N = \iota_{\mathcal{B}}(t \cup (N/u))$. Then either $t \sqsubseteq s_M$ or else $s_M \sqsubseteq t$. This second alternative is impossible since it implies that $s_M \subsetneq u$, and both are in the Sperner family $\mathcal{B}$. Now it is easy to produce a Δ-sequence (s_n) of elements of $\mathcal{B}$ with root s.

(b): It is clear that $\overline{\mathcal{B} \upharpoonright N}^{\subseteq} \subseteq \overline{\mathcal{B}}^{\subseteq} \upharpoonright N$. Now suppose that $t \in \overline{\mathcal{B}}^{\subseteq} \upharpoonright N$. Let $u \in \mathcal{B}$ be such that $t \subseteq u$. Then $t \sqsubseteq \iota_{\mathcal{B}}(t \cup (N/t)) \in \mathcal{B} \upharpoonright N$ (otherwise $\iota_{\mathcal{B}}(t \cup (N/t)) \subsetneq u$ both in the Sperner family $\mathcal{B}$, impossible).

(c): Fix an infinite subset N of M such that $P = M \setminus N$ is infinite as well. By (a) we have to prove that $\mathcal{B}[N] = \overline{\mathcal{B} \upharpoonright N}^{\sqsubseteq}$. By (b) and Proposition 1 (c),

$$\mathcal{B}[N] \subseteq \overline{\mathcal{B}}^{\subseteq}[N] = \overline{\mathcal{B}}^{\subseteq} \upharpoonright N = \overline{\mathcal{B} \upharpoonright N}.$$

Now, let us show that $\overline{\mathcal{B} \upharpoonright N}^{\subseteq} \subseteq \mathcal{B}[N]$: Fix $t \in \overline{\mathcal{B} \upharpoonright N}^{\subseteq}$. One can argue as before to show that $t \sqsubseteq \iota_{\mathcal{B}}(t \cup (P/t))$, and so $t = \iota_{\mathcal{B}}(t \cup (P/t)) \cap N \in \mathcal{B}[N]$.

■

The next is a well known result. We extract its proof from [5].

Lemma 2 *Suppose that $\mathcal{B}$ and $\mathcal{C}$ are two barriers on M. Then there is some infinite $N \subseteq M$ such that either $\overline{\mathcal{B} \upharpoonright N} \subseteq \overline{\mathcal{C} \upharpoonright N}$ or else $\overline{\mathcal{C} \upharpoonright N} \subseteq \overline{\mathcal{B} \upharpoonright N}$.*

PROOF. Define $\varphi : \mathcal{B} \to \mathrm{FIN}$ by $\varphi(s) = \iota_{\mathcal{C}}(s \cup (M/s))$, i.e. $\varphi(s) \in \mathcal{C}$ is such that $\varphi(s) \sqsubseteq s \cup (M/s)$. By the Ramsey property of $\mathcal{B}$ there is some $N \subseteq M$ such that either
(a) $s \sqsubseteq \varphi(s)$ for every $s \in \mathcal{B} \upharpoonright N$, or else
(b) $\varphi(s) \sqsubseteq s$ for every $s \in \mathcal{B} \upharpoonright N$. Suppose that the first alternative holds.

We claim that in this case $\mathcal{B} \upharpoonright N \subseteq \overline{\mathcal{C} \upharpoonright N}$: Fix $s \in \mathcal{C} \upharpoonright N$, let $t = \iota_{\mathcal{C}}(s \cup (N/s))$. As $s \sqsubseteq \varphi(s)$ we have necessarily that $s \sqsubseteq t$ (because otherwise $t \sqsubset \varphi(s)$ both in $\mathcal{C}$, a contradiction). So, $s \in \overline{\mathcal{C} \upharpoonright N}$.

Finally, suppose that the second alternative (b) holds. We claim that in this case we have that $\mathcal{C} \upharpoonright N \subseteq \overline{\mathcal{B} \upharpoonright N}$: Let $t \in \mathcal{C} \upharpoonright N$, and let $s = \iota_{\mathcal{B}}(t \cup (N/t))$. As $\varphi(s) \sqsubseteq s$, we have that necessarily $t \sqsubseteq s$. So, $t \in \overline{\mathcal{B} \upharpoonright N}$. ■

Corollary 1 *Suppose that $\mathcal{B}$ and $\mathcal{C}$ are respectively α and β-uniform some M, and suppose that $\alpha < \beta$. Then there is $N \subseteq M$ such that $\mathcal{B} \upharpoonright N \subseteq \overline{\mathcal{C}}$.*

PROOF. It follows from Lemma 2 and Remark 1 (d). ■

3. Mapping on Barriers

As pointed out in the introduction, many problems about sequences in Banach spaces can be coded as problems about mappings defined on barriers. In this section we consider the simple particular case of such mappings with ranges included in FIN. We shall later see that even this case will lead us to some interesting results (see Theorem 2). We start with the following two natural definitions.

Definition 6 *Let $\mathcal{F} \subseteq \mathrm{FIN}$ and $\varphi : \mathcal{F} \to \mathrm{FIN}$.*
(a) We say that φ is uniform *iff*

$$\text{for every } s, u \in \mathcal{F} \text{ and every } t \sqsubseteq s, u \text{ we have that } \min(s \setminus t) \in \varphi(s) \Leftrightarrow \min(u \setminus t) \in \varphi(u). \tag{2}$$

(b) We say that φ is Lipschitz *iff*

$$\text{for every } s, u \in \mathcal{F}, \text{ if } t \sqsubseteq s, u \text{ then } \varphi(s) \cap t = \varphi(u) \cap t. \tag{3}$$

So, uniform mappings are those $\varphi : \mathcal{F} \to \mathrm{FIN}$ such that given $s \in \mathcal{F}$ and $n \in \mathbb{N}$, the value of $\chi_{\varphi(s)}(n) \in \{0,1\}$ only depends on the initial part $s \cap [0,n)$ of s, while Lipschitz mappings are those that the value of $\varphi(s) \cap t$ only depends on t for every $t \sqsubseteq s \in \mathcal{F}$. This notion of lipschitzness has the following metric interpretation.

Remark 2 *Recall that standard metric d on* FIN *defined by*

$$d(s,t) = \frac{1}{2^{\min(s \triangle t)}},$$

where $s\triangle t = (s \setminus t) \cup (t \setminus s)$ is the symmetric difference *of s and t. This metric defines the topology on* FIN *we explained in the introduction. With this metric it is easy to see that the Lipschitz notion defined above coincides with the metric 1-Lipschitz condition associated to d. In subsection 3.4. we will extend those notions to mappings from $\mathcal{F} \subseteq$* FIN *into c_0.*

Proposition 3 *If φ is uniform, then φ is Lipschitz.*

PROOF. The proof is an easy induction on $|t|$, where $t \sqsubseteq s, u \in \mathcal{F}$. ■

Proposition 4 *Suppose that $\mathcal{B}$ a barrier on M and $\varphi : \mathcal{B} \to$* FIN *is an arbitrary mapping. Then there is $N \subseteq M$ such that $\varphi \upharpoonright (\mathcal{B} \upharpoonright N)$ is uniform.*

PROOF. Find a fusion sequence (see definition 5) (M_k) of subsets of M, $m_k = \min M_k$, such that for every k, and every $t \in \{m_0, \dots, m_k\}$ the mapping

$$f_t : \mathcal{B}_t \upharpoonright M_{k+1} \to \{0,1\}$$

defined by $f_t(u) = \chi_{\varphi(t \cup u)}(\min u)$ is constant. Then the fusion limit $\{m_k\}$ is our set. ■

A consequence of this is that the selection of an initial part of every element of a barrier, defines essentially a new barrier. More precisely,

Corollary 2 *Suppose that $\mathcal{B}$ is a barrier on M and suppose that $\varphi : \mathcal{B} \to$* FIN *is such that $\varphi(s) \sqsubseteq s$ for every $s \in \mathcal{B}$. Then there is $N \subseteq M$ such that φ"$(\mathcal{B} \upharpoonright N)$ is a barrier on N.*

PROOF. Let $P \subseteq M$ be such that φ is uniform when restricted to $\mathcal{B} \upharpoonright P$. We claim that this implies that $\mathcal{F} = \varphi$"$(\mathcal{B} \upharpoonright P)$ is a thin family: Suppose that otherwise that $t \sqsubset \bar{t}$ both in $\mathcal{F}$. Let $s, \bar{s} \in \mathcal{B} \upharpoonright P$ be such that $\varphi(s) = t$, $\varphi(\bar{s}) = \bar{t}$. As φ is Lipschitz on $\mathcal{B} \upharpoonright P$ and $\min(\bar{s} \setminus t) \in \bar{t} = \varphi(\bar{s})$, we obtain that $\min(s \setminus t) \in \varphi(s) = t$, impossible. Now let $N \subseteq P$ be such that $\mathcal{F} \upharpoonright N$ is either empty or a uniform barrier on N. Note that the first alternative is impossible as φ"$(\mathcal{B} \upharpoonright N) \subseteq \mathcal{F} \upharpoonright N$. We finish the proof by checking that indeed $\mathcal{F} \upharpoonright N = \varphi$"$(\mathcal{B} \upharpoonright N)$: The reverse inclusion is trivial. Suppose that $t \in \mathcal{F} \upharpoonright N$, and fix $s \in \mathcal{B} \upharpoonright P$ such that $t = \varphi(s)$. Let $u = \iota_{\mathcal{B}}(t \cup (N/t)) \in \mathcal{B} \upharpoonright N$. By uniformity of φ on $\mathcal{B} \upharpoonright P$ we have that $\varphi(u) = t$, and we are done. ■

Definition 7 *We say that a mapping $\varphi : \mathcal{F} \to$* FIN *is* internal *iff $\varphi(s) \subseteq s$ for every $s \in \mathcal{B}$.*

We prove now that every mapping $\varphi : \mathcal{B} \to \mathrm{FIN}$ defined on a barrier whose range is pre-compact is "almost" internal. In the next, given a set $B \subseteq \mathbb{N}$, we define the mapping $\chi_B \cdot : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ by $\chi_B \cdot A = A \cap B$.

Lemma 3 *Let $\mathcal{B}$ be a uniform barrier on M, and suppose that $\varphi : \mathcal{B} \to$* FIN *is such that its range is a pre-compact family. Then there is some infinite subset $N \subseteq M$ such that $\chi_N \cdot \varphi \upharpoonright (\mathcal{B} \upharpoonright N)$ is internal (i.e. $\varphi(s) \cap N \subseteq s$ for every $s \in \mathcal{B} \upharpoonright N$).*

PROOF. Let $h : \mathcal{B} \to \mathrm{FIN}$ be defined by $h(s) = \varphi(s) \setminus s$ ($s \in \mathcal{B}$). It is clear that $h"\mathcal{B}$ is a pre-compact family, and, by definition, $h(s) \cap s = \emptyset$. We are going to show that there is some $N \subseteq M$ such that $h(s) \cap N = \emptyset$ for every $s \in \mathcal{B} \upharpoonright N$, that gives the desired conclusion for f. The proof is by induction on the rank of $\mathcal{B}$. For every $m \in M$, let $h_m : \mathcal{B}_{\{m\}} \to \mathrm{FIN}$ be naturally defined by

$$h_m(s) = h(\{m\} \cup s) \text{ for every } s \in \mathcal{B}_{\{m\}}.$$

It is clear that $h_m : \mathcal{B}_{\{m\}} \to \mathrm{FIN}$ fulfills (a) and (b) above, so, by inductive hypothesis, we can find a fusion sequence $(M_k)_{k\in\mathbb{N}}$, $M_k = M$, and such that, setting $m_k = \min M_k$ ($k \in \mathbb{N}$), we have that

$$h_{m_k}(s_k) \cap M_{k+1} = \emptyset \text{ for every } k \in \mathbb{N} \text{ and every } s \in \mathcal{B}_{\{m_k\}} \upharpoonright M_{k+1}.$$

Let $M_\infty = \{m_k\}$. It is easy to check that

$$h(s) \cap M_\infty \subseteq \{m_0, \dots m_{k-1}\} \text{ for every } s \in \mathcal{B} \upharpoonright M_\infty,$$

and where k is such that $m_k = \min s$. For $m \in M_\infty$, we define

$$\begin{aligned} g_m : \mathcal{B}_{\{m\}} \upharpoonright M_\infty &\to \mathcal{P}(M \cap \{0, \dots, m-1\}) \\ s &\mapsto g_m(s) = h_m(s) \cap M_\infty. \end{aligned}$$

Since the image of g_m has only finitely many possibilities, we can find another fusion sequence (N_k), $N_0 = M_\infty$, such that, setting $n_k = \min N_k$, for every k the mapping g_{n_k} is constant on $\mathcal{B}_{\{n_k\}} \upharpoonright N_{k+1}$ with value $s_{n_k} < n_k$. Let $N_\infty = \{n_k\}$. Notice that, by the properties of this last fusion sequence, we know that $h(s) \cap N_\infty \subseteq h(s) \cap M_\infty = s_{\min s}$ for every $s \in \mathcal{B} \upharpoonright N_\infty$. Since the range of h is a pre-compact family, there is some infinite set $I \subseteq N_\infty$ such that $(s_i)_{i\in I}$ is a Δ-sequence with root r. Take a thinner $N \subseteq I$ such that $N \cap \bigcup_{n\in N} s_n = \emptyset$. Then for every $s \in \mathcal{B} \upharpoonright N$ we have that $h(s) \cap N \subseteq s_{\min s} \cap N = \emptyset$, as desired. ∎

The next generalizes Lemma 3 and it will be very important in the understanding of mappings from barriers into c_0.

Lemma 4 *Suppose that $\{\mathcal{B}_l\}_{l\in\mathbb{N}}$ is a collection of uniform barriers on M, and suppose that for every $k \in \mathbb{N}$ we have $\varphi_l : \mathcal{B}_l \to \mathrm{FIN}$ with pre-compact range. Then there is some infinite subset N of M such that*

$$(\varphi_l(s) \setminus s) \cap N \subseteq N \cap [0, n] \tag{4}$$

for every $n \in N$, $l \leq n$, and every $s \in \mathcal{B}_l \upharpoonright N$.

PROOF. For each $l \in \mathbb{N}$, Let $\psi_l : \mathcal{B}_l \to \mathrm{FIN}$ be defined by $\psi_l(s) = \varphi_l(s) \setminus s$ for every $s \in \mathcal{B}_l$. Using previous Lemma 3 we can find a fusion sequence (N_k) of M such that, setting $n_k = \min N_k$ ($k \in \mathbb{N}$), we have that for every $k \in \mathbb{N}$,

$$\psi_l(s) \cap N_{k+1} = \emptyset \text{ for every } l \leq n_k \text{ and every } s \in \mathcal{B}_l \upharpoonright (\{n_0, \dots, n_k\} \cup N_{k+1}). \tag{5}$$

Then the fusion set $N = \{n_k\}_k$ fulfills the desired requirements: Fix $n \in N$, $l \in \mathbb{N}$ such that $l \leq n$, and $s \in \mathcal{B}_l \upharpoonright N$. Let $k \in \mathbb{N}$ be such that $n = n_k$. Observe that the fusion set N satisfies that $N \subseteq \{n_0, \dots, n_k\} \cup N_{k+1}$, so $s \in \mathcal{B}_l \upharpoonright \{n_0, \dots, n_k\} \cup N_{k+1}$. Hence, by (5), $\psi_l(s) \cap N_{k+1} = \emptyset$, so $\psi_l(s) \cap N \subseteq \{n_0, \dots, n_k\}$, that is equivalent to (4).

Let us explain how to find this fusion sequence. Suppose we have found $N_k \subseteq N_{k-1} \subseteq \cdots \subseteq N_0$. For every $t \subseteq \{n_0, \dots, n_k\}$, and $l \in \mathbb{N}$ let $\psi_{l,t} : (\mathcal{B}_l)_t \to \mathrm{FIN}$ be naturally defined by

$$\psi_{l,t}(u) = \psi_l(s \cup t)$$

for each $u \in (\mathcal{B}_k)_t$. Using repeatedly Lemma 3 to each $h_{l,t}$ with $l \leq n_k$ and $t \subseteq \{n_0, \dots, n_k\}$ we can find $N_{k+1} \subseteq N_k$ with the property that for every $s \in \mathcal{B}_k \upharpoonright (\{n_0, \dots, n_k\} \cup N_{k+1})$ and every $l \leq n_k$, the intersection $\psi_l(s) \cap M_{k+1} = \emptyset$, as desired. ∎

3.1. Reducing problems about families of finite sets to problems about barriers

As a first result of this section we show that an arbitrary family $\mathcal{F} \subseteq \mathrm{FIN}$ is related either to a uniform barrier on M, or the downwards closure of $\mathcal{F}$ includes a family $M^{[<\infty]}$ of all finite subsets of a given infinite set M. More precisely, we prove the following:

Theorem 2 *For every family $\mathcal{F} \subseteq \mathrm{FIN}$ there is an infinite set M such that either*
(a) $\mathcal{F}[M]$ is the closure of a uniform barrier on M, or else
(b) $M^{[<\infty]} \subseteq \overline{\mathcal{F}}^{\subseteq}$.

Observe that if (a) holds, then the trace $\mathcal{F}[M]$ of $\mathcal{F}$ on M is hereditary. Note also that if $\mathcal{F}$ is pre-compact then, (a) must hold. The next readily follows from Theorem 2.

Corollary 3 *Suppose that $\mathcal{F}_0, \mathcal{F}_1 \subseteq \mathrm{FIN}$ are such that*

$$M^{[<\infty]} \subseteq \{s_0 \cup s_1 \,:\, s_0 \in \mathcal{F}_0,\, s_1 \in \mathcal{F}_1\}. \tag{6}$$

Then there is an infinite $N \subseteq M$ and $i = 0, 1$ such that $N^{[<\infty]} \subseteq \overline{\mathcal{F}_i}^{\subseteq}$.

PROOF. First note that the union mapping $\mathrm{FIN} \times \mathrm{FIN} \to \mathrm{FIN}$, $(s,t) \mapsto s \cup t$ is continuous. So if two families $\mathcal{F}_0$ and $\mathcal{F}_1$ are pre-compact, then $\{s_0 \cup s_1 \,:\, s_i \in \mathcal{F}_i,\, i = 0, 1\}$ is also pre-compact. This implies, by Theorem 2, that there is $N \subseteq M$ such that either $N^{[<\infty]} \subseteq \overline{\mathcal{F}_0}^{\subseteq}$ or $N^{[<\infty]} \subseteq \overline{\mathcal{F}_1}^{\subseteq}$. ■

We are going to link Theorem 2 for pre-compact families with a property concerning a simple family of weakly-null sequences. We need to introduce two natural semi-norms.

Definition 8 *Given an arbitrary family $\mathcal{F} \subseteq \mathrm{FIN}$ we define on c_{00} the following two semi-norms:*

$$\begin{aligned} \|\boldsymbol{a}\|_{\mathcal{F}} &= \sup_{t \in \mathcal{F}} \|\boldsymbol{a} \upharpoonright t\|_{\ell_1} \\ \|\boldsymbol{a}\|_{\mathcal{F}}^{1} &= \sup_{t \in \mathcal{F}} |\langle \boldsymbol{a}, \chi_t \rangle|. \end{aligned}$$

Let $X_{\mathcal{F}}$ and $X_{\mathcal{F}}^1$ be the corresponding completions.

We give some examples to illustrate the previous definition:
(a) The space $X_{\mathcal{S}}$ for the *Schreier* family $\mathcal{S}$ consisting on those finite sets of integers s with $|s| \leq \min s$, $X_{\mathcal{S}}$ is the so-called Schreier space introduced by Schreier [29] to provide the first example of a normalized weakly-null sequence without Cesaro summable subsequence. In case that $\mathcal{F}$ is compact and hereditary and $N^{[1]} \subseteq \mathcal{F}$, then $\boldsymbol{e}$ is a normalized weakly-null unconditional Schauder basis of $X_{\mathcal{F}}$. We call $X_{\mathcal{F}}$ the *$\mathcal{F}$-Schreier space*.
(b) It is easy to see that if $\mathcal{F}$ is pre-compact, then Hamel basis $\boldsymbol{e}$ of c_{00} is a weakly-null sequence of both $X_{\mathcal{F}}$ and $X_{\mathcal{F}}^1$.
(c) $X_{\mathrm{FIN}} = \ell_1$, and in general, the sequence $\boldsymbol{e} \upharpoonright M$ of $X_{M^{[<\infty]}}$ is 1-equivalent to the natural basis of ℓ_1. Also, it is easy to see that the norm on X_{FIN}^1 is 2-equivalent to the ℓ_1-norm.

We see now few properties of the two spaces introduced above.

Proposition 5 *(a) $\|\boldsymbol{a}\|_{\mathcal{F}}^1 \leq \|\boldsymbol{a}\|_{\mathcal{F}}$.*
(b) If $\mathcal{B}$ is a barrier on M, then $\|\boldsymbol{a}\|_{\overline{\mathcal{B}}} = \|\boldsymbol{a}\|_{\mathcal{B}}$ and $\|\boldsymbol{a}\|_{\overline{\mathcal{B}}}^1 = \|\boldsymbol{a}\|_{\mathcal{B}}^1$ for every $\boldsymbol{a} \in c_{00} \upharpoonright M$.
(c) If $\mathcal{F}$ is a barrier on M or if $\mathcal{F} = M^{[<\infty]}$, then

$$\|\boldsymbol{a}\|_{\mathcal{F}} \leq 2\|\boldsymbol{a}\|_{\mathcal{F}}^1$$

for every $\boldsymbol{a} \in c_{00} \upharpoonright M$.

