Monografías de la Real Academia de Ciencias de Zaragoza. **25:** 251–268, (2004).

Symmetries and convergence of normal form transformations

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Abstract

We present a survey of results about Poincaré–Dulac normal forms and normalizing transformations, with emphasis on the close connection between symmetry properties and convergence of normal form transformations. Some open problems and questions are also mentioned. No particular prerequisites are expected from the reader.

Key words and expressions: Normal forms of vector fields, convergent transformations, eigenvalues and resonances.

MSC: 34C20, 37C15, 34C14, 37C80

1 Introduction

Normal forms are among the most important tools for the local analysis and classification of vector fields and maps near a stationary point. The theory of normal forms was initiated by Poincaré, and later extended by Dulac, and by Birkhoff to Hamiltonian vector fields. Contemporary accounts on normal forms and their applications can be found in Arnold [1], Arnold et al. [2], Bibikov [4], Bruno [6], Chow et al. [8], Cicogna and Gaeta [11], and Iooss and Adelmeyer [18]. There are various types of normal forms in use, depending on the specific problem one wants to address. One canonical type of normal form (with a uniqueness property in mind) was introduced and discussed by Elphick et al. [15]. Poincaré-Dulac normal forms (see also Bruno [5], [6]) are defined only with respect to the semisimple part of the linearization, and thus are not unique (the normalization can often be refined), but they are very important since they have certain built-in symmetries, and thus admit a well-defined reduction procedure.

One purpose of normal forms is to aid in the analysis of the local dynamics, and for this it is frequently sufficient to compute and analyze a finite portion of the normal form. (A recent paper on computations in general circumstances is [27]). But in the analytic setting the complete normal form is also of interest for certain stability problems, and for classification. In this context, convergence issues arise.

In this paper we first give a quick introduction to normalization procedures and normal forms, and then discuss convergence issues and their connection to symmetries. This is mostly a survey of existing results, but some new perspectives, problems and suggestions were added.

2 Transformations of Vector Fields

Our objects are local ordinary differential equations (over $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$)

$$\dot{x} = f(x), \qquad f(0) = 0$$

with f analytic, thus we have a Taylor expansion

$$f(x) = Bx + \sum_{j \geq 2} f_j(x) = Bx + f_2(x) + f_3(x) + \cdots \quad \text{near} \quad 0 \in \mathbf{K}^n.$$

Here $B = Df(0)$ is linear, and each f_j is homogeneous of degree j.

Our objective is to simplify the Taylor expansion of f. For this purpose, take an analytic "near-identity" map

$$H(x) = x + h_2(x) + \cdots$$

Since H is locally invertible, there is a unique

$$f^*(x) = Bx + \sum_{j \geq 2} f_j^*(x)$$

such that

$$DH(x)f^*(x) = f(H(x)) \qquad (\dagger)$$

for all x. (One then says that f^* and f are related by H, and H "preserves solutions" in the sense that parameterized solutions of $\dot{x} = f^*(x)$ are mapped to parameterized solutions of $\dot{x} = f(x)$ by H.)

It is convenient to introduce the following notation:

$$f^* \xrightarrow{H} f \quad \text{if } (\dagger) \text{ holds.}$$

3 Simplification

Consider the Taylor expansion

$$f(x) = Bx + f_2(x) + \cdots + f_{r-1}(x) + f_r(x) + \cdots,$$

and assume that $f_2, \ldots, f_{r-1}$ are already "satisfactory" (according to some specified criterion).

Then the ansatz $H(x) = x + h_r(x) + \cdots$ yields

$$f^*(x) = Bx + f_2(x) + \cdots + f_{r-1}(x) + f_r^*(x) + \cdots$$

(thus terms of degree $< r$ are unchanged), and at degree r one finds the so-called homological equation:

$$[B, h_r] = f_r - f_r^*.$$

(As usual, $[p, q](x) = Dq(x)p(x) - Dp(x)q(x)$ is the Lie bracket.)

Here we have an equation on the space $\mathcal{P}_r$ of homogeneous vector polynomials of degree r. This is a finite dimensional space, and $\mathrm{ad}B = [B, \cdot]$ sends $\mathcal{P}_r$ to $\mathcal{P}_r$.

How can the degree r term f_r be simplified? Let $\mathcal{W}$ be any subspace of $\mathcal{P}_r$ such that

$$\mathrm{image}\,(\mathrm{ad}B) + \mathcal{W} = \mathcal{P}_r.$$

Then one may choose $f_r^* \in \mathcal{W}$. If the sum is direct then $f_r^* \in \mathcal{W}$ is uniquely determined by f_r.

4 Poincaré-Dulac Normal Form

One may say that the type of simplification is specified by the choice of a subspace $\mathcal{W}_r$ for each degree r such that $\mathrm{image}\,(\mathrm{ad}B) + \mathcal{W}_r = \mathcal{P}_r$.

The Poincaré-Dulac choice is as follows: If

$$B = B_s + B_n$$

is the decomposition into semisimple and nilpotent part, then $\mathrm{ad}\,B = \mathrm{ad}\,B_s + \mathrm{ad}\,B_n$ is the corresponding decomposition on $\mathcal{P}_r$. Choose $\mathcal{W}_r = \mathrm{Ker}(\mathrm{ad}\,B_s)$. By linear algebra

$$\mathcal{W}_r + \mathrm{image}(\mathrm{ad}\,B) = \mathcal{P}_r;$$

and the sum is direct if B is semisimple. Elphick *et al.* [15] proceeded to choose a suitable subspace of $\mathrm{Ker}(\mathrm{ad}\,B_s)$ (constructed from the Bargmann scalar product), to achieve a direct sum decomposition, and thus uniqueness, even in the non-semisimple case. Anyway, we have $[B_s, f_r^*] = 0$.

