Monografías de la Real Academia de Ciencias de Zaragoza. **25:** 59–70, (2004).

On orbital stability of planar motions of symmetric satellites in cases of first and second order resonances

B. S. Bardin

Department of Theoretical Mechanics.
Moscow Aviation Institute (Technical University).
Volokolamskoe Shosse 4. 125871 Moscow, Russia.

Abstract

We consider a dynamically symmetric satellite, whose center of mass moves in a circular orbit. In the cases of the first and second order resonances we study orbital stability of planar periodic motions of the satellite about its center of mass.

By using normal form method we carry out a nonlinear analysis of the Hamiltonian of the problem and apply the stability theorem developed for the resonant cases. We represent the results in the form of stability diagrams.

Key words and expressions: Hamiltonian, orbital stability, normalization, resonances, action-angle variables.

MSC: 34D20, 34D35, 37C27.

1 Introduction

We consider a satellite, whose center of mass moves in a circular orbit. We assume that the satellite is a dynamically symmetric rigid body, i.e. its ellipsoid of inertia is an ellipsoid of revolution. The equations of motion of the symmetric satellite about its center of mass admit a family of particular solutions corresponding to the so-called planar motions, when the symmetry axis of the inertia ellipsoid moves in the orbital plane. The planar motions of the satellite are similar to the planar motions of a pendulum in the gravitational field, and so the most of them are periodic. The planar periodic motions are unstable with respect to planar perturbations of coordinates and velocities because their period depends on the initial conditions. Markeev [3] was the first one to study the so-called orbital stability of the planar periodic motions, i.e. the stability with respect to spatial perturbations and perturbations of the period. In particular, he studied the orbital stability of the planar periodic motions for an oblate dynamically symmetric satellite. Unfortunately, diagrams of stability obtained by Markeev contain some imperfections because of a low accuracy of numerical calculations. This was pointed out by Akulenko *et al.* [1].

A more complete investigation of the orbital stability of the above motions has been recently done by Markeev and Bardin [4]. They have studied the problem in rigorous nonlinear formulation for the most of parameter values with the exception of some special cases. In particular, they did not study the orbital stability of the planar periodic motions on boundaries of instability domains.

2 Formulation of the Problem

To describe the motion of the satellite about its center of mass, we introduce two orthonormal frames: orbital frame (axes OX, OY, OZ directed along the radius vector of the center of mass O, the transversal and normal of the orbit, respectively) and principal axes frame $Oxyz$ (axis Oz is directed along the symmetry axis of the inertia ellipsoid). The equations of motion can be written in the canonical form. We use the Euler angles ψ, θ, φ (Fig.1) and the corresponding dimensionless momenta $p_\psi, p_\theta, p_\varphi$ as canonical variables. The angle φ is a cyclic coordinate and, hence, $p_\varphi =$ constant is an integral of motion. In what follows we put $p_\varphi = 0$, when the planar motions are only possible. The Hamiltonian of the problem reads (see [3] for details)

$$H = \frac{1}{2\sin^2\theta}(p_\psi + 1)^2 - p_\psi + \frac{1}{2}p_\theta^2 + \frac{3}{2}(\alpha - 1)\sin^2\psi \sin^2\theta, \tag{1}$$

where $\alpha = C/A \quad (0 \leq \alpha \leq 2)$ and $A = B, C$ are the principal moments of inertia corresponding to the axes Ox, Oy, Oz, respectively. We use the true anomaly ν as the independent variable. For convenience, in (1), the momentum p_ψ is changed for momentum $p_\psi - 1$ which corresponds to the relative motion in the orbital frame.

The planar motions ($\theta = \pi/2$, $p_\theta = 0$) of the satellite are represented either by periodic motions (oscillations or rotations of the satellite symmetry axis in the orbital plane) or by a separatrix motion, which is a limiting motion for the oscillations and rotations. The partial solution describing the planar periodic motions can be expressed in terms of elliptic functions [4]. In the domains of the planar periodic motions the so-called action-angle variables I, w can be introduced. The explicit form of the transformation $\psi, p_\psi \to w, I$ is given in Appendix. In the phase space of the canonical variables w, I, θ, p_θ the planar periodic motions are represented by the following families of periodic orbits:

$$w = \omega(I_0)(\nu - \nu_0) + w_0, \quad I = I_0 = \text{constant},$$

$$\theta = \frac{\pi}{2}, \qquad p_\theta = 0,$$

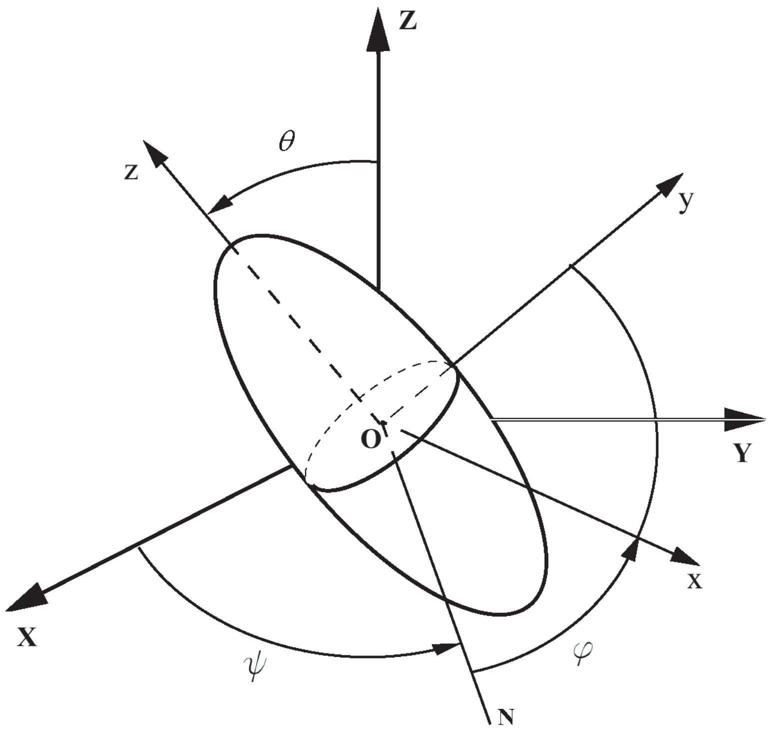

Figure 1: Frames of reference.

where $\omega(I_0)$ is the frequency of planar oscillations (or the averaged angular velocity of rotations). The value I_0 is a parameter of the above family.

We regard the orbital stability of the planar periodic motions as a nonlinear stability of the corresponding periodic orbit with respect to the spatial variables θ, p_θ and the value $I - I_0$.

The stability problem of the planar periodic motion of symmetric satellite has been considered by many authors from different points of view [1, 3, 4, 5], however, for some special cases it has not been solved yet. The aim of this paper is to study one of such cases. Namely, we study the problem of orbital stability for the parameter values corresponding to the first and second order resonances.

3 Isoenergetic Reduction

Let us introduce the following local canonical variables

$$r_1 = I - I_0, \quad q_2 = \theta - \frac{\pi}{2}, \quad p_2 = p_\theta.$$

and expand the Hamiltonian into a power series in r_1, q_2, p_2

$$H = H_2 + H_3 + H_4 + \dots. \tag{2}$$

In (2) the constant term is dropped. We denote by H_n $(n = 2, 3, 4, \dots)$ the terms whose order of smallness is equal to n. Taking into account that the variable r_1 has the second order of smallness as compared with q_2, p_2, we have the following explicit form for H_n ($n = 2,\ 3,\ 4$):

$$
\begin{aligned}
H_2 &= \omega(I_0)r_1 + \frac{1}{2}[p_2^2 + f_2(I_0, w)q_2^2], \\
H_3 &= 0, \\
H_4 &= \frac{1}{2}\frac{\partial \omega}{\partial I_0}r_1^2 + \frac{\partial f_2}{\partial I_0}r_1 q_2^2 + f_4(I_0, w)q_2^4.
\end{aligned} \tag{3}
$$

To avoid bulky formulae, we write down the expressions for f_2, f_4 with respect to variables ψ, p_ψ

$$
f_2 = (p_\psi + 1)^2 - 3(\alpha - 1)\sin^2\psi, \qquad f_4 = \frac{1}{3}(p_\psi + 1)^2 + \frac{1}{2}(\alpha - 1)\sin^2\psi. \tag{4}
$$

The explicit form of f_2, f_4 with respect to w, I can be obtained by substituting (22) (or (23)) into (4).

Hamiltonian (2) depends on two parameters, α and I_0. At fixed values of α and I_0 the equilibrium $r_1 = q_2 = p_2 = 0$ of the canonical system with Hamiltonian (2) corresponds to the planar periodic motion (oscillation or rotation) with $I = I_0$. The planar periodic motion is orbitally stable if and only if the equilibrium $r_1 = q_2 = p_2 = 0$ is stable in the sense of Liapunov .