PROOF. (a) is trivial. To show (b), use that for every $\boldsymbol{a} \in c_{00} \upharpoonright M$, and every $t \in \overline{\mathcal{B}}$ there is $s \in \mathcal{B}$ such that $\operatorname{supp} \boldsymbol{a} \cap s = \operatorname{supp} \boldsymbol{a} \cap t$. Finally, we prove (c): The result for $\mathcal{F} = M^{[<\infty]}$ is trivial. So, suppose that $\mathcal{B} = \mathcal{F}$ is a barrier on M, and fix $\boldsymbol{a} \in c_{00} \upharpoonright M$. Choose first $s \in \mathcal{B}$ such that $\|\boldsymbol{a}\|_{\mathcal{B}} = \|\boldsymbol{a} \upharpoonright s\|_{\ell_1}$. Now pick $t \subseteq \operatorname{supp} \boldsymbol{a} \cap s$ such that $\boldsymbol{a} \upharpoonright t$ has constant sign and $2\|\boldsymbol{a} \upharpoonright t\|_{\ell_1} \geq \|\boldsymbol{a} \upharpoonright s\|_{\ell_1}$. Finally let $u = \iota_{\mathcal{B}}(t \cup (M/\operatorname{supp} \boldsymbol{a}))$. Then $t \sqsubseteq u$ (otherwise, $u \sqsubset t \subseteq s$, impossible since both are in the barrier $\mathcal{B}$), so $|\sum_{n \in s} \boldsymbol{a}(n)| \geq 2\|\boldsymbol{a} \upharpoonright s\|_{\ell_1}$, and we are done. ■

We are ready to present the equivalence we promised.

Theorem 3 *The following are equivalent:*
(a) For every pre-compact family $\mathcal{F} \subseteq \mathrm{FIN}$ there is an infinite set M such that $\mathcal{F}[M]$ is the closure of a uniform barrier on M.
(b) For every weakly-null infinite sequence $\boldsymbol{x}$ of $C(K)$, K compact, consisting of characteristic functions there is some infinite set M and a uniform barrier $\mathcal{B}$ on $\mathbb{N}$ such that the subsequence $\boldsymbol{x} \upharpoonright M$ is 1-equivalent to the natural basis $(e_n)_{n \in \mathbb{N}}$ of the space $X^1_{\mathcal{B}}$.

Before we give the proof, let us point out, that this Theorem, together with Theorem 2 and Proposition 5 gives the following generalization of a result of Rosenthal stating that every normalized weakly-null sequence consisting on characteristic functions of $C(K)$ has a 1-unconditional subsequence (see [22]).

Corollary 4 *Suppose that $\boldsymbol{x}$ is an arbitrary weakly-null sequence consisting on characteristic functions of $C(K)$, K compact. Then there is some an infinite set M and some uniform barrier $\mathcal{B}$ on $\mathbb{N}$ such that $\boldsymbol{x} \upharpoonright M$ is 2-equivalent to the natural basis of some Schreier space $X_{\mathcal{B}}$ with $\mathcal{B}$ a uniform barrier on $\mathbb{N}$.* ■

We will present a generalization of previous Corollary in Subsection 4.4.. Let us go now to the proof of Theorem 3.
PROOF. (a) implies (b): For every $\xi \in K$, let

$$\operatorname{supp} \xi = \{n \in \mathbb{N} \,:\, x_n(\xi) \neq 0\} = \{n \in \mathbb{N} \,:\, x_n(\xi) = 1\}. \tag{7}$$

As $\boldsymbol{x}$ is weakly-null, (7) implies that the set

$$\mathcal{F} = \{\operatorname{supp} \xi \,:\, \xi \in K\}$$

is a pre-compact family. Let M be such that the $\subseteq$-maximal elements of $\mathcal{F}[M]$ is a barrier $\mathcal{C}$ on M.

Claim 1 *For every $\boldsymbol{a} \in c_{00}$,*

$$\|\boldsymbol{a} \cdot \boldsymbol{x} \upharpoonright M\|_K = \sup_{s \in \mathcal{F}[M]} |\sum_{n \in s} \boldsymbol{a}(n)| = \sup_{s \in \mathcal{C}} |\sum_{n \in s} \boldsymbol{a}(n)|. \tag{8}$$

Before we show this Claim, let us see how it implies (b): Let $\pi : \mathbb{N} \to M$ be the unique order-preserving mapping from $\mathbb{N}$ onto M, and we set $\mathcal{B} = \pi^{-1}"\mathcal{C} = \{\pi^{-1}"s \,:\, s \in \mathcal{C}\}$, then it is easy to show from (8) that

$$\|\boldsymbol{a} \cdot \boldsymbol{e}\|^1_{\mathcal{B}} = \sup_{s \in \mathcal{C}} |\sum_{n \in s} a_{\pi(n)}| = \|\sum_{n \in M} a_{\pi(n)} x_{\pi(n)}\|_K.$$

We prove now the Claim. First of all we observe that the last equality in (8) follows from Proposition 5 (b). To show the first one, notice that for a given $\xi \in K$ and $\boldsymbol{a} \in c_{00}$

$$|(\boldsymbol{a} \cdot \boldsymbol{x} \upharpoonright M)(\xi)| = |(\boldsymbol{a} \upharpoonright (M \cap \operatorname{supp} \xi) \cdot \boldsymbol{x})(\xi)| = |\sum_{n \in \operatorname{supp} \xi \cap M} a_n| \leq \sup_{s \in \mathcal{F}[M]} |\sum_{n \in s} \boldsymbol{a}(n)|.$$

Now pick $u \in \mathcal{F}[M]$ such that

$$\sup_{t \in \mathcal{F}[M]} |\sum_{n \in t} \boldsymbol{a}(n)| = |\sum_{n \in u} \boldsymbol{a}(n)|. \tag{9}$$

As $\mathcal{F}[M]$ is hereditary, (9) implies that $\boldsymbol{a} \upharpoonright u$ has constant sign. Find $\xi \in K$ such that $\operatorname{supp} \xi \cap M = u$. Then

$$|(\boldsymbol{a} \cdot \boldsymbol{x} \upharpoonright M)(\xi)| = |(\boldsymbol{a} \upharpoonright (\operatorname{supp} \xi \cap M) \cdot \boldsymbol{x})(\xi)| = |\sum_{n \in u} \boldsymbol{a}(n)|$$

because $\boldsymbol{x}$ consist on characteristic functions.

(b) implies (a): Fix an arbitrary pre-compact family $\mathcal{F}$, and let

$$K = \overline{\mathcal{F}} \subseteq \text{FIN}.$$

This is a compact family, because $\mathcal{F}$ is pre-compact. Consider the sequence $\boldsymbol{x} = (x_n)$ of $C(K)$ defined for $n \in \mathbb{N}$ and $s \in K$ by

$$x_n(s) = 1 \text{ iff } n \in s, \text{ and } x_n(s) = 0 \text{ otherwise.}$$

Observe that $x_n = 0$ or $\|x_n\|_K = 1$ for every n. Now if there is some M such that $x_n = 0$ for every $n \in M$, then we have clearly that $\mathcal{F}[M] = \{\emptyset\}$ that is the closure of the 0-uniform family $\{\emptyset\}$. So, we may assume that $\boldsymbol{x}$ is infinite, and hence normalized. By (b) there is an infinite set M such that $\boldsymbol{x} \upharpoonright M$ is 1-equivalent to the natural basis of $X^1_{\mathcal{G}}$, where $\mathcal{G}$ is the closure of a barrier on $\mathbb{N}$. Let $\pi : \mathbb{N} \to M$ be the unique order-preserving onto mapping, and set $\mathcal{H} = \pi"\mathcal{G}$. Note that $\mathcal{H}$ is hereditary. Since the corresponding sequences are equivalent, we have that for every $\boldsymbol{a} \in c_{00} \upharpoonright M$

$$\|\boldsymbol{a} \cdot \boldsymbol{x}\|_K = \|\pi^{-1}(\boldsymbol{a})\|^1_{\mathcal{G}} = \sup_{t \in \mathcal{G}} |\langle \pi^{-1}(\boldsymbol{a}), \chi_t \rangle| = \sup_{s \in \mathcal{H}} |\langle \boldsymbol{a}, \chi_s \rangle|. \tag{10}$$

Fix $s \in \mathcal{F}[M]$. Observe that $\|\chi_s \cdot \boldsymbol{x}\|_K = |s|$, because $s = t \cap M$ for some $t \in \mathcal{F}$, and so

$$|s| \geq \|\chi_s \cdot \boldsymbol{x}\|_K \geq |(\chi_s \cdot \boldsymbol{x})(t)| = |s|.$$

By (10), we can find $u \in \mathcal{H}$ such that $|\langle \chi_s, \chi_u \rangle| = |s|$. This clearly means that $s \subseteq u$, so, since $\mathcal{H}$ is hereditary, $s \in \mathcal{H}$. We have just shown that

$$\mathcal{F}[M] \subseteq \mathcal{H}. \tag{11}$$

Suppose now that $s \in \mathcal{H}$ is non-empty. Let $t = \pi"s \in \mathcal{G}$. Notice that, by (10),

$$|s| = |t| = \|\sum_{n \in t} e_n - \sum_{n \notin t} \frac{1}{2^{n+1}} e_n\|^1_{\mathcal{G}} = \|\sum_{n \in s} x_n - \sum_{n \in M \setminus s} \frac{1}{2^{\pi^{-1}(n)+1}} x_n\|_K.$$

This means that $s \in \mathcal{F}[M]$. So,

$$\mathcal{H} \setminus \{\emptyset\} \subseteq \mathcal{F}[M]. \tag{12}$$

Let $N \subseteq M$ be arbitrary such that $M \setminus N$ is infinite. Note that $\emptyset \in \mathcal{F}[N]$: Fix $n \in M \setminus N$. Then $\{n\} \in \mathcal{F}[M]$, so $\emptyset = \{n\} \cap N \in \mathcal{F}[M][N] = \mathcal{F}[N]$. So,

$$\mathcal{H}[N] \subseteq \mathcal{F}[M][N] = \mathcal{F}[N] \subseteq \mathcal{H}[N].$$

This gives (a): We know that $\mathcal{H}$ is the closure of a uniform barrier $\mathcal{B}$ on M, so, by Proposition 2 (c), $\mathcal{F}[N] = \mathcal{H}[N]$ is the closure of $\mathcal{B} \upharpoonright N$. ∎

We start now the proof of Theorem 2 by analyzing first the case of pre-compact families.

Lemma 5 *Suppose that $\mathcal{F} \subseteq$ FIN is a pre-compact family. Then there is an infinite set M such that $\mathcal{F}[M]$ is the closure of a uniform barrier on M.*

PROOF. We split the proof into two cases:
CASE 1. $\mathcal{F}$ is compact and hereditary. Let $\alpha = r(\mathcal{F})$, and let $\mathcal{B}$ be an arbitrary $\alpha+1$-uniform family on $\mathbb{N}$. Color each $s \in \mathcal{B}$ by either 0 if there is some $t \in \mathcal{F}$ such that $s \sqsubseteq t$ or 1 otherwise. By the Ramsey property of $\mathcal{B}$ there is some N such that $\mathcal{B} \upharpoonright N$ is monochromatic. Observe that the constant color there has to be 1 since otherwise $\overline{\mathcal{B} \upharpoonright N} \subseteq \overline{\mathcal{F}} = \mathcal{F}$ (as $\mathcal{F}$ is hereditary) and hence $\alpha = r(\mathcal{F}) \geq r(\overline{\mathcal{B} \upharpoonright N}) = \alpha + 1$, a contradiction. Define $\varphi : \mathcal{B} \upharpoonright N \to \mathcal{F}$ by choosing a $\sqsubseteq$-maximal $\varphi(s) \in \mathcal{F}$ such that $\varphi(s) \sqsubseteq s$. Use Corollary 2 to find $M \subseteq N$ such that $\varphi"(\mathcal{F} \upharpoonright M)$ is a uniform barrier $\mathcal{B}$ on M. We claim that $\mathcal{F} \upharpoonright M = \overline{\mathcal{B} \upharpoonright N}$: Suppose that $t \in \mathcal{F} \upharpoonright M$. Let $s = \iota_{\mathcal{B}}(t \cup (M/t))$. As the color of s is 1, $t \sqsubseteq s$, so, $t \sqsubseteq \varphi(s)$, by maximality of $\varphi(s)$. Hence $t \in \overline{\mathcal{B} \upharpoonright N}$. The inverse inclusion is trivial.
CASE 2. $\mathcal{F}$ arbitrary pre-compact family. By case 1 applied to the compact hereditary family $\mathcal{G} = \overline{\mathcal{F}}^{\subseteq}$ there is some M and some uniform barrier $\mathcal{B}$ on M such that $\mathcal{G}[M] = \overline{\mathcal{B}}$. First we have that $\mathcal{F}[M] \subseteq \mathcal{G}[M] = \overline{\mathcal{B}}$. Now we show that

$$\mathcal{B} \subseteq \mathcal{F}[M]:$$

Fix $s \in \mathcal{B}$. Then $s \in \mathcal{G}[M]$, so there is $t \in \mathcal{G}$ such that $s = t \cap M$. Hence there is $u \in \mathcal{F}$ such that $t \subseteq u$, and therefore $s \subseteq u \cap M \in \mathcal{F}[M]$. As $\mathcal{F}[M] \subseteq \overline{\mathcal{C}}$, there is $\bar{s} \in \mathcal{B}$ such that $s \subseteq u \cap M \sqsubseteq \bar{s}$. This means, by the Sperner property of $\mathcal{B}$, that $s = u \cap M = \bar{s}$, and so $s \in \mathcal{F}[M]$. Finally, use Proposition 2 to find $N \subseteq M$ such that $\overline{\mathcal{B} \upharpoonright N} = \mathcal{B}[N]$. Then we have that

$$\overline{\mathcal{B} \upharpoonright N} = \mathcal{B}[N] \subseteq \mathcal{F}[N] \subseteq \overline{\mathcal{B}}[N] = \overline{\mathcal{B} \upharpoonright N},$$

as demanded. ■

We are ready to give a proof of Theorem 2.
PROOF. Fix $\mathcal{F} \subseteq \mathrm{FIN}$. Let

$$\mathcal{G} = \mathrm{FIN} \setminus \overline{\mathcal{F}}^{\subseteq}.$$

We apply Galvin's Lemma to it to obtain an infinite $M \subseteq \mathbb{N}$ such that either
(a) $\mathcal{G} \upharpoonright M$ contains a barrier on M, or else,
(b) $\mathcal{G} \upharpoonright M = \emptyset$.

Suppose that (a) holds. Then we claim that $\mathcal{F}[M]$ is pre-compact. Let $N \subseteq M$ be arbitrary infinite set. Since $\mathcal{G} \upharpoonright M$ contains a barrier on M there is $s \in \mathcal{G} \upharpoonright N$, i.e., $s \notin (\overline{\mathcal{F}}^{\subseteq}) \upharpoonright M$. Since $\overline{(\overline{\mathcal{F}}^{\subseteq}) \upharpoonright M}$ is hereditary, we obtain that $N \notin \overline{(\overline{\mathcal{F}}^{\subseteq}) \upharpoonright M}$, hence $N \notin \overline{\mathcal{F}[M]}$. Now we apply Lemma 5 to find $N \subseteq M$ such that $\mathcal{F}[N]$ is the closure of a uniform barrier on N.

If (b) holds, then it is clear that $M^{[<\infty]} \subseteq \overline{\mathcal{F}}^{\subseteq}$. ■

Remark 3 *(a)*

3.2. Rosenthal's ℓ_1-theorem

In this subsection we give a proof of Rosenthal's ℓ_1-theorem [27] using the techniques introduced so far. Indeed we are going to see that this dichotomy is in fact closely related to the dichotomy appearing in Theorem 2 above, where the only difference is in replacing families of finite sets by ordered families of finite sets of doubletons. In order to make our approach more transparent we start with a topological characterization of when a given sequence $\boldsymbol{x}$ is weakly-null. Later on, we shall apply a similar idea to characterize weakly-Cauchy sequences.

Definition 9 *Let $\boldsymbol{x}$ be a sequence in a Banach space X. We define $\mathcal{W}^0(\boldsymbol{x}) \subseteq \mathrm{FIN}$ as follows:*

$$s \in \mathcal{W}^0(\boldsymbol{x}) \text{ iff there is } x^* \in B_{X^*} \text{ such that } |x^*(x_n)| \geq \frac{1}{2^{\min s}} \text{ for every } n \in s.$$

Proposition 6 *(a)* $\mathcal{W}^0(\boldsymbol{x} \upharpoonright M) = (\mathcal{W}^0(\boldsymbol{x}))[M]$,
(b) $\mathcal{W}^0(\boldsymbol{x})$ *is hereditary, so* $\mathcal{W}^0(\boldsymbol{x} \upharpoonright M) = (\mathcal{W}^0(\boldsymbol{x})) \upharpoonright M$.
(c) A sequence $\boldsymbol{x}$ *is weakly-null iff* $\mathcal{W}^0(\boldsymbol{x})$ *is pre-compact.*

PROOF. Since (a) and (b) are straightforward, we concentrate on (c). Suppose that $\boldsymbol{x}$ is not weakly-null. Then there is some $x^* \in B_{X^*}$ and $\varepsilon > 0$ such that $A = \{n \ : \ |x^*(x_n)| \geq \varepsilon\}$ is infinite. Then $\{n \in A \ : \ n \geq n_0\}$ is clearly in the closure of $\mathcal{W}^0(\boldsymbol{x})$, for n_0 such that $2^{n_0}\varepsilon \geq 1$. Suppose that $\boldsymbol{x}$ is weakly-null. Let A be in the closure of $\mathcal{W}^0(\boldsymbol{x})$. Fix $(s_n) \subseteq \mathcal{W}^0(\boldsymbol{x})$ such that $s_n \to A$. Let $(x_n^*)_n \subseteq B_{X^*}$ be such that $|x_n^*(x_m)| \geq 2^{-\min s_n}$ for every $m \in s_n$ and every n. Since $s_n \to A$, we assume that $\min s_n = \min A = n_0$ for every n. We assume also that $(x_n^*)_n$ is weak*-convergent with limit $x^* \in B_{X^*}$ (by Alaoglu's Theorem). It is easy to see that $|x^*(x_n)| \geq 2^{-n_0}$ for every $n \in A$, so, since $\boldsymbol{x}$ is weakly-null, A must be finite. ■

Definition 10 *Let* $(\mathbb{N}^{[2]})^{[\leq\infty]} \subseteq \mathcal{P}(\mathbb{N}^{[2]})$ *be the set of* block sets of doubletons *, i.e. the set of those* $A \subseteq P(\mathbb{N}^{[2]})$ *such that for every* $s, t \in A$*, either* $s < t$ *or* $t < s$*. If we consider* $\mathbb{N}^{[2]}$ *with its discrete topology, then* $(\mathbb{N}^{[2]})^{[\leq\infty]}$ *is a closed subspace of* $\mathcal{P}(\mathbb{N}^{[2]})$*, this with its product topology, so* $(\mathbb{N}^{[2]})^{[\leq\infty]}$ *is a compact space. Let*

$$\mathrm{FIN}_2 = \{A \in (\mathbb{N}^{[2]})^{[\leq\infty]} \ : \ A \text{ is finite}\}.$$

We say that $\mathcal{U} \subseteq \mathrm{FIN}_2$ *is pre-compact iff* $\overline{\mathcal{U}} \subseteq \mathrm{FIN}_2$*. We say that* $\mathcal{U}$ *is hereditary iff* $A \subseteq B \in \mathcal{U}$ *implies that* $A \in \mathcal{U}$*. Given* $\mathcal{U} \subseteq \mathrm{FIN}_2$ *and* $M \subseteq \mathbb{N}$ *infinite, we define*

$$\begin{aligned} \mathcal{U}[M] &= \{A \cap M^{[2]} \ : \ A \in \mathcal{U}\} \\ \mathcal{U} \upharpoonright M &= \mathcal{U} \cap \mathcal{P}(M^{[2]}) \\ \overline{\mathcal{U}}^{\subseteq} &= \{B \in \mathrm{FIN}_2 \ : \ B \subseteq A \in \mathcal{U}\}. \end{aligned}$$

Given a sequence $\boldsymbol{x}$ *in a Banach space, and* $\mathcal{U} \subseteq \mathrm{FIN}_2$*, we define the sequence* $\boldsymbol{x}_{\mathcal{U}}$ *indexed by* $\mathcal{U}$ *by*

$$x_s = x_{\min s} - x_{\max s}.$$

for every $s \in \mathcal{U}$*. Define now* $\mathcal{W}^c(\boldsymbol{x}) \subseteq \mathrm{FIN}_2$ *by*

$$A \in \mathcal{W}^c(\boldsymbol{x}) \text{ iff there is } x^* \in B_{X^*} \text{ such that } |x^*(x_s)| \geq \frac{1}{2^{\min \cup A}} \text{ for every } s \in A.$$

Definition 11 *Recall that a sequence* $\boldsymbol{x}$ *in a Banach space is called* weakly-Cauchy *iff for every* $x^* \in B_{X^*}$ *the corresponding numerical sequence* $x^*(\boldsymbol{x})$ *is Cauchy.*

We give a characterization of the fact of being weakly-Cauchy in terms of pre-compactness of $\mathcal{W}^c(\boldsymbol{x})$.

Proposition 7 *(a)* $\mathcal{W}^c(\boldsymbol{x} \upharpoonright M) = \mathcal{W}^c(\boldsymbol{x})[M]$.
(b) $\mathcal{W}^c(\boldsymbol{x})$ *is hereditary, hence* $\mathcal{W}^c(\boldsymbol{x} \upharpoonright M) = (\mathcal{W}^c(\boldsymbol{x})) \upharpoonright M$.
(c) $\boldsymbol{x}$ *is weakly-Cauchy iff* $\mathcal{W}^c(\boldsymbol{x})$ *is pre-compact.*

PROOF. Again, only (c) requires a proof. Suppose that $\boldsymbol{x}$ is not weakly-Cauchy. Then there is $A \in (\mathbb{N}^{[2]})^{[\infty]}$ and $\varepsilon > 0$ such that $|x_s| \geq \varepsilon$ for every $s \in A$. Let n_0 be such that $\varepsilon 2^{n_0} \geq 1$. Then $\{s \in A \ : \ \min s \geq n_0\}$ is infinite and in the closure of $\mathcal{G}(\boldsymbol{x})$. Suppose now that $\boldsymbol{x}$ is weakly-Cauchy. Let A be limit point of $\mathcal{G}(\boldsymbol{x})$. Fix $(A_n)_n \subseteq \mathcal{W}^c(\boldsymbol{x})$, and $x_n^* \in B_{X^*}$ such that $|x_n^*(x_s)| \geq 2^{-\min \cup A_n}$ for every $s \in A_n$ and every n. As $A_n \to A$, we may assume that $\min A = \min A_n$ for every n. We assume also that (x_n^*) is weak*-convergent with limit x^*. Then $|x^*(x_s)| \geq 2^{-\min \cup A}$ for every $s \in A$. Since $\boldsymbol{x}$ is weakly-Cauchy, A is finite, as desired. ■

The combinatorial theory of FIN_2 is very similar to that of FIN. We only present here the following analogue to Theorem 2.

