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Symmetric planar non–collinear relative equilibria for the Lennard–Jones potential 3–body problem with two equal masses

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Abstract

In this paper we study the planar relative equilibria for a system of three point particles with only two equal masses moving under the action of a Lennard–Jones potential. A central configuration is a special position of the particles where the position and acceleration vectors of each particle with respect to the center of mass are proportional, and the constant of proportionality is the same for all particles. Since the Lennard–Jones potential depends only on the mutual distances among the particles, it is invariant under rotations. In a convenient rotating frame the orbits coming from central configurations become equilibrium points, the relative equilibria. Due to the form of the potential, the relative equilibria depend on the size of the system, that is, depend strongly of the momentum of inertia I of the system. In this work we characterize the symmetric planar non–collinear relative equilibria and we give the values of I depending on the parameters of the Lennard–Jones potential for which the number of relative equilibria changes.

Key words and expressions: Central configurations, Lennard–Jones potential, relative equilibria.

MSC: 70F10, 70H05, 34C23.

1 Introduction

In order to get an accurate model to study the action of the intermolecular and gravitational forces at the same time, many authors from physics, astrophysics, astronomy and chemistry have introduced new kinds of potentials, with a structure different from the classical Newtonian and Coulombian potentials. In this way, one potential that has been used very often in those branches of science is the Lennard–Jones potential, which is the one studied in this paper. For instance, it is used to model the nature and stability of small clusters of interacting particles in crystal growth, random geometry of liquids, or in the theory of homogeneous nucleation, see [4] and [8]. This potential also appears in molecular dynamics to simulate many particle systems ranging from solids, liquids, gases, and biomolecules on Earth. Also it appears in the study of the motion of stars and galaxies in the Universe among other applications.

This work is based on the previous work [2]. In [2] the authors studied the equilibria and the relative equilibria of the planar Lennard–Jones 2– and 3–body problem when all the particles have equal masses. A *relative equilibrium solution* is a solution such that the configuration of the three particles remains invariant under a convenient rotation. This configuration is *central*, this is equivalent to say that the position and acceleration vectors of each particle with respect to the center of mass are proportional with the same constant of proportionality. Since the Lennard–Jones potential is invariant under rotations but it is not invariant under homothecies, the relative equilibria depend on the size of the system, that is, depend on the momentum of inertia I of the system. This does not happen in other planar problems, like for instance the planar Newtonian 3–body problem (see for a definition [3] or [6]).

In this paper we consider the planar Lennard–Jones 3–body problem with masses m_1, m_2 and m_3 when we have only two particles with equal masses. Without loss of generality we shall take $m_1 = m_2$. In particular, we find the planar non–collinear equilibrium points (see Theorem 2), we analyze the symmetric planar non–collinear central configurations and we give the bifurcation values of I, depending on the parameters of the Lennard–Jones potential, for which the number of central configurations changes (see Theorems 6 and 9). When we say a *symmetric planar non–collinear central configuration* we mean that the triangle formed by the three particles of a central configuration has an axis of symmetry passing through m_3. In [2] it has been proved that when the three particles are equal, all planar non–collinear central configurations are symmetric. Here we show that when the three particles are not equal there exist also non–symmetric central configurations.

2 Equations of Motion

We consider two particles m_1 and m_2 with the same mass and a third one m_3 with different mass that are moving in the Euclidean plane. The forces between every pair of particles are given by a Lennard–Jones potential energy. Let $\mathbf{q}_i \in \mathbb{R}^2$ denote the position of the particle i in an inertial coordinate system, and let $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$.

The Lennard–Jones potential is a spherically symmetric non–bounded interaction between two particles given by

$$\phi(r) = D_0 \left[\left(\frac{R_0}{r} \right)^{12} - 2 \left(\frac{R_0}{r} \right)^{6} \right] , \tag{1}$$

where r is the distance between the particles, R_0 is the equilibrium separation of two interacting particles and it corresponds to the minimum of $\phi(r)$, and $D_0 = -\phi(R_0)$ is sometimes called the *well depth.* The function (1) is equivalent to the following one by using the relationships $R_0 = 2^{1/6}\sigma$ and $D_0 = \varepsilon$

$$\phi(r) = 4\varepsilon \left[\left(\frac{\sigma}{r} \right)^{12} - \left(\frac{\sigma}{r} \right)^{6} \right] ,$$

where σ is the arithmetic mean of the two van der Waals radii of the two interacting particles.

Let σ_{11} and ε_{11} be the parameters of the Lennard–Jones potential corresponding to interactions between the two particles m_1 and m_2 with equal masses; and let σ_{22} and ε_{22} be the parameters corresponding to interactions between a pair of particles with mass m_3. Cross interactions between particles of different masses are computed using the Lorentz–Berthelot combining rules: (see [1])

$$\sigma_{12} = \frac{1}{2}(\sigma_{11} + \sigma_{22}) , \qquad \varepsilon_{12} = \sqrt{\varepsilon_{11}\varepsilon_{22}} .$$

Choosing the units of mass, length and time conveniently we can think that the particles with mass $m_1 = m_2$ have mass 1, radius $\sigma = 1/(2^{1/6})$ (which corresponds to take an equilibrium separation equal to 1), and an interaction energy $\varepsilon = 1$; and the particle m_3 has mass m, radius σ (equilibrium separation equal to R_0), and an interaction energy ε. Using the Lorentz–Berthelot combining rules we have that

$$\sigma_{12} = \frac{1}{2} \left(\frac{1}{2^{1/6}} + \frac{R_0}{2^{1/6}} \right) , \qquad \varepsilon_{12} = \sqrt{\varepsilon} .$$

Then, denoting by ρ the R_0 corresponding to (1) for an interaction between a pair of particles with masses 1 and m, we get that $\rho = \frac{1}{2}(1 + R_0)$. In short, from (1), the potential energy of the three particles is given by

$$U = \frac{1}{r_{12}^{12}} - 2\frac{1}{r_{12}^{6}} + \sqrt{\varepsilon} \left[\left(\frac{\rho}{r_{13}} \right)^{12} - 2 \left(\frac{\rho}{r_{13}} \right)^{6} \right] + \sqrt{\varepsilon} \left[\left(\frac{\rho}{r_{23}} \right)^{12} - 2 \left(\frac{\rho}{r_{23}} \right)^{6} \right] , \tag{2}$$

where $r_{ij} = |\mathbf{q}_i - \mathbf{q}_j|$ is the distance between the particles i and j.

The Newton's equations of the planar motion associated to potential (2) are

$$M\ddot{\mathbf{q}} = -\nabla U(\mathbf{q}) \ , \tag{3}$$

where M =diag$(1,1,1,1,m,m)$ is a 6×6 diagonal matrix, and the dot denotes derivative with respect to the time t. Equations (3) are only defined on the *configuration space* $\triangle = \{(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3)\in\mathbb{R}^6 \ : \ \mathbf{q}_i \neq \mathbf{q}_j\}$.

The *center of mass* of the particles is $\mathbf{R} = \frac{1}{2+m}(\mathbf{q}_1+\mathbf{q}_2+m\mathbf{q}_3)$. In what follows we will assume that the center of mass of the particles is fixed at the origin.

3 Equilibrium Solutions

The simplest type of solutions of system (3) are the *equilibrium points*; that is, when the 3 particles are at rest for all $t\in\mathbb{R}$. Then, an equilibrium point of (3) is a solution satisfying the equation $\nabla U(\mathbf{q}) = 0$.

We denote by $R_{12} = r_{12}$, $R_{13} = r_{13}/\rho$ and $R_{23} = r_{23}/\rho$, then the potential energy (2) becomes

$$U(R_{12},R_{13},R_{23}) = \frac{1}{R_{12}^{12}} - \frac{2}{R_{12}^{6}} + \sqrt{\varepsilon}\left(\frac{1}{R_{13}^{12}} - \frac{2}{R_{13}^{6}}\right) + \sqrt{\varepsilon}\left(\frac{1}{R_{23}^{12}} - \frac{2}{R_{23}^{6}}\right) \ . \tag{4}$$

We note that R_{12}, R_{13} and R_{23} are functions of the variables $\mathbf{q}_i$, for $i=1,2,3$.

In order to solve equation $\nabla U(\mathbf{q}) = 0$ we will use the following lemma proved in [2].

Lemma 1 *Let $u = f(\mathbf{x})$ be a function with $\mathbf{x} = (x_1,x_2,\ldots,x_n)$, $x_1 = g_1(\mathbf{y})$, $x_2 = g_2(\mathbf{y})$,…, $x_n = g_n(\mathbf{y})$, $\mathbf{y} = (y_1,y_2,\ldots,y_m)$ and $m \geqslant n$.*

If rank $(A) = n$ being

$$A = \begin{pmatrix} \dfrac{\partial x_1}{\partial y_1} & \cdots & \dfrac{\partial x_n}{\partial y_1} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial x_1}{\partial y_m} & \cdots & \dfrac{\partial x_n}{\partial y_m} \end{pmatrix} ,$$

then $\nabla f(\mathbf{x}) = \mathbf{0}$ if and only if $\nabla u(\mathbf{y}) = \mathbf{0}$.

