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Keplerian systems: orbital elements and reductions

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Abstract

Can we present what is behind symmetries and reductions of the Kepler flow in terms of orbital elements, in such a way it may be easily used in astrodynamics? With that goal we propose coordinates for the orbital space $S^2_L \times S^2_L$ of bounded Keplerian orbits with a given semimajor axis, as an alternative to Cartan coordinates. They have the property of 'separating' the orientation of the orbital plane from the position of Laplace vector in that plane. Considering the momentum mappings, these coordinates allow to illustrate, in a straightforward manner, the result of a second reduction when H or G are integrals.

Key words and expressions: Bounded Keplerian orbits, axial symmetries, reduced orbit spaces.

MSC: 70F15, 70F05, 37J15, 58D19.

1 Introduction

Orbital elements and the canonical version of them (Delaunay variables) are well known; any text book on astrodynamics is expected to give a full account of them. A mathematical oriented reader perhaps may prefer to consult the book by Abraham and Marsden (Chap. 9) [1], or the recent paper by Chang and Marsden [2] but it is not necessary for the reading of this note.

Orbital elements, excluding the anomalies, are no more than functions defined in a reduced phase space $\mathcal{O}_L$ called the *orbital space*, result of the *Kepler action* (see [14, 5]) on the phase space $T^*\mathbb{R}^3$ which is a regular reduction. All the orbits in this space $\mathcal{O}_L$ have the same semimajor axis. This, but in a different language, is one of the fundamental facts on which hinges studies in dynamical astronomy and astrodynamics, both in the design and the analysis and evolution of nominal orbits in specific missions, in dynamical models where the 'elimination of the mean anomaly' has been carried out. Here we will constrain to orbits with negative energy.

A classical result states that *the set of bounded Keplerian orbits with a fixed semimajor axis $\mathcal{O}_L$ is in 1-1 correspondence with* $S^2 \times S^2$. A proof of this was given by Moser (1970, Lemma 2) using quaternions [15]. More recently Cushman [4] and Coffey *et al.* [3] gave another elegant proof making use of the angular momentum and Laplace-Runge-Lenz vectors, referring that proof to Jauch and Hill and even to Elie Cartan. Here we present this result based on the splitting of the set of orbits in three classes, and a straightforward identification of each element of them with the sextuples defining $S^2 \times S^2$. The interest of the one we give is that it helps in an easy characterization of families or orbits connected with the process of the second (singular) reduction in the case of some classical symmetries.

Based on invariant theory, Cushman [4, 5, 6, 8] has made a complete and beautiful study of the axial reduction when H, the third component of the angular momentum, is an integral. But the way he does asks for paying not a cheap price. Indeed, as a first step he leaves the ambient space $\mathbb{R}^3$, defined by the configuration space, and he moves to the sphere S^3 and the corresponding cotangent space. We may see it as a tribute to J. Moser and it certainly makes wonders dealing with regularizations and the like (see exercises in Chap. 2 of his book with L. Bates) [7]. Nevertheless, it seems no essential for the purposes of reduction. The second and tougher step requires to enter the quarters of differential and algebraic geometry where *invariant theory* is rooted.

Although after 30 years of maintained efforts (regular reduction in the 70s and singular reduction in the 80s and 90s) things in this field may look to be set up, recent publications suggest there are still aspects to go after. As an example, from Cushman and Sniatycki [8] we quote: "the paper offers a new approach to singular reduction of Hamiltonian systems with symmetries. The main difference between this and other approaches present in the wide literature on the subject is the use of two main sets of tools in their analysis. The first is the category of differential spaces of Sikorski. Working in this category the authors obtain a finer description of the local differential geometry of the stratified orbit space. The second tool is a theorem of Stefan and Sussmann, which ensures that accessible sets of the generalized distribution spanned by the Hamiltonian vector fields of invariant functions are immersed submanifolds of the symplectic manifold. This theorem is used to investigate the structure of the orbit space induced by a coadjoint equivariant momentum map. The main result of the paper is the identification of accessible sets of this generalized distribution with singular reduced spaces. The authors are also able to describe the differential structure of a singular reduced space corresponding to a coadjoint orbit which need not be locally closed." We hope it will not take too long to see these ideas leading to new techniques at the level for applications.

But meanwhile, with the solid ground given by all the studies made in the last decades we refer above, with this note we would like to contribute to answer the question which opens the abstract. Although some will say that this was already done by Coffey *et al.* [3], what they did was to propose S^2 (for $H \neq 0$) as a space isomorphic to the double reduced space, sending the reader to a paper by Cushman [4] for its justification; working with orbital elements, here we point out some features of the orbital space which help to see what is at the core of the reduction process. We do not make use of invariant theory, although any one used to it will easily track that it is behind.

The note may be seen as made of two parts. The first, on which the note hinges, proposes a different 'sorting' of the orbits in $S^2 \times S^2$ as an alternative to Cartan coordinates. They have the property of separating the orientation of the orbital plane from the position of Laplace vector in that plane. Considering the momentum mappings defined by the modulus and third component of the angular momentum vector, we easily identify families of orbits relevant in the second part, when we attempt the description of the reduction process. We finish with two appendices. Full details will be given elsewhere [11].

