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Quadratic Hamiltonians on pinched spheres

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Abstract

Among other dynamical systems, two normalized perturbed elliptic oscillators in resonance 1:1 may be represented by a parametric quadratic form in a set of Cartesian variables whose Poisson structure is the one of the rigid body and the topology of the phase space is the one of the S^2 sphere. However, when the resonance is p:q, the Poisson structure is different and the phase portrait lies on a "pinched" sphere.

In this communication, we show how the loss of the spherical symmetry affects the phase flow and parametric bifurcations for the bi-parametric cases.

Key words and expressions: Hamiltonians, resonances, phase spaces, equilibria, bifurcations.

MSC: 37J20, 37J35, 70K30.

1 Introduction

There are some problems in Dynamics represented by a Hamiltonian that is a quadratic function in some variables that lie on the S^2 sphere, which presents several advantages, among them that a sphere is simple to represent and that the phase portrait may be obtained as the intersection of the two surfaces, the quadratic function and the sphere, with no need of numerical integration for several values of the energy constant. Among these systems, the motion of the rigid body in free rotation is one of the simplest [5], although in the classical literature an ellipsoid of variable size is preferred to the sphere. Indeed, by denoting by g_1, g_2 and g_3 the components of the angular momentum $\boldsymbol{G}$ in the body frame, there results that its norm is an integral, thus we have the sphere

$$g_1^2 + g_2^2 + g_3^2 = G^2,$$

and the energy —or the Hamiltonian in this case— is the quadratic form

$$\mathcal{H} = \frac{1}{2}(a_1 g_1^2 + a_2 g_2^2 + a_3 g_3^2),$$

where a_i are the reciprocal of the moments of inertia. In this case, the quadratic form is very simple: a triaxial ellipsoid. This problem has a Poisson structure given by the relations

$$\{g_1, g_2\} = -g_3, \qquad \{g_2, g_3\} = -g_1, \qquad \{g_3, g_1\} = -g_2. \tag{1}$$

Quadratic Hamiltonians on the S^2 sphere also appear as a representation of the phase flow of perturbed elliptic oscillators in resonance 1:1 obtained after a normalization. It was proven [19, 3] that the reduced phase space in Hopf coordinates is a sphere. In this context, many problems of Galactic Dynamics [2, 4, 9], Cosmology [1], Molecular Physics [10, 11, 12], etc. belong to the class of Hamiltonians of the type

$$\mathcal{H} = \mathcal{H}(u, v, w; \boldsymbol{\alpha}),$$

with the variables on the sphere

$$S^2 \equiv u^2 + v^2 + w^2 = 1 \tag{2}$$

and the Poisson structure

$$\{u, v\} = w, \quad \{v, w\} = u, \quad \{w, u\} = v, \tag{3}$$

where $\boldsymbol{\alpha} \in \mathbb{R}^n$ are real parameters. Obviously, Eqs. (1) and (3) represent the same symplectic structure.

Lanchares and Elipe [15] and later on Frauendiener [13] proceeded to classify all possible parametric quadratic Hamiltonians

$$\mathcal{H} = Au^2 + Bv^2 + Cw^2 + 2Fuv + 2Guw + 2Hvw + Lu + Mv + Nw \tag{4}$$

with the above Poisson structure (3). By virtue of the spherical symmetry, the Hamiltonian and the Poisson brackets enjoy properties like the invariance by translations, rotations and time reversing, which reduce the possibilities to only four cases:

Cases depending on the number of parameters	
1	$\mathcal{H} = \frac{1}{2}u^2 + Rv$
2.a	$\mathcal{H} = \frac{1}{2}u^2 + Qu + Rv$
2.b	$\mathcal{H} = \frac{1}{2}u^2 + \frac{1}{2}Pv^2 + Qu$
3	$\mathcal{H} = \frac{1}{2}u^2 + \frac{1}{2}Pv^2 + Qu + Rv$
4	$\mathcal{H} = \frac{1}{2}u^2 + \frac{1}{2}Pv^2 + Qu + Rv + Sw$

The determination of parametric bifurcations and the evolution of the phase flow for cases 1, 2.a, 2.b and 3 was done by Elipe and coworkers [16, 17, 18].