Using Lemma 1 we have that if rank $(A) = 3$ being

$$A = \begin{pmatrix} \dfrac{\partial R_{12}}{\partial q_{11}} & \dfrac{\partial R_{13}}{\partial q_{11}} & \dfrac{\partial R_{23}}{\partial q_{11}} \\ \dfrac{\partial R_{12}}{\partial q_{12}} & \dfrac{\partial R_{13}}{\partial q_{12}} & \dfrac{\partial R_{23}}{\partial q_{12}} \\ \dfrac{\partial R_{12}}{\partial q_{21}} & \dfrac{\partial R_{13}}{\partial q_{21}} & \dfrac{\partial R_{23}}{\partial q_{21}} \\ \dfrac{\partial R_{12}}{\partial q_{22}} & \dfrac{\partial R_{13}}{\partial q_{22}} & \dfrac{\partial R_{23}}{\partial q_{22}} \\ \dfrac{\partial R_{12}}{\partial q_{31}} & \dfrac{\partial R_{13}}{\partial q_{31}} & \dfrac{\partial R_{23}}{\partial q_{31}} \\ \dfrac{\partial R_{12}}{\partial q_{32}} & \dfrac{\partial R_{13}}{\partial q_{32}} & \dfrac{\partial R_{23}}{\partial q_{32}} \end{pmatrix} = \begin{pmatrix} \dfrac{q_{11}-q_{21}}{r_{12}} & \dfrac{1}{\rho}\dfrac{q_{11}-q_{31}}{r_{13}} & 0 \\ \dfrac{q_{12}-q_{22}}{r_{12}} & \dfrac{1}{\rho}\dfrac{q_{12}-q_{32}}{r_{13}} & 0 \\ -\dfrac{q_{11}-q_{21}}{r_{12}} & 0 & \dfrac{1}{\rho}\dfrac{q_{21}-q_{31}}{r_{23}} \\ -\dfrac{q_{12}-q_{22}}{r_{12}} & 0 & \dfrac{1}{\rho}\dfrac{q_{22}-q_{32}}{r_{23}} \\ 0 & -\dfrac{1}{\rho}\dfrac{q_{11}-q_{31}}{r_{13}} & -\dfrac{1}{\rho}\dfrac{q_{21}-q_{31}}{r_{23}} \\ 0 & -\dfrac{1}{\rho}\dfrac{q_{12}-q_{32}}{r_{13}} & -\dfrac{1}{\rho}\dfrac{q_{22}-q_{32}}{r_{23}} \end{pmatrix} ,$$

then

$$\nabla U(\mathbf{q}) = 0 \qquad \text{if and only if} \qquad \nabla U(R_{12}, R_{13}, R_{23}) = 0 \ .$$

Here, $\mathbf{q}_i = (q_{i1}, q_{i2})$ for $i = 1, 2, 3$.

After some computations we see that rank $(A) = 3$ if and only if

$$\det \begin{pmatrix} q_{11} & q_{12} & 1 \\ q_{21} & q_{22} & 1 \\ q_{31} & q_{32} & 1 \end{pmatrix} \neq 0 \ .$$

This determinant is twice the oriented area of the triangle formed by the 3 particles. In short, if $\mathbf{q}_1$, $\mathbf{q}_2$ and $\mathbf{q}_3$ are not collinear, then $\nabla U(R_{12}, R_{13}, R_{23}) = \mathbf{0}$ if and only if $\nabla U(\mathbf{q}) = \mathbf{0}$. In this paper we do not consider the collinear case. Therefore, the planar non–collinear equilibrium points of the Lennard–Jones 3–body problem (3) are given by the solutions of the equation

$$\nabla U(R_{12}, R_{13}, R_{23}) = \begin{pmatrix} 12\left(R_{12}^{-13} - R_{12}^{-7}\right) \\ 12\sqrt{\varepsilon}\left(R_{13}^{-13} - R_{13}^{-7}\right) \\ 12\sqrt{\varepsilon}\left(R_{23}^{-13} - R_{23}^{-7}\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \ .$$

We solve the equation $R^{-13} - R^{-7} = 0$ obtaining a unique positive real root $R = 1$. So, $\nabla U(R_{12}, R_{13}, R_{23}) = \mathbf{0}$ if and only if $R_{12} = R_{13} = R_{23} = 1$. Therefore, we have infinitely many planar non–collinear equilibrium points of the Lennard–Jones 3–body problem (3) which are characterized by the following result.

Theorem 2 *The planar non–collinear equilibrium points of the Lennard–Jones* 3*–body problem (3) are given by the set*

$$\left\{ (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) \in \mathbb{R}^6 \ : \ |\mathbf{q}_1 - \mathbf{q}_2| = 1, \ |\mathbf{q}_1 - \mathbf{q}_3| = |\mathbf{q}_2 - \mathbf{q}_3| = \rho, \ \mathbf{q}_3 = -\frac{1}{m}(\mathbf{q}_1 + \mathbf{q}_2) \right\} \ .$$

We note that in a planar equilibrium point of the Lennard–Jones 3–body problem (3), the three particles are at the vertices of an isosceles triangle, equilateral if $\rho = 1$.

4 Relative Equilibrium Solutions

Another simple type of solutions are the *relative equilibrium solutions*; that is, solutions of (3) that become an equilibrium point in a uniformly rotating coordinate system. These solutions are characterized as follows.

Let $R(\theta)$ and J denote the 6×6 block diagonal matrices with 3 blocks of size 2×2 of the form

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} , \qquad \text{and} \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} ,$$

respectively, over the diagonal of the 6×6 matrix. We define a new coordinate vector $\mathbf{x} \in \mathbb{R}^6$ by $\mathbf{q} = R(\omega t)\mathbf{x}$, where the constant ω is the angular velocity of the uniform rotating coordinate system. In this new coordinate system the equation of motion (3) becomes

$$M\ddot{\mathbf{x}} + 2\omega JM\dot{\mathbf{x}} = -\nabla U(\mathbf{x}) + \omega^2 M\mathbf{x}\ . \tag{5}$$

Then a configuration $\mathbf{x}$ is central if and only if $\mathbf{x}$ is an equilibrium point of system (5). That is, if and only if

$$-\nabla U(\mathbf{x}) + \omega^2 M\mathbf{x} = \mathbf{0}\ ,$$

for some ω. If $\mathbf{x}$ is a central configuration, then $\mathbf{q} = R(\omega t)\mathbf{x}$ is a relative equilibrium solution of system (3). Moreover $\mathbf{q} = R(\omega t)\mathbf{x}$, is a periodic solution of system (3) with period $T = 2\pi/|\omega|$.

The study of central configurations can be seen as a problem of Lagrange multipliers where we are looking for critical points of the potential U on the "ellipsoid" $\{\mathbf{x} \in \triangle : (1/2)\mathbf{x}^T M\mathbf{x} = I\}$ where $I > 0$ is a constant. Thus, $\mathbf{x}$ is a central configuration if it is a solution of system

$$\nabla F(\mathbf{x}) = \mathbf{0}\ , \qquad\qquad i(\mathbf{x}) - I = 0, \tag{6}$$

where $F(\mathbf{x}) = -U(\mathbf{x}) + \omega^2(i(\mathbf{x}) - I)$ and $i(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T M\mathbf{x}$ is the *moment of inertia* of the configuration.

Since we have chosen the origin of the coordinates at the center of mass of the three particles, $i(\mathbf{x})$ can be written in terms of the mutual distances r_{ij}, i.e.

$$i(\mathbf{x}) = \frac{1}{2(2+m)}(r_{12}^2 + m r_{13}^2 + m r_{23}^2) = \frac{1}{2(2+m)}(R_{12}^2 + m\rho^2 R_{13}^2 + m\rho^2 R_{23}^2)\ .$$

The potential U depends on $\mathbf{x}$ through the mutual distances r_{ij}, and it is given by (4). Therefore, we can think that F depends on $\mathbf{x}$ through R_{ij}.

Proceeding as in Section 3 we can see that if $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ are not collinear, then $\nabla F(R_{12}, R_{13}, R_{23}) = \mathbf{0}$ if and only if $\nabla F(\mathbf{x}) = \mathbf{0}$. Therefore, from (6), the planar non–collinear central configurations of the Lennard–Jones 3–body problem (3) are given by the solutions of system

$$\begin{array}{ll} 12\left(\dfrac{1}{R_{12}^{13}} - \dfrac{1}{R_{12}^{7}}\right) + \dfrac{1}{2+m}R_{12}\omega^2 = 0\ , & 12\sqrt{\varepsilon}\left(\dfrac{1}{R_{13}^{13}} - \dfrac{1}{R_{13}^{7}}\right) + \dfrac{m\rho^2}{2+m}R_{13}\omega^2 = 0\ , \\ 12\sqrt{\varepsilon}\left(\dfrac{1}{R_{23}^{13}} - \dfrac{1}{R_{23}^{7}}\right) + \dfrac{m\rho^2}{2+m}R_{23}\omega^2 = 0\ , & \dfrac{1}{2(2+m)}(R_{12}^2 + m\rho^2 R_{13}^2 + m\rho^2 R_{23}^2) = I\ . \end{array} \tag{7}$$

From the first three equations of (7) we have

$$-\frac{\omega^2}{12(2+m)} = \frac{1}{R_{12}^{14}} - \frac{1}{R_{12}^{8}} = \frac{\sqrt{\varepsilon}}{m\rho^2}\left(\frac{1}{R_{13}^{14}} - \frac{1}{R_{13}^{8}}\right) = \frac{\sqrt{\varepsilon}}{m\rho^2}\left(\frac{1}{R_{23}^{14}} - \frac{1}{R_{23}^{8}}\right)\ . \tag{8}$$

We set $s_{12} = R_{12}^2$, $s_{13} = R_{13}^2$, $s_{23} = R_{23}^2$, $\alpha = m\rho^2$ and $C = 2(2+m)I$, then using (8) a solution of (7) is a solution of the system

$$\frac{1}{s_{12}^7} - \frac{1}{s_{12}^4} = \frac{\sqrt{\varepsilon}}{\alpha}\left(\frac{1}{s_{13}^7} - \frac{1}{s_{13}^4}\right),\ \frac{1}{s_{12}^7} - \frac{1}{s_{12}^4} = \frac{\sqrt{\varepsilon}}{\alpha}\left(\frac{1}{s_{23}^7} - \frac{1}{s_{23}^4}\right),\ s_{12} + \alpha s_{13} + \alpha s_{23} = C, \tag{9}$$

satisfying that $\omega^2 > 0$. Next we analyze the solutions of (9) depending on C, α and ε.

We consider the first two equations of (9). Let $f(a) = a^{-7} - a^{-4}$. We note that $\lim_{a\to 0^+} f(a) = \infty$, $\lim_{a\to\infty} f(a) = 0$, $f(1) = 0$, $f(a) < 0$ when $a > 1$, and $f(a)$ has a minimum at the point $a = a^* = \left(\frac{7}{4}\right)^{1/3} = 1.20507\ldots$ with $\beta = f(a^*) = -\frac{12}{49}\left(\frac{4}{7}\right)^{1/3} = -0.203222\ldots$ (see Figure 1).