Lemma 6 *For every $\mathcal{U} \subseteq \mathrm{FIN}_2$ there is $M \subseteq \mathbb{N}$ infinite such that either*
(a) $\mathcal{U}[M]$ is pre-compact or else,
(b) $\mathrm{FIN}_2 \upharpoonright M \subseteq \overline{\mathcal{U}}^{\subseteq}$.

PROOF. Let

$$\mathcal{G} = \{\bigcup A \ : \ A \in \mathrm{FIN}_2 \setminus \overline{\mathcal{U}}^{\subseteq}\}.$$

By Galvin's Lemma, there is M such that either $\mathcal{G} \upharpoonright M$ contains a barrier on M or else $\mathcal{G} \upharpoonright M = \emptyset$. Suppose first that $\mathcal{G} \upharpoonright M$ contain a barrier on M. We claim that in this case $\mathcal{U}[M]$ is pre-compact. Indeed we show that $\overline{U}^{\subseteq}[M]$ is compact: Let $A \in (M^{[2]})^{[\infty]}$, and set $N = \bigcup A \subseteq N$. As $\mathcal{G} \upharpoonright M$ contain a barrier, there is $s \in \mathcal{G} \upharpoonright M$ such that $s \sqsubseteq N$. Let $B \in \mathrm{FIN}_2 \setminus \overline{\mathcal{U}}^{\subseteq}$ such that $s = \bigcup B$. We note that $\overline{\overline{U}^{\subseteq}}$ is also hereditary, so $A \notin \overline{\overline{U}^{\subseteq}}$, because $B \subseteq A$ and $B \notin \overline{\overline{U}^{\subseteq}}$.

Suppose now that $\mathcal{G} \upharpoonright M = \emptyset$. It is clear that, by definition of $\mathcal{G}$, we have that $\mathrm{FIN}_2 \upharpoonright M \subseteq \overline{\mathcal{U}}$. ■

As a consequence we obtain the following. Its proof is very similar to that of Corollary 3.

Corollary 5 *Suppose that $\mathcal{U}_0, \mathcal{U}_1 \subseteq \mathrm{FIN}_2$ and $M \subseteq \mathbb{N}$ are such that*

$$\mathrm{FIN}_2 \upharpoonright M \subseteq \{A_0 \cup A_1 \ : \ A_0 \in \mathcal{U}_0, \ A_1 \in \mathcal{U}_1\}.$$

Then there is infinite $N \subseteq M$ and $i = 0, 1$ such that $\mathrm{FIN}_2 \upharpoonright M \subseteq \overline{\mathcal{U}_i}^{\subseteq}$.

PROOF. The mapping $\mathrm{FIN}_2 \times \mathrm{FIN}_2 \to \mathrm{FIN}_2$, $(A, B) \mapsto A \cup B$ is continuous, so the desired result follows from Lemma 6. ■

Theorem 4 (Rosenthal's ℓ_1-Theorem) *Suppose that $\boldsymbol{x}$ is a bounded sequence in a Banach space X. Then there is infinite M such that either*
(a) $\boldsymbol{x} \upharpoonright M$ is weakly-Cauchy or else
(b) $\boldsymbol{x} \upharpoonright M$ is equivalent to the natural basis of ℓ_1.

PROOF. We apply Proposition 6 to $\mathcal{W}^c(\boldsymbol{x})$ to obtain some infinite set M such that either $\mathcal{W}^c(\boldsymbol{x})[M] = \mathcal{W}^c(\boldsymbol{x} \upharpoonright M)$ (by Proposition 7) is pre-compact or else $\mathcal{W}^c(\boldsymbol{x} \upharpoonright M) = \mathrm{FIN}_2 \upharpoonright M$. In the first case, we have, by Proposition 7 (c) that $\boldsymbol{x} \upharpoonright M$ is weakly-Cauchy. Now suppose that $\mathcal{W}^c(\boldsymbol{x} \upharpoonright M) = \mathrm{FIN}_2 \upharpoonright M$. Let $N = \{m_k\}_{k \geq 2}$, where $\{m_k\}_{k \geq 0}$ is the increasing enumeration of M. Notice that for every $A \in \mathrm{FIN}_2 \upharpoonright N$ we have that $\{\{m_0, m_1\}\} \cup A \in \mathcal{W}^c(\boldsymbol{x} \upharpoonright M)$, so there is $x^* \in B_{X^*}$ such that $|x^*(x_s)| \geq \varepsilon$ for every $s \in A$, where $\varepsilon = 2^{-m_0}$.

Now we claim that there is infinite $P \subseteq N$ and two real numbers $d_0 < d_1$ such that for every $A \in \mathrm{FIN}_2 \upharpoonright P$ there is $x^* \in B_{X^*}$ such that

$$x^*(x_{\min s}) \leq p_0 \text{ and } x^*(x_{\max s}) \geq p_1 \text{ for every } s \in A:$$

Let D be a finite $\varepsilon/3$-net of the interval $[-\sup_n \|x_n\|, \sup_n \|x_n\|]$. We define, for $(d_0, d_1) \in D^{[2]}$, the sets

$$\mathcal{U}_{(d_0,d_1)} = \{A \in \mathrm{FIN}_2 \upharpoonright N \ : \ \text{there is } x^* \in B_{X^*} \text{ with } x^*(x_{\min s}) \leq d_0 \text{ and } x^*(x_{\max s}) \geq d_1 \ \forall s \in A\}.$$

Observe that every $A \in \mathrm{FIN}_2 \upharpoonright N$ is the union of elements of $\mathcal{U}_{(d_0,d_1)}$'s, and that each $\mathcal{U}_{(d_0,d_1)}$ is hereditary. By Corollary 5 there is $P \subseteq N$ and $(d_0, d_1) \in D^{[2]}$ such that $\mathrm{FIN}_2 \upharpoonright P = \mathcal{U}_{(d_0,d_1)}[P] = \mathcal{U}_{(d_0,d_1)} \upharpoonright P$, as desired.

Now set $Q = \{p_{2k+1}\}_k$, where $\{p_k\}_k$ is the increasing enumeration of P. We claim that for every disjoint s, t subsets of N there is x^* such that

$$x^*(x_n) \leq d_0 \text{ and } x^*(x_m) \geq d_1 \text{ for every } n \in s \text{ and } m \in t:$$

This follows from the fact that for every disjoint and finite $s, t \subseteq P$ there is $A \in \mathrm{FIN}_2 \upharpoonright N$ such that $s = P \cap \{\min u : u \in A\}$ and $t = P \cap \{\max u : u \in A\}$.

Finally, the proof will be finished if we show the following: For every $\boldsymbol{a} \in c_{00} \upharpoonright P$,

$$\|\boldsymbol{a} \cdot \boldsymbol{x}\|_X \geq \frac{d_1 - d_0}{2} \|\boldsymbol{a}\|_{\ell_1} : \tag{13}$$

Fix $\boldsymbol{a} \in c_{00} \upharpoonright P$. Let

$$\begin{aligned} s_0 =& \{n \in \operatorname{supp} \boldsymbol{a} : \boldsymbol{a}(n) > 0\} \\ s_1 =& \operatorname{supp} \boldsymbol{a} \setminus s_0. \end{aligned}$$

Since s_0 and s_1 are disjoint, we can find $x_0^*, x_1^* \in B_{X^*}$ such that for every $i \neq j = 0, 1$,

$$\begin{aligned} x_0^*(x_n) \leq& d_0 \text{ and } x_0^*(x_m) \geq d_1 \text{ for every } n \in s_0 \text{ and } m \in s_1 \\ x_1^*(x_n) \leq& d_0 \text{ and } x_1^*(x_m) \geq d_1 \text{ for every } n \in s_1 \text{ and } m \in s_0. \end{aligned}$$

We compute:

$$\begin{aligned} |(x_1^* - x_0^*)(\boldsymbol{a} \cdot \boldsymbol{x})| =& |(x_1^* - x_0^*)(\boldsymbol{a} \upharpoonright s_0 \cdot \boldsymbol{x}) + (x_1^* - x_0^*)(\boldsymbol{a} \upharpoonright s_1 \cdot \boldsymbol{x})| = \\ =& \sum_{n \in s_0} \boldsymbol{a}(n)(x_1^* - x_0^*)(x_n) + \sum_{n \in s_1} \boldsymbol{a}(n)(x_1^* - x_0^*)(x_n) \geq \\ \geq& (d_1 - d_0) \sum_{n \in s_0} \boldsymbol{a}(n) + (d_0 - d_1) \sum_{n \in s_1} \boldsymbol{a}(n) = (d_1 - d_0)(\sum_{n \in s_0} \boldsymbol{a}(n) - \sum_{n \in s_1} \boldsymbol{a}(n)) = \\ =& (d_1 - d_0)\|\boldsymbol{a}\|_{\ell_1}. \end{aligned}$$

Since $(x_1^* - x_0^*)/2 \in B_{X^*}$, we obtain (13). ■

3.3. Matching pairs of finite sets from a given barrier

When trying to solve a given partial unconditionality problem that typically calls for the existence if a constant that measures the extent of this unconditionality, one is lead to consider a new kind of combinatorial problems about barrier. Roughly speaking, one typically ends up with an assignment that gives a subset t_s to every element s of a barrier and one is required to find (among other things) a couple s and u of elements of the barrier for which we assigned the same subset t and such that the intersection of s and u is t. In this section we consider this purely combinatorial problem leaving its exact relationship to the partial unconditionality problem to some later point. We should point out that the first place where a "matching Lemma" appears explicitly is in [10], where it is used for the similar purpose of getting results about partial unconditionality. We shall show that, in particular, the matching problem has a positive solution if the chosen sets t_s lie in another barrier, while in general the answer is negative, but instead we prove that there are two elements s, u of the barrier such that t_s is initial part of t_u and the intersection of s and u is t_s.

Proposition 8 *Suppose that $\mathcal{B}$ and $\mathcal{C}$ are two barriers on M and $\varphi : \mathcal{B} \to \mathcal{C}$ is an internal mapping. Then there is an infinite subset N of M and a mapping $\sigma : \mathcal{B} \upharpoonright N \to \mathcal{B}$ such that:*

$$\chi_N \cdot \sigma = \varphi \upharpoonright (\mathcal{B} \upharpoonright N) = \varphi \circ \sigma.$$

In particular, for every $s \in \mathcal{B} \upharpoonright N$, there is $t \in \mathcal{B}$ (not necessarily a subset of N) such that
(a) $\varphi(s) = \varphi(t)$, and
(b) $s \cap t = \varphi(s)$.

PROOF. First of all, color each $t \in \mathcal{C}$ by 1 iff there is $s \in \mathcal{B}$ such that $\varphi(s) = t$, and 0 otherwise. By the Ramsey property of $\mathcal{C}$ there is some $P \subseteq M$ such that $\mathcal{C} \upharpoonright P$ is monochromatic, with color $i = 0, 1$. As for every $s \in \mathcal{B} \upharpoonright P$, $\varphi(s) \in \mathcal{C} \upharpoonright P$ is colored by 1, i must be equal to 1. Define now $\psi : \mathcal{C} \upharpoonright P \to \mathcal{B}$ by $\psi(t) \in \mathcal{B}$ is such that $\varphi(\psi(t)) = t$. Apply Lemma 3 to ψ to get some $N \subseteq P$ such that $\psi(t) \cap N \subseteq t$ for every $t \in \mathcal{C} \upharpoonright N$. Observe that this is equivalent to say that $\psi(t) \cap N = t$ ($t \subseteq \psi(t)$ because $t = \varphi(\psi(t)) \subseteq \psi(t)$ by the properties of φ). Finally define $\sigma : \mathcal{B} \upharpoonright N \to \mathcal{B}$ by $\sigma(s) = \psi(\varphi(s))$ for each $s \in \mathcal{B} \upharpoonright N$. Then, for $s \in \mathcal{B} \upharpoonright N$ we have that

$$\varphi(\sigma(s)) = \varphi(\psi(\varphi(s))) = \varphi(s),$$

and

$$\sigma(s) \cap N = \psi(\varphi(s)) \cap N = \varphi(s),$$

as desired. ■

In the next we use the notation $f \sqsubseteq g$, for $f, g : \mathcal{B} \to \mathrm{FIN}$, to denote that $f(s) \sqsubseteq g(s)$ for every $s \in \mathcal{B}$.

Corollary 6 *Suppose that $\mathcal{B}$ is a barrier on M and that $\varphi : \mathcal{B} \to \mathrm{FIN}$ is an internal mapping. Then there is an infinite subset N of M and $\sigma : \mathcal{B} \upharpoonright N \to \mathcal{B}$ such that*

$$\chi_N \cdot \sigma = \varphi \circ \sigma \sqsubseteq \varphi.$$

In particular, for every $s \in \mathcal{B} \upharpoonright N$ there is $t \in \mathcal{B}$ such that
(a') $\varphi(t) \sqsubseteq \varphi(s)$, and
(b') $s \cap t = \varphi(t)$.

PROOF. Let $\mathcal{G}$ be the set of $\sqsubseteq$-minimal elements of φ"$\mathcal{B}$. Observe that $\mathcal{G} \upharpoonright P \neq \emptyset$ for every $P \subseteq M$: Fix such P, and let $s \in \mathcal{B} \upharpoonright P$. Then, there must be some $t \in \mathcal{G}$ such that $t \sqsubseteq \varphi(s)$. Such t belongs to $\mathcal{G} \upharpoonright P$. As $\mathcal{G}$ is a thin family, by Theorem 1, there is some $P \subseteq M$ such that $\mathcal{C} = \mathcal{G} \upharpoonright P$ is a barrier on P. Now define $\psi : \mathcal{B} \upharpoonright P \to \mathcal{C}$ by picking for every $s \in \mathcal{B} \upharpoonright P$ some $\psi(s) \in \mathcal{C}$ such that $\psi(s) \sqsubseteq \varphi(s)$ (well defined by minimality of elements of $\mathcal{G}$). Define also $\varpi : \mathcal{B} \upharpoonright P \to \mathcal{B}$ by choosing for $s \in \mathcal{B} \upharpoonright P$ some $\varpi(s) \in \mathcal{B}$ such that $\psi(s) = \varphi(\varpi(s))$. Now we apply Lemma 3 to ϖ to obtain $R \subseteq P$ such that

$$\varpi(s) \cap R \subseteq s \text{ for every } s \in \mathcal{B} \upharpoonright R. \tag{14}$$

Finally we apply previous Proposition 8 to $\psi \upharpoonright (\mathcal{B} \upharpoonright R)$ to obtain $N \subseteq R$ and $\bar{\sigma} : \mathcal{B} \upharpoonright N \to \mathcal{B} \upharpoonright R$ with the property that

$$\bar{\sigma}(s) \cap N = \psi(s) \text{ and } \psi(\bar{\sigma}(s)) = \psi(s) \text{ for every } s \in \mathcal{B} \upharpoonright N. \tag{15}$$

Define $\sigma = \varpi \circ \bar{\sigma}$. We claim that σ has the desired properties. Fix $s \in \mathcal{B} \upharpoonright N$. Then, by (14) and (15),

$$\varphi(\sigma(s)) = \varphi(\varpi(\bar{\sigma}(s))) = \psi(\bar{\sigma}(s)) = \psi(s) \sqsubseteq \varphi(s),$$

and hence

$$\varphi(\sigma(s)) \subseteq \sigma(s) \cap N = (\varpi(\bar{\sigma}(s)) \cap R) \cap N \subseteq \bar{\sigma}(s) \cap N = \psi(s) = \varphi(\sigma(s)),$$

as desired. ■

The dual result of the previous Corollary:

Corollary 7 *Suppose that $\mathcal{B}$ is a barrier on M and that $\varphi : \mathcal{B} \to \mathrm{FIN}$ is an internal mapping. Then there is an infinite subset N of M and $\sigma : \mathcal{B} \upharpoonright N \to \mathcal{B}$ such that*

$$\chi_N \cdot \sigma = \varphi \upharpoonright (\mathcal{B} \upharpoonright N) \sqsubseteq \varphi \circ \sigma.$$

In particular, for every $s \in \mathcal{B} \upharpoonright N$ there is $t \in \mathcal{B}$ such that
(a') $\varphi(s) \sqsubseteq \varphi(t)$, and
(b') $s \cap t = \varphi(s)$.

PROOF. Let $P \subseteq M$ be such that φ is uniform when restricted to $\mathcal{B} \upharpoonright P$ (Proposition 4). Let $\mathcal{G}$ be the set of $\sqsubseteq$-maximal nodes of φ"$(\mathcal{B} \upharpoonright P)$. This is clearly a thin family. Moreover, $\mathcal{G} \upharpoonright Q \neq \emptyset$ for every $Q \subseteq P$: Let $s_0 \in \mathcal{B} \upharpoonright Q$, then if $\varphi(s_0) \in \mathcal{G}$ we are done. Otherwise, there is $t_0 \in \mathcal{G}$, $\varphi(s_0) \sqsubset t_0$. Let $\bar{s}_1 \in \mathcal{B} \upharpoonright P$ be such that $t_0 = \varphi(\bar{s}_1)$. Set $n_0 = \min(t_0 \setminus \varphi(s_0))$. Find $s_1 \in \mathcal{B} \upharpoonright P$, $s_1 \sqsubseteq (\bar{s}_1 \cap [0, n_0)) \cup (Q \cap [n_0, \infty))$. By maximality of elements of $\mathcal{B}$, $\bar{s}_1 \sqsubseteq s_1$. By uniformity of φ, $\min(s_1 \setminus \bar{s}_1) \in \varphi(s_1)$, and so $\varphi(s_0) \sqsubset \varphi(s_1)$, and $\varphi(s_1) \subseteq Q$. If $\varphi(s_1) \in \mathcal{G}$, then we are done; otherwise we can keep producing $s_1, \dots, s_k$ such that $\varphi(s_{i-1}) \sqsubset \varphi(s_i)$ and $\varphi(s_i) \subseteq Q$ $(1 \leq i \leq k)$. There must be some k such that $s_k \in \mathcal{G}$ because otherwise, we find an infinite set $\bigcup_{k=0}^{\infty} \varphi(s_k)$ in the closure of φ"$\mathcal{B}$ which is included in $\overline{\mathcal{B}} \subseteq$ FIN, a contradiction.

Find $Q \subseteq P$ such that $\mathcal{C} = \mathcal{B} \upharpoonright Q$ is a barrier on Q. Let

$$\mathcal{D} = \{s \cup u \,:\, s \in \mathcal{B} \upharpoonright Q,\, u \in \mathcal{C} \text{ such that } \varphi(s) \sqsubseteq u \text{ and } s < (u \setminus \varphi(s))\}.$$

It is easy to see that $\mathcal{D}$ is a front on Q. So, let $R \subseteq Q$ be such that $\mathcal{D} \upharpoonright R$ is a barrier on R. Observe that every $t \in \mathcal{D}$ has attached $s(t) \in \mathcal{B} \upharpoonright Q$ and $u(t) \in \mathcal{C} \upharpoonright Q$ such that $t = s(t) \cup u(t)$ with $s(t) \sqsubseteq t$. Define $\varpi : \mathcal{D} \upharpoonright R \to \mathcal{B}$ by picking $\varpi(t) \in \mathcal{B}$ such that $\varphi(\varpi(t)) = u(t)$. Find $S \subseteq R$ such that $\varpi(t) \cap S \subseteq t$ for every $t \in \mathcal{D} \upharpoonright S$. Apply Proposition 8 to $u : \mathcal{D} \upharpoonright S \to \mathcal{C} \upharpoonright S$ to find $T \subseteq S$ and $\bar{\sigma} : \mathcal{D} \upharpoonright T \to \mathcal{D} \upharpoonright S$ such that $u \circ \bar{\sigma} = u$, and $\bar{\sigma}(t) \cap T = u(t)$ for every $t \in \mathcal{D} \upharpoonright T$.

For each $s \in \mathcal{B} \upharpoonright T$ choose $u_s \in \mathcal{C} \upharpoonright T$ such that $\varphi(s) \sqsubseteq u_s$ and $s < (u_s / \varphi(s))$, and define $\sigma : \mathcal{B} \upharpoonright T \to \mathcal{B} \upharpoonright S$ by $\sigma(s) = \bar{\sigma}(s \cup u_s)$. Then for every $s \in \mathcal{B} \upharpoonright T$,

$$\varphi(\sigma(s)) = \varphi(\varpi(\bar{\sigma}(s \cup u_s))) = u(\bar{\sigma}(s \cup u_s)) = u(s \cup u_s) = u_s \sqsupseteq \varphi(s),$$

so

$$\varphi(s) \subseteq \sigma(s) \cap s = (\varpi(\bar{\sigma}(s \cup u_s)) \cap N) \cap s \subseteq \bar{\sigma}(s \cup u_s) \cap s \subseteq u_s \cap s = \varphi(s). \tag{16}$$

Finally, apply Lemma 3 to σ to get $N \subseteq T$ such that $\sigma(s) \cap N \subseteq s$ for every $s \in \mathcal{B} \upharpoonright N$. Then, by (16),

$$\varphi(s) \subseteq \sigma(s) \cap N \subseteq \sigma(s) \cap s = \varphi(s).$$

■

We finish this subsection, presenting the following well-known structural result of Pudlak-Rödl [26] concerning mappings defined on barriers.