The equations of motion possess integral of energy $H = h$, where h is a constant. We perform the so-called isoenergetic reduction of our system by considering the motion on the energy level $H = 0$. On the fixed energy level the evaluation of the variables q_2, p_2 can be described by the following system of the canonical equations (Whittaker equations)[6]

$$
\frac{dq_2}{dw} = \frac{\partial K}{\partial p_2}, \qquad \frac{dp_2}{dw} = -\frac{\partial K}{\partial q_2}. \tag{5}
$$

We obtain the Hamiltonian K of system (5) by solving the equation $H = 0$ with respect to r_1

$$
r_1 = -K = -K_2 - K_4 + O_6,
$$

where

$$
K_2 = \frac{1}{2\omega}(p_2^2 + f_2(I_0, w)q_2^2), \tag{6}
$$

$$
\begin{aligned}
K_4 &= \frac{1}{8\omega^3}\left(\frac{\partial \omega}{\partial I_0}f_2^2(I_0, w) - 2\omega\frac{\partial f_2}{\partial I_0}f_2(I_0, w) + 8\omega^2 f_4(I_0, w)\right)q_2^4 \\
&+ \frac{1}{4\omega^3}\left(\frac{\partial \omega}{\partial I_0}f_2(I_0, w) - \omega\frac{\partial f_2}{\partial I_0}\right)p_2^2 q_2^2 + \frac{1}{8\omega^3}\frac{\partial \omega}{\partial I_0}p_2^4.
\end{aligned} \tag{7}
$$

By O_6 we denote the terms of order six and higher with respect to canonical variables q_2, p_2. The functions $f_2(I_0, w)$ and $f_4(I_0, w)$ have period T with respect to w, where T is equal to 2π in the case of oscillations and π in the case of rotations.

The criterion for stability of the equilibrium $q_2 = p_2 = 0$ of system (5) and criterion for stability of the equilibrium $r_1 = q_2 = p_2 = 0$ of the canonical system with Hamiltonian (2) are equivalent. Taking this fact into account in what follows we consider system (5), which has one and a half degrees of freedom and is more convenient for the stability study than the system with Hamiltonian (2).

4 Linear System

We start our study from the analysis of the linear system with Hamiltonian (6). The characteristic equation of the linear system reads

$$\rho^2 - 2\varkappa\rho + 1 = 0, \tag{8}$$

where $2\varkappa = [q_2^{(1)}(T) + p_2^{(2)}(T)]$. The functions $q_2^{(i)}(w), p_2^{(i)}(w)$ $(i = 1, 2)$ are the solutions of the linear system satisfying the following initial conditions:

$$q_2^{(1)}(0) = p_2^{(2)}(0) = 1, \qquad q_2^{(2)}(0) = p_2^{(1)}(0) = 0. \tag{9}$$

If $|\varkappa| > 1$, then the equilibrium $q_2 = p_2 = 0$ is unstable. In this case from the instability of the equilibrium $q_2 = p_2 = 0$ of the linear system it follows that the corresponding planar periodic orbit is also unstable. The instability zones for both the planar rotations and oscillations have been constructed in [4].

In what follows we assume that values of the parameters α, I_0 belong to the boundaries of the instability zones. In this case $|\varkappa| = 1$ and equation (8) has multiple root $\rho = 1$ or $\rho = -1$. If $\rho = 1$, then characteristic exponents $\pm i\lambda$ satisfy the relation $\lambda = N$, where N is an integer. It means that the so-called first order resonance takes place. If $\rho = -1$, then the relation $2\lambda = 2N + 1$ is fulfilled, i.e. the second order resonance takes place. At $|\varkappa| = 1$ the equilibrium of the linear system is generally unstable, but the equilibrium of the corresponding nonlinear system can be both stable and unstable. This is the reason why in the case of the first and second order resonances we have to take into account nonlinear terms in the equations of motion to study the stability problem.

5 Normalization and Stability Conditions

The stability conditions can be written by using the coefficients of the Hamiltonian normal form. In this section we describe the normalization procedure and obtain the normal form of the Hamiltonian up to fourth order terms.

First we introduce new canonical variables u, v so that the quadratic part K_2 of the Hamiltonian takes the following normal form

$$K_2 = \sigma\frac{1}{2}v^2. \tag{10}$$

The linear transformation $q_2, p_2 \to u, v$ reads

$$q_2 = a_{11}u + a_{12}v, \qquad p_2 = a_{21}u + a_{22}v. \tag{11}$$

The coefficients a_{ij} are the elements of the matrix A. At the resonance of the first order the matrix A is T-periodic with respect to variable w and it can be calculated by means the following formula [2]

$$A = MPN, \tag{12}$$

where

$$M = \begin{pmatrix} q_2^{(1)}(w) & q_2^{(2)}(w) \\ p_2^{(1)}(w) & p_2^{(2)}(w) \end{pmatrix}, \qquad N = \begin{pmatrix} 1 & -\sigma w \\ 0 & 1 \end{pmatrix}. \tag{13}$$