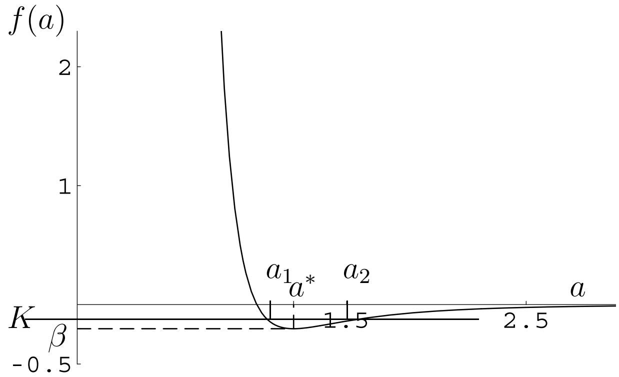

Figure 1: Plot of $f(a)$.

We note that the first two equations of (9) can be written as

$$f(s_{12}) = \frac{\sqrt{\varepsilon}}{\alpha}K, \qquad\qquad f(s_{13}) = f(s_{23}) = K\ , \tag{10}$$

for some $K \in \mathbb{R}$. Since in order to have a solution of (7) we need that $\omega^2 > 0$, we are only interested in values of s_{ij} such that $f(s_{ij}) < 0$. Then, the solutions of (7) come from solutions of (10) such that $K \in [\beta, 0)$ (see Figure 1) and additionally $\frac{\sqrt{\varepsilon}}{\alpha}K \in [\beta, 0)$. Therefore, if $\sqrt{\varepsilon} \leqslant \alpha$, then we can take values of $K \in [\beta, 0)$. Whereas if $\sqrt{\varepsilon} > \alpha$, then we can take values of $K \in [\alpha\,\beta/\sqrt{\varepsilon}, 0)$. Fixed an admissible value of K, we can find two values $a_1 \leqslant a_2$ satisfying that $f(a_i) = K$ (see Figure 1), and two values $\overline{a}_1 \leqslant \overline{a}_2$ satisfying that $f(\overline{a}_i) = \frac{\sqrt{\varepsilon}}{\alpha}K$. Combining these values we obtain eight types of solutions of (10) which are detailed in cases (1)–(8) of Table 1.

We note that the case $\omega = 0$ corresponds to the equilibrium points of the planar Lennard–Jones 3–body problem (3) given by Theorem 2. These equilibrium points have moment of inertia $I = \frac{C}{2(2+m)} = \frac{1+2\alpha}{2(2+m)}$.

The planar non–collinear central configurations of the Lennard–Jones 3–body problem (3) are triangles with sides r_{12}, r_{13} and r_{23} that could be equilateral, isosceles or scalene. In Table 1 we give the solutions of (10) in the variables $r_{12} = R_{12} = \sqrt{s_{12}}$, $r_{13} = \rho\,R_{13} = \rho\sqrt{s_{13}}$ and $r_{23} = \rho\,R_{23} = \rho\sqrt{s_{23}}$. We see that if the solutions of (10) define a triangle (i.e. the mutual distances r_{ij} satisfy the conditions $r_{12} < r_{13} + r_{23}$, $r_{13} < r_{12} + r_{23}$ and $r_{23} < r_{12} + r_{13}$), then the solutions of types (1)–(4) give central configurations that are isosceles triangles, whereas the solutions of types (5)–(8) give central configurations that are scalene triangles, except perhaps for a particular set of values of m, ρ and ε.

In the variables s_{12}, s_{13}, s_{23}	In the variables r_{12}, r_{13}, r_{23}
(1) $s_{12} = \overline{a}_1, \quad s_{13} = s_{23} = a_1$	$r_{12} = \sqrt{\overline{a}_1}, \quad r_{13} = r_{23} = \rho\sqrt{a_1}$,
(2) $s_{12} = \overline{a}_2, \quad s_{13} = s_{23} = a_1$	$r_{12} = \sqrt{\overline{a}_2}, \quad r_{13} = r_{23} = \rho\sqrt{a_1}$,
(3) $s_{12} = \overline{a}_1, \quad s_{13} = s_{23} = a_2$	$r_{12} = \sqrt{\overline{a}_1}, \quad r_{13} = r_{23} = \rho\sqrt{a_2}$,
(4) $s_{12} = \overline{a}_2, \quad s_{13} = s_{23} = a_2$	$r_{12} = \sqrt{\overline{a}_2}, \quad r_{13} = r_{23} = \rho\sqrt{a_2}$,
(5) $s_{12} = \overline{a}_1, \quad s_{13} = a_1, \quad s_{23} = a_2$	$r_{12} = \sqrt{\overline{a}_1}, \quad r_{13} = \rho\sqrt{a_1}, \quad r_{23} = \rho\sqrt{a_2}$,
(6) $s_{12} = \overline{a}_2, \quad s_{13} = a_1, \quad s_{23} = a_2$	$r_{12} = \sqrt{\overline{a}_2}, \quad r_{13} = \rho\sqrt{a_1}, \quad r_{23} = \rho\sqrt{a_2}$,
(7) $s_{12} = \overline{a}_1, \quad s_{13} = a_2, \quad s_{23} = a_1$	$r_{12} = \sqrt{\overline{a}_1}, \quad r_{13} = \rho\sqrt{a_2}, \quad r_{23} = \rho\sqrt{a_1}$,
(8) $s_{12} = \overline{a}_2, \quad s_{13} = a_2, \quad s_{23} = a_1$	$r_{12} = \sqrt{\overline{a}_2}, \quad r_{13} = \rho\sqrt{a_2}, \quad r_{23} = \rho\sqrt{a_1}$.

Table 1: Types of solutions of (10).

In this work we consider only symmetric planar non–collinear central configurations given by solutions of (7) of the form $r_{13} = r_{23}$, or equivalently solutions of (9) with $s_{13} = s_{23}$. System (9) when $s_{13} = s_{23}$ becomes

$$\frac{1}{s_{12}^7} - \frac{1}{s_{12}^4} = \frac{\sqrt{\varepsilon}}{\alpha}\left(\frac{1}{s_{13}^7} - \frac{1}{s_{13}^4}\right) , \qquad s_{12} + 2\alpha s_{13} = C \ . \tag{11}$$

5 Symmetric Planar Non–Collinear Central Configurations When $\sqrt{\varepsilon} = \alpha$

The case $\varepsilon = \alpha = m = 1$ has been studied in [2]. Here, we analyze the case $\sqrt{\varepsilon} = \alpha$ and $m \neq 1$. The results that we obtain in this case are similar to the results obtained when $\varepsilon = \alpha = m = 1$.

It is easy to see that if $\sqrt{\varepsilon} = \alpha$, then $\overline{a}_1 = a_1$ and $\overline{a}_2 = a_2$. Then, the types of symmetric solutions of (10) given in Table 1 when $\sqrt{\varepsilon} = \alpha$ are the ones in Table 2.

(1) $s_{12} = s_{13} = s_{23} = a_1$	$\Longrightarrow$	$C \in (1 + 2\alpha, (1 + 2\alpha)a^*]$,
(2) $s_{12} = a_2$, $\quad s_{13} = s_{23} = a_1$	$\Longrightarrow$	$C \in (a^* + 2\alpha, \infty)$,
(3) $s_{12} = a_1$, $\quad s_{13} = s_{23} = a_2$	$\Longrightarrow$	$C \in (1 + 2\alpha a^*, \infty)$,
(4) $s_{12} = s_{13} = s_{23} = a_2$	$\Longrightarrow$	$C \in [(1 + 2\alpha)a^*, \infty)$.

Table 2: Symmetric solutions of (10) when $\sqrt{\varepsilon} = \alpha$.

From Figure 1 we see that $a_1 \in (1, a^*]$ and $a_2 \in [a^*, \infty)$. Then we can find lower and upper bounds for the value of C for each type of solutions, which are specified in Table 2. We note that the lower bounds given in Table 2 are not necessarily the infima.

It is easy to check, from Table 2, that for any admissible value of C there are at most 3 different central configurations. On the other hand, from Figure 1, we see that a_1 as a function of K is a decreasing function, whereas a_2 as a function of K is an increasing function. Therefore, for each $C \in (1+2\alpha, (1+2\alpha)a^*]$ we have a solution of (11) of type (1) and for each $C \in [(1+2\alpha)a^*, \infty)$ we have a solution of (11) of type (4). We note that $a_2 \to \infty$ when $K \to 0$. Then we can find $C^* \geqslant \max(a^* + 2\alpha, 1 + 2\alpha a^*, (1+2\alpha)a^*) = (1+2\alpha)a^*$ such that there exist three different solutions of (11) for all $C > C^*$. On the other hand if $C \in (1+2\alpha, C_*)$ with $C_* = \min(a^* + 2\alpha, 1 + 2\alpha a^*, (1+2\alpha)a^*)$ there exists only one solution of (11). For values of $C \in (C_*, C^*)$ we can have one, two or three different solutions of (11) depending on the value of C, ε and α. It is not difficult to see that $C_* = 1 + 2\alpha a^*$ for $0 < \alpha < 1/2$ and $C_* = a^* + 2\alpha$ for $\alpha > 1/2$.

Now we analyze the bifurcation values of C for which the number of solutions of (11) changes. From the above discussion we know that there exists at least one bifurcation value $C \in (C_*, C^*)$.