Incidentally, let us mention that the problem of the gyrostat in free motion is the most general case belonging to this kind of Hamiltonians [8]. Indeed, the expression of the Hamiltonian for this problem is

$$\mathcal{H} = \frac{1}{2}(a_1 g_1^2 + a_2 g_2^2 + a_3 g_3^2) - (a_1 f_1 g_1 + a_2 f_2 g_2 + a_3 f_3 g_3)$$

where g_i, $(i = 1, 2, 3)$ are the components of the total angular momentum, and f_i the components of the angular moment of the rotors, both expressed in the system of principal axes of inertia. In this system the angular momentum vector is an integral, and the Poisson structure is the given in (1).

Hamiltonians of the class (4) on the sphere with the Poisson structure (3) appear also in several physical problems, and among them, the problem of normalized harmonic oscillators in resonance 1:1. But, there are problems (for instance, the motion of a star under the field of an elliptic galaxy) where the resonance is p:q, with p, q coprime integers. For this resonant case the topology of the phase flow [14, 7] is a one-pinched sphere if $p = 1, q \neq 1$, a two-pinched sphere if both $p, q \neq 1$ and a sphere when $p, q = 1$, see Fig. 1. In fact, these surfaces are given by the equation:

$$C_2^2 + S_2^2 = (M_1 + M_2)^q \, (M_1 - M_2)^p \tag{5}$$

where M_1 is an integral (that from here on will be taken equal to the unit) and the three coordinates $(C_2, S_2, M_2) \in \mathbb{R}^3$. The Poisson structure is given by the relations:

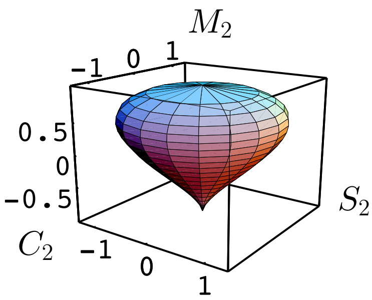
$$\{M_2; C_2\} = pqS_2, \quad \{C_2; S_2\} = pq\, f(M_2), \quad \{S_2; M_2\} = pqC_2, \tag{6}$$

with $f(M_2) = \frac{1}{2}(M_1 + M_2)^{q-1}(M_1 - M_2)^{p-1}[(q + p)M_2 + (p - q)M_1]$.

In this case there is no spherical symmetry but just axial symmetry about the M_2 axis and, consequently, the classification of the possible cases depending on the number of parameters is more complex than in the classical case and will appear elsewhere. Notwithstanding with this classification, we will present how this lack of symmetry affects the parametric bifurcations in some particular cases. In this communication we will only consider the biparametric case. A more complete analysis is in progress.

2 Biparametric Case: Two Known Examples

For quadratic Hamiltonians on the sphere (2) with the Poisson structure (1), Lanchares and Elipe [16, 17] proved that there are two and only two independent Hamiltonians, the



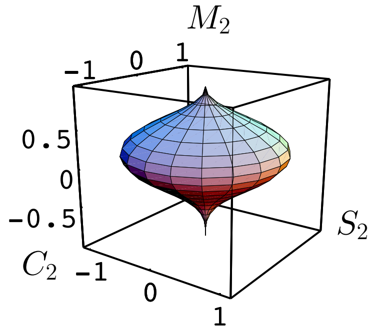


Figure 1: Phase space: pinched spheres. Left $p = 1, q - 3$; right $p = 2, q = 3$. By meridian we mean the intersection of the surface with any of the planes $C_2 = 0$ or $S_2 = 0$. By equator, we mean the intersection of the surface with the plane $M_2 = 0$.

ones dubbed as 2.a and 2.b in the above table. Now, a question is in order. What happens when the quadratic Hamiltonian is on the revolution surface (5) with the structure (6) with $1 \leq p < q$?

For the Hamiltonians,

$$\mathcal{H} = \tfrac{1}{2}C_2^2 + \tfrac{1}{2}AS_2^2 + BC_2, \tag{7}$$

$$\mathcal{H} = \tfrac{1}{2}C_2^2 + AC_2 + BS_2, \tag{8}$$

some of the results here obtained are analogous to the ones studied in [16, 17], but there are more Hamiltonian cases and different bifurcations, as we will show.