2 Cartan Coordinates and Orbital Elements in $S^2_L \times S^2_L$ Reduced Space

2.1 Basic vector and real functions in phase space

Let $(\boldsymbol{x}, \boldsymbol{X})$ (with $\boldsymbol{x} \neq 0$) be an element in cotangent space T^*R^3. Then, in what follows, we will deal with the well known vectorial functions

$$\mathbf{G} : T^*\mathbb{R}^3 \longrightarrow \mathbb{R}^3\,; (\boldsymbol{x}, \boldsymbol{X}) \rightarrow \boldsymbol{x} \times \boldsymbol{X}, \tag{1}$$

$$\boldsymbol{A} : T^*\mathbb{R}^3 \longrightarrow \mathbb{R}^3\,; (\boldsymbol{x}, \boldsymbol{X}) \rightarrow L(\boldsymbol{X} \times (\boldsymbol{x} \times \boldsymbol{X}) - \mu \boldsymbol{x}/\|\boldsymbol{x}\|), \tag{2}$$

(*i.e.* the angular momentum and Laplace-Runge-Lenz vectors) where L is defined below, and real functions

$$G : T^*\mathbb{R}^3 \longrightarrow \mathbb{R}\ ;\ (\boldsymbol{x}, \boldsymbol{X}) \rightarrow \|\boldsymbol{x} \times \boldsymbol{X}\|, \tag{3}$$

$$H = G_3 : T^*\mathbb{R}^3 \longrightarrow \mathbb{R}\ ;\ (\boldsymbol{x}, \boldsymbol{X}) \rightarrow x_1 X_2 - x_2 X_1, \tag{4}$$

joint to the Hamiltonian function

$$\mathcal{H} : T^*\mathbb{R}^3 \longrightarrow \mathbb{R}\ ;\ (\boldsymbol{x}, \boldsymbol{X}) \rightarrow \frac{1}{2}\|\boldsymbol{X}\|^2 - \frac{\mu}{\|\boldsymbol{x}\|} = -\frac{\mu^2}{2L^2}. \tag{5}$$

With this notation the classical function a ('semi-major axis') is the quadratic function $a = L^2/\mu$.

As we shall see, the functions G and H will play a key role in the study of families of orbits. Indeed, when $G \neq 0$ we will refer to *"the plane of the orbit"*. Two families of orbits receive special consideration: when $G = |H| > 0$ we have "*equatorial orbits*". If $H = 0$ ($G \neq 0$), we refer to "*polar orbits*". When $G = 0$, we have *rectilinear orbits* (also called *degenerate orbits*).

Proposition 1 *On each manifold $L^2 = \mu a$ constant, the set of bounded orbits consists of a product two dimensional spheres $S_L^2 \times S_L^2$.*

Proof: The angular momentum and Laplace vectors functions are independent of ℓ, and so are

$$\boldsymbol{\sigma} = \tfrac{1}{2}(\boldsymbol{G} + \boldsymbol{A}) \qquad \text{and} \qquad \boldsymbol{\delta} = \tfrac{1}{2}(\boldsymbol{G} - \boldsymbol{A}). \tag{6}$$

The identities $||\boldsymbol{\sigma}||^2 = ||\boldsymbol{\delta}||^2 = \frac{1}{4}L^2$ defines the two spheres recognized by Cartan. □

Poisson brackets of $\boldsymbol{\sigma}$ and $\boldsymbol{\delta}$ given by

$$(\sigma_i, \sigma_j) = \omega_{ijk}\sigma_k, \quad (\delta_i, \delta_j) = \omega_{ijk}\delta_k, \quad (\sigma_i, \delta_j) = 0, \tag{7}$$

where ω_{ijk} is the Levi-Civita symbol for permutations of the indices (i, j, k).

2.2 Delaunay variables as functions of Cartan coordinates on $S_L^2 \times S_L^2$

Denoting generically by D any of the Delaunay variables (G, H, g, h), they are functions

$$D : \Delta \subset S_L^2 \times S_L^2 \longrightarrow \mathbb{R}$$

defined on the open set Δ of non-circular non-equatorial orbits. From (6) we easily obtain

$$\begin{cases} G = \frac{1}{2}\sqrt{(\sigma_1 + \delta_1)^2 + (\sigma_2 + \delta_2)^2 + (\sigma_3 + \delta_3)^2}, \\ H = \frac{1}{2}(\sigma_3 + \delta_3), \\ \sin g = \dfrac{\sigma_3 - \delta_3}{2seL}, \qquad \cos g = \dfrac{\sigma_1\sigma_2 - \delta_1\delta_2}{2seGL}, \\ \sin h = \dfrac{\sigma_1 + \delta_1}{2sG}, \qquad \cos h = \dfrac{\sigma_2 + \delta_2}{2sG}. \end{cases} \tag{8}$$

where $e^2 = 1 - \eta^2$, $s^2 = 1 - c^2$ with $\eta = G/L$ and $c = H/G$.

The inverse transformation is $\boldsymbol{\sigma} = \frac{1}{2}(\boldsymbol{G} + \boldsymbol{A})$ and $\boldsymbol{\delta} = \frac{1}{2}(\boldsymbol{G} - \boldsymbol{A})$ with

$$G_1 = \ \ G \sin I \sin h, \qquad A_1 = Le(\cos g \cos h - \sin g \cos I \sin h), \tag{9}$$

$$G_2 = -G \sin I \cos h, \qquad A_2 = Le(\cos g \sin h + \sin g \cos I \cos h), \tag{10}$$

$$G_3 = \ \ H, \qquad\qquad A_3 = Le \sin g \sin I, \tag{11}$$

Note that *rectilinear orbits* correspond to $(\sigma_1, \sigma_2, \sigma_3, -\sigma_1, -\sigma_2, -\sigma_3)$.

3 Orbital Elements and a New Sorting of Orbits in $S^2_L \times S^2_L$

Leaving the (σ, δ) representation, we propose a 'new reordering' of the orbital space. It is based on the fact that bounded Keplerian orbits are split in three types of orbits, $\mathcal{O} = \Delta_c \bigcup \Delta_e \bigcup \Delta_r$: circular orbits $\Delta_c = \{G = L\}$; elliptic orbits $\Delta_e = \{G \,|\, 0 < G < L\}$, and rectilinear $\Delta_r = \{G = 0\}$.