Definition 1 *f^* is in Poincaré-Dulac normal form if $[B_s, f_j^*] = 0$ for all j; equivalently $[B_s, f^*] = 0$.*

An immediate consequence of the considerations above is:

Proposition 1 *For any $f = B + \cdots$ there are formal power series $H(x) = x + \cdots$, $f^*(x) = Bx + \cdots$ such that $f^* \xrightarrow{H} f$ and f^* is in Poincaré-Dulac normal form.*

As it turns out, convergence is a quite different, and quite difficult, matter.

5 The Role of the Eigenvalues

Let f be as above, and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of B. With no loss of generality, we may assume

$$B_s = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)$$

A simple computation for the "vector monomial" $p(x) = x_1^{m_1} \cdots x_n^{m_n} e_j$ shows that

$$[B_s, p] = (m_1\lambda_1 + \cdots + m_n\lambda_n - \lambda_j) \cdot p$$

Thus, the vector monomials form an eigenbasis of $\mathrm{ad}\, B_s$ on the space $\mathcal{P}_r$, with eigenvalues $m_1\lambda_1 + \cdots + m_n\lambda_n - \lambda_j$ (m_j nonnegative integers, $\sum m_j = r$). The eigenvalues play a crucial role both for a first classification of formal normal forms and for convergence issues. The following distinction is pertinent here:

- One calls $(\lambda_1, \ldots, \lambda_n)$ *non-resonant* if the equations

  $$m_1\lambda_1 + \cdots + m_n\lambda_n - \lambda_k = 0$$

  for integers $m_j \geq 0$, $\sum m_j \geq 2$, have no solution for $k = 1, \ldots, n$.

- One calls $(\lambda_1, \ldots, \lambda_n)$ *resonant* otherwise.

Examples.

- If $(\lambda_1, \ldots, \lambda_n)$ is non-resonant then necessarily the normal form f^* equals B.

- For $(\lambda_1, \lambda_2) = (1, -1)$ a general vector field in normal form is given by

  $$f^*(x) = Bx + \sum_{j \geq 1} (x_1 x_2)^j \cdot (\alpha_j x + \beta_j Bx) \quad \text{with} \quad \alpha_j, \beta_j \in \mathbf{K}.$$

- *Simple resonance:* Given $B = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)$, assume that there are nonnegative integers $d_1, \ldots, d_n$, $\sum d_j > 0$ such that $d_1\lambda_1 + \cdots + d_n\lambda_n = 0$, and furthermore that $n - 1$ of the λ_j are linearly independent over the rational number field $\mathbf{Q}$. With $\psi(x) := x_1^{d_1} \cdots x_n^{d_n}$, every normal form is of type

  $$f^*(x) = Bx + \sum_{l \geq 1} \psi(x)^l C_l x,$$

  with diagonal matrices C_l.

The eigenvalues are relevant for convergence questions because products of terms $m_1\lambda_1 + \cdots + m_n\lambda_n - \lambda_j$ $(\neq 0)$ occur in the denominators of the coefficients for a normalizing transformation, as the homological equation shows. Thus one may expect problems with "small denominators".

6 Poincaré-Dulac: Special Properties

In this section let $\dot{x} = f(x)$ be in Poincaré-Dulac normal form, thus

$$[B_s, f] = 0 \qquad (\text{and } B_s \neq 0)$$

Such systems have built-in symmetries: The Lie bracket condition implies that the flow of $\dot{x} = B_s x$ yields symmetries of $\dot{x} = f(x)$. This, in turn, opens a path to reduction. But systems in Poincaré-Dulac normal form have properties beyond being symmetric, which might be given the working title of "rigidity":

Proposition 2 *Let $\dot{x} = f(x)$ be in Poincaré-Dulac normal form. Then:*

- $[g, f] = 0 \;\Rightarrow\; [g, B_s] = 0$

  (A vector field that commutes with f necessarily commutes with B_s.)

- $L_f(\varphi) = 0 \;\Rightarrow\; L_{B_s}(\varphi) = 0$

  (An integral of f is necessarily an integral of B_s.)

(We recall that for scalar-valued φ the Lie derivative $L_f(\varphi)$ is defined by $L_f(\varphi)(x) = D\varphi(x)f(x)$.) For a proof we refer to [32]. "Rigidity" is a strong structural property, and it turns out to be essential for normal forms.

7 Convergence Problems

We have seen: Given an analytic vector field f, there are formal series H and f^* such that $f^* \xrightarrow{H} f$ and f^* is in Poincaré-Dulac normal form. This raises the question: Does there always exist a convergent H? (Here "convergent" means: convergent in some neighborhood of 0.)

An early positive result is due to Poincaré (about 1890): If $\lambda_1, \ldots, \lambda_n$ lie in some open half-plane in $\mathbf{C}$ that does not contain 0 (e.g. the open left half-plane), then there is a convergent transformation. The proof is relatively easy, using suitable majorants.

An early negative result is due to Horn (about 1890): The system

$$\begin{aligned} \dot{x}_1 &= x_1^2 \\ \dot{x}_2 &= x_2 - x_1 \end{aligned}$$

(with eigenvalues $(0,1)$ for the linear part) admits no convergent transformation to normal form.

(The ansatz for a transformation leads to the differential equation

$$x^2 \cdot y' = y - x,$$

with divergent solution $\sum_{k\geq 1}(k-1)!x^k$. See [12] for a detailed discussion.)

Note that this is a frustrating example: There are no problems with "small denominators" here, but still convergence is elusive.

The following theorem may be seen as the starting point for the "modern phase" of convergence results.

Theorem 1 *(C.L. Siegel, 1952, see [28])*

Assume that there are constants $C > 0$, $\nu > 0$ such that for all nonnegative integer tuples (m_i), $\sum m_i > 1$ the following inequality holds:

$$\left|\sum_{i=1}^{n} m_i\lambda_i - \lambda_j\right| \geq C \cdot (m_1 + \cdots + m_n)^{-\nu}. \qquad \text{(S)}$$

Then there is a convergent transformation to normal form.