Definition 12 *Suppose that $\mathcal{B}$ is a barrier on M, and suppose that $f : \mathcal{B} \to A$, A is an arbitrary set. f is called* canonical *if there is a barrier $\mathcal{C}$ on M, $\bar{f} : \mathcal{C} \to A$ and an internal $\varphi : \mathcal{B} \to \mathcal{C}$ such that*
(a) $\bar{f}$ is 1-1, and
(b) $f = \bar{f} \circ \varphi$.

Theorem 5 (Pudlak-Rödl) *For every barrier $\mathcal{B}$ on M and every $f : \mathcal{B} \to A$, A an arbitrary set, there is $N \subseteq M$ such that f is canonical when restricted to $\mathcal{B} \upharpoonright N$.*

In the next section we will use the following consequence of Pudlak-Rödl's Theorem. Before we introduce some useful notation: Given $f : \mathcal{F} \to Y$, where $\mathcal{F} \subseteq$ FIN and Y is an arbitrary set, and $t \in$ FIN, we define $f_t : \mathcal{F}_t \to \mathbb{N}$ by $f_t(s) = f(t \cup s)$.

Corollary 8 *Suppose that $\mathcal{B}$ is a barrier on M, $f : \mathcal{B} \to \mathbb{R}$. Then for every $\varepsilon > 0$ there is an infinite subset N of M such that one of the following two conditions happens:*
(a) $\operatorname{osc}(f \upharpoonright (\mathcal{B} \upharpoonright N)) \leq \varepsilon$, i.e. $|f(s) - f(\bar{s})| \leq \varepsilon$ for every $s, \bar{s} \in \mathcal{B} \upharpoonright N$.
(b) For every integer k there is a finite subset t of N such that $\operatorname{osc}(f_t \upharpoonright (\mathcal{B}_t \upharpoonright N)) \leq \varepsilon$ and $|f_t(s)| > k|t|$ $(s \in \mathcal{B}_t \upharpoonright N)$.

PROOF. We may assume (by the Ramsey property of $\mathcal{B}$) that $f : \mathcal{B} \to \mathbb{R}^+$. Let $(I_n)_{n \in \mathbb{N}}$ be any partition of $\mathbb{R}^+$ into disjoint intervals $I_n = [a_n, b_n)$ of diameter at most ε. Let $g : \mathcal{B} \to \mathbb{N}$ be defined by $g(s) = n$ iff $f(s) \in I_n$. By Pudlak-Rödl Theorem there is $N_0 \subseteq M$ such that g restricted to $\mathcal{B} \upharpoonright N_0$ is a canonical mapping. So, fix a barrier $\mathcal{C}$ on N_0, $\sigma : \mathcal{B} \upharpoonright N_0 \to \mathcal{C}$ ($\sigma(s) \subseteq s$) and a 1-1 mapping $\bar{g} : \mathcal{C} \to \mathbb{N}$ such that $g = \bar{g} \circ \sigma$. If $\mathcal{C} = \{\emptyset\}$, then g is clearly constant, and so we obtain (a). Suppose that $\mathcal{C} \neq \{\emptyset\}$. Find $N \subseteq N_0$ such that the mapping $\varpi : \mathcal{B} \to c_{00}$ defined by $\varpi(s) = \chi_{\{\max \sigma(s)\}}$ and σ (i.e. χ_σ) are uniform on $\mathcal{B} \upharpoonright N$. We claim that N satisfies (b): Fix an integer k. Let $s_0 \in \mathcal{B} \upharpoonright N$ and $t_0 = s_0 \cap [0, \max \sigma(s_0))$. By definition, $\varpi(s_0)(\max \sigma(s_0)) = 1$. So, since σ and ϖ are uniform we have that

$$\sigma(t_0 \cup u) = (\sigma(s_0) \setminus \{\max \sigma(s_0)\}) \cup \{\min u\} \quad (u \in \mathcal{B}_{t_0} \upharpoonright N).$$

Using this and the fact that $\bar{g}$ is 1-1 we may find $u_0 \in \mathcal{B}_{t_0} \upharpoonright N$ such that

$$a_{\bar{g}(\sigma(t_0 \cup u_0))} > k(|t_0| + 1).$$

Finally set $t = t_0 \cup \{\min u_0\}$. Then, by Lipschitzness of σ, we have that for every $u \in \mathcal{B}_t \upharpoonright N$, $\sigma(t \cup u) = \sigma(t_0 \cup u_0)$, so $g(t \cup u) = g(t_0 \cup u_0)$ and hence

$$f(t \cup u) \geq a_{g(t \cup u)} = a_{g(t_0 \cup u_0)} > k(|t_0| + 1) = k|t|.$$

■

3.4. Mapping from barriers into c_0

In this subsection we continue with our study of mapping defined on barriers but we make a weaker restriction on the nature of their ranges.

Definition 13 *Let $\mathcal{F} \subseteq$ FIN and $\varphi : \mathcal{F} \to c_0$ be given.*
(a) φ is called semi-Lipschitz *iff for every $t \in$ FIN*

$$\{\varphi(s) \upharpoonright t \,:\, t \sqsubseteq s, \text{ and } s \in \mathcal{F}\} \text{ is finite.} \tag{17}$$

(b) φ is called semi-uniform *iff for every $t \in$ FIN*

$$\{\varphi(s)(\min(s/t)) \,:\, t \sqsubseteq s, \text{ and } s \in \mathcal{F}\} \text{ is finite.} \tag{18}$$

We extend now the corresponding definitions of the beginning of this section.
(c) φ is called Lipschitz *iff for every $t \in$ FIN*

$$|\{\varphi(s) \upharpoonright t \,:\, t \sqsubseteq s, \text{ and } s \in \mathcal{F}\}| = 1. \tag{19}$$

(d) φ is called uniform *iff for every $t \in$ FIN*

$$|\{\varphi(s)(\min(s/t)) \,:\, t \sqsubseteq s, \text{ and } s \in \mathcal{F}\}| = 1. \tag{20}$$

(e) φ is called internal *iff* $\operatorname{supp} \varphi(s) \subseteq s$ *for every $s \in \mathcal{F}$.*
(f) φ is called a L-mapping *iff it is internal and Lipschitz.*
(g) φ is called a U-mapping *iff it is internal and uniform.*

Remark 4 *(a) Observe that the above conditions* (17), (18), (19), *and* (20) *is non-vacuous if $t \in \overline{\mathcal{F}}^{\sqsubseteq}$.*
(b) We have the following implications:

$$\begin{aligned} \textit{uniform} &\Rightarrow \textit{semi-uniform} \Rightarrow \textit{semi-Lipschitz, and} \\ \textit{Lipschitz} &\Rightarrow \textit{semi-Lipschitz.} \end{aligned}$$

(c) If $\varphi : \mathcal{F} \to c_0$ is a L-mapping, then we can naturally extend it to the compact family $\overline{\mathcal{F}}^{\sqsubseteq}$ just by declaring $\varphi(t) = \varphi(s) \upharpoonright t$ where $s \in \mathcal{B}$ is such that $t \sqsubseteq s$. Note that the extension is continuous, so if in addition $\mathcal{F}$ is pre-compact, φ"$(\overline{\mathcal{F}}^{\sqsubseteq})$ is a weakly-compact subset of c_{00}.

We have the following reverse implications of (b) before:

Proposition 9 *(a) For every barrier $\mathcal{B}$ on M and every semi-Lipschitz mapping $\varphi : \mathcal{B} \to c_{00}$ there is $N \subseteq M$ such that φ is a Lipschitz mapping when restricted to $\mathcal{B} \restriction N$.*
(b) For every barrier $\mathcal{B}$ on M and every semi-uniform mapping $\varphi : \mathcal{B} \to c_{00}$ there is $N \subseteq M$ such that φ is a uniform mapping when restricted to $\mathcal{B} \restriction N$.

PROOF. (a): Find a fusion sequence (M_k) of subsets of M such that for every k we have that for every $t \subseteq \{m_0, \dots, m_k\}$, $t \in \overline{\mathcal{B}}^{\sqsubseteq}$ the mapping

$$f_t : \mathcal{B}_t \restriction M_{k+1} \to c_{00}$$
$$u \mapsto f_t(u) = \varphi(t \cup u) \restriction t.$$

is constant. Observe that this is possible because the range of f_t is finite $\mathcal{B}_t$ is a barrier on M/t. Then the fusion set M_∞ of (M_k) has the desired property: Fix $t \in \overline{\mathcal{B} \restriction M_\infty}^{\sqsubseteq}$. Let k be the first integer such that $t \subseteq \{m_0, \dots, m_k\}$. Observe that, by definition of M_∞, we know that $\mathcal{B}_t \restriction M_\infty \subseteq \mathcal{B}_t \restriction M_{k+1}$, so for every $s, u \in \mathcal{B} \restriction M_\infty$, if $t \sqsubseteq s, u$, then

$$\varphi(s) \restriction t = f_t(s \setminus t) = f_t(u \setminus t) = \varphi(u) \restriction t,$$

that proves that φ restricted to $\mathcal{B} \restriction M_\infty$ is Lipschitz.

(b): The proof is quite similar than for (a), so we only sketch it: Find a fusion sequence (M_k) of subsets of M such that for every k we have that for every $t \subseteq \{m_0, \dots, m_k\}$, $t \in \overline{\mathcal{B}}^{\sqsubseteq}$ the mapping

$$g_t : \mathcal{B}_t \restriction M_{k+1} \to \mathbb{R}$$
$$u \mapsto g_t(u) = \varphi(t \cup u)(\min u).$$

is constant. Then the fusion set M_∞ has the desired property. ■

Definition 14 *Given $\boldsymbol{\varepsilon} \downarrow 0$, let $I_n(\boldsymbol{\varepsilon}) = \{k\varepsilon_n \,:\, k \in \mathbb{Z}\}$. We define $\tau_{\boldsymbol{\varepsilon}} : c_0 \to c_0$ by $\tau_{\boldsymbol{\varepsilon}}(\boldsymbol{a})(n) = k\varepsilon_n \in I_n(\boldsymbol{\varepsilon})$ iff*

$$|k|\varepsilon_n \leq |\boldsymbol{a}(n)| < |k+1|\varepsilon_n.$$

*We say that $\varphi : \mathcal{F} \to c_0$ is $\boldsymbol{\varepsilon}$-*Lipschitz *iff $\tau_{\boldsymbol{\varepsilon}} \circ \varphi$ is Lipschitz. We say that φ is* almost-Lipschitz *if for every $\boldsymbol{\varepsilon}$ and every M there is $N \subseteq M$ such that φ is $\boldsymbol{\varepsilon}$-Lipschitz when restricted to $\mathcal{F} \restriction N$.*

Remark 5 *(a) $\tau_{\boldsymbol{\varepsilon}}$ preserves the supports, and $\tau_{\boldsymbol{\varepsilon}}(\boldsymbol{a} \restriction s) = \tau_{\boldsymbol{\varepsilon}}(\boldsymbol{a} \restriction s)$.*
(b) If $\tau_{\boldsymbol{\varepsilon}}(\boldsymbol{a}) = \pi_{\boldsymbol{\varepsilon}}(\boldsymbol{b})$, then $|\boldsymbol{a}(n) - \boldsymbol{b}(n)| \leq \varepsilon_n$ for every n.
(c) For every bounded $\varphi : \mathcal{F} \to c_0$, the corresponding composition $\tau_{\boldsymbol{\varepsilon}} \circ \varphi$ is semi-Lipschitz: Let $\lambda = \sup_{s \in \mathcal{F}} \|\varphi(s)\|_\infty$. For every $t \in \overline{\mathcal{F}}$ we have that

$$\{\tau_{\boldsymbol{\varepsilon}}(\varphi(s)) \restriction t \,:\, s \in \mathcal{F}\} \subseteq \{\boldsymbol{a} \in c_{00} \,:\, \operatorname{supp} \boldsymbol{a} \subseteq t, \text{ and } \boldsymbol{a}(n) \in I_n(\boldsymbol{\varepsilon}) \cap [-\lambda, \lambda] \text{ for every } n\},$$

and $\{\boldsymbol{a} \in c_{00} \,:\, \operatorname{supp} \boldsymbol{a} \subseteq t, \text{ and } \boldsymbol{a}(n) \in I_n(\boldsymbol{\varepsilon}) \cap [-\lambda, \lambda] \text{ for every } n\}$ is a finite set.
(d) If $\varphi : \mathcal{F} \to c_0$ is $\boldsymbol{\varepsilon}$-Lipschitz, then for every $t \in \overline{\mathcal{F}}$ and every $s, u \in \mathcal{F}$ such that $t \sqsubseteq s, u$

$$|\varphi(s)(n) - \varphi(u)(n)| \leq \varepsilon_n \text{ for every } n \in t:$$

As $\tau_{\boldsymbol{\varepsilon}} \circ \varphi$ is Lipschitz, $\tau_{\boldsymbol{\varepsilon}}(\varphi(s)) \restriction t = \tau_{\boldsymbol{\varepsilon}}(\varphi(u)) \restriction t$, so, by (a),

$$\tau_{\boldsymbol{\varepsilon}}(\varphi(s) \restriction t) = \tau_{\boldsymbol{\varepsilon}}(\varphi(u)) \restriction t = \tau_{\boldsymbol{\varepsilon}}(\varphi(u) \restriction t).$$

And by (b) we are done.
(e) If φ is Lipschitz, then φ is $\boldsymbol{\varepsilon}$-Lipschitz for every $\boldsymbol{\varepsilon}$, and hence almost-Lipschitz.
(f) If φ is almost-Lipschitz then for every $\varepsilon > 0$ there is N such that for every $t \in \overline{\mathcal{F} \restriction N}$ and every $s, u \in \mathcal{F} \restriction N$ with $t \sqsubseteq s, u$ we have that

$$\|\varphi(s) \restriction t - \varphi(u) \restriction t\|_{\ell_1} \leq \varepsilon.$$

Corollary 9 *Suppose that $\mathcal{B}$ is a barrier on M and $\varphi : \mathcal{B} \to c_0$ is bounded. Then*
(a) φ is almost-Lipschitz.
(b) For every $\boldsymbol{\varepsilon} \downarrow 0$ there is an infinite set $N \subseteq M$ and a Lipschitz $\varpi : \mathcal{B} \upharpoonright N \to c_0$ such that for every $s \in \mathcal{B} \upharpoonright N$

$$|\varphi(s)(n) - \varpi(s)(n)| \leq \varepsilon_n \text{ for every } n.$$

PROOF. (a) and (b): $\tau_{\boldsymbol{\varepsilon}} \circ \varphi$ is semi-Lipschitz, so, by Proposition 9 (a), there is $N \subseteq M$ such that $\tau_{\boldsymbol{\varepsilon}} \circ \varphi$ is Lipschitz restricted to $\mathcal{B} \upharpoonright N$, i.e. φ is $\boldsymbol{\varepsilon}$-Lipschitz when restricted to $\mathcal{B} \upharpoonright N$, and $|\tau_{\boldsymbol{\varepsilon}}(\varphi(s))(n) - \varphi(s)(n)| \leq \varepsilon_n$ for every $s \in \mathcal{B}$ and every n. ■

In the next given an infinite $N \subseteq \mathbb{N}$ and $n \in N$, n^+ denotes the immediate successor of n in N defined by $n^+ = \min(N/n)$.

Proposition 10 *Let $\mathcal{B}$ be a uniform barrier on M and $\varphi : \mathcal{B} \to c_0$ with weakly-pre-compact range. Then for every $\boldsymbol{\varepsilon} \downarrow 0$ there is an infinite set $N \subseteq M$ such that $\chi_N \cdot (\tau_{\boldsymbol{\delta}} \circ \varphi)$ is internal i.e., for every $s \in \mathcal{B} \upharpoonright N$ and $n \in N \setminus s$*

$$|\varphi(s)(n)| \leq \delta_n,$$

and where $\boldsymbol{\delta}$ is defined by $\delta_{\min N} = \varepsilon_0$, and $\delta_{n^+} = \varepsilon_n$ for every $n \in N$.

PROOF. For $k \in \mathbb{N}$ define $\varphi_k : \mathcal{B} \to \mathrm{FIN}$ for $s \in \mathcal{B}$ by

$$\varphi_k(s) = \mathrm{supp}_{\varepsilon_k} \varphi(s)$$

which is an element of the pre-compact subset $\mathrm{supp}_{\varepsilon_k}(\varphi"\mathcal{B})$ of FIN. So φ_k fulfills the conditions for Lemma 4 to be applied. So, there is some $N \subseteq M$ such that for every $n \in N$ and every $s \in \mathcal{B} \upharpoonright N$

$$\begin{aligned} \varphi_n(s) \cap (N \setminus s) &\subseteq [0, n], \\ \varphi_0(s) \cap (N \setminus s) &= \emptyset. \end{aligned}$$

Hence for every $s \in \mathcal{B} \upharpoonright N$ and every $n^+ \in N \setminus s$, $n^+ \notin \varphi_n(s)$, so $|\varphi(s)(n^+)| < \varepsilon_n$, while if $n = \min N \notin s$, $|\varphi(s)(n)| < \varepsilon_0$. ■

Corollary 10 *For every $\varphi : \mathcal{B} \to K$ with $\mathcal{B}$ a uniform barrier on M and $K \subseteq c_0$ weakly-pre-compact and every $\varepsilon > 0$ there is $N \subseteq M$ such that for every $s \in \mathcal{B} \upharpoonright N$,*

$$\|\varphi(s) \upharpoonright (N \setminus s)\|_{\ell_1} \leq \varepsilon.$$

PROOF. Apply previous Proposition 10 to a sequence $\boldsymbol{\varepsilon} \downarrow 0$ such that $\sum_n \varepsilon_n < \varepsilon$. ■

The previous result gives the following consequence concerning weakly-null sequences. The earliest result of this sort appearing in the literature seems Lemma 4.6 of [28].

Corollary 11 *Suppose that $\boldsymbol{x}$ is a weakly-null sequence of a Banach space X, $\mathcal{B}$ is a barrier on some infinite set M, and $f : \mathcal{B} \to B_{X^*}$, $f(s) = x_s^*$. Then for every $\varepsilon > 0$ there is an infinite subset N of M such that for every $s \in \mathcal{B} \upharpoonright N$*

$$\|x_s^*(\boldsymbol{x}) \upharpoonright (N \setminus s)\|_{\ell_1} \leq \varepsilon. \tag{21}$$

PROOF. Apply Corollary 10 to the mapping $s \in \mathcal{B} \mapsto x_s^*(\boldsymbol{x}) \in K(\boldsymbol{x}) \subseteq c_0$. ■

Remark 6 *By applying the previous Corollary to the simple case of $\mathcal{B} = \mathbb{N}^{[1]}$, we obtain the following:*
(a) If for every n we pick $x_n^* \in B_{X^*}$ then there is N such that

$$\|x_n^*(\boldsymbol{x}) \upharpoonright (N \setminus \{n\})\|_{\ell_1} \le \varepsilon,$$

for all $n \in N$, i.e., $(x_n)_{n\in N}$ and (x_n^*) are ε-orthogonal.
(b) If $\boldsymbol{x}$ is a semi-normalized weakly-null sequence then for every $\varepsilon > 0$ there is a subsequence $\boldsymbol{x} \upharpoonright M$ such that

$$\|\boldsymbol{a} \cdot \boldsymbol{x} \upharpoonright M\| \ge (1-\varepsilon)(\inf_{n\in\mathbb{N}} \|x_n\|)\|\boldsymbol{a}\|_\infty$$

for every $\boldsymbol{a} \in c_{00} \upharpoonright M$.

We finish with the following approximation result concerning U-mappings.

Proposition 11 *Suppose that $\mathcal{B}$ is a barrier on M, and $\varphi : \mathcal{B} \to K$ is a bounded mapping with weakly-pre-compact range $\subseteq c_0$.*
(a) Suppose that $\boldsymbol{\varepsilon} = (\varepsilon_n) \downarrow 0$ is such that

$$\sum_{n>m} \varepsilon_n < \varepsilon_m \text{ for every } m \in \mathbb{N}. \tag{22}$$

Then there is $N \subseteq M$ and a U-mapping $\psi : \mathcal{B} \upharpoonright N \to c_{00}$ such that for every $s \in \mathcal{B} \upharpoonright N$

$$\|\psi(s) - \varphi(s) \upharpoonright s\|_{\ell_1} \le \varepsilon_{\min s}.$$

(b) For every $\varepsilon > 0$ there is $N \subseteq M$ a U-mapping $\psi : \mathcal{B} \upharpoonright N \to c_{00}$ such that for every $s \in \mathcal{B} \upharpoonright N$

$$\|\psi(s) - \varphi(s) \upharpoonright N\|_{\ell_1} \le \varepsilon.$$

PROOF. Let $\lambda = \sup_{\boldsymbol{a}\in K} \|\boldsymbol{a}\|_\infty$. Choose for each n a finite subset I_n of $[-\lambda, \lambda]$ that contains 0 and that is $\varepsilon_n/2$-dense set. Now define an internal $\delta : \mathcal{B} \to c_{00}$ by for $s \in \mathcal{B}$ and $n \in s$,

$$\delta(s)(n) = \min s + |s \cap [0,n)|$$

Observe that the natural composition $\boldsymbol{\varepsilon} \circ \delta$ defined by $(\boldsymbol{\varepsilon} \circ \delta)(s)(n) = \boldsymbol{\varepsilon}(\delta(n))$ has the property that for every $s \in \mathcal{B}$

$$\|(\boldsymbol{\varepsilon} \circ \delta)(s)\|_{\ell_1} = \sum_{n\in s} \boldsymbol{\varepsilon}(\delta(s)(n)) \le 2\varepsilon_{\min s}. \tag{23}$$

Define the internal mapping $\psi : \mathcal{B} \to c_{00}$ as follows: For $s \in \mathcal{B}$ and $n \in s$ pick $\psi(s)(n) \in I_{k(s,n)}$ such that

$$|\varphi(s)(n) - \psi(s)(n)| \le \varepsilon_{k(s,n)}, \tag{24}$$

and $\psi(s)(n) = 0$ if $\varphi(s)(n) = 0$. Notice that for every $t \in \overline{\mathcal{B}}$ and every $s, u \in \mathcal{B}$ such that $t \sqsubseteq s, u$ we have that

$$\begin{aligned} k(s, \min(s\setminus t)) =& \min s + |s \cap [0, \min(s \setminus t))| = \min t + |t| = \min u + |u \cap [0, \min(s\setminus t))| = \\ =& k(u, \min(u \setminus t)), \end{aligned}$$

so, by definition of ψ,

$$\psi(s)(\min(s \setminus t)), \psi(u)(\min(u\setminus t)) \in I_{\min t + |t|}.$$

This implies that ψ is semi-uniform mapping because each I_n is finite. Then we apply Proposition 9 (b) to ψ to obtain $N \subseteq M$ such that ψ is uniform when restricted to $\mathcal{B} \upharpoonright N$. Then, for every $s \in \mathcal{B} \upharpoonright N$ we have, by (23) and (24), that

$$\|\psi(s) - \varphi(s) \upharpoonright s\|_{\ell_1} \le \frac{1}{2}\|(\boldsymbol{\varepsilon} \circ \delta)(s)\|_{\ell_1} \le \varepsilon_{\min s},$$

as desired.