Proposition 2 *Orbital elements can be used to express the set $S^2_L \times S^2_L$ as the union of three disjoint subsets: circular, elliptic and rectilinear orbits.*

Proof: The idea is to 'separate' orbital elements in the way we 'locate' the orbits in $S^2_L \times S^2_L \;\equiv\; S^2_1 \times S^2_2$: We use the first sphere S^2_1 for the argument of periaxis and eccentricity. Each 'small circle of latitude' of S^2_1 relates to orbits with the same eccentricity. We associate the north pole of S^2_1 with the circular orbits, and the south pole with the rectilinear. The second sphere S^2_2 gives the positions of the orbital plane. Each 'small circle of latitude' S^2_2 corresponds to orbits with the same inclination; north and south poles of S^2_2 relate to direct and retrograde equatorial orbits. Coordinates are denoted by $S^2_1 = \{(\lambda_1, \lambda_2, \lambda_3) \,|\, \sum \lambda_i^2 = \frac{1}{4}L^2\}$ and $S^2_2 = \{(\zeta_1, \zeta_2, \zeta_3) \,|\, \sum \zeta_i^2 = \frac{1}{4}L^2\}$. By n_i and s_i we refer to the 'north' and 'south' poles of S^2_i respectively.

More precisely, the sorting is as follows:

- $\Delta_c =$ circular orbits $(G = L) \,:\, n_1 \times S^2_2$. The point (n_1, n_2) corresponds to the direct equatorial circular orbit, and (n_1, s_2) is the retrograde equatorial circular orbit. Moreover the set $(0, 0, \frac{1}{2}L, \zeta_1, \zeta_2, \zeta_3)$, $\zeta_3 \neq \pm\frac{1}{2}L$ are circular orbits, with the inclination $\cos I = H/L$.

  Thus, we obtain the argument of the node and inclination (h, I) by inverting the following expressions

$$\zeta_1 = \tfrac{1}{2}\sqrt{L^2 - H^2}\cos h, \qquad \zeta_2 = \tfrac{1}{2}\sqrt{L^2 - H^2}\sin h, \qquad \zeta_3 = \tfrac{1}{2}H. \tag{12}$$

  Finally, the set $\frac{1}{2}L(0, 0, 1, \cos h, \sin h, 0)$ are circular polar orbits, with $0 < h \leq 2\pi$.

- $\Delta_e =$ elliptic orbits $(0 < G < L) \,:\, (S^2_1 - \{n_1, s_1\}) \times S^2_2$.

  'Equatorial ellipses'. First, we make correspond first the equatorial elliptic orbits (direct and retrograde) to $(S^2_1 - \{n_1, s_1\}) \times \{n_2, s_2\}$, i.e. to sextuples $(\lambda_1, \lambda_2, \lambda_3, 0, 0, \pm\frac{1}{2}L)$, with $\lambda_3 \neq \pm\frac{1}{2}L$.

  We obtain the argument of periaxis and eccentricity from the expressions

$$\lambda_1 = \frac{G}{L}\sqrt{L^2 - G^2}\,\cos g, \qquad \lambda_2 = \frac{G}{L}\sqrt{L^2 - G^2}\,\sin g, \qquad \lambda_3 = \frac{1}{L}(G^2 - \tfrac{1}{2}L^2).$$

Notice that we have two disks:

$$\{(\lambda_1, \lambda_2, \lambda_3, 0, 0, \tfrac{1}{2}L) \,|\, \lambda_3 \neq -\tfrac{1}{2}L\} \quad \text{and} \quad \{(\lambda_1, \lambda_2, \lambda_3, 0, 0, -\tfrac{1}{2}L) \,|\, \lambda_3 \neq -\tfrac{1}{2}L\},$$

which correspond to direct orbits ($H = G$), and to retrograde orbits ($H = -G$). For the relation of these orbits with coordinates for plane motions see equations (25) in Appendix I.

'Non-equatorial ellipses'. We make correspond the elliptic inclined (non-equatorial) orbits with the sextuples $(S_1^2 - \{n_1, s_1\}) \times (S_2^2 - \{n_2, s_2\})$. We establish the correspondence between the Delaunay set of variables (G, H, g, h), and our coordinates on $S_1^2 \times S_2^2$ as follows

$$\begin{aligned}
\lambda_1 &= \frac{G}{L}\sqrt{L^2 - G^2}\,\cos g, & \zeta_1 &= \tfrac{1}{2}\frac{L}{G}\sqrt{G^2 - H^2}\,\cos h,\\
\lambda_2 &= \frac{G}{L}\sqrt{L^2 - G^2}\,\sin g, & \zeta_2 &= \tfrac{1}{2}\frac{L}{G}\sqrt{G^2 - H^2}\,\sin h,\\
\lambda_3 &= \frac{1}{L}(G^2 - \tfrac{1}{2}L^2), & \zeta_3 &= \tfrac{1}{2}\frac{L}{G}H.
\end{aligned} \tag{13}$$

Delaunay variables $D_i \equiv D_i(g, h, G, H)$ as functions of $\boldsymbol{\zeta}$ and $\boldsymbol{\lambda}$ are given by

$$\left\{\begin{array}{lll}
G = \sqrt{L(\lambda_3 + \tfrac{1}{2}L)}, & \cos g = \dfrac{\lambda_1}{\sqrt{\lambda_1^2 + \lambda_2^2}}, & \sin g = \dfrac{\lambda_2}{\sqrt{\lambda_1^2 + \lambda_2^2}},\\
H = 2\zeta_3\sqrt{\dfrac{\lambda_3}{L} + \dfrac{1}{2}}, & \cos h = \dfrac{\zeta_1}{\sqrt{\zeta_1^2 + \zeta_2^2}}, & \sin h = \dfrac{\zeta_2}{\sqrt{\zeta_1^2 + \zeta_2^2}}.
\end{array}\right. \tag{14}$$

- Δ_r = rectilinear orbits ($G = 0$): $s_1 \times S_2^2$.

  Rectilinear orbits are defined by all the rays with a given length, function of L, in ambient space $\mathbf{R}^3$: S_L^2. Thus, this set is in 1-1 correspondence to the subset $s_1 \times S_2^2$ in $S_1^2 \times S_2^2$. In other words, we locate rectilinear with sextuples $(0, 0, -\tfrac{1}{2}L, A_1, A_2, A_3)$ which uses the south pole of the first sphere and the whole second sphere. Note that the A_i are the components of the Laplace-Runge-Lenz vector, defining the direction of the rectilinear motion.