From the second equation of (11) we have that

$$s_{12} = C - 2\,s_{13}\,\alpha\ . \tag{12}$$

Substituting s_{12} into the first equation of (11) with $\sqrt{\varepsilon} = \alpha$, we obtain the equation

$$g_1(s_{13}, \alpha, C)\, g_2(s_{13}, \alpha, C) = 0\ , \tag{13}$$

where $g_1(s_{13}, \alpha, C) = C - (1+2\alpha)\,s_{13}$, and

$$\begin{aligned}
g_2(s_{13}, \alpha, C) = {} & -C^6 + s_{13}\left(-C^5 + 12\,C^5\,\alpha\right) + {s_{13}}^2\left(-C^4 + 10\,C^4\,\alpha - 60\,C^4\,\alpha^2\right) + \\
& {s_{13}}^3\left(-C^3 + C^6 + 8\,C^3\,\alpha - 40\,C^3\,\alpha^2 + 160\,C^3\,\alpha^3\right) + \\
& {s_{13}}^4\left(-C^2 + C^5 + 6\,C^2\,\alpha - 12\,C^5\,\alpha - 24\,C^2\,\alpha^2 + 80\,C^2\,\alpha^3 - 240\,C^2\,\alpha^4\right) + \\
& {s_{13}}^5\left(-C + C^4 + 4\,C\,\alpha - 10\,C^4\,\alpha - 12\,C\,\alpha^2 + 60\,C^4\,\alpha^2 + 32\,C\,\alpha^3 - 80\,C\,\alpha^4 + 192\,C\,\alpha^5\right) + \\
& {s_{13}}^6\left(-1 + C^3 + 2\,\alpha - 8\,C^3\,\alpha - 4\,\alpha^2 + 40\,C^3\,\alpha^2 + 8\,\alpha^3 - 160\,C^3\,\alpha^3 - 16\,\alpha^4 + 32\,\alpha^5 - 64\,\alpha^6\right) + \\
& {s_{13}}^7\left(-6\,C^2\,\alpha + 24\,C^2\,\alpha^2 - 80\,C^2\,\alpha^3 + 240\,C^2\,\alpha^4\right) + \\
& {s_{13}}^8\left(12\,C\,\alpha^2 - 32\,C\,\alpha^3 + 80\,C\,\alpha^4 - 192\,C\,\alpha^5\right) + {s_{13}}^9\left(-8\,\alpha^3 + 16\,\alpha^4 - 32\,\alpha^5 + 64\,\alpha^6\right)\ .
\end{aligned}$$

Solving equation $g_1(s_{13}, \alpha, C) = 0$ with respect to s_{13} we get $s_{13} = \frac{C}{1+2\,\alpha}$. Then, from (12) we have that $s_{12} = \frac{C}{1+2\,\alpha}$. We note that the solution $s_{12} = s_{13} = \frac{C}{1+2\,\alpha}$ is either of type (1) or of type (4) in Table 2. We can see easily that the solution of (11) $s_{12} = s_{13} = \frac{C}{1+2\,\alpha}$, satisfies that $\omega^2 > 0$ if and only if $C > 1 + 2\alpha$. Moreover, in order to have a symmetric planar non–collinear central configuration of the Lennard–Jones 3–body problem (3) we need that $r_{12} < 2\,r_{13}$, or equivalently, we need that $s_{12} < \frac{4\alpha}{m} s_{13}$. We note that for the solution $s_{12} = s_{13} = \frac{C}{1+2\,\alpha}$ this condition is satisfied only when $m < 4\alpha$ (i.e. when $\rho > 1/2$). In short, we have proved the following result.

Proposition 3 *Fixed $\rho > 1/2$, there exists a symmetric planar non–collinear central configuration $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ of the Lennard–Jones 3–body problem (3) for each $C > 1 + 2m\rho^2$ satisfying that*

$$|\mathbf{x}_1 - \mathbf{x}_2| = \sqrt{\frac{C}{1 + 2\,m\rho^2}}, \quad |\mathbf{x}_1 - \mathbf{x}_3| = |\mathbf{x}_2 - \mathbf{x}_3| = \rho\sqrt{\frac{C}{1 + 2\,m\rho^2}}, \quad \mathbf{x}_3 = -\frac{1}{m}(\mathbf{x}_1 + \mathbf{x}_2)\ ,$$

and having moment of inertia $I = \frac{C}{2(2+m)}$.

Next we analyze the bifurcation values $C(\alpha)$ for which the number of positive real solutions $s_{13}(C,\alpha)$ of equation $g_2(s_{13},C,\alpha)=0$ changes. We have seen that there exists at least one bifurcation value $C(\alpha) \in (C_*, C^*)$. In particular, this bifurcation value satisfies that $C(\alpha) > 1+2\alpha$. In order to simplify the computations we will take $1+2\alpha$ as a lower bound for the bifurcation values $C(\alpha)$.

Proposition 4 *For every $\alpha > 0$ there is a unique bifurcation value $C_1(\alpha)$ for the number of positive real solutions s_{13} of the equation $g_2(s_{13},C,\alpha)=0$.*

Proof: The number of positive real solutions of $g_2(s_{13},C,\alpha)=0$ changes either when a negative real solution is transformed into a positive one, by passing through the solution $s_{13}=0$, or when a pair of complex conjugate solutions are transformed to a pair of positive real solutions, by passing through a double positive real solution.

Since the coefficient of the independent term of $g_2(s_{13},C,\alpha)$ is always negative for $C > 1+2\alpha$, there is no bifurcation value $C_1(\alpha) > 1+2\alpha$ coming from passing a negative real solution of $g_2(s_{13},C,\alpha)=0$ to a positive one. On the other hand, a solution $s_{13}(C,\alpha)$ of $g_2(s_{13},C,\alpha)=0$ is a double solution if it is a solution of system

$$g_2(s_{13},C,\alpha)=0 \ , \qquad g_3(s_{13},C,\alpha)=\frac{d\,g_2}{d\,s_{13}}(s_{13},C,\alpha)=0 \ . \tag{14}$$

Therefore, the bifurcation values $C_1(\alpha)$, coming from passing a pair of complex conjugate solutions to a pair of positive real solutions, are given by the solutions $C=C(\alpha)$ of system (14).

In order to solve system (14) we compute the resultant (see [5] and [7] for more information about resultants) of $g_2(s_{13},C,\alpha)$ and $g_3(s_{13},C,\alpha)$ with respect to s_{13} obtaining the polynomial

$$P(C,\alpha) = -512\,C^{30}\,\alpha^9\,\left(-1+2\,\alpha\right)\left(1+4\,\alpha^2\right)\,P_1(C,\alpha) \ ,$$

where $P_1(C,\alpha)$ is a polynomial of degree 30 in the variable C. We note that if $\alpha=1/2$, then $P(C,\alpha)=0$, so, this case is treated aside. The solutions $C_1(\alpha)$ of $P_1(C,\alpha)=0$ are possible bifurcation values. We know that $P_1(C,\alpha)=0$ has at least one solution $C_1(\alpha) \in (C_*,C^*)$. Now we shall see that, for all $\alpha > 0$, $P_1(C,\alpha)=0$ has a unique real solution $C_1(\alpha)$ satisfying that $C_1(\alpha) > 1+2\alpha$.

First we analyze the case $\alpha = 1/2$. If we compute the resultant of $g_2(s_{13},C,1/2)$ and $g_3(s_{13},C,1/2)$ with respect to s_{13} we obtain the polynomial

$$P(C) = 8\,C^{32}\,\left(-14+C^3\right)\left(-49+2016\,C^3+2322\,C^6+432\,C^9+27\,C^{12}\right)^2.$$

This polynomial has a unique positive real solution with $C > 1+2\alpha = 2$ which is given by $C = C_1 = 14^{\frac{1}{3}}$. Analyzing the solutions of $g_2(s_{13},C,1/2)=0$ for C near C_1, we see that C_1 is a bifurcation value for the solutions of $g_2(s_{13},C,1/2)=0$.

Now we analyze the bifurcation values α for which the number of real solutions $C_1(\alpha) > 1+2\alpha$ of equation $P_1(C,\alpha) = 0$ changes. As above, the number of real solutions of $P_1(C,\alpha) = 0$ with $C_1(\alpha) > 1+2\alpha$ changes, either when a solution $C_1(\alpha) < 1+2\alpha$ is transformed to a solution $C_1(\alpha) > 1+2\alpha$ by passing through the solution $C_1(\alpha) = 1+2\alpha$, or when a pair of complex conjugate solutions are transformed to two positive real solutions by passing through a double positive real solution with $C_1(\alpha) > 1+2\alpha$.

We see that $P_1(1+2\alpha,\alpha) < 0$ for all $\alpha > 0$. Then, there is no bifurcation values α coming from passing a solution $C_1(\alpha)$ of $P_1(C,\alpha) = 0$ through the value $1+2\alpha$ in the positive sense. The bifurcation values α coming from passing a pair of complex solutions of $P_1(C,\alpha) = 0$ to two positive real solutions through a double positive real solution are given by the α's solutions of the system

$$P_1(C,\alpha) = 0 \ , \qquad \frac{d\,P_1}{d\,C}(C,\alpha) = 24C^2\,P_2(C,\alpha) = 0 \ . \tag{15}$$

As above we solve system (15) by computing the resultant of $P_1(C,\alpha)$ and $P_2(C,\alpha)$ with respect to C (we note that we are not interested in solutions of (15) with $C = 0$). We obtain a polynomial $G(\alpha)$ of degree 762 in the variable α, whose real positive roots are

$$\begin{array}{llll} \alpha_1 = 0.0185509\ldots, & \alpha_2 = 0.333133\ldots, & \alpha_3 = 0.33395\ldots, & \alpha_4 = 0.5, \\ \alpha_5 = 0.748615\ldots, & \alpha_6 = 0.750450\ldots, & \alpha_7 = 13.4764\ldots\ . & \end{array}$$

We have analyzed the solutions of $P_1(C,\alpha) = 0$ near the bifurcation values α_i and we have seen that for all $\alpha > 0$ there exists a unique real solution $C_1(\alpha)$ of $P_1(C,\alpha) = 0$ satisfying that $C_1(\alpha) > 1+2\alpha$. Moreover, this solution corresponds to a bifurcation value for the solutions of equation $g_2(s_{13}, C, \alpha) = 0$.

□

Up to here, we have analyzed the number of solutions of $g_1(s_{13}, C, \alpha) = 0$ and $g_2(s_{13}, C, \alpha) = 0$ separately. Since we are interested in the bifurcation values of (13), we must find also the values of C for which the solutions of those two equations coincide. We can see that there exists a unique value $C = C_2(\alpha) = \frac{7^{\frac{1}{3}}\,(1+2\,\alpha)}{2^{\frac{2}{3}}}$ satisfying this condition. This value is obtained by substituting the solution $s_{13} = \frac{C}{1+2\alpha}$ into $g_2(s_{13}, C, \alpha)$ and solving the resulting equation $g_2\left(\frac{C}{1+2\alpha}, C, \alpha\right) = 0$ with respect to C.

Analyzing the solutions of (13) near the bifurcation values $C_1(\alpha)$ and $C_2(\alpha)$ we obtain de following result.

