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The invariant manifolds of a finite straight segment

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Abstract

We give a method to compute explicitly the asymptotic expressions of some invariant manifolds in the vicinity of the collinear equilibrium points corresponding to the Hamiltonian system defined by the motion of a particle orbiting around a finite straight segment. For this we use normal form methods so that we make two different Lie transformations of the original Hamiltonian: (i) either we calculate the Hamilton function corresponding to the centre manifold of one of the collinear equilibrium points or (ii) either we determine the Hamiltonian related to the stable–unstable direction. By means of (i) we are able to parametrise the centre, the stable and the unstable manifolds of the original system using the direct changes of coordinates of the two transformations. Using (ii) we compute some $2D$–tori and quasiperiodic orbits.

Key words and expressions: Generalised normal forms, straight segment, invariant manifolds.

MSC: 34K19, 37G05, 37J15, 70H33.

1 Scope of the Paper

1.1 Introduction

The computation of asymptotic (formal) integrals in Hamiltonian systems has received special attention during the last decades due to its utility from a qualitative standpoint. The approach of extending an integral of the principal part of the Hamiltonian to higher orders can be looked up in [8]. It generalizes the standard theory of normal forms for polynomial Hamiltonians, see for instance [7]. The use of generalised normal forms allows one to get a deeper insight of a Hamilton function through its different normal forms.

Specifically, one can use this theory to approximate some invariant manifolds to decide on the stability of an equilibrium solution or to analyse the monodromy of a dynamical system [5]. Applications appear in various fields such as astrodynamics [8] or atomic physics [13]. Theory on this subject has been developed in [8], see also [4, 5]. An algorithm for the case of Hamiltonians with m DOF together with some applications exhibiting the benefit of our approach has been devised in [5].

In this paper we consider Hamiltonians of the form

$$\mathcal{H}(\boldsymbol{x}) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}\, \mathcal{H}_n(\boldsymbol{x}), \tag{1}$$

where $\boldsymbol{x}$ is a $2m$-dimensional vector in the coordinates $x_1, x_2, \ldots, x_m$ and respective momenta $X_1, X_2, \ldots, X_m$. Each $\mathcal{H}_n$ is a homogeneous polynomial in $\boldsymbol{x}$ of degree $n+2$ and ε stands for a small parameter.

First of all, our purpose is to simplify (1) by reducing its number of degrees of freedom by at least one unit. This goal is achieved through the introduction of a formal integral, that is, an integral up to a certain order of approximation after truncating the tail of the normal form Hamiltonian. Furthermore, when the standard normal form does not imply a decrease of the number of degrees of freedom, then our generalised normal forms do it and they can be used to reduce the number of degrees of freedom of the original system.

Both classical and generalised normal forms are managed through normalising procedures that involve Lie transformations. After the reduction process is performed, the transformed Hamiltonian obtained from (1) gives us information on the dynamics associated with the original system from a qualitative point of view. We have taken advantage of this fact to find out some quasiperiodic orbits in the original system.

The article is divided into three sections. First of all, in Section 1 we give a summary of the normal form theory. Secondly, Section 2 contains the case study of the Hamiltonian modelling the rotating straight segment where the theory is applied to the search of the invariant manifolds related to the collinear equilibria of the problem. Finally, Section 3 is devoted to the conclusions of the work.

An extension of this work analysing more invariant sets using other normal forms is currently in progress and will appear elsewhere [9].

1.2 Theorems on normalisation

This section deals with the reduction of Hamiltonian systems through the construction of formal integrals. To achieve this target we use the well–known normal form theorem [7, 14] and a generalization of normal forms explained in [8].

Let $\mathcal{H}$ be an m–DOF Hamiltonian of the type (1). It represents an analytic function whose quadratic terms are given by $\mathcal{H}_0(\boldsymbol{x}) = \frac{1}{2}\,\boldsymbol{x}^t\, B\, \boldsymbol{x}$, where B corresponds to a sym-

metric $2m$–matrix. Let $\mathcal{J}$ be the standard skew–symmetric matrix of dimension $2m$ and let $A = \mathcal{J}\, B$ be the matrix associated with the system defined by $\mathcal{H}_0$.