By analogy with the sphere, we will denote North and South Poles to the points $(0, 0, +1)$ and $(0, 0, -1)$, respectively. By equator we mean here the maximum circle perpendicular to the axis of rotation M_2; its radius is R, with

$$R^2 = 2^{p+q}\left(\frac{p}{p+q}\right)^p\left(\frac{q}{p+q}\right)^q.$$

2.1 Case $\mathcal{H} = \frac{1}{2}C_2^2 + \frac{1}{2}AS_2^2 + BC_2$

For the first case, the equations of motion are

$$\dot{C}_2 = \{\mathcal{H}\,;\,C_2\} = -\frac{1}{2}A(1 - M_2)^{-1+p}(1 + M_2)^{-1+q}\Big[p(1 + M_2) - q(1 - M_2)\Big]pqS_2,$$

$$\dot{S}_2 = \{\mathcal{H}\,;\,S_2\} = -\frac{1}{2}(B + C_2)(1 - M_2)^{-1+p}(1 + M_2)^{-1+q}\Big[p(1 + M_2) - q(1 - M_2)\Big]pq,$$

$$\dot{M}_2 = \{\mathcal{H}\,;\,M_2\} = -(B + C_2 - AC_2)S_2.$$

We are looking for equilibria of this system with the constraint (5) since the points must be on the surface.

For $A = 0$ and $C_2 = -B$, the above three equations vanish; therefore the closed curve

$$\begin{cases} C_2 = -B, \qquad S_2^2 = -B^2 + (1+M_2)^q(1-M_2)^p, \\ \text{and} \quad -R \leq B \leq +R \end{cases}$$

is made of equilibria, resulting, hence, a degeneracy. Note that the bound R for the value B comes from the fact that the both C_2, S_2 are, at most, equal to the radius R above defined.

On the other hand, for $A = 1$, $B = 0$, the equator

$$M_2 = -\frac{p-q}{p+q}, \qquad C_2^2 + S_2^2 = R^2,$$

is made of equilibria.

Besides these particular cases, the isolated equilibria are of four kinds, namely, on the equator and $S_2 = 0$

$$E_{2,4} = \left(\pm R, 0, -\frac{p-q}{p+q} \right);$$

on the equator and $S_2 \neq 0$

$$E_e = \left(\frac{B}{A-1}, S_{2,e}, -\frac{p-q}{p+q} \right);$$

with

$$\left| \frac{B}{A-1} \right| < R,$$

and $S_{2,e}$ the two roots of the equation

$$S_{2,e}^2 = R^2 - \left(\frac{B}{A-1} \right)^2.$$

On the meridian $E_m = (-B, 0, M_{2,m})$, with $|B| < R$, and $M_{2,m}$ the roots of the equation

$$(1+M_2)^q \, (1-M_2)^p = B^2.$$

Lastly, the south pole $E_s = (0, 0, -1)$ is equilibrium everywhere since it is a singular point in the phase space.

The bifurcation lines are quite similar to the classical case ($p = q = 1$), but now the value R plays a important role, as we can see in Figure 2. Note that in the classical case, $R = 1$. In this figure, the parallel lines correspond to $B = \pm R$ and the crossing lines to $B = \pm R\,(A-1)$. In the particular case $p = q = 1$ we have the same parametric plane obtained by Lanchares and Elipe [17].

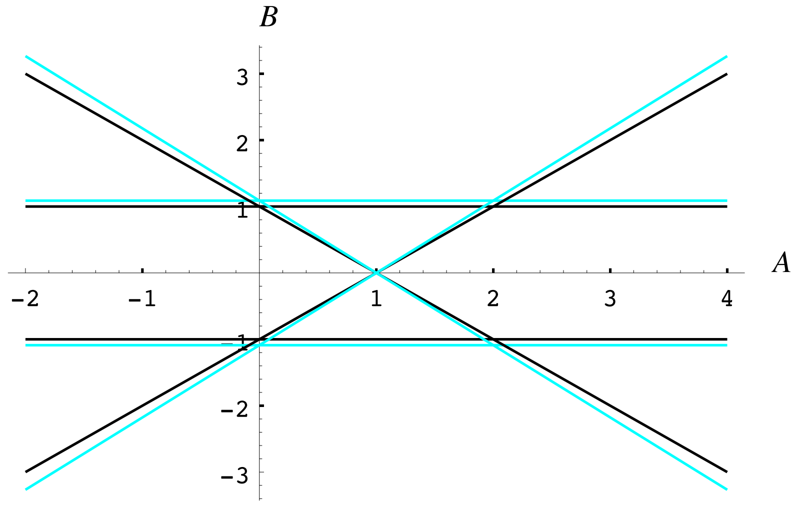


Figure 2: Bifurcation lines in the parametric plane AB for the Hamiltonian (7). In black the classical case (3); in grey the case (6). In the latter, the vertical segment $A = 0,\ |B| \leq R$, corresponds to non-isolated equilibria.