Proposition 5 *Suppose that $\sqrt{\varepsilon} = \alpha$. Fixed $\alpha > 0$, we can find two bifurcation values $C_1 = C_1(\alpha)$ and $C_2 = C_2(\alpha) = \frac{7^{\frac{1}{3}}\,(1+2\,\alpha)}{2^{\frac{2}{3}}}$, with $C_0 = 1+2\alpha < C_1(\alpha) \leqslant C_2(\alpha)$, at which the number of positive real solutions of equation (13) changes. Fixed $\alpha > 0$, the number(#) of solutions $C > C_0$ of equation (13) is given in Table 3. We also give their type (T), according to Table 2, and their multiplicity (M).*

		#	T	M
$\alpha > 1/2$	$C \in (C_0, C_1)$	1	(1)	1
	$C = C_1$	1	(1)	1
		1	(2)	2
	$C \in (C_1, C_2)$	1	(1)	1
		2	(2)	1
	$C = C_2$	1	(2)	1
		1	(4)	2
	$C \in (C_2, \infty)$	1	(2)	1
		1	(3)	1
		1	(4)	1
$\alpha = 1/2$	$C \in (C_0, C_1)$	1	(1)	1
	$C = C_1 = C_2$	1	(4)	3
	$C > C_2$	1	(2)	1
		1	(3)	1
		1	(4)	1
$\alpha < 1/2$	$C \in (C_0, C_1)$	1	(1)	1
	$C = C_1$	1	(1)	1
		1	(3)	2
	$C \in (C_1, C_2)$	1	(1)	1
		2	(3)	1
	$C = C_2$	1	(3)	1
		1	(4)	2
	$C \in (C_2, \infty)$	1	(2)	1
		1	(3)	1
		1	(4)	1

Table 3: The number (#), type (T) and multiplicity (M) of solutions $C > C_0$ of (13).

The solutions of (13) give planar non–collinear central configuration of the Lennard–Jones 3–body problem (3) if $s_{12} < \frac{4\alpha}{m} s_{13}$; that is, if $\varphi(s_{13}, C, \alpha) < 0$ with $\varphi(s_{13}, C, \alpha) = C - 2\alpha s_{13} - \frac{4\alpha}{m} s_{13}$. We have analyzed previously the central configurations coming from solutions of equation $g_1(s_{13}, C, \alpha) = 0$, now we analyze the central configurations coming from solutions of equation $g_2(s_{13}, C, \alpha) = 0$. We see that there is a unique value of C for which the solution of equation $\varphi(s_{13}, C, \alpha) = 0$ is a solution of equation $g_2(s_{13}, C, \alpha) = 0$. This value is given by

$$C_{m\alpha} = \frac{(2+m)}{2\,m} \left(\frac{m^6 + 4\,m^5\,\alpha + 16\,m^4\,\alpha^2 + 64\,m^3\,\alpha^3 + 256\,m^2\,\alpha^4 + 1024\,m\,\alpha^5 + 4096\,\alpha^6}{(m + 4\,\alpha)\,\left(m^2 + 16\,\alpha^2\right)} \right)^{\frac{1}{3}},$$

and it corresponds to a collinear central configuration. So, if $s_{13}(C, \alpha)$ is a solution of equation $g_2(s_{13}, C, \alpha) = 0$ such that $\varphi(s_{13}(C_{m\alpha}, \alpha), C_{m\alpha}, \alpha) = 0$, then the number of planar non–collinear central configurations coming from solutions of $g_2(s_{13}, C, \alpha) = 0$ changes at $C = C_{m\alpha}$.

Fixed $\alpha > 0$, we can see that $C_{m\alpha} \to \infty$ when $m \to 0$ and when $m \to \infty$, and $C_{m\alpha}$ has a minimum at $m = \mu$ and $C_{\mu\alpha} = C_1(\alpha)$. The number of central configurations coming from the solutions of (16) given by Proposition 5 is summarized in the following result.

Theorem 6 *Suppose that $\sqrt{\varepsilon} = \alpha$. Let $C_1 = C_1(\alpha)$ and $C_2 = C_2(\alpha)$, with $C_0 = 1 + 2\alpha < C_1(\alpha) \leqslant C_2(\alpha)$, be the bifurcation values given in Proposition 5; and let $\mu = \mu(\alpha)$ be the minimum of $C_{\mu\alpha}$. Fixed $\alpha > 0$ and $m > 0$, the number of symmetric planar non–collinear central configurations of the Lennard–Jones 3–body problem (3) having moment of inertia*

$I = \frac{C}{2(2+m)}$ *changes at* $C = C_1 = C_1(\alpha)$, $C = C_2 = C_2(\alpha)$ *and* $C = C_m = C_{m\alpha}$. *The number (#) of symmetric planar non–collinear central configurations for fixed* $\alpha > 0$, $C > C_0$ *and* $m > 0$ *is summarized in Table 4. In this table,* $\mu_1^* = \mu_1^*(\alpha) < \mu$ *and* $\mu_2^* = \mu_2^*(\alpha) > \mu$ *are values such that* $C_{\mu_i^*\alpha} = C_2$.

$\alpha > 1/2$	$m \in (0, \mu_1^*)$	C	(C_0, C_1)	C_1	(C_1, C_2)	C_2	(C_2, C_m)	$[C_m, \infty)$
		#	1	2	3	2	3	2
	$m = \mu_1^*$	C	(C_0, C_1)	C_1	(C_1, C_2)	$C_2 = C_{\mu_1^*}$	(C_2, ∞)	
		#	1	2	3	2	2	
	$m = (\mu_1^*, \mu)$	C	(C_0, C_1)	C_1	(C_1, C_m)	$[C_m, C_2)$	C_2	(C_2, ∞)
		#	1	2	3	2	1	2
	$m = \mu$	C	(C_0, C_1)	$C_1 = C_\mu$	(C_1, C_2)	C_2	(C_2, ∞)	
		#	1	1	2	1	2	
	$m = (\mu, \mu_2^*)$	C	(C_0, C_1)	C_1	$(C_1, C_m]$	(C_m, C_2)	C_2	(C_2, ∞)
		#	1	1	1	2	1	2
	$m = \mu_2^*$	C	(C_0, C_1)	C_1	(C_1, C_2)	$C_2 = C_{\mu_2^*}$	(C_2, ∞)	
		#	0	0	0	1	1	
	$m = (\mu_2^*, \infty)$	C	(C_0, C_1)	C_1	(C_1, C_2)	C_2	$(C_2, C_m]$	(C_m, ∞)
		#	0	0	0	0	0	1
$\alpha = 1/2$	$m \in (0, \mu)$	C	(C_0, C_1)	C_1	(C_1, C_m)	$[C_m, \infty)$		
		#	1	1	3	2		
	$m = \mu$	C	(C_0, C_1)	$C_1 = C_\mu$	(C_1, ∞)			
		#	0	0	1			
	$m \in (\mu, \infty)$	C	(C_0, C_1)	C_1	$(C_1, C_m]$	(C_m, ∞)		
		#	0	0	0	1		
$\alpha < 1/2$	$m \in (0, \mu_1^*)$	C	(C_0, C_1)	C_1	(C_1, C_2)	C_2	(C_2, C_m)	$[C_m, \infty)$
		#	1	2	3	2	3	2
	$m = \mu_1^*$	C	(C_0, C_1)	C_1	(C_1, C_2)	$C_2 = C_{m_1^*}$	(C_2, ∞)	
		#	0	1	2	1	1	
	$m = (\mu_1^*, \mu)$	C	(C_0, C_1)	C_1	(C_1, C_m)	$[C_m, C_2)$	C_2	(C_2, ∞)
		#	0	1	2	1	1	1
	$m = \mu$	C	(C_0, C_1)	$C_1 = C_\mu$	(C_1, C_2)	C_2	(C_2, ∞)	
		#	0	0	1	1	1	
	$m = (\mu, \mu_2^*)$	C	(C_0, C_1)	C_1	$(C_1, C_m]$	(C_m, C_2)	C_2	(C_2, ∞)
		#	0	0	0	1	1	1
	$m = \mu_2^*$	C	(C_0, C_1)	C_1	(C_1, C_2)	$C_2 = C_{\mu_2^*}$	(C_2, ∞)	
		#	0	0	0	0	1	
	$m = (\mu_2^*, \infty)$	C	(C_0, C_1)	C_1	(C_1, C_2)	C_2	$(C_2, C_m]$	(C_m, ∞)
		#	0	0	0	0	0	1

Table 4: The number (#) of central configurations for $\sqrt{\varepsilon} = \alpha$.

6 Symmetric Planar Non–Collinear Central Configurations When $\sqrt{\varepsilon} \neq \alpha$

Now we analyze the case $\sqrt{\varepsilon} \neq \alpha$. We start giving some preliminary results, and then we will find the bifurcation values of C for which the number of central configurations changes for a fixed value $\alpha = 2$. From numerical experiments, it seems that the bifurcation pattern for C is qualitatively the same for all $\alpha > 0$, but this problem is still open.

When $\sqrt{\varepsilon} < \alpha$, let $\overline{a}_1^* < \overline{a}_2^*$ be the values satisfying that $f(a_i^*) = \frac{\sqrt{\varepsilon}}{\alpha}\beta$, let $a_1 \leqslant a_2$ denote the values such that $f(a_i) = K$ for some $K \in [\beta, 0)$, and let $\overline{a}_1 < \overline{a}_2$ denote the values of a, such that $f(\overline{a}_i) = \frac{\sqrt{\varepsilon}}{\alpha}K$. Since $\sqrt{\varepsilon} < \alpha$ it is easy to see that $a_1 \in (1, a^*]$, $a_2 \in [a^*, \infty)$, $\overline{a}_1 \in (1, \overline{a}_1^*]$ and $\overline{a}_2 \in [\overline{a}_2^*, \infty)$ with $\overline{a}_1 < \overline{a}_2$.

When $\sqrt{\varepsilon} > \alpha$, let $a_1^* < a_2^*$ be the values satisfying $f(a_i^*) = \frac{\alpha}{\sqrt{\varepsilon}}\beta$, let $a_1 < a_2$ denote the values such that $f(a_i) = K$ for some $K \in [\frac{\alpha}{\sqrt{\varepsilon}}\beta, 0)$, and let $\overline{a}_1 \leqslant \overline{a}_2$ denote the values such that $f(\overline{a}_i) = \frac{\sqrt{\varepsilon}}{\alpha}K$. In this case $a_1 \in (1, a_1^*]$, $a_2 \in [a_2^*, \infty)$, $\overline{a}_1 \in (1, a^*]$ and $\overline{a}_2 \in [a^*, \infty)$.

Finding lower and upper bounds for the value of C for each type of symmetric solutions of Table 1, we obtain Table 5.