If $q_2^{(2)}(T) \neq 0$, then the matrix P and constant σ read

$$P = \begin{pmatrix} \sqrt{\dfrac{|q_2^{(2)}(T)|}{T}} & 0 \\ \sigma \dfrac{p_2^{(2)}(T) - 1}{\sqrt{T|q_2^{(2)}(T)|}} & \sqrt{\dfrac{T}{|q_2^{(2)}(T)|}} \end{pmatrix}, \qquad \sigma = \operatorname{sgn}(q_2^{(2)}(T)), \tag{14}$$

and if $p_2^{(1)}(T) \neq 0$, then the matrix P and constant σ read

$$P = \begin{pmatrix} \sigma \dfrac{q_2^{(2)}(T) - 1}{\sqrt{T|p_2^{(1)}(T)|}} & \sqrt{\dfrac{T}{|p_2^{(1)}(T)|}} \\ -\sqrt{\dfrac{|p_2^{(1)}(T)|}{T}} & 0 \end{pmatrix}, \qquad \sigma = -\operatorname{sgn}(p_2^{(1)}(T)). \tag{15}$$

At the resonance of the second order the matrix A is $2T$-periodic with respect to variable w. It can be calculated by means of formulae (12)–(15), where the period T is replaced by $2T$.

In the variables u, v the form K_4 reads

$$K_4 = k_{40}u^4 + k_{31}u^3v + k_{22}u^2v^2 + k_{13}uv^3 + k_{04}v^4,$$

where the coefficients k_{ij} are periodic in w with the period T. We write down the explicit form only for the coefficient k_{40}

$$k_{40} = \frac{1}{8\omega^3}\left[8\omega^2 a_{11}^4 f_4(w, I_0) - 2\omega a_{11}^2 \frac{\partial f_2}{\partial I_0}\left(a_{11}^2 f_2(w, I_0) + a_{21}^2\right) + \frac{\partial \omega}{\partial I_0}\left(a_{11}^2 f_2(w, I_0) + a_{21}^2\right)^2\right].$$

The explicit forms of other coefficients are not necessary for the stability analysis.

To bring the Hamiltonian into the normal form up to the fourth order terms, we perform a periodic (with respect to w) smooth canonical transformation $u, v \to \xi, \eta$, which differs from the identity only by terms that are small of third order. In the new variables the Hamiltonian reads [2]

$$K = \frac{1}{2}\sigma\eta^2 + a\xi^4 + O_6, \tag{16}$$

where

$$a = \int_0^T k_{40} dw.$$

According to the stability theorem for the case of the first and second order resonance [2], the equilibrium $u = v = 0$ is stable if $\sigma a > 0$ and unstable if $\sigma a < 0$. At $a = 0$ we have to consider terms of order higher than fourth in Hamiltonian (16) to solve the stability problem.

6 Stability of the Planar Periodic Motions

In this section we apply the theorem mentioned for the stability study of the planar periodic motions of the satellite, when the parameters values correspond to the boundaries of instability zones. Let us start with the stability analysis of the planar oscillations. For