Through the application of the normal form theorem [7, 14], we reduce by at least one the number of degrees of freedom of Hamiltonian systems provided that the matrix A is not nilpotent. More precisely, if $A = S+N$ is decomposed as the sum of its semisimple ($S \neq 0$) and nilpotent (N) components, then the quadratic Hamiltonian $\mathcal{I}_S(\boldsymbol{x}) = -\frac{1}{2}\, \boldsymbol{x}^t\, \mathcal{J}\, S\, \boldsymbol{x}$ becomes a formal integral of the reduced system. Applying a more general theorem [8] one can reduce, by at least one, the number of degrees of freedom of any Hamiltonian system, even if the associated matrix A is nilpotent, i.e. $S = 0$. This is usually achieved whether the Hamiltonian $\mathcal{H}$ fulfills some non-resonant hypotheses and the process is carried out up to a certain order $L \geq 1$. On this occasion, the formal integral is not necessarily $\mathcal{I}_S(\boldsymbol{x})$, but an integral $\mathcal{G}$ of $\mathcal{H}_0$ one chooses previously. In all situations, one obtains a symplectic change of variables $\boldsymbol{x} \to \boldsymbol{y}$ that transforms $\mathcal{H}$ into the normalised Hamiltonian $\mathcal{K}$, with

$$\mathcal{K}(\boldsymbol{y}; \varepsilon) = \sum_{i=0}^{L} \frac{\varepsilon^n}{n!}\, \mathcal{K}_n(\boldsymbol{y}), \tag{2}$$

where $\mathcal{K}_0(\boldsymbol{y}) \equiv \mathcal{H}_0(\boldsymbol{x})$ and each $\mathcal{K}_n$ is a homogeneous polynomial of degree $n+2$ in $\boldsymbol{y}$. Truncating at order L the error committed is of the type $\mathcal{O}(\varepsilon^{L+1})$.

The construction of $\mathcal{K}$ is done order by order in an ascendent way from $n = 1$ to $n = L$. The homology equation

$$\{\, \mathcal{W}_n\, ,\, \mathcal{H}_0\, \} + \mathcal{K}_n \;=\; \widetilde{\mathcal{H}}_n \tag{3}$$

needs to be solved with the extra condition $\{\, \mathcal{K}_n\, ,\, \mathcal{G}\, \} = 0$ for $n = 1, \ldots, L$. The operator $\{\ ,\ \}$ denotes the usual Poisson bracket and the terms $\widetilde{\mathcal{H}}_n$ are those known from the previous orders. The solution of (3) is the pair $(\mathcal{W}_n, \mathcal{K}_n)$, where $\mathcal{W}_n$ denotes the generating function determined at order n. The Poisson bracket $\{\, \mathcal{W}_n\, ,\, \mathcal{H}_0\, \}$ is also written as $\mathcal{L}_{\mathcal{H}_0}(\mathcal{W}_n)$. This is the so–called Lie–Deprit method, see [1] for more details.

The above can be extended somewhat if we assume that $\mathcal{H}_n = 0$, for $n = 1, \ldots, k-1$ and $\mathcal{H}_k \neq 0$ represents a homogeneous polynomial of degree $k+2$. In this situation one still may consider generalised normal forms though the theory becomes more involved as we cannot split $\mathcal{H}_k$ into semisimple and nilpotent terms.

The setback of the generalised method is that now $\mathcal{W}_n$ is not necessarily a polynomial function of degree $n+2$, as it occurs for the standard approach, but $\mathcal{W}_n$ can involve rational, logarithmic or arctangent functions. As a consequence we have to exclude the singularities from the domain of validity of the normal form.

1.3 Choice of $\mathcal{G}$

Taking into account the decomposition of A into its semisimple and nilpotent components we classify the types of reduction into three remarkable cases: (a) Semisimple case, $A = S$;

(b) Semisimple plus nilpotent case, $A = S + N$ with $S, N \neq 0$; (c) Nilpotent case, $A = N$. Next, according to the type of matrix one has at hand we proceed as follows.

(a) We apply the normal form theorem with $\mathcal{G}(\boldsymbol{x}) = \mathcal{H}_0(\boldsymbol{x})$. Therefore, the procedure yields a generating function which is polynomial. If $m > 1$ and the reduced Hamiltonian defines a system of zero degrees of freedom we choose another $\mathcal{G}$ and apply the generalised method.

(b) We proceed with the normal form theorem taking $\mathcal{G}(\boldsymbol{x}) = \mathcal{I}_S(\boldsymbol{x})$.

(c) We perform the reduction taking $\mathcal{G}(\boldsymbol{x}) = \mathcal{H}_0$. Obviously, we make use of the generalised method exposed above. Besides, if $\mathcal{H}_0 = 0$, then we even could select $\mathcal{G}$ among the integrals of $\mathcal{H}_k$, provided that $\mathcal{H}_n = 0$, for $n = 1, \ldots, k-1$.