				$\sqrt{\varepsilon} < \alpha$	$\sqrt{\varepsilon} > \alpha$
(1)	$s_{12} = \overline{a}_1$	$s_{13} = s_{23} = a_1$	$\Longrightarrow$	$C \in (1 + 2\alpha, \overline{a}_1^* + 2\alpha\, a^*]$	$C \in (1 + 2\alpha, a^* + 2\alpha\, a_1^*]$,
(2)	$s_{12} = \overline{a}_2$	$s_{13} = s_{23} = a_1$	$\Longrightarrow$	$C \in (\overline{a}_2^* + 2\alpha, \infty)$	$C \in (a^* + 2\alpha, \infty)$,
(3)	$s_{12} = \overline{a}_1$	$s_{13} = s_{23} = a_2$	$\Longrightarrow$	$C \in (1 + 2\alpha\, a^*, \infty)$	$C \in (1 + 2\alpha\, a_2^*, \infty)$,
(4)	$s_{12} = \overline{a}_2$	$s_{13} = s_{23} = a_2$	$\Longrightarrow$	$C \in [\overline{a}_2^* + 2\alpha\, a^*, \infty)$	$C \in [a^* + 2\alpha\, a_2^*, \infty)$.

Table 5: Symmetric solutions of (10) when $\sqrt{\varepsilon} \neq \alpha$.

As in the case $\sqrt{\varepsilon} = \alpha$, a_1 and $\overline{a}_1$ are decreasing functions as functions of K, whereas a_2 and $\overline{a}_2$ are increasing functions as functions of K. Therefore, when $\sqrt{\varepsilon} < \alpha$, for each $C \in (1+2\alpha, \overline{a}_1^* + 2\alpha\, a^*]$ we have a solution of (11) of type (1), and for each $C \in [\overline{a}_2^* + 2\alpha\, a^*, \infty)$ we have a solution of (11) of type (4). When $\sqrt{\varepsilon} < \alpha$, for each $C \in (1 + 2\alpha, a^* + 2\alpha\, a_1^*]$ we have a solution of (11) of type (1) and for each $C \in [a^* + 2\alpha\, a_2^*, \infty)$ we have a solution of (11) of type (4).

On the other hand, if $K \to 0$, then $a_2 \to \infty$ and $\overline{a}_2 \to \infty$. Therefore, we can find $C^* \geqslant \max(\overline{a}_1^* + 2\alpha\, a^*, \overline{a}_2^* + 2\alpha, \overline{a}_2^* + 2\alpha\, a^*)$ when $\sqrt{\varepsilon} < \alpha$, and $C^* \geqslant \max(a^* + 2\alpha\, a_1^*, a^* + 2\alpha, 1 + 2\alpha\, a_2^*)$ when $\sqrt{\varepsilon} > \alpha$ such that there exist three different solutions of (11) for all $C > C^*$. Let $C_* = \min(\overline{a}_1^* + 2\alpha\, a^*, \overline{a}_2^* + 2\alpha, \overline{a}_2^* + 2\alpha\, a^*)$ when $\sqrt{\varepsilon} < \alpha$, and $C_* = \min(a^* + 2\alpha\, a_1^*, a^* + 2\alpha, 1 + 2\alpha\, a_2^*)$ when $\sqrt{\varepsilon} > \alpha$. If $C \in (1 + 2\alpha, C_*)$, then there exists only one solution of (11). For values of $C \in (C_*, C^*)$ we can have one, two or three different solutions of (11) depending on the values of C, ε and α.

6.1 Bifurcation values for $\sqrt{\varepsilon} \neq \alpha$ and $\alpha = 2$

Now we analyze the bifurcation values of C for the solutions of (11) fixed the value $\alpha = 2$. Proceeding in a similar way that in the case $\sqrt{\varepsilon} = \alpha$ we know that there exists at least one bifurcation value $C(\varepsilon) \in (C_*, C^*)$. Moreover, this bifurcation value satisfies that $C(\varepsilon) > 1 + 2\alpha = 5$, and 5 is taken as a lower bound for $C(\varepsilon)$.

In order to simplify the computations, we set $\epsilon = \sqrt{\varepsilon}$. From the second equation of (11) we have that $s_{12} = C - 2\, s_{13}\, \alpha$. Substituting s_{12} into the first equation of (11), for $\alpha = 2$ we obtain equation

$$h_1(s_{13}, \epsilon, C) = 0 \ , \tag{16}$$

where

$$h_1(s_{13}, \epsilon, C) = -C^7\, \epsilon + 28\, C^6\, s_{13}\, \epsilon - 336\, C^5\, {s_{13}}^2\, \epsilon + {s_{13}}^3\, \left(2240\, C^4\, \epsilon + C^7\, \epsilon\right) +$$

$$s_{13}{}^4\,\left(-8960\,C^3\,\epsilon-28\,C^6\,\epsilon\right)+s_{13}{}^5\,\left(21504\,C^2\,\epsilon+336\,C^5\,\epsilon\right)+$$
$$s_{13}{}^6\,\left(-28672\,C\,\epsilon-2240\,C^4\,\epsilon\right)+s_{13}{}^7\,\left(2-2\,C^3+16384\,\epsilon+8960\,C^3\,\epsilon\right)+$$
$$s_{13}{}^8\,\left(24\,C^2-21504\,C^2\,\epsilon\right)+s_{13}{}^9\,\left(-96\,C+28672\,C\,\epsilon\right)+s_{13}{}^{10}\,\left(128-16384\,\epsilon\right)\ .$$

Next we will analyze the bifurcation values $C(\epsilon)$ for which the number of positive real solutions of equation (16) changes. We will proceed in a similar way as in the proof of Proposition 4.

The number of positive real solutions of $h_1(s_{13}, C, \epsilon) = 0$ changes, either when a negative real solution is transformed to a positive one, or when a pair of complex conjugate solutions are transformed to a pair of positive real solutions.

Since the coefficient of the independent term of $h_1(s_{13}, C, \epsilon) = 0$ is always negative for $C > 5$ and $\epsilon > 0$, there is no bifurcation value $C(\epsilon) > 5$ coming from passing a negative real solution of $h_1(s_{13}, C, \epsilon) = 0$ to a positive one. On the other hand, the bifurcation values $C(\epsilon)$ coming from passing a pair of complex conjugate solutions to a pair of real solutions are given by the values of C as functions of ϵ of the solutions of system

$$h_1(s_{13}, C, \epsilon) = 0\ , \qquad h_2(s_{13}, C, \epsilon) = \frac{d\,h_1}{d\,s_{13}}(s_{13}, \epsilon, C) = 0\ . \tag{17}$$

In order to solve system (17) we compute the resultant of $h_1(s_{13}, C, \epsilon)$ and $h_2(s_{13}, C, \epsilon)$ with respect to s_{13} obtaining the polynomial

$$Q(C, \epsilon) = -2147483648\,C^{42}\,\epsilon^6\,\left(-1+128\,\epsilon\right)\,Q_1(C, \epsilon)\ ,$$

where $Q_1(C, \epsilon)$ is a polynomial of degree 36 in the variable C. We note that if $\epsilon = 1/128$, then $Q(C, \epsilon) = 0$, so, this case is treated aside. The solutions $C(\epsilon)$ of $Q_1(C, \epsilon) = 0$ provide the bifurcation values $C(\epsilon)$.

When $\epsilon = 1/128$, the resultant of the polynomials $h_1(s_{13}, C, 1/128)$ and $h_2(s_{13}, C, 1/128)$ is a polynomial $Q(C)$ of degree 77 in the variable C. This polynomial has a unique real root satisfying that $C > 5$ which is $C = 10.5853\ldots$. Analyzing the solutions of equations (16) with $\epsilon = 1/128$ for C near $10.5853\ldots$ we see that $C = 10.5853\ldots$ is a bifurcation value where one positive real solution of (16) bifurcates to three real positive solutions of (16).

We analyze the bifurcation values ϵ for which the number of real positive solutions $C(\epsilon)$, with $C(\epsilon) > 5$, of equation $Q_1(C, \epsilon) = 0$ changes. We see that $Q_1(5, \epsilon) < 0$ for all $\epsilon > 0$, then there is no bifurcation value ϵ coming from solutions $C(\epsilon) < 5$ which pass to solutions $C(\epsilon) > 5$. The bifurcation values ϵ coming from passing a pair of complex solutions of $Q_1(C, \epsilon) = 0$ to a pair of real solutions of $Q_1(C, \epsilon) = 0$ are given by the values of ϵ solution of system

$$Q_1(C, \epsilon) = 0\ , \qquad \frac{d\,Q_1}{d\,C}(C, \epsilon) = 24C^2\,Q_2(C, \epsilon) = 0\ . \tag{18}$$

We solve system (18) by computing the resultant of $Q_1(C,\epsilon)$ and $Q_2(C,\epsilon)$ with respect to C. We obtain a polynomial $H(\epsilon)$ of degree 357 in the variable ϵ, whose real positive roots are

$$\begin{array}{llll} \epsilon_1 = 0.0000155637\ldots, & \epsilon_2 = 0.0000223661\ldots, & \epsilon_3 = 0.00024942\ldots, & \epsilon_4 = 0.00232545\ldots, \\ \epsilon_5 = 0.0169526\ldots, & \epsilon_6 = 0.043392\ldots, & \epsilon_7 = 2, & \epsilon_8 = 2.2494\ldots\ . \end{array}$$

Analyzing the solutions of $Q_1(C,\epsilon) = 0$ for values of ϵ close to ϵ_i, we obtain the following result.

Lemma 7 *Fixed $\epsilon > 0$, we have the following number of solutions $C(\epsilon)$ of equation $Q_1(C,\epsilon) = 0$ such that $C(\epsilon) > 5$.*

(a) For $\epsilon < 2$ there exists a unique solution, $C_1(\epsilon)$, with multiplicity 1.

(b) For $\epsilon = 2$ there exist two different solutions, $C_1(\epsilon) < C_2(\epsilon)$; $C_1(\epsilon)$ with multiplicity 1 and $C_2(\epsilon)$ with multiplicity 2.

(c) For $\epsilon \in (2, 2.2494\ldots)$ there exist three different solutions, $C_1(\epsilon) < C_2(\epsilon) < C_3(\epsilon)$, with multiplicity 1.

(d) For $\epsilon = 2.2494\ldots$ there exist two different solutions, $C_1(\epsilon) < C_3(\epsilon)$; $C_1(\epsilon)$ with multiplicity 2 and $C_3(\epsilon)$ with multiplicity 1.