Example. Let $\dot{x} = Bx + \ldots$ be given in dimension two, and assume that the eigenvalues λ_1, λ_2 of B are nonresonant, and algebraic but not rational numbers. (This is the case when the entries of B are rational but the characteristic polynomial is irreducible over the rationals.) Then λ_2/λ_1 is algebraic but not rational, and (λ_1, λ_2) satisfies a condition of type (S), due to a number-theoretic result of Thue, Siegel and Roth. Thus there exists a convergent transformation to normal form.

Siegel's result is strong in the sense that a condition of type (S) is satisfied by Lebesgue - almost all tuples $(\lambda_1, \ldots, \lambda_n) \in \mathbf{C^n}$. But there is a drawback: The condition forces the normal form to be uninteresting: One necessarily has $f^* = B$. An extension to some resonant scenarios is as follows.

Theorem 2 *(V.A. Pliss, 1964, see [26])*

Assume that:

(i) *The nonzero elements among the $\sum_{i=1}^{n} m_i\lambda_i - \lambda_j$ satisfy condition* (S).

(ii) *Some formal normal form of f is equal to B.*

Then there is a convergent transformation to normal form.

Although it seems that the drawback to Siegel's theorem has not really been removed, Pliss' result will turn out to be very useful in the following.

Fundamental insights into convergence and divergence problems were achieved by A.D. Bruno around 1970 and later; see [5], [6]. Bruno introduced two conditions:

- *Condition* ω is a sharper version of Siegel's (S) (for $\sum m_i\lambda_i - \lambda_j \neq 0$)
- *Condition* A (in a simplified version):
  For some formal normal form one has
  $$f^* = \sigma \cdot B \qquad \text{(with } \sigma \text{ a scalar function)}$$
  This condition clearly extends Pliss' condition.

Theorem 3 *(Bruno) If Condition ω and Condition A are satisfied then a convergent normal form transformation exists.*

Theorem 4 *(Bruno) Assume that some formal normal form f^* is convergent and does not satisfy (a weaker version of) Condition A, or that $(\lambda_1, \ldots, \lambda_n)$ does not satisfy a weaker version of Condition ω.*
Then there exists an analytic f with formal normal form f^*, but no convergent transformation sending f^* to f.

Bruno's criteria are still the standard against which later results are measured. A recent substantial improvement, including criteria for divergence of the formal normal form, is due to L. Stolovich [29].

Despite the deep results cited above, the following basic question remains to be addressed: Given a local analytic vector field f, characterize properties that ensure the existence of a convergent normal form transformation. (Note that Bruno's convergence and divergence theorems deal with a different problem!)

A possible approach to an answer is based on the earlier observation that symmetry properties play a role. Let us formalize this observation:

Proposition 3 *If f admits a convergent normalizing transformation then there is a nontrivial g (i.e., $g \notin \mathbf{K} \cdot f$) such that $[g, f] = 0$.*

Proof. There is a convergent Ψ and a convergent f^* in normal form such that

$$f^* \xrightarrow{\Psi} f$$

Then Ψ sends B_s to some analytic $g = B_s + \cdots$, and $[B_s, f^*] = 0$ implies $[g, f] = 0$. If $f^* \neq B_s$, we are done. If $f^* = B_s$ take some linear map commuting with B_s. □

One can try to turn this around and thus obtain sufficient convergence criteria, which has been done, with some success, since the early 1990s. Some of the relevant contributions are due to Markhashov [22], Bruno and Walcher [7], Cicogna [9], [10], Gramchev and Yoshino [17]. See also the paper [12], where some results are surveyed and discussed in a general framework.

The objects to deal with are $\mathcal{C}_{for}(f)$ and $\mathcal{C}_{an}(f)$, i.e., the formal, respectively analytic, centralizer of f, which by definition consists of all formal, respectively analytic, vector fields h such that $[h,\, f] = 0$.

The Non-resonant Case

Here all λ_i are distinct, and $\sum_{j=1}^{n} m_i\lambda_i - \lambda_j \neq 0$ for all nonnegative integer tuples $(m_1, \ldots, m_n)$ with $\sum m_i \geq 2$.

Any normal form of $f = B + \cdots$ is given by $\widehat{f} = B$. Let us consider the centralizer:

$\mathcal{C}_{for}(\widehat{f})$ consists only of linear vector fields (and thus equals $\mathcal{C}_{an}(\widehat{f})$), and $\dim \mathcal{C}_{for}(\widehat{f}) = n$. To verify this, let $B = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)$, then the centralizer consists of all diagonal matrices.

The next result may be called of "Markhashov type"; in the form given here it is due to Bambusi et al. [3].

Theorem 5 *There is a convergent transformation to normal form if and only if* $\dim \mathcal{C}_{an}(f)$ $= n$.

Proof. For necessity, note that such a convergent transformation sends $\mathcal{C}_{an}(\widehat{f})$ to $\mathcal{C}_{an}(f)$.

As for sufficiency, first take a formal transformation Ψ such that $\widehat{f} \xrightarrow{\Psi} f$, and $\widehat{f}$ is in normal form. Then $\mathcal{C}_{for}(f)$ is mapped to $\mathcal{C}_{for}(\widehat{f})$ via Ψ.
By the dimension assumption, $\mathcal{C}_{for}(f) = \mathcal{C}_{an}(f)$. Since $\widehat{g}(x) = x \in \mathcal{C}_{for}(\widehat{f})$, there exists $g(x) = x + \ldots \in \mathcal{C}_{an}(f)$. According to Poincaré, there is a convergent transformation Φ such that $\widehat{g} \xrightarrow{\Phi} g$.
Now define $\widetilde{f}$ via $\widetilde{f} \xrightarrow{\Phi} f$. Then $\widetilde{f} = B + \cdots$, and $[\widetilde{f}, \widehat{g}] = 0$ (so $\widetilde{f}$ is linear), whence $\widetilde{f} = B$ is in normal form. □

This result can be extended to the case when $\dim \mathcal{C}_{for}(B)$ is finite (note that $\dim \mathcal{C}_{for}(f)$ is then necessarily also finite, due to Proposition 2): There is a convergent normalizing transformation if and only if $\mathcal{C}_{an}(f)$ and $\mathcal{C}_{for}(f)$ have the same dimension. (Problems with Condition ω can be avoided by suitable choice of centralizer elements.)