Once we have chosen (or determined) $\mathcal{G}$ the next step consists in performing the normal form transformation, calculating $\mathcal{K}$ and $\mathcal{W}$. Now $\mathcal{G}$ corresponds to an integral of $\mathcal{K}$ and

$$\mathcal{I}(\boldsymbol{x};\varepsilon) = \mathcal{G}(\boldsymbol{x}) + \sum_{n=1}^{L} \frac{\varepsilon^n}{n!} \mathcal{L}_{-\mathcal{W}}^n [\mathcal{G}(\boldsymbol{x})]$$

becomes an asymptotic integral of $\mathcal{H}$ functionally independent of it, and up to an approximation of $\mathcal{O}(\varepsilon^{L+1})$. Here $\mathcal{L}_{-\mathcal{W}}$ refers to the Lie operator $\mathcal{L}_{-\mathcal{W}} : F \longrightarrow \{\mathcal{W}, F\}$ whereas the composition operator $\mathcal{L}_{-\mathcal{W}}^n(\boldsymbol{y})$ denotes the recurrence $\mathcal{L}_{-\mathcal{W}}(\mathcal{L}_{-\mathcal{W}}^{n-1}(\boldsymbol{y}))$ for $n \geq 2$. For a complete description of the method, see [5, 8] and references therein.

The main features of the approach described above are: (i) The algorithm is valid for any degree of freedom and works with real and complex coefficients. (ii) The polynomial Hamiltonian may be of any degree. We do not need to start with $\mathcal{H}_0$ being a quadratic polynomial, see [5]. (iii) If $\mathcal{H}_0$ is a quadratic polynomial, its associated matrix does not need to be in diagonal form. (iv) We can make some qualitative analysis of a certain system, such as the study of the monodromy of a system, the nonlinear stability character of equilibria, the computation of periodic solutions and other invariant manifolds or the determination of versal deformations.

2 Application to a Finite Straight Segment

2.1 Hamiltonian of the problem

The case study we have chosen to apply the theory of Section 1 corresponds to a mechanical system dealing with the motion of a point mass under the gravitational field of a massive finite segment. We broach this analysis within a 3–DOF frame.

The dynamics around an elongated celestial body can be represented approximately by using a massive segment as the representation of this kind of celestial objects, see

for instance [6]. Within this context, the finite straight segment is a model used as an approximation to the gravitational field of irregular shaped bodies, such as asteroids, comet nuclei and planets's moons. For this potential, Riaguas [10], Riaguas *et al.* [11] and Elipe *et al.* [2] have computed several families of periodic orbits and bifurcations. In addition, Riaguas *et al.* [12] and Elipe and Riaguas [3] have analysed the nonlinear stability of the equilibria in 2–DOF and 3–DOF respectively.

We consider a straight segment of length $2\,\ell$ and mass M that rotates uniformly with constant angular velocity ω about an axis perpendicular to the segment and fixed in the space. Then we fix the origin of a reference frame $Ox_1x_2x_3$ at the centre of mass O with the segment lying on the axis Ox_1, identifying the axis of rotation with Ox_3. We stress that the reference frame rotates with the straight segment with angular velocity ω.

We follow the approach of Riaguas [10], Riaguas *et al.* [11] and Elipe *et al.* [2] to get the initial Hamiltonian. After making some arrangements and rescaling, the problem is represented in closed form by the Hamilton function given by:

$$\mathcal{H}(\boldsymbol{x}) = \tfrac{1}{2}\left(X_1^2 + X_2^2 + X_3^2\right) - (x_1X_2 - x_2X_1) + U(\boldsymbol{r}),$$

where $\boldsymbol{r} = (x_1, x_2, x_3)$ stands for the position of the particle, while (X_1, X_2, X_3) represent their velocities or conjugate momenta. The potential U is defined through

$$U(\boldsymbol{r}) = -k \log\left(\frac{r_1 + r_2 + 1}{r_1 + r_2 - 1}\right),$$

where r_1, r_2 are the distances of the particle to the end–points of the segment given by

$$r_1 = \sqrt{(x_1 - 1/2)^2 + x_2^2 + x_3^2}, \quad r_2 = \sqrt{(x_1 + 1/2)^2 + x_2^2 + x_3^2}$$

and $k = GM/(\omega^2(2\ell)^3) \in (0, \infty)$ stands for a dimensionless parameter that represents the ratio of the gravitational acceleration to centrifugal acceleration, such that $0 < k < 1$ means fast rotation of the segment, whereas $k > 1$ means slow rotation.

For our study, we have selected $k = 3$, that is, we are in the slow–rotation regime. In this situation, the system has four equilibrium points, two of them located on the axis Ox_1 at the points $(\pm 3/2, 0, 0)$, the so–called collinear equilibria, and the other two placed in the axis Ox_2, specifically at the points $(0, \pm 3/2, 0)$. The momenta of the equilibria resting at the axis Ox_1 are $X_1 = X_3 = 0$ and $X_2 = \pm 3/2$ whereas the momenta corresponding to the equilibria in the axis Ox_2 are $X_1 = \mp 3/2$ and $X_2 = X_3 = 0$.

