MODEL REDUCTION METHODS IN OPTION PRICING*

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ABSTRACT

In this work we introduce the Proper Orthogonal Decomposition (POD) approach to the valuation of contingent claims for one–dimensional price models. First, we present the POD in the context of an abstract Hilbert space and we give an application for the numerical pricing of Double Barrier Options. In a finite dimension setting, we show the model reduction method for Finite Difference schemes of implicit type. In particular, we construct the reduced version of the Crank–Nicolson scheme and some numerical examples are given.

Keywords: Model Reduction, Proper Orthogonal Decomposition, Finite Difference Schemes, Crank–Nicolson Scheme.

1 Introduction

The aim of this paper is to introduce the Proper Orthogonal Decomposition (POD) for model reduction in the framework of Partial Differential Equations arising in Option Pricing Models. The Proper Orthogonal Decomposition is a powerful and elegant method for deriving low order models of dynamical systems. The POD provides a basis for the modal decomposition of an ensamble of functions, composed of theoretical, experimental o computer data. Its properties suggest that it is the preferred basis to use in various applications.

It was successfully used in different fields including signal analysis and pattern recognition (see for example [5]), fluid dynamics (see [3] and [8]) and more recently in control theory (see [2]). The process is also known as Principal Component Analysis (PCA) (see [5]) or the Karhunen–Loève expansion the latter name comes from two individual papers by Karhunen [6] and Loève [7].

The most striking feature of the POD is its optimality. It provides the most efficient way of capturing the dominant components of an infinite–dimensional process with only a finite number of "modes" and often there are surprinsigly few. The main idea is to find a set of ordered orthonormal basis vector in a subspace associated to a set of data. This set of data can be composed of theoretical, experimental, or computed data. The optimal ordering of the basis is such that the first basis element best represents the data in the input collection, the second basis element is the next best, and so forth. For the purposes of model reduction, one type of input collection is a set of time snapshots, where each snapshot contains spatial data obtained from a numerical simulation at a fixed time. These snapshots are chosen so that the basis reflects the system dynamics.

In the present work we examine the feasibility and efficiency of the POD method for pricing numerically an European type derivative. To this end we introduce the reduced finite difference scheme of implicit type.

The paper is organized as follows. In Section 2 we introduce the POD in the context of a general Hilbert space and we give an example of application to the valuation of Double Barrier Options by using a set of data composed by data of theoretical type. In Section 3, by using the POD in the context of a finite Hilbert space, we introduce the reduced finite difference scheme of implicit type. We applied it by using the reduced version of the Crank–Nicolson scheme to study, at the numerical level, the valuation of the European Put in the Black–Scholes model. Finally, in 4 we give some concluding remarks.

2 Model Reduction with Proper Orthogonal Decomposition

The tools of POD and Galerkin method provides a systematic way for producing reduced–order models from data. In this section, we give an overview of these methods, first we introduce the proper orthogonal decomposition in the context of an abstract Hilbert space. The abstract setting is used due to the fact that, in this paper we will consider different inner products, and keeping the exposition general allows one to see precisely where the dependence on the inner products lies. Next, we introduce the POD in the context of pricing Double Barrier Options

2.1 The Proper Orthogonal Decomposition

Let $\mathcal{H}$ be a real Hilbert space with inner product $(\cdot,\cdot)_{\mathcal{H}}$ and norm $||\cdot||_{\mathcal{H}}$. For

$$\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n \in \mathcal{H}$$

we set

$$\mathbf{V}^{\text{snap}} = \text{span}\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$$

and we refer to $\mathbf{V}^{\text{snap}}$ as ensamble consisting of the snapshots, at least one of which is assumed to be non–zero. Assume that $\dim \mathbf{V}^{\text{snap}} = p$ and let $\{\mathbf{\Psi}_1, \mathbf{\Psi}_2, \ldots, \mathbf{\Psi}_p\}$ denote an orthonormal basis of $\mathbf{V}^{\text{snap}}$. Then each member of the ensamble can be expressed as

$$\mathbf{y}_j = \sum_{k=1}^{p} (\mathbf{y}_j, \mathbf{\Psi}_k)_{\mathcal{H}} \mathbf{\Psi}_k \tag{1}$$

for $j = 1, 2, \ldots, n$. The proper orthogonal decomposition consists in choosing the orthonormal basis such that for every $l \in \{1, 2, \ldots, p\}$ the mean square error between the elements of $\mathbf{V}^{\text{snap}}$ and the corresponding l–th partial sum of (1) is minimized on averaged, that is,

$$\begin{array}{cc} \min_{\{\mathbf{\Psi}_1, \mathbf{\Psi}_2, \ldots, \mathbf{\Psi}_l\}} & \frac{1}{n} \sum_{j=1}^{n} \left|\left| \mathbf{y}_j - \sum_{k=1}^{l} (\mathbf{y}_j, \mathbf{\Psi}_k)_{\mathcal{H}} \mathbf{\Psi}_k \right|\right|_{\mathcal{H}} \\ \text{subject to} & (\mathbf{\Psi}_i, \mathbf{\Psi}_j)_{\mathcal{H}} = \delta_{i,j} \\ & \text{for } 1 \leq i \leq l,\ 1 \leq j \leq i. \end{array} \tag{2}$$

A solution $\{\mathbf{\Psi}_1, \mathbf{\Psi}_2, \ldots, \mathbf{\Psi}_l\}$ to (2) is called a POD basis of rank l. The solution of (2) is characterized by the first order necessary conditions of optimality, which can be written as an eigenvalue problem. To see this let us define the bounded linear operator $L_n : \mathbb{R}^n \longrightarrow \mathcal{H}$ by

$$L_n\left([u_1\ u_2\ \cdots\ u_n]^{\mathbf{T}}\right) = \sum_{i=1}^{n} u_i \mathbf{y}_i.$$

Then the adjoint $L_n^* : \mathcal{H} \longrightarrow \mathbb{R}^n$ is given by

$$L_n^*(\mathbf{v}) = [(\mathbf{v}, \mathbf{y}_1)_{\mathcal{H}} \ \cdots \ (\mathbf{v}, \mathbf{y}_n)_{\mathcal{H}}]^T.$$

Let us next define $R_n = L_n L_n^*$ and $K_n = L_n^* L_n$, given by

$$R_n(\mathbf{v}) = \sum_{i=1}^{n} (\mathbf{v}, \mathbf{y}_i)_{\mathcal{H}} \mathbf{y}_i$$

and

$$(K_n)_{i,j} = (\mathbf{y}_j, \mathbf{y}_i)_{\mathcal{H}},$$

respectively. It is now easy to show that the solution to (2) is characterized by the following first order necessary condition of optimality

$$R_n \mathbf{\Psi} = \lambda \mathbf{\Psi},$$

where R_n is a linear self–adjoint compact operator. Therefore, by the Hilbert–Schmidt theory, there exists a complete orthonormal basis $\{\mathbf{\Psi}_i\}_{i=1}^{\infty}$ so that

$$R_n \mathbf{\Psi}_i = \lambda_i \mathbf{\Psi}_i,$$

and $\lambda_1 \geq \lambda_2 \geq \cdots$ and $\lambda_i = 0$ for $i > k$. Moreover, we have the following error formula

$$\sum_{j=1}^{n} \left\| \mathbf{y}_j - \sum_{i=1}^{p} (\mathbf{y}_j, \mathbf{\Psi}_i)_{\mathcal{H}} \mathbf{\Psi}_i \right\|_{\mathcal{H}} = \sum_{i=p+1}^{k} \lambda_i.$$

In practical applications the POD basis of rank l may be computed as follows:

Step 1 Solve the eigenvalue problem

$$K_n \mathbf{w} = \lambda \mathbf{w}$$

where $(K_n)_{i,j} = (\mathbf{y}_j, \mathbf{y}_i)_X$ and the nonnegative eigenvalues satisfy $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$.

Step 2 Find

$$\mathbf{\Psi}_i = \frac{1}{\sqrt{\lambda_i}} \sum_{j=1}^{n} w_i^j \mathbf{y}_j$$

where w_i^j denotes the j–th component of the eigenfunction $\mathbf{w}_i$, for $i = 1, 2, \ldots, k$.