8 Dimension Two

Theorem 6 *(Bruno and Walcher, see [7]). In dimension $n = 2$, there is a convergent transformation of f to normal form if and only if f admits a nontrivial centralizer element.*

Proof. We sketch the argument for the case $(\lambda_1, \lambda_2) = (1, -1)$, to keep notation simple. The formal normal form is given by $\widehat{f}(x) = Bx + \sum_{j\geq 1} \varphi(x)^j (\alpha_j x + \beta_j Bx)$ with $\varphi(x) = x_1 x_2$, and $\alpha_j, \beta_j \in \mathbf{K}$.
From $[g, f] = 0$ and $\widehat{f} \xrightarrow{\Psi} f$ one gets $\widehat{g} \xrightarrow{\Psi} g$ and $[\widehat{g}, \widehat{f}] = 0$.
Thus $[\widehat{g}, B] = 0$ (see Proposition 2) and $\widehat{f}, \widehat{g}$ are reducible by φ to dimension one:

$$\widehat{f} \xrightarrow{\varphi} f^*, \quad \widehat{g} \xrightarrow{\varphi} g^*,$$

f^*, g^* one-dimensional vector fields with $[f^*, g^*] = 0$.

Case 1: If $f^* = 0$ (this is the case if and only if $\widehat{f} = \sigma(\varphi) \cdot B$ for some series σ) then $\widehat{f}$ satisfies Condition A, and a convergent transformation to normal form exists.

Case 2: If $f^* \neq 0$ then $g^* = \nu \cdot f^* \in \mathbf{K} \cdot f^*$ by properties of one-dimensional vector fields. Thus

$$\widehat{g} = \nu \cdot \widehat{f} + \sigma(\varphi) \cdot B$$

for some series σ, and

$$\widehat{g} \in \mathbf{K} \cdot \widehat{f} + \mathbf{K} \cdot B$$

follows by evaluating $[\widehat{g}, \widehat{f}] = 0$. Thus we may assume $\widehat{g} = B$.

Due to Pliss, there is a convergent Γ with $\widehat{g} \xrightarrow{\Gamma} g$, and $\widetilde{f} \xrightarrow{\Gamma} f$ yields a convergent transformation to normal form $\widetilde{f} : [\widetilde{f}, \widehat{g}] = 0$.

□

There are other characterizations of resonant vector fields (for λ_2/λ_1 a negative rational number) admitting a convergent normalization, which can be drawn from the work of Martinet and Ramis [23]. Building on work of Ecalle [13], [14] and Voronin [30], Martinet and Ramis succeeded in giving an analytical classification of germs of such vector fields, and those admitting a convergent normalizing transformation can be characterized by the vanishing of infinitely many analytical invariants. In this sense, the convergence problem was settled, at least for the interesting cases, prior to Theorem 6. But Theorem 6 deals with the question from a different perspective, gives an algebraic characterization and provides structural insight that is not readily available from [23].

9 Finite Centralizer Dimension

Theorem 7 *(See Cicogna [9, 10], Walcher [33]). Let f be given with formal normal form $\widehat{f}$, and assume*

$$\dim \mathcal{C}_{for}(\widehat{f}) = k < \infty.$$

If the eigenvalues of B satisfy Condition ω and $\dim \mathcal{C}_{an}(f) = k$ then there is a convergent transformation to normal form.

Proof. We have $\dim \mathcal{C}_{for}(f) = \dim \mathcal{C}_{for}(\widehat{f})$, so $\dim \mathcal{C}_{an}(f) = k$ implies $\mathcal{C}_{an}(f) = \mathcal{C}_{for}(f)$. Given a formal transformation Ψ with $\widehat{f} \xrightarrow{\Psi} f$, there exists h such that $B \xrightarrow{\Psi} h$ and $[h, f] = 0$, since $B \in \mathcal{C}_{for}(\widehat{f})$.
Note that $h = B + \cdots$, so B is a normal form of h. Due to Pliss, there is a convergent Φ with $B \xrightarrow{\Phi} h$. Now $\widetilde{f} \xrightarrow{\Phi} f$ for some $\widetilde{f}$, and $[B, \widetilde{f}] = 0$, so $\widetilde{f}$ is in normal form. □

The requirement on Condition ω can be somewhat relaxed; see [9], [33] and [12].

Applicability

One has to start with a quite frustrating observation: Given an analytic f, there is *no* algorithmic procedure to decide about the existence of some nontrivial analytic h with $[h, f] = 0$. (This is the "exceptional case" for Lie point symmetries.) But nevertheless the theorems relating symmetries and convergence provide structural insight, and they explain what makes convergence possible or impossible. Moreover, the existence of symmetries is frequently known from additional requirements or a priori information.

Example. The analytic vector fields

$$f(x) = \begin{pmatrix} -x_2 + (x_1 + x_2)x_1 \\ x_1 + (x_1 + x_2)x_2 \end{pmatrix} \quad \text{and} \quad h(x) = \begin{pmatrix} x_1 + (x_2 - x_1)x_1 \\ x_2 + (x_2 - x_1)x_2 \end{pmatrix}$$

commute, as a simple verification shows. Therefore f admits a convergent transformation to normal form. This example is constructed from a priori knowledge: The vector field is of the special type $f(x) = Bx + \mu(x) \cdot x$ with a linear map B and a linear form μ, and therefore is contained in the Lie algebra of the projective group in dimension two. The commuting vector field h is contained in the same Lie algebra.