Next we choose the point $\boldsymbol{r}_0 = (3/2, 0, 0)$ (e.g. the point $(3/2, 0, 0, 0, 3/2, 0)$ in the full phase space) and translate it to the origin by means of a linear (and canonical) change of variables, say ψ. After this translation, we keep the same name for the variables and for the Hamiltonian. Thereafter, we Taylor–expand $U(\boldsymbol{r})$ around the equilibrium up to degree eight in x_1, x_2 and x_3, yielding that:

$$\mathcal{H}(\boldsymbol{r}) = \mathcal{H}_0 + \sum_{i=1}^{6} \mathcal{H}_i(\boldsymbol{r}).$$

For each i, $\mathcal{H}_i$ is a homogeneous polynomial in $\boldsymbol{x}$ of degree $i+2$, for $i = 0, \ldots, 6$.

The next step consists in diagonalizing the main part of the quadratic term $\mathcal{H}_0$. This is achieved by computing the eigenvalues of the matrix associated with $\mathcal{H}_0$. They are:

$$\pm\mu_1\imath, \quad \pm\mu_3\imath, \quad \pm\mu_5 \qquad \text{where} \qquad \mu_1 = \frac{3\sqrt{2}}{4}, \quad \mu_3 = \frac{\sqrt{7+3\sqrt{17}}}{4}, \quad \mu_5 = \frac{\sqrt{-7+3\sqrt{17}}}{4}.$$

Now we can conclude that our point has a linearisation of the type centre $\times$ centre $\times$ saddle and is, therefore, unstable. The same applies to the equilibrium $(-3/2, 0, 0, 0, -3/2, 0)$. On the other hand, the two equilibria resting on the axis Ox_2 have a linearisation of the type centre $\times$ centre $\times$ centre and are linearly stable. Their nonlinear stability is analysed in [12] and [3]. As a previous step, we perform a symplectic change of variables that we call φ, such that $\mathcal{H}_0$ is transformed into

$$\mathcal{H}_0(\varphi(\boldsymbol{x})) = \tfrac{1}{2}\left(X_1^2 + X_2^2 + X_3^2\right) + \tfrac{1}{2}\left(\mu_1^2 x_1^2 + \mu_3^2 x_2^2 - \mu_5^2 x_3^2\right).$$

By doing so, the quadratic Hamiltonian $\mathcal{H}_0$ is in normal form and the subsequent computations will be carried out in an easier way. We denote $\mathcal{H}(\varphi) \equiv \mathcal{H}$ in order to avoid cumbersome notation and maintain the same name for the variables.

2.2 The standard normalization

Now we choose $\mathcal{G} = \mathcal{H}_0$ and our Hamiltonian is ready to apply the Lie transformations explained in Section 1. Thus, we may calculate the normal form Hamiltonian up to order six, that is, up to polynomials of degree eight. We call this Hamiltonian $\mathcal{K}$. In diagonal complex coordinates $(q_1, q_2, q_3, p_1, p_2, p_3)$ defined through the linear change of coordinates:

$$\begin{aligned} &x_1 = \tfrac{1}{\sqrt{2}}(q_1 + \imath\, p_1), \quad x_2 = \tfrac{1}{\sqrt{2}}(q_2 + \imath\, p_2), \quad x_3 = \tfrac{1}{\sqrt{2}}(q_3 - p_3), \\ &X_1 = \tfrac{\mu_1}{\sqrt{2}}(\imath\, q_1 + p_1), \quad X_2 = \tfrac{\mu_3}{\sqrt{2}}(\imath\, q_2 + p_2), \quad X_3 = \tfrac{\mu_5}{\sqrt{2}}(q_3 + p_3), \end{aligned} \tag{4}$$

$\mathcal{K}$ reads:

$$\mathcal{K} = \sum_{\substack{0 \le j,k,\ell \le 4 \\ 1 \le j+k+\ell \le 4}} a(j,k,\ell)(p_1 q_1)^j (p_2 q_2)^k (p_3 q_3)^\ell, \tag{5}$$

where the coefficients $a(j,k,\ell)$ are given in Table 1.

2.3 Invariant manifolds

Once we have carried out the transformation, $\mathcal{H}(\boldsymbol{x}) \to \mathcal{K}(\boldsymbol{y})$, we obtain the explicit expressions for the direct and inverse changes of coordinates. If the new (i.e., the transformed) coordinates are denoted by $\boldsymbol{y} = (y_1, y_2, y_3, Y_1, Y_2, Y_3)$, the direct change is given by $\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{y})$ whereas the inverse change is $\boldsymbol{y} = \boldsymbol{Y}(\boldsymbol{x})$. Note that $\mathcal{H}(\boldsymbol{x}(\boldsymbol{X}(\boldsymbol{y}))) = \mathcal{K}(\boldsymbol{y})$.