To show (b), first use (a) for a fast decreasing sequence $\boldsymbol{\varepsilon}$ such that $\boldsymbol{\varepsilon} \le \varepsilon/2$, and then apply Proposition 10 to the corresponding restriction of φ. ∎

3.5. Mappings defined on infinite cubes and with countable range

By a "cube" we mean the set $M^{[\infty]}$ of all infinite subsets of some infinite set $M \subseteq \mathbb{N}$. In this section we apply the theory developed so far to treat mappings with domains of the form $M^{[\infty]}$ and with countable range. We start with the following well known notions.

Definition 15 *The topology on $\mathbb{N}^{[\infty]}$ is the induced by the topology on $2^{\mathbb{N}}$, i.e. $U \subseteq \mathbb{N}^{[\infty]}$ is open iff for every $A \in U$ there is a finite $s \sqsubseteq A$ such that $\langle s\rangle \subseteq U$, where*

$$\langle s\rangle = \{A \in \mathbb{N}^{[\infty]} \,:\, s \sqsubseteq A\}.$$

Every cube $M^{[\infty]}$ is a closed subset of $\mathbb{N}^{[\infty]}$. We say that a mapping $F : M^{[\infty]} \to C$, where C is an arbitrary set, is continuous, if it is topologically continuous when $M^{[\infty]}$ carries its natural topology and C is with its discrete topology. We say that F is Borel if F is a topologically Borel mapping.

We recall the following well-known result of Galvin-Prikry [15].

Theorem 6 (Galvin-Prikry) *Every Borel subset of $\mathbb{N}^{[\infty]}$ is* Ramsey *i.e., if $\mathcal{X} \subseteq \mathbb{N}^{[\infty]}$ is Borel then there is some infinite set M such that either $M^{[\infty]} \subseteq \mathfrak{X}$ or else $M^{[\infty]} \cap \mathfrak{X} = \emptyset$.* ■

This was latter extended independently by Ellentuck [11] and Silver [30] to analytic subsets, and in the first case giving a topological interpretation of the Ramsey property.

Remark 7 *The result of Nash-Williams in [25] stating that every barrier is Ramsey (see Definition 1) is equivalent to the fact that clopen subsets of $\mathbb{N}^{[\infty]}$ are Ramsey. It is worth to point out that the Ramsey property for open subsets is equivalent to Galvin's Lemma.*

The following shows that Borel mappings with countable range are, when restricted to some cube, automatically continuous.

Proposition 12 *Suppose that $F : M^{[\infty]} \to C$, is a Borel mapping, C countable. Then there is $N \in M^{[\infty]}$ such that F is continuous when restricted to $N^{[\infty]}$.*

PROOF. Enumerate $C = \{c_k\}_{k\in\mathbb{N}}$. Find a fusion sequence (M_k) consisting on subsets of M such that for every k and every $t \subseteq \{m_0, \dots, m_k\}$ the coloring

$$\begin{aligned} \Phi_{t,k} : M_{k+1}^{[\infty]} &\to \{0,1\} \\ N &\mapsto 1 \text{ iff } \Phi(s \cup N) = c_k \end{aligned} \tag{25}$$

is constant. It is clear that this can be done, as $F^{-1}\{c_k\} \cap \langle t\rangle$ is Borel. Let $M_\infty = \{m_k\}_{k\in\mathbb{N}}$ be the fusion limit. We claim that F is continuous when restricted to $M^{[\infty]}$: Fix $k \in \mathbb{N}$, and suppose that $N \in M_\infty^{[\infty]}$ is such that $F(N) = c_k$. Set $t = N \cap \{m_0, \dots, m_k\}$. Then $N \setminus t \in M_{k+1}$, and $\Phi_{t,k}(N) = 1$, so $F(P) = c_k$ for every $P \subseteq M_\infty$ such that $t \sqsubseteq P$. ■

Corollary 12 *Suppose that $F : M^{[\infty]} \to C$ is a Borel mapping with countable range C. Then there is $N \subseteq M$, a barrier $\mathcal{B}$ on N and $f : \mathcal{B} \to$ FIN such that $F \restriction N^{[\infty]} = f \circ \iota_{\mathcal{B}}$.*

PROOF. First use Proposition 12 to obtain $N \subseteq M$ such that F is continuous when restricted to $N^{[\infty]}$. Let

$$\mathcal{F} = \{s \in N^{[<\infty]} \,:\, F \restriction (\langle s\rangle \cap N) \text{ is constant}\}.$$

Let $\mathcal{G}$ be the set $\mathcal{F}^{\sqsubseteq-\min}$ of $\sqsubseteq$-minimal elements of $\mathcal{F}$. This is a thin family. So by Theorem 1, there is $P \subseteq N$ such that $\mathcal{G} \restriction N$ is either a barrier on P or empty. The second possibility is impossible since it implies that $\mathcal{F} \restriction N$ is also empty and this is contradictory with the continuity of F on $P^{[\infty]}$. Let $\mathcal{B} = \mathcal{G} \restriction P$. Define $f : \mathcal{B} \to C$ by $f(s) = F(s \cup (P/s))$. It is easy to check now that $F \restriction P^{[\infty]} = f \circ \iota_{\mathcal{B}}$. ■

We can extend now, using previous Corollary, most of the result proved for barriers.

Corollary 13 *Suppose that $F : \mathbb{N}^{[\infty]} \to \mathrm{FIN}$ is Borel. Then there is an infinite set $M \subseteq \mathbb{N}$ such that $\chi_M \cdot F \upharpoonright M^{[\infty]}$ is internal, i.e. $F(N) \cap M \subseteq N$ for every $N \in M^{[\infty]}$.*

PROOF. Apply Lemma 3 to the corresponding f given by Corollary 12 when applied to F. ■

Corollary 14 *$\Phi : M^{[\infty]} \to \mathrm{FIN}$ Borel and internal. Then there is an infinite set M and $\Sigma : M^{[\infty]} \to M^{[\infty]}$ such that*
(a) $\Phi \circ \Sigma \sqsubseteq \Phi$, and
(b) $\chi_M \cdot \Sigma = \Phi$. ■

Corollary 15 *$\Phi : M^{[\infty]} \to \mathrm{FIN}$ Borel and internal. Then there is an infinite set M and $\Sigma : M^{[\infty]} \to M^{[\infty]}$ such that*
(a) $\Phi \sqsubseteq \Phi \circ \Sigma$, and
(b) $\chi_M \cdot \Sigma = \Phi$. ■

Corollary 16 *Suppose that $\boldsymbol{x}$ is a weakly-null sequence of a Banach space X, $\Phi : \mathbb{N}^{[\infty]} \to B_{X^*}$ is an arbitrary Borel mapping with countable range. Then for every $\varepsilon > 0$ there is an infinite M such that for every $N \subseteq M$, $\sum_{n\in M\setminus N} |\Phi(N)(x_n)| \leq \varepsilon$.* ■

4. Partial unconditionality of weakly-null sequences

In this section we present the main results of this paper. We introduce and study an abstract notion of partial unconditionality and relate them to some known ones. As we shall see, our abstract notion of partial unconditionality will cover most the result about partial unconditionality found in the literature.

Definition 16 *Following the corresponding notions for families of finite sets, for a given $\mathfrak{F} \subseteq \mathrm{FIN} \times c_{00}$ such that if $(t, \boldsymbol{a}) \in \mathfrak{F}$ then $t \subseteq \operatorname{supp} a$ and $M \in \mathbb{N}^{[\infty]}$, we define the restriction $\mathfrak{F} \upharpoonright M$ and trace $\mathfrak{F}[M]$ of $\mathfrak{F}$ by*

$$\mathfrak{F} \upharpoonright M = \{(s, \boldsymbol{a}) \in \mathfrak{F} \,:\, \operatorname{supp} \boldsymbol{a} \subseteq M\}.$$
$$\mathfrak{F}[M] = \{(s \cap M, \boldsymbol{a} \upharpoonright M) \,:\, (s, \boldsymbol{a}) \in \mathfrak{F}\}.$$

The following definition is motivated by a similar definition that appears in [10].

Definition 17 *Let $\mathfrak{F} \subseteq \mathrm{FIN} \times c_{00}$ be such that $t \subseteq \operatorname{supp} \boldsymbol{a}$ for every $(t, \boldsymbol{a}) \in \mathfrak{F}$, and let $w : \mathrm{FIN} \times c_{00} \to \mathbb{R}^+$ be an arbitrary mapping. A sequence $\boldsymbol{x} = (x_n)_{n\in M}$ indexed on a set $N \subseteq \mathbb{N}$ of a given Banach space X is called $(\mathfrak{F}_w, C)$-unconditional ($C > 0$) iff for every pair $(s, \boldsymbol{a}) \in \mathfrak{F} \upharpoonright M$ we have that*

$$\|\boldsymbol{a} \upharpoonright t \cdot \boldsymbol{x}\|_X \leq C w(t, \boldsymbol{a}) \|\boldsymbol{a} \cdot \boldsymbol{x}\|_X. \tag{26}$$

We call the mapping w a weight. *The sequence $\boldsymbol{x}$ is $\mathfrak{F}_w$-unconditional iff it is $(\mathfrak{F}, C)$-unconditional for some $C > 0$.*

Define

$$\boldsymbol{C}(\mathfrak{F}_w, \boldsymbol{x}) = \inf\{C \,:\, \text{ there is a } (\mathfrak{F}_w, C)\text{-unconditional subsequence of } \boldsymbol{x}\}$$
$$\boldsymbol{C}(\mathfrak{F}_w) = \sup\{\boldsymbol{C}(\mathfrak{F}_w, \boldsymbol{x}) \,:\, \boldsymbol{x} \text{ is a semi-normalized weakly-null sequence}\}.$$

Definition 18 *Given $\mathfrak{F} \subseteq \mathrm{FIN} \times c_{00}$, a weight w and a sequence $\boldsymbol{x}$ indexed on a set N and $C > 0$, define*

$$\mathfrak{X}_C(\mathfrak{F}_w, \boldsymbol{x}) = \{M \in N^{[\infty]} \,:\, \boldsymbol{x} \upharpoonright M \text{ is } (\mathfrak{F}_w, C)\text{-unconditional}\}.$$

Remark 8 *(a) It is easy to show that*

$$\boldsymbol{C}(\mathfrak{F}_w, \boldsymbol{x}) = \sup\{C \,:\, \mathfrak{X}_C(\mathfrak{F}_w, \boldsymbol{x}) = \emptyset\}. \tag{27}$$

Observe that $\{C \,:\, \mathfrak{X}_C(\mathfrak{F}_w, \boldsymbol{x}) = \emptyset\}$ *is an initial interval of* $\mathbb{R}^+$ *(i.e. if* $C' < C$ *and* $\mathfrak{X}_C(\mathfrak{F}_w, \boldsymbol{x}) = \emptyset$*, then* $\mathfrak{X}_{C'}(\mathfrak{F}_w, \boldsymbol{x}) = \emptyset$*).*
(b) For every C*, if* $\boldsymbol{C}(\mathfrak{F}_w, \boldsymbol{x}) > C$*, then* $\mathfrak{X}_C(\mathfrak{F}_w, \boldsymbol{x}) = \emptyset$*. This fact will be used quite often.*

We give some examples to illustrate the previous definition of $\mathfrak{F}_w$-unconditionality:

Example 1 *(1) Let*

$$\mathfrak{F}^{triv} = \{(\operatorname{supp} \boldsymbol{a}, \boldsymbol{a}) \,:\, \boldsymbol{a} \in c_{00}\}.$$

Then clearly $\boldsymbol{C}(\mathfrak{F}^{triv}) = 1$*. We call this set the* trivial *set.*
(2) Let

$$\mathfrak{F}^{\infty} = \{(t, \boldsymbol{a}) \in \mathrm{FIN} \times c_{00} \,:\, t \subseteq \operatorname{supp} \boldsymbol{a} \textit{ and } |t| = 1\},$$

and let $w \equiv \lambda^{-1}$*, with* $0 < \lambda \leq 1$*. Suppose that* $\boldsymbol{x}$ *is a semi-normalized sequence of* X *with* $\lambda = \inf_{n\in\mathbb{N}} \|x_n\|$*. Then* $\boldsymbol{x}$ *is* $(\mathfrak{F}^{\infty}_w, C)$*-unconditional iff*

$$\|\boldsymbol{a} \cdot \boldsymbol{x}\|_X \geq \frac{\lambda}{C} \|\boldsymbol{a}\|_\infty,$$

for every $\boldsymbol{a} \in c_{00}$*. So, in Remark 6 (b) we have just proved that* $\boldsymbol{C}(\mathfrak{F}^{\infty}_w, \boldsymbol{x}) = 1$ *for those sequences* $\boldsymbol{x}$ *with* $\inf_{n\in\mathbb{N}} \|x_n\| = \lambda$*.*
(3) Let

$$\mathfrak{F}^{in} = \{(t, \boldsymbol{a}) \in \mathrm{FIN} \times c_{00} \,:\, t \sqsubseteq \operatorname{supp} \boldsymbol{a}\}.$$

Then being $\mathfrak{F}^{in}$*-unconditional just means being a basic sequence.*
(4) Let

$$\mathfrak{F}^{bi} = \{(t, \boldsymbol{a}) \in \mathrm{FIN} \times c_{00} \,:\, t \textit{ is an interval of } \operatorname{supp} \boldsymbol{a}\}.$$

Then $(\mathfrak{F}^{bi}, 1)$*-unconditionality just means that the corresponding sequence is a bi-monotone basis.*
(5) Let

$$\mathfrak{U} = \{(s, \boldsymbol{a}) \in \mathrm{FIN} \times c_{00} \,:\, s \subseteq \operatorname{supp} \boldsymbol{a}\}.$$

Then $\mathfrak{F}$*-unconditionality just means unconditionality.*
(6) Suppose that $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *are two sequences of* X *and* Y *respectively. Let*

$$w(t, \boldsymbol{a}) = \frac{\|\boldsymbol{a} \cdot \boldsymbol{x}\|_X}{\|\boldsymbol{a} \cdot \boldsymbol{y}\|_Y}.$$

Then $\boldsymbol{x}$ *is* $(\mathfrak{T}_w, C)$ *unconditional iff the linear extension of the mapping* $y_n \mapsto x_n$ *defines a bounded mapping with norm at most* C *from the closed linear span of* $\boldsymbol{y}$ *into the one of* $\boldsymbol{x}$*.*

4.1. Test of $\mathfrak{F}_w$-unconditionality

In this section we treat the problem of when a given weakly-null sequence have a subsequence that is $\mathfrak{F}$-unconditional and show how to estimate the corresponding constant C. First, we start with the following classical result that will illustrate the strategy we follow.

We give a proof of the classical result of Bessaga and Pełczyński [7] that states that every semi-normalized weakly-null sequence has a basic subsequence. Observe that this is equivalent to say that $\boldsymbol{C}(\mathfrak{I}_{in}, \boldsymbol{x}) < \infty$ for every semi-normalized weakly-null sequence $\boldsymbol{x}$.

Proposition 13 *Suppose that* $\boldsymbol{x}$ *is a semi-normalized w-null sequence. Then for every* $\varepsilon > 0$ *there is some subsequence* $\boldsymbol{x} \upharpoonright M$ *that is a* $1+\varepsilon$*-basic sequence such that in addition* $\|\boldsymbol{a} \cdot \boldsymbol{x} \upharpoonright M\| \geq (1-\varepsilon) \inf_{n\in\mathbb{N}} \|x_n\| \cdot \|\boldsymbol{a}\|_\infty$ *for every* $\boldsymbol{a} \in c_{00}$*.*

PROOF. Fix a semi-normalized weakly-null sequence $\boldsymbol{x}$ of a Banach space X. The last required property follows from Remark 6 (b). So, we assume, by going to a subsequence if needed, that for a fixed λ, $\|\boldsymbol{a}\cdot\boldsymbol{x}\| \geq \lambda\|\boldsymbol{a}\|_\infty$ for every $\boldsymbol{a}\in c_{00}$. Now, as mentioned before, to be a $(1+\varepsilon)$-basic sequence is equivalent to be $(\mathfrak{F}_{in}, 1+\varepsilon)$-unconditional, where $\mathfrak{F}^{in}$ is the set of pairs $(t,\boldsymbol{a})\in \mathrm{FIN}\times c_{00}$ such that t is an initial subset of $\operatorname{supp}\boldsymbol{a}$. Our goal is to show that the corresponding unconditional constant $\mathfrak{C}(\mathfrak{F}_{in},\boldsymbol{x})$ is 1. Otherwise, let $1<C<\mathfrak{C}(\mathfrak{F}^{in})$. Then $\mathfrak{X}_C(\mathfrak{F}^{in},\boldsymbol{x})=\emptyset$ (see Remark 8). This means that for every infinite $M\subseteq\mathbb{N}$ we can find $(t_M,\boldsymbol{a}^M)\in\mathfrak{F}^{in}\restriction M$ such that

$$\|\boldsymbol{a}^M\restriction t_M\cdot\boldsymbol{x}\| > C\|\boldsymbol{a}^M\cdot\boldsymbol{x}\|.$$

It is clear that we may assume, after normalizing, that $\|\boldsymbol{a}^M\cdot\boldsymbol{x}\|=1$ (the set $\mathfrak{F}^{in}$ is closed under multiplication by scalars). Let

$$\mathcal{F}=\{\operatorname{supp}\boldsymbol{a}^M \,:\, M\in\mathbb{N}^{[\infty]}\}.$$

The family $\mathcal{G}=\mathcal{F}^{\subseteq-\min}$ of minimal set of $\mathcal{F}$ is clearly a Sperner family (see Definition 1), hence by Theorem 1, there is an infinite set M such that $\mathcal{G}\restriction M$ is either a barrier on M or empty. The last possibility is impossible since it implies that $\mathcal{F}\restriction M=\emptyset$ and we know that $\operatorname{supp}\boldsymbol{a}^M\subseteq M$. Set $\mathcal{B}=\mathcal{G}\restriction M$. We have naturally defined the mapping $s\mapsto(t_s,\boldsymbol{a}^s,x_s^*)$ from $\mathcal{B}$ into $\mathfrak{F}\restriction M\times S_{X^*}$ with the property that

$$|x_s^*(\boldsymbol{a}^s\restriction t_s\cdot\boldsymbol{x})| = \|\boldsymbol{a}^s\restriction t_s\cdot\boldsymbol{x}\| > C,$$

where $t_s=t_M\sqsubseteq s$ and $\boldsymbol{a}^s=\boldsymbol{a}^M$ for some M such that $s=\operatorname{supp}\boldsymbol{a}^M$. The sequence $\boldsymbol{x}$ is seminormalized and weakly-null, so the mapping $s\in\mathcal{B}\mapsto x_s^*(\boldsymbol{x})\in c_0$ is, by Corollary 9, almost-Lipschitz. Fix then and infinite subset $N\subseteq M$ such that for every $t\in\overline{\mathcal{B}\restriction N}$ and every $s,u\in\mathcal{B}\restriction N$ such that $t\sqsubseteq s,u$ we have that

$$\|x_s^*(\boldsymbol{x})\restriction t - x_u^*(\boldsymbol{x})\restriction t\|_{\ell_1}\leq (C-1)\lambda.$$

Now fix $s\in\mathcal{B}$, let $v\in(\mathcal{B}_{t_s})\restriction(N/s)$, and set $u=t_s\cup v\in\mathcal{B}\restriction N$. Then $\|x_s^*(\boldsymbol{x})\restriction t_s - x_u^*(\boldsymbol{x})\restriction t_s\|_{\ell_1}\leq (C-1)\lambda$, and so,

$$\begin{aligned}1=\|\boldsymbol{a}^s\cdot\boldsymbol{x}\| \geq&|x_u^*(\boldsymbol{a}^s\cdot\boldsymbol{x})| = |x_u^*(\boldsymbol{a}^s\restriction t_s\cdot\boldsymbol{x})| \geq |x_s^*(\boldsymbol{a}^s\restriction t_s\cdot\boldsymbol{x})| - (C-1)\lambda\|\boldsymbol{a}^s\restriction t_s\|_\infty > \\ &> C\|\boldsymbol{a}^s\cdot\boldsymbol{x}\| - (C-1)\|\boldsymbol{a}^s\cdot\boldsymbol{x}\| = 1,\end{aligned}$$

hence $1>1$, a contradiction. ■

The following result is the test of $\mathfrak{F}_w$-unconditionality. Again, the kind of assignment in the condition (b) of the following lemma is modeled after similar ones appearing in [10], Theorem 12.

Lemma 7 *Fix $\mathfrak{F}\subseteq\mathrm{FIN}\times c_{00}$, a weight w, an arbitrary sequence $\boldsymbol{x}$ of a Banach space X, indexed on a set $N\in\mathbb{N}^{[\infty]}$. The following are equivalent:*
(a) $\mathfrak{X}_C(\mathfrak{F}_w,\boldsymbol{x})=\emptyset$.
(b) For every $P\subseteq N$ there is $M\subseteq P$, a uniform barrier $\mathcal{B}$ on M and

$$\varphi:\mathcal{B}\to\mathfrak{F}\restriction M\times S_{X^*},$$

$\varphi(s)=(t_s,\boldsymbol{a}^s,x_s^*)$ such that for every $s\in\mathcal{B}$:
(b.1) $t_s\subseteq s=\operatorname{supp}\boldsymbol{a}^s$, and
(b.2) $|x_s^*(\boldsymbol{a}^s\restriction t_s\cdot\boldsymbol{x})| = \|\boldsymbol{a}^s\restriction t_s\cdot\boldsymbol{x}\| > Cw(t_s,\boldsymbol{a}^s)\|\boldsymbol{a}^s\cdot\boldsymbol{x}\|$.