Thus, pasting the three subsets we obtain

$$\Delta_c \bigcup \Delta_e \bigcup \Delta_r \longleftrightarrow S_1^2 \times S_2^2,$$

which ends the description of the proposed parametrization. □

3.1 Comparing with Cartan coordinates

It is worth comparing the new sorting with the corresponding in Cartan coordinates (Eqs. (8)). In particular, from equations (14) we get the eccentricity as $e = e(\lambda_3)$ and inclination $I = I(\zeta_3)$. We just consider two cases:

(i) $(0, 0, \frac{1}{2}L, 0, 0, -\frac{1}{2}L)$. In Cartan coordinates: the *rectilinear* orbit in the negative OZ axis. In our parametrization: the retrograde *equatorial circular* orbit.

(ii) $(\alpha_1, \alpha_2, \alpha_3, -\alpha_1, -\alpha_2, -\alpha_3,)$, with $\alpha_3 \neq \pm\frac{1}{2}L$. In Cartan coordinates: another *rectilinear* orbit. In our parametrization it represents an *ellipse* with elements (e, I, g, h) given by

$$e = \sqrt{\frac{1}{2} - \frac{\alpha_3}{L}}, \quad \cos I = -\frac{2\alpha_3,}{L},$$

$$\cos g = \frac{\alpha_1}{\sqrt{\alpha_1^2 + \alpha_2^2}}, \quad \sin g = \frac{\alpha_2}{\sqrt{\alpha_1^2 + \lambda_2^2}}, \quad \cos h = -\cos g, \quad \sin h = -\sin g.$$

4 Orbits on the Momentum Space

In what follows, we give special attention to the functions G and H defined in Section 2, dubbed as *momentum mappings*

$$\mathcal{M}_{L,G} : S_L^2 \times S_L^2 \longrightarrow \mathbb{R};\, (\boldsymbol{\lambda}, \boldsymbol{\zeta}) \;\to\; \sqrt{L(\lambda_3 + \tfrac{1}{2}L)},$$

$$\mathcal{M}_{L,H} : S_L^2 \times S_L^2 \longrightarrow \mathbb{R};\, (\boldsymbol{\lambda}, \boldsymbol{\zeta}) \;\to\; \zeta_3\sqrt{2(2\lambda_3 + L)/L},$$

and $\mathcal{M}_L = (\mathcal{M}_{L,H}, \mathcal{M}_{L,G}) : S_L^2 \times S_L^2 \to \mathbb{R} \times \mathbb{R}$ which are invariant under the Kepler flow. A more detailed analysis is included in Ferrer [10] considering the work of Llibre and Pragana [13].

As a consequence of the proposed parametrization, it is easy to show that

Proposition 3 *Giving the momentum mapping*

$$\mathcal{M}_L \,:\, S_L^2 \times S_L^2 \longrightarrow \Delta = \{(H, G) \,|\, -L \leq H \leq L,\, |H| \leq G \leq L\} \subset \mathbb{R}^2,$$

then, for each value (H_0, G_0)*, the inverse image*

$$\mathcal{M}_L^{-1}(H_0, G_0) \subset S_L^2 \times S_L^2$$

is in correspondence with: (i) one point (the cases $|H_0| = G_0 = L$*); (ii) a circle (when* $|H_0| = G_0$ *or* $G_0 = L$*); (iii) a two-torus (when* $|H_0| < G_0 < L$*), or (iv) a two-sphere (when* $G_0 = 0$*).*

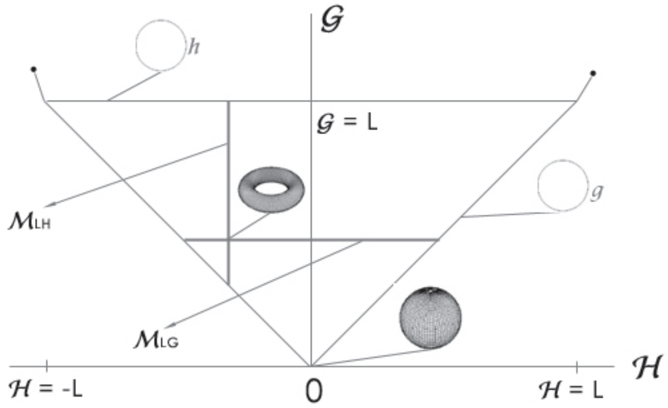

Figure 1: *An illustration of the momentum mapping $\mathcal{M}_L$. The drawing was made by J. Egea.*

Proof: It is straightforward using the proposed parametrization. Note that the two-torus $S^1(R^*) \times S^1(\bar{R})$ is given by

$$\begin{aligned} S^1_{L,G}(R^*) &= \{(\lambda_1,\lambda_2)\,|\,\lambda_1^2+\lambda_2^2 = R^{*2} = (L^2-G^2)G^2/L^2\}, \\ S^1_{L,G,H}(\bar{R}) &= \{(\zeta_1,\zeta_2)\,|\,\zeta_1^2+\zeta_2^2 = \bar{R}^2 = \tfrac{1}{4}L^2\left(1-\tfrac{H^2}{G^2}\right)\} \end{aligned} \tag{15}$$

and, when G takes the two extreme values of the interval ($i.e.$ $G = |H|$ and $G = L$), the two torus shrinks to a circle with radius $R^* = |H|\sqrt{L^2-H^2}/L$ or $\bar{R} = \frac{1}{2}\sqrt{L^2-H^2}$ respectively. These are precisely the circles of *(ii)*. Both shrink to a point when $H = \pm L$. $\square$

Proposition 4 *Giving the momentum mapping*

$$\mathcal{M}_{L,H} : S^2_L \times S^2_L \longrightarrow \tilde{\Delta} \subset \mathbb{R} : (\boldsymbol{\lambda},\boldsymbol{\zeta}) \to \zeta_3\sqrt{2(2\lambda_3+L)/L}, \tag{16}$$

where $\tilde{\Delta} = \{H\,|\,-L \le H \le L\}$, then

(i) if $H = \pm L$, then the inverse image $\mathcal{M}^{-1}_{L,H}(\pm L)$, for each sign, is made of one point of $S^2_L \times S^2_L$.