2.2 Case $\quad \mathcal{H} = \frac{1}{2}C_2^2 + AC_2 + BS_2$

Now, the equations of motion are

$$\dot{C}_2 = \{\mathcal{H}\,;\,C_2\} = -\tfrac{1}{2}B(1-M_2)^{-1+p}(1+M_2)^{-1+q}[p(1+M_2) - q(1-M_2)]pq,$$

$$\dot{S}_2 = \{\mathcal{H}\,;\,S_2\} = \tfrac{1}{2}(A+C_2)(1-M_2)^{-1+p}(1+M_2)^{-1+q}[p(1+M_2) - q(1-M_2)]pq,$$

$$\dot{M}_2 = \{\mathcal{H}\,;\,M_2\} = pq(BC_2 - (A+C_2)S_2).$$

For this system, together with the constraint (5), we find that the North and South Poles $(0,0,\pm 1)$ make zero the system (provided $p > 1$, otherwise the point $(0,0,+1)$ is not solution).

When the parameter $B = 0$, we have two types of equilibria.

On the one hand, the points on the equator

$$\left(\pm R, 0, -\frac{p-q}{p+q}\right),$$

are equilibria and, on the other, all points on the closed curve

$$\begin{cases} C_2 = -A, \quad S_2^2 = -A^2 + (1+M_2)^q(1-M_2)^p, \\ \text{with} \quad -R \leq A \leq +R \end{cases}$$

are equilibria. We have a degeneracy.

If $B \neq 0$, there are equilibria on the equator $M_2 = -(p-q)/(p+q)$. Indeed, for this value of M_2, the first two equations of the motion vanish, and the third one is equal to zero for $S_2 = BC_2/(A + C_2)$. But the point must be on the surface (5), hence, must be a root of the quadratic equation

$$B^2 C_2^2 + (A + C_2)^2 (C_2^2 - R^2) = 0. \tag{9}$$

For the classical case ($p = q = 1$), Lanchares and Elipe [16] obtained that the above equation (with $R^2 = 1$) had either 2 or 3 or 4 roots (equivalently there were 2, 3 or 4 equilibria) depending on whether the values of the parameters A and B are, respectively, in the interior, on the curve or the exterior of the astroid $A^{2/3} + B^{2/3} = 1$.

By the same token, one can prove that for the resonance p:q, the bifurcation curve on the parametric plane A, B is the astroid or the four-cusps-hypocycloid represented in Fig. 3 and given by

$$A^{2/3} + B^{2/3} = R^{2/3}$$

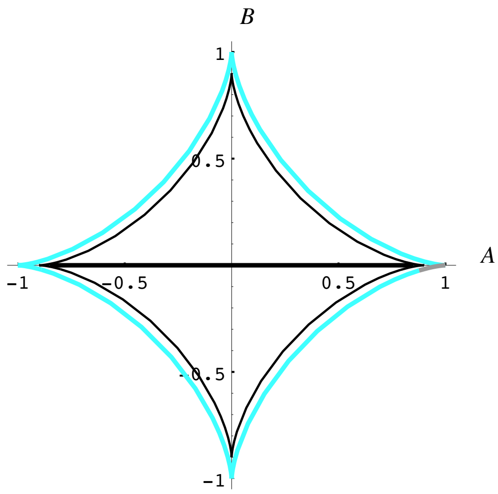


Figure 3: Bifurcation lines in the parametric plane A, B for the Hamiltonian (8). In black the classical case (3); in grey the case (6). The horizontal segment $B = 0$, $-R \leq A \leq R$ corresponds to the degenerate case (non-isolated equilibria).