If in addition the sequence $\boldsymbol{x}$ is weakly-null, then for every $\varepsilon>0$ we can obtain φ with the extra property that

$$\|x_s^*(\boldsymbol{x})\restriction(M\setminus s)\|_{\ell_1}\leq\varepsilon \text{ for every } s\in\mathcal{B}.$$

PROOF. Suppose that (b) holds. Then for every $P \subseteq N$ there is $(t, \boldsymbol{a}) \in \mathfrak{F} \upharpoonright P$ such that

$$\|\boldsymbol{a} \upharpoonright t \cdot \boldsymbol{x}\| = \|\boldsymbol{a} \upharpoonright t \cdot \boldsymbol{x}\| > Cw(t, \boldsymbol{a})\|\boldsymbol{a} \cdot \boldsymbol{x}\| = Cw(t, \boldsymbol{a})\|\boldsymbol{a} \cdot \boldsymbol{x}\|,$$

so $\boldsymbol{x} \upharpoonright P$ is not $(\mathfrak{F}_w, C)$-unconditional.

Suppose now that (a) holds. We follow the lines used in the proof of Proposition 13. Fix $P \subseteq N$. It is clear that $\mathfrak{X}_C(\mathfrak{F}_w, \boldsymbol{x} \upharpoonright P) = \emptyset$, so for every infinite subset $M \subseteq P$ there is $(t_M, \boldsymbol{a}^M) \in \mathfrak{F} \upharpoonright M$ such that $\|\boldsymbol{a}^M \upharpoonright t_M \cdot \boldsymbol{x}\| > Cw(t_M, \boldsymbol{a}^M)\|\boldsymbol{a}^M \cdot \boldsymbol{x}\|$. Define

$$\mathcal{F} = \{\operatorname{supp} \boldsymbol{a}^M \ : \ M \in P^{[\infty]}\}.$$

The family $\mathcal{G} = \mathcal{F}^{\sqsubseteq-\min}$ is Sperner, so by Theorem 1 there is an infinite set $M \subseteq P$ such that $\mathcal{G} \upharpoonright M$ is either a uniform barrier or empty. Observe that this last possibility is not possible as it implies that $\mathcal{F} \upharpoonright M = \emptyset$ while $\operatorname{supp} \boldsymbol{a}^M \subseteq M$. Call $\mathcal{B} = \mathcal{G} \upharpoonright M$. Now for each $s \in \mathcal{B}$ choose $N(s) \in M^{[\infty]}$ such that $\operatorname{supp} \boldsymbol{a}^{N(s)} = s$. This leads us to the mapping $\varphi : \mathcal{B} \to \mathfrak{F} \upharpoonright M \times S_{X^*}$ defined for $s \in \mathcal{B}$ by

$$\varphi(s) = (t_{N(s)}, \boldsymbol{a}^{N(s)}, x_s^*),$$

where $x_s^* \in S_{X^*}$ is such that

$$|x_s^*(\boldsymbol{a}^{N(s)} \upharpoonright t_{N(s)} \cdot \boldsymbol{x})| = \|\boldsymbol{a}^{N(s)} \upharpoonright t_{N(s)} \cdot \boldsymbol{x}\|.$$

It is clear that φ has the desired properties.

If in addition $\boldsymbol{x}$ is weakly-null, then we apply Corollary 11 to $s \mapsto x_s^*$ to obtain the additional property. ■

In some cases the next is useful.

Corollary 17 *Fix $\mathfrak{F} \subseteq \mathrm{FIN} \times c_{00}$, and a weight w with the property that if $(t_0, \boldsymbol{a}), (t_1, \boldsymbol{a}) \in \mathfrak{F}$ are such that $t_0 \subseteq t_1$, then $w(t_0, \boldsymbol{a}) \leq w(t_1, \boldsymbol{a})$. Suppose that $\boldsymbol{x}$ is a sequence of a Banach space X, indexed on a set $N \subseteq \mathbb{N}$. The following are equivalent:*
(a) $\mathfrak{X}_C(\mathfrak{F}_w, \boldsymbol{x}) = \emptyset$.
(b) For every $P \subseteq N$ there is $M \subseteq P$, a uniform barrier $\mathcal{B}$ on M and

$$\varphi : \mathcal{B} \to M^{[<\infty]} \times c_{00} \upharpoonright M \times S_{X^*},$$

$\varphi(s) = (t_s, \boldsymbol{a}^s, x_s^*)$ with the property that for every $s \in \mathcal{B}$:
(b.1) $\operatorname{supp} \boldsymbol{a}^s = s$, $t_s \subseteq s$,
(b.2) $t_s \subseteq u_s$ for some u_s such that $(u_s, \boldsymbol{a}^s) \in \mathfrak{F}$,
(b.3) $\boldsymbol{a}^s \upharpoonright t_s$ and $x_s^*(\boldsymbol{x}) \upharpoonright t_s$ are sequences of constant signs, independents of s, and finally
(b.4) $|x_s^*(\boldsymbol{a}^s \upharpoonright t_s \cdot \boldsymbol{x})| = \|\boldsymbol{a}^s \upharpoonright t_s \cdot \boldsymbol{x}\| > (C/2)w(t_s, \boldsymbol{a}^s)\|\boldsymbol{a}^s \cdot \boldsymbol{x}\|$.

If in addition the sequence $\boldsymbol{x}$ is weakly-null, then for every $\varepsilon > 0$ we can obtain φ with the extra property that $\|x_s^*(\boldsymbol{x}) \upharpoonright (M \setminus s)\|_{\ell_1} \leq \varepsilon$ for every $s \in \mathcal{B}$.

PROOF. Use Lemma 7 to find corresponding $\mathcal{B}$, M and φ defined on $\mathcal{B}$. Now fix $s \in \mathcal{B}$, and observe that

$$|x_s^*(\boldsymbol{a}^s \upharpoonright t_s \cdot \boldsymbol{x})| \leq \max\{|x_s^*(\sum_{n \in t_s, a_n^s x_s^*(x_n) > 0} a_n^s x_n)|, |x_s^*(\sum_{n \in t_s, a_n^s x_s^*(x_n) < 0} a_n^s x_n)|\}. \tag{28}$$

Let $\bar{t}_s$ be either equal to $\{n \in t_s \ : \ a_n^s x_s^*(x_n) > 0\}$ or to $\{n \in t_s \ : \ a_n^s x_s^*(x_n) < 0\}$ and it has the property that $|x_s^*(\boldsymbol{a}^s \upharpoonright t_s \cdot \boldsymbol{x})| \leq |x_s^*(\boldsymbol{a}^s \upharpoonright \bar{t}_s \cdot \boldsymbol{x})|$. Let $i_s = 1, -1$ be the sign of $a_n^s x_s^*(x_n)$ in $\bar{t}_s$. Now use that

$$|x_s^*(\boldsymbol{a}^s \upharpoonright \bar{t}_s \cdot \boldsymbol{x})| \leq 2\max\{|x_s^*(\sum_{n \in \bar{t}_s, a_n^s, i_s x_s^*(x_n) > 0} a_n^s x_n)|, |x_s^*(\sum_{n \in \bar{t}_s, a_n^s, i_s x_s^*(x_n) < 0} a_n^s x_n)|\}. \tag{29}$$

Finally choose $\bar{\bar{t}}_s \subseteq \bar{t}_s$ such that both $\boldsymbol{a}^s \upharpoonright \bar{\bar{t}}_s$ and $x_s^*(\boldsymbol{x}) \upharpoonright \bar{\bar{t}}_s$ have constant signs and with the property that $|x_s^*(\boldsymbol{a}^s \upharpoonright \bar{t}_s \cdot \boldsymbol{x})| \leq 2|x_s^*(\boldsymbol{a}^s \upharpoonright \bar{\bar{t}}_s \cdot \boldsymbol{x})|$. Use the Ramsey property of $\mathcal{B}$ to find an infinite subset $R \subseteq M$ such that these two signs are independent of s. The desired mapping is $s \in \mathcal{B} \upharpoonright R \mapsto (\bar{\bar{t}}_s, \boldsymbol{a}^s, x_s^*) \in R^{[<\infty]} \times c_{00} \upharpoonright R \times S_{X^*}$. ■

We present now known results of partial unconditionality.

4.2. Schreier unconditionality

Recall that $\mathcal{S}$ is the family of finite sets $s \subseteq \mathbb{N}$ such that $|s| \le \min s$. This is the closure of the ω-uniform barrier on $\mathbb{N}$ consisting on the $\subseteq$-maximal elements of $\mathcal{S}$ (i.e. the sets s such that $|s| = \min s$). We are going to show that for every weakly-null sequence $\boldsymbol{x}$ and every $\varepsilon > 0$ there is a subsequence $\boldsymbol{x} \upharpoonright M$ such that

$$\|\boldsymbol{a} \upharpoonright t \cdot \boldsymbol{x} \upharpoonright M\| \le (2+\varepsilon)\|\boldsymbol{a} \cdot \boldsymbol{x} \upharpoonright M\|$$

for every $t \in \mathcal{S} \upharpoonright M$ and every $\boldsymbol{a} \in c_{00}$. This was first announced in [24] and a proof is given in [28].

Finally, observe that the previous result is equivalent to say that $\boldsymbol{C}(\mathfrak{S}, \boldsymbol{x}) \le 2$, where

$$\mathfrak{S} = \{(s, \boldsymbol{a}) \,:\, s \in \mathcal{S} \text{ and } s \subseteq \operatorname{supp} \boldsymbol{a}\}.$$

Proposition 14 $\boldsymbol{C}(\mathfrak{S}) = 2$.

PROOF. In the Subsection 4.6. we give a normalized weakly-null sequence with no unconditional subsequence. This sequence has the additional property that $\boldsymbol{C}(\mathfrak{S}, \boldsymbol{x}) \ge 2$. This shows that $\boldsymbol{C}(\mathfrak{S}) \ge 2$. Let us prove now that $\boldsymbol{C}(\mathfrak{S}) \le 2$. Otherwise, fix a semi-normalized weakly-null sequence $\boldsymbol{x}$ and C such that $\boldsymbol{C}(\mathfrak{S}, \boldsymbol{x}) > C > 2$. Fix also $\varepsilon > 0$ arbitrary. We assume, by Proposition 13, that $\boldsymbol{x}$ is a $(1+\varepsilon)$-basic sequence such that $\|\boldsymbol{a} \cdot \boldsymbol{x}\| \ge \lambda(1-\varepsilon)\|\boldsymbol{a}\|_\infty$ for every $\boldsymbol{a} \in c_{00}$, and where $0 < \lambda = \inf_n \|x_n\|$.

By Lemma 7, there is a barrier $\mathcal{B}$ on some M, and a mapping $s \mapsto (t_s, \boldsymbol{a}^s, x_s^*)$ such that for every $s \in \mathcal{B}$,
(a) $t_s \in \mathcal{S}$, $t_s \subseteq s = \operatorname{supp} \boldsymbol{a}^s$, $\|\boldsymbol{a}^s\|_\infty \le 1$
(b) $\|x_s^*(\boldsymbol{x}) \upharpoonright (M \setminus s)\|_{\ell_1} \le \varepsilon$, and
(c) $|x_s^*(\boldsymbol{a}^s \upharpoonright t_s \cdot \boldsymbol{x})| > C\|\boldsymbol{a}^s \cdot \boldsymbol{x}\|$.

As this last inequality is linear (i.e. changing $\boldsymbol{a}^s$ by $\lambda \boldsymbol{a}^s$ makes no difference) we assume that $\|\boldsymbol{a}^s\|_\infty \le 1$ ($s \in \mathcal{B}$). We also assume, by Proposition 4, that $s \mapsto t_s$ is Lipschitz. Now fix arbitrary $s_0 \in \mathcal{B}$ and set $k = \min t_{s_0}$, $v = s_0 \cap [0,k]$. Observe that if $w \in \mathcal{B}_v$, then $\min t_{v\cup w} = \min t_{s_0} = k$, so , as $t_{v \cup w} \in \mathcal{S}$, $|t_{v\cup w}| \le k$. By the Ramsey property of $\mathcal{B}_v$ we can assume that there is some fixed $0 \le l_0 \le k$ such that $|t_{v \cup w}| = l_0$ for every $w \in \mathcal{B}_v$. Let D be a finite ε-dense subset of $[-1,1]^{l_0}$ with the ℓ_1-norm. For each $w \in \mathcal{B}_w$ find $(c_i)_{i<l_0} \in D$ in a way that $\sum_{i<l_0} |x^*_{(v\cup w)}(x_{n_i}) - d_i| \le \varepsilon$, where $\{n_i\}_{i<l_0}$ is the increasing enumeration of $t_{v\cup w}$. As D is finite, we assume that it is constant. This implies that if $w_0, w_1 \in \mathcal{B}_v$ are such that $t_{v \cup w_0} = t_{v\cup w_1}$, then

$$\|x^*_{(v\cup w_0)}(\boldsymbol{x}) \upharpoonright t_{(v\cup w_0)} - x^*_{(v\cup w_0)}(\boldsymbol{x}) \upharpoonright t_{(v \cup w_1)}\|_{\ell_1} \le 2\varepsilon. \tag{30}$$

Finally, we apply Proposition 8 to the mapping $w \in \mathcal{B}_u \mapsto t_{v\cup w} \cap w \in M^{[l_0]}$ to obtain $w_0, w_1 \in \mathcal{B}_v$ such that
(d) $\rho(w_0) = \rho(w_1)$ and
(e) $w_0 \cap \bar{w}_1 = \rho(w)$.

Set $s = v \cup w_0$, $u = v \cup w_1$, and $\bar{v} = v \setminus \{\min v\}$. Then (c) means that $t_s = t_u$, while (e) means that $s \cap u = v \cup t_s$.

Using what we know so far, we compute:

$$\begin{aligned}|x_s^*(\boldsymbol{a}^u) \upharpoonright (u \setminus \bar{v})| \ge & |x_s^*(\boldsymbol{a}^u) \upharpoonright ((u \cap s) \setminus \bar{v})| - |x_s^*(\boldsymbol{a}^u) \upharpoonright ((u \setminus s)| \ge \\ \ge & |x_s^*(\boldsymbol{a}^u) \upharpoonright ((u\cap s) \setminus \bar{v})| - \varepsilon\|\boldsymbol{a}^u\|_\infty = |x_s^*(\boldsymbol{a}^u) \upharpoonright t_s| - \varepsilon \|\boldsymbol{a}^u\|_\infty \ge \\ \ge & |x_u^*(\boldsymbol{a}^u) \upharpoonright t_u| - 3\varepsilon\|\boldsymbol{a}^u\|_\infty > (C - \frac{3\varepsilon}{\lambda(1-\varepsilon)})\|\boldsymbol{a}^u \cdot \boldsymbol{x}\|.\end{aligned} \tag{31}$$

Hence, as $\boldsymbol{x}$ is a $(1+\varepsilon)$-basic sequence and $\bar{v} \sqsubseteq u$,

$$\begin{aligned}\|\boldsymbol{a}^u \cdot \boldsymbol{x}\| \ge & \frac{1}{2+\varepsilon}\|\boldsymbol{a}^u \upharpoonright (u \setminus \bar{v}) \cdot \boldsymbol{x}\| \ge \frac{1}{2+\varepsilon}|x_s^*(\boldsymbol{a}^u) \upharpoonright (u \setminus \bar{v})| > \\ > & \frac{1}{2+\varepsilon}(C - \frac{3\varepsilon}{\lambda(1-\varepsilon)})\|\boldsymbol{a}^u \cdot \boldsymbol{x}\|,\end{aligned} \tag{32}$$

so

$$C < 2 + \varepsilon + \frac{3\varepsilon}{\lambda(1-\varepsilon)}. \tag{33}$$

As λ is fix and ε is arbitrary, we conclude that $C \leq 2$, a contradiction. ■

4.3. Near and convex unconditionality

In this subsection we present applications of our methods to treat Elton's notion of near unconditionality [12] and the notion convex unconditionality of Argyros, Mercourakis and Tsarpalias [4]. In fact we are going to see that there is a single combinatorial results that lies behind both of these two notions of partial unconditionality. Recall that in the literature these two forms of partial unconditionality are usually treated differently though sometimes convex unconditionality is referred as the "dual version" of near unconditionality. Our approach will give an explanation of this duality phenomenon as well. Another explanation of this duality appears in [10] where it is shown that the corresponding unconditional constants behave similarly.

Recall that a sequence $\boldsymbol{x}$ is called *δ-near unconditional* $(0 < \delta \leq 1)$ iff there is a constant $C \geq 1$ depending on δ such that for every $\boldsymbol{a} \in c_{00}$ with $\|\boldsymbol{a}\|_\infty \leq 1$ and every $t \subseteq \operatorname{supp} \boldsymbol{a}$ such that $\delta \leq \min_{n\in t} |\boldsymbol{a}(n)|$ we have that $\|\boldsymbol{a} \upharpoonright t \cdot \boldsymbol{x}\| \leq C\|\boldsymbol{a}\cdot\boldsymbol{x}\|$. A result of Elton ([12]) states that for every $0 < \delta \leq 1$ every normalized weakly-null sequence has a δ-near unconditional subsequence.

We say that $\boldsymbol{x}$ is *δ-convex unconditional* iff there is a constant $C \geq 1$ such that for every $\boldsymbol{a} \in c_{00}$ and every $t \subseteq \operatorname{supp} a$ with $\delta\|\boldsymbol{a} \upharpoonright t\|_{\ell_1} \leq \|\boldsymbol{a} \upharpoonright t \cdot \boldsymbol{x}\|$, then we have that $\|\boldsymbol{a} \upharpoonright t \cdot \boldsymbol{x}\| \leq C\|\boldsymbol{a}\cdot\boldsymbol{x}\|$. Argyros, Mercourakis and Tsarpalias ([4]) have shown that for every $0 < \delta \leq 1$ every normalized weakly-null sequence has a δ-convex unconditional subsequence.

For a real number $r \geq 1$ let $\log_2(r) = \min\{n \in \mathbb{N} : r \leq 2^n\}$. We introduce two notions of oscillation of vectors of c_{00}.

Definition 19 *Fix a non-zero $\boldsymbol{a} \in c_{00}$*

$$\begin{aligned}\operatorname{osc}_0(\boldsymbol{a}) =& \frac{\|\boldsymbol{a}\|_\infty}{\min_{n\in\operatorname{supp} a}|a_n|}\\ \operatorname{osc}_1^{\boldsymbol{x}}(\boldsymbol{a}) =& \frac{\|\boldsymbol{a}\|_{\ell_1}}{\|\boldsymbol{a}\cdot\boldsymbol{x}\|_X},\end{aligned}$$

where $\boldsymbol{x}$ is a fixed semi-normalized weakly-null sequence of a Banach space X.

Two simple observations:

Remark 9 *(a) Both notions of oscillation are invariants under multiplication by scalars, i.e. $\operatorname{osc}_0(\lambda\boldsymbol{a}) = \operatorname{osc}_0(\boldsymbol{a})$ and $\operatorname{osc}_1^{\boldsymbol{x}}(\lambda\boldsymbol{a}) = \operatorname{osc}_1^{\boldsymbol{x}}(\boldsymbol{a})$ for non-zero $\boldsymbol{a} \in c_{00}$ and $\lambda \in \mathbb{R}$.*
(b) $\operatorname{osc}_0(\boldsymbol{a} \upharpoonright s) \leq \operatorname{osc}_0(\boldsymbol{a})$ and $\operatorname{osc}_1^{\boldsymbol{x}}(\boldsymbol{a} \upharpoonright s) \leq \operatorname{osc}_1^{\boldsymbol{x}}(\boldsymbol{a})$ for every $\boldsymbol{a} \in c_{00}$ and $s \in$ FIN.

The aim of this subsection is to prove the following

Theorem 7 (Near and Convex unconditionality) *Suppose that $\boldsymbol{x}$ is a semi-normalized weakly-null sequence of a Banach space X. Then for every $\varepsilon > 0$ there is some M such that for every sequence of scalars $\boldsymbol{a}$ and every finite subset $s \subseteq M$,*

$$\|\boldsymbol{a} \upharpoonright s \cdot \boldsymbol{x} \upharpoonright M\| \leq (8+\varepsilon)\max\{1, \log_2(\operatorname{osc}_0(\boldsymbol{a} \upharpoonright s))\}\, \|\boldsymbol{a}\cdot\boldsymbol{x} \upharpoonright M\| \text{ and} \tag{34}$$

$$\|\boldsymbol{a} \upharpoonright s \cdot \boldsymbol{x} \upharpoonright M\| \leq (16+\varepsilon)\,(1 + \log_2(\operatorname{osc}_1^{\boldsymbol{x}}(\boldsymbol{a} \upharpoonright s)))\, \|\boldsymbol{a}\cdot\boldsymbol{x} \upharpoonright M\|. \tag{35}$$

Before we give a proof, we need two more combinatorial results.

Theorem 8 *Let $\mathcal{B}$ be a barrier on M, and let $s \mapsto (t_s, \boldsymbol{a}^s, \boldsymbol{b}^s)$ is a mapping such that for every $s \in \mathcal{B}$*
(a) $t_s, \operatorname{supp} \boldsymbol{a}^s, \operatorname{supp} \boldsymbol{b}^s \subseteq s$,
(b) $\|\boldsymbol{a}^s\|_\infty, \|\boldsymbol{b}^s\|_\infty \leq 1$, and
(c) $\boldsymbol{a}^s \upharpoonright t_s$ and $\boldsymbol{b}^s \upharpoonright t_s$ have constant signs.