(e) For $\epsilon > 2.2494\ldots$ there exist a unique solution, $C_3(\epsilon)$, with multiplicity 1.

Analyzing the solutions of equation (16) near the bifurcation values $C_i(\epsilon)$ given in Lemma 7 we obtain the following result.

Proposition 8 *Suppose that $\alpha = 2$ and $\epsilon = \sqrt{\varepsilon}$. Let $C_i = C_i(\epsilon)$ be the bifurcation values given in Lemma 7. Fixed $\epsilon > 0$ the number(#) of solutions of equation (16) for $C > 5$ is given in Table 6.*

$\epsilon < 2$	C	$(5,C_1)$	C_1	(C_1,∞)
	#	1	2	3

$\epsilon = 2$	C	$(5,C_1)$	C_1	(C_1,C_2)	C_2	(C_2,∞)
	#	1	2	3	2	3

$\epsilon \in (2, 2.2494\ldots)$	C	$(5,C_1)$	C_1	(C_1,C_2)	C_2	(C_2,C_3)	C_3	(C_3,∞)
	#	1	2	3	2	1	2	3

$\epsilon = 2.2494\ldots$	C	$(5,C_1)$	C_1	(C_1,C_3)	C_3	(C_3,∞)
	#	1	1	1	2	3

$\epsilon > 2.2494\ldots$	C	$(5,C_3)$	C_3	(C_3,∞)
	#	1	2	3

Table 6: The number (#) of solutions of equation (16) with $C > 5$.

In order to count the number of planar non–collinear central configurations of the Lennard–Jones 3–body problem (3) that come from solutions of (16), we proceed in a similar way as in Section 5.

The solutions of (16) give planar non–collinear central configuration of the Lennard–Jones 3–body problem (3) if $s_{12} < \frac{8}{m}s_{13}$; that is, if $\varphi(s_{13}, C) < 0$ with $\varphi(s_{13}, C) = C - 4s_{13} - \frac{8}{m}s_{13}$. We see that there is a unique value of C for which the solution of equation $\varphi(s_{13}, C) = 0$ is a solution of equation (16). This value is given by

$$C_{m\epsilon} = \frac{2+m}{2\,m}\left(\frac{m^7 - 1048576\,\epsilon}{m^4 - 2048\,\epsilon}\right)^{\frac{1}{3}},$$

and it corresponds to a collinear central configuration. So, if $s_{13}(C, \epsilon)$ is a solution of equation (16) such that $\varphi(s_{13}(C_{m\epsilon}), C_{m\epsilon}) = 0$, then the number of planar non–collinear central configurations coming from solutions of (16) changes at $C = C_{m\epsilon}$.

Fixed $\epsilon > 0$ and $\epsilon \neq 2$, we analyze the properties of the function $C_{m\epsilon}$ as a function of m (see Figure 2 for details). When $\epsilon = 2$ we have that $\sqrt{\varepsilon} = \alpha = 2$, and this case has been studied in Section 5.

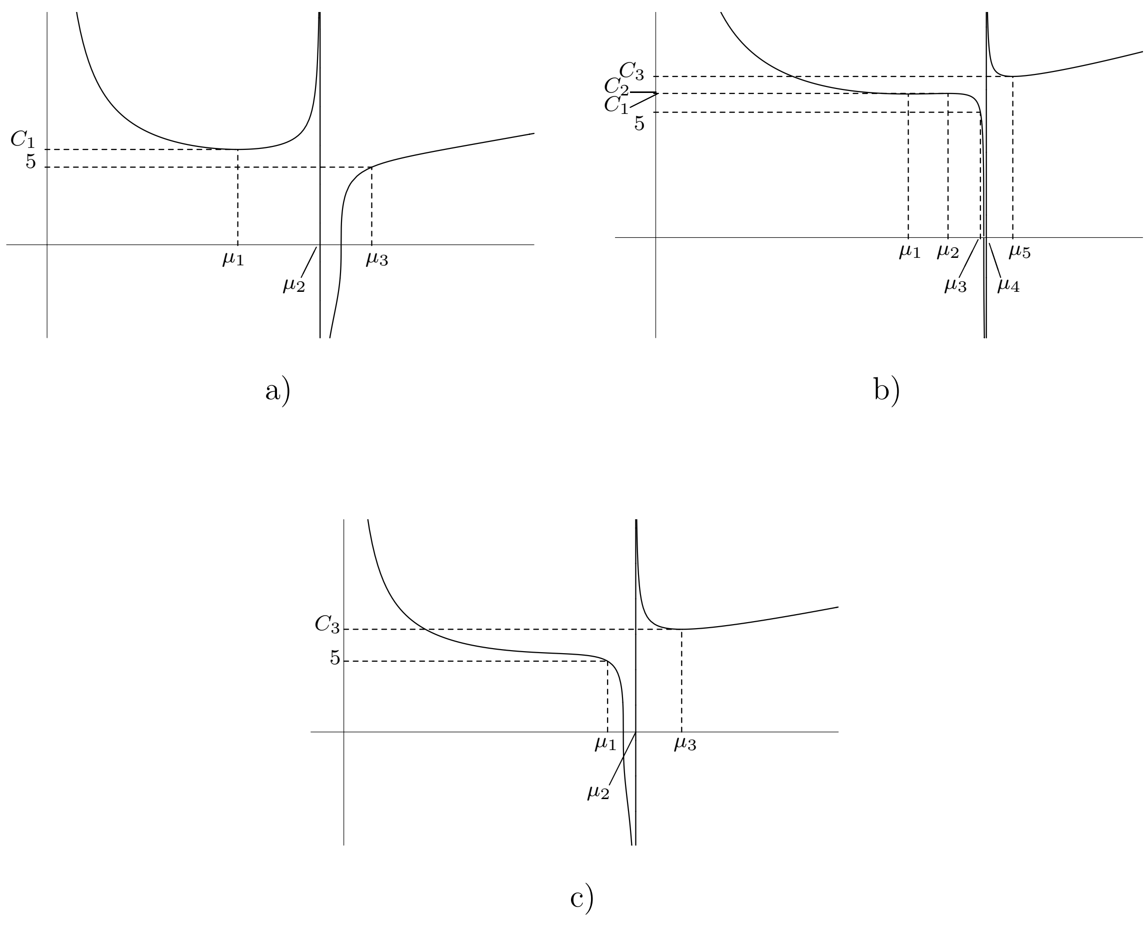

Figure 2: Plot of $C_{m\epsilon}$.

If $\epsilon \in (0,2)$, then $C_{m\epsilon} \to \infty$ when $m \to 0^+$; $C_{m\epsilon}$ has a minimum at a point $m = \mu_1 = \mu_1(\epsilon)$ with $C_{\mu_1\epsilon} = C_1(\epsilon)$, where $C_1(\epsilon)$ is the bifurcation value given in Lemma 7; $C_{m\epsilon} \to \infty$ when $m \to \mu_2^-$ with $\mu_2 = \mu_2(\epsilon) = 4\,2^{\frac{3}{4}}\,\epsilon^{\frac{1}{4}}$. For $m > \mu_2$, $C_{m\epsilon}$ is an increasing function such that $C_{m\epsilon} \to -\infty$ when $m \to \mu_2^+$; $C_{m\epsilon} = 0$ when $m = 4\,2^{\frac{6}{7}}\,\epsilon^{\frac{1}{7}}$; $C_{m\epsilon} < 5$ when $m \in (\mu_2, \mu_3)$ with $\mu_3 = 8$; and $C_{m\epsilon} \to \infty$ when $m \to \infty$ (see Figure 2 a)). We note that $\mu_1 < \mu_2 < 4\,2^{\frac{6}{7}}\,\epsilon^{\frac{1}{7}} < \mu_3$.

If $\epsilon \in (2, 2.2494\ldots)$ where $\epsilon = 2.2494\ldots$ is the bifurcation value for ϵ given in Lemma 7, then $C_{m\epsilon} \to \infty$ when $m \to 0^+$; $C_{m\epsilon}$ has a minimum at a point $m = \mu_1 = \mu_1(\epsilon)$ with $C_{\mu_1\epsilon} = C_1(\epsilon)$; $C_{m\epsilon}$ has a maximum at a point $m = \mu_2 = \mu_2(\epsilon)$ with $C_{\mu_2\epsilon} = C_2(\epsilon)$; $C_{m\epsilon} = 5$ when $m = \mu_3 = 8$; $C_{m\epsilon} = 0$ when $m = 4\,2^{\frac{6}{7}}\,\epsilon^{\frac{1}{7}}$; $C_{m\epsilon} \to -\infty$ when $m \to \mu_4^-$ with $\mu_4 = \mu_4(\epsilon) = 4\,2^{\frac{3}{4}}\,\epsilon^{\frac{1}{4}}$; $C_{m\epsilon} \to \infty$ when $m \to \mu_4^+$; $C_{m\epsilon}$ has a minimum at a point $m = \mu_5 = \mu_5(\epsilon)$ with $C_{\mu_5\epsilon} = C_3(\epsilon)$; and $C_{m\epsilon} \to \infty$ when $m \to \infty$ (see Figure 2 b)). Here $C_1(\epsilon)$, $C_2(\epsilon)$ and $C_3(\epsilon)$ are the bifurcation values given in Lemma 7. We note that $\mu_1 < \mu_2 < \mu_3 < 4\,2^{\frac{6}{7}}\,\epsilon^{\frac{1}{7}} < \mu_4 < \mu_5$.

If $\epsilon = 2.2494\ldots$, then $C_{m\epsilon} \to \infty$ when $m \to 0^+$; $C_{m\epsilon}$ has an inflection point at $m = \mu(\epsilon) = \mu(\epsilon)$ with $C_{\mu\epsilon} = C_1(\epsilon)$; $C_{m\epsilon} = 5$ when $m = \mu_1 = 8$; $C_{m\epsilon} = 0$ when $m = 4\,2^{\frac{6}{7}}\,\epsilon^{\frac{1}{7}}$; $C_{m\epsilon} \to -\infty$ when $m \to \mu_2^-$ with $\mu_2 = \mu_2(\epsilon) = 4\,2^{\frac{3}{4}}\,\epsilon^{\frac{1}{4}}$; $C_{m\epsilon} \to \infty$ when $m \to \mu_2^+$; $C_{m\epsilon}$ has a minimum at a point $m = \mu_3 = \mu_3(\epsilon)$ with $C_{\mu_3\epsilon} = C_3(\epsilon)$; and $C_{m\epsilon} \to \infty$ when $m \to \infty$. Here $C_1(\epsilon)$ and $C_3(\epsilon)$ are the bifurcation values given in Lemma 7. We note that $\mu < \mu_1 < 4\,2^{\frac{6}{7}}\,\epsilon^{\frac{1}{7}} < \mu_2 < \mu_3$.