(ii) If $0 < |H| < L$, for each value $H = c$ the inverse image $\mathcal{M}^{-1}_{L,H}(c)$, is in correspondence with

$$\mathcal{M}^{-1}_{L,H}(c) = \bigcup_{|H|\le G\le L} \mathcal{M}^{-1}_{L,H}(G) \longleftrightarrow S^2_{L,H} \times S^1_{L,H}.$$

(iii) If $H = 0$, the set $\mathcal{M}^{-1}_{L,H}(0)$ is in correspondence with

$$\mathcal{C}^2_{L,0} \times \tilde{S}^1_L,$$

where $\mathcal{C}^2_{L,0}$ is homeomorphic to a 2-sphere, and the radius of the circle $\tilde{S}^1_L$ reduces to zero in two points of $\mathcal{C}^2_{L,0}$ (These two points correspond to two rectilinear orbits in both directions of the z axis).

Proof: *(i)* and *(ii)* are straightforward.

(iii) If $H=0$, we can consider the stratum $\mathcal{M}^{-1}_{L,H}(0)$, which according to (16), is made of two disjoint subsets of $S^2_L\times S^2_L$: $\mathcal{A}=\{(\lambda,\zeta)\,|\,G\neq 0\}$ and $\mathcal{B}=\{(\lambda,\zeta)\,|\,G=0\}$. In other words, $\mathcal{A}$ is the set of 'non-rectilinear polar orbits' and $\mathcal{B}$ the set of rectilinear orbits.

Then, from the parametrization (13) we may write

$$\mathcal{M}^{-1}_{L,H}(0)=\{(\boldsymbol{\lambda},\boldsymbol{\zeta})\,|\,-\tfrac{1}{2}L<\lambda_3\leq\tfrac{1}{2}L,\,\zeta_3=0\}\bigcup\{(\boldsymbol{\lambda},\boldsymbol{\zeta})\,|\,-\tfrac{1}{2}L=\lambda_3\}.$$

In order to identify $\mathcal{M}^{-1}_{L,H}(0)$ with a set similar to what we have done for $H\neq 0$, we proceed as follows: First we establish a correspondence

$$\{(\boldsymbol{\lambda},\boldsymbol{\zeta})\,|\,-\tfrac{1}{2}L<\lambda_3\leq\tfrac{1}{2}L,\,\zeta_3=0\}\longrightarrow S^2_+\times S^1,$$

where S^2_+ is the positive half 2-sphere, including the border, of radius $L/2$ and $S^1=\{(\zeta_1,\zeta_2,0)\}$. Then, we define the surface $\mathcal{C}^2_{L,0}=\{\boldsymbol{\omega}\}\subset\mathbb{R}^3$ as the result of taking S^2_+ and pasting its border by identifying points $(\omega_1,\omega_2,0)$ with $(\omega_1,-\omega_2,0)$. Then, we will denote

$$(\mathcal{C}^2_{L,0})^+=\{\boldsymbol{\omega}\in\mathcal{C}^2_{L,0}\,|\,0<\omega_3\},$$

$$(\mathcal{C}^2_{L,0})^0=\{\boldsymbol{\omega}\in\mathcal{C}^2_{L,0}\,|\,-\tfrac{1}{2}L\leq\omega_1\leq\tfrac{1}{2}L,\,\omega_2=\omega_3=0\}.$$

Now, as a second step, we consider the application

$$\mathcal{M}_0\,:\,\mathcal{M}^{-1}_{L,H}(0)\subset S^2_L\times S^2_L\longrightarrow\mathcal{C}^2_{L,0}\times S^1_{L,0}$$

in the following way

$$\mathcal{M}_0=\begin{cases}\{(\boldsymbol{\lambda},\boldsymbol{\zeta})\,|\,-\tfrac{1}{2}L<\lambda_3\leq\tfrac{1}{2}L,\,\zeta_3=0\}\;\rightarrow\;(\mathcal{C}^2_{L,0})^+\times S^1_{L,0},\\ \qquad\{(\boldsymbol{\lambda},\boldsymbol{\zeta})\,|\,-\tfrac{1}{2}L=\lambda_3\}\;\rightarrow\;(\mathcal{C}^2_{L,0})^0\times S^1(r=\sqrt{\tfrac{1}{4}L^2-\omega_1^2})\}.\end{cases}$$

Let us note that attached to each point of $\mathcal{C}^2_{L,0}$ there is a circle $S^1(r=\sqrt{\tfrac{1}{4}L^2-\omega_1^2})$, except for two points of $\mathcal{C}^2_{L,0}$ namely $(\tfrac{1}{2}L,0,0)$ and $(-\tfrac{1}{2}L,0,0)$, which correspond to $s_1\times n_2$ and $s_1\times s_2$ of $S^2_L\times S^2_L$. (As we will see below, they are fixed points in the reduction). □

Note: As an alternative to $\mathcal{C}^2_{L,0}$ we can take the double cone $\mathcal{V}^2_{L,0}$ proposed by Cushman [4], which he finds working with invariant theory.

Complementing the previous Proposition we give, without proof, the following

Proposition 5 *Giving the momentum mapping*

$$\mathcal{M}_{L,G} : S^2_L \times S^2_L \longrightarrow \bar{\Delta} \subset \mathbb{R} : (\boldsymbol{\lambda}, \boldsymbol{\zeta}) \to \sqrt{L(\lambda_3 + \tfrac{1}{2}L)}, \tag{17}$$

with $\bar{\Delta} = \{G \,|\, 0 \leq G \leq L\}$ *then,*

(i) for $G = L$ *or* $G = 0$ *the inverse images* $\mathcal{M}^{-1}_{L,G}(L)$ *and* $\mathcal{M}^{-1}_{L,G}(0)$ *are in 1-1- correspondence with a 2-sphere;*

(ii) if $0 < G < L$*, for each value* $G = c$ *the inverse image* $\mathcal{M}^{-1}_{L,G}(c)$ *is in 1-1 correspondence with*

$$\mathcal{M}^{-1}_{L,G}(c) = \bigcup_{-G<H<G} \mathcal{M}^{-1}_{L,G}(H) \longleftrightarrow S^2_{L,G} \times S^1_{L,G}.$$

When $G = |H|$ *we get set isomorphic to two circles.*

Note: Details on the relation of $\mathcal{M}^{-1}_{L,G}(c)$ with S^3, mentioned by Cushman, will be given in [11].