$a(1,0,0)$	$1.125000000000000000000000000000000000000000000000000000000000\imath$
$a(0,1,0)$	$1.210582304803311353091514347995139442215099854757553831449744\imath$
$a(0,0,1)$	$0.335582304803311353091514347995139442215099854757553831449744$
$a(2,0,0)$	$0.012030004392096116064681880006045035514583648178933051231676 0$
$a(0,2,0)$	$0.066308730845837153713360708735727160836961568549103183918924\imath$
$a(0,0,2)$	$-0.086345894109368605622561330565536170463916665996215216217999$
$a(1,1,0)$	$-0.009126002113334923119912467663874443664951847740191280320854 9$
$a(1,0,1)$	$-0.095227013792664774628311803153716609682486568460503898763322\imath$
$a(0,1,1)$	$-0.297252742477241851459623674892761250708685001551084154338222\imath$
$a(3,0,0)$	$0.000318398384857662229656622037595485756558153744503556688113 57\imath$
$a(0,3,0)$	$0.027145488584395491344331454450057847512821281983895958135504\imath$
$a(0,0,3)$	$-0.043692379682477334385578543306482135635541873260412713783267$
$a(2,1,0)$	$0.056036986263531688900510154521095706235718505605565692123365\imath$
$a(2,0,1)$	$-0.007181048971515578356816427396050399929645195594391462774472 0$
$a(1,2,0)$	$-0.095445019140551020485391679319426103114361856110216448035834\imath$
$a(0,2,1)$	$0.065870615091316453833825737336039001085835666557418312095763$
$a(1,0,2)$	$-0.006927637253694715777650473729012212898463707168702333696577\imath$
$a(0,1,2)$	$-0.106768591640445723910056439151573202462634068271821935291025\imath$
$a(1,1,1)$	$0.026918284311774477744419934897706660459522404408789952169848$
$a(4,0,0)$	$0.000084404313541226691175880660396023467860733144840748899313 48$
$a(0,4,0)$	$-0.021531391821351515200894612206090041439329157836071593974346$
$a(0,0,4)$	$-0.037384835495131112197231667394596164922688566168369438454374$
$a(3,1,0)$	$0.044014722983883639440442802126176410631197682361266557162 44$
$a(2,2,0)$	$-0.449792973986013498949950441884140843874921228545777482442 57$
$a(1,3,0)$	$0.322239390096914200741027812468130093375354515213973560596 98$
$a(3,0,1)$	$0.000677097195404915755713532529007278891929145967028075974023 65\imath$
$a(2,0,2)$	$0.001868700475158339364905017413122312686093007170127794480057 6$
$a(1,0,3)$	$-0.009596542258008953249177316578688008450271406582342195703274\imath$
$a(0,3,1)$	$0.084259851454671320518081116092249924037833608904071251017 73\imath$
$a(0,2,2)$	$0.188070142041829899961152640131552788531372869697402299296 19$
$a(0,1,3)$	$-0.149829779314173684571163483469201654267690712898453781711 50\imath$
$a(2,1,1)$	$0.013974751918553899340566685848616490545694007428879981221591\imath$
$a(1,2,1)$	$-0.026153278416863235085351618444028681673631314704894494837680\imath$
$a(1,1,2)$	$-0.022167757568694328712245961256391163513889189566623263026867$

Table 1: Coefficients of the normal form Hamiltonian $\mathcal{K}$.

Thus, the expressions of the local invariant manifolds in the initial variables are computed as follows:

- The $1D$–stable manifold is obtained by doing $\boldsymbol{x}_s = \psi \, \varphi \, \boldsymbol{X}(0,0,y_3,0,0,-\mu_1 y_3)$.
- The $1D$–unstable manifold is computed through $\boldsymbol{x}_u = \psi \, \varphi \, \boldsymbol{X}(0,0,y_3,0,0,\mu_1 y_3)$.
- The $4D$–centre manifold is computed as $\boldsymbol{x}_c = \psi \, \varphi \, \boldsymbol{X}(y_1,y_2,0,Y_1,Y_2,0)$.

The intersection of the centre manifold with the energy surface $\mathcal{H} = h$ for a fixed of h gives the normally hyperbolic invariant manifold (NHIM) [15], which is $3D$. This invariant set has also its stable and unstable invariant manifolds which are $4D$. The NHIM bounds a $4D$–surface, called a "transition state" in chemical reaction dynamics, which partitions the energy surface into volumes characterized as "before" and "after" the transformation. The $4D$–stable and unstable manifolds associated with the $3D$–NHIM are impenetrable barriers with the topology of multidimensional spherical cylinders. All these invariants have been approximated using the normal form Hamiltonian $\mathcal{K}$, see [13].