The optimality properties of POD state among all linear combinations the one which correspond to POD is the best in the sense that it will capture the most information possible in the average sense. Since the eigenvalues can be used to find how close the low–dimensional subspace approximant is to the date set, one can seek the number of modes l such that such that the fraction

$$\sum_{i=1}^{l} \lambda_i / \sum_{i=1}^{n} \lambda_i$$

is close to one and yet $l << n$. The l–dimensional reduced basis subspace is defined as

$$\mathbf{V}^{\mathrm{POD}} = \mathrm{span}\{\mathbf{\Psi}_1, \mathbf{\Psi}_2, \ldots, \mathbf{\Psi}_l\}.$$

2.2 A case study: the Double Barrier Option

Now we would to use the POD method to valuate an European style derivative. Assume that the payoff of this derivative at expiration time T, is given by a function f of a single state variable X, which follows a scalar diffusion process under the risk neutral measure:

$$dX_t = b(X_t)dt + a(X_t)dW_t,\ X_0 = x. \tag{3}$$

Suppose that the derivative contract has a double barrier provision given by $0 < L < U < \infty$, such that if at any time between the contract inception and expiration time either the lower barrier L or the upper one U is reached, the contract is canceled. It is well–known that the value u of the derivative security, solves the Partial Differential Equation (PDE):

$$-u_t(x,t) + \mathcal{A}(u(t,x)) = 0, \tag{4}$$

where $\mathcal{A}$ is the negative of the infinitesimal generator of X given by

$$\mathcal{A}(\phi(x)) = -\frac{1}{2}a^2(x)\phi_{xx}(x) - b(x)\phi_x(x) + r(x)\phi(x).$$

and with boundary conditions $u(x,T) = f(x)$ and $u(L,t) = u(U,t) = 0$. Our main goal is to construct an approximation $\hat{u}(x,t)$ of the true solution $u(x,t)$, by using the Proper Orthogonal Decomposition.

Consider the speed density of the diffusion process X given by

$$\mathcal{M}(x) = \frac{2}{a^2(x)\mathcal{S}(x)} \tag{5}$$

where

$$\mathcal{S}(x) = \exp\left(-\int_{\cdot}^{x} \frac{2b(y)}{a^2(y)}\right), \tag{6}$$

is the scale density of X. Let $\mathcal{H} = \mathcal{L}^2([L,U],\mathcal{M})$ be the Hilbert space of square integrable functions on (L,U) with the speed density $\mathcal{M}$ and endowed with the inner product

$$(g(x),h(x))_{\mathcal{H}} = \int_L^U g(x)h(x)\mathcal{M}(x)dx, \tag{7}$$

(see Proposition 1 in [4]).

Let $u(x,t)$ be a flow defined on $[L,U]\times[0,T]$. For now, imagine a time average over an ensamble with members $\{u^{(k)}(x)\} = \{u(x,t_k)\}$, obtained from successive measurements during a single run for

$$0 = t_1 < t_2 < \cdots < t_n = T,$$

a grid in the trading interval $[0,T]$, represents the "snapshots" or a sampled data representation of the flow. Assume that we obtain as snapshots the following finite sequence of functions

$$u(x,t_k) = u^{(k)}(x) = \sum_{j=1}^{k} c_j e^{-\mu_j(T-t_j)}\phi_j(x) \tag{8}$$

for $k = 1,2,\ldots,n$, where $\{\phi_j(x)\}_{j=1}^{\infty}$ is a complete orthonormal basis of $\mathcal{L}^2([L,U],\mathcal{M})$, of eigenvectors with associated eigenvalues $0 < \mu_1 < \mu_2 < \cdots < \mu_j < \cdots$ obtained as solution of the regular Sturm–Liouville boundary value problem

$$\mathcal{A}(\phi(x)) \quad = \quad \mu\phi(x), \quad \phi(U) = \phi(L) = 0, \tag{9}$$

and

$$c_j = (f(x),\phi_j(x))_{\mathcal{H}}$$

for $j = 1,2,\ldots,n$. Then, it follows that

$$K_n = L_{K_n} L_{K_n}^{\mathbf{T}}$$

where

$$L_{K_n} = \begin{bmatrix} c_1 e^{-\mu_1(T-t_1)} & 0 & 0 & \ldots & 0 \\ c_1 e^{-\mu_1(T-t_2)} & c_2 e^{-\mu_2(T-t_2)} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 e^{-\mu_1(T-t_n)} & c_2 e^{-\mu_2(T-t_n)} & c_3 e^{-\mu_3(T-t_n)} & \ldots & c_n e^{-\mu_n(T-t_n)} \end{bmatrix}.$$

We denote by

$$\mathbf{w}_i = [w_i^1, w_i^2, \ldots, w_i^n]^{\mathbf{T}}$$

the i–th eigenvector of K_n with associate eigenvalue equal to λ_i for $i = 1, 2, \ldots, l$ and where l is choosed by means the condition

$$\sum_{i=1}^{l} \lambda_i / \sum_{i=1}^{n} \lambda_i > 0.99.$$

Recall that

$$\mathcal{A}(\phi_i(x)) = \mu_i \phi_i(x)$$

for $i = 1, 2, \ldots$. Thus the modal basis is given by

$$\psi_i(x) = \frac{1}{\sqrt{\mu_i}} \sum_{j=1}^{n} w_i^j u^{(j)}(x)(x),$$

for $i = 1, 2, \ldots, l$.

The approximate solution $\hat{u}$ of (4) can be expanded in terms of these POD basis functions as

$$\hat{u}(x,t) = \sum_{i=1}^{l} a_i(t)\psi_i(x) \tag{10}$$

and forms the Galerkin expansion. We derive the reduced order model by employing the Galerkin weighted residual discretization of the model (4). The model residual can be expressed as

$$\mathbf{R}(\hat{u}(x,t)) = -\hat{u}_t(x,t) + \mathcal{A}(\hat{u}(x,t))$$

Applying the Galerkin projection which enforces the residual to be orthogonal to each basis functions

$$(\mathbf{R}(\hat{u}(x,t)), \psi_i(x))_{\mathcal{H}} = 0 \tag{11}$$

for $i = 1, 2, \ldots, l$ leads to the reduced order model of l ordinary differential equations for the amplitude coefficients

$$\frac{da_i(t)}{dt} = \sum_{j=1}^{l} a_j(t) \left(\mathcal{A}(\psi_j(x)), \psi_i(x)\right)_{\mathcal{H}} \tag{12}$$

for $i = 1, 2, \ldots, l$. Recall that $(\psi_i(x), \psi_j(x))_{\mathcal{H}} = \delta_{i,j}$. Then it is easy to see that

$$(\mathcal{A}(\psi_i(x)), \psi_j(x))_{\mathcal{H}} = \frac{\mathbf{w}_i^{\mathbf{T}}}{\sqrt{\mu_i}} Z \frac{\mathbf{w}_j}{\sqrt{\mu_j}},$$

where Z is an $n \times n$ matrix satisfying that

$$Z_{i,j} = (u^{(i)}(x), \mathcal{A}(u^{(j)}(x)))_{\mathcal{H}}.$$

Then it follows that

$$Z = L_{\sqrt{\mu}} L_{\sqrt{\mu}}^{\mathbf{T}}.$$

where

$$L_{\sqrt{\mu}} = \begin{bmatrix} c_1\sqrt{\mu_1}e^{-\mu_1(T-t_1)} & 0 & \dots & 0 \\ c_1\sqrt{\mu_1}e^{-\mu_1(T-t_2)} & c_2\sqrt{\mu_2}e^{-\mu_2(T-t_2)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c_1\sqrt{\mu_1}e^{-\mu_1(T-t_n)} & c_2\sqrt{\mu_2}e^{-\mu_2(T-t_n)} & \dots & c_n\sqrt{\mu_n}e^{-\mu_n(T-t_n)} \end{bmatrix}.$$

Moreover, if consider the residual at time T given by

$$\mathbf{R}(\hat{u}(x,T)) = f(x) - \sum_{j=1}^{l} a_j(T)\psi_j(x), \tag{13}$$

and in a similar way we enforce the residual to be orthogonal to each basis functions

$$(\mathbf{R}(\hat{u}(x,T)), \psi_i(x))_{\mathcal{H}} = 0 \tag{14}$$

for $i = 1, 2, \dots, l$. We obtain that

$$a_i(T) = (f(x), \psi_i(x))_{\mathcal{H}} \tag{15}$$

for $i = 1, 2, \dots, l$.