3 Biparametric Case: A New Example

Let us take, for instance, the following Hamiltonian:

$$\mathcal{H} = \frac{1}{2} C_2^2 + \frac{1}{2} A S_2^2 + B M_2. \tag{10}$$

Since there is spherical symmetry neither on the surface (5) nor in the Poisson structure (6), this problem is not equivalent to the previous ones. Hence, one should expect that the number of equilibria and the partition of the parametric plane would be different to any of the above cases.

The equations of the motion are now

$$\dot{C}_2 = \frac{pq}{2(1-M_2^2)}\Big\{2B(1-M_2^2) - A(1-M_2)^p(1+M_2)^q\Big[p(1+M_2) - q(1-M_2)\Big]\Big\}S_2,$$
$$\dot{S}_2 = -\frac{pq}{2(1-M_2^2)}\Big\{2B(1-M_2^2) + (1-M_2)^p(1+M_2)^q\Big[p(1+M_2) - q(1-M_2)\Big]\Big\}C_2,$$
$$\dot{M}_2 = (-1+A)\,pq\,C_2S_2.$$

Since for the time being we are interested in showing the differences of this new Hamiltonian with the previous ones, we restrict ourselves to the case $p=1$, $q=2$. In that case, the equations of motion are reduced to

$$\dot{C}_2 = ((1-2M_2-3M_2^2)A+2B)S_2,$$
$$\dot{S}_2 = -(1-2M_2-3M_2^2+2B)C_2,$$
$$\dot{M}_2 = 2(-1+A)C_2S_2.$$

Both North and South poles $(0,0,\pm1)$ make zero the equations; thus they are equilibria everywhere.

The meridian $S_2=0$ contains two equilibria. The coordinate M_2 is given by the roots of the second degree equation $1-2M_2-3M_2^2+2B=0$, that is,

$$M_2^{(0+)} = \frac{1}{3}(-1+\sqrt{4+6B}), \qquad \text{only for} \qquad -\frac{2}{3}\le B\le 2,$$
$$M_2^{(0-)} = \frac{1}{3}(-1-\sqrt{4+6B}), \qquad \text{only for} \qquad -\frac{2}{3}\le B\le 0,$$

and C_2 is determined from the equation of the surface (5).

The meridian $C_2=0$ contains other two equilibria. M_2 is given by the roots of the equation $(1-2M_2-3M_2^2)A+2B=0$, that is,

$$M_2^{(0+)} = \frac{1}{3}(-1+\sqrt{4+6B/A}), \qquad \text{only for} \qquad -\frac{2}{3}\le B/A\le 2,$$
$$M_2^{(0-)} = \frac{1}{3}(-1-\sqrt{4+6B/A}), \qquad \text{only for} \qquad -\frac{2}{3}\le B/A\le 0,$$

and S_2 is obtained by solving the equation (5) for $S_2=0$ and M_2 each one of the above roots.

But for the particular case $A=1$, there results that the two parallels

$$M_2^{(1+)} = \frac{1}{3}(-1+\sqrt{4+6B}), \qquad \text{only for} \qquad -\frac{2}{3}\le B\le 2,$$
$$M_2^{(1-)} = \frac{1}{3}(-1-\sqrt{4+6B}), \qquad \text{only for} \qquad -\frac{2}{3}\le B\le 0,$$

and C_2 and S_2 on the corresponding minor circle, make zero the equations; thus we have either none, one or two circles of no-isolated equilibria.

The partition of the parametric plane AB and the number of equilibria in each region are shown in Figure 4.

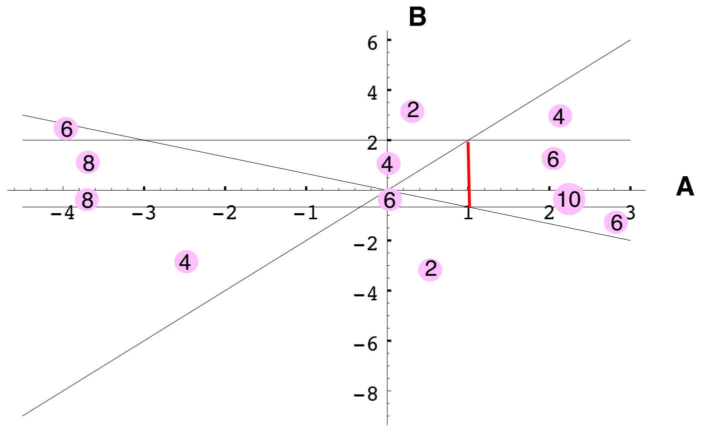


Figure 4: Parametric bifurcations for the Hamiltonian (10) and $p = 1$, $q = 2$. The vertical segment $A = 1$, $-2/3 \leq B \leq 2$ corresponds to the degenerate cases of non-isolated equilibria.