As far as computations are concerned, there is a not-so-frustrating observation pertinent to Theorem 7: The formal centralizer $\mathcal{C}_{for}(\widehat{f})$ is often computationally accessible (and only a finite portion of the Taylor series is required, so $\mathcal{C}_{for}(f)$ is also accessible).

If $\widehat{f} = B + \cdots$ is in normal form then $\widehat{f}$ and B are elements of $\mathcal{C}_{for}(\widehat{f})$. But $\mathcal{C}_{for}(\widehat{f})$ is frequently bigger than that. For instance, if $(\lambda_1, \ldots, \lambda_n)$ is non-resonant then $\mathcal{C}_{for}(\widetilde{f})$ has

dimension n (and consists of linear vector fields).
Another example is given by $(\lambda_1, \ldots, \lambda_4) = (1, -1, \sqrt{2}, -\sqrt{2})$. The resonance conditions are of the type, e.g.

$$m_1 - m_2 + \sqrt{2}\, m_3 - \sqrt{2}\, m_4 = 1$$

and always yield separate conditions for (m_1, m_2) and (m_3, m_4).

A simple verification shows: Let $C_1 = \operatorname{diag}(1, -1, 0, 0)$; $C_2 = \operatorname{diag}(0, 0, 1, -1)$. Then $[B, \widehat{f}] = 0$ if and only if $[C_1, \widehat{f}] = [C_2, \widehat{f}] = 0$. Thus $\mathcal{C}_{for}(\widehat{f})$ has dimension ≥ 3. The underlying reason for this phenomenon is that $\lambda_1, \ldots, \lambda_4$ span a vector space of dimension > 1 over $\mathbf{Q}$.

Conjecture: If $(\lambda_1, \ldots, \lambda_n)$ are resonant, and the nonlinear part of $\widehat{f}$ is "sufficiently generic" then $\mathcal{C}_{for}(\widehat{f})$ is spanned by $\widehat{f}$ and linear vector fields.

(This is true, and verifiable by computations, in many cases; see [33].)

Hamiltonian Systems

Most of the results on symmetries and convergence presented so far do not apply to Hamiltonian systems, since for local Hamiltonian vector fields $f(x) = Bx + f_2(x) + \cdots$ the formal centralizer always has infinite dimension: There exists a nonconstant integral ψ, and for any series σ in one variable, $\sigma(\psi) \cdot f$ centralizes f. In view of the correspondence between integrals of f and Hamiltonian vector fields commuting with f, it is natural to consider integrals in this setting. There are two fundamental results by H. Ito, around 1990:

Theorem 8 *(See Ito [19])*
Let f be Hamiltonian and $(\omega_1, -\omega_1, \ldots, \omega_r, -\omega_r)$ be the eigenvalues of B. Moreover assume that $\omega_1, \ldots, \omega_r$ are non-resonant, thus $\sum m_j \omega_j = 0$ for integers $m_1, \ldots, m_r$ implies $m_1 = \cdots = m_r = 0$. If f possesses r independent integrals in involution (i.e. with vanishing Poisson brackets) then there exists a convergent canonical transformation of f to Birkhoff normal form.

Note that this condition is also necessary in the non-resonant case: If there is a convergent transformation to analytic normal form $\widehat{f}$ then there are r linearly independent linear Hamiltonian vector fields that commute with $\widehat{f}$, and these, in turn, correspond to r independent quadratic integrals of $\widehat{f}$. For the "single resonance" case one has:

Theorem 9 *(See Ito [20])*
Let f be Hamiltonian and $(\omega_1, -\omega_1, \ldots, \omega_r, -\omega_r)$ be the eigenvalues of B. Moreover assume that there are nonzero integers n_1, n_2 such that $n_1\omega_1 + n_2\omega_2 = 0$, but there are no

further resonances. If f possesses r independent integrals in involution then there exists a convergent canonical transformation to Birkhoff normal form.

Again, the condition is also necessary. Ito's proofs are quite long and complicated, and since their publication it seems that there has only been one substantial improvement: Kappeler, Kodama and Nemethi [21] proved a generalization of Theorem 9 for more general single-resonance cases. Moreover, there is a natural obstacle to improvements, since there exist non-integrable (polynomial) Hamiltonian systems in normal form. Thus, a complete integrability condition is not generally necessary for convergence.
In a recent paper Pérez-Marco [25] established a theorem about convergence or generic divergence of the normal form (rather than of the normalizing transformation) in the non-resonant scenario. Although numerical computations indicate the existence of analytic Hamiltonian vector fields which admit only divergent normal forms, there still seems to be no example known.

10 Open Questions and Problems

Much remains open in the scenario when f is analytic and $\mathcal{C}_{for}(f)$ has infinite dimension. A simple argument shows that $\mathcal{C}_{an}(f)$ then must be a proper subset of $\mathcal{C}_{for}(f)$: If $\mathcal{C}_{an}(f)$ has infinite dimension then choose analytic $g_1, g_2, \ldots$ commuting with f such that the orders satisfy

$$o(g_1) < o(g_2) < \cdots$$

(As usual, the *order* of a nonzero power series is defined as the minimal degree of the nonzero terms in its expansion.) Then for suitable $\alpha_j \in \mathbf{K}$ the formal power series $\sum_{j\geq 1} \alpha_j g_j$ is not convergent. Thus there is no hope to generalize the arguments from Theorems 5, 6 and 7 directly.

Perhaps the following generalization of the finite-dimensional approach would be worth pursuing:
On formal power series, one has the familiar valuation υ defined by

$$\upsilon(g) = \upsilon\left(\sum_{j\geq 0} g_j\right) = \begin{cases} 2^{-l} & \text{if} \quad g_l \neq 0, \quad \text{all} \quad g_j = 0 \quad \text{for} \quad j < l \\ 0 & \text{if} \quad g = 0 \end{cases}$$

and a metric defined by

$$d(g,h) = \upsilon(g-h)$$

(Clearly, the convergent power series form a dense subset of this metric space.)