Let the $n \times l$ matrix

$$W_\mu = \begin{bmatrix} \frac{\mathbf{w}_1}{\sqrt{\mu_1}} & \frac{\mathbf{w}_2}{\sqrt{\mu_2}} & \cdots & \frac{\mathbf{w}_l}{\sqrt{\mu_l}} \end{bmatrix}$$

and

$$\mathbf{a}(t) = \begin{bmatrix} a_1(t) & a_2(t) & \cdots & a_l(t) \end{bmatrix}^{\mathbf{T}}.$$

From (15), it follows that

$$\mathbf{a}(T) = \left(\mathbf{1}^{\mathbf{T}} L_c W_\mu\right)^{\mathbf{T}}. \tag{16}$$

where

$$L_c = \begin{bmatrix} c_1^2 e^{-\mu_1(T-t_1)} & 0 & 0 & \dots & 0 \\ c_1^2 e^{-\mu_2(T-t_2)} & c_2^2 e^{-\mu_2(T-t_2)} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1^2 e^{-\mu_1(T-t_n)} & c_2^2 e^{-\mu_2(T-t_n)} & c_3^2 e^{-\mu_3(T-t_n)} & \dots & c_n^2 e^{-\mu_n(T-t_n)} \end{bmatrix}$$

and $\mathbf{1} = [1\ 1\ \cdots\ 1]^{\mathbf{T}} \in \mathbb{R}^n$. Then the reduced model can be written as

$$\begin{aligned} \tfrac{d}{dt}\mathbf{a}(t) &= W_\mu^{\mathbf{T}} Z W_\mu \mathbf{a}(t), \\ \mathbf{a}(T) &= \left(\mathbf{1}^{\mathbf{T}} L_c W_\mu\right)^{\mathbf{T}}, \end{aligned} \tag{17}$$

with solution

$$\mathbf{a}(t) = \exp\left(-W_\mu^{\mathbf{T}} Z W_\mu (T-t)\right)\left(\mathbf{1}^{\mathbf{T}} L_c W_\mu\right)^{\mathbf{T}}. \tag{18}$$

Thus, we can write

$$\hat{u}(x,t) = \mathbf{a}(t)^{\mathbf{T}} \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \\ \vdots \\ \psi_l(x) \end{bmatrix}.$$

2.2.1 Numerical Results

We consider that X follows a Geometric Brownian Motion with $b(x) = rx$ and $a(x) = \sigma x$. Assume that $r(x) = r$ and let

$$\nu = \frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right)$$

Then, from [4], we known that

$$\phi_j(x) = \frac{\sigma}{\sqrt{\ln(U/L)}} x^{-2\nu/\sigma} \sin\left(\frac{\pi j \ln(x/L)}{\ln(U/L)}\right) \tag{19}$$

for $j = 1, 2, \ldots,$ and $x \in [L, U]$, form the complete orthonormal basis in $\mathcal{H}$ obtained for the regular Sturm–Liouville boundary value problem (9) and where

$$\mu_j = r + \frac{\nu^2}{2} + \frac{\sigma^2 \pi^2 j^2}{2\ln^2(U/L)}.$$

for $j = 1, 2, \ldots$. We consider the payoff of an European Call $f(x) = (x-K)^+$ on $[L, U]$. Then it follows from [4], that

$$c_j = \frac{L^{\nu/\sigma}}{\sqrt{\ln(U/L)}}\left[L\gamma_j(\nu+\sigma) - K\gamma_j(\nu)\right],$$

where

$$\gamma_j(a) = \frac{2}{\omega_j^2 + a^2}\left[e^{a\kappa}(\omega_j \cos(\omega_j k) - a\sin(\omega_j k)) - (-1)^j \omega_j e^{a\eta}\right],$$

with

$$\eta = \frac{1}{\sigma}\ln(U/L),\ \omega_j = \frac{j\pi}{u},\ \kappa = \frac{1}{\sigma}\ln(K/L).$$

for $j = 1, 2, \ldots$.

Figure 1 plots call prices for $L = 9$, $U = 12$, $r = 0.1$, $\sigma = 0.6$, $K = 10$ $T = 1/12$ by using the representation of the call price by means 20 eigenfunctions against the call price computed by $\hat{u}(x,t)$ with $n = 4$.

We remark that we need to known previously a basis function obtained by solving a Sturm–Liouville problem, that allow to us to construct a solution of the problem under consideration. Thus, in practical application when no closed solution is available the above methodology doesn't runs. The main problem in real applications is that the price $u(x,t)$ is known in a discrete form, that is, at the nodes of a spatial grid or mesh and for some times. In this way, in the next section we apply the model reduction to a numerical scheme by using the information of either a previous simulation or a empirical data set.

3 Reduced Finite Difference Schemes of implicit type in Option Pricing Models

In this section we would to introduce the Model Reduction in the context of Finite Difference schemes of implicit type. To this end consider the boundary value problem (4) where

$$u(x,t) = g(x,t),\ \text{for } (x,t) \in [0,T] \times \{0,a\},$$

and

$$u(x,T) = f(x) \text{ for } x \in [0,a].$$

Note that $f(T) = g(0,T)$ and $f(a) = g(a,T)$. Here, f and g are given data. We assume that the problem under consideration has a unique solution u with certain smoothness that makes the following calculation meaningful.

To develop a finite difference method, we need to introduce grid points. Let N_x and N_t be integers, $h_x = a/N_x$, $k_t = T/N_t$ and define the partition points

$$x_j = jh_x, \qquad j = 0, 1, \ldots, N_x,$$

$$t_m = T - mh_t \quad m = 0, 1, \ldots, N_t.$$

A point of the form (x_j, t_m) is called grid point and we are interested in computing approximate solutions values at the grid points. We use the notation U_j^m for an

approximation to $u_j^m = u(x_j, t_m)$. We remark that

$$U_0^m = u_0^m = g(0, t_m),\ U_{N_x}^m = u_{N_x}^m = g(a, t_m),\ U_j^0 = u_0(x_j),$$

for $m = 0, 1, \ldots, N_t$ and $j = 0, 1, \ldots, N_x$ are known values. In order to compute the approximate values for the solution of (4) at the grid points we introduce the following vectors. Let

$$\mathbf{u}^m = [u_1^m\ u_2^m\ \cdots\ u_{N_x-1}^m]^{\mathbf{T}} \in \mathbb{R}^{N_x-1}$$

and the corresponding the approximation vector

$$\mathbf{U}^m = [U_1^m\ U_2^m\ \cdots\ U_{N_x-1}^m]^{\mathbf{T}} \in \mathbb{R}^{N_x-1},$$

for $m = 0, 1, \ldots, N_t$.

Now, assume that by using a typical finite difference scheme of implicit type for equation (4) we are to be able to construct the following two–level difference equation

$$\begin{array}{rcl} Q\mathbf{U}^{m+1} & = & \mathbf{f}(\mathbf{U}^m) + \mathbf{g}^m, \\ \mathbf{U}^0 & = & \mathbf{u}^0. \end{array} \tag{20}$$

for $m = 0, 1, \ldots, N_t - 1$ (see for example [9] and Chapter 3 of [1]). The vector $\mathbf{g}^m$ is usually constructed from values $\{u_j^m : j = 0, N_x,\ m = 0, 1, \ldots, N_t\}$ and $\mathbf{f} : \mathbb{R}^{N_x-1} \longrightarrow \mathbb{R}^{N_x-1}$. Here the matrix Q may depend on k_t and h_x. Thus, by solving this difference equation we obtain the matrix

$$\mathbf{U} = [\mathbf{U}^1\ \mathbf{U}^2\ \cdots \mathbf{U}^{N_t}]$$

which represents the approximate solution to the matrix

$$\mathbf{u} = [\mathbf{u}^1\ \mathbf{u}^2\ \cdots \mathbf{u}^{N_t}].$$

3.1 An example: The Crank–Nicolson Scheme

Now, we apply the Crank–Nicolson scheme to the PDE (4), for $(x, t) \in (0, a) \times (0, T)$ and where

$$\begin{array}{l} u\,(x, T) = f(x), \\ u\,(x, t) = g(t, x),\ t \in [0, T],\ x \in \{0, a\}. \end{array} \tag{21}$$

In this case we introduce the following parameters

$$\nu_1 = \frac{k_t}{h_x^2},\ \nu_2 = \frac{k_t}{h_x}.$$

Then, by using the boundary conditions we have

$$\begin{aligned} U_j^0 &= f(x_j) \text{ for } 0 \le j \le N_x; \\ U_0^m &= g(0,t_m), \text{ and } U_{N_x}^m = g(a,t_m), \text{ for } 0 \le m \le N_t. \end{aligned}$$