4 Conclusions

For parametric quadratic Hamiltonians with the Poisson structure (6), the loss of spherical symmetry with respect to the classical structure (3), makes that the number of independent parametric cases be different from the classical case. In the same manner, the number of equilibria and the parametric bifurcations in the phase space are different.

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References

[1] Blanco, S., Domenich, G. and Rosso, O.A.: 1994, 'Chaos in classical cosmology (II)', *General Relativity and Gravitation* **27,** 1295–1307.

[2] Contopoulos, G. and Moutsoulas, M.: 1966, 'Resonance cases and small divisors in a third integral of motion II', *The Astronomical Journal* **70,** 687–698.

[3] Deprit, A.: 1991, 'The Lissajous transformation I. Basics', *Celestial Mechanics* **51,** 201–225.

[4] Deprit, A. and Elipe, A.: 1991, 'The Lissajous transformation II. Normalization', *Celestial Mechanics* **51,** 227–250.

[5] Deprit, A. and Elipe, A.: 1993, 'Complete reduction of the Euler-Poinsot problem', *The Journal of Astronautical Sciences* **51,** 603–628.

[6] Deprit, A. and Elipe, A.: 1999, 'Oscillators in resonance', *Mechanics Research Communications* **26,** 635–640.

[7] Elipe, A.: 2000, 'Complete reduction of oscillators in resonance $p{:}q$', *Physical Review E* **61**, 6477–6484.

[8] Elipe, A., Arribas, M. and Riaguas, A.: 1997, 'Complete analysis of bifurcations in the axial gyrostat problem', *Journal of Physics A* **30**, 587–601.

[9] Elipe, A., Miller, B. R. and Vallejo, M.: 1995, 'Bifurcations in a non symmetric cubic potential', *Astronomy and Astrophysics* **300**, 722–725.

[10] Farrelly, D.: 1986, 'Lie algebraic approach to quantization of nonseparable system with internal nonlineal resonance', *Journal of Chemical Physics* **85,** 2119–2131.

[11] Farrelly, D. and Uzer, T.: 1986, 'Semiclassical quantization of slightly nonresonant systems: Avoided crossings, dynamical tunneling, and molecular spectra', *Journal of Chemical Physics* **85,** 308–318.

[12] Fermi, E., Pasta, J. and Ulam, S.: 1955, 'Studies of nonlinear problems I', *Los Alamos Scientific Laboratory Report* **LA-1940** 20 pp.

[13] Frauendiener, J.: 1995, 'Quadratic Hamiltonians on the unit sphere', *Mechanics Research Communications* **22,** 313–317.

[14] Kummer, M.: 1984, 'On resonant Hamiltonian systems with finitely many degrees of freedom'. Eds. A. V. Sáenz, W. W. Zachary and R. Cawley, *Local and Global Methods in Nonlinear Dynamics*, *Lecture Notes in Physics* **252,** 19–31.

[15] Lanchares, V. and Elipe, A.: 1994, 'Biparametric quadratic Hamiltonians on the unit sphere: complete classification', *Mechanics Research Communications* **21,** 209–214.

[16] Lanchares, V. and Elipe, A.: 1995, 'Bifurcations in biparametric quadratic potentials', *Chaos* **5,** 367–373.

[17] Lanchares, V. and Elipe, A.: 1995, 'Bifurcations in biparametric quadratic potentials. II', *Chaos* **5,** 531–535.

[18] Lanchares, V., Iñarrea, M., Salas, J.P., Sierra, J.D. and Elipe, A.: 1995, 'Surfaces of bifurcation in a triparametric quadratic Hamiltonian.', *Physical Review E* **52,** 5540–5548.

[19] Moser, J.: 1970, 'Regularization of Kepler's problem and the averaging method on a manifold', *Communications on Pure and Applied Mathematics* **XXIII,** 609–636.