In order to estimate the error committed after truncating the Lie transformation we have used MATHEMATICA, Version 5.0 with precision 10^{-60}. All computations involved in the linear changes of coordinates, in the Taylor expansions and in the Lie transformations,

have been performed within this precision.The global error committed when dropping the tail of the transformation has been studied by estimating the difference $E = |\mathcal{H}(\boldsymbol{x}) - \mathcal{K}(\boldsymbol{Y}(\boldsymbol{x}))|$ within $B_{(0,\delta)}$, a ball centered at the origin of radius δ. Our study has been developed in a neighborhood of radius $\delta = 10^{-2}$ around the equilibrium. We give the results in Table 2.

Order 1	$E = 0.000052784698125$
Order 2	$E = 0.000015958799870$
Order 3	$E = 7.175092835102288 \times 10^{-7}$
Order 4	$E = 1.561351100193047 \times 10^{-7}$
Order 5	$E = 1.1439383299375211 \times 10^{-8}$
Order 6	$E = 7.6559789904728368 \times 10^{-9}$

Table 2: Error committed by the Lie transformation process, after truncating the tail of the normal form at orders one to six, that is, after dropping the polynomials of degrees three and higher to degrees nine and higher.

An approximation of the centre manifold is drawn in Fig. 1.

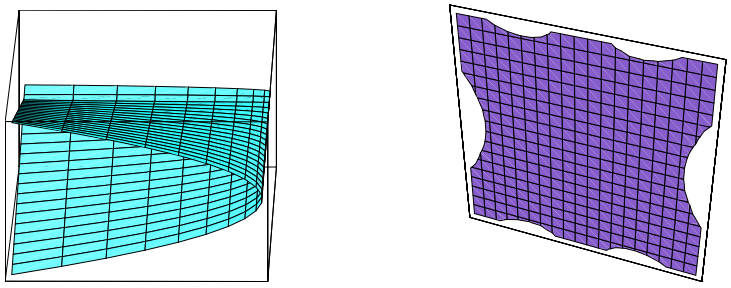

Figure 1: Surfaces defined by the components x_1, x_2, x_3 of the centre manifold, on the left we set $Y_1, Y_2 = 0$ and on the right $y_1, y_2 = 0$.

2.4 Another reduction: quasiperiodic orbits

We choose this time $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2$, where $\mathcal{G}_1 = (X_1^2 + \mu_1^2 x_1^2)/2$ and $\mathcal{G}_2 = (X_2^2 + \mu_3^2 x_2^2)/2$. Thus we calculate the generalised normal form, say $\mathcal{S}$, and the corresponding change of coordinates $\boldsymbol{x} = \boldsymbol{X}^*(\boldsymbol{y}^*)$, up to degree four (that is, taking into account second–order terms).

In the diagonal complex coordinates defined by (4), the Hamilton function $\mathcal{S}$ reads:

$$\mathcal{S} = \sum_{\substack{0\le j,k\le 2,\ 0\le \ell,m\le 4 \\ 2\le 2j+2k+\ell+m\le 4}} b(j,k,\ell,m)(p_1q_1)^j(p_2q_2)^k p_3^\ell q_3^m, \tag{6}$$

where the coefficients $b(j,k,\ell,m)$ are given in Table 3.

$b(1,0,0,0)$	$1.125000000000000000000000000000000000000000000000000000000000000 \imath$
$b(0,1,0,0)$	$1.210582304803311353091514347995139442215099854757553831449744 \imath$
$b(0,0,1,1)$	$0.335582304803311353091514347995139442215099854757553831449744$
$b(1,0,1,0)$	$3.634824707673186493250495102357885829267333915512489889 1180 \imath$
$b(1,0,0,1)$	$-0.080726447867715688281910958496296389887292109746952783700 0 \imath$
$b(0,1,1,0)$	$-1.903560365373189888904465653290128829568561376574067173589 88 \imath$
$b(0,1,0,1)$	$0.042276499957192756217406856401626700233983520033737888171 13 \imath$
$b(0,0,3,0)$	$129.040864005748859507016198556538528233553736784577101088946$
$b(0,0,0,3)$	$-0.001413592970556240158304896755127595167352354829610932212$
$b(0,0,2,1)$	$4.215891942174566702354984170999338413113368744656003421 62 \imath$
$b(0,0,1,2)$	$-0.093631470141442053571191676928864752587088433271232600 02$
$b(2,0,0,0)$	$0.886410098551843434610669406920829413813780008185841007880 21$
$b(0,2,0,0)$	$0.306118364885931497060286299488490305426981219701420771417 97$
$b(1,1,0,0)$	$-0.924952995365646170377586332965110260608061049136296356612 80$
$b(1,0,2,0)$	$-60.261329873355174731798372121243539876539773317904542556589 \imath$
$b(0,1,2,0)$	$-15.260589952741833305824483579409751879291548890353264223706 9 \imath$
$b(1,0,1,1)$	$-4.151864997554668685742992182078910134548761524351851838826 \imath$
$b(0,1,1,1)$	$1.827211544962906181230889568250744289955895507169110007984 \imath$
$b(1,0,0,2)$	$-0.029723740064180193376835465423773894953343523575710591735 \imath$
$b(0,1,0,2)$	$-0.007527245248895515811126375175495963013469046812401059291 9 \imath$
$b(0,0,4,0)$	$-1213.703910843576960464925416370618260065745658108317223064 15$
$b(0,0,0,4)$	$-0.000295285578902026766305639023929043006529343675486903 90$
$b(0,0,3,1)$	$-134.715170011016148143213724031526552598153315876730687414 7$
$b(0,0,2,2)$	$-5.245897609280132845418090838201297781196957119689698490 6$
$b(0,0,1,3)$	$-0.066447897922673947076292371140097546195265559351750805 5$