Now, set

$$\begin{aligned} b_j &= b(x_j), \\ a_j &= a(x_j) \text{ and} \\ r_j &= -r(x_j) \end{aligned}$$

for $0 \le j \le N_x$. From the Crank–Nicolson scheme we obtain that

$$\begin{aligned} &\left(\frac{1}{k_t}\left[U_j^m - U_j^{m+1}\right]\right) \\ &+\frac{b_j^2}{4}\left(\frac{1}{h_x^2}\left[U_{j+1}^{m+1} - 2U_j^{m+1} + U_{j-1}^{m+1}\right]\right) + \frac{b_j^2}{4}\left(\frac{1}{h_x^2}\left[U_{j+1}^m - 2U_j^m + U_{j-1}^m\right]\right) \\ &+\frac{a_j}{2}\left(\frac{1}{2h_x}\left[U_{j+1}^{m+1} - U_{j-1}^{m+1}\right]\right) + \frac{a_j}{2}\left(\frac{1}{2h_x}\left[U_{j+1}^m - U_{j-1}^m\right]\right) \\ &+\frac{r_j}{2}U_j^{m+1} + \frac{r_j}{2}U_j^m = 0, \end{aligned}$$

for $1 \le j \le N_x - 1$ and $0 \le m \le N_t - 1$. This can be written as

$$-A_j U_{j-1}^{m+1} + (1 - B_j)\, U_j^{m+1} - C_j U_{j+1}^{m+1} = A_j U_{j-1}^m + (1 + B_j)\, U_j^m + C_j U_{j+1}^m$$

where

$$\begin{aligned} A_j &= \tfrac{1}{4}\nu_1 b_j^2 + \tfrac{1}{4}\nu_2 a_j, \\ B_j &= -\tfrac{1}{2}\nu_1 b_j^2 + \tfrac{1}{2}k r_j, \\ C_j &= \tfrac{1}{4}\nu_1 b_j^2 - \tfrac{1}{4}\nu_2 a_j. \end{aligned}$$

Written in matrix form the above expressions we have the difference equation

$$\begin{bmatrix} -A_1 & 1-B_1 & -C_1 & 0 & \cdot & \cdot & \cdot \\ 0 & -A_2 & 1-B_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1-B_{N_x-2} & -C_{N_x-2} & 0 \\ \cdot & \cdot & \cdot & 0 & -A_{N_x-1} & 1-B_{N_x-1} & -C_{N_x-1} \end{bmatrix} \begin{bmatrix} U_0^{m+1} \\ U_1^{m+1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ U_{N_x-1}^{m+1} \\ U_{N_x}^{m+1} \end{bmatrix}$$

$$= \begin{bmatrix} A_1 & 1+B_1 & C_1 & 0 & \cdot & \cdot & \cdot \\ 0 & A_2 & 1+B_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1+B_{N_x-2} & C_{N_x-2} & 0 \\ \cdot & \cdot & \cdot & 0 & A_{N_x-1} & 1+B_{N_x-1} & C_{N_x-1} \end{bmatrix} \begin{bmatrix} U_0^{m} \\ U_1^{m} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ U_{N_x-1}^{m} \\ U_{N_x}^{m} \end{bmatrix}.$$

Since the values U_0^m and $U_{N_x}^m$ for $0 \leq m \leq N_t$ are known, then the above system is equivalent to (20) where

$$\mathbf{U}^m = \begin{bmatrix} U_1^m \\ \vdots \\ U_{N_x-1}^m \end{bmatrix}, \; Q = \begin{bmatrix} 1-B_1 & -C_1 & 0 & \cdot & \cdot \\ -A_2 & 1-B_2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & 1-B_{N_x-2} & -C_{N_x-2} \\ \cdot & \cdot & 0 & -A_{N_x-1} & 1-B_{N_x-1} \end{bmatrix},$$

$$\mathbf{g}^m = \begin{bmatrix} A_0 U_0^m + A_0 U_0^{m+1} \\ 0 \\ \vdots \\ 0 \\ C_{N_x} U_{N_x}^m + C_{N_x} U_{N_x}^{m+1} \end{bmatrix}$$

and

$$\mathbf{f}(\mathbf{U}^m) = \begin{bmatrix} (1+B_1)\, U_1^m + C_1 U_2^m \\ A_2 U_1^m + (1+B_2)\, U_2^m + C_2 U_3^m \\ \vdots \\ A_{N_x-2} U_{N_x-3}^m + (1+B_{N_x-2})\, U_{N_x-2}^m + C_{N_x-2} U_{N_x-1}^m \\ A_{N_x-1} U_{N_x-2}^m + (1+B_{N_x-1})\, U_{N_x-1}^m \end{bmatrix}.$$

We need to solve N_t linear systems with a $(N_x-1)\times(N_x-1)$ coefficient matrix. We would to reduce the dimension of the associated linear system. To this end we construct an orthogonal basis of dimension l in order to restrict the map Q onto this subspace.

3.2 Proper Orthogonal Decomposition

Now, we introduce the POD method to be used to reduce the finite–difference scheme of implicit type (20). Let consider a grid in $[0,a]\times[0,T]$ of points

$$\{(\hat{x}_j,\hat{t}_m) : j=1,2,\ldots,N_x-1,\ m=0,1,\ldots,\hat{N}_t\},$$

here $\hat{N}_t$ is the snapshots number. We assume that in this case we can obtain the snapshots

$$\mathbf{u}_s^{(m)} = [u(\hat{x}_1,\hat{t}_m)\ u(\hat{x}_2,\hat{t}_m)\ \cdots\ u(\hat{x}_{N_x-1},\hat{t}_m)]^{\mathbf{T}} \in \mathbb{R}^{N_x-1}.$$

for $m=0,1,\ldots,\hat{N}_t$, either by experimental data or by previous computer simulation of a similar problem. In this situation we consider the Hilbert space $\mathbb{R}^{N_x-1}$ with the usual inner product. It easy to see that the map R_n is given by

$$R_n(\mathbf{v}) = \sum_{i=1}^{\hat{N}_t} (\mathbf{v},\mathbf{u}_s^{(i)})_{\mathbb{R}^{N_x-1}}\mathbf{u}_s^{(i)} = \mathbf{u}_s\mathbf{u}_s^{\mathbf{T}}\mathbf{v}$$

where

$$\mathbf{u}_s = \left[\mathbf{u}_s^{(1)}\ \mathbf{u}_s^{(k)}\ \cdots\ \mathbf{u}_s^{(\hat{N}_t)}\right] \in \mathbb{R}^{(N_x-1)\times\hat{N}_t}.$$

Thus, the l–modal basis, that we denote by $\{\mathbf{\Psi}_1,\ldots,\mathbf{\Psi}_l\}$, can be computed solving the eigenvalue problem

$$\mathbf{u}_s\mathbf{u}_s^{\mathbf{T}}\mathbf{v} = \lambda\mathbf{v}. \tag{22}$$

This is called *the direct method*.

The normalized eigenvectors in (22) are the left singular vectors in the Singular Value Decomposition (SVD) of $\mathbf{u}_s$, that is, the eigenvectors $\mathbf{\Psi}_j$, for $j=1,2,\ldots,k$ are the columns of $\mathbf{\Psi}$ where

$$\mathbf{u}_s = \mathbf{\Psi}\Sigma V^{\mathbf{T}}. \tag{23}$$

Here $\mathbf{\Psi}$ and V have orthonormal columns and Σ is a square diagonal matrix. The diagonal elements of Σ are the non–zero singular values, that is, $\sqrt{\lambda}$. The matrix representation $\mathbf{\Psi}$ for the POD basis vectors can also be computed by finding the right singular vectors of $\mathbf{u}_s$ as described below. The eigenvalues and eigenvectors for

$$\mathbf{u}_s^{\mathbf{T}}\mathbf{u}_s\mathbf{v} = \lambda\mathbf{v} \tag{24}$$

correspond to the right singular vectors and singular values of (22), because

$$\mathbf{u}_s^{\mathbf{T}} = V\Sigma\boldsymbol{\Psi}^{\mathbf{T}}.$$

Since, $\Sigma V^{\mathbf{T}} V\Sigma^{-1} = I$, then, from (23), we obtain

$$\boldsymbol{\Psi} = \mathbf{u}_s V\Sigma^{-1}. \tag{25}$$

The latter method is called the *snapshot method or the sample method*. We note that if the data $\mathbf{u}_s^{(i)}$ are linearly independent and if $\hat{N}_t < N_x - 1$ then it is convenient to use this methodology.