It would be interesting to know under which circumstances density of $\mathcal{C}_{an}(f)$ in $\mathcal{C}_{for}(f)$ with respect to this metric is necessary or sufficient for the existence of a convergent normalizing transformation. (If $\mathcal{C}_{for}(f)$ is finite dimensional, this is the case because density then means equality.)

In general, more detailed knowledge of the structure of the centralizer of f (both formal and analytic) seems to be a prerequisite for further progress in matters of symmetry and convergence. At present, too little is known about this structure, although Proposition 2 provides strong restrictions. But at least the simple resonance case is well-understood. The following is an extended version of Example 2 in [33].

Proposition 4 *Assume the simple resonance case. Let $\widehat{f} = B + \sum_{l\geq 1} \psi^l C_l$ be a vector field (analytic or formal) in normal form. Then $\mathcal{C}_{for}(\widehat{f})$ is infinite dimensional if and only if $\widehat{f}$ admits a (nonconstant) formal first integral, and this is the case if and only if $L_{\widehat{f}}(\psi) = 0$, equivalently if all $L_{C_l}(\psi) = 0$.*
Given this case, every vector field $D_0 + \sum_{j\geq 1} \psi^j D_j$ which satisfies $L_{D_l}(\psi) = 0$, all $j \geq 0$, commutes with $\widehat{f}$.

Proof. Let C and D be diagonal matrices. From the definition $\psi(x) = x_1^{d_1} \cdots x_n^{d_n}$ one sees that $L_C(\psi) = \gamma\psi$, $L_D(\psi) = \delta\psi$ for suitable $\gamma, \delta \in \mathbf{K}$.
Thus, for nonnegative integers l, j:

$$\begin{aligned}\left[\psi^l C, \psi^j D\right] &= -l\psi^{l+j-1} L_D(\psi) \cdot C + j\psi^{l+j-1} L_C(\psi) \cdot D + \psi^{l+j}[C, D] \\ &= -l\delta\psi^{l+j} C + j\gamma\psi^{l+j} \cdot D.\end{aligned}$$

This shows the assertion about the centralizer in case $L_{\widehat{f}}(\psi) = 0$. Now assume $L_{\widehat{f}}(\psi) \neq 0$. Then there is a one-dimensional vector field $g \neq 0$ such that $D\psi(x)\widehat{f}(x) = g(\psi(x))$, and moreover $[\widehat{f}, q] = 0$ for some vector field $q \neq 0$ implies $[B, q] = 0$ and $D\psi(x)q(x) = r(\psi(x))$ for some one-dimensional vector field r, with $[g, r] = 0$.
By properties of one-dimensional vector fields, we may assume $r = 0$, thus $q = \sum_{j\geq m} \psi^j D_j$, with all $L_{D_j}(\psi) = 0$, and $D_m \neq 0$.
Now let s be the smallest index such that $L_{C_s}(\psi) = \gamma_s \cdot \psi \neq 0$. Then, for $m > 0$,

$$[\psi^s C_s, \psi^m D_m] = -m\gamma_s\psi^{s+m} D_m \neq 0,$$

and this forces $[\widehat{f}, q] \neq 0$; a contradiction. □

Remarks. (a) The formal result remains true for arbitrary vector fields $f = B + \cdots$ in the simple resonance situation, because formal transformations respect centralizers and integrals.
(b) From the Proposition one sees that, in the simple resonance case, density of $\mathcal{C}_{an}(f)$ in $\mathcal{C}_{for}(f)$ is a necessary condition for the existence of a convergent normalizing transformation.

In the simple resonance case, infinite dimension of $\mathcal{C}_{for}(f)$ is equivalent to the existence of a nonconstant formal first integral, as we just have seen. This equivalence also holds for vector fields f with normal form $\widehat{f} = B$ (see [32]). In general the following can be proven:

Proposition 5 *Let f be a formal vector field.*
(a) If f admits a nonconstant formal first integral then $\mathcal{C}_{for}(f)$ is infinite dimensional.
(b) If $\mathcal{C}_{for}(f)$ is infinite dimensional then there are formal series $\mu, \vartheta_0, \vartheta_1, \ldots$ such that all $\vartheta_i \neq 0$, $o(\vartheta_i) \to \infty$ for $i \to \infty$ and $L_f(\vartheta_i) = \mu \cdot \vartheta_i$ for all i.
Thus $L_f(\vartheta_i/\vartheta_0) = 0$; f admits infinitely many "meromorphic" integrals.

(Note: The statements carry over directly to the analytic case.)

Proof. (See [31] for details). If ψ is a first integral of f then $\psi^l \cdot f$ centralizes f for all l. This proves (a).
As for (b), let $q_1, q_2, \ldots \in \mathcal{C}_{for}(f)$, and let $q_1, \ldots, q_s$ be a maximal subset such that their values $q_1(y), \ldots, q_s(y)$ are linearly independent in $\mathbf{K}^n$ for some y. (The infinitely many vector fields $q_1, q_2, \ldots$ form a linearly independent set, but their values must be linearly dependent, due to finite dimension of $\mathbf{K}^n$.) By Cramer's rule there are σ_0, σ_{jk} such that

$$q_j = \sum_{k=1}^{s} \frac{\sigma_{jk}}{\sigma_0} \cdot q_k,$$

and one verifies $L_f(\sigma_{jk}/\sigma_0) = 0$. Dividing out common factors yields $\sigma_{jk} = \varrho_{jk} \cdot \beta$, $\sigma_0 = \varrho_0 \cdot \beta$ and

$$0 = L_f(\varrho_{jk}/\varrho_0) = \left(\varrho_0 L_f(\varrho_{jk}) - L_f(\varrho_0)\varrho_{jk}\right)/\varrho_0^2,$$

whence $L_f(\varrho_{jk}) = \nu \cdot \varrho_{jk}$ and $L_f(\varrho_0) = \nu \cdot \varrho_0$ for some ϱ_0, by considering prime factors. □

In the case $\widehat{f} = B$ one can carry this further and prove the existence of a formal power series first integral; see [32].