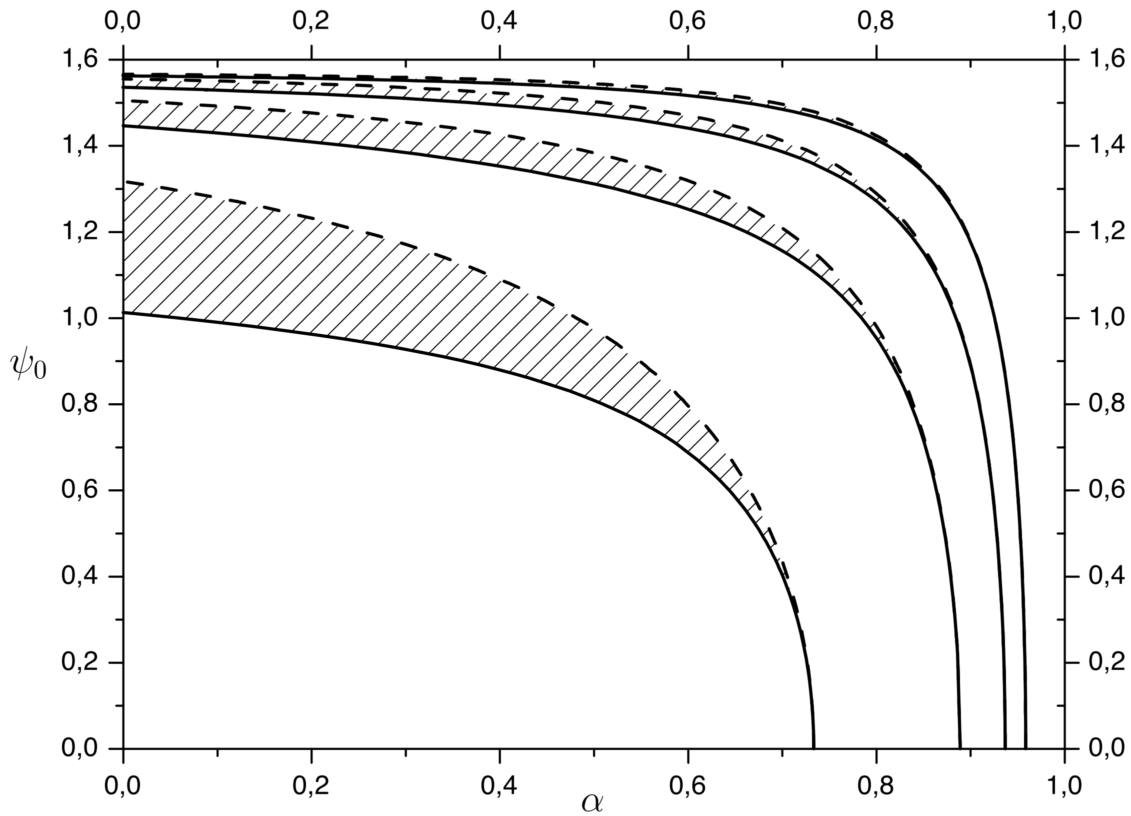

Figure 2: Stability diagram for the planar oscillations ($0 < \alpha < 1$).

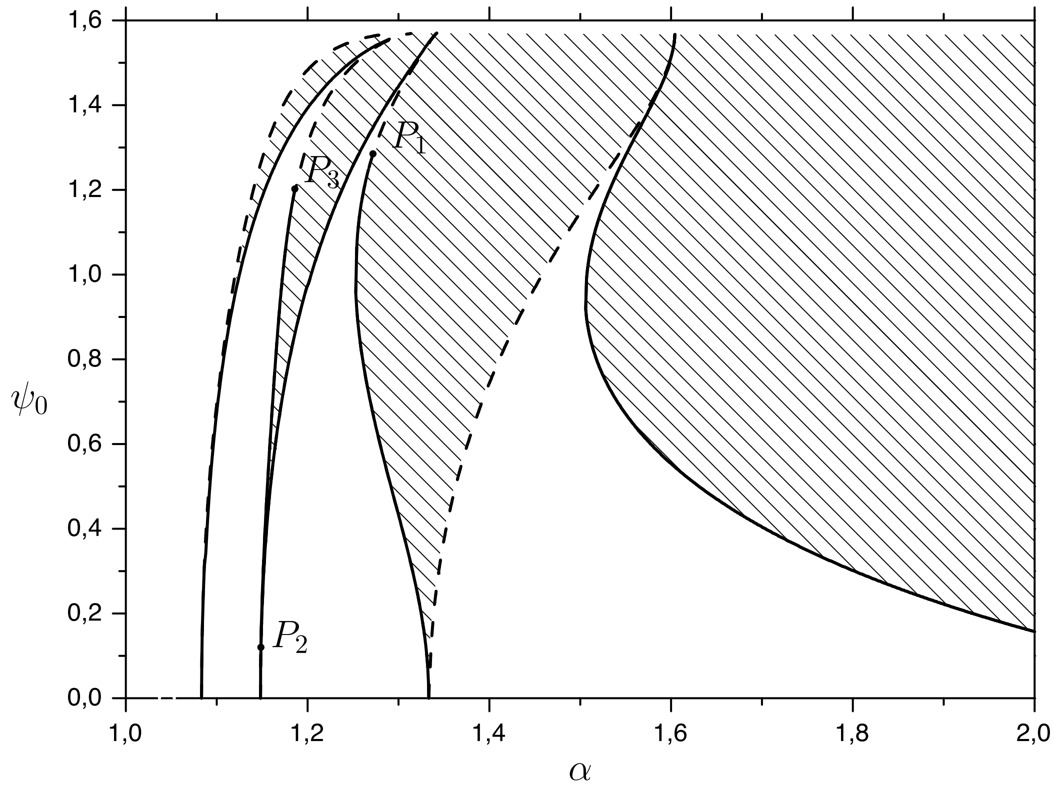

Figure 3: Stability diagram for the planar oscillations ($1 < \alpha < 2$).

convenience, we use the amplitude of oscillations ψ_0 instead of I_0 to parameterize the family of periodic orbits. The linear system with Hamiltonian (6) has been studied in [4] in detail. The results of the above study are as follows:

- In the plane of parameters α, ψ_0 there is an infinite set of instability zones (parametric resonance zones).