3.3 A reduced finite–difference scheme of implicit type

Now we construct a reduced finite–difference scheme by using the representative modes computed either by the direct method or by the snapshot method. To this end we consider the matrix of representative modes

$$\boldsymbol{\Psi}^* = [\boldsymbol{\Psi}_1 \ \boldsymbol{\Psi}_2 \ \cdots \ \boldsymbol{\Psi}_l]$$

where $l << N_x - 1$ is choosed by means the condition

$$\sum_{i=1}^{l} \lambda_i / \sum_{i=1}^{N_x - 1} \lambda_i > 0.99.$$

Then, we would to search an approximate solution $\hat{\mathbf{U}}^{m+1}$ of the difference equation (20) satisfying that

$$\hat{\mathbf{U}}^{m+1} = \boldsymbol{\Psi}^* \mathbf{a}^{m+1} = \sum_{j=1}^{l} a_j^{m+1} \boldsymbol{\Psi}_j, \tag{26}$$

where $\mathbf{a}^{m+1} = [a_1^{m+1} \ a_2^{m+1} \ \cdots \ a_l^{m+1}]^{\mathbf{T}}$, for $m = 0, 1, \ldots, N_t - 1$. Observe that (26) is equivalent to the Galerkin expansion (10). Moreover,

$$Q|_{\mathrm{Col}\,\boldsymbol{\Psi}^*} = (\boldsymbol{\Psi}^*)^{\mathbf{T}} Q \boldsymbol{\Psi}^*,$$

where Col $\boldsymbol{\Psi}^* = \mathrm{span}\{\boldsymbol{\Psi}_1, \boldsymbol{\Psi}_2, \ldots, \boldsymbol{\Psi}_l\}$. Thus, $\mathbf{a}^{m+1}$ is the solution of the reduced model given by the difference equation

$$\begin{aligned} Q|_{\mathrm{Col}\,\boldsymbol{\Psi}^*} \mathbf{a}^{m+1} &= (\boldsymbol{\Psi}^*)^{\mathbf{T}} \left(\mathbf{f}(\hat{\mathbf{U}}^m) + \mathbf{g}^m \right), \\ \hat{\mathbf{U}}^0 &= \mathbf{u}^0. \end{aligned} \tag{27}$$

for $m = 0, 1, \ldots, N_t - 1$. Solving (27) we obtain the matrix

$$\mathbf{a} = [\mathbf{a}^1 \ \mathbf{a}^2 \ \cdots \mathbf{a}^{N_t}]$$

and finally we have that

$$\hat{\mathbf{U}} = \mathbf{\Psi}^* \mathbf{a}.$$

Note that in the reduced version we solve N_t linear systems with a $l \times l$ coefficient matrix.

3.4 Numerical Results

We consider a vanilla European put with strike $K = 8$ and maturity $T = 1/3$ and we assume that the volatility and interest rates are constant $\sigma = 0.30$ and $r = 0.05$. Moreover, we consider $N_x = 200$ and $N_t = 400$. Plotting the pointwise error for the Crank–Nicolson scheme Figure 2 shows that the error is concentrated around the singularity, that is $t = 0$ and $x = K$. Thus if we would to improve the accuracy the grid should be highly refined near $t = 0$ and $x = K$. However, finite difference methods of implicit type presented so far rely on uniform grids in the x–variable. Therefore, at this point, we keep in mind to use a reduced Crank–Nicolson scheme.

First, we generate the snapshots by using the closed form solution for the European Put in a grid where $x_j = \hat{x_j}$ for $j = 1, 2, \ldots, N_x - 1$ and $\hat{N}_t = 10$. We obtained, in the numerical experiments implemented for different values of r and σ and $\hat{N}_t \geq 10$, only one basis function, that is, $l = 1$. In Figures 3 and 4 we plot the pointwise errors produced at $t = 0$ from the reduced Crank–Nicolson scheme using the direct method or the snapshot method respectively, against the Crank–Nicolson scheme. As we can see in all cases the Reduced scheme has more accuracy near $x = K$. A similar result is obtained when we consider the snapshots generated by a previous Crank–Nicolson taking $\hat{t}_m = 40m$ for $m = 1, 2, \ldots, 10$; (see Figures 5 and 6). Note, that due to the fact that the modal basis is composed by only one vector, we solve on each iteration, in the case of the Reduced Crank–Nicolson scheme, a one–dimensional linear equation.

Now, we compute by means a Crank–Nicolson with parameter values $r_{POD} = 0.10$ and $\sigma_{POD} = 0.3$ a modal basis by using the snapshot method. Then we use this modal basis to run a Reduced Crank–Nicolson with parameter values $r = 0.05$ and $\sigma = 0.4$. In Figures 7 and 8 we show as the accuracy near $x = K$ is, in both cases, unexpected good. In Figure 9 we plot $||\mathbf{u}^{N_t} - \mathbf{U}^{N_t}||_{\mathbb{R}^{N_x}}$ against $||\mathbf{u}^{N_t} - \hat{\mathbf{U}}^{N_t}||_{\mathbb{R}^{N_x}}$ as functions of (r, σ), where $\hat{\mathbf{U}}^{N_x}$ was computed by using the Reduced Crank–Nicolson scheme by means the modal basis with parameter values r_{POD} and σ_{POD}. From this we can conclude that it is possible to use a modal basis obtained from a previous computer simulation in order to valuate, by using the reduced finite difference scheme, options with a different parameter values.

4 Concluding Remarks

We have presented an effective method for the numerical solution of a Pricing Partial Differential Differential Equation of European type, due to the use of POD–based reduced order models applied to a Finite Difference scheme of implicit type. Moreover we observe from the numerical experiments that the efficiency of the reduced scheme, constructed from a previously computed basis, remains for a wide set of parameters values.

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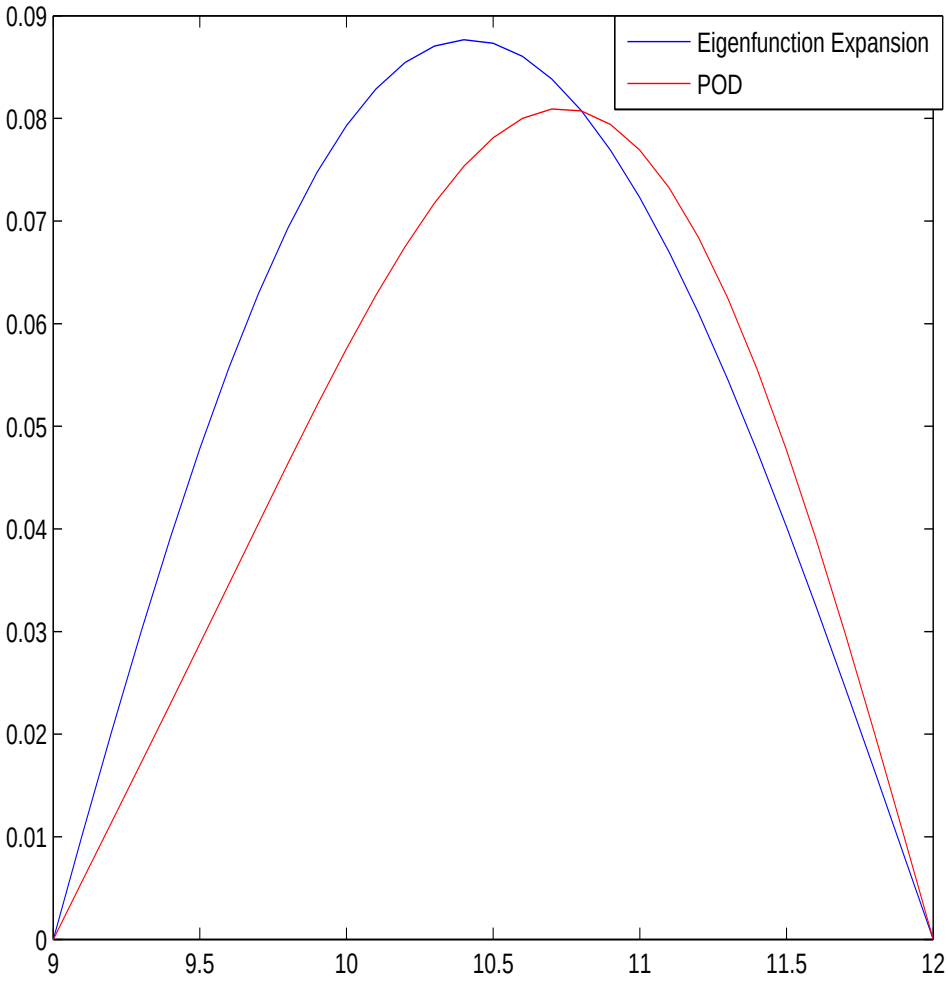

Figure 1: The maps $u(x,0) = \sum_{j=1}^{20} c_j e^{-\mu_j T}\phi_j(x)$ and $\hat{u}(x,0)$ with $n = 4$.