Remark. An other approach to the (formal) centralizer of a vector field $\widehat{f}$ in normal form is via reduction; see [32] for details: For suitable invariants $\psi_1, \ldots, \psi_r$ of B_s, the map $\Psi = (\psi_1, \ldots, \psi_r)$ will be solution-preserving from $\widehat{f}$ to some vector field $\widehat{g}$: $\widehat{f} \xrightarrow{\Psi} \widehat{g}$.
If $[h, \widehat{f}] = 0$ then h will also be reducible by $\Psi : h \xrightarrow{\Psi} k$, and moreover $[k, \widehat{g}] = 0$. (This is not quite true if the invariant algebra of B_s admits no independent set of generators, but we will not consider this case.)
Now $\widehat{g}$ has a degenerate stationary point, and in many cases one knows that $\mathcal{C}_{for}(\widehat{g}) = \mathbf{K}\cdot\widehat{g}$ is trivial (see [31] for some results in that direction). Then we may assume $k = 0$, and *every vector field that commutes with $\widehat{f}$ admits all integrals of B_s as integrals.* This is a good starting point for further investigations.
Perhaps two "concrete" examples are in order here:

- In the simple resonance case we used just this argument in the proof of Proposition 4 .

- $B = B_s = \mathrm{diag}(\lambda_1, -\lambda_1, \ldots, \lambda_r, -\lambda_r)$, with $\lambda_1, \ldots, \lambda_r$ linearly independent over the rationals $\mathbf{Q}$. Then $\Psi = (\psi_1, \ldots, \psi_r)$, with $\psi_k = x_{2k-1}x_{2k}$, is a reduction map, and one can verify
$$\widehat{g}(x) = \begin{pmatrix} x_1(\sum_{j=1}^r \alpha_{1j}x_j) & + & \text{h.o.t.} \\ & \vdots & \\ x_r(\sum_{j=1}^r \alpha_{rj}x_j) & + & \text{h.o.t.} \end{pmatrix}$$
If the constants α_{ij} are "sufficiently generic" (concrete conditions, which are computationally verifiable, can be found in [31]) then $\mathcal{C}_{for}(\widehat{g})$ is trivial. Hence any vector field commuting with $\widehat{f}$ admits the integrals $\psi_1, \ldots, \psi_r$. One can proceed from this to show that the centralizer of $\widehat{f}$ is spanned by $\widehat{f}$ and linear vector fields, given suitable genericity assumptions.

The role of integrals was already highlighted in the context of Hamiltonian systems. A particular result in the general setting is as follows.

Proposition 6 *Assume that $B = B_s$ admits $n-1$ independent formal first integrals, and $f = B + \cdots$ is analytic. If f also admits $n-1$ independent formal integrals then there exists a convergent normalizing transformation for f.*

Proof. We may assume that $B = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)$. The condition on the first integrals implies that the equation $\sum m_j\lambda_j = 0$ has $n-1$ linearly independent solutions in $\mathbf{Z}^n$. (There are some details involved here to reduce matters from formal power series to monomials; see [32].)
But then there are integers $q_1, \ldots, q_n$ and some $\lambda \in \mathbf{K}$ such that $(\lambda_1, \ldots, \lambda_n) = \lambda \cdot (q_1, \ldots, q_n)$, and Condition ω is satisfied.
Now an elementary argument, using $n-1$ independent integrals $\psi_1, \ldots, \psi_{n-1}$ of a normal form $\widehat{f}$ of f (hence also of B), shows that $\widehat{f} = \sigma \cdot B$ for some scalar function σ: For every y, $\widehat{f}(y)$ and By are both "orthogonal to the gradients $\mathrm{grad}\,\psi_1(y), \ldots, \mathrm{grad}\,\psi_{n-1}(y)$", hence must be linearly dependent over $\mathbf{K}$. (Details are left to the reader.) This means that $\widehat{f}$ satisfies Condition A, and by Bruno's theorem we are finished. □

Example. Let ϕ_1 and ϕ_2 be (convergent or formal) series in $\mathbf{K}^3$, and
$$f = \mathrm{grad}\,\phi_1 \times \mathrm{grad}\,\phi_2$$
with the vector product $\times$ in $\mathbf{K}^3$. Then f admits the integrals ϕ_1 and ϕ_2, and (assuming $f(0) = 0$ and suitable genericity conditions) Proposition 6 is applicable. This scenario

includes topologically interesting cases: For suitable ϕ_1 and ϕ_2 the linearization of f has eigenvalues 0 and $\pm i\omega$, with nonzero $\omega \in \mathbf{R}$.

In summary, there are still many open questions in the interplay of symmetries, integrals and convergence issues, and they may well be worth pursuing.

Finally, we briefly address the situation for maps. Poincaré-Dulac normal forms of maps near a fixed point can be defined analogous to the vector field case, but the connection between maps and their normal forms is more tenuous than for vector fields: Even germs of one-dimensional maps may exhibit very complicated behavior. There is, however, still a quite strong relation between the existence of symmetries and the existence of convergent normal form transformations. The appropriate notion for "symmetries" here involves Lie algebras of infinitesimal symmetries (i.e. vector fields that generate one-parameter groups of symmetries). This will be discussed in detail in the forthcoming paper [16].

Acknowledgements

The paper is an extended version of a talk given at the Jornadas de Trabajo en Mecánica Celeste, Señorío de Bértiz, Navarra, Spain, in July 2003. The author expresses his thanks to the organizers for the invitation to participate, and for creating a stimulating and congenial atmosphere during the workshop. Partial support was given by Project # BFM2002-03157 of Ministerio de Ciencia y Tecnología (Spain) and Project Resolución 92/2002 of Departamento de Educación y Cultura, Gobierno de Navarra (Spain).

References

[1] Arnold, V. I.: 1982, *Geometrical Methods in the Theory of Ordinary Differential Equations*, Springer–Verlag, New York.