Table 3: Coefficients of the normal form Hamiltonian $\mathcal{S}$.

with the following features: (i) the reduction associated with the normal form is regular and the resulting phase space is a plane; (ii) we have already introduced two symmetries in the system, i.e., $\mathcal{G}_1$ and $\mathcal{G}_2$; (iii) fixing a ball of radius $\delta = 10^{-2}$ and calling $\boldsymbol{y}^* = \boldsymbol{Y}^*(\boldsymbol{x})$ the inverse change of $\boldsymbol{x} = \boldsymbol{X}^*(\boldsymbol{y}^*)$, the error up to terms of degree four is $E = |\mathcal{H}(\boldsymbol{x}) - \mathcal{S}(\boldsymbol{Y}^*(\boldsymbol{x}))| = 0.000056212888314$.

Next, after fixing the value for the integrals: $\mathcal{G}_1 = j_1 \geq 0$ and $\mathcal{G}_2 = j_2 \geq 0$ we arrive at a Hamiltonian of 1 DOF. Now, the corresponding reduced and truncated Hamiltonian obtained from $\mathcal{S}$ is $\mathcal{Q}(y_3^*, Y_3^*; j_1, j_2)$. To calculate its equilibrium points we need to determine the roots of the algebraic system

$$(\partial\mathcal{Q}/\partial Y_3^*,\ \partial\mathcal{Q}/\partial y_3^*) = (0,0).$$

The solution of this system of equations yields four non–degenerate equilibrium points:

$$\begin{cases} (1,2) & y_3^{*1,2} = \left(p_1 + p_2\sqrt{p_3} \pm p_4\sqrt{p_3\,p_5 + p_6\sqrt{p_3}} \pm p_7\sqrt{p_5 + \frac{p_6}{\sqrt{p_3}}}\right)/(p_8\sqrt{p_3}), \\ & Y_3^{*1,2} = c - \frac{1}{2}\sqrt{p_3} \pm \frac{1}{2}\sqrt{p_5 + \frac{p_6}{\sqrt{p_3}}}, \\ (3,4) & y_3^{*3,4} = \left(-p_1 + p_2\sqrt{p_3} \pm p_4\sqrt{p_3\,p_5 - p_6\sqrt{p_3}} \mp p_7\sqrt{p_5 - \frac{p_6}{\sqrt{p_3}}}\right)/(p_8\sqrt{p_3}), \\ & Y_3^{*3,4} = c + \frac{1}{2}\sqrt{p_3} \pm \frac{1}{2}\sqrt{p_5 - \frac{p_6}{\sqrt{p_3}}}, \end{cases}$$

where the p_i are polynomials of degree one or two in j_1, j_2, for $i = 1, \ldots, 8$, and c is a negative–real constant. Now, we assume that $p_3, p_8 \neq 0$. Depending on the signs of p_3, p_5 and p_6 we arrive at these situations:

- Whenever $p_3 > 0$ and $p_5\sqrt{p_3} > |p_6|$, we get four different equilibria $(y_3^{*i}, Y_3^{*i}), i = 1, \ldots, 4$, which correspond to four families of $2D$–invariant tori of $\mathcal{H}$.

- If $p_3 > 0$ and $-|p_6| < p_5\sqrt{p_3} < |p_6|$, we find two different equilibrium points: either $(y_3^{*1,2}, Y_3^{*1,2})$ when $p_6 > 0$ or $(y_3^{*3,4}, Y_3^{*3,4})$ if $p_6 < 0$. They correspond to two families of $2D$–invariant tori of the original system.

- If $p_3 > 0$ and $p_5\sqrt{p_3} = |p_6|$ or $p_3 > 0$ and $p_5\sqrt{p_3} = -|p_6|$, there are three equilibria. More specifically, if $p_5\sqrt{p_3} = p_6$ we have that (y_3^{*3}, Y_3^{*3}) coincides with (y_3^{*4}, Y_3^{*4}) whereas if $p_5\sqrt{p_3} = -p_6$, the point (y_3^{*1}, Y_3^{*1}) is the same as (y_3^{*2}, Y_3^{*2}).