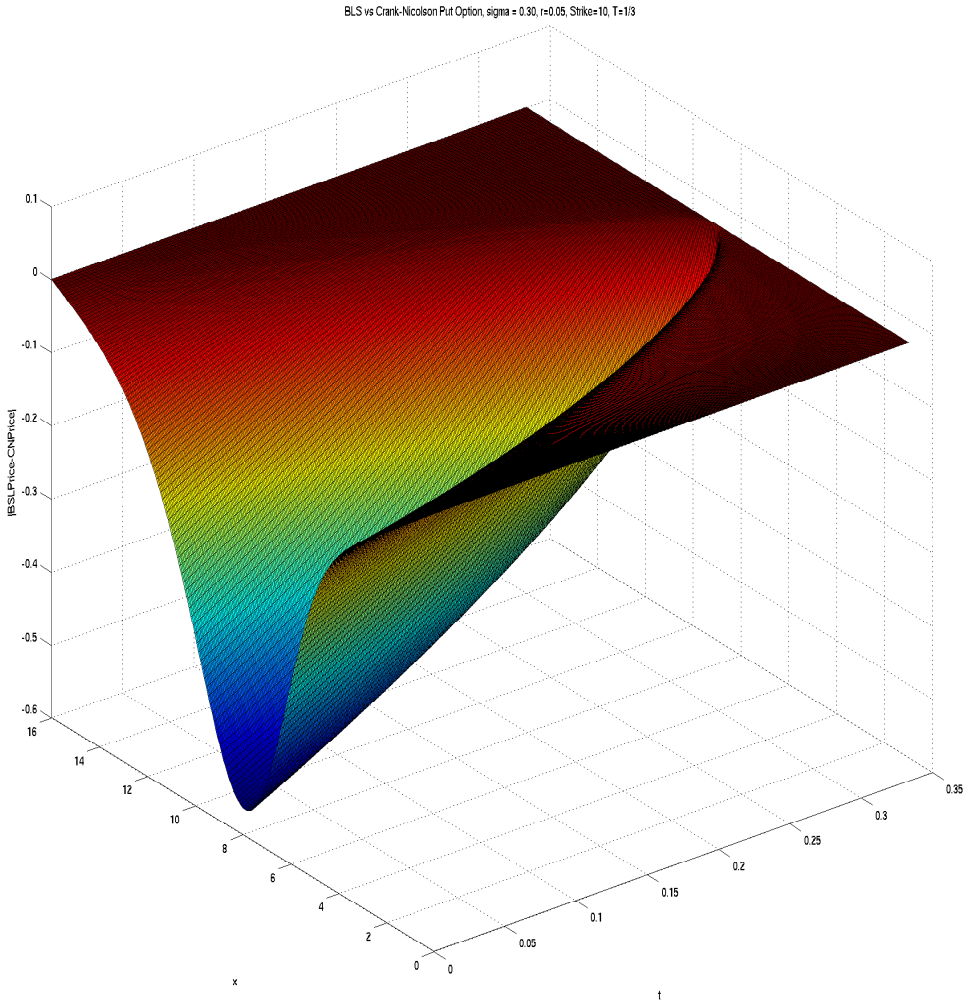

Figure 2: The pointwise error produced by the Crank–Nicolson scheme for an European Put in $[0, 16] \times [0, 1/3]$, $K = 8$, $\sigma = 0.30$, and $r = 0.05$.

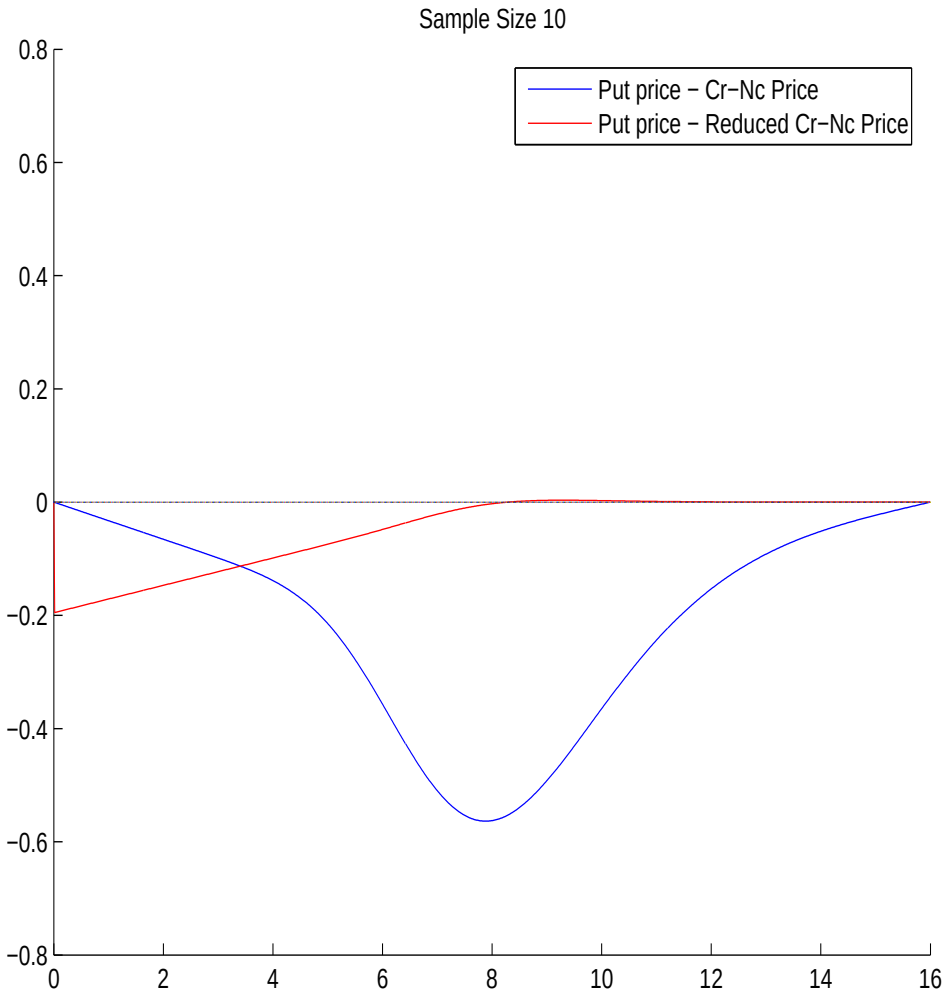

Figure 3: The pointwise error at $t = 0$ produced by the Crank–Nicolson and Reduced Crank–Nicolson (direct method) schemes for an European Put in $[0, 16] \times [0, 1/3]$, $K = 8$, $\sigma = 0.30$, and $r = 0.05$.

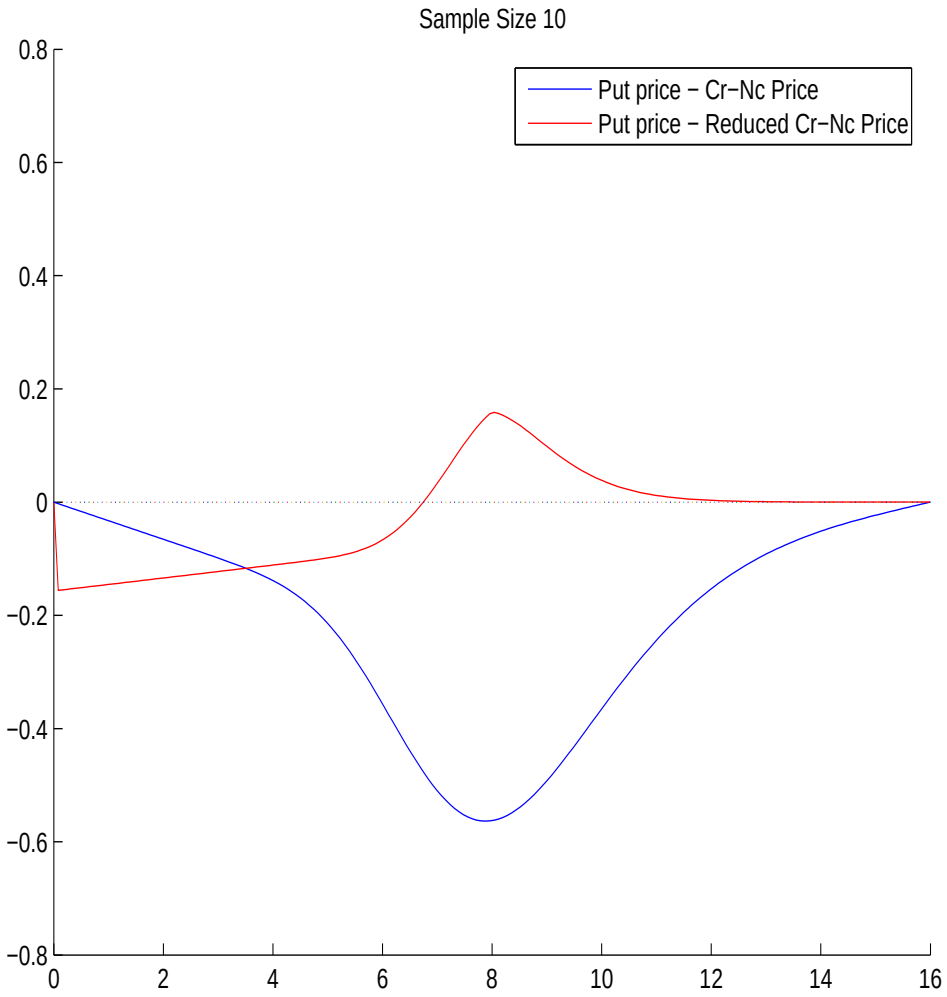

Figure 4: The pointwise error at $t = 0$ produced by the Crank–Nicolson and Reduced Crank–Nicolson (snapshot method) schemes for an European Put in $[0, 16] \times [0, 1/3]$, $K = 8$, $\sigma = 0.30$, and $r = 0.05$.