- The instability zones emanate from the following points of the axis α: $\alpha_n^{(1)} = 1 - 4/3(n^2+4n)$ for $\alpha < 1$ and $\alpha_n^{(2)} = 1+4/3(n+1)^2$ for $\alpha > 1$, $n \in \mathbb{N}$. For $n = 1, 2, 3, 4$ the instability zones are the shaded regions in Fig. 2 and Fig. 3.

- The instability zones become narrower and are located closer to the straight lines $\alpha = 1$ and $\psi_0 = \pi/2$ as n increases.

On the boundaries of the instability zones the resonances of the first and second order take place. In this case we solve the stability problem by calculating the coefficients σ, a and verifying the stability condition $\sigma a > 0$. The formulas for the coefficients σ and a contain the function $q_2^{(i)}(w)$ and $p_2^{(i)}(w)$ $(i = 1, 2)$. In general case, they can be calculated only numerically by integrating the canonical system with Hamiltonian (6). The right-hand sides of the above system have singularities at $\alpha = 1$ (the ellipsoid of inertia is a sphere) and at $\psi_0 = \pi/2$ (the separatrix motion). To avoid difficulties of the numerical integration, we omit the stability study on the boundaries of the instability zones located close to the straight lines $\alpha = 1$ and $\psi_0 = \pi/2$.

The results of the calculations are represented in Fig. 2 and Fig. 3. By solid and dashed curves are denoted the boundaries, where the planar oscillations are stable and unstable, respectively. In points P_1(1.2680,1.26206), P_2(1.1487,0.1186), P_3(1.1861,1.2014) the coefficient a of normal form (16) is equal to zero. These points separate stable parts of the corresponding boundaries from unstable ones. The point P_2 belongs to the right boundary of the instability domain.

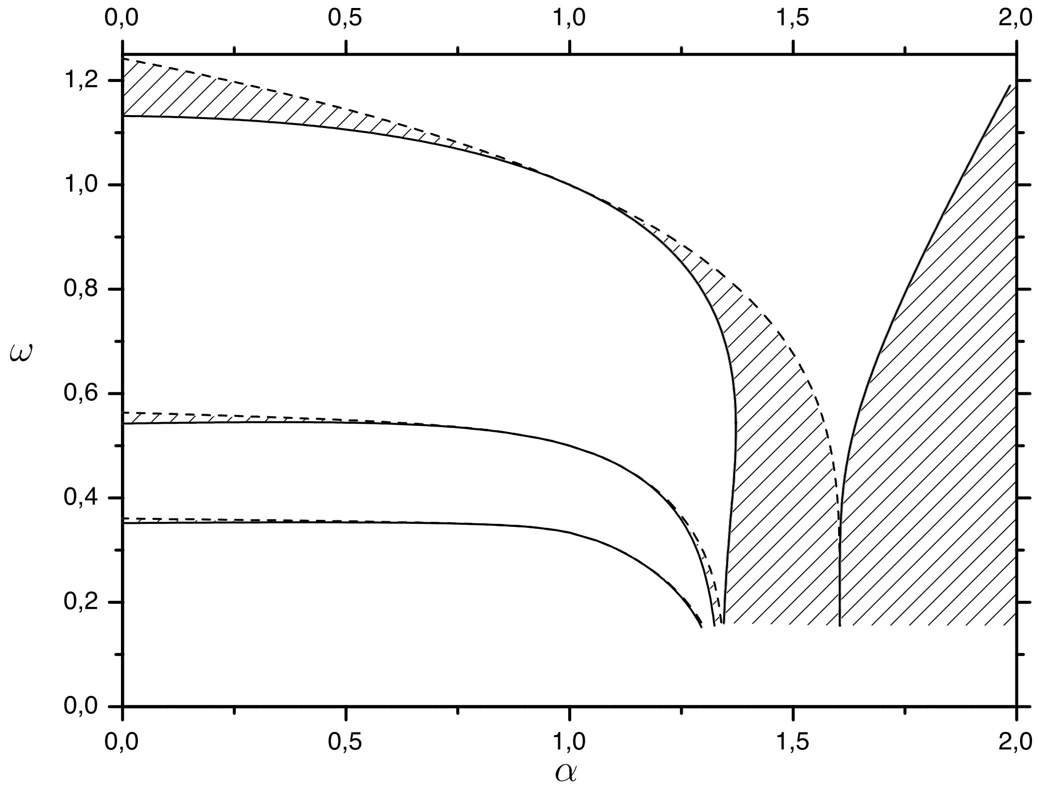

Figure 4: Stability diagram for the planar rotations ($\omega > 0$).