- Finally, when $p_3 < 0$ or $p_5\sqrt{p_3} < -|p_6|$, there is no isolated critical point.

Now since the p_i depend on j_1 and j_2, the conditions $p_5\sqrt{p_3} = |p_6|$ and $p_5\sqrt{p_3} = -|p_6|$ correspond to bifurcation curves in the plane of parameters defined by j_1 and j_2. By going back to the initial variables, undoing the Lie transformation and the other changes, these curves correspond to bifurcations of invariant tori in the original system.

The approximation of the invariant tori and the quasiperiodic trajectories can be calculated explicitly as follows. First we compose the different changes of coordinates $\boldsymbol{x} = \psi\varphi\boldsymbol{X}^*(\boldsymbol{y}^*)$. Then we make:

$$\mu_1\, y_1^* = \sqrt{j_1}\,\cos t, \quad Y_1^* = \sqrt{j_1}\,\sin t, \quad \mu_3\, y_2^* = \sqrt{j_2}\,\cos u, \quad Y_2^* = \sqrt{j_2}\,\sin u,$$

with $t, u \in [0, 2\pi)$. Thus, we arrive at an expression of the form $\boldsymbol{y}^*(t, u; j_1, j_2)$. It represents a family of $2D$–invariant tori in the phase space $\mathbf{R}^2$. To obtain a very accurate approximation of the $2D$–tori we carry out the Lie transformation to a high order L. The $2D$–invariant tori are depicted in Fig. 2.

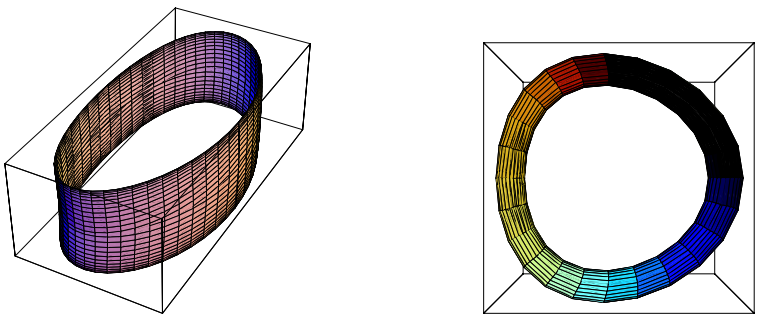

Figure 2: Left: a torus in the coordinate space. Right: another view of the same torus.

Finally, given a certain $2D$–invariant torus, fixing either the angle u or the angle t or putting one of them in terms of the other, we obtain some quasiperiodic orbits confined in the torus. We have drawn a couple of examples in Fig. 3.

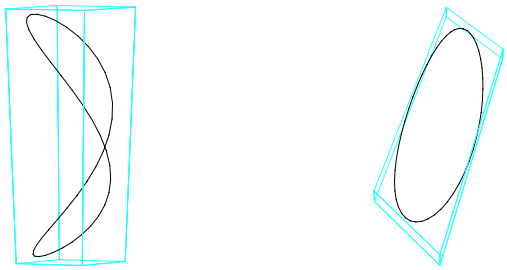

Figure 3: On the left, a quasiperiodic orbit parametrised by t, after fixing $u = 0$. On the right, a quasiperiodic orbit parametrised by u after fixing $t = 0$.

3 Conclusions

In the context of generalised normal forms for Hamiltonians, we apply different Lie transformations to the problem of a particle orbiting around a straight segment to determine some invariant manifolds in the vicinity of the (unstable) collinear equilibrium points.

With the choice $\mathcal{G} = \mathcal{H}_0$ we calculate high–order approximations of the centre, the stable and the unstable manifolds of one of the collinear equilibrium points. We also compute the NHIM associated with such equilibrium, its stable and unstable manifolds.

Using the function $\mathcal{G} = (X_1^2 + \mu_1^2 x_1^2)/2 + (X_2^2 + \mu_3^2 x_2^2)/2$, we have determined some $2D$–tori and quasiperiodic orbits of the original Hamiltonian, obtaining furthermore the conditions that some parameters must hold to achieve various bifurcations.

As we give the expressions of the two Hamiltonians, $\mathcal{K}$ and $\mathcal{S}$, corresponding to the two normalisations, and since it is straightforwardly to determine $\mathcal{H}$ (the Hamiltonian previous to the two normalisations) up to terms of degree eight, one could obtain the direct and the inverse changes of coordinates. One can recover the quasiperiodic orbits, the $2D$–tori, the NHIM and its associated manifolds.

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