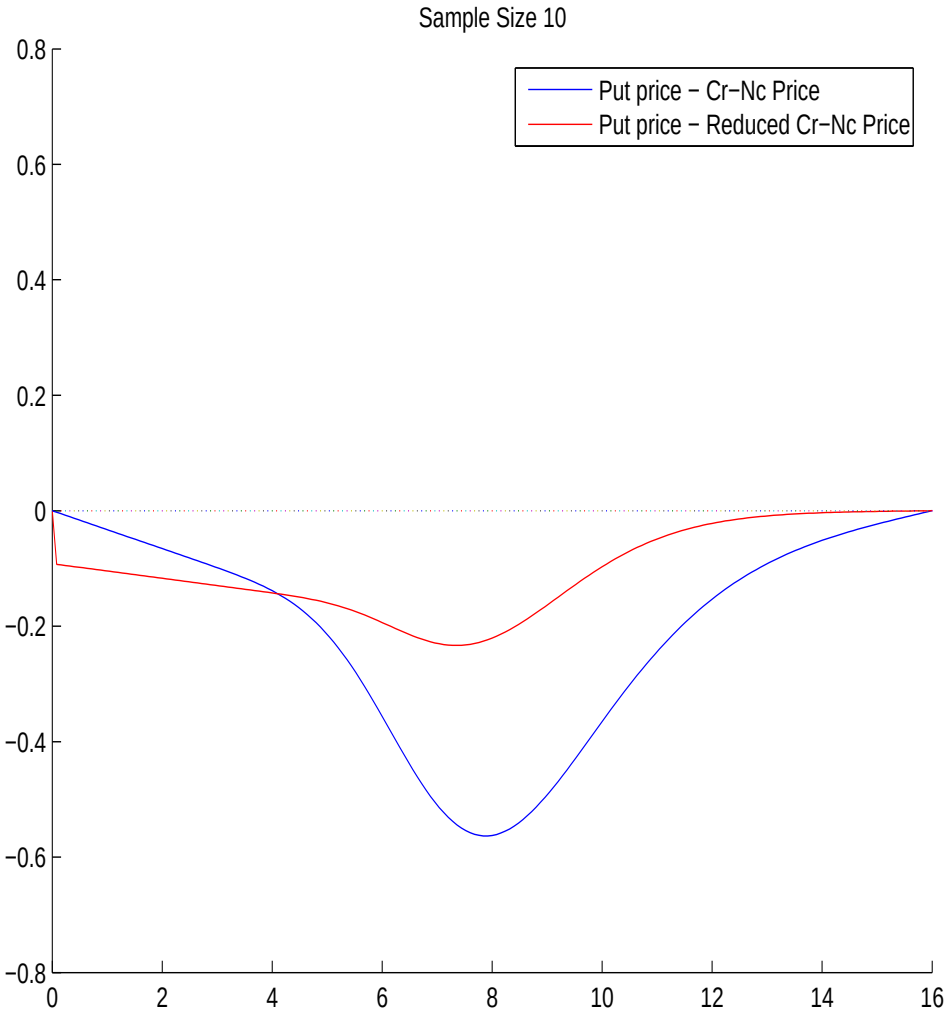

Figure 5: The pointwise error at $t = 0$ produced by the Crank–Nicolson and Reduced Crank–Nicolson (direct method) schemes for an European Put in $[0, 16] \times [0, 1/3]$, $K = 8$, $\sigma = 0.30$, and $r = 0.05$.

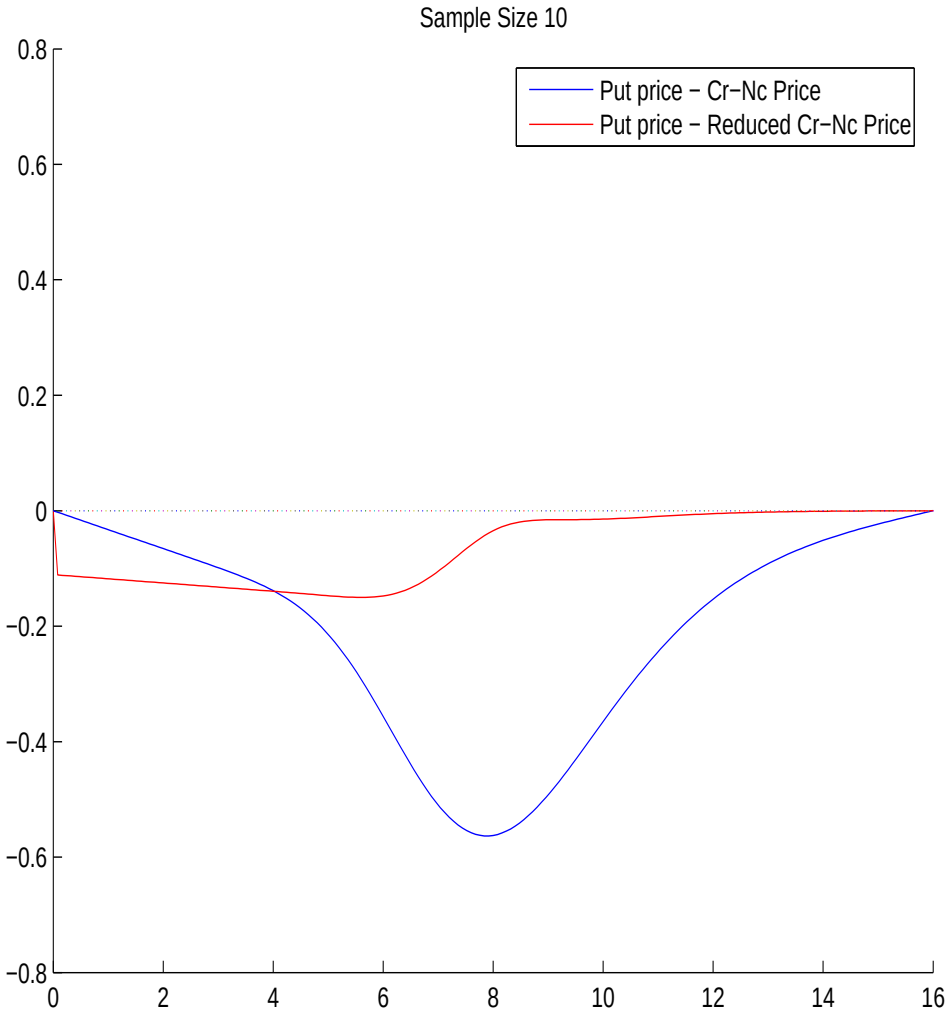

Figure 6: The pointwise error at $t = 0$ produced by the Crank–Nicolson and Reduced Crank–Nicolson (snapshot method) schemes for an European Put in $[0, 16] \times [0, 1/3]$, $K = 8$, $\sigma = 0.30$, and $r = 0.05$.