5 Reductions by Axial Rotations: a Singular Reduction

So far we have presented some subsets of $S^2 \times S^2$ related with momentum mappings. Now, if we consider symmetries, is the time to look for reductions of the orbit space; reductions related with those subsets. That is well known in classical terms; for example, the 'ignorable variable' related to an axial symmetry. It took some time to realize that *reduction theory* of the seventies did not cover this case: it was necessary to distinguish between *regular* versus *singular* reduction.

As we have said in the Introduction, Cushman [6] has given a full explanation of this process, making use of *actions* and *invariants*; we mention here just some of the concepts and results involved. Nevertheless, the main interest of this section is to show that, again making use of Delaunay variables, the second reduction can be presented in a simpler form, noticing the close relation of the actions with the *node* and *perigee* angles.

5.1 Cushman: actions and invariants

The flow induced in $\mathcal{E}_L = S^2_L \times S^2_L$ by a *rotation* $\mathbf{R}_\alpha(\epsilon)$ *around an 'axis'* α will leave invariant a subset of the orbital space $\mathcal{E}_L$. To construct the corresponding *reduced space* Cushman considers the S^1 action Φ on $S^2_\ell \times S^2_\ell$ defined by

$$\begin{array}{rccc} \Phi : & S^1 \times (S^2_L \times S^2_L) & \to & S^2_L \times S^2_L \\ & (\epsilon, \boldsymbol{\lambda}, \boldsymbol{\zeta}) & \longrightarrow & (\mathbf{R}_\epsilon \boldsymbol{\lambda}, \mathbf{R}_\epsilon \boldsymbol{\zeta}), \end{array}$$

where

$$\mathbf{R}_\epsilon = \begin{pmatrix} \cos\epsilon & -\sin\epsilon & 0 \\ \sin\epsilon & \cos\epsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The action Φ produces a *singular reduction*: *it leaves 4 orbits invariant.* Indeed, the *circular equatorial direct and retrograde, and the rectilinear in the positive and negative Oz axis.*

Lemma 6 *(Cushman, 1984) The algebra of polynomials on $\mathbb{R}^3 \times \mathbb{R}^3$ invariant under the linear S^1 action defined by (5.1) is generated by*

$$\sigma_1 = x_1, \qquad \sigma_2 = x_2 y_2 + x_3 y_3, \qquad \sigma_3 = x_3 y_2 - x_2 y_3,$$

$$\sigma_4 = y_1, \qquad \sigma_5 = x_2^2 + x_3^2, \qquad \sigma_6 = y_2^2 + y_3^2,$$

subject to the relation

$$\sigma_2^2 + \sigma_3^2 = \sigma_5 \sigma_6, \qquad \sigma_5 \geq 0, \qquad \sigma_6 \geq 0. \tag{18}$$

In addition, every polynomial relation among the generators $\sigma_1, \ldots, \sigma_6$ is a consequence of (18).

The reduced space: Denoting by $M_{L,H}$ the invariant manifold induced by the action, the reduced space $M_{L,H}/S^1$ is given by the *semialgebraic variety* $V_{L,H}$ defined by

$$\sigma_2^2 + \sigma_3^2 = (L^2 - \sigma_1^2)(L^2 - (2H - \sigma_1)^2).$$

For $0 < H < L$, the $V_{L,H}$ is a smooth manifold, diffeomorphic to a two sphere $S^2_{L,H}$. For $H = 0$, the variety $V_{L,0}$ has two singular points $\pm(1,0,0)$. Therefore $V_{L,0}$ is homeomorphic but not diffeomorphic to a two sphere.

The invariant σ_i and Delaunay variables: Cushman established the relation of his invariants with Delaunay variables as follows

$$\sigma_1 = LGes\cos g, \quad \sigma_2 = \tfrac{1}{2}(2G^2 - L^2 - H^2 + L^2 e^2 s^2 \sin^2 g), \quad \sigma_3 = G.$$

5.2 Delaunay variables and the second reduction

Instead of the way followed by Cushman, based on invariant theory, for the purpose of reduction in the presence of symmetries, we propose an alternative procedure, making use of Delaunay variables, arriving to another proof of the previous Lemma. More precisely, it is convenient to consider a new parametrization for the open set of 'elliptic inclined' orbits: $(S_1^2 - \{n_1, s_1\}) \times (S_2^2 - \{n_2, s_2\})$.

We propose the following one

$$
\begin{aligned}
\bar\lambda_1 &= \frac{G}{L}\sqrt{L^2-G^2}\,\cos\frac{h+g}{2}, & \bar\zeta_1 &= \tfrac{1}{2}\frac{L}{G}\sqrt{G^2-H^2}\,\cos\frac{h-g}{2},\\
\bar\lambda_2 &= \frac{G}{L}\sqrt{L^2-G^2}\,\sin\frac{h+g}{2}, & \bar\zeta_2 &= \tfrac{1}{2}\frac{L}{G}\sqrt{G^2-H^2}\,\sin\frac{h-g}{2},\\
\bar\lambda_3 &= \frac{1}{L}(G^2-\tfrac{1}{2}L^2), & \bar\zeta_3 &= \tfrac{1}{2}\frac{L}{G}H,
\end{aligned}
\tag{19}
$$

maintaining circular, equatorial and rectilinear located as before.