Then for every $\varepsilon > 0$ there are $s, u \in \mathcal{B}$ such that

$$|\langle \boldsymbol{a}^s, \boldsymbol{b}^u \rangle| \geq \frac{1}{4+\varepsilon} \frac{|\langle \boldsymbol{a}^s \upharpoonright t_s, \boldsymbol{b}^s \rangle|}{\max\{1, \log_2(\operatorname{osc}_0(\boldsymbol{a}^s \upharpoonright t_s))\}} - \varepsilon. \tag{36}$$

PROOF. The proof has two parts depending on the following to functions $f_\varphi, g_\varphi, h_\varphi : \mathcal{B} \to \mathbb{N}$ defined as follows

$$\begin{aligned} f_\varphi(s) =& \max\{1, \log_2 (\operatorname{osc}_0(\boldsymbol{a}^s \upharpoonright t_s))\} \\ g_\varphi(s) =& \|\boldsymbol{a}^s \upharpoonright t_s\|_\infty \\ h_\varphi(s) =& \frac{f(s)}{g(s)}. \end{aligned}$$

Notice that $g \leq 1 \leq f$ and $h \geq 1$. Observe that h is bounded (i.e. $\sup_{s\in\mathcal{B}} f(s) < \infty$) iff f is upper-bounded and g is lower bounded.

CASE 1. h bounded. In this case we the following stronger result: For every $\varepsilon > 0$ there are $s, u \in \mathcal{B}$ such that

$$|\langle \boldsymbol{a}^s, \boldsymbol{b}^u \rangle| \geq \frac{1}{2+\varepsilon} \frac{|\langle \boldsymbol{a}^s \upharpoonright t_s, \boldsymbol{b}^s \rangle|}{\max\{1, \log_2(\operatorname{osc}_0(\boldsymbol{a}^s \upharpoonright t_s))\}}. \tag{37}$$

As $f : \mathcal{B} \to \mathbb{N}$ is bounded, By the Ramsey property of $\mathcal{B}$, we may assume that f is constant with value L. Let $\bar{\varepsilon} > 0$ be such that $(2+\varepsilon)(1-\bar{\varepsilon}) > 2$. Let $r_0 \in \mathbb{N}$ be such that $(1-\bar{\varepsilon})^{r_0} < \inf_{s\in\mathcal{B}} g(s)$, and color each $s \in \mathcal{B}$ by $1 \leq r \leq r_0$ iff $(1-\bar{\varepsilon})^r < g(s) \leq (1-\bar{\varepsilon})^{r-1}$. By Ramsey, we assume also that $\mathcal{B}$ is monochromatic with constant color r. Observe that this implies that for every $s, u \in \mathcal{B}$

$$\frac{g(s)}{g(u)} \geq 1 - \bar{\varepsilon}. \tag{38}$$

For a given $s \in \mathcal{B}$ and $1 \leq l \leq L$, let

$$t_s^l = \{n \in t_s \, : \, \frac{g(s)}{2^l} \leq |a_n^s| \leq \frac{g(s)}{2^{l-1}}\}.$$

Observe that if $s, u \in \mathcal{B}$ and $n \in t_s^l \cap t_u^l$, then

$$\frac{a_n^s}{a_n^u} \geq \frac{\frac{g(s)}{2^l}}{\frac{g(u)}{2^{l-1}}} \geq \frac{(1-\bar{\varepsilon})}{2}. \tag{39}$$

We color each $s \in \mathcal{B}$ by $1 \leq l \leq L$ iff (l is the first such that)

$$\langle \boldsymbol{a}^s \upharpoonright t_s^l, \boldsymbol{b}^s \rangle \geq \frac{1}{L} \langle \boldsymbol{a}^s \upharpoonright t_s, \boldsymbol{b}^s \rangle,$$

(this is well defined because $t_s = \bigcup_{l=1}^L t_s^l$). By Ramsey, we may assume that $\mathcal{B}$ is monochromatic, with color l_0.

Now consider the mapping from $\mathcal{B}$ into FIN defined by $s \mapsto t_s^{l_0}$. By Corollary 6, there are $s, u \in \mathcal{B}$ such that
(c) $t_s^{l_0} \sqsubseteq t_u^{l_0}$ and
(d) $s \cap u = t_s^{l_0}$.

We compute:

$$\langle \boldsymbol{a}^u, \boldsymbol{b}^s \rangle = \sum_{n \in t_s^{l_0}} a_n^u b_n^s \geq \frac{1-\bar{\varepsilon}}{2} \sum_{n \in t_s^{l_0}} a_n^s b_n^s \geq \frac{1}{(2+\varepsilon)L} \langle \boldsymbol{a}^s \restriction t_s, \boldsymbol{a}^s \rangle$$

as desired.
CASE 2. h is unbounded. Since by Corollary 9 (a) the mapping $s \mapsto \boldsymbol{b}^s$ is almost-Lipschitz, we assume, after restricting to some appropriate subset, that for every $t \in \overline{\mathcal{B}}$ and every $s, u \in \mathcal{B}$ such that $t \sqsubseteq s, u$ we have that

$$\|\boldsymbol{b}^s \restriction t - \boldsymbol{b}^u \restriction t\|_{\ell_1} \leq \varepsilon.$$

Now use Corollary 8 to find some $N \subseteq M$ and $v \in \overline{\mathcal{B} \restriction N}$ such that
(a) $|h(s) - h(u)| \leq \varepsilon$ if $v \sqsubseteq s, u \in \mathcal{B} \restriction N$ and
(b) $|v| < \varepsilon h(s)$ for every $s \in \mathcal{B} \restriction N$ such that $v \sqsubseteq s$.

Notice that this implies that for every $s \in \mathcal{B} \restriction N$ such that $v \sqsubseteq s$

$$|\langle \boldsymbol{a}^s \restriction (t_s \cap v), \boldsymbol{b}^s \rangle| \leq |v| g_\varphi(s) \leq \varepsilon f(s). \tag{40}$$

Define $\bar{\varphi} : \mathcal{B}_v \restriction N \to \mathrm{FIN} \times c_{00} \times c_{00}$ by $\bar{\varphi}(w) = (t_{v\cup w} \cap w, \boldsymbol{a}^{v\cup w} \restriction w, \boldsymbol{b}^s \restriction v \cup w)$. This new mapping fulfills the conditions of Case 1 (i.e $H_{\bar{\varphi}}$ is bounded), so, there is $w, z \in \mathcal{B}_v \restriction N$ such that

$$|\langle \boldsymbol{a}^{v\cup w} \restriction w, \boldsymbol{b}^{v\cup z} \restriction z \rangle| \geq \frac{1}{2+\varepsilon} \frac{|\langle \boldsymbol{a}^{v\cup w} \restriction t_{v\cup w} \cap w, \boldsymbol{b}^{v\cup w} \rangle|}{f_{\bar{\varphi}}(w)}. \tag{41}$$

From this and (40) we have that

$$\begin{aligned} |\langle \boldsymbol{a}^{v\cup w} \restriction w, \boldsymbol{b}^{v\cup z} \restriction z \rangle| \geq & \frac{1}{2+\varepsilon} \left(\frac{|\langle \boldsymbol{a}^{v\cup w} \restriction (t_{v\cup w} \cap w), \boldsymbol{b}^{v\cup w} \rangle|}{f_{\bar{\varphi}}(w)} - \varepsilon \right) \geq \\ \geq & \frac{1}{2+\varepsilon} \left(\frac{|\langle \boldsymbol{a}^{v\cup w} \restriction (t_{v\cup w} \cap w), \boldsymbol{b}^{v\cup w} \rangle|}{f_{\varphi}(v \cup w)} - \varepsilon \right). \end{aligned} \tag{42}$$

Finally we consider two cases:
SUBCASE 2.1 $|\langle \boldsymbol{a}^{v\cup w}, \boldsymbol{b}^{v\cup z} \rangle| \geq (1/2) |\langle \boldsymbol{a}^{v\cup w} \restriction w, \boldsymbol{b}^{v\cup z} \restriction z \rangle|$. The desired inequality in Claim 1 follows immediately from (41).
SUBCASE 2.2 $|\langle \boldsymbol{a}^{v\cup w}, \boldsymbol{b}^{v\cup z} \rangle| < (1/2) |\langle \boldsymbol{a}^{v\cup w} \restriction w, \boldsymbol{b}^{v\cup z} \restriction z \rangle|$. This means that

$$|\langle \boldsymbol{a}^{v\cup w} \restriction v, \boldsymbol{b}^{v\cup z} \restriction v \rangle| > (1/2) |\langle \boldsymbol{a}^{v\cup w} \restriction w, \boldsymbol{b}^{v\cup z} \restriction z \rangle|. \tag{43}$$

Let $u \in \mathcal{B}$ be such that $v \sqsubseteq u$ and $u \cap s = v$. As we are assuming that $s \mapsto \boldsymbol{b}^s$ is $(1+\varepsilon)$-Lipschitz, we have that

$$|\langle \boldsymbol{a}^{v\cup w}, \boldsymbol{b}^u \rangle| = |\langle \boldsymbol{a}^{v\cup w} \restriction v, \boldsymbol{b}^u \rangle| \geq |\langle \boldsymbol{a}^{v\cup w} \restriction v, \boldsymbol{b}^{v\cup z} \rangle| - \varepsilon, \tag{44}$$

that together with (41) and (43) Gives the inequality in Claim 1. ∎

Corollary 18 *Suppose that $s \in \mathcal{B} \mapsto (t_s, \boldsymbol{a}^s, \boldsymbol{b}^s)$ is as in Theorem 8. Then for every $\varepsilon > 0$ there are $s, u \in \mathcal{B}$ such that*

$$|\langle \boldsymbol{a}^s, \boldsymbol{b}^u \rangle| \geq \frac{1}{8+\varepsilon} \frac{|\langle \boldsymbol{a}^s \restriction t_s, \boldsymbol{b}^s \rangle|}{1 + \log_2 \left(\frac{\|\boldsymbol{a}^s \restriction t_s\|_{\ell_1}}{|\langle \boldsymbol{a}^s, \boldsymbol{b}^s \restriction t_s \rangle|} \right)} - \varepsilon. \tag{45}$$

PROOF. Define

$$\bar{t}_s = \{ n \in t_s \, : \, b_n \geq \frac{1}{2} \frac{|\langle \boldsymbol{a}^s, b^s \restriction t_s \rangle|}{\|\boldsymbol{a}^s \restriction t_s\|_{\ell_1}} \}.$$

Observe that

$$|\langle \boldsymbol{a}^s \upharpoonright (t_s \setminus \bar{t}_s), \boldsymbol{b}^s\rangle| < \frac{1}{2}\frac{|\langle \boldsymbol{a}^s, \boldsymbol{b}^s \upharpoonright t_s\rangle|}{\|\boldsymbol{a}^s \upharpoonright t_s\|_{\ell_1}}\|\boldsymbol{a}^s \upharpoonright t_s\|_{\ell_1} = \frac{1}{2}|\langle \boldsymbol{a}^s, \boldsymbol{b}^s \upharpoonright t_s\rangle|, \tag{46}$$

so $|\langle \boldsymbol{a}^s \upharpoonright \bar{t}_s, \boldsymbol{b}^s\rangle| \geq (1/2)|\langle \boldsymbol{a}^s \upharpoonright t_s, \boldsymbol{b}^s\rangle|$. Also, by definition of $\bar{t}_s$,

$$\log_2(\mathrm{osc}_0(\boldsymbol{b}^s \upharpoonright t_s)) \leq \log_2\left(\frac{1}{m(\boldsymbol{b}^s \upharpoonright \bar{t}_s)}\right) \leq \log_2\left(\frac{\|\boldsymbol{a}^s \upharpoonright t_s\|_{\ell_1}}{|\langle \boldsymbol{a}^s, \boldsymbol{b}^s \upharpoonright t_s\rangle|}\right) + 1. \tag{47}$$

We apply Theorem 8 to $s \mapsto (\bar{t}_s, \boldsymbol{b}^s, \boldsymbol{a}^s)$ and $\bar{\varepsilon} = \varepsilon/2$, to obtain $s, u \in \mathcal{B}$ such that

$$|\langle \boldsymbol{b}^s, \boldsymbol{a}^u\rangle| \geq \frac{1}{8+\bar{\varepsilon}}\frac{|\langle \boldsymbol{b}^s \upharpoonright \bar{t}_s, \boldsymbol{a}^s\rangle|}{\max\{1, \log_2(\mathrm{osc}_0(\boldsymbol{b}^s \upharpoonright t_s))\}}.$$

This, together with (46) and (47) give the desired inequality. ■

We are now ready to give a proof of Theorem 7.

PROOF. First of all, observe that (34) is invariant under multiplication by scalars, i.e. the corresponding inequalities for $\boldsymbol{a}$ and for $\lambda\boldsymbol{a}$ are the same ($\lambda \in \mathbb{R}$). So we may assume that we deal with sequences of scalars $\boldsymbol{a}$ with $\|\boldsymbol{a}\|_\infty = 1$. Consider

$$\begin{aligned}\mathfrak{F} &= \{(s, \boldsymbol{a}) \in \mathrm{FIN} \times c_{00} \,:\, s \subseteq \mathrm{supp}\, a,\ \|\boldsymbol{a}\|_\infty = 1\}\\ w(t, \boldsymbol{a}) &= \min\{\max\{1, \log_2(\mathrm{osc}_0(\boldsymbol{a} \upharpoonright t))\}, 2(1 + \log_2(\mathrm{osc}_1^{\boldsymbol{x}}(\boldsymbol{a} \upharpoonright t)))\}.\end{aligned}$$

We show that $\boldsymbol{C}(\mathfrak{F}_w, \boldsymbol{x}) \leq 8$. Otherwise, fix C such that $8 < C < \boldsymbol{C}(\mathfrak{F}_w, \boldsymbol{x})$ and work to produce a contradiction. Fix $\varepsilon > 0$. As $\boldsymbol{x}$ is a semi-normalized weakly-null sequence we may assume, by Proposition 13 that $\|\boldsymbol{a} \cdot \boldsymbol{x}\| \geq (1-\varepsilon)L\|\boldsymbol{a}\|_\infty$ for every $\boldsymbol{a} \in c_{00}$, and where $L = \inf_{n\in\mathbb{N}} \|x_n\| \in (0, 1]$ is independent of ε. Observe also that $w(t_0, \boldsymbol{a}) \leq w(t_1, \boldsymbol{a})$ if $t_0 \subseteq t_1$, so we can apply By Corollary 17 to obtain an infinite set M, a uniform barrier $\mathcal{B}$ on M and $\varphi_0 : \mathcal{B} \to \mathrm{FIN} \times c_{00} \times S_{X^*}$ such that for every $s \in \mathcal{B}$, setting $\varphi_0(s) = (t_s, \boldsymbol{a}^s, x_s^*)$, we have that

(a) $t_s \subseteq \mathrm{supp}\, \boldsymbol{a}^s = s$,

(b) $\boldsymbol{a}^s \upharpoonright t_s$ and $(x_s^*(x_n))_{n\in t_s}$ have constant signs independent of s,

(c) $\|\boldsymbol{a}^s\|_\infty = 1$, and $\sum_{n\in M\setminus s} |x_s^*(x_n)| \leq \varepsilon L$ and

(d)

$$|x_s^*(\boldsymbol{a}^s \upharpoonright t_s \cdot \boldsymbol{x})| = \|\boldsymbol{a}^s \upharpoonright t_s \cdot \boldsymbol{x}\| > \frac{C}{2}w(t_s, \boldsymbol{a}^s)\|\boldsymbol{a}^s \cdot \boldsymbol{x}\|. \tag{48}$$

Also, by the Ramsey property of $\mathcal{B}$ we may assume that either

(e) $w(t_s, \boldsymbol{a}^s) = \max\{1, \log_2(\mathrm{osc}_0(\boldsymbol{a}^s \upharpoonright t_s))\}$ for every $s \in \mathcal{B}$, or else

(f) $w(t_s, \boldsymbol{a}^s) = 2(1 + \log_2(\mathrm{osc}_1^{\boldsymbol{x}}(\boldsymbol{a} \upharpoonright t))$ for every $s \in \mathcal{B}$.

Define on $\mathcal{B}$ the mapping φ by $\varphi(s) = (t_s, \boldsymbol{a}^s, (x^*(x_n))_{n\in s})$. We apply to φ and $\bar{\varepsilon} = \varepsilon L$ either Theorem 8 if (e) above holds, or else Corollary 18 to obtain $s, u \in \mathcal{B}$ such that

$$\begin{aligned}|x_u^*(\sum_{n\in s\cap u} a_n x_n)| &= |\langle \boldsymbol{a}^s, (x_u^*(x_n))_{n\in u}\rangle| \geq \frac{1}{4+\bar{\varepsilon}}\frac{\|\boldsymbol{a}^s \upharpoonright t_s \cdot \boldsymbol{x}\|}{w(t_s, \boldsymbol{a}^s)} - \bar{\varepsilon} > \frac{C}{8+2\bar{\varepsilon}}\|\boldsymbol{a}^s \cdot \boldsymbol{x}\| - \bar{\varepsilon} \geq\\ &\geq \frac{C}{8+2\bar{\varepsilon}}\|\boldsymbol{a}^s \cdot \boldsymbol{x}\| - \frac{\varepsilon}{1-\varepsilon}\|\boldsymbol{a}^s \cdot \boldsymbol{x}\| = \left(\frac{C}{8+2\bar{\varepsilon}} - \frac{\varepsilon}{1-\varepsilon}\right)\|\boldsymbol{a}^s \cdot \boldsymbol{x}\|,\end{aligned} \tag{49}$$

where we used that $\|\boldsymbol{a}^s \upharpoonright t_s \cdot \boldsymbol{x}\| = |\langle x_s^*(\boldsymbol{a}^s \upharpoonright t_s \cdot \boldsymbol{x})\rangle|$ for every $s \in \mathcal{B}$, and inequality (48). On the other hand we have that

$$\begin{aligned}\|\boldsymbol{a}^s \cdot \boldsymbol{x}\| &\geq |x_u^*(\sum_{n\in s} a_n^s x_n)| \geq |x_u^*(\sum_{n\in s\cap u} a_n^s x_n)| - \|\boldsymbol{a}^s\|_\infty \sum_{n\in s\setminus u} |x_u^*(x_n)| =\\ &= |x_u^*(\sum_{n\in s\cap u} a_n^s x_n)| - \frac{\varepsilon}{1-\varepsilon}\|\boldsymbol{a}^s \cdot \boldsymbol{x}\|.\end{aligned} \tag{50}$$

So, by (49) and (50),

$$(1+\frac{\varepsilon}{1-\varepsilon})\|\boldsymbol{a}^s\cdot\boldsymbol{x}\| \geq \left(\frac{C}{8+2\bar{\varepsilon}} - \frac{\varepsilon}{1-\varepsilon}\right)\|\boldsymbol{a}^s\cdot\boldsymbol{x}\|,$$

that clearly implies that $C<(8+2\varepsilon L)(1+\varepsilon)(1-\varepsilon)^{-1}$. As ε was arbitrary we conclude that $C\leq 8$, a contradiction. ■

Remark 10 *Observe that from Theorem 7 one obtains that for every $0<\delta\leq 1$ every normalized weakly-null sequence has a δ-unconditional subsequence with constant $C\leq 9\log_2(\delta^{-1})$. It is an open question if there is a universal constant C (i.e. independent of δ) such that for every $0<\delta\leq 1$ every normalized weakly-null sequence has a δ-unconditional subsequence with constant C. This conjecture is closely related to the corresponding fact for δ-convex unconditionality (see [10] for more details).*

4.4. Sequences bounded away from zero.

The intention of this subsection is to generalize the dichotomy concerning sequences of characteristic functions of $C(K)$ presented in Corollary 4 to sequences $\boldsymbol{x}\subseteq C(K)$ bounded away from zero, i.e.

$$\inf\{|x_n(\xi)| \,:\, |x_n(\xi)|\neq 0,\, \xi\in K\}>0.$$

More precisely, we have the following result proved independently and in different forms by A. Arvanitakis [6], I. Gasparis, E. Odell, and B. Wahl in [17] and the authors in [21].

Theorem 9 *Suppose that K is a compact space and suppose that $\boldsymbol{x}$ is a semi-normalized weakly-null sequence of $C(K)$ with the property that*

$$\inf\{|x_n(\xi)| \,:\, |x_n(\xi)|\neq 0,\, n\in\mathbb{N},\, \xi\in K\}=\delta>0. \tag{51}$$

Then there is some M and a uniform barrier $\mathcal{B}$ on $\mathbb{N}$ such that $\boldsymbol{x}\upharpoonright M$ is $2/\delta$-equivalent to the natural basis (e_n) of the Schreier space $X_{\mathcal{B}}$. In particular, $\boldsymbol{x}\upharpoonright M$ is unconditional.

PROOF. First of all, the set

$$L=\{\boldsymbol{x}(\xi)\,:\,\xi\in K\}\subseteq c_{00} \text{ is weakly-pre-compact.}$$

So, as we mentioned in the introduction, $\operatorname{supp}_\delta L$ is also pre-compact, but by (51), we have that

$$\operatorname{supp}_\delta L=\operatorname{supp} L=\{\operatorname{supp}(x_n(\xi))_{n\in\mathbb{N}}\,:\,\xi\in K\},$$

where $\operatorname{supp}\xi=\{n\in\mathbb{N}\,:\,x_n(\xi)\neq 0\}$. Use Theorem 2 to find an infinite $N\subseteq\mathbb{N}$ such that $\mathcal{F}=(\operatorname{supp} L)[N]$ is the closure of a uniform barrier $\mathcal{C}$ on N. Observe that for every $\boldsymbol{a}$,

$$\|\boldsymbol{a}\cdot\boldsymbol{x}_N\|\leq\sup\{\|\boldsymbol{a}\upharpoonright s\|_{\ell_1}\,:\,s\in\mathcal{F}\}: \tag{52}$$

Let $\xi\in K$ be such that $\|\boldsymbol{a}\cdot\boldsymbol{x}_N\|=|(\boldsymbol{a}\cdot\boldsymbol{x}_N)(\xi)|$. As $(\operatorname{supp}\xi)\cap N\in\mathcal{F}$ we obtain that $\|\boldsymbol{a}\cdot\boldsymbol{x}_N\|=|(\boldsymbol{a}\cdot\boldsymbol{x}_N)(\xi)|=|\sum_{n\in\operatorname{supp}\xi\cap N}a_nx_n(\xi)|$, so we are done because $\|x_n\|\leq 1$.