If $\epsilon > 2.2494\ldots$, then, for $C \in (0, \mu_2)$ with $\mu_2 = 4\,2^{\frac{3}{4}}\,\epsilon^{\frac{1}{4}}$, $C_{m\epsilon}$ is a decreasing function such that $C_{m\epsilon} \to \infty$ when $m \to 0^+$; $C_{m\epsilon} = 5$ when $m = \mu_1 = 8$; $C_{m\epsilon} = 0$ when $m = 4\,2^{\frac{6}{7}}\,\epsilon^{\frac{1}{7}}$; $C_{m\epsilon} \to -\infty$ when $m \to \mu_2^-$ with $\mu_2 = 4\,2^{\frac{3}{4}}\,\epsilon^{\frac{1}{4}}$. Moreover, $C_{m\epsilon} \to \infty$ when $m \to \mu_2^+$; $C_{m\epsilon}$ has a minimum at a point $m = \mu_3$ with $C_{\mu_3\epsilon} = C_3(\epsilon)$ where $C_3(\epsilon)$ is the bifurcation value given in Lemma 7; and $C_{m\epsilon} \to \infty$ when $m \to \infty$ (see Figure 2 c)). We note that $\mu_1 < 4\,2^{\frac{6}{7}}\,\epsilon^{\frac{1}{7}} < \mu_2 < \mu_3$.

The number of central configurations coming from the solutions of (16) given by Proposition 8 is summarized in the following result.

Theorem 9 *Let $C_1 = C_1(\epsilon)$, $C_2 = C_2(\epsilon)$ and $C_3 = C_3(\epsilon)$, with $5 < C_1(\epsilon) \leqslant C_2(\epsilon) \leqslant C_3(\epsilon)$, be the bifurcation values given in Proposition 8; and let $\mu_i = \mu_i(\epsilon)$ be the values defined above. Fixed $\epsilon > 0$ and $m > 0$, the number of symmetric planar non–collinear central configurations of the Lennard–Jones 3–body problem (3) having moment of inertia $I = \frac{C}{2(2+m)}$ changes at $C = C_1 = C_1(\epsilon)$, $C = C_2 = C_2(\epsilon)$, $C = C_3 = C_3(\epsilon)$ and $C = C_m = C_{m\epsilon}$.*

(a) Fixed $\epsilon \in (0,2)$, the number (#) of symmetric planar non–collinear central configurations for $C > 5$ and $m > 0$ is summarized in Table 7. In this table, $\mu^* = \mu^*(\epsilon) > \mu_3$ is a value such that $C_{\mu^*\epsilon} = C_1$.

$m \in (0, \mu_1)$	C	$(5, C_1)$	C_1	(C_1, C_m)	$[C_m, \infty)$
	#	1	2	3	2
$m = \mu_1$	C	$(5, C_1)$	$C_1 = C_{\mu_1}$	(C_1, ∞)	
	#	1	1	2	
$m = (\mu_1, \mu_2)$	C	$(5, C_1)$	C_1	$(C_1, C_m]$	(C_m, ∞)
	#	1	1	1	2
$m = [\mu_2, \mu_3]$	C	$(5, C_1)$	C_1	(C_1, ∞)	
	#	1	1	1	
$m = (\mu_3, \mu^*)$	C	$(5, C_m]$	(C_m, C_1)	C_1	(C_1, ∞)
	#	0	1	1	1
$m = \mu^*$	C	$(5, C_1)$	$C_1 = C_{\mu^*}$	(C_1, ∞)	
	#	0	0	1	
$m = (\mu^*, \infty)$	C	$(5, C_1)$	C_1	$(C_1, C_m]$	(C_m, ∞)
	#	0	0	0	1

Table 7: The number (#) of central configurations fixed $\epsilon \in (0, 2)$.

(b) *If $\epsilon = 2$, then $\sqrt{\varepsilon} = \alpha = 2$. Therefore the number (#) of symmetric planar non–collinear central configurations for $C > 5$ and $m > 0$ is given by Theorem 6 (see Table 4).*

(c) *Fixed $\epsilon \in (2, 2.2494\ldots)$, the number (#) of symmetric planar non–collinear central configurations for $C > 5$ and $m > 0$ is summarized in Table 8. In this table, $\mu_1^* = \mu_1^*(\epsilon)$, $\mu_2^* = \mu_2^*(\epsilon)$ and $\mu_3^* = \mu_3^*(\epsilon)$ are values such that $\mu_1^* < \mu_2^* < \mu_2 < \mu_2 < \mu_3^* < \mu_4 < \mu_5$ and $C_{\mu_1^*\epsilon} = C_3$, $C_{\mu_2^*\epsilon} = C_2$ and $C_{\mu_3^*\epsilon} = C_1$.*

(d) *Fixed $\epsilon \geqslant 2.2494\ldots$, the number (#) of symmetric planar non–collinear central configurations for $C > 5$ and $m > 0$ is summarized in Table 9. In this table, $\mu^* = \mu^*(\epsilon) < \mu_1$ is a value such that $C_{\mu^*\epsilon} = C_3$.*

$m \in (0,\mu_1^*)$	C	$(5,C_1)$	C_1	(C_1,C_2)	C_2	(C_2,C_3)	C_3	(C_3,C_m)	$[C_m,\infty)$
	#	1	2	3	2	1	2	3	2

$m = \mu_1^*$	C	$(5,C_1)$	C_1	(C_1,C_2)	C_2	(C_2,C_3)	$C_3 = C_{\mu_1^*}$	(C_3,∞)
	#	1	2	3	2	1	1	2

$m = (\mu_1^*,\mu_2^*)$	C	$(5,C_1)$	C_1	(C_1,C_2)	C_2	(C_2,C_m)	$[C_m,C_3)$	C_3	(C_3,∞)
	#	1	2	3	2	1	0	1	2

$m = \mu_2^*$	C	$(5,C_1)$	C_1	(C_1,C_2)	$C_2 = C_{\mu_2^*}$	(C_2,C_3)	C_3	(C_3,∞)
	#	1	2	3	1	0	1	2

$m = (\mu_2^*,\mu_1)$	C	$(5,C_1)$	C_1	(C_1,C_m)	$[C_m,C_2)$	C_2	(C_2,C_3)	C_3	(C_3,∞)
	#	1	2	3	2	1	0	1	2

$m = \mu_1$	C	$(5,C_1)$	$C_1 = C_{\mu_1}$	(C_1,C_2)	C_2	(C_2,C_3)	C_3	(C_3,∞)
	#	1	1	2	1	0	1	2

$m = (\mu_1,\mu_2)$	C	$(5,C_1)$	C_1	$(C_1,C_m]$	(C_m,C_2)	C_2	(C_2,C_3)	C_3	(C_3,∞)
	#	1	1	1	2	1	0	1	2

$m = \mu_2$	C	$(5,C_1)$	C_1	(C_1,C_2)	$C_2 = C_{\mu_2}$	(C_2,C_3)	C_3	(C_3,∞)
	#	1	1	1	0	0	1	2

$m = (\mu_2,\mu_3^*)$	C	$(5,C_1)$	C_1	(C_1,C_m)	$[C_m,C_2)$	C_2	(C_2,C_3)	C_3	(C_3,∞)
	#	1	1	1	0	0	0	1	2

$m = \mu_3^*$	C	$(5,C_1)$	$C_1 = C_{\mu_3^*}$	(C_1,C_2)	C_2	(C_2,C_3)	C_3	(C_3,∞)
	#	1	0	0	0	0	1	2

$m = (\mu_3^*,\mu_3)$	C	$(5,C_m)$	$[C_m,C_1)$	C_1	(C_1,C_2)	C_2	(C_2,C_3)	C_3	(C_3,∞)
	#	1	0	0	0	0	0	1	2

$m = [\mu_3,\mu_4]$	C	$(5,C_1)$	C_1	(C_1,C_2)	C_2	(C_2,C_3)	C_3	(C_3,∞)
	#	0	0	0	0	0	1	2

$m = (\mu_4,\mu_5)$	C	$(5,C_1)$	C_1	(C_1,C_2)	C_2	(C_2,C_3)	C_3	(C_3,C_m)	$[C_m,\infty)$
	#	0	0	0	0	0	1	2	1

$m = \mu_5$	C	$(5,C_1)$	C_1	(C_1,C_2)	C_2	(C_2,C_3)	$C_3 = C_{\mu_5}$	(C_3,∞)
	#	0	0	0	0	0	0	1

$m = (\mu_5,\infty)$	C	$(5,C_1)$	C_1	(C_1,C_2)	C_2	(C_2,C_3)	C_3	$(C_3,C_m]$	$C_m,\infty)$
	#	0	0	0	0	0	0	0	1

Table 8: The number (#) of central configurations fixed $\epsilon \in (2, 2.2494\ldots)$.

$m \in (0, \mu^*)$	C	$(5, C_3)$	C_3	(C_3, C_m)	$[C_m, \infty)$
	#	1	2	3	2
$m = \mu^*$	C	$(5, C_3)$	$C_3 = C_{\mu^*}$	(C_3, ∞)	
	#	1	1	2	
$m = (\mu^*, \mu_1)$	C	$(5, C_m)$	$[C_m, C_3)$	C_3	(C_3, ∞)
	#	1	0	1	2
$m = [\mu_1, \mu_2]$	C	$(5, C_3)$	C_3	(C_3, ∞)	
	#	0	1	2	
$m = (\mu_2, \mu_3)$	C	$(5, C_3)$	C_3	(C_3, C_m)	$[C_m, \infty)$
	#	0	1	2	1
$m = \mu_3$	C	$(5, C_3)$	$C_3 = \mu_3$	(C_3, ∞)	
	#	0	0	1	
$m = (\mu_3, \infty)$	C	$(5, C_3)$	C_3	$(C_3, C_m]$	(C_m, ∞)
	#	0	0	0	1

Table 9: The number (#) of central configurations fixed $\epsilon \geqslant 2.2494\ldots$.

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