In an explicit form Delaunay variables are functions given by the inverse transformation

$$
\begin{cases}
G = \sqrt{\frac{\bar\lambda_3}{L}+\frac{1}{2}}\,L,\\
H = 2\sqrt{(\frac{\bar\lambda_3}{L}+\frac{1}{2})}\,\bar\zeta_3,\\
\cos h = 2\,\dfrac{\bar\lambda_1\bar\zeta_1-\bar\lambda_2\bar\zeta_2}{LGes}, \qquad \sin h = 2\,\dfrac{\bar\lambda_2\bar\zeta_1+\bar\lambda_1\bar\zeta_2}{LGes},\\
\cos g = 2\,\dfrac{\bar\lambda_1\bar\zeta_1+\bar\lambda_2\bar\zeta_2}{LGes}, \qquad \sin g = 2\,\dfrac{\bar\lambda_2\bar\zeta_1-\bar\lambda_1\bar\zeta_2}{LGes}.
\end{cases}
\tag{20}
$$

It is worth comparing equations (20) with (8). In particular, from equations (20) we get the eccentricity as $e = e(\lambda_3)$ and inclination $I = I(\zeta_3)$.

With these new coordinates, some quadratic functions $\pi_i : S^2_L \times S^2_L \longrightarrow \mathbb{R}$ which separate Delaunay variables are

$$
\begin{aligned}
\bar\pi_1 &= \bar\lambda_1^2+\bar\lambda_2^2,\\
\bar\pi_2 &= \bar\zeta_1^2+\bar\zeta_2^2,\\
\bar\pi_3 &= \pi_3(\bar\lambda_3),\\
\bar\pi_4 &= \pi_4(\bar\zeta_3),\\
\bar\pi_5 &= 2(\bar\lambda_2\bar\zeta_1+\bar\lambda_1\zeta_2) = LGes\,\sin h,\\
\bar\pi_6 &= 2(\bar\lambda_2\bar\zeta_1-\bar\lambda_1\bar\zeta_2) = LGes\,\sin g,\\
\bar\pi_7 &= 2(\bar\lambda_1\bar\zeta_1-\bar\lambda_2\bar\zeta_2) = LGes\,\cos h,\\
\bar\pi_8 &= 2(\bar\lambda_1\bar\zeta_1+\bar\lambda_2\bar\zeta_2) = LGes\,\cos g,\\
\bar\pi_9 &= \bar\pi_1-\bar\pi_2 = \bar\lambda_1^2+\bar\lambda_2^2-\bar\zeta_1^2-\bar\zeta_2^2 = \bar\zeta_3^2-\bar\lambda_3^2,\\
\bar\pi_{10} &= \bar\pi_1+\bar\pi_2 = \bar\lambda_1^2+\bar\lambda_2^2+\bar\zeta_1^2+\bar\zeta_2^2 = \tfrac{1}{2}L^2-\bar\lambda_3^2-\bar\zeta_3^2.
\end{aligned}
$$

Note that $\bar\pi_8$ and $\bar\pi_6$ are the variables ξ_1, ξ_2 proposed by Coffey *et al.* Among the relations of the functions $\bar\pi$ they verify the following

$$\bar\pi_5^2+\bar\pi_7^2+\bar\pi_9^2 = \bar\pi_{10}^2, \tag{21}$$

$$\bar\pi_6^2+\bar\pi_8^2+\bar\pi_9^2 = \bar\pi_{10}^2 \tag{22}$$

and

$$4\bar{\pi}_1\bar{\pi}_2 = \bar{\pi}_6^2 + \bar{\pi}_8^2 = \bar{\pi}_5^2 + \bar{\pi}_7^2. \tag{23}$$

We should note that our functions $\bar{\pi}_i$, $(5 \le i \le 8)$ are multiplied by 2, a factor not present in the corresponding expressions of Cushman (Lemma 5.2, p. 85) [6]. Moreover, for rectilinear orbits we have $\bar{\pi}_1 = \bar{\pi}_5 = \bar{\pi}_6 = \bar{\pi}_7 = \bar{\pi}_8 = 0$. Moreover, $\bar{\pi}_2 + \bar{\pi}_3^2 = \frac{1}{4}L^2$, i.e. a portion of a parabola $\bar{\pi}_2 = -\bar{\pi}_3^2 + \frac{1}{4}L^2$, in the plane $(\bar{\pi}_3, \bar{\pi}_2)$ because $\bar{\pi}_2 \ge 0$. Each point represents a family of rectilinear orbits 'at the same latitude'; the two extreme values of the parabola are in correspondence with the two polar rectilinear. More details in [10, 11].

Thinking in systems with $H \ne 0$ as an integral, and following Cushman, we choose $\bar{\pi}_3 = \lambda_3$ and $\bar{\pi}_4 = \zeta_3$ (the invariant polynomial of lower degree). Fixing a value for the momentum mapping $(c = H)$, we have $c = \bar{\pi}_4\sqrt{2(2\bar{\pi}_3 + L)/L}$. Then, from

$$\bar{\pi}_6^2 + \bar{\pi}_8^2 = 4\bar{\pi}_1\bar{\pi}_2$$

and the two sphere constraints $\bar{\pi}_1 + \bar{\pi}_3^2 = \frac{1}{4}L^2$ and $\bar{\pi}_2 + \bar{\pi}_4^2 = \frac{1}{4}L^2$, we may put the right member as function of $\bar{\pi}_3$. After some computations we write

$$\bar{\pi}_6^2 + \bar{\pi}_8^2 = 4(\tfrac{1}{4}L^2 - \bar{\pi}_3^2)(\tfrac{1}{4}L^2 - \bar{\pi}_4^2) = 4(\tfrac{1}{4}L^2 - \bar{\pi}_3^2)(\tfrac{1}{4}L^2 - \frac{L\,c^2}{2(2\bar{\pi}_3 + L)})$$

and also

$$\bar{\pi}_6^2 + \bar{\pi}_8^2 = \frac{1}{4}(L^2 - 2\,\bar{\pi}_3 L)(L^2 + 2\,\bar{\pi}_3 L - 2\,c^2)$$

or

$$\bar{\pi}_6^2 + \bar{\pi}_8^2 = \frac{1}{4}(L^4 - 2\,c^2L^2 - 4\,\bar{\pi}_3^2L^2 + 4\bar{\pi}_3Lc^2).$$