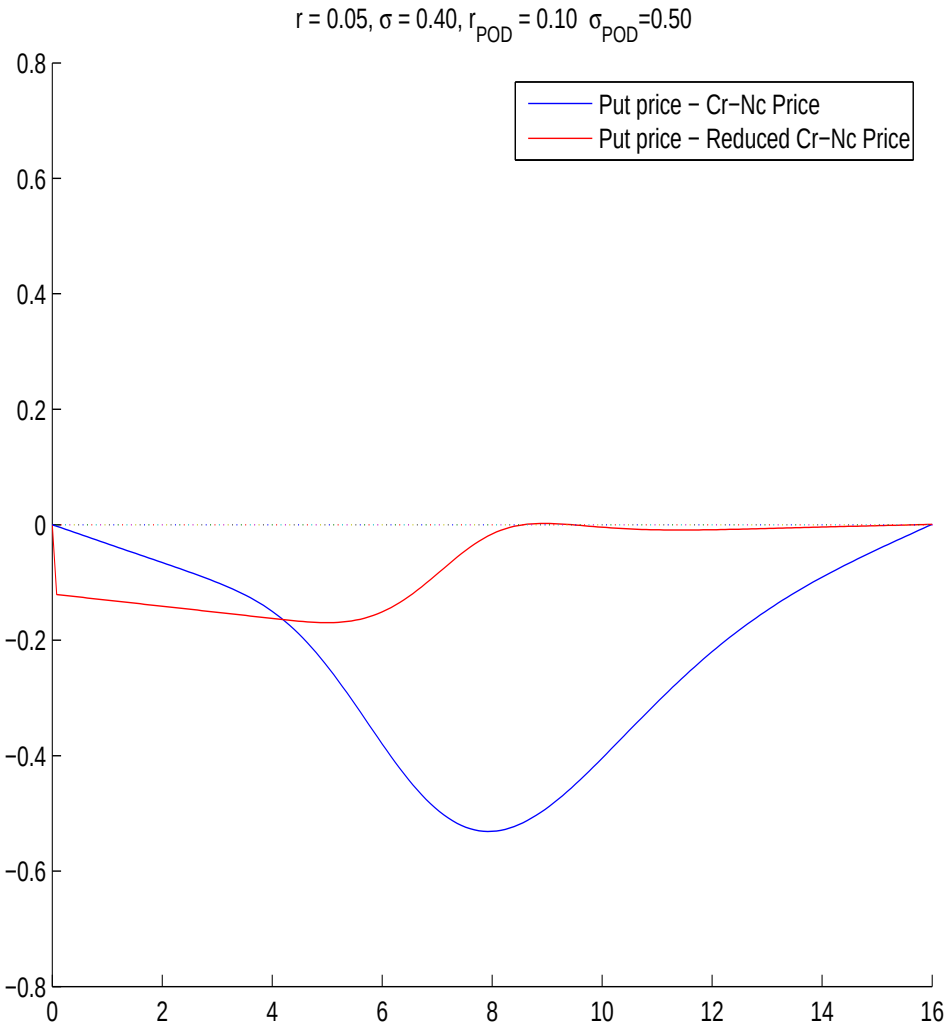

Figure 7: The pointwise error at $t = 0$ produced by the Crank–Nicolson and Reduced Crank–Nicolson (snapshot method) schemes for an European Put in $[0, 16] \times [0, 1/3]$, $K = 8$.

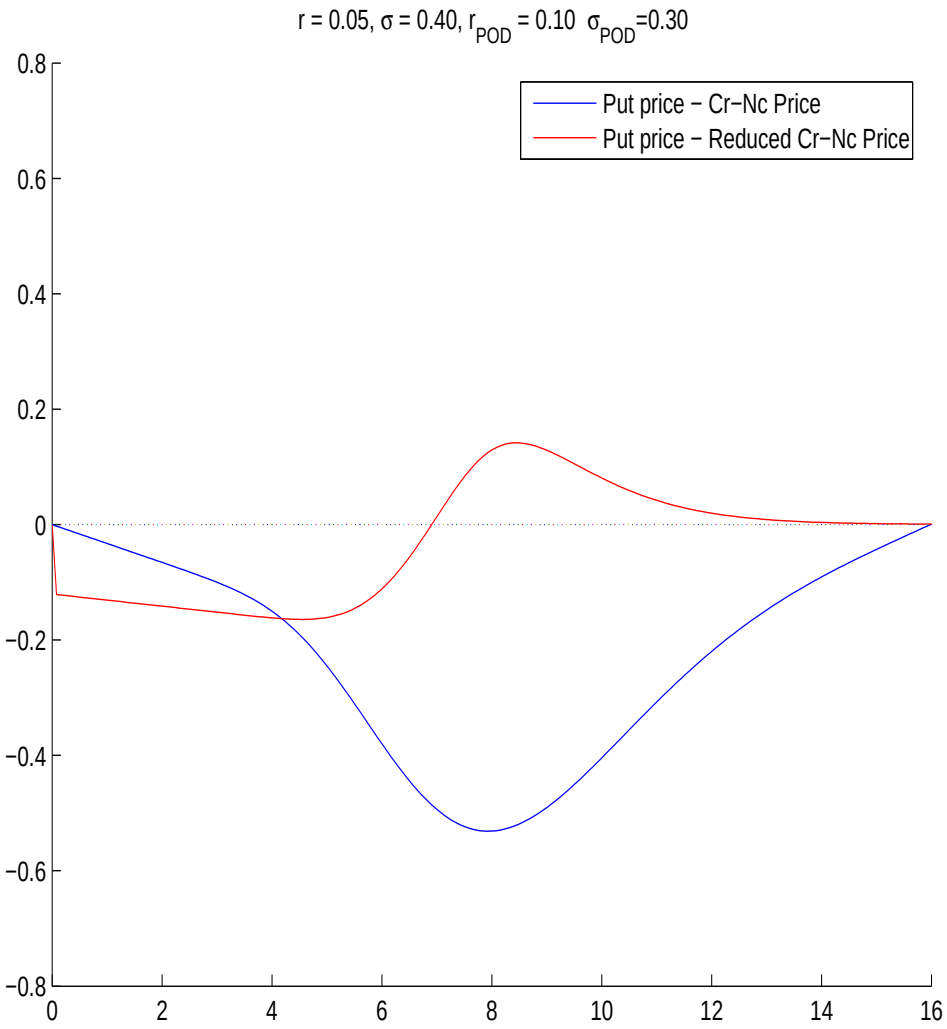

Figure 8: The pointwise error at $t = 0$ produced by the Crank–Nicolson and Reduced Crank–Nicolson (snapshot method) schemes for an European Put in $[0, 16] \times [0, 1/3]$, $K = 8$.

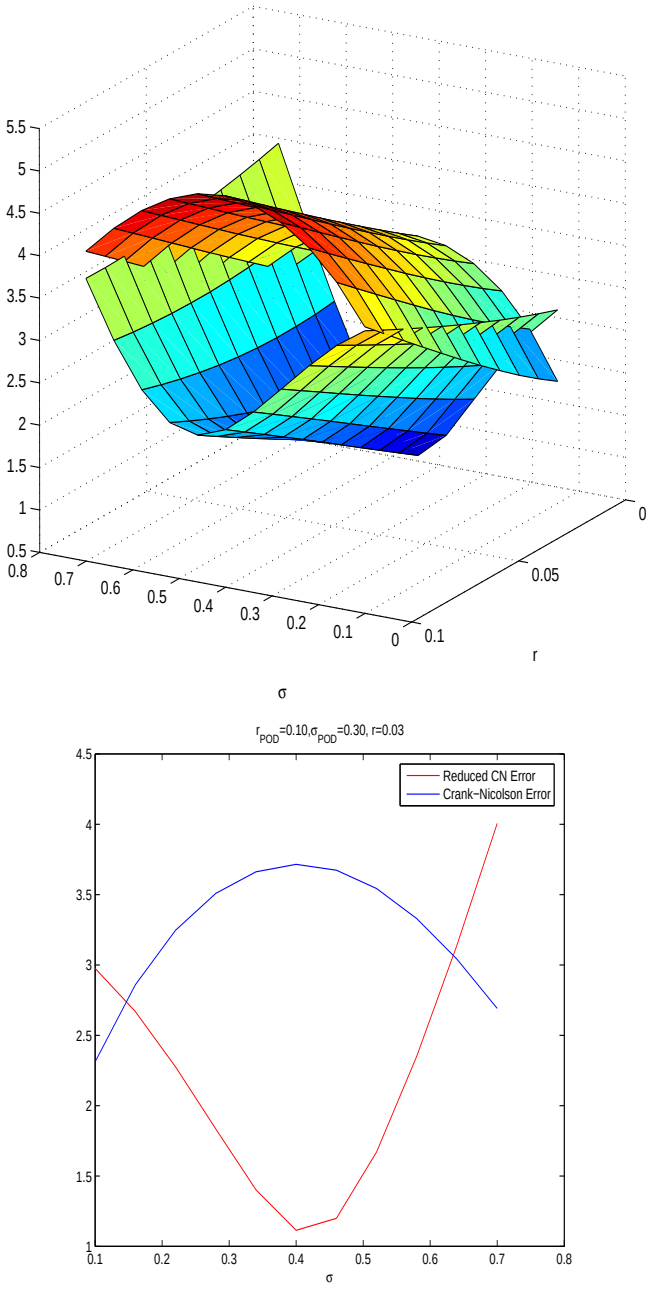

Figure 9: The Crank–Nicolson Error function $||\mathbf{u}^{N_t} - \mathbf{U}^{N_t}||_{\mathbb{R}^{N_x}}$ against the Reduced Crank–Nicolson Error function $||\mathbf{u}^{N_t} - \hat{\mathbf{U}}^{N_t}||_{\mathbb{R}^{N_x}}$ and the cross section at $r = 0.03$.