Let

$$\begin{aligned}\mathfrak{F}=&\{(\operatorname{supp}\boldsymbol{a},\boldsymbol{a})\,:\,\boldsymbol{a}\in c_{00},\operatorname{supp} a\subseteq N\}\\ w(t,\boldsymbol{a})=&\frac{\|\boldsymbol{a}\cdot\boldsymbol{x}\|}{\sup_{s\in\mathcal{F}}\|\boldsymbol{a}\upharpoonright s\|_{\ell_1}}.\end{aligned}$$

Note that, for every $M\subseteq N$ the sequence $\boldsymbol{x}\upharpoonright M$ is $(\mathfrak{F}_w,C)$-unconditional iff for every $\boldsymbol{a}$ we have that

$$\sup_{s\in\mathcal{F}\upharpoonright M}\|\boldsymbol{a}\upharpoonright s\|_{\ell_1}\leq C\|\boldsymbol{a}\cdot\boldsymbol{x}\upharpoonright M\|. \tag{53}$$

Claim 2 $\mathfrak{X}_{2/\delta}(\mathfrak{F}_w, \boldsymbol{x}_N) \neq \emptyset$.

Before we give a proof of this fact, let us see how to find the desired subsequence: Let $M \in \mathfrak{X}_{2/\delta}(\mathfrak{F}_w, \boldsymbol{x}_N)$, i.e. $\boldsymbol{x} \upharpoonright M$ is $(\mathfrak{F}_w, 2/\delta)$-unconditional. Let $\pi : M \to \mathbb{N}$ be the unique order-preserving onto mapping, and let

$$\mathcal{B} = \pi"(\mathcal{C} \upharpoonright M) = \{\pi" s \,:\, s \in \mathcal{C} \upharpoonright M\}.$$

This is a uniform barrier on $\mathbb{N}$ (see remark 1). Let $\boldsymbol{e} = (e_n)$ be the natural basis of the Schreier space $X_{\mathcal{B}}$. Fix a sequence $\boldsymbol{a} = (a_n)_{n\in M}$ of scalars. Observe that, by Proposition 5 (b),

$$\sup_{s\in\mathcal{F}\upharpoonright M} \|\boldsymbol{a} \upharpoonright s\|_{\ell_1} = \sup_{s\in\mathcal{C}\upharpoonright M} \|\boldsymbol{a} \upharpoonright s\|_{\ell_1} = \sup_{t\in\mathcal{B}} |\sum_{n\in t} a_{\pi^{-1}(n)}| = \|\sum_{n\in\mathbb{N}} a_{\pi^{-1}(n)} e_{\pi^{-1}(n)}\|_{\mathcal{B}}. \tag{54}$$

Now using (52), (53) and (54) we obtain that

$$\frac{2}{\delta}\|\sum_{n\in\mathbb{N}} a_{\pi^{-1}(n)} e_n\|_{\mathcal{B}} \leq \|\sum_{n\in M} a_n x_n\|_X \leq \|\sum_{n\in\mathbb{N}} a_{\pi^{-1}(n)} e_n\|_{\mathcal{B}},$$

as desired. Finally, let us prove the Claim: Suppose not, i.e. $\mathfrak{X}_{2/\delta}(\mathfrak{F}_w, \boldsymbol{x}_N) = \emptyset$. The idea is to use, as for the previous results, Corollary 17. However this is not possible because the weight w we use here does not satisfy the hypothesis of that corollary. So we use Lemma 7 instead to find $M \subseteq N$, a barrier $\mathcal{D}$ on M and $\varphi : \mathcal{B} \to \mathcal{F} \upharpoonright M \times c_{00}$, $s \mapsto (\bar{t}_s, \boldsymbol{a}^s)$, such that $\bar{t}_s \subseteq \operatorname{supp} \boldsymbol{a}^s = s$, $\bar{t}_s \in \mathcal{F} \upharpoonright M$ ($\mathcal{F}$ is hereditary), and such that

$$\|\boldsymbol{a}^s \upharpoonright \bar{t}_s\|_{\ell_1} > \frac{2}{\delta}\|\boldsymbol{a}^s \cdot \boldsymbol{x} \upharpoonright N\|. \tag{55}$$

Now let $t_s \subseteq \bar{t}_s$ be such that $\boldsymbol{a}^s \upharpoonright t_s$ has constant sign and such that $\|\boldsymbol{a}^s \upharpoonright t_s\|_{\ell_1} \geq \|\boldsymbol{a}^s \upharpoonright \bar{t}_s\|_{\ell_1}$.

For each $s \in \mathcal{D}$ let $\xi_s \in K$ be such that $(\operatorname{supp} \xi_s) \cap M = t_s$ ($\mathcal{F}$ is downwards closed). For $i = 0, 1$, let $t_s^{(i)} = \{n \in t_s \,:\, (-1)^i x_n(\xi_s) > 0\}$, and choose now $i_s = 0, 1$ such that

$$\|\boldsymbol{a}^s \upharpoonright t_s^{(i_s)}\|_{\ell_1} \geq \frac{1}{2}\|\boldsymbol{a}^s \upharpoonright t_s\|_{\ell_1}. \tag{56}$$

By the Ramsey property of $\mathcal{D}$, we may assume that $s \mapsto i_s \in \{0, 1\}$ is constant with value k. We have then naturally defined a new mapping from $\mathcal{D}$ into FIN, $s \mapsto t_s^{(k)}$. Use the matching result in Proposition 6 to find $s, u \in \mathcal{D}$ such that
(a) $t_s^{(k)} \sqsubseteq t_u^{(k)}$, and
(b) $s \cap u = t_s^{(k)}$.

Finally, using that $(x_n(\xi_u))_{n\in t_u^{(k)}}$ have constant sign and conditions (51) and (56), we obtain

$$\|\boldsymbol{a}^s \cdot \boldsymbol{x} \upharpoonright N\| \geq |(\boldsymbol{a}^s \cdot \boldsymbol{x} \upharpoonright N)(\xi_u)| = |(\boldsymbol{a}^s \upharpoonright (\operatorname{supp} \xi_u) \cdot \boldsymbol{x} \upharpoonright N)(\xi_u)| = |(\boldsymbol{a}^s \upharpoonright t_s^{(k)} \cdot \boldsymbol{x} \upharpoonright N)(\xi_u)| = \tag{57}$$

$$= |\sum_{n\in t_s^{(k)}} a_n^s x_n(\xi_u)| \geq \delta\|\boldsymbol{a}^s \upharpoonright t_s^{(k)}\|_{\ell_1} \geq \frac{\delta}{2}\|\boldsymbol{a}^s \upharpoonright t_s\|_{\ell_1}, \tag{58}$$

which is contradictory with (55). ∎

4.5. Some canonical examples

In this subsection we show that mappings on barriers naturally lead to weakly null sequences in Banach spaces showing thus that our approach to study weakly null sequences using mappings on barriers is in some sense necessary.

Definition 20 *Suppose that $\mathcal{B}$ is a uniform barrier on M, $\varphi : \mathcal{B} \to c_{00}$ is an L-mapping, $\|\varphi\|_\infty \le 1$ and $0 < \lambda \le 1$. We define on $c_{00} \upharpoonright M$ the following norm $\|\cdot\|_{\varphi,\lambda}$: for $\boldsymbol{a} \in c_{00} \upharpoonright M$, let*

$$\|\boldsymbol{a}\|_{\varphi,\lambda} = \max\{\|\boldsymbol{a}\|_\infty, \sup_{s\in\mathcal{B}} |\langle \boldsymbol{a}, \varphi(s)\rangle|\}.$$

Let $X_{\varphi,\lambda}$ be its completion. It is not difficult to see that the subsequence $\boldsymbol{x}^{(\varphi,\lambda)} = \boldsymbol{e}_M$ of the Hamel basis $\boldsymbol{e}$ of c_{00} is a semi-normalized weakly-null basic sequence of $X_{\varphi,\lambda}$. We call such sequences L-sequences.

We call a sequence $\boldsymbol{x}$ a U-sequence *if it is a L-sequence whose mapping φ is a U-mapping and $\lambda = 1$.*

Remark 11 *Another interpretation of $X_{\varphi,\lambda}$ is the following: Let L be the weak-closure of φ"$\mathcal{B}$ (i.e. $L = \{\varphi(s) \upharpoonright t \, : \, t \sqsubseteq \operatorname{supp} s\}$), and let*

$$K_{\varphi,\lambda} = L \cup \{\lambda e_n\}_{n\in M}.$$

This is a weakly-compact subset of c_{00}, and then $X_{\varphi,\lambda}$ is the closed linear span of the sequence $\boldsymbol{e} \subseteq C(K_{\varphi,\lambda})$.

Theorem 10 *Let $\mathfrak{F} \subseteq \mathrm{FIN} \times c_{00}$ and w be as in Definition 17 with the additional requirement that w is bounded away from zero, i.e. $\inf w(t, \boldsymbol{a}) > 0$. If there is some semi-normalized weakly-null sequence $\boldsymbol{x}$ with $\boldsymbol{C}(\mathfrak{F}_w, \boldsymbol{x}) = \infty$ (i.e. with no $\mathfrak{F}_w$-unconditional subsequence) then there is a U-sequence $\boldsymbol{y}$ such that $\boldsymbol{C}(\mathfrak{F}_w, \boldsymbol{y}) = \infty$.*

PROOF. Fix all the ingredients in the statement. We assume that $\|\boldsymbol{a} \cdot \boldsymbol{x}\|_X \ge \lambda \|\boldsymbol{a}\|_\infty$ for every $\boldsymbol{a} \in c_{00}$. Let $C_n \uparrow_n \infty$. As $\mathfrak{X}_{C_n}(\mathfrak{F}_w, \boldsymbol{x}) = \emptyset$, we can repeatedly use Lemma 7 to find a fusion sequence (M_k), uniform barriers $\mathcal{B}_k$ on M_{k+1} and $\varphi_k : \mathcal{B}_k \to \mathfrak{F} \upharpoonright M_k$, $\varphi_k(s) = (t_{(s,k)}, \boldsymbol{a}^{(s,k)})$, such that for every $k \in \mathbb{N}$ and every $s \in \mathcal{B}_n$

(1) $t_{(s,k)} \subseteq \operatorname{supp} \boldsymbol{a}^{(s,k)} = s$, and

(2) $\|\boldsymbol{a}^{(s,k)} \upharpoonright t_{(s,k)} \cdot \boldsymbol{x}\| > C_{\min M_k} w(t_{(s,k)}, \boldsymbol{a}^{(s,k)}) \|\boldsymbol{a}^{(s,k)} \cdot \boldsymbol{x}\|$.

Consider the sequence $(\alpha_k)_k$ of ranks of $\mathcal{B}_k$'s, and find a subset $I \subseteq \mathbb{N}$ such that $(\alpha_k)_{k\in I}$ is either constant or strictly increasing, and let $\alpha = \sup_{k\in I} \alpha_k$. Let $M = \{m_k\}_{k\in I}$, where $m_k = \min M_k$. Define now $\mathcal{B} \subseteq \mathcal{P}(M)$ by $s \in \mathcal{B}$ iff ${}_*s \in \mathcal{B}_k$, where $k \in I$ is such that $\min s = m_k$. This is an uniform family on M (it is $\alpha+1$-uniform if $(\alpha_k)_{k\in I}$ is constant with value α and α-uniform otherwise). Let $N \subseteq M$ be such that $\mathcal{B} \upharpoonright N$ is in addition a barrier on N. Define $\varphi : \mathcal{B} \upharpoonright N \to \mathfrak{F} \upharpoonright N$ by

$$\varphi(s) = (t_{(s,k)}, \boldsymbol{a}^{(s,k)})$$

where $k \in I$ is such that $\min s = m_k$. Set $\varphi(s) = (t_s, \boldsymbol{a}^s)$ for $s \in \mathcal{B} \upharpoonright N$. Then for every $s \in \mathcal{B} \upharpoonright N$, if $k \in I$ is such that $\min s = m_k$, then we have that,

$$t_s = t_{({}_*s,k)} \subseteq \operatorname{supp} \boldsymbol{a}^{({}_*s,k)} = {}_*s \subseteq s$$

so $t_s \subseteq \operatorname{supp} \boldsymbol{a}^s \subseteq s$, and also,

$$\|\boldsymbol{a}^s \upharpoonright t_s \cdot \boldsymbol{x}\| > C_{\min s} w(t_s, \boldsymbol{a}^s) \|\boldsymbol{a}^s \cdot \boldsymbol{x}\|. \tag{59}$$

Now for every $s \in \mathcal{B} \upharpoonright N$ pick $x_s^* \in S_{X^*}$ such that

$$\|\boldsymbol{a}^s \upharpoonright t_s \cdot \boldsymbol{x}\| = |x_s^*(\boldsymbol{a}^s \upharpoonright t_s \cdot \boldsymbol{x})|.$$

Define $\psi : \mathcal{B} \upharpoonright N \to c_0$ by $\psi(s) = (x_s^*(x_n))_{n\in N}$. We are going to "perturb" ψ to make it uniform: Apply Proposition 11 (b) to ψ and λ to produce some $P \subseteq N$, and some U-mapping $\varpi : \mathcal{B} \upharpoonright P \to c_{00}$ such that

$$\|\varpi(s) - \psi(s) \upharpoonright P\|_{\ell_1} \le \lambda \bar{w} \text{ for every } s \in \mathcal{B} \upharpoonright P \,, \tag{60}$$

and where $\bar{w} = \inf_{(t,\boldsymbol{a})} w(t, \boldsymbol{a})$. Let $\boldsymbol{y}$ be the U-sequence associated to ϖ. First of all, from (60) we obtain that for every $s \in \mathcal{B} \upharpoonright P$ and every $\boldsymbol{a} \in c_{00} \upharpoonright N$,

$$|\langle \boldsymbol{a}, \varpi(s)\rangle - \langle \boldsymbol{a}, \psi(s)\rangle| \le \|\boldsymbol{a}\|_\infty \|\varphi(s) - \psi(s) \upharpoonright P\|_{\ell_1} \le \lambda\bar{w}\|\boldsymbol{a}\|_\infty. \tag{61}$$

Hence

$$\begin{aligned}\|\boldsymbol{a}^s\|_\varpi =& \max\{\|\boldsymbol{a}^s\|_\infty, \sup_{u\in\mathcal{B}\upharpoonright P} |\langle \boldsymbol{a}^s, \varpi(u)\rangle|\} \le \max\{\|\boldsymbol{a}^s\|_\infty, \lambda\bar{w}\|\boldsymbol{a}^s\|_\infty + \sup_{u\in\mathcal{B}\upharpoonright P} |\langle \boldsymbol{a}, \psi(u)\rangle|\}\\ \le& \max\{\|\boldsymbol{a}^s\|_\infty, \lambda\bar{w}\|\boldsymbol{a}^s\|_\infty + \|\boldsymbol{a}\cdot\boldsymbol{x}\|_X\} \le \max\{\frac{1}{\lambda}, 1+\bar{w}\}\|\boldsymbol{a}\cdot\boldsymbol{x}\|_X.\end{aligned} \tag{62}$$

Set $\bar{\lambda} = \max\{\lambda^{-1}, 1+\bar{w}\}$. Using (59), (61) and (62), we get that for every $s \in \mathcal{B} \upharpoonright P$

$$\begin{aligned}\|\boldsymbol{a}^s \upharpoonright t_s \cdot \boldsymbol{y}\|_\varpi \ge& |\langle \boldsymbol{a}^s \upharpoonright t_s, \varpi(s)\rangle| \ge |\langle \boldsymbol{a}^s \upharpoonright t_s, \psi(s)\rangle| - \lambda\bar{w}\|\boldsymbol{a}^s \upharpoonright t_s\|_\infty =\\ =& |x_s^*(\boldsymbol{a}^s \upharpoonright t_s \cdot \boldsymbol{x})| - \lambda\bar{w}\|\boldsymbol{a}^s \upharpoonright t_s\|_\infty > C_{\min s} w(t_s, \boldsymbol{a}^s)\|\boldsymbol{a}^s\cdot\boldsymbol{x}\|_X - \lambda\bar{w}\|\boldsymbol{a}^s \upharpoonright t_s\|_\infty \ge\\ \ge& \frac{C_{\min s} w(t_s, \boldsymbol{a}^s)}{\bar{\lambda}}\|\boldsymbol{a}^s\|_\varpi - \lambda\bar{w}\|\boldsymbol{a}^s \upharpoonright t_s\|_\infty \ge (\frac{C_{\min s}}{\bar{\lambda}} - \lambda) w(t_s, \boldsymbol{a}^s)\|\boldsymbol{a}^s\|_\varpi.\end{aligned}$$

Since $C_{\min s} \uparrow \infty$ if $\min s \uparrow \infty$, we obtain that

$$\boldsymbol{C}(\mathfrak{F}_w, \boldsymbol{y}) \ge \sup_{s\in\mathcal{B}\upharpoonright P} (\frac{C_{\min s}}{\bar{\lambda}} - \lambda) = \infty,$$

as desired. ■

Remark 12 *(a) In addition, one can get, with similar proof, the following: for arbitraries $\mathfrak{F}$ and w, and semi-normalized weakly-null sequence $\boldsymbol{x}$ there is an L-sequence $\boldsymbol{y}$ such that $\boldsymbol{C}(\mathfrak{F}, \boldsymbol{x}) \le \boldsymbol{C}(\mathfrak{F}, \boldsymbol{y})$.*
(b) Observe that to every unconditionality notion $(\mathfrak{F}, w)$ such that there is some semi-normalized weakly null sequence with no unconditional subsequence, we can naturally define a rank as

$$\alpha(\mathfrak{F}_w) = \min\{\alpha(\boldsymbol{x}) \,:\, \boldsymbol{x} \text{ is U-sequence with } \boldsymbol{C}(\mathfrak{F}_w, \boldsymbol{x}) = \emptyset\},$$

and where $\alpha(\boldsymbol{x})$ is such that the barrier associated to $\boldsymbol{x}$ is $\alpha(\boldsymbol{x})$-uniform. For example, we show in the next subsection that $\alpha(\mathfrak{U}) = \omega^2$, i.e. every U-sequence of rank strictly smaller than ω^2 has an unconditional subsequence, while there is a U-sequence of rank ω^2 with no unconditional subsequence (Maurey-Rosenthal example).

4.6. Maurey-Rosenthal example

Let us return back to an even more basic problem that must be solved before even considering various notions of partial unconditionality: Does every normalized weakly-null sequences has a subsequence which is unconditional? If not, then by Theorem 10 there must be a U-sequence with no unconditional subsequence. We present now (as in [21]) the famous example of Maurey-Rosenthal [24] of a normalized weakly-null sequence with no unconditional subsequence (see also [23]):

Example 2 *First of all, for a fixed $0 < \varepsilon < 1$ choose a fast increasing sequence (m_i) such that*

$$\sum_{i=0}^{\infty}\sum_{j\neq i} \min((\frac{m_i}{m_j})^{1/2}, (\frac{m_j}{m_i})^{1/2}) \le \frac{\varepsilon}{2}. \tag{63}$$

Let $\mathrm{FIN}^{[<\infty]}$ be the collection of all finite block sequences $s_0 < s_1 < \cdots < s_k$ of nonempty finite subsets of $\mathbb{N}$. Now choose a $1-1$ function

$$\sigma : \mathrm{FIN}^{[<\infty]} \to \{m_i\} \tag{64}$$

such that $\sigma((s_i)_{i=0}^n) > s_n$ *for all* $(s_i) \in \mathrm{FIN}^{[<\infty]}$ *Now let* $\mathcal{B}_{\mathrm{MR}}$ *be the family of unions* $s_0 \cup s_1 \cup \cdots \cup s_n$ *of finite sets such that*
(a) $s_0 = \{n\}$.
(b) (s_i) *is block and, and*
(c) $|s_i| = \sigma(s_0, \ldots, s_{i-1})$ $(1 \leq i \leq n)$.

It is not difficult to see that $\mathcal{B}_{\mathrm{MR}}$ *is a* ω^2*-uniform barrier on* $\mathbb{N}$. *Observe that by definition, every* $s \in \mathcal{B}_{\mathrm{MR}}$ *has a unique decomposition* $s = s_0 \cup \cdots \cup s_n$ *satisfying (a), (b) and (c) above. Now define the mapping* $\varphi : \mathcal{B}_{\mathrm{MR}} \to c_{00}$,

$$\varphi(s) = \sum_{i=0}^{n} \frac{1}{|s_i|^{\frac{1}{2}}} \chi_{s_i}. \tag{65}$$

Observe that φ *is a U-mapping (i.e. it is internal and uniform). Notice also that for every* $\boldsymbol{a} \in c_{00}$ $\|\boldsymbol{a}\|_\varphi \geq \|\boldsymbol{a}\|_\infty$. *The corresponding U-sequence* $\boldsymbol{x}$ *it has no unconditional subsequence. Moreover* $\boldsymbol{x}$ *has the property that the summing basis* (S_i) *of c, the Banach space of* convergent sequences of reals, *is finitely-block representable in the linear span of every subsequence of* $\boldsymbol{x}$ *(and so the summing basis of* c_0*), more precisely, for every* M, *every* $n \in \mathbb{N}$ *and every* $\varepsilon > 0$ *there is a normalized block subsequence* $(y_i)_{i=0}^{n-1}$ *of* $\boldsymbol{x} \restriction M$ *such that for every sequence of scalars* $(a_i)_{i=0}^{n-1}$,

$$\max\{|\sum_{i=0}^{m} a_i| \,:\, m < n\} \leq \|\sum_{i=0}^{n-1} a_i y_i\|_\varphi \leq (1+\varepsilon)\max\{|\sum_{i=0}^{m} a_i| \,:\, m < n\}.$$

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