Completing squares by adding and subtracting c^4, and introducing the variable $\xi_3 = \frac{1}{2}(2\,\bar{\pi}_3L - c^2)$, we obtain finally

$$\bar{\pi}_6^2 + \bar{\pi}_8^2 + \xi_3^2 = \frac{1}{4}(L^2 - c^2)^2.$$

In other situations (when G is an integral), a different reordering may be done eliminating $\bar{\pi}_3$. As before, considering $4\bar{\pi}_1\bar{\pi}_2 = \bar{\pi}_5^2 + \bar{\pi}_7^2$. Introducing the variable $\bar{\pi}_{12} = \bar{\pi}_{12}(\bar{\lambda}_3, \bar{\zeta}_3) = H\sqrt{L^2 - G^2}$, and after some computations, we get finally

$$\bar{\pi}_5^2 + \bar{\pi}_7^2 + \bar{\pi}_{12}^2 = G^2(L^2 - G^2). \tag{24}$$

6 Appendix I. On the Submanifold of Orbits in a Fixed Plane

Along this Appendix we refer to the sorting presented in Section 3. Some subsets of $S^2 \times S^2$ receive special attention; because there are dynamical systems which have them as invariant submanifolds. Usually these subsets of $S^2 \times S^2$ will consist of orbits coming from the three parts made in the partition: circular, elliptic and rectilinear orbits. The set $\mathcal{E}$ of 'equatorial orbits' (orbits on the $\{O, \boldsymbol{i}, \boldsymbol{j}\}$ plane) is one of them.

Proposition 7 *The orbit space of bounded Kepler orbits on a fixed plane is in 1-1 correspondence to S^2.*

Proof: Without loss of generality, choosing $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ properly, we always may refer to these orbits as $\mathcal{E}$, i.e. the ones corresponding to $G = |H| > 0$, joint with the rectilinear orbits in the plane OXY. According to Proposition 2, this subset $\mathcal{E} \subset \mathcal{O}$ is given by

$$\mathcal{E} = \Delta_1 \bigcup \Delta_2 \bigcup \Delta_3$$

where

$$\Delta_1 = \{(\lambda_1, \lambda_2, \lambda_3, 0, 0, \tfrac{L}{2}) \mid \textstyle\sum_{i\geq 1}^{3} \lambda_i^2 = \tfrac{1}{4}L^2,\ \lambda_3 \neq -\tfrac{L}{2}\} : \qquad \text{direct orbits,}$$

$$\Delta_2 = \{(\lambda_1, \lambda_2, \lambda_3, 0, 0, -\tfrac{L}{2}) \mid \textstyle\sum_{i\geq 1}^{3} \lambda_i^2 = \tfrac{1}{4}L^2,\ \lambda_3 \neq -\tfrac{L}{2}\} : \qquad \text{retrograde orbits,}$$

$$\Delta_3 = \{(0, 0, -\tfrac{L}{2}, \zeta_1, \zeta_2, 0) \mid \zeta_1^2 + \zeta_2^2 = \tfrac{L^2}{4}\} : \qquad \text{rectilinear orbits.}$$

We can apply $\mathcal{E} \longrightarrow S^2$ in a 1-1 correspondence. Indeed, considering the sphere $S^2 = \{(\alpha_1, \alpha_2, \alpha_3) | \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = L^2\}$, we may project $\Delta_1 \longrightarrow S_+^2$, $\Delta_2 \longrightarrow S_-^2$ and $\Delta_3 \longrightarrow \{(\alpha_1, \alpha_2, 0) | \alpha_1^2 + \alpha_2^2 = L^2\}$. It remains to paste these three sets and we obtain S^2. $\square$

There is a convenient set of coordinates for dealing with orbits on a plane. With $\boldsymbol{G} = (G_1, G_2, G_3) = (0, 0, G_3)$, $G = |G_3|$, and the Laplace vector $\boldsymbol{A} = (A_1, A_2, A_3) = (A_1, A_2, 0)$, we know that $A_1^2 + A_2^2 + G_3^2 = L^2$. Thus Deprit [9] proposed we take $(\alpha_1, \alpha_2, \alpha_3) = (A_1, A_2, G_3)$ as the coordinates for the sphere. We will locate the circular direct and retrograde, in the north and south poles of the sphere respectively, and orbits with a given eccentricity in the same small circle of latitude. The rectilinear orbits will occupy the 'equator' of the sphere.

Then, the transformation to Delaunay variables $(g, G_3) \longrightarrow (\alpha_1, \alpha_2, \alpha_3)$

$$\alpha_1 = \sqrt{L^2 - G_3^2}\, \cos g, \qquad \alpha_2 = \sqrt{L^2 - G_3^2}\, \sin g, \qquad \alpha_3 = G_3. \tag{25}$$

We have the following Poisson brackets $(\alpha_i, \alpha_j) = \delta_{ijk}\alpha_k$, which we can obtain directly or from (25). When $G = G_3 = 0$ the Laplace vector reduces to a fixed vector defining the direction on the rectilinear motion and g gives the position of that vector.

7 Appendix II. Comments on a Previous Paper

In 1992 Miller and the author published a note [12] where an attempt was made in order to present the axial reduction in terms of Delaunay variables. There, the idea was to describe the orbital space $S_L^2 \times S_L^2$ by 'separating coordinates' as it appears in Fig. 1 of that note. Based on the assumption of an axial symmetry, we put together concepts related to the first reduction and the invariant sets of the second reduction. For details on how the present research clarifies and correct some of those issues, see [10].

8 Conclusions and Future Work

We have proposed coordinates for the orbital space $S^2_L \times S^2_L$ of bounded Keplerian orbits with a given semimajor axis, as an alternative to Cartan coordinates. They have the property of separating the orientation of the orbital plane from the position of Laplace vector in that plane. Considering the momentum mappings we easily identify families of orbits in bijective correspondence with S^3. This allows to illustrate, in a straightforward manner, the result of a second reduction when H or G are integrals. Details in a forthcoming paper.

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