Following the paper [4] we use the average angular velocity ω to parameterize the family of the planar rotations. It is more natural and convenient than I_0. The linear study of stability of the planar rotations shows [4]:

- In the plane of parameters α, ω there is a countable set of instability zones (parametric resonance zones)
- The instability zones emanate from the points (1,1/n), $n \in \mathbb{Z}\backslash\{0\}$. For $n = \pm1, \pm2, \pm3$ the instability zones are the shaded regions in Fig. 4 and Fig. 5.

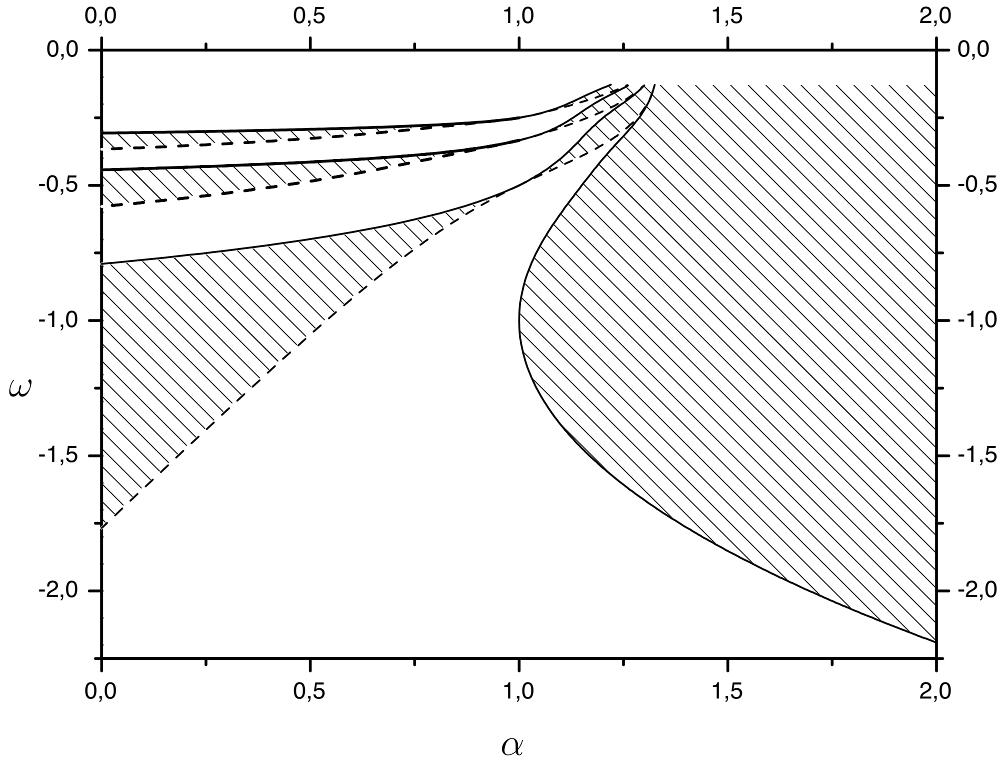

Figure 5: Stability diagram for the planar rotations ($\omega < 0$).

- The instability zones become narrower and are located closer to the axis α as n increases.

By analogy with the case of oscillations, we studied the stability problem at the boundaries of the instability zones where resonances of the first and second order take place. The results of this study are shown in Fig. 4 and Fig. 5. As above, solid curves denote stable boundaries and dashed curves denote unstable boundaries.

Appendix

If $2h_0 < 3|\alpha - 1|$, then the planar motions of the satellite involve oscillations of the symmetry axis of the inertia ellipsoid, which are described by the following formulas:

$$\begin{aligned} \psi &= \arcsin\{k_1 \mathrm{sn}[\sqrt{3|\alpha-1|}(\nu-\nu_0)+K(k_1);k_1]\} + \frac{\pi}{4}(\mathrm{sgn}(\alpha-1)-1), \\ p_\psi &= k_1\sqrt{3|\alpha-1|}\mathrm{cn}[\sqrt{3|\alpha-1|}(\nu-\nu_0)+K(k_1);k_1], \end{aligned} \tag{17}$$

where $K(k_1)$ is the complete elliptic integral of the first kind, whose modulus k_1 is calculated by the following formula

$$k_1^2 = \frac{2h_0}{3|\alpha-1|}. \tag{18}$$

Let us note that $k_1 = \sin\psi(\nu_0)$. Without loss of generality we can assume that $\psi(\nu_0) = \psi_0$, where ψ_0 is the amplitude of oscillations.