[2] Arnold, V. I., Kozlov, V. V. and Neishtadt, A. I.: 1993, *Mathematical Aspects of Classical and Celestial Mechanics.* Ed. V. I. Arnold, *Dynamical Systems III*, Encyclopaedia Mathematical Science vol. **3,** 2nd ed., Springer–Verlag, New York.

[3] Bambusi, D., Cicogna, G., Gaeta, G. and Marmo, G.: 1998, 'Normal forms, symmetry and linearization of dynamical systems', *Journal of Physics A* **31,** 5065–5082.

[4] Bibikov, Yu. N.: 1979, *Local Theory of Nonlinear Analytic Ordinary Differential Equations*, Lecture Notes in Mathematics vol. **702,** Springer Verlag, New York.

[5] Bruno, A. D.: 1971, 'Analytical form of differential equations', *Transactions of the Moscow Mathematical Society* **25,** 131–288.

[6] Bruno, A. D.: 1989, *Local Methods in Nonlinear Differental Equations*, Springer Verlag, New York.

[7] Bruno, A. D. and Walcher, S.: 1994, 'Symmetries and convergence of normalizing transformations', *Journal of Mathematical Analysis and Applications* **183,** 571–576.

[8] Chow, S.-N., Li, C. and Wang, D.: 1994, *Normal Forms and Bifurcations of Planar Vector Fields*, Cambridge University Press, Cambridge.

[9] Cicogna, G.: 1996, 'On the convergence of normalizing transformations in the presence of symmetries', *Journal of Mathematical Analysis and Applications* **199,** 243–255.

[10] Cicogna, G.: 1997, 'Convergent normal forms of symmetric dynamical systems', *Journal of Physics A* **30,** 6021–6028.

[11] Cicogna, G. and Gaeta, G.: 1999, *Symmetry and Perturbation Theory in Nonlinear Dynamics*, Lecture Notes in Physics vol. **57,** Springer Verlag, New York.

[12] Cicogna, G. and Walcher, S.: 2002, 'Convergence of normal form transformations: The role of symmetries', *Acta Applicandae Mathematicae* **70,** 95–111.

[13] Ecalle, J.: 1981, *Sur les fonctions résurgentes. Tome I.* Publications Mathématiques d'Orsay **81 (5),** Orsay.

[14] Ecalle, J.: 1981, *Sur les fonctions résurgentes. Tome II.* Publications Mathématiques d'Orsay **81 (6),** Orsay.

[15] Elphick, C., Tirapegui, E., Brachet, M. E., Coullet, P. and Iooss, G.: 1987, 'A simple global characterization for normal forms of singular vector fields', *Physica D* **29,** 95–127.

[16] Gramchev, T. and Walcher, S.: 2003, 'Normal forms of maps: Formal and algebraic aspects', *preprint.*

[17] Gramchev, T. and Yoshino, M.: 1999, 'Rapidly convergent iteration method for simultaneous normal forms of commuting maps', *Mathematische Zeitschrift* **231,** 745–770.

[18] Iooss, G. and Adelmeyer, M.: 1992, *Topics in Bifurcation Theory and Applications*, World Scientific, Singapore.

[19] Ito, H.: 1989, 'Convergence of Birkhoff normal forms for integrable systems', *Commentarii Mathematici Helvetici* **64,** 412–461.

[20] Ito, H.: 1992, 'Integrability of Hamiltonian systems and Birkhoff normal forms in the simple resonance case', *Mathematische Annalen* **292,** 411–444.

[21] Kappeler, T., Kodama, Y. and Nemethi, A.: 1998, 'On the Birkhoff normal form of a completely integrable system near a fixed point in resonance', *Annali della Scuola Normale Superiore di Pisa. Classe di Scienze* **26,** 623–661.

[22] Markhashov, L. M.: 1974, 'On the reduction of an analytic system of differential equations to the normal form by an analytic transformation', *Journal of Applied Mathematics and Mechanics* **38,** 788–790.

[23] Martinet, J. and Ramis, J.-P.: 1983, 'Classification analytique des équations différentielles non linéaires résonnantes du premier ordre', *Annales Scientifiques de l'École Normale Supérieure* **16,** 571–621.

[24] Olver, P. J.: 1986, *Applications of Lie Groups to Differential Equations*, Springer Verlag, New York.

[25] Pérez–Marco, R.: 2003, 'Convergence and generic divergence of the Birkhoff normal form', *Annals of Mathematics* **157,** 557–574.

[26] Pliss, V. A.: 1965, 'On the reduction of an analytic system of differential equations to linear form', *Differential Equations* **1,** 153–161.

[27] Scheurle, J. and Walcher, S.: 2002, 'On normal form computations', *Geometry, Dynamics and Mechanics*. Eds. P. K. Newton, P. Holmes and A. Weinstein, Springer Verlag, New York, 309–325.

[28] Siegel, C. L.: 1952, 'Über die Normalform analytischer Differentialgleichungen in der Nähe einer Gleichgewichtslösung', *Nachrichten der Akademie der Wissenschaften in Gttingen. I. Mathematisch-Physikalische Klasse* **1952,** 21–30.

[29] Stolovitch, L.: 2000, 'Singular complete integrability', *Institut des Hautes Études Scientifiques. Publications Mathématiques* **91,** 133–210.

[30] Voronin, S. M.: 1981, 'Analytic classification of germs of conformal mappings $(\mathbf{C},\mathbf{0}) \to (\mathbf{C},\mathbf{0})$', *Functional Analysis and Applications* **15,** 1–13.

[31] Walcher, S.: 1991, *Algebras and Differential Equations*, Hadronic Press, Palm Habor.

[32] Walcher, S.: 1991, 'On differential equations in normal form', *Mathematische Annalen* **291,** 293–314.

[33] Walcher, S.: 2000, 'On convergent normal form transformations in presence of symmetries', *Journal of Mathematical Analysis and Applications* **244,** 17–26.