If $2h_0 > 3|\alpha - 1|$, then the planar motions of the satellite involve oscillations of the symmetry axis of the inertia ellipsoid, which are described by the following formulas:

$$\begin{aligned} \psi &= \mathrm{am}\left[\frac{\sqrt{3|\alpha-1|}}{k_2}(\nu-\nu_0) + F(\psi_0, |k_2|); |k_2|\right], \\ p_\psi &= \frac{\sqrt{3|\alpha-1|}}{k_2}\mathrm{dn}\left[\frac{\sqrt{3|\alpha-1|}(\nu-\nu_0)}{k_2} + F(\psi_0, |k_2|); |k_2|\right], \end{aligned} \tag{19}$$

where $F(k_2, \psi_0)$ is the elliptic integral of the first kind. Its modulus k_2 satisfies the following relation:

$$k_2^2 = \frac{3|\alpha-1|}{2h_0}. \tag{20}$$

If the rotation of the satellite with respect to orbital frame and the motion of its center of mass have the same direction, then k_2 is positive; otherwise k_2 is negative. The frequency of oscillations and the averaged angular velocity of rotations read, respectively

$$\omega_1 = \frac{\pi\sqrt{3|\alpha-1|}}{2K(k_1)}, \qquad \omega_2 = \frac{\pi\sqrt{3|\alpha-1|}}{2k_2K(|k_2|)}. \tag{21}$$

In the case of the planar oscillations, the action-angle variables w, I are introduced by the following formulas:

$$\psi = \arcsin\left\{k_1\mathrm{sn}\left[\frac{2K(k_1)}{\pi}w; k_1\right]\right\}, \qquad p_\psi = k_1\sqrt{3|\alpha-1|}\mathrm{cn}\left[\frac{2K(k_1)}{\pi}w; k_1\right]. \tag{22}$$

In the case of the planar rotations, the above transformation reads

$$\psi = \mathrm{am}\left[\frac{2K(|k_2|)}{\pi}w; |k_2|\right], \qquad p_\psi = \frac{\sqrt{3|\alpha-1|}}{k_2}\mathrm{dn}\left[\frac{2K(|k_2|)}{\pi}w; |k_2|\right], \tag{23}$$

where $k_i (i = 1, 2)$ and the action I are connected by the following relations:

$$I = \frac{2}{\pi}\sqrt{3|\alpha-1|}[E(k_1) - (1-k_1^2)K(k_1)], \qquad I = \frac{2\sqrt{3|\alpha-1|}}{\pi k_2}E(|k_2|). \tag{24}$$

Acknowledgments

This work was partially supported by the Russian Foundation for Basic Research, grant No. 02-01-00831, Project # BFM2002-03157 of Ministerio de Ciencia y Tecnología (Spain) and Project Resolución 92/2002 of Departamento de Educación y Cultura, Gobierno de Navarra (Spain). The author would like to thank Professor A.P. Markeev, Institute for Problems of Mechanics, Russian Academy of Sciences, for useful discussions.

References

[1] Akulenko, L. D., Nesterov, S. V. and Shmatkov, A. M.: 1999, ‘Generalized parametric oscillations of the mechanical systems’, *Prikladnaya Matematika i Mekhanika* **63,** 746–756.

[2] Ivanov, A. P. and Sokol’skii, A. G.: 1980, ‘On stability of non-autonomous system in the case of parametric resonance of fundamental type’, *Prikladnaya Matematika i Mekhanika* **44,** 963–970.

[3] Markeev, A. P.: 1975, ‘Stability of planar oscillations and rotations of a satellite in a circular orbit’, *Kosmicheskie Issledovaniia* **13,** 322–336.

[4] Markeev, A. P. and Bardin, B. S.: 2003, ‘On the stability of planar oscillations and rotations of a satellite in a circular orbit’, *Celestial Mechanics and Dynamical Astronomy* **85,** 51–66.

[5] Neishtadt, A. I., Simó, C. and Sidorenko, V. V.: 2000, ‘Stability of long-period planar satellite motions in a circular orbit’, *Proceedings of the US/European Celestial Mechanics Workshop*, Poznań, Poland, 227–233.

[6] Whittaker, E. T.: 1927, *A Treatise on Analytical Dynamics of Particles and Rigid Bodies*, University Press